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Litvak, Nelly

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Collecting n Items Randomly Located on a Circle

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prof.dr. J. Wessels
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prof.dr. W.H.M. Zijm

Copromotor:
dr.ir. I.J.B.F. Adan

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Contents

Chapter 1

Introduction

This work is devoted to the following problem. Consider n items located randomly on a circle of length 1. The locations of the items are assumed to be independent and uniformly distributed on $[0, 1)$. A picker starts at point 0, and he has to collect all the n items moving along the circle at unit speed in either direction. The illustration of this model and one of the possible routes are displayed in Figure 1.1.

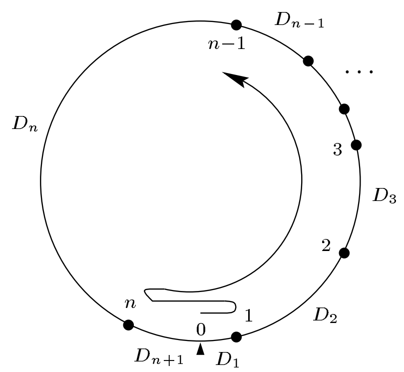

Figure 1.1: Collecting n items on a circle

We are interested in the travel time of the picker. Obviously, the travel time highly depends on the pick strategy. For example, if the picker just proceeds in the same arbitrarily chosen direction (say, clockwise), then the distribution function $F_{CW}(t)$ of the travel time simply equals t^n, $0 < t \leq 1$. However, we would like to study strategies that provide smaller travel times. In this sense, a better algorithm that one can think of is the 'greedy' strategy, or the *Nearest Item (NI)* heuristic: always travel to the nearest item to be picked (as in Figure 1.1). The NI strategy indeed performs very well, but the question is:

What is the distribution of the travel time under the NI heuristic?

This problem is not at all trivial. At least, straightforward methods, such as conditioning on possible items' locations, don't lead to feasible calculations.

Furthermore, despite of its nice properties, the NI heuristic does not guarantee the optimal, i.e. the shortest, route. For instance, in Figure 1.1 the optimal picking sequence is $(n, 1, 2, \dots, n-1)$ rather than $(1, n, 2, 3, \dots, n-1)$. Thus, the naturally arising questions are:

Are there other simple strategies which are better than the NI heuristic?

What is the distribution of their travel time?

How close are they to the optimal strategy?

And, finally, there is always the problem:

What can we say about the properties of the optimal route?

In this monograph, we develop methods that enable us to answer the questions above and to derive many other (sometimes counterintuitive) results on collecting n items on a circle. In our approach, we use well-known relations between exponential random variables and uniform spacings $D_1, D_2, \dots, D_{n+1}$ which are the distances between adjacent points (the $n+1$ points are, of course, the picker's starting position plus the n items' locations). We introduce probabilistic arguments based on the memory-less property to prove two distributional identities which imply further results on the travel time distribution. By themselves, these identities are of pure mathematical interest as new peculiar properties of exponentials.

1.1 Motivation: performance analysis of carousel systems

A carousel is an automated storage and retrieval system (AS/RS) which is widely used in modern warehouses. This order picking system consists of a large number of shelves or drawers rotating in a closed loop in either direction. Orders are represented by a list of items. The list specifies the type and retrieval quantity of each item. The picker has a fixed position in front of the carousel, which rotates the required items to the picker. The advantage of such systems is that the picker has time for sorting, packing, labeling etc., while the carousel is rotating.

'In the family of material-handling equipment, carousels have long been viewed as the clumsy under-achiever. But carousels have matured... A variety of configurations, sizes and applications make carousels versatile – for manual picking of tiny items to automated full-case picking', according to T.A. Foster [33]. In the last two years *Modern Materials Handling*, *Material Handling Management*, *Warehousing Management* and other journals published a lot of articles about efficiency of carousels implemented in warehouses of leading companies such as Boeing and Ford. Nowadays, carousels are successfully used for storage and retrieval of health and beauty products, repair parts of boilers for space heating, parts of vacuum cleaners and sewing machines, books, shoes and many other goods.

Basically, carousels come in two designs: horizontal and vertical [33, 68]. Both bring items to the operator. 'Benefits include improved control over materials, reduced levels of inventory, greater utilization of available floor space, less repetitive or unnecessary handling of staged/stored loads, and more efficient use of labor', T. Feare [31]. In other articles (e.g. [33, 34, 68, 81]), the authors discuss the growing popularity of carousels, their various implementations and the tasks where such AS/RS may be especially useful. One of the newest tasks is order picking in e-commerce. E-commerce start-up companies 'just want to fill orders and move product through very fast. For split-case, high-speed and fast-flow orderpicking, carousels fill the bill', says Larry Strayhorn, president of Diamond Phoenix, a carousel supplier [81].

From a mathematical point of view, the performance analysis of carousel systems delivers many interesting problems. For example, Jacobs et al. [55] assume a fixed number of possible orders, and they propose a heuristic defining how many cases of each item should be stored on the carousel in order to maximize the number of orders that can be retrieved without reloading. Egbelu and Wu [28] study the problem of pre-positioning of the extractor in anticipation of storage/retrieval requests. Some related mathematical models arising in manufacturing have also been discussed in literature. For example, consider numerically controlled punch press operations where each type of hole requires a different tool mounted on a fixed rotating carousel. Holes are punched on a metallic bar in several passes, using each time a contiguous subset of tools. The problem is to find an optimal partition of the tool set to minimize the expected completion time [36, 61] or to maximize the profit [52]. Another assignment problem in the design of tool carousels for flexible manufacturing systems was considered in [35] where the objective was to maximize the sum of the adjacency ratings of tools. Common techniques that are used to analyze performance of carousel systems are combinatorial optimization methods or computer simulations. The same is true for most AS/RSs. A stochastic analysis of a unit-load AS/RS by using a single-server queueing model was presented by Lee [63]. For a recent overview on mathematical analysis of carousel systems, as part of a general overview on planning and control of warehousing systems, we refer to Van den Berg [12].

It is natural to model a carousel as a circle. Stern [79] and Ghosh and Wells [42] consider a discrete model where the circle consists of a fixed number of locations. Bartholdi and Platzman [6] and Van den Berg [10, 11] propose a continuous version where the circle has length 1 and the locations of the required items are represented as arbitrary points on the circle.

One of the most important performance characteristics of carousel systems is the total time needed to pick an order. Ideally, the items should be picked in a sequence minimizing the total pick time, which is the travel time plus the pure pick time. The latter obviously does not depend on the pick strategy. Hence, we only have to consider the travel time in order to minimize the total pick time.

Van den Berg [10, 11] studies the problem of picking several orders in a row. There are two possible approaches: a single order picking (the orders are picked one after another), and a batch order picking (the orders are picked together and then must be sorted and accumulated). The author presents efficient algorithms minimizing the total travel time for the single order picking. When the sequence of the orders is fixed, he derives a dynamic programming algorithm which finds the optimal sequence within every order. In the case when the sequence of orders is free, he reduces the problem to a Traveling Salesman Problem (see e.g. [37, 38, 69]) on a circle. The proposed heuristic is motivated by the approach from [6].

From a practical point of view, it is also interesting to study the optimal pick sequence within a single order consisting of n items to pick. In their paper, Bartholdi and Platzman [6] consider some simple heuristics for carousel systems. One of these heuristics is the Nearest Item (NI) heuristic, where the next item to be picked is always the nearest one. This algorithm is very often used in practice. The NI heuristic usually performs close to optimal, except in some pathological cases, and it produces solutions that are guaranteed to be never too far from optimal. In particular, the authors prove that the travel time under the NI heuristic is never greater than one rotation of the carousel and never greater than twice the optimal travel time. Litvak et al. [64, 65] give an exhaustive analysis of the NI heuristic in the case that the positions of the items are independent and uniformly distributed. Their results, including the travel time distribution, are presented in Chapter 3 of this monograph.

Bartholdi and Platzman [6], Stern [79] and Ghosh and Wells [42] study the optimal pick strategy for carousels. They show that the shortest route admits at most one turn. Intuitively, this follows even by observing Figure 1.1. Indeed, it is never optimal to cover the same part of the circle three times or more, because in such a case there always exists a shorter route which covers this part at most twice. For example, in Figure 1.1 the displayed route can be shortened by skipping the part $(0, 1, 0)$. Thus, to find an optimal sequence, one has to consider only $2n$ candidate routes which turn at most once. This implies that an optimal route can always be found in linear time. By using simple observations on the optimal pick sequence, Ghosh and Wells [42] derive an algorithm which reduces a number of evaluations needed to get to the optimum.

The carousel model that we explore in this monograph is the following:

The carousel model addressed in the thesis: We study the rotation (travel) time of a carousel while picking one order. The carousel is modeled as a circle of length 1. The order is represented by the list of n items whose positions are assumed to be independent and uniformly distributed on $[0, 1)$. For ease of presentation, we act as if the picker travels to the pick positions instead of the other way around. Also, we assume that the acceleration time of the carousel is negligible or that it is assigned to the pick time, and therefore the travel distance can be identified with the travel time.

In this set up, the problem of efficient order picking in carousel systems reduces, up to some extent, to the problem of collecting n items randomly located on a circle.

The model above was considered by Rouwenhorst et al. [77] who provided some stochastic upper bounds for the minimal travel time. The idea of their upper bounds is based on the fact that (i) the optimal route implies no more than one turn; and (ii) tours with not too many items collected before the turn, if any, already constitute a tight approximation for the optimal route. Thus, they considered the shortest route among the ones that allow to turn only once and only after collecting at most m items. In the paper of Litvak and Adan [66] such algorithms are called the 'm-step' strategies. Rouwenhorst et al. [77] study the three simplest cases, namely, $m = 0, 1, 2$. Simulations show that for $m = 2$ the performance of the heuristic is very close to optimal. After lengthy calculations, the authors also derive a distribution of the travel time for $m = 1$. It turned out that for $n \geq 3$ the distribution function $F_1(t)$ of the travel time is given by

$$F_1(t) = 3t_+^n - \frac{9}{4}(2t-1)_+^n - (3t-2)_+^n + \frac{3}{2}(4t-3)_+^n - \frac{1}{4}(6t-5)_+^n, \quad 0 \leq t \leq 1, \tag{1.1}$$

where $x_+ = x$ if $x > 0$ and $x_+ = 0$ otherwise. This expression has a certain nice structure, though from the calculations it is not at all clear where this structure comes from. Litvak and Adan [66] analyse the m-strategies using their approach from [64, 65, 67] based on the properties of exponential random variables and uniform spacings. This approach is described in detail in Chapter 2, and the results from [66] are put in Chapter 4. For the case $2m < n$, Litvak and Adan [66] derive the distribution of the travel time under the m-step strategies. In particular, they give a simple proof and an explanation for (1.1) (see Example 4.1). Also, they compare the m-step strategies with the optimal route and provide analytical and numerical results proving that these strategies perform almost optimally already for $m \geq 2$.

In the literature, one can find only some algorithms and estimates for the optimal picking strategy. If the positions of the items are independent uniform random variables, then the methods presented in this work seem to be applicable for studying the optimal route. Such possibilities are discussed in Chapter 5. However, only a few results are available so far. A complete analysis of the minimal travel time remains a challenging open problem.

1.2 Uniform spacings and exponential random variables

Let the random variable $U_0 = 0$ be the picker's starting point and the random variable U_i, where $i = 1, 2, \ldots, n$, be the position of the ith item. We suppose that the U_i's, $i = 1, 2, \ldots, n$, are independent and uniformly distributed on $[0, 1)$. Let $U_{(1)}, U_{(2)}, \ldots U_{(n)}$ denote the order statistics of $U_1, U_2, \ldots U_n$. Put $U_{(0)} = 0$, $U_{(n+1)} = 1$. Then the uniform spacings are defined as

$$D_i = U_{(i)} - U_{(i-1)}, \quad 1 \leq i \leq n + 1.$$

If we consider n items randomly located on a circle, then the spacings $D_2, D_3, \ldots, D_n$ are the distances between two neighbor items, and the spacings D_1 and D_{n+1} are the distances between the starting point and the two items adjacent to it (see Figure 1.1). Whatever strategy the picker takes, he always has to cover one or more uniform spacings on his way from one location to another. Hence, in general, the travel time can be expressed as a function of the uniform spacings. In this monograph we explore such representation, and we use probabilistic properties of the spacings to derive the distribution of the travel time under various strategies.

All the spacings have the same marginal distribution given by

$$\mathrm{P}(D_i < t) = 1 - (1-t)^n, \quad 1 \le i \le n+1.$$

It was shown by Huang et al. [51] that under certain conditions the identical distribution of D_1 and D_k, where $k = 2, 3, \ldots, n+1$, is sufficient to guarantee that the distribution of the U_i's is uniform. The random vector $(D_1, D_2, \ldots, D_{n+1})$ has the density function

$$f_{(D_1,\ldots,D_{n+1})}(d_1, \ldots, d_{n+1}) = \begin{cases} n!, & \text{if } d_i \ge 0 \text{ and } d_1 + d_2 + \cdots + d_{n+1} = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Uniform spacings have received an extensive analysis in classical review papers of Pyke [72, 73]. In his paper [72], the author gives four useful constructions for uniform spacings. In fact, all these constructions establish a connection between uniform spacings and exponential random variables.

A random variable X has an exponential distribution with parameter λ, if its distribution function $H(\cdot)$ is given by

$$H(t) = \begin{cases} 0, & t \le 0, \\ 1 - \exp(-\lambda t), & t > 0. \end{cases} \tag{1.2}$$

Such a distribution has mean $1/\lambda$ and variance $1/\lambda^2$. The exponential distribution has many nice properties. The most famous and crucial one is the *memory-less property*, or *Markov property* (see e.g. Section I.3 in [32]). This property says that 'Whatever present age, the residual lifetime is unaffected by the past and has the same distribution as the lifetime itself'. Formally,

$$\mathrm{P}(X > t + s | X > s) = \mathrm{P}(X > t), \quad t, s > 0. \tag{1.3}$$

Moreover, (1.2) is the only continuous distribution for which this property holds. Another very well-known property is the following. If $X_1, X_2, \ldots, X_N$ are independent exponential random variables with parameters $\lambda_1, \lambda_2, \ldots, \lambda_N$, respectively, then

$$\begin{aligned} \mathrm{P}(\min\{X_1, X_2, \ldots, X_N\} > t) &= \mathrm{P}(X_1 > t, X_2 > t, \ldots, X_N > t) \qquad (1.4) \\ &= \prod_{i=1}^{N} \mathrm{P}(X_i > t) = \exp(-(\lambda_1 + \cdots + \lambda_N)t), \quad t > 0, \end{aligned}$$

i.e. the minimum of $X_1, X_2, \ldots, X_N$ has again an exponential distribution with parameter equal to $\lambda_1 + \lambda_2 + \cdots + \lambda_N$. Obviously, due to its nice properties, the exponential distribution often becomes the most analytically tractable. For example, in queueing theory, systems with Poisson arrivals and exponential service times are the easiest to study. Very often the distribution of the queue lengths and waiting times in such systems can be obtained in a closed form, whereas for general arrival process or general service time distributions such closed-form results are available only in a few special cases (see e.g. [59]). We refer to [5] for an introduction and an extended list of references on the exponential distribution and its applications. See also Section 2.1.

From the above discussion it appears that relations between exponential random variables and uniform spacings might be very helpful in order to analyze the latter. Let $X_1, X_2, \ldots$ be independent exponential random variables with mean 1. Denote $S_0 = 0$; $S_i = X_1 + X_2 + \cdots + X_i$, $i \geq 1$. Then, according to Pyke [72], uniform spacings can be represented as follows.

1. *Uniform spacings as exponentials proportional to their sum:*

$$(D_1, D_2, \ldots, D_{n+1}) \stackrel{d}{=} (X_1/S_{n+1}, X_2/S_{n+1}, \ldots, X_{n+1}/S_{n+1})\,. \tag{1.5}$$

In particular, Pyke says that this equality in distribution is convenient 'to show that an ordering of uniform spacings may be considered as an ordering of the exponential random variables'. In the present manuscript we use this construction with similar purposes.

2. *Uniform spacings as exponentials conditioned on their sum.* Since

$$f_{(X_1,\ldots,X_n)|S_{n+1}}(x_1, \ldots, x_n|s) = n!s^{-n},$$

one can see that the conditional distribution function of $(X_1, \ldots, X_{n+1})$ given $S_{n+1} = 1$, is the same as that of $(n+1)$-uniform spacings $(D_1, D_2, \ldots, D_{n+1})$. This construction was used in [72] to prove limit theorems for uniform spacings applying the limit theorem of LeCam [62] for exponential random variables.

3. *Uniform spacings as inter-event times in a Poisson process.* Let $\{N(t) : t \geq 0\}$ be a Poisson process with parameter $\lambda = E\{N(1)\}$. A systematic treatment of Poisson processes can be found in [58]. Let $T_1 \leq T_2 \leq \ldots$ denote the successive times of events in the process. For some fixed $t > 0$, set $X_i = T_i - T_{i-1}$, $(T_0 = 0)$, for $1 \leq i \leq N(t)$, and set $X_{N(t)+1} = t - T_{N(t)}$. Then the conditional distribution of $(t^{-1}X_1, t^{-1}X_2, \ldots, t^{-1}X_{n+1})$, given $N(t) = n$, is the same as the distribution of the $(n+1)$-uniform spacings determined by a set of n independent uniform random variables on $[0, 1)$. According to Pyke [72], 'This is possibly the oldest construction of uniform spacings and one which represents the most natural relationship between the Poisson Process ('random' points on a line) and the uniform distribution ('random' points on an interval)'.

4. *Uniform spacings as a function of exponential spacings.* If X is a random variable with continuous distribution function $F(\cdot)$, then the random variable $U = F(X)$ is uniformly distributed on $[0, 1]$. Now let $H(t) = 1 - e^{-t}$, $t \geq 0$, be a probability distribution function of the exponential distribution. For $1 \leq i \leq n$, set

$$Z_i = \sum_{j=1}^{i} (n - j + 1)^{-1} X_j, \qquad U_{(i)} = H(Z_i) = 1 - \exp(-Z_i).$$

Here the Z_i's are distributed as order statistics of exponentials. This is a well-known result on the exponential distribution (see e.g. the paper of Rényi [75]). This result is a direct consequence of (1.3) and (1.4). Now it follows that the $U_{(i)}$'s are distributed as uniform order statistics on $[0, 1]$. The resulting construction of the spacings is:

$$\begin{aligned} D_i = U_{(i)} - U_{(i-1)} &= \exp(-Z_{i-1}) - \exp(-Z_i) \\ &= (1 - U_{(i-1)})[1 - \exp(-X_i/(n - i + 1))]. \end{aligned}$$

Note that by construction both factors are independent. Rényi [75] used this construction 'to advantage to prove both exact and limiting results about order statistics'.

Besides the constructions above, various methods of analyzing probabilistic properties of uniform spacings were reviewed by Borovikov [13].

Uniform spacings play an important role in mathematical statistics. Mainly, they are used for goodness-of-fit tests which examine how well the sample of data agrees with a given distribution F_0 as its population. The idea of using uniform spacings is based on the integral transformation $x \to F_0(x)$ which reduces the problem to testing of uniformity of the transformed sample. There is a vast literature on the distributions, limiting behavior, approximations and bounds for various goodness-of-fit test statistics and empirical processes based on uniform spacings. These investigations are of great mathematical and practical interest. Considerable progress in the area has been achieved in the eighties, but there are still many open problems motivating new studies. We don't attempt to give an extended literature survey on this subject, since the results of the present manuscript are of different type and their applications lie in different areas. Nevertheless, we will mention some key papers as well as several overviews and recent contributions.

In his detailed review, Pyke [72] distinguishes two main types of goodness-of-fit statistics based on a function of uniform spacings: a sum of the form

$$G_n = \sum_{i=1}^{n} g_n(D_i),$$

or a function of the ordered spacings and their ranks. See also Chapters 8 and 10 of [18] for a more recent review with concrete examples of various tests. For later developments and further references on the first type of tests see the papers of Does et al. [27], Shao and Hahn [78], Gatto and Jammalamadaka [39], Ghosh and Jammalamadaka [43, 44]. The second type of tests requires knowledge on properties of ordered spacings. Here considerable contributions are made by Devroye [23, 24, 25] and Deheuvels [19, 20]. The problem of limiting behavior of the maximal uniform spacing finally has been solved by Einmahl and Van Zuijlen [29]. An original discrete version of the problem is studied by Henze [50] who derives the distribution of the maximal and minimal spacings in lottery tickets. The examples of the goodness-of-fit tests based on ordered spacings as well as the references for other works can be found in [56, 74, 30]. Besides of the tests mentioned above, there are also tests based on m-spacings which are the gaps between the order statistics $U_{(i)}$ and $U_{(i+m)}$. Such test statistics and their asymptotic properties were studied by Del Pino [71], Hall [48, 49] and Deheuvels [21]. The ordered m-spacings were studied in [8, 22]. A review on this subject can be found in [45]. For analysis and applications of various empirical processes based on spacings see Pyke [72], Beirlant et al. [7, 9], Csörgő and Révész [17], Csörgő and Horvath [16], Einmahl and Van Zuijlen [29] and references therein.

Some other interesting stochastic processes based on n points randomly located on a circle have been studied in literature. Itoh et al. [54] introduce a random generation model with cyclic dominance relations. They put n species as points $Q_1, Q_2, \dots, Q_n$ on a circle and say that Q_i dominates Q_j, if the counterclockwise way from Q_i to Q_j is shorter than the clockwise way. The authors investigate the properties of the described oriented graph and they find a limit distribution for the number of existing species assuming a Lotka-Volterra cascade model for food webs. Coffman et al. [15] study the following selection-replacement process for n points uniformly and independently located on a circle: (i) select a new point P uniformly at random, (ii) remove the first point encountered in a counterclockwise sweep around the circle starting at P, and (iii) add a new point selected uniformly at random. This process arises in computer disk scheduling. When the number of iterations tends to infinity, the authors determine the limit for the expected length of the counterclockwise sweep at step (ii). It turns out that the joint distribution of the n points on a circle approaches a stationary limit which is, surprisingly, different from the uniform distribution, provided $n \geq 3$.

As we already mentioned, the results of this monograph differ from the above-listed works. We are aiming to find the distribution of the travel time while collecting n items randomly located on a circle. In order to do it, we express the travel time as a function of uniform $(n+1)$-spacings. For example, the expression for the optimal strategy can be found simply by looking at Figure 1.1. Indeed, remember that the optimal route admits no more than one turn. Further, $D_j - \sum_{l=1}^{j-1} D_l$ or $D_j - \sum_{l=j+1}^{n+1} D_l$ is the gain in travel time (compared to one full rotation) obtained by skipping the spacing D_j and going back instead, ending in clockwise or counterclockwise direction, respectively. The optimal strategy provides the largest gain (see also Section 5.1). Thus, the minimal travel time is distributed as

$$1 - \max\left\{ \max_{1\leq j\leq n} \left\{ D_j - \sum_{l=1}^{j-1} D_l \right\}, \max_{1\leq j\leq n} \left\{ D_{n+2-j} - \sum_{l=1}^{j-1} D_{n+2-l} \right\} \right\}.$$

In the analysis, we use construction (1.5) that treats the spacings as i.i.d. exponentials $X_1, X_2, \dots, X_{n+1}$ divided by their sum S_{n+1}. Then the minimal travel time would be distributed as

$$1 - \frac{1}{S_{n+1}} \max\left\{ \max_{1\leq j\leq n} \{X_j - S_{j-1}\}, \max_{1\leq j\leq n} \{X_{n+2-j} - (S_{n+1} - S_{n+2-j})\} \right\}.$$

Furthermore, in Chapter 3 we will prove that the travel time under the Nearest Item heuristic is distributed as

$$1 - \frac{1}{S_{n+1}} \sum_{j=1}^{n+1} (X_j - S_{j-1})_+$$

(see formula (3.28)). It follows that the sum and the maximum of the terms $(X_j - S_{j-1})_+$ play an important role in the analysis of the travel time distribution. Therefore, in Chapter 2 we consider the terms $(X_j - bS_{j-1})_+$, $b \geq 0$, and we prove that their sum as well as their maximum is distributed as a linear combination of exponentials. In the proof, we use probabilistic arguments based on the memory-less property and the distribution of the minimum of exponential random variables. Further, we extend our results to the case of uniform spacings providing, with $b = 1$, the distribution of the travel time under the Nearest Item heuristic and some other strategies. Due to the principal difficulties described in Section 5.4, we will not be able to derive the distribution of the minimal travel time. However, our methods enable us to completely analyse the m-step strategies which are very close to optimal (see Chapter 4).

1.3 Outline of the thesis

In this thesis, we study the travel time needed to collect n items randomly located on a circle. Chapter 1 was devoted to the motivation of this research and the literature survey on relevant works. The problem arises in performance analysis of carousel systems. A carousel is an automated warehousing system consisting of a large number of shelves or drawers rotating in a closed loop in either direction. A picker has a fixed position in front of the carousel that rotates the required items to the picker. Nowadays, such systems are widely used for storage and retrieval of various goods – from beauty and health products to repair details for airplanes. We reviewed the literature on carousels taking into account both theoretical and practical aspects. In this monograph, we model a carousel as a circle of length 1. The objective is to characterize the travel time needed to pick one order represented by a list of n items whose positions are independent and uniformly distributed on the circle. Our methods are based on the properties of uniform spacings. Their relations with exponential random variables make the spacings accessible for analytical studying. We listed some properties of the spacings following the fundamental work of Pyke [72]. Further, we gave a brief overview on the spacings and their applications. We did not aim to fully survey the enormous literature on the subject referring mostly to substantial papers, reviews and recent findings. Besides, we mentioned some other interesting problems concerning stochastic processes generated by n random points on a circle. At the end, we explained the motivation for the analytical studies presented in Chapter 2.

Chapter 2 is based on the paper [67]. We consider the sum and the maximum of the terms $(X_j - bS_{j-1})_+$, $j = 1, 2, \ldots, N$, where $b \geq 0$; $X_1, X_2, \ldots$ are independent exponential random variables with mean 1; $S_0 = 0$; $S_i = X_1 + X_2 + \cdots + X_i$,

$i \geq 1$. We develop an approach based on the memory-less property and we prove that both the sum and the maximum are distributed as linear combinations of exponential random variables. This is a generalization of well-known facts about the distribution of the sum and the maximum of the X_j's. We also prove some auxiliary assertions and corollaries. Then we show that similar results hold for uniform spacings. Finally, we explore the distribution and the moments of linear combinations of the spacings.

In the further chapters we study various order picking strategies (or, in other words, the strategies to collect n items on a circle). Chapter 3 mostly contains the results from [64, 65] on the Nearest Item (NI) heuristic where the next item to be picked is always the nearest one. This algorithm is often used in practice. For any realization of the items' locations, we derive a tight upper bound for the travel time. Then we develop a recursive procedure to obtain a closed-form expression for the mean and the variance of the travel time conditioned on the size of an empty space at one side of the picker's position. Further, we use a similar procedure to find the conditional, the unconditional and the limiting distribution for the number of turns. Then we express the travel time as a function of uniform spacings, and we use the results from Chapter 2 to prove that the travel time under the NI heuristic is distributed as a linear combination of uniform spacings. That enables us to derive the closed-form expression for the distribution and the moments of the travel time. Then we give an exhaustive analysis of the limiting behavior of the travel time distribution. Finally, we show an alternative way to derive the distribution of the number of turns. Moreover, we prove that the travel time and the number of turns are independent random variables.

In Chapter 4 we address the strategies related to the optimal route. This chapter starts with the analysis of the One-Side Optimal (OSO) strategy. This algorithm chooses the best picking sequence providing that the picker ends in a given direction (say, clockwise). Applying the results of Chapter 2, we derive the distribution of the travel time under the OSO strategy. We show that this travel time is stochastically bigger than the travel time under the NI heuristic. The major part of Chapter 4 is based on the paper [66]. This part of the thesis is devoted to so-called m-step strategies: the picker chooses the shortest route among the ones that change direction only once, and only do so after collecting no more than m items. Since it is never optimal to turn more than once, the optimal strategy is in fact an $(n-1)$-step strategy. We derive, for any $m \geq 0$, explicit expressions for the distribution and all moments of the travel time under the m-step strategy, provided $2m < n$. The analysis is based on the arguments from Chapter 2. The performance of m-step strategies is compared with the performance of the optimal pick strategy. Numerical results show that, already for small values of m, the performance of the m-step strategy is very close to optimal. In fact, with high probability, the optimal strategy coincides with the 2-step strategy. Furthermore, m-step strategies are compared with the NI heuristic. It appears that, on average, the m-step strategy performs better than the NI heuristic already for $m = 2$.

Chapter 5 is devoted to the optimal pick sequence. We first study the probability that under the optimal strategy the picker would collect exactly k items before the turn. We prove that, for fixed $k = 0, 1, \ldots,$ this probability tends to $1/2^{k+1}$, when the number of items tends to infinity. Further, we find a tight upper bound of the travel time for any realization of the items' locations. Also, we obtain a stochastic upper bound and make a conjecture about a stochastic lower bound for the minimal travel time. Further, we discuss possibilities to find the distribution of the travel time under the optimal strategy. Although the methods from Chapter 2 can be applied again, we discover difficulties that don't allow us to solve the problem in general. Therefore, we can only provide some conjectures. As an example, we derive the distribution of the minimal travel time for $n = 3$. We complete the last chapter with conclusions and discussion.

Chapter 2

Peculiar properties of exponential random variables and uniform spacings

2.1 Introduction

In this chapter we find out some peculiar properties of exponential random variables and uniform spacings. In fact, this material forms the theoretical basis of the monograph.

The major part of this chapter presents the analysis of Litvak [67]. The arguments that we use were initially introduced by Litvak et al. [64], Litvak and Adan [65] and generalized in [67] in order to derive two distributional identities for exponential random variables (see Theorem 2.1 below). The exponential distribution is given by (1.2). It has a large number of nice properties, for example, the memory-less property (1.3) and the property of the minimum (1.4). The exponential distribution often naturally arises in practice, for example, as the distribution of the inter-event times of the Poisson process which can be used to model numerous natural processes like the sequence of phone calls (see e.g. [59]) or the traffic flow on a highway [46]. Furthermore, for analytical studies it is important that Markov processes have exponentially distributed state times. Therefore, the exponential distribution has been applied extensively in many fields such as life testing, reliability theory and applications, queueing theory with applications and survival analysis, to mention some major fields. For a survey of the theoretical aspects of the distribution and its applications with some view of historical developments we refer to the volume edited by Balakrishnan and Asit [5].

There has been done a lot of work on the characterization of the exponential distribution. The book of Azlarov and Volodin [3] is dedicated to this problem. They consider, among others, characterizations by the memory-less property, by some properties of order statistics and by the geometric distribution. Dimitrov and Khalil [26] find a property of exponentials, closely related to a single-service queue with an unreliable server. They show that this property is a characterization of the exponential distribution. Van Harn and Steutel [47] improve this characterization by substantially weakening the conditions. Gilat [40, 41] extends the well-known fact that the standard deviation of the exponential distribution is equal to its mean, to means and deviations of all orders and to generalized means on convex increasing functions. He analyses the converse of this result as an unsettled conjecture. Bairamov [4] gives a new characterization of the exponential distribution in terms of record values and probabilities of finite sums of i.i.d. nonnegative random variables. Wilf [83] presents elegant methods to obtain values that have an exponential distribution given a standard random number generator.

This chapter is also devoted to some specific properties of exponential random variables. Let $X_1, X_2, \ldots$ be independent exponential random variables with mean 1. Denote

$$S_0 = 0; \qquad S_i = \sum_{j=1}^{i} X_j, \quad i \geq 1.$$

In Section 2.2 we consider the sum and the maximum of the terms $(X_j - bS_{j-1})_+$, $j = 1, 2, \ldots, N$, where $x_+ = x$ if $x > 0$ and $x_+ = 0$ if $x \leq 0$. As we explained at the end of Section 1.2, this kind of expressions play an important role in the analysis of the travel time on a circle. The sum is only interesting when $b > 0$ whereas the maximum is non-trivial if $b > -1$ (in fact, we could assume $b > 0$ in the maximum without loss of generality). Our main result is that for such values of b, both the sum and the maximum are distributed as a linear combination of exponentials.

Note that the distribution of the sum and the maximum of the X_j's is well-known. For the sum, it is the Erlang (Gamma) distribution with shape parameter N and scale parameter 1 whose probability density is given by

$$f_{S_N}(t) = \frac{t^{N-1}}{(N-1)!} \exp(-t).$$

For the maximum, there is a classical result on exponential order statistics:

$$\max_{1 \leq j \leq N} X_j \stackrel{d}{=} \sum_{j=1}^{N} \frac{1}{j} X_j, \quad N \geq 1. \tag{2.1}$$

Thus, our results can be seen as a sort of generalization of these well-known properties of exponentials: we consider the terms $(X_j - bS_{j-1})_+$ instead of the terms X_j providing the classical results when $b \to 0$. In fact, one could anticipate this kind of generalization. Indeed, the distribution of the term $(X_i - bS_{i-1})_+$ is determined by the event $[X_i > bS_{i-1}]$ which is known to be independent of the subsequent terms $(X_j - bS_{j-1})_+$, $j > i$ (see also Remark 2.2). Therefore, one can expand all the terms one by one finally deriving the distribution of the whole expression. Nevertheless, the simplicity of the resulting distributional identities is surprising.

In Section 2.2.1 we describe the method and obtain some preliminary results. The main theorem is proved in Section 2.2.2. Further, in Section 2.2.3 we derive several corollaries that follow from the proof of the main result. In Section 2.3 we extend the main theorem to the case of the uniform spacings. In Section 2.4 we explore the result of Ali [1] on the distribution of linear combinations of uniform spacings. In particular, we derive a closed-form expression for the case of coinciding coefficients. Also, we give formulas for calculating the moments of linear combinations of uniform spacings. Finally, in the last section we briefly summarize the results.

2.2 Main theorem for exponential random variables

In this section we prove the following theorem.

Theorem 2.1 *For any* $N = 1, 2, \dots,$

$$\sum_{j=1}^{N} (X_j - bS_{j-1})_+ \stackrel{d}{=} \sum_{j=1}^{N} \frac{1}{(b+1)^{j-1}} X_j, \quad b \geq 0; \tag{2.2}$$

$$\max_{1 \leq j \leq N} \{X_j - bS_{j-1}\} \stackrel{d}{=} \sum_{j=1}^{N} \frac{b}{(b+1)^j - 1} X_j, \quad b > -1. \tag{2.3}$$

Remark 2.1 Note that, by letting $b \to 0$ in (2.3), we obtain (2.1). Below in Section 2.2.3 we will show that in (2.3) we can, without loss of generality, assume $b \geq 0$.

2.2.1 Preliminary results

Below we prove Theorem 2.1 by induction, expanding the terms in the left-hand side of (2.2) and (2.3) one by one. As we will show in Section 2.2.2, at the i-th induction step there always arises a term

$$\sum_{j=1}^{i-1} C_{i-j-1} X_j + (X_i - S_{i-1}(\mathbf{c}))_+ \,, \tag{2.4}$$

where for $\mathbf{c} = (c_1, c_2, \dots)$ and $c_i \geq 0$ for $i \geq 1$, the sum $S_{i-1}(\mathbf{c})$ is defined as

$$S_0(\mathbf{c}) = 0; \qquad S_{i-1}(\mathbf{c}) = \sum_{j=1}^{i-1} c_{i-j} X_j, \quad i \geq 2,$$

and coefficients C_m for $m \geq 0$ are given by

$$C_0 = 1; \qquad C_m = \prod_{j=1}^{m} (1 + c_j)^{-1}, \quad m \geq 1.$$

Although, in order to prove (2.2) and (2.3), we have only to consider specific c_j's (see (2.10) and (2.12), respectively), in this section we explore (2.4) for arbitrary **c**. Lemma 2.1 states that expression (2.4) can be simplified. This will be done by conditioning on events under which the last term in (2.4) can be explicitly determined. Then both assertions of Theorem 2.1 will follow from the same general arguments introduced in the proof of Lemma 2.1.

Lemma 2.1 *For any* $i = 1, 2, \dots,$

$$\sum_{j=1}^{i-1} C_{i-j-1} X_j + (X_i - S_{i-1}(\mathbf{c}))_+ \stackrel{d}{=} \sum_{j=1}^{i} C_{i-j} X_j. \tag{2.5}$$

Proof. For fixed i, partial sums of $c_{i-j}X_j$'s partition the time-axis into i disjoint intervals $[0, c_{i-1}X_1)$, $[c_{i-1}X_1, c_{i-1}X_1 + c_{i-2}X_2)$, $\dots$, $[S_{i-1}(\mathbf{c}), \infty)$. It is natural to compare X_i and $S_{i-1}(\mathbf{c})$ by conditioning on the interval in which X_i falls. Let $E_{i,k}(\mathbf{c})$, $k = 1, 2, \dots, i$, denote the event that X_i falls in the k-th interval, i.e.:

$$\begin{aligned} E_{i,1}(\mathbf{c}) &= [X_i < c_{i-1}X_1]; \\ E_{i,k}(\mathbf{c}) &= [c_{i-1}X_1 + c_{i-2}X_2 + \cdots + c_{i-k+1}X_{k-1} \le X_i \\ &\qquad < c_{i-1}X_1 + c_{i-2}X_2 + \cdots + c_{i-k}X_k], \qquad 2 \le k \le i-1; \\ E_{i,i}(\mathbf{c}) &= [c_{i-1}X_1 + c_{i-2}X_2 + \cdots + c_1 X_{i-1} \le X_i]. \end{aligned}$$

Given the event $E_{i,k}(\mathbf{c})$ for some $k = 2, 3, \dots, i-1$, the random variable $c_{i-1}X_1$ is the minimum of $c_{i-1}X_1$ and X_i, and thus it is exponential with mean $c_{i-1}/(c_{i-1}+1)$ (see also Figure 2.1). Due to the memory-less property the overshoot of X_i is again

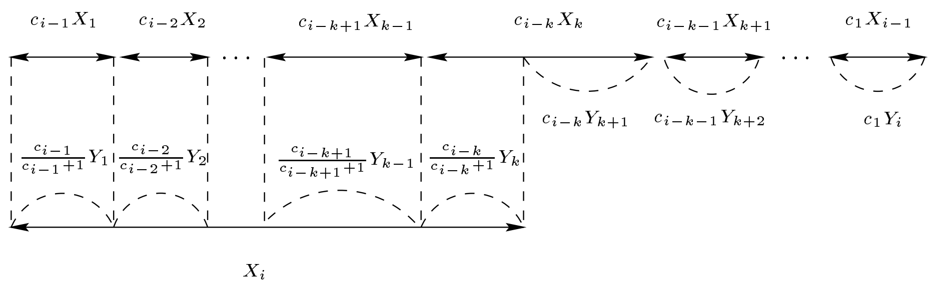

Figure 2.1: Coupling of the random variables $X_1, \dots, X_i$ under event $E_{i,k}(\mathbf{c})$

exponential with mean 1. Hence, we can repeat the argument for $c_{i-2}X_2$ and so on. Eventually, the random variable

$$X_i - (c_{i-1}X_1 + c_{i-2}X_2 + \cdots + c_{i-k+1}X_{k-1})$$

is less than $c_{i-k}X_k$, so it is exponential with mean $c_{i-k}/(c_{i-k}+1)$, and the overshoot of $c_{i-k}X_k$ is exponential with mean c_{i-k}. Since the event $E_{i,k}(\mathbf{c})$ does not provide any information on the other random variables, their distribution remains the same. Similar arguments apply under the events $E_{i,1}(\mathbf{c})$ and $E_{i,i}(\mathbf{c})$. Hence, assuming $E_{i,k}(\mathbf{c})$, where $k = 1, 2, \ldots, i$, the random variables $X_1, X_2, \ldots$ can be coupled in the following way:

$$\begin{aligned}
X_j &= \frac{1}{c_{i-j}+1} Y_j + \mathbf{1}_{[k=j]} Y_{k+1}, \quad 1 \le j \le \min\{k, i-1\}; \\
X_j &= Y_{j+1}, \quad k < j < i; \\
X_i &= \sum_{j=1}^{\min\{k,i-1\}} \frac{c_{i-j}}{c_{i-j}+1} Y_j + \mathbf{1}_{[k=i]} Y_i; \qquad X_j = Y_j, \quad j > i,
\end{aligned} \tag{2.6}$$

where $Y_1, Y_2, \ldots$ are independent exponentials with mean 1.

Now to see (2.5) we apply (2.6) under the event $E_{i,k}(\mathbf{c})$, where $k = 1, 2, \ldots, i$, yielding

$$\begin{aligned}
&\sum_{j=1}^{i-1} C_{i-j-1} X_j + (X_i - S_{i-1}(\mathbf{c}))_+ \\
&\quad = \sum_{j=1}^{\min\{k,i-1\}} C_{i-j-1}(c_{i-j}+1)^{-1} Y_j + \sum_{j=k}^{i-1} C_{i-j-1} Y_{j+1} + \mathbf{1}_{[k=i]} Y_i \\
&\quad = \sum_{j=1}^{i} C_{i-j} Y_j.
\end{aligned} \tag{2.7}$$

Here the crucial and surprising feature is that the last expression in (2.7) is the same for any $k = 1, 2, \ldots, i$. Thus, the resulting expression is independent of $E_{i,k}(\mathbf{c})$, although it was obtained by using the transformation, which was defined by $E_{i,k}(\mathbf{c})$. Now equality (2.5) immediately follows from (2.7), the law of total probability and the identical joint distribution of X_j's and Y_j's. □

For this work, another key issue is that after applying (2.6) under the event $E_{i,k}(\mathbf{c})$, where $k = 1, 2, \ldots, i$, the sum S_i remains of 'the same form' for any $k = 1, 2, \ldots, i$:

$$S_i = \sum_{j=1}^{k} \left(\frac{1}{c_{i-j}+1} + \frac{c_{i-j}}{c_{i-j}+1} \right) Y_j + \sum_{j=k}^{i-1} Y_{j+1} = Y_1 + Y_2 + \cdots + Y_i, \tag{2.8}$$

where $c_0 = 0$. We will need this property of transformation (2.6) in the proof of Theorem 2.1.

2.2.2 Proof of Theorem 2.1

We prove (2.2) by expanding the terms on the left-hand side one by one, and then using induction. Specifically, we show that if for some $i = 2, 3, \dots, N$

$$\sum_{j=1}^{N}(X_j - bS_{j-1})_+ \stackrel{d}{=} \sum_{j=1}^{i-1} \frac{1}{(b+1)^{i-j-1}} X_j + \sum_{j=i}^{N}(X_j - bS_{j-1})_+ \tag{2.9}$$
$$= \sum_{j=1}^{i-1} \frac{1}{(b+1)^{i-j-1}} X_j + (X_i - bS_{i-1})_+ + \sum_{j=i+1}^{N}(X_j - bS_{j-1})_+,$$

then it is also valid for $i+1$. Note that (2.9) holds trivially for $i = 2$. Now we put

$$c_1 = c_2 = \cdots = b \tag{2.10}$$

and, as in the proof of Lemma 2.1, apply (2.6) under the event $E_{i,k}(\mathbf{c})$ to the last expression of equality (2.9). The first two terms give $\sum_{j=1}^{i}(b+1)^{-(i-j)}Y_j$ according to (2.7), where $C_j = (b+1)^{-j}$, $j \geq 1$. Furthermore, it follows from (2.8) that the form of the third term does not change. Hence, for any $k = 1, 2, \dots, i$ the whole expression becomes

$$\sum_{j=1}^{i} \frac{1}{(b+1)^{i-j}} Y_j + \sum_{j=i+1}^{N}(Y_j - b(Y_1 + Y_2 + \cdots + Y_{j-1}))_+$$
$$\stackrel{d}{=} \sum_{j=1}^{i} \frac{1}{(b+1)^{i-j}} X_j + \sum_{j=i+1}^{N}(X_j - bS_{j-1})_+,$$

as required.

The proof of (2.3) is very similar. We assume that for some $i = 2, 3, \dots, N$

$$\max_{1\leq j\leq N}\{X_j - bS_{j-1}\}$$
$$\stackrel{d}{=} \max\left\{\sum_{j=1}^{i-1} \frac{b}{(b+1)^{i-j}-1} X_j, \max_{i\leq j\leq N}\{X_j - bS_{j-1}\}\right\} \tag{2.11}$$
$$= \max\left\{\max\left\{\sum_{j=1}^{i-1} \frac{b}{(b+1)^{i-j}-1} X_j, X_i - bS_{i-1}\right\}, \max_{i+1\leq j\leq N}\{X_j - bS_{j-1}\}\right\}$$
$$= \max\left\{\sum_{j=1}^{i-1} \frac{b}{(b+1)^{i-j}-1} X_j + (X_i - S_{i-1}(\mathbf{c}))_+, \max_{i+1\leq j\leq N}\{X_j - bS_{j-1}\}\right\},$$

where

$$c_j = \left(b + \frac{b}{(b+1)^j - 1}\right) = \frac{b(b+1)^j}{(b+1)^j - 1}, \quad j \geq 1. \tag{2.12}$$

Equation (2.11) is trivial for $i = 2$. Now it suffices to show that if (2.11) holds for i then it also holds for $i+1$. In order to see that, we apply transformation (2.6) under the event $E_{i,k}(\mathbf{c})$ to the last expression of equality (2.11). Then we use (2.7) and (2.8) yielding

$$\max\left\{\sum_{j=1}^{i}\frac{b}{(b+1)^{i-j+1}-1}Y_j, \max_{i+1\le j\le N}\{Y_j - b(Y_1+Y_2+\cdots+Y_{j-1})\}\right\}$$

$$\stackrel{d}{=} \max\left\{\sum_{j=1}^{i}\frac{b}{(b+1)^{i-j+1}-1}X_j, \max_{i+1\le j\le N}\{X_j - bS_{j-1}\}\right\}. \qquad (2.13)$$

for any $k = 1, 2, \ldots, i$. This proves the statement of induction. □

2.2.3 Corollaries of Theorem 2.1

This section contains several corollaries of the results obtained above. First of all, note that the proof of (2.2) as well as the proof of (2.3) establishes N distributional identities, not just two. That is, the random variables

$$\sum_{j=1}^{i}\frac{1}{(b+1)^{i-j}}X_j + \sum_{j=i+1}^{N}(X_j - bS_{j-1})_+, \quad 1\le i\le N,$$

are identically distributed. The same holds for the random variables

$$\max\left\{\sum_{j=1}^{i}\frac{b}{(b+1)^{i-j+1}-1}X_j, \max_{i+1\le j\le N}\{X_j - bS_{j-1}\}\right\}, \quad 1\le i\le N.$$

Let us now consider the proof of (2.3). Clearly, given an event $E_{i,k}(\mathbf{c})$, where the coordinates of the vector $\mathbf{c}$ are given by (2.12), we know whether the i-th term of the expression in the left-hand side of (2.3) is bigger than the first $i-1$ terms or not. On the other hand, this expression is always distributed as (2.13) under any event $E_{i,k}(\mathbf{c})$. So the events $E_{i,k}(\mathbf{c})$ do not provide information on the distribution of the maximum of the first i, nor of all N terms in the left-hand side of (2.3). Hence, we may conclude that the events A_i defined as

$$A_i = \left[\arg\max_{1\le j\le i}\{X_j - bS_{j-1}\} = i\right], \qquad i = 1, 2, \ldots,$$

have the properties as formulated in Corollary 2.1.

Result (*iii*) in the corollary follows from

$$\mathrm{P}(A_i) = \mathrm{P}(E_{i,i}(\mathbf{c})) = \prod_{j=1}^{i-1}\frac{(b+1)^j-1}{(b+1)^{j+1}-1} = \frac{b}{(b+1)^i-1}.$$

Corollary 2.1

(*i*) *The events $A_1, A_2, \dots$ are independent;*

(*ii*) *The distribution of (2.3) is independent of the events A_i, $i = 1, \dots, N$;*

(*iii*) $\mathrm{P}(A_i) = b/((b+1)^i - 1), \quad i = 1, 2, \dots.$

(*iv*) *For any $i = 1, 2, \dots, N$,*

$$\mathrm{P}\left(\arg\max_{1\le j\le N}\{X_j - bS_{j-1}\} = i\right) \tag{2.14}$$
$$= \mathrm{P}\left(A_i \bigcap_{j=i+1}^{N} \bar{A}_j\right) = \frac{b(b+1)^{N-i}}{(b+1)^N - 1}.$$

We will need this corollary in Chapter 4 (see derivation of Theorem 4.2). Another corollary is provided by (2.8):

Corollary 2.2 *Let $f(\cdot,\cdot)$ be a function defined on the positive quadrant of $\mathbb{R}^2$. Then for any $b \ge 0$,*

$$f\left(\sum_{j=1}^{N}(X_j - bS_{j-1})_+,\, S_N\right) \stackrel{d}{=} f\left(\sum_{j=1}^{N}\frac{1}{(b+1)^{j-1}}X_j,\, S_N\right).$$

Similarly, for any $b > -1$,

$$f\left(\max_{1\le j\le N}\{X_j - bS_{j-1}\},\, S_N\right) \stackrel{d}{=} f\left(\sum_{j=1}^{N}\frac{b}{(b+1)^j - 1}X_j,\, S_N\right).$$

The proof of this corollary is the same as the proof of Theorem 2.1 with the only additional remark that according to (2.8) the term S_N under transformation (2.6) always becomes $Y_1 + Y_2 + \cdots + Y_N$.

Corollary 2.2 allows us, without loss of generality, to assume that $b \ge 0$ in (2.3). Indeed, observe that

$$\begin{aligned} S_N - \max_{1\le j\le N}\{X_j - bS_{j-1}\} &= S_N - \max_{1\le j\le N}\{S_j - (b+1)S_{j-1}\} \\ &\stackrel{d}{=} (b+1)\left(S_N - \max_{1\le j\le N}\left\{S_j - (b+1)^{-1}S_{j-1}\right\}\right), \end{aligned}$$

where the last equality is obtained by renumbering X_j, $1 \le j \le N$, in the reverse order.

In the next section we will use the arguments from Corollary 2.2 to show that an analogous result to Theorem 2.1 also holds for uniform spacings. Before closing this section we make two more remarks concerning (2.8).

Remark 2.2 From (2.8) one can immediately derive a weaker assertion:

$$\mathrm{P}(S_l < t | E_{i,k}(\mathbf{c})) = \mathrm{P}(S_l < t), \quad 1 \le k \le i \le l, \quad t \in \mathbb{R}. \tag{2.15}$$

From (2.15) it follows, for example, that

$$\mathrm{P}(X_1 + X_2 + X_3 < t | X_3 > 1000X_1 + 1000X_2) = \mathrm{P}(X_1 + X_2 + X_3 < t).$$

Formula (2.15) itself has simpler proofs. For instance,

$$\begin{aligned}&\mathrm{P}\left(S_3 < t | X_3 > 1000X_1 + 1000X_2\right) \\ &= \mathrm{P}\left(S_3 < t | X_3/S_3 > 1000\left(X_1/S_3\right) + 1000\left(X_2/S_3\right)\right) = \mathrm{P}(S_3 < t),\end{aligned}$$

where the last equality holds, since X_1/S_3, X_2/S_3 and X_3/S_3 are independent of S_3. For the proof of independence we refer to Pyke [72] (see also Section 2.3). Note that the similar property does not hold for, e.g., independent uniform random variables $U_1, U_2, \dots$ on the interval $(0,1)$. It is easily verified that in this case we have, for instance,

$$\mathrm{P}\left(U_1 + U_2 + U_3 < 1.5\right) = \frac{1}{2}$$

and

$$\mathrm{P}\left(U_1 + U_2 + U_3 < 1.5 | U_3 > 1000U_1 + 1000U_2\right) = 1.$$

Remark 2.3 A property like (2.8) does not hold for an arbitrary sum

$$a_{i-1}X_1 + a_{i-2}X_2 + \cdots + a_1X_{i-1} + a_0X_i = S_{i-1}(\mathbf{a}) + a_0X_i,$$

where $a_j \ge 0$, $j \ge 0$, and $\mathbf{a} = (a_1, a_2, \dots)$. Indeed, after applying (2.6) under the event $E_{i,k}(\mathbf{c})$, where $k = 1, 2, \dots, i$, this sum becomes

$$S_{i-1}(\mathbf{a}) + a_0X_i = \sum_{j=1}^{k}\left(\frac{a_{i-j}}{c_{i-j}+1} + \frac{a_0c_{i-j}}{c_{i-j}+1}\right)Y_j + \sum_{j=k}^{i-1} a_{i-j}Y_{j+1}, \tag{2.16}$$

where we put $c_0 = 0$. Note that if the a_j's are not constant, then the right-hand side of (2.16) will depend on k. This fact will impose the condition $2m < n$, when considering the m-step strategies in Chapter 4 (see the derivation of Theorem 4.1). Also, as we will see in Section 5.4, the same issue makes it difficult to use our methods for analyzing the minimal travel time.

2.3 Main theorem for uniform spacings

Let $U_1, U_2, \dots, U_n$ be independent random variables uniformly distributed on the interval $[0,1)$, and let $U_{(1)}, U_{(2)}, \dots, U_{(n)}$ denote their order statistics. Put $U_{(0)} = 0$, $U_{(n+1)} = 1$. We define uniform spacings $D_1, D_2, \dots, D_{n+1}$ as

$$D_i = U_{(i)} - U_{(i-1)}, \quad 1 \le i \le n+1. \tag{2.17}$$

In Section 1.2 we gave a brief overview on the properties and applications of the uniform spacings. One of the classical properties is that those spacings are distributed as i.i.d. exponentials divided by their sum:

$$(D_1, D_2, \dots, D_{n+1}) \stackrel{d}{=} (X_1/S_{n+1}, X_2/S_{n+1}, \dots, X_{n+1}/S_{n+1}). \tag{2.18}$$

Pyke [72] gives a simple proof of (2.18). This proof also implies the well-known result that the normalized exponentials are independent of their sum. We already used this fact in Remark 2.2.

Property (2.18) allows us to easily prove a theorem similar to Theorem 2.1 for the uniform spacings. Indeed, assume that $N \le n+1$. As in Corollary 2.2, consider the function $f(x, y) = x/y$, where the first argument is the left-hand side of (2.2) or (2.3), and the second argument is S_{n+1}. Then Corollary 2.2 implies the following result.

Theorem 2.2 *For any* $N \le n+1$; $n = 1, 2, \dots,$

$$\sum_{j=1}^{N} \frac{(X_j - bS_{j-1})_+}{S_{n+1}} \stackrel{d}{=} \sum_{j=1}^{N} \left(\frac{1}{(b+1)^{j-1}} \right) \frac{X_j}{S_{n+1}}, \quad b \ge 0; \tag{2.19}$$

$$\max_{1 \le j \le N} \left\{ \frac{X_j - bS_{j-1}}{S_{n+1}} \right\} \stackrel{d}{=} \sum_{j=1}^{N} \left(\frac{b}{(b+1)^j - 1} \right) \frac{X_j}{S_{n+1}}, \quad b \ge 0. \tag{2.20}$$

Remark 2.4 An alternative way to prove Theorem 2.2 is to show that (2.2) and (2.19) are equivalent as well as (2.3) and (2.20). Note that in general none of the equalities

$$f(X_1, \dots, X_{n+1}) \stackrel{d}{=} g(X_1, \dots, X_{n+1})$$

and

$$f(X_1, \dots, X_{n+1})/S_{n+1} \stackrel{d}{=} g(X_1, \dots, X_{n+1})/S_{n+1}$$

implies the other. For example, let $F(\cdot)$ be the distribution function of S_{n+1}. Since the random variable $1 - \exp(-X_1)$ is uniformly distributed on $[0, 1)$, we have

$$F^{-1}(1 - \exp(-X_1)) \stackrel{d}{=} S_{n+1}.$$

However,

$$F^{-1}(1 - \exp(-X_1))/S_{n+1} \stackrel{d}{\neq} S_{n+1}/S_{n+1} \equiv 1.$$

Nevertheless, let us show the equivalence of, say, (2.2) and (2.19). Denote the left-hand side and the right-hand side of (2.19) by $V_{n,N}$ and $W_{n,N}$, respectively. Then (2.19) becomes

$$V_{n,N} \stackrel{d}{=} W_{n,N}. \tag{2.21}$$

Also, (2.2) can be rewritten as

$$S_{n+1}V_{n,N} \stackrel{d}{=} S_{n+1}W_{n,N}. \tag{2.22}$$

Equation (2.22) follows from (2.21), because $V_{n,N}$ and $W_{n,N}$ are functions of normalized exponentials X_1/S_{n+1}, X_2/S_{n+1}, ..., X_{n+1}/S_{n+1} which are independent of their sum S_{n+1}. Furthermore, since all moments of S_{n+1}, $V_{n,N}$ and $W_{n,N}$ exist, it follows from (2.22) that

$$E(V_{n,N}^k) = E(W_{n,N}^k), \quad k \geq 1.$$

Clearly, the distribution of each of the random variables $V_{n,N}$ and $W_{n,N}$ has a finite support. Such a distribution is uniquely defined by its moments, because its Laplace transform is analytic on the whole complex plane (see also Section VII.3 and Section XV.4 of Feller [32]). From the above we conclude that (2.22) and (2.21) are equivalent.

The distributional identities from Theorem 2.2 (with $b = 1$) provide further results on collecting n items randomly located on a circle. Specifically, the result (2.19) will be directly used in the next chapter to derive the distribution of the travel time under the Nearest Item heuristic. Further, formula (2.20) is needed in Chapter 4 where we discuss the One-Side-Optimal strategy and the m-step strategies.

2.4 Distribution of linear combinations of uniform spacings

In the right-hand side of (2.19), (2.20) we have random variables distributed as linear combinations of uniform spacings. Such random variables have been studied by many authors (see e.g. Steutel [80] and references therein). The distribution of linear combinations of uniform spacings was derived by Ali [1]. We use this result in the form presented in Theorem 2 of Ali and Obaidullah [2].

Consider the n-th divided differences $[f(x)|x = a_0, a_1, \dots, a_n]$ of a function $f(x)$. For distinct values of $a_0, a_1, \dots, a_n$ the definition of $[f(x)|x = a_0, a_1, \dots, a_n]$ is given in Isaacson and Keller [53] as follows. Let $Q_n(x)$ be the unique interpolation polynomial of degree at most n, with respect to $n+1$ distinct points $a_0, a_1, \dots, a_n$. Then $[f(x)|x = a_0, a_1, \dots, a_n]$ equals the leading coefficient, the coefficient at x^n, of $Q_n(x)$. The divided differences can be also seen as a discrete analogue of derivatives. For more details we refer to Section 6.1 of [53].

By definition, if the values of $a_0, a_1, \dots, a_n$ are distinct, then

$$[f(x)|x = a_0, a_1, \dots, a_n] = \sum_{k=0}^{n} f(a_k) \prod_{\substack{j=0 \\ j\neq k}}^{n} (a_k - a_j)^{-1}. \tag{2.23}$$

For coinciding values of $a_0, a_1, \dots, a_n$ we quote the following formula from [53, p. 254]. Suppose that $a_0, a_1, \dots, a_n$ are relabelled as $b_0, b_1, \dots, b_m$ ($b_i \neq b_j$ for

$i \neq j$) where b_ν is repeated $p_\nu + 1$ times, $p_\nu \geq 0$; $\nu = 0, 1, \ldots, m$, so that $p_0 + p_1 + \cdots + p_m + m = n$. If $f(x) = (x_+)^n \exp\{-\alpha/x\}$ or $f(x) = (x_-)^n \exp\{-\alpha/x\}$, where $x_- = x$ if $x < 0$ and $x_- = 0$ otherwise, then

$$[f(x)|x = a_0, a_1, \ldots, a_n] = \prod_{\nu=0}^{m} \frac{1}{p_\nu!} \left\{ \prod_{\nu=0}^{m} \left(\frac{\partial}{\partial b_\nu} \right)^{p_\nu} \right\} [f(x)|x = b_0, b_1 \ldots, b_m] \,. \tag{2.24}$$

Now the distribution of

$$T_n = \sum_{i=1}^{n} a_i D_i$$

can be found from the following theorem.

Theorem 2.3 *(Ali [1]) Let D_1, D_2, $\ldots$, D_{n+1} be the uniform spacings as defined by (2.17). Then*

$$\mathrm{P}\,(T_n < t) = [\{(x - t)_-\}^n | x = a_0, a_1, \ldots, a_n] \,, \tag{2.25}$$

where $a_0 = 0$ and the divided difference is as defined in (2.23) and (2.24).

Below we explore (2.25) to derive closed-form expressions for the distribution function of T_n with distinct and coinciding coefficients.

2.4.1 Distinct coefficients

Assume that all the coefficients $a_0, a_1, \ldots, a_n$ are distinct. Then (2.25) gives

$$\mathrm{P}(T_n < t) = \sum_{k=0}^{n} (t - a_k)_+^n \prod_{\substack{j=0 \\ j \neq k}}^{n} (a_j - a_k)^{-1}, \quad 0 \leq t \leq 1. \tag{2.26}$$

In the applications we will consider only $a_j \geq 0$, $j = 0, 1, \ldots, n$. Suppose that $0 = a_0 < a_1 < \cdots < a_n$. Then formula (2.26) gives different expressions for the distribution function of T_n in the sequence of intervals $(0, a_1], (a_1, a_2], \ldots, (a_{n-1}, a_n]$. The expression in $(0, a_1]$ is the simplest one:

$$\mathrm{P}(T_n < t) = t^n \prod_{j=1}^{n} a_j^{-1}, \qquad 0 < t \leq a_1.$$

This formula has a clear probabilistic interpretation. Let $t \in (0, a_1]$. Then it follows from (2.18) and from the memory-less property that

$$\begin{aligned} \mathrm{P}\left(\sum_{j=1}^{n} a_j D_j < t \right) &= \mathrm{P}\left(\sum_{j=1}^{n} a_j X_j < t S_{n+1} \right) = \mathrm{P}\left(\sum_{j=1}^{n} (a_j - t) X_j < t X_{n+1} \right) \\ &= \prod_{j=1}^{n} \mathrm{P}\left((a_j - t) X_j < t X_{n+1} \right) = t^n \prod_{j=1}^{n} a_j^{-1}. \end{aligned}$$

In the interval $(a_1, a_2]$ we get one additional term:

$$\mathrm{P}(T_n < t) \quad = \quad t^n \prod_{j=1}^{n} a_j^{-1} + (t-a_1)^n (-a_1)^{-1} \prod_{j=2}^{n} (a_j - a_1)^{-1}, \quad a_1 < t \le a_2.$$

Further an extra term appears in each of the following intervals. For $t > a_n$, all the $n+1$ terms are present in the sum. Moreover, in this case all these terms sum up to 1, because by definition of T_n,

$$\mathrm{P}(T_n < t) = 1, \quad t > a_n.$$

In fact, it holds that

$$\sum_{k=0}^{n} (t-a_k)^n \prod_{\substack{j=0 \\ j \ne k}}^{n} (a_j - a_k)^{-1} = 1, \quad t \in \mathbb{R}. \tag{2.27}$$

Let us give a simple proof of (2.27). Consider the function

$$f(s,t) = (1-st)^n \prod_{j=0}^{n} \frac{c_j}{c_j - s}, \quad t \in \mathbb{R},\ s \in \mathbb{C}.$$

It has the following expansion in rational fractions of s:

$$f(s,t) = \sum_{k=0}^{n} (1-c_k t)^n \frac{c_k}{c_k - s} \prod_{\substack{j=0 \\ j \ne k}}^{n} \frac{c_j}{c_j - c_k}, \quad t \in \mathbb{R},\ s \in \mathbb{C}.$$

Putting $s = 0$ in both sides of the last equation and substituting $a_j = 1/c_j$ in the right-hand side we get (2.27). Here $a_0 = 0$ corresponds to $c_0 \to \infty$.

Combining (2.26) with (2.27) we obtain another formula for the distribution of T_n:

$$\mathrm{P}(T_n < t) = 1 - \sum_{k=0}^{n} (t-a_k)_-^n \prod_{\substack{j=0 \\ j \ne k}}^{n} (a_j - a_k)^{-1}, \quad 0 \le t \le 1. \tag{2.28}$$

This formula provides the simplest expression $\mathrm{P}(T_n < t) = 1$ for $t > a_n$. Further, a new term appears in each of the intervals $(a_{n-1}, a_n], (a_{n-2}, a_{n-1}], \ldots, (0, a_1]$. Comparing it with (2.26), in (2.28) the new terms appear in the opposite order and with the sign multiplied by $(-1)^{n+1}$. Also, note that in (2.28) a term

$$(t-a_k)^n \prod_{\substack{j=0 \\ j \ne k}}^{n} (a_j - a_k)^{-1},$$

where $k = 0, 1, \ldots, n$, arises for the first time on the interval $(a_{k-1}, a_k]$, whereas in (2.26) it appears on the interval $(a_k, a_{k+1}]$ (here we assume that $a_{-1} = -\infty$ and $a_{n+1} = +\infty$). Use of (2.28) yields an expression with less terms than in (2.26) for $a_{k-1} < t \le a_k$, when k is greater than $n/2$.

2.4.2 Coinciding coefficients

The case of coinciding coefficients is more complicated, because we have to use formula (2.24) which includes partial derivatives. In the present work we derive the following closed-form expression for $\mathrm{P}(T_n < t)$:

$$\mathrm{P}(T_n < t) = \sum_{k=0}^{m} (t-b_k)_+^{n-p_k} \prod_{\substack{j=0\\ j\neq k}}^{m} (b_j - b_k)^{-p_j-1} \tag{2.29}$$
$$\times \left\{ \sum_{l=0}^{p_k} (-1)^l \binom{n}{p_k - l} (t-b_k)_+^l \sum_{\substack{l_0,\dots,l_m\geq 0\\ l_0+\dots+l_m=l\\ l_k=0,\ l_\nu\leq p_\nu+1}} \prod_{\substack{\nu=0\\ \nu\neq k}}^{m} \binom{p_\nu+1}{l_\nu} (b_\nu - b_k)^{-l_\nu} \right\},$$

where $b_0, b_1, \dots, b_m$ being repeated $p_\nu + 1$, $\nu = 0, 1, \dots, m$, times coefficients $a_0, a_1, \dots, a_n$. The motivation for this formula is the following. Assume that $0 = b_0 < b_1 < \cdots < b_m$ and let $t \in (0, b_1]$. Suppose that the coefficient b_0 has a multiplicity greater than 1, i.e., $p_0 > 0$. This actually means that not all the spacings $D_1, D_2, \dots, D_n$ are present in T_n. Then, exactly as for distinct coefficients,

$$\mathrm{P}\left(\sum_{j=1}^{n} a_j D_j < t\right) = \mathrm{P}\left(\sum_{\nu=1}^{m} b_\nu \sum_{j=0}^{p_\nu} X_{\nu,j} < t \sum_{\nu=0}^{m}\sum_{j=0}^{p_\nu} X_{\nu,j}\right)$$
$$= \mathrm{P}\left(\sum_{\nu=1}^{m} (b_\nu - t) \sum_{j=0}^{p_\nu} X_{\nu,j} < t(X_{0,0} + X_{0,1} + \dots + X_{0,p_0})\right)$$
$$= t^{n-p_0} \prod_{j=1}^{m} (b_j)^{-p_j-1} \left\{ \sum_{l=0}^{p_0} (-1)^l \binom{n}{p_0 - l} t^l \sum_{\substack{l_1,\dots,l_m\geq 0\\ l_1+\dots+l_m=l\\ l_\nu\leq p_\nu+1}} \prod_{\nu=1}^{m} \binom{p_\nu+1}{l_\nu} b_\nu^{-l_\nu} \right\},$$

where $X_{\nu,j}$, $\nu = 0, 1, \dots, m$; $j \geq 0$ are again independent exponentials with mean 1. The last expression follows from lengthy combinatorial arguments. This expression is identical to the term with $k = 0$ in the right-hand side of (2.29). Since in case of distinct coefficients the terms for all $k = 0, 1, \dots, n$ have the same form (cf. (2.26)), then from considerations of continuity this should also hold for coinciding coefficients providing (2.29).

We prove (2.29) by induction in the following way. If all coefficients are distinct, i.e. $m = n$; $p_\nu = 0$, $\nu = 0, 1, \dots, n$, then formula (2.29) is just (2.26). Now let some coefficients in T_n coincide. Without loss of generality assume that $p_m > 0$. In T_n we replace one of the coefficients b_m by $b_m - \varepsilon$, where ε is arbitrarily small. Denote the new linear combination by $T_{n,\varepsilon}$. If ε is small enough, then the coefficient $b_m - \varepsilon$ in $T_{n,\varepsilon}$ is distinct from the others, and, obviously,

$$\lim_{\varepsilon\to 0} \mathrm{P}(T_{n,\varepsilon} < t) = \mathrm{P}(T_n < t).$$

In order to prove (2.29) assume that it holds for the distribution of $T_{n,\varepsilon}$ and let $\varepsilon \to 0$. After laborious algebra it gives the desired result.

If $0 = b_1 < b_2 < \cdots < b_m$, then a new term in the right-hand side of (2.29) appears on each of the intervals $(0, b_1]$, $(b_1, b_2]$, ..., $(b_{m-1}, b_m]$. Also, (2.29) can be rewritten in a form analogous to (2.28):

$$\mathrm{P}(T_n < t) = 1 - \sum_{k=0}^{m} (t - b_k)_-^{n-p_k} \prod_{\substack{j=0\\ j\neq k}}^{m} (b_j - b_k)^{-p_j-1} \tag{2.30}$$
$$\times \left\{ \sum_{l=0}^{p_k} (-1)^l \binom{n}{p_k - l} (t - b_k)_-^l \sum_{\substack{l_0,\dots,l_m \geq 0\\ l_0+\dots+l_m = l\\ l_k = 0,\ l_\nu \leq p_\nu + 1}} \prod_{\substack{\nu=0\\ \nu\neq k}}^{m} \binom{p_\nu + 1}{l_\nu} (b_\nu - b_k)^{-l_\nu} \right\}.$$

Sometimes, formula (2.30) can be very useful for calculating $\mathrm{P}(T_n < t)$. Assume, for example, that $0 = b_0 < b_1 < \ldots < b_m$; $p_0 = n - m > 0$; $p_\nu = 0$, $\nu = 1, 2, \ldots, m$. In other words, only the spacings $D_1, D_2, \ldots, D_m$ are present in T_n, and they have distinct positive coefficients. This occurs, for example, in the right-hand side of (2.19) and (2.20) when $m = N < n$. In this case already the first term appearing in (2.29) is very complicated whereas in (2.30) for any $t \geq 0$ all the terms are of the form

$$(t - b_k)_-^n (-b_k)^{-n+m-1} \prod_{\substack{j=1\\ j\neq k}}^{m} (b_j - b_k)^{-1},$$

where $k = 1, 2, \ldots, m$. The complicated term would appear in (2.30) only for $t < 0$, when we may just write $\mathrm{P}(T_n < t) = 0$.

2.4.3 The moments

In this short section we will give a formula for calculating the moments of T_n. First, we write

$$\begin{aligned} \mathrm{E}\left(T_n^k\right) &= \mathrm{E}\left[\left(\sum_{i=1}^{n} a_i D_i\right)^k\right] \qquad (2.31)\\ &= \sum_{\substack{k_1,\dots,k_n \geq 0\\ k_1+\dots+k_n = k}} \frac{k!}{k_1! k_2! \ldots k_n!} \mathrm{E}\left(D_1^{k_1} D_2^{k_2} \ldots D_n^{k_n}\right) \prod_{i=1}^{n} a_i^{k_i}. \end{aligned}$$

For any collection $k_1, k_2, \ldots, k_{n+1}$ of nonnegative integers it holds that

$$\mathrm{E}\left(D_1^{k_1} D_2^{k_2} \ldots D_{n+1}^{k_{n+1}}\right) = \frac{k_1! k_2! \ldots k_{n+1}! n!}{(n + k_1 + k_2 + \cdots + k_{n+1})!}. \tag{2.32}$$

Indeed, under the condition that $D_3 = d_3, \dots, D_{n+1} = d_{n+1}$, the random variable D_1 is uniform on the interval $[0, 1 - d_3 - \cdots - d_{n+1}]$ (cf. Sec. 13.1 in Karlin and Taylor [57]). Hence, by conditioning and partial integration we obtain

$$\mathrm{E}\left(\frac{D_1^{k_1}}{k_1!}\frac{D_2^{k_2}}{k_2!}\frac{D_3^{k_3}}{k_3!}\cdots\frac{D_{n+1}^{k_{n+1}}}{k_{n+1}!}\right) = \mathrm{E}\left(\frac{D_1^{k_1+1}}{(k_1+1)!}\frac{D_2^{k_2-1}}{(k_2-1)!}\frac{D_3^{k_3}}{k_3!}\cdots\frac{D_{n+1}^{k_{n+1}}}{k_{n+1}!}\right).$$

By symmetry and repeatedly applying this equality we find

$$\mathrm{E}\left(\frac{D_1^{k_1}}{k_1!}\frac{D_2^{k_2}}{k_2!}\cdots\frac{D_{n+1}^{k_{n+1}}}{k_{n+1}!}\right) = \mathrm{E}\left(\frac{D_1^{k_1+\cdots+k_{n+1}}}{(k_1+\cdots+k_{n+1})!}\right).$$

Using $\mathrm{E}(D_1^k/k!) = n!/(n+k)!$ yields (2.32). In fact, formula (2.32) has been derived many times in different ways. For example, (2.32) follows from more general formula (9) in [80]. By substituting (2.32) into (2.31) we finally arrive at

$$\mathrm{E}\left(T_n^k\right) \quad = \quad \binom{n+k}{k}^{-1} \sum_{\substack{k_1,k_2,\dots,k_n \geq 0 \\ k_1+k_2+\cdots+k_n=k}} \prod_{i=1}^{n} a_i^{k_i}. \tag{2.33}$$

Obviously, for $k = 1$ we have:

$$\mathrm{E}\left(T_n\right) = \frac{1}{n+1}\sum_{i=1}^{n} a_i. \tag{2.34}$$

Further, from (2.32) it follows that

$$\mathrm{E}(D_1^2) = \frac{2}{(n+1)(n+2)}, \quad \mathrm{E}(D_1 D_2) = \frac{1}{(n+1)(n+2)}.$$

Hence,

$$\mathrm{E}\left(T_n^2\right) = \frac{1}{(n+1)(n+2)}\left[\sum_{i=1}^{n} a_i^2 + \left(\sum_{i=1}^{n} a_i\right)^2\right], \tag{2.35}$$

$$\mathrm{Var}(T_n) = \frac{1}{(n+1)(n+2)}\sum_{i=1}^{n} a_i^2 - \frac{1}{(n+1)^2(n+2)}\left(\sum_{i=1}^{n} a_i\right)^2. \tag{2.36}$$

2.5 Concluding remarks

In this chapter we studied some curious properties of exponential random variables and uniform spacings.

First, we considered two functions whose arguments are i.i.d. exponential random variables and we showed that these functions are distributed as linear combinations of exponentials. To prove this result we developed a method based on the memory-less property. We pointed out the most important features of the proof and derived some auxiliary results and corollaries.

Further, we proved similar results for the same two functions when the arguments are uniform spacings. Our methods worked again, because uniform spacings are distributed as i.i.d. exponentials divided by their sum. Since a linear combination of uniform spacings is not an object just as simple as a linear combination of exponentials, we studied its probabilistic properties in detail. Using a result of Ali [1] we derived a closed form expression for the distribution function in case of distinct and coinciding coefficients. Also, we gave formulas for calculating the moments of linear combinations of the spacings.

In the sequel we will use the results of this chapter to elaborate an exhaustive probabilistic analysis for the travel time while collecting n items randomly located on a circle.

Chapter 3

The Nearest Item (NI) heuristic

3.1 The model

In this chapter we study the problem of collecting n items on a circle under the Nearest Item (NI) heuristic. As above, we consider a circle of length 1. The picker starts at position $U_0 = 0$, and he has to collect n items located at positions $U_1, U_2, \ldots, U_n$, where the U_i's are independent and uniformly distributed on $[0, 1)$. The NI heuristic is defined as follows (*cf.* Bartholdi and Platzman [6]):

Nearest Item (NI) heuristic: Always travel to the nearest item to be retrieved.

An example of the route under the NI heuristic is given in Figure 3.1. Note that

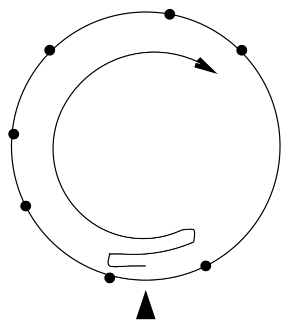

Figure 3.1: A route under the NI heuristic.

this route is not optimal, because the picker has to turn twice whereas the optimal route admits no more than one turn (see Section 1.1). Below in Example 3.1 we will meet another situation where the travel time under the NI heuristic is significantly bigger than the optimal travel time (see Remark 3.3). Moreover, in Chapter 4 we propose quite simple m-step strategies whose travel time has smaller mean and variance than the travel time under the NI heuristic (see Section 4.6). Nevertheless, the NI heuristic usually performs quite well except in some pathological cases, and it produces solutions that are guaranteed to be never too far from optimal.

The NI heuristic is related to the *greedy server* model studied in the framework of queueing theory; see, e.g., Kroese and Schmidt [60] and the references therein. In this model customers arrive according to a Poisson process randomly distributed on a circle and wait to be served by a single server. The server travels on the circle and he is greedy in the sense that the next customer to be served is always the nearest one. In fact, this model describes a carousel picking orders (for one item) *on-line* under the NI heuristic.

In practice, carousel systems often operate under the NI heuristic, because it provides reasonable control without much (computational) effort. Furthermore, simple heuristics are also useful in *on-line* (dynamic) situations. In fact, in such situations, they usually perform much better than '(static) optimal' strategies.

Analytically the performance of the NI heuristic for carousel systems has been partly investigated by Bartholdi and Platzman [6]. They prove that the travel time under the NI heuristic is never greater than one rotation of the carousel. Litvak *et al.* [64] improve this upper bound and show that the new upper bound is tight. Using an analytical approach, they find the mean and variance of the remaining travel time under the NI heuristic, i.e., the travel time, when there is an empty space at one side of the picker's position. They also introduce a probabilistic approach to determine the mean total travel time. Litvak and Adan [65] elaborate this probabilistic approach and show that it enables us to completely analyze the travel time. The method itself and the results that it yields were discussed in detail in Chapter 2.

The present chapter mainly contains the analysis from Litvak *et al.* [64] and Litvak and Adan [65]. In Section 3.2 we improve the upper bound of Bartholdi and Platzman [6] for the travel time and we show that the new upper bound is tight. In Section 3.3 we develop a recursive procedure to derive a closed-form expression for the mean and the variance of the travel time conditioned on the size of the empty space at one side of the picker's position. In Section 3.4 we recursively find the conditional, the unconditional and the limiting distribution for the number of turns. In the further analysis we use the methods and the results from Chapter 2. In Section 3.5 we prove that the travel time under the NI heuristic can be represented as a linear combination of uniform spacings, and we give a closed form expression for the distribution and the moments of the travel time. In Section 3.6 we give an exhaustive analysis of the limiting behavior of the travel time distribution. Finally, in Section 3.7 we show an alternative way to derive the distribution of the number of turns. Moreover, we prove that the travel time is independent of the number of turns. In Section 3.8 we briefly summarize the results.

3.2 Upper bounds for the travel time

The main object in this section is to establish an upper bound for the travel time under the NI heuristic and to prove its tightness. We will use the following important feature of the NI heuristic that we call a 'recursive' property:

Property 3.1 *The remaining part of the NI heuristic is equal to the NI heuristic for the rest of the items with the picker's current position as starting point.*

To obtain the upper bounds for the NI heuristic we will compare it with the Shorter Direction (SD) heuristic:

Shorter Direction (SD) heuristic: The picker evaluates the length of the route that simply rotates clockwise and the length of the route that simply rotates counter-clockwise. Then he chooses the shorter of these two routes.

Since the NI heuristic seems to be slightly more subtle than the SD heuristic, one may expect that it performs better with high probability. In fact, we will prove that the NI heuristic is never worse than the SD heuristic.

Let

$$\omega = (\omega_1, \omega_2, \dots, \omega_n) \in [0,1)^n$$

be a realization of the random vector $(U_1, U_2, \dots, U_n)$. By applying the NI heuristic to retrieve a list of n items, the picker will subsequently visit the positions ω_{i_1}, ω_{i_2}, ..., ω_{i_n}. For convenience we denote

$$x_l = \omega_{i_l},\ l = 1, 2, \dots, n;\ x_0 = 0.$$

Further, denote

T_n^{NI} – the travel time to retrieve n items under the NI heuristic.

T_n^{SD} – the travel time to retrieve n items under the SD heuristic.

The random variables T_n^{NI} and T_n^{SD} are of course functions of the elementary random event $\omega \in [0,1)^n$. The following lemma establishes a relation between these two functions for arbitrary ω.

Lemma 3.1 *For any $\omega \in [0,1)^n$ it holds that $T_n^{NI}(\omega) \le T_n^{SD}(\omega)$.*

Proof. We will present a proof by induction to n. It is clear that for any $\omega \in [0,1)$ we have $T_1^{NI}(\omega) = T_1^{SD}(\omega) = \rho(x_0, x_1)$, where $\rho(y,z)$ is the shortest distance between the positions y and z. Now suppose that for some $n = 1, 2, \dots$ we have $T_n^{NI}(\omega) \le T_n^{SD}(\omega)$, $\omega \in [0,1)^n$. Then we will prove that $T_{n+1}^{NI}(\omega) \le T_{n+1}^{SD}(\omega)$, $\omega \in [0,1)^{n+1}$. The proof is illustrated in Figure 3.2. First, recall that under the SD heuristic the picker always proceeds in the same direction. There are only two possible routes of that kind, and their lengths differ only in the first segment. Therefore, choosing the shorter direction actually means choosing the shorter first interval. Hence, the algorithm for the SD heuristic can be formulated as follows:

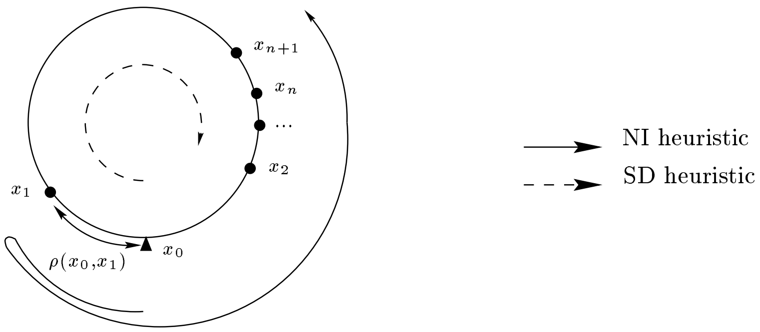

Figure 3.2: An illustration for the proof of Lemma 3.1.

Step 1: Rotate to the nearest item.

Step 2: Proceed further in the same direction.

It means that the NI and SD heuristic start with the same segment of length $\rho(x_0, x_1)$. After the first step the picker is at position x_1 and n items remain to be picked. If we now consider x_1 as a starting point, then the current situation can be described by some $\omega' \in [0,1)^n$. The remaining travel time under the SD heuristic cannot be shorter than $T_n^{SD}(\omega')$, since by definition $T_n^{SD}(\omega')$ is the minimum travel time needed to pick n items by proceeding in the same direction. Hence,

$$\rho(x_0, x_1) + T_n^{SD}(\omega') \le T_{n+1}^{SD}(\omega). \tag{3.1}$$

Further, due to Property 3.1 we have

$$T_{n+1}^{NI}(\omega) = \rho(x_0, x_1) + T_n^{NI}(\omega'). \tag{3.2}$$

From (3.2), the induction assumption and (3.1) it follows that

$$T_{n+1}^{NI}(\omega) = \rho(x_0, x_1) + T_n^{NI}(\omega') \le \rho(x_0, x_1) + T_n^{SD}(\omega') \le T_{n+1}^{SD}(\omega),$$

which completes the proof. □

In order to pick n items under the NI heuristic, n segments of the circle should be covered. Their lengths are $\rho(x_0, x_1)$, $\rho(x_1, x_2)$, $\ldots$, $\rho(x_{n-1}, x_n)$. Note that they do not necessarily coincide with spacings between two adjacent items, since under the NI heuristic the picker can travel in different directions (see Figures 3.1, 3.2). Bartholdi and Platzman [6] showed that T_n^{NI} is always less than 1 for all n. Now we will use Lemma 3.1 to prove the following stronger assertion.

Theorem 3.2 *For any $\omega \in [0,1)^n$ and any $k = 1, 2, \ldots, n$, the total length of k arbitrarily chosen segments that arise under the NI heuristic never exceeds $1 - 1/2^k$.*

Proof. Consider the NI heuristic starting at $x_0 = 0$. Let $1 \le l_1 < l_2 < \ldots < l_k \le n$ be the indices of k arbitrarily chosen segments in the order we cover them, and $\rho(x_{l_1-1}, x_{l_1})$, $\rho(x_{l_2-1}, x_{l_2})$, ..., $\rho(x_{l_k-1}, x_{l_k})$ are their corresponding lengths.

We proceed with the NI heuristic until facing the first segment l_1. Now the picker is at point x_{l_1-1}, and there are still $n - l_1 + 1$ positions to be visited.

Consider the case that $\rho(x_{l_1-1}, x_{l_1}) \ge 1/2^k$. If we pick the remaining $n - l_1 + 1$ items under the SD heuristic starting at point x_{l_1-1}, then the travel time cannot exceed $1 - 1/2^k$. Then, from Property 3.1 and Lemma 3.1 it follows that the remaining travel time under the NI heuristic also does not exceed $1 - 1/2^k$. Recall that l_1 is the first one of the k chosen segments faced under the NI heuristic. Hence, all k segments under consideration are included in the remaining path. So, their total length cannot be greater than $1 - 1/2^k$.

Now, assume that $\rho(x_{l_1-1}, x_{l_1}) < 1/2^k$. Then we proceed further until segment l_2 is faced. If $\rho(x_{l_2-1}, x_{l_2}) \ge 1/2^{k-1}$, then we can use similar arguments as above to conclude that the total length of the remaining $k-1$ of the k chosen segments is not greater than $1 - 1/2^{k-1}$, and it immediately follows that the total length of k chosen segments does not exceed

$$\rho(x_{l_1-1}, x_{l_1}) + 1 - 1/2^{k-1} < 1/2^k + 1 - 1/2^{k-1} = 1 - 1/2^k.$$

If $\rho(x_{l_2-1}, x_{l_2}) < 1/2^{k-1}$, then we proceed with the NI heuristic and repeat the same arguments. Finally, two cases are possible:

(1) There exists an $i = 2, 3, \ldots, k$ such that $\rho(x_{l_j-1}, x_{l_j}) < 1/2^{k-j+1}$, $j = 1, 2, \ldots, i-1$, and $\rho(x_{l_i-1}, x_{l_i}) \ge 1/2^{k-i+1}$. In this case the remaining path under the NI heuristic is not longer than $1 - 1/2^{k-i+1}$, and therefore the total length of k chosen segments does not exceed

$$\begin{aligned} \sum_{j=1}^{i-1} \rho(x_{l_j-1}, x_{l_j}) + 1 - \frac{1}{2^{k-i+1}} &< \frac{1}{2^k} + \frac{1}{2^{k-1}} + \ldots + \frac{1}{2^{k-i+2}} + 1 - \frac{1}{2^{k-i+1}} \\ &= 1 - \frac{1}{2^k}. \end{aligned}$$

(2) For each $i = 2, 3, \ldots, k$ we have $\rho(x_{l_i-1}, x_{l_i}) < 1/2^{k-i+1}$. Then the total length of the k chosen segments is less than

$$\frac{1}{2^k} + \frac{1}{2^{k-1}} + \ldots + \frac{1}{2} = 1 - \frac{1}{2^k}.$$

Thus, in both cases the assertion of the theorem holds. □

Since the complete travel time is identical to the total length of the n segments, an upper bound for the travel time under the NI heuristic immediately follows from Theorem 3.2.

Corollary 3.1 *For each $\omega \in [0,1)^n$ the travel time under the NI heuristic satisfies*

$$T_n^{NI}(\omega) \le 1 - 1/2^n.$$

Let us give an example to show that Corollary 3.1 provides a tight upper bound.

Example 3.1 Let $n = 5$, and let the starting position of the picker be $x_0 = 0$. The items to be picked are located at the positions 1/32, 3/32, 7/32, 15/32 and $31/32 - \varepsilon$, where ε is positive and arbitrarily small (see Figure 3.3).

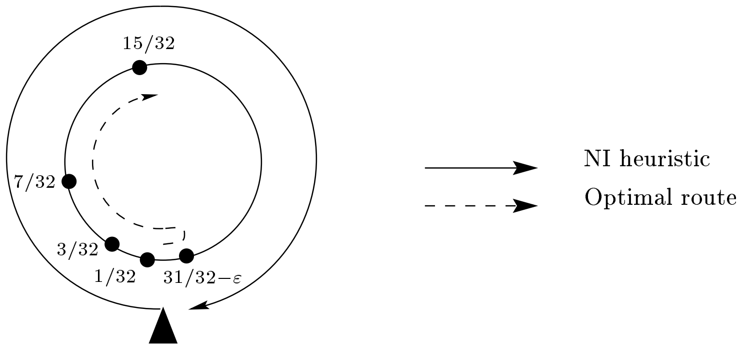

Figure 3.3: The largest travel time under the NI heuristic when $n = 5$.

Then the travel distance under the NI heuristic is

$$\frac{1}{32} + \frac{2}{32} + \frac{4}{32} + \frac{8}{32} + \left(\frac{16}{32} - \varepsilon\right) = \frac{31}{32} - \varepsilon = 1 - \frac{1}{2^5} - \varepsilon.$$

The upper bound $1 - 1/2^5$ is tight, since ε is arbitrarily small. A similar example can be easily constructed for any n.

Remark 3.1 In Example 3.1 the travel time does not really achieve its upper bound. However, if the picker starts at point $x_0 = 0$ and needs to pick only one item at point $x_1 = 1/2$, or two items at points $x_1 = 1/4$ and $x_2 = 3/4$, then the travel time is equal to its upper bound (1/2 and 3/4 respectively). For $n > 2$ the upper bound can also be achieved, if we assume that when the travel times to the nearest items clockwise and counterclockwise are exactly the same, the picker always proceeds, say, clockwise. Now, if we put $\varepsilon = 0$ in the example above, then the travel time will be exactly $1 - 1/2^5$.

Remark 3.2 Note that Example 3.1 is the only one we can construct to show that the upper bound is tight. Indeed, from the proof of Theorem 3.2 it follows that if the first segment is smaller or greater than $1/2^n$, then the travel time to pick n items under the NI heuristic is less than $1 - 1/2^n$. The only case when the upper bound can be achieved is when $\rho(x_0, x_1) = 1/2^n$. Then after the first step, the picker is at position x_1 and $n - 1$ items remain to be picked. Due to Property 3.1 we can use similar arguments to show that the upper bound can only be achieved if $\rho(x_1, x_2) = 1/2^{n-1}$. The same can be done for each of the n steps under the NI heuristic. It implies that the upper bound can be achieved if and only if the l-th segment has length $1/2^{n-l+1}$ for all $l = 1, 2, \ldots, n$.

Remark 3.3 Figure 3.3 also shows that the NI strategy is sometimes far from optimal. Indeed, in the case under consideration the optimal sequence is: $31/32-\varepsilon$, $1/32$, $3/32$, $7/32$, $15/32$. The total length of this route is

$$\left(\frac{1}{32}+\varepsilon\right)+\left(\frac{1}{32}+\varepsilon\right)+\frac{1}{32}+\frac{2}{32}+\frac{4}{32}+\frac{8}{32}=\frac{17}{32}+2\varepsilon,$$

which is much less than $31/32-\varepsilon$, when ε is small.

3.3 Mean and variance of the remaining travel time

In this section we expose an analytical approach to determine the moments of the *remaining travel time* after picking some items, i.e., conditioned on the size of the *known* empty space at one side of the picker's current position.

To derive a formula for the mean remaining travel time under the NI heuristic we will develop a procedure exploiting Property 3.1. According to this property the remaining part of the NI heuristic after the first step is equal to the NI heuristic for the other $n-1$ items with the picker's current position as starting point. The expected travel time of the first step can be found straightforwardly. However, the expectation of the remaining travel time is not just the mean travel time under the NI heuristic for $n-1$ items, because we also need to take into consideration the size of the empty space at one side of the picker's position. So, we can obtain a recursive equation for the mean travel time conditioned on the size of the empty space at one side of the picker's position. Denote by $\mathrm{E}\left(T_n^{NI}|\, x\right)$ the mean travel time under the NI heuristic, given that at one side of the picker's starting point there is an empty space of size x. Then the mean travel time under the NI heuristic is just equal to $\mathrm{E}\left(T_n^{NI}|\, 0\right)$:

$$\mathrm{E}\left(T_n^{NI}\right)=\mathrm{E}\left(T_n^{NI}|\, 0\right).$$

Our objective now is to derive a formula for $\mathrm{E}\left(T_n^{NI}|\, x\right)$, $0\le x<1$.

The case $1/2\le x<1$ is trivial, since in this case the picker will proceed in one direction only. It is easy to see that there are n segments to cover, and the average length of each segment is $(1-x)/(n+1)$. Thus, we have:

$$\mathrm{E}\left(T_n^{NI}|\, x\right)=\frac{n}{n+1}(1-x),\quad 1/2\le x<1. \tag{3.3}$$

Let us now consider $0\le x<1/2$. We will derive a recursive equation for $\mathrm{E}\left(T_n^{NI}|\, x\right)$ by conditioning on the location of the nearest item. Let $f_n(y|\, x)$ denote the density of the travel time to the nearest item given that there is an empty space of size x near the starting point. There are two possible cases, which are shown in Figure 3.4. For $y\le x$ we have

$$f_n(y|\, x)=n(1-x-y)^{n-1}/(1-x)^n,$$

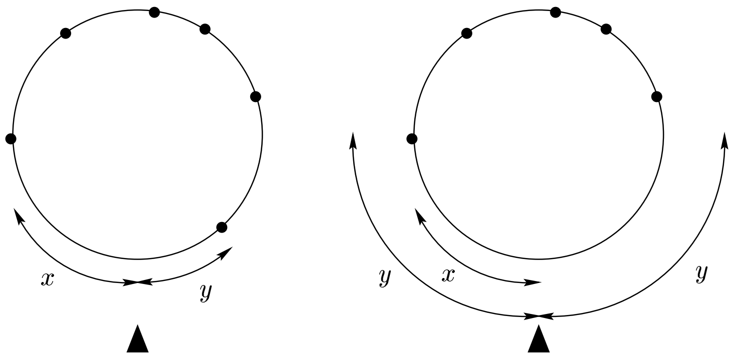

Figure 3.4: Two possible locations of the nearest item.

and after this step there will be an empty space of size $x+y$. For $x < y < 1/2$ it holds that

$$f_n(y|\,x) = 2n(1-2y)^{n-1}/(1-x)^n,$$

and after such a step, there will be an empty space of size $2y$. Now we use the full expectation formula:

$$\begin{aligned}\mathrm{E}\left(T_n^{NI}|\,x\right) &= \int_0^x \frac{n(1-x-y)^{n-1}}{(1-x)^n}\left[\mathrm{E}\left(T_{n-1}^{NI}|\,x+y\right)+y\right]\,dy \\ &+ \int_x^{1/2} \frac{2n(1-2y)^{n-1}}{(1-x)^n}\left[\mathrm{E}\left(T_{n-1}^{NI}|\,2y\right)+y\right]\,dy, \quad 0 \le x < 1/2. \qquad (3.4)\end{aligned}$$

To find a solution for equation (3.4) we first introduce the functions

$$G_n(x) = \mathrm{E}\left(T_n^{NI}|\,x\right)(1-x)^n, \quad 0 \le x < 1.$$

Now we can rewrite equation (3.4) in the following form:

$$\begin{aligned}G_n(x) &= \int_0^x nG_{n-1}(x+y)\,dy + \int_x^{1/2} 2nG_{n-1}(2y)\,dy \qquad (3.5)\\ &+ \int_0^x n(1-x-y)^{n-1}y\,dy + \int_x^{1/2} 2n(1-2y)^{n-1}y\,dy, \quad 0 \le x < 1/2.\end{aligned}$$

The last two integrals in (3.5) can be easily calculated, yielding

$$\int_0^x n(1-x-y)^{n-1}y\,dy + \int_x^{1/2} 2n(1-2y)^{n-1}y\,dy = \frac{(1-x)}{n+1}^{n+1} - \frac{(1-2x)}{2(n+1)}^{n+1}.$$

Putting $\tau = y+x$ in the first integral and $\tau = 2y$ in the second one, we simplify equation (3.5) to:

$$G_n(x) = \int_x^1 nG_{n-1}(\tau)\,d\tau + \frac{(1-x)}{n+1}^{n+1} - \frac{(1-2x)}{2(n+1)}^{n+1}, \quad 0 \le x < 1/2. \qquad (3.6)$$

In this case, the change of variables simplifies the recursion significantly. However, as we will see in Section 3.4, it does not always help that much. There we need to consider each of the intervals $0 \le x < 1/2^n$, $1/2^n \le x < 1/2^{n-1}$, ..., $1/4 \le x < 1/2$ separately, which makes the calculations much more complicated.

From (3.6) it is seen that one needs to know $G_{n-1}(x)$ at $1/2 \le x < 1$ to calculate $G_n(x)$ at $0 \le x < 1/2$. From (3.3) we have

$$G_n(x) = \frac{n}{n+1}\,(1-x)^{n+1}, \quad 1/2 \le x < 1. \tag{3.7}$$

Since

$$G_0(x) = \mathrm{E}\left(T_0^{NI} \,|\, x\right)(1-x)^0 \equiv 0,$$

the solution of (3.6) should be of the form

$$G_n(x) = a_n(1-x)^{n+1} + b_n(1-2x)^{n+1}, \quad 0 \le x < 1/2, \tag{3.8}$$

where

$$\begin{aligned} a_n &= \frac{n}{n+1}\,a_{n-1} + \frac{1}{n+1}; \quad a_0 = 0, \\ b_n &= \frac{n}{2(n+1)}\,b_{n-1} - \frac{1}{2(n+1)}; \quad b_0 = 0. \end{aligned}$$

Denoting $a'_n = (n+1)a_n$, $b'_n = (n+1)b_n$ we have:

$$\begin{aligned} a'_n &= a'_{n-1} + 1 \;=\; a'_0 + n \;=\; n, \\ b'_n &= \frac{1}{2}\,b'_{n-1} - \frac{1}{2} \;=\; \frac{1}{2^n}\,b'_0 - \sum_{i=1}^{n} \frac{1}{2^i} \;=\; \frac{1}{2^n} - 1, \end{aligned}$$

which gives

$$G_n(x) = \frac{n}{n+1}\,(1-x)^{n+1} - \frac{1}{n+1}\left(1 - \frac{1}{2^n}\right)(1-2x)^{n+1}, \quad 0 \le x < 1/2. \tag{3.9}$$

Function (3.9) satisfies both the recursion (3.6) and the initial condition $G_0(x) = 0$.

Remark 3.4 We could immediately say that if $G_n(x)$ satisfies (3.8), then a_n should necessarily be $n/(n+1)$. Otherwise, the function defined by (3.7) and (3.8) is not continuous at $x = 1/2$.

Our results are summarized in the following theorem:

Theorem 3.3 *For all* $n = 1, 2, \ldots$,

$$\mathrm{E}\left(T_n^{NI} \,|\, x\right) = \frac{n}{n+1}\,(1-x) - \left(1 - \frac{1}{2^n}\right)\frac{(1-2x)_+^{n+1}}{(n+1)(1-x)^n}, \quad 0 \le x < 1. \tag{3.10}$$

When we set $x = 0$ in (3.10), then we obtain the formula for the (unconditional) mean travel time:

$$\mathrm{E}\left(T_n^{NI}\right) = \frac{n}{n+1} - \frac{1}{(n+1)}\left(1 - \frac{1}{2^n}\right). \qquad (3.11)$$

Below in Figure 3.5 we show the conditional expectation of the travel time as a function of the empty space x for n equal to 2, 5 and 10. Surprisingly, we see that the graphs slightly increase for small x. This is very well seen for $n = 2$. It means that information about the empty space can be 'negative'. This may be explained by the fact that this information reduces the probability that items must be retrieved nearby the picker's position. Another observation is that the conditional expectation tends very fast to a linear function, which is of course also apparent from formula (3.10).

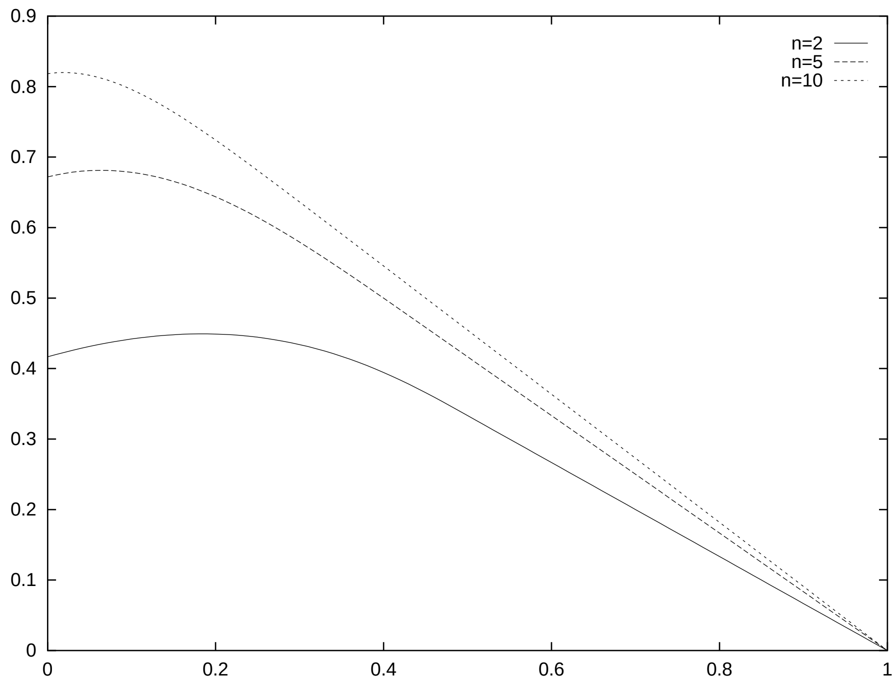

Figure 3.5: The conditional mean travel time as a function of the empty space x.

For the second moment we need to consider the conditional expectation $\mathrm{E}\left([T_n^{NI}]^2 \,|\, x\right)$. One can easily see that

$$\mathrm{E}\left([T_n^{NI}]^2 \,|\, x\right) = \frac{n}{n+2}\,(1-x)^2, \quad 1/2 \le x < 1, \qquad (3.12)$$

and that a recursive equation similar to (3.4) holds for $0 \le x < 1/2$:

$$\begin{aligned} \mathrm{E}\left([T_n^{NI}]^2 | \, x\right) &= \int_0^x \frac{n(1-x-y)^{n-1}}{(1-x)^n} \mathrm{E}\left(\left[T_{n-1}^{NI} + y\right]^2 \mid x+y\right) dy \\ &+ \int_x^{1/2} \frac{2n(1-2y)^{n-1}}{(1-x)^n} \mathrm{E}\left(\left[T_{n-1}^{NI} + y\right]^2 \mid 2y\right) dy, \quad 0 \le x < 1/2. \end{aligned} \tag{3.13}$$

By introducing the functions

$$G_n^{(2)}(x) = \mathrm{E}\left([T_n^{NI}]^2 | \, x\right)(1-x)^n,$$

and changing variables, we can rewrite (3.13) in the following form:

$$\begin{aligned} G_n^{(2)}(x) &= \int_x^1 n G_{n-1}^{(2)}(\tau)\, d\tau \\ &+ \int_0^x 2ny G_{n-1}(x+y)\, dy + \int_x^{1/2} 4ny G_{n-1}(2y)\, dy \\ &+ \int_0^x ny^2(1-x-y)^{n-1}\, dy + \int_x^{1/2} 2ny^2(1-2y)^{n-1}\, dy, \quad 0 \le x < 1/2. \end{aligned} \tag{3.14}$$

Substituting (3.7) and (3.9) into (3.14) for $1/4 \le x < 1/2$ we obtain

$$\begin{aligned} G_n^{(2)}(x) &= \int_x^1 n G_{n-1}^{(2)}(\tau)\, d\tau + \frac{2n(1-x)^{n+2}}{(n+1)(n+2)} \\ &- \frac{x(1-2x)^{n+1}}{n+1} - \left(n + \left(1 - \frac{1}{2^n}\right)\right) \frac{(1-2x)^{n+2}}{(n+1)(n+2)}, \end{aligned} \tag{3.15}$$

and for $0 \le x < 1/4$ we get

$$\begin{aligned} G_n^{(2)}(x) &= \int_x^1 n G_{n-1}^{(2)}(\tau)\, d\tau + \frac{2n(1-x)^{n+2}}{(n+1)(n+2)} - \frac{x(1-2x)^{n+1}}{n+1} \\ &- \left(n + \left(1 - \frac{1}{2^n}\right)\right) \frac{(1-2x)^{n+2}}{(n+1)(n+2)} + \frac{(1-4x)^{n+2}}{4(n+1)(n+2)} \left(1 - \frac{1}{2^{n-1}}\right). \end{aligned} \tag{3.16}$$

From (3.12) we know that

$$G_n^{(2)}(x) = \frac{n}{n+2}(1-x)^{n+2}, \quad 1/2 \le x < 1.$$

Thus, we can first solve the recursion (3.15) and then (3.16) exactly as it was done before. However, the calculations become much more complicated. Note that our method performs the calculations ‘backwards’ (see also Remark 3.5). This finally leads to the following result.

Theorem 3.4 *For all* $n = 1, 2, \ldots; 0 \le x < 1$,

$$\begin{aligned} \mathrm{E}\left([T_n^{NI}]^2 \,|\, x\right) &= \frac{n}{n+2}(1-x)^2 - \frac{2}{n+1}\left(1-\frac{1}{2^n}\right)\frac{x\,(1-2x)_+^{n+1}}{(1-x)^n} \\ &- \frac{2n+1}{(n+1)(n+2)}\left(1-\frac{1}{2^n}\right)\frac{(1-2x)_+^{n+2}}{(1-x)^n} \\ &+ \frac{1}{(n+1)(n+2)}\left(\frac{1}{3}-\frac{1}{2^n}+\frac{2}{3\cdot 4^n}\right)\frac{(1-4x)_+^{n+2}}{(1-x)^n}. \end{aligned}$$

Putting $x = 0$ in Theorem 3.4 we readily see that for all $n = 1, 2, \ldots$,

$$\mathrm{E}\left([T_n^{NI}]^2\right) = \frac{1}{(n+1)(n+2)}\left(n^2 - n - \frac{2}{3} + \frac{n}{2^{n-1}} + \frac{2}{3\cdot 4^n}\right)$$

and

$$\mathrm{Var}\left(T_n^{NI}\right) = \frac{1}{(n+1)^2(n+2)}\left(\frac{4n}{3} - \frac{n}{3\cdot 4^n} - \frac{8}{3} + \frac{1}{2^{n-2}} - \frac{1}{3\cdot 4^{n-1}}\right). \qquad (3.17)$$

Remark 3.5 Our method determines the second moment of T_n^{NI} according to a backward recursion: it subsequently solves $G_n^{(2)}(x)$ on the intervals $[1/2, 1)$, $[1/4, 1/2)$ and $[0, 1/4)$. To determine the k-th moment we will have to consider the sequence of the intervals $[1/2, 1), [1/4, 1/2), \ldots, [0, 1/2^k)$.

3.4 Conditional, unconditional and limiting distribution of the number of turns

In this section we first derive an upper bound for the number of remaining turns after picking the i-th item. Next we will obtain the mean, variance and the distribution of the number of remaining turns conditioned on an empty space of size x at one side of the picker's position. From this we shall derive the unconditional and the limiting distribution for the number of turns.

Bartholdi and Platzman [6] mention that a route to pick n items under the NI heuristic actually consists of a number of segments of uninterrupted clockwise and counterclockwise movements. Denote the number of segments by N (so the number of turns is $N-1$) and let I_j denote the length along the j-th segment to the first item retrieved on that segment (see Figure 3.6). Then they notice that

$$I_j \ge 2I_{j-1}, \qquad 2 \le j \le N.$$

From this observation it immediately follows that

$$I_1 \le 1/2 I_2 \le \ldots \le 1/2^{N-1} I_N \le 1/2^N.$$

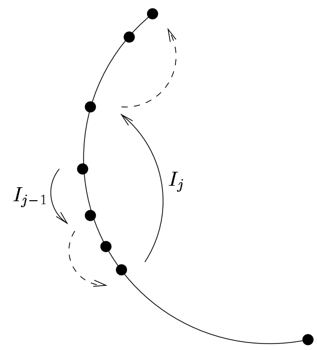

Figure 3.6: Segments of uninterrupted clockwise and counterclockwise movements.

Here the last inequality follows from the simple fact that any step under the NI heuristic is never greater than 1/2. Thus we have proved the following lemma, which allows us to estimate the number of remaining turns after picking the i-th item:

Lemma 3.2 *The number of remaining turns under the NI heuristic after picking the i-th item in the j-th segment,* $1 \le j \le i \le n$, *is never greater than* $\log_{1/2} I_j - 1$.

Let K_n be a number of turns under the NI heuristic. We will use the procedure from the previous section to obtain the mean and variance of K_n conditioned on an empty space of size x at one side of the picker's position. Denote by $\mathrm{E}(K_n|\, x)$ the expected number of turns conditioned on the empty space x. If the size of the empty space is greater than 1/2, then no turns are possible:

$$\mathrm{E}(K_n|\, x) = 0, \qquad 1/2 \le x < 1. \tag{3.18}$$

For $x < 1/2$ note that changing direction actually implies crossing the known empty space. The probability of this event is

$$\mathrm{P}[\text{crossing an empty space of size } x] = \int_x^{1/2} \frac{n(1-2y)^{n-1}}{(1-x)^n}\, dy = \frac{(1-2x)^n}{2(1-x)^n}.$$

However, if $x = 0$, then this probability becomes 1/2. This is in contradiction with the natural assumption that the first step is never a turn. Hence, the case $x = 0$ becomes exceptional. To avoid that we introduce an artificial random variable

$$K_n' = K_n + K, \tag{3.19}$$

where K is a random variable independent of K_n, and

$$\begin{aligned} \mathrm{P}(K = 0|\, x) &= 0, \qquad 0 < x < 1, \\ \mathrm{P}(K = 0|\, 0) &= \mathrm{P}(K = 1|\, 0) \;=\; 1/2. \end{aligned}$$

The conditional characteristics of K'_n are continuous at $x = 0$, and for $0 < x < 1$ they coincide with conditional characteristics of K_n. Thus, we are first going to find the conditional mean, variance and the conditional distribution of K'_n. Then we can retrieve formulas for the mean, variance and distribution of K_n, by putting $x = 0$ and applying (3.19).

We use K'_n mostly as an auxiliary random variable. However, it has a reasonable interpretation itself. Indeed, the picker's starting point is often just the location of the last item from the previous order, and the picker reaches this point following a certain direction. To pick the first item of the next order the picker changes the direction with probability 1/2. If this event is also considered as a turn, then the total number of turns is actually distributed as K'_n (instead of K_n).

For the conditional expectation $\mathrm{E}(K'_n|\,x)$, where $0 \le x < 1/2$, we have the recursion

$$\begin{aligned} \mathrm{E}(K'_n|\,x) &= \int_0^x \frac{n(1-x-y)^{n-1}}{(1-x)^n}\,\mathrm{E}(K'_{n-1}|\,x+y)\,dy \\ &+ \int_x^{1/2} \frac{n(1-2y)^{n-1}}{(1-x)^n}\,\mathrm{E}(K'_{n-1}|\,2y)\,dy \\ &+ \int_x^{1/2} \frac{n(1-2y)^{n-1}}{(1-x)^n}\,\big[\mathrm{E}(K'_{n-1}|\,2y)+1\big]\,dy, \qquad 0 \le x < 1/2. \end{aligned} \tag{3.20}$$

Denoting

$$C_n(x) = \mathrm{E}(K'_n|\,x)(1-x)^n,$$

and taking into consideration (3.18) we can rewrite (3.20) as

$$C_n(x) = \int_x^{1/2} nC_{n-1}(\tau)\,d\tau + \frac{(1-2x)^n}{2}.$$

This recursion can be solved in the same way as done in the previous section. The outcome is presented in the following theorem.

Theorem 3.5 *For all* $n = 1, 2, \ldots,$

$$\mathrm{E}(K'_n|\,x) = \left(1 - \frac{1}{2^n}\right)\frac{(1-2x)_+^n}{(1-x)^n}, \quad 0 \le x < 1.$$

The mean number of turns can now be obtained as follows:

$$\mathrm{E}(K_n) = \mathrm{E}(K'_n|\,0) - \mathrm{E}(K|\,0) = \frac{1}{2} - \frac{1}{2^n}. \tag{3.21}$$

For the conditional second moment $\mathrm{E}([K'_n]^2|\,x)$ we again apply the same procedure as for $\mathrm{E}([T_n^{NI}]^2|\,x)$ in Section 3.3. This yields

Theorem 3.6 *For all $n = 1, 2, \dots,$*

$$\begin{aligned} \mathrm{E}([K_n']^2|\,x) &= \left(1-\frac{1}{2^n}\right)\frac{(1-2x)_+^n}{(1-x)^n} \\ &+ 2\left(\frac{1}{3}-\frac{1}{2^n}+\frac{2}{3\cdot 4^n}\right)\frac{(1-4x)_+^n}{(1-x)^n}, \quad 0 \le x < 1. \end{aligned}$$

From Theorem 3.6 it follows that

$$\mathrm{Var}(K_n) = \mathrm{Var}(K_n'|\,0) - \mathrm{Var}(K|\,0) = \frac{5}{12} - \frac{1}{2^n} + \frac{1}{3\cdot 4^n}. \tag{3.22}$$

Using the recursive procedure, we can also derive the conditional distribution for the random variable K_n'. Let $\mathrm{P}(K_n' = k|\,x)$ be the probability that K_n' equals k, if there is an empty space of size x at one side of the picker's position. We will first determine $\mathrm{P}(K_n' = 0|\,x)$. Clearly,

$$\mathrm{P}(K_n' = 0|\,x) = 1, \qquad 1/2 \le x < 1. \tag{3.23}$$

Further, we obtain

$$\begin{aligned} \mathrm{P}(K_n' = 0|\,x) &= \int_0^x \mathrm{P}(K_{n-1}' = 0|\,x+y)\,\frac{n(1-x-y)^{n-1}}{(1-x)^n}\,dy \\ &+ \int_x^{1/2} \mathrm{P}(K_{n-1}' = 0|\,2y)\,\frac{n(1-2y)^{n-1}}{(1-x)^n}\,dy, \qquad 0 \le x < 1/2. \end{aligned}$$

By introducing

$$L_n^{(0)}(x) = \mathrm{P}(K_n' = 0|\,x)(1-x)^n, \qquad 0 \le x < 1,$$

the last expression becomes

$$L_n^{(0)}(x) = \int_x^{2x} nL_{n-1}^{(0)}(\tau)\,d\tau + \frac{1}{2}\int_{2x}^{1} nL_{n-1}^{(0)}(\tau)\,d\tau. \tag{3.24}$$

Note that this time change of variables does not help that much, because now we do not only face a recursion in n, but also one in x. Indeed, if we naturally put

$$L_0^{(0)}(x) = \mathrm{P}(K_0' = 0|\,x)(1-x)^0 \equiv 1, \qquad 0 \le x < 1,$$

then for $n = 1$ the equations (3.24) and (3.23) immediately yield:

$$L_1^{(0)}(x) = \begin{cases} 1/2 \;=\; (1-x) - \dfrac{(1-2x)}{2}, & 0 \le x < 1/2, \\ 1-x, & 1/2 \le x < 1. \end{cases}$$

This expression can also be verified directly. Now, to solve $L_2^{(0)}(x)$ from (3.24) we have to distinguish two cases: $0 \le 2x < 1/2$ and $1/2 \le 2x < 1$. Hence, we will have different expressions for $L_2^{(0)}(x)$ at $0 \le x < 1/4$ and $1/4 \le x < 1/2$. Proceeding this way, we conclude that it is necessary to consider the intervals $0 \le x < 1/2^n$, $1/2^n \le x < 1/2^{n-1}$, ..., $1/2 \le x < 1$ to find $L_n^{(0)}(x)$. As before, the calculations have to be executed 'backwards' (cf. Remark 3.5), because we need to know $L_{n-1}^{(0)}(\tau)$ at $\tau \in [x, 1)$ in order to find $L_n^{(0)}(x)$.

Solving the recursion (3.24) one can see that the function $L_n^{(0)}(x)$ has the form

$$L_n^{(0)}(x) = \sum_{i=0}^{n} (-1)^i k_{n,i} (1 - 2^i x)_+^n, \quad 0 \le x < 1,$$

and it only remains to find the coefficients $k_{n,i}$, $n = 0, 1, \ldots$; $i = 0, 1, \ldots, n$, which turn out to satisfy the following recursion:

$$\begin{aligned} k_{n,0} &= 1, \quad n = 0, 1, \ldots \\ k_{n,i} &= \sum_{l=i-1}^{n-1} k_{l,i-1} \left(\frac{1}{2^i}\right)^{n-l}, \quad n = 1, 2, \ldots; \quad i = 1, 2, \ldots, n. \end{aligned}$$

Thus, those coefficients are just geometric sums:

$$\begin{aligned} k_{n,0} &= 1, \qquad n \ge 0 \\ k_{n,1} &= 1 - \frac{1}{2^n}, \qquad n \ge 1 \\ k_{n,2} &= \frac{1}{3} - \frac{1}{2^n} + \frac{2}{3} \cdot \frac{1}{4^n}, \qquad n \ge 2, \end{aligned}$$

and so on. Note that we have seen the same coefficients for the conditional mean and variance of the travel time and the number of turns (see Theorems 3.3, 3.4, 3.5 and 3.6).

Now we can apply similar methods to calculate the conditional probabilities $\mathrm{P}(K_n' = k|\, x)$, $k = 1, 2, \ldots, n$, conditioned on the empty space x. Since a turn necessarily provides a step, which is greater than the size of the empty space, we may conclude from Lemma 3.2 that the number of turns can only achieve k if $x < 1/2^k$:

$$\mathrm{P}(K_n' = k|\, x) = 0, \qquad x \ge 1/2^k.$$

Thus, denoting

$$L_n^{(k)}(x) = \mathrm{P}(K_n' = k|\, x)(1 - x)^n, \qquad 0 \le x < 1, \quad 1 \le k \le n,$$

we obtain the following recursion:

$$L_n^{(k)}(x) \;=\; \int_0^x n L_{n-1}^{(k)}(x + y)\, dy + \int_x^{1/2^k} n L_{n-1}^{(k)}(2y)\, dy + \int_x^{1/2^k} n L_{n-1}^{(k-1)}(2y)\, dy,$$

which can be rewritten as

$$L_n^{(k)}(x) \;=\; \int_x^{2x} nL_{n-1}^{(k)}(\tau)\,d\tau + \frac{1}{2}\int_{2x}^{1/2^k} nL_{n-1}^{(k)}(\tau)\,d\tau + \frac{1}{2}\int_{2x}^{1/2^{k-1}} nL_{n-1}^{(k-1)}(\tau)\,d\tau.$$

This recursion can be solved subsequently for x in $[1/2^{k+1}, 1/2^k)$, $[1/2^{k+2}, 1/2^{k+1}), \ldots,$ $[0, 1/2^n)$. The results are presented in the following theorem.

Theorem 3.7 *For all $n = 1, 2, \ldots$; $0 \le k \le n$,*

$$\begin{aligned}
\mathrm{P}(K_n' = k|\, x) &= \frac{1}{(1-x)^n}\sum_{i=k}^{n}(-1)^{i+k}\binom{i}{k}k_{n,i}(1-2^i x)_+^n, \\
\mathrm{P}(K_n' = k) &= \sum_{i=k}^{n}(-1)^{i+k}\binom{i}{k}k_{n,i}.
\end{aligned}$$

Putting $x = 0$ we can immediately find the distribution for K_n.

Theorem 3.8 *For all $n = 1, 2, \ldots$; $0 \le k < n$,*

$$\begin{aligned}
\mathrm{P}(K_n = k) &= 2\sum_{l=0}^{k}(-1)^l \mathrm{P}(K_n' = k-l) \\
&= 2\sum_{l=0}^{k}\sum_{i=l}^{n}(-1)^{i+k}\binom{i}{l}k_{n,i}, \qquad 0 \le k \le n. \qquad (3.25)
\end{aligned}$$

In Section 3.7 we will derive (3.25) in another form (see formula (3.36)).

Figure 3.7 shows the conditional probability $\mathrm{P}(K_n = 0|\, x)$ of no turns as a function of the empty space x for $n = 10$ (observe the discontinuity at $x = 0$). We see that it rapidly goes to 1 as x increases. Hence, the picker only oscillates near his starting position. Once he has picked a few items, it becomes very unlikely that he will turn.

In Figure 3.8 we show the distribution of K_n for $n = 4, 5, 7$ and 10. We see that it rapidly converges to the limiting distribution for K_n as $n \to \infty$. Let the random variable K_∞ have this limiting distribution. Letting $n \to \infty$ in (3.25) and denoting

$$k_{\infty,i} = \lim_{n\to\infty} k_{n,i} = \prod_{j=1}^{i}\frac{1}{2^j - 1}, \quad i = 1, 2, \ldots,$$

we obtain after some simplification the following expression for the limiting distribution.

Theorem 3.9 *The limiting distribution of K_n as $n \to \infty$ is given by:*

$$\mathrm{P}(K_\infty = k) \;=\; 2\cdot(-1)^k \sum_{i=k+2}^{\infty}(-1)^i k_{\infty,i}\sum_{l=0}^{k}\binom{i}{l}. \qquad (3.26)$$

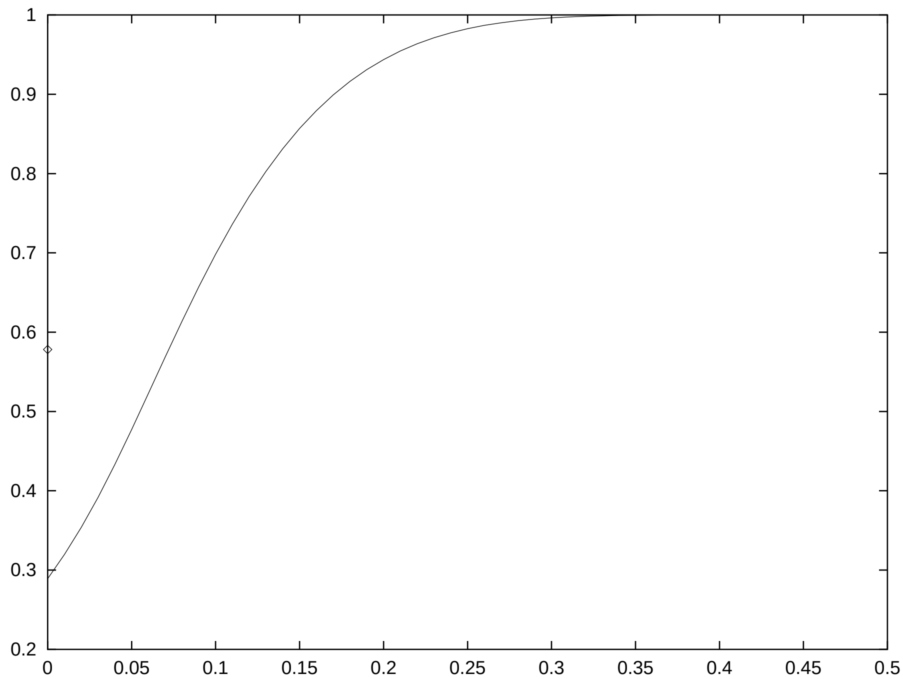

Figure 3.7: The conditional probability $\mathrm{P}(K_n = 0|\, x)$ of no turns as a function of the empty space x for $n = 10$.

In particular, (3.26) gives

$$\begin{aligned} \mathrm{P}(K_\infty = 0) &= 2 \cdot \left\{ \frac{1}{3} - \frac{1}{3 \cdot 7} + \frac{1}{3 \cdot 7 \cdot 15} - \dots \right\} \approx 0.5776, \\ \mathrm{P}(K_\infty = 1) &= 2 \cdot \left\{ \frac{1+3}{3 \cdot 7} - \frac{1+4}{3 \cdot 7 \cdot 15} + \frac{1+5}{3 \cdot 7 \cdot 15 \cdot 31} \dots \right\} \approx 0.3504, \\ \mathrm{P}(K_\infty = 2) &= 2 \cdot \left\{ \frac{1+4+6}{3 \cdot 7 \cdot 15} - \frac{1+5+10}{3 \cdot 7 \cdot 15 \cdot 31} + \dots \right\} \approx 0.0666, \end{aligned} \quad (3.27)$$

and so on. An alternative expression for the distribution of K_∞ will be derived in Section 3.7. Note that the NI heuristic with no turns is actually the SD heuristic. One might intuitively expect that for large n it is very unlikely that the picker will change direction. However, we see that the probability that the NI and SD heuristics coincide does not tend to 1 as $n \to \infty$, but it decreases to approximately 0.5776. So, in the limit the NI heuristic oscillates with quite high probability. However, the oscillation is very modest, since the limiting probability of 4 turns is about 0.0002, and the probability of more than 4 turns is negligible.

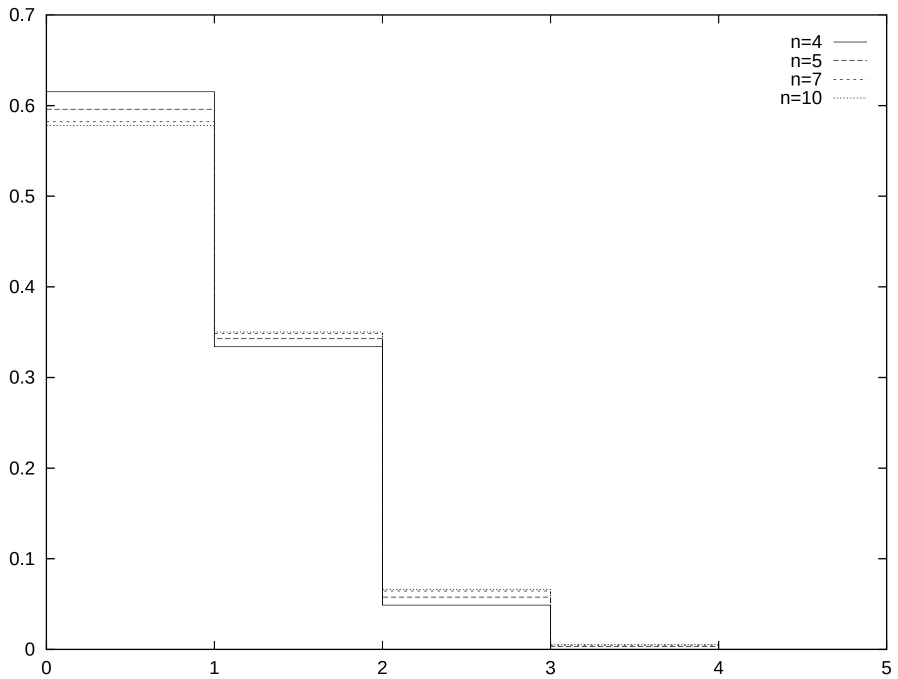

Figure 3.8: Distribution for the number of turns.

3.5 Distribution and moments of the travel time

In this section we present the travel time under the NI heuristic in such a form that Theorem 2.2 can be applied to obtain its distribution and moments.

As was discussed above, the positions of the items partition the circle in $n+1$ uniform spacings $D_1, D_2, \ldots, D_{n+1}$ defined by (2.17). These spacings are distributed as i.i.d. exponentials divided by their sum (see (2.18)). Under the NI heuristic the picker however does not have to know all spacings at once. He first considers the two spacings adjacent to his starting position and then travels to the nearest item. Next he also looks at the other spacing adjacent to that item and compares the distance to the item located at the endpoint of that spacing and the distance to the first item in the other direction, which is the sum of the spacings previously considered. Then he travels again to the nearest item, and so on. Furthermore, we may act as if the picker faces non-normalized exponential spacings, and afterwards divide the travel time by the sum of all spacings. Then it is clear that each new spacing faced by the picker is independent of the ones already observed. Now let X_i, where $i = 1, \ldots, n+1$, denote the i-th non-normalized exponential spacing faced by the picker. So the spacings are numbered as observed by the picker operating

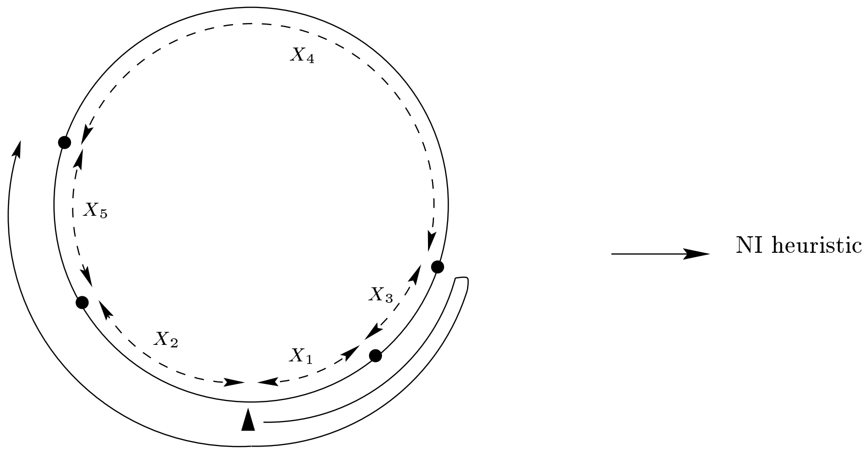

Figure 3.9: The NI route of the picker facing 5 exponential spacings.

under the NI heuristic (see Figure 3.9). Then T_n^{NI} can be expressed as

$$T_n^{NI} = \sum_{i=2}^{n+1} \frac{\min\{X_i, S_{i-1}\}}{S_{n+1}} = 1 - \sum_{i=1}^{n+1} \frac{(X_i - S_{i-1})_+}{S_{n+1}}, \tag{3.28}$$

where X_i's are i.i.d. exponentials with mean 1; $S_0 = 0$; $S_i = X_1 + X_2 + \cdots + X_i$, $i \geq 1$. The sum in the right-hand side of (3.28) coincides with the left-hand side of (2.19), where $N = n + 1$; $b = 1$. Hence, Theorem 2.2 immediately yields the following result.

Theorem 3.10 *For all* $n = 1, 2, \ldots,$

$$T_n^{NI} \stackrel{d}{=} \sum_{i=1}^{n} \left(1 - \frac{1}{2^i}\right) D_i. \tag{3.29}$$

Thus, the travel time under the NI heuristic is distributed as a linear combination of uniform spacings with distinct coefficients. Now we can use results of Section 2.4 to find its distribution and moments. The distribution of T_n^{NI} follows directly from (2.26).

Theorem 3.11 *For all* $n = 1, 2, \ldots,$

$$\mathrm{P}(T_n^{NI} < t) = \sum_{i=0}^{n} \left(2^i t - 2^i + 1\right)_+^n \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{2^j}{2^j - 2^i}, \quad 0 < t \leq 1. \tag{3.30}$$

As was discussed in Section 2.4.1, formula (3.30) gives different expressions for the distribution function of T_n^{NI} in the sequence of intervals

$(0,1/2],(1/2,3/4],\ldots,(1-1/2^{n-1},1-1/2^n]$. The expression in $(0,1/2]$ is the simplest one:

$$\mathrm{P}(T_n^{NI}<t)=2\cdot\frac{4}{3}\cdots\frac{2^n}{2^n-1}t^n,\qquad 0<t\le 1/2.$$

In the next interval we get one additional term:

$$\mathrm{P}(T_n^{NI}<t)=2\cdot\frac{4}{3}\cdots\frac{2^n}{2^n-1}t^n-2\cdot\frac{4}{3}\cdots\frac{2^{n-1}}{2^{n-1}-1}(2t-1)^n,\qquad 1/2<t\le 3/4.$$

Further an extra term appears in each of the following intervals.

According to (2.28) the distribution of T_n^{NI} can be also written in another form:

$$\mathrm{P}(T_n^{NI}<t)=1-\sum_{i=0}^{n}\left(2^it-2^i+1\right)_-^n\prod_{\substack{j=0\\ j\neq i}}^{n}\frac{2^j}{2^j-2^i},\qquad 0<t\le 1,$$

which yields an expression with less terms than in (3.30) for $1-1/2^{k-1}<t\le 1-1/2^k$, when k is greater than $n/2$.

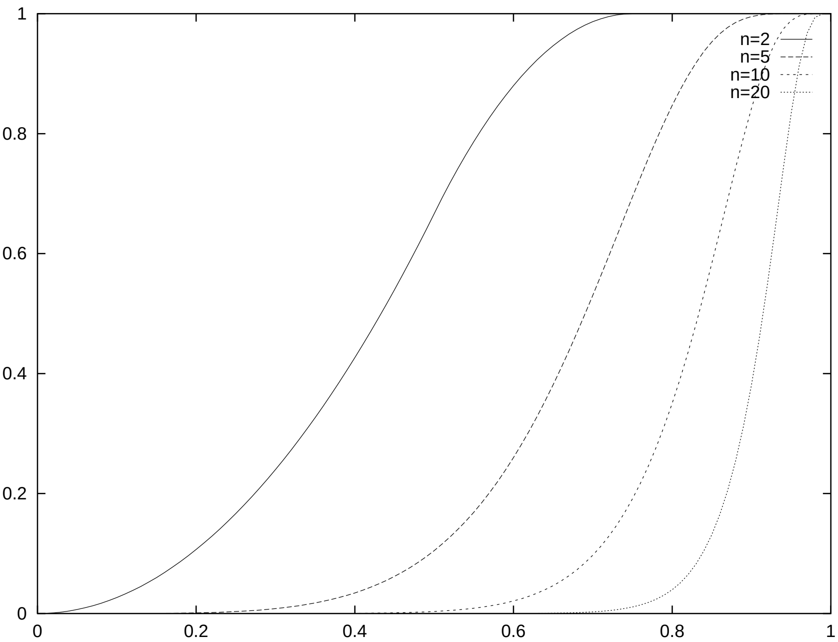

Figure 3.10: The distribution of T_n^{NI} for $n=2,5,10,20$.

In Figure 3.10 we show the distribution of the travel time for several values of n.

Now we use representation (3.29) and equation (2.33) to directly calculate the moments of the travel time T_n^{NI}:

$$\mathrm{E}\left([T_n^{NI}]^k\right) = \binom{n+k}{k}^{-1} \sum_{\substack{k_1,k_2,\dots,k_n \geq 0\\ k_1+k_2+\dots+k_n=k}} \prod_{j=1}^{n} \left(1-\frac{1}{2^j}\right)^{k_j}.$$

For example, for $k=1$ we have:

$$\mathrm{E}\left(T_n^{NI}\right) = \frac{n}{n+1} - \frac{1}{n+1}\left(1-\frac{1}{2^n}\right).$$

This equation has been already derived recursively in Section 3.3 (see formula (3.11)).

3.6 Asymptotic results

In this section we analyze the distribution of the travel time under the NI heuristic, when the number of items n tends to infinity. In fact, we consider $1 - T_n^{NI}$, which is the difference between the travel time under the NI heuristic and one complete rotation. It is clear that $1 - T_n^{NI}$ converges in distribution to zero as $n \to \infty$. However, since

$$\mathrm{E}\left(1 - T_n^{NI}\right) = \frac{2}{n+1}\left(1 - \frac{1}{2^{n+1}}\right),$$

one may expect that $(n+1)(1-T_n^{NI})$ has a proper limiting distribution.

We will use the common notation $Z_n \xrightarrow{d} Z$, if the sequence $Z_1, Z_2, \dots$ converges in distribution to Z. The limiting distribution of $(n+1)(1-T_n^{NI})$ is presented in the following theorem.

Theorem 3.12 *Let X_1, X_2, ... be independent exponentials with mean* 1. *Then*

$$(n+1)\left(1-T_n^{NI}\right) \xrightarrow{d} \sum_{j=1}^{\infty} \frac{1}{2^{j-1}} X_j, \tag{3.31}$$

and the limiting distribution is given by

$$\mathrm{P}\left(\sum_{j=1}^{\infty} \frac{1}{2^{j-1}} X_j < t\right) = \left(\prod_{\nu=1}^{\infty} \frac{2^\nu}{2^\nu - 1}\right) \sum_{j=0}^{\infty} (-1)^j \left(1 - \exp(-2^j t)\right) \prod_{l=1}^{j} \frac{1}{2^l - 1}, \quad t > 0. \tag{3.32}$$

Proof. Denote

$$\xi_n = \sum_{j=1}^{n} \frac{1}{2^{j-1}} X_j, \qquad \xi = \sum_{j=1}^{\infty} \frac{1}{2^{j-1}} X_j.$$

According to the Monotone Convergence Theorem we have $\mathrm{E}(\xi) = \lim_{n\to\infty} \mathrm{E}(\xi_n) = 2$, which in particular implies $\mathrm{P}(\xi < \infty) = 1$.

To prove (3.31) we only need to use (2.18) in order to rewrite (3.29) as

$$(n+1)(1 - T_n^{NI}) \stackrel{d}{=} \frac{(n+1)\xi_{n+1}}{S_{n+1}}.$$

By definition the sequence $\{\xi_n\}$ converges a.s. to ξ. Further, according to the strong law of large numbers, the sequence $\{S_n/n\}$ converges a.s. to 1. Thus, the sequence $\{n\xi_n/S_n\}$ converges a.s. to ξ, which immediately gives (3.31).

The distribution of ξ can be determined via inversion of its Laplace-Stieltjes transform $\varphi(s)$, which is given by

$$\varphi(s) = \mathrm{E}(\exp(-s\xi)) = \prod_{j=0}^{\infty} \frac{2^j}{s+2^j}.$$

It is readily verified that $\varphi(s)$ is a meromorphic function with simple poles $s_j = -2^j$, $j = 0, 1, \ldots$. The residues r_j at these poles are given by

$$r_j = \left(\prod_{l=1}^{\infty} \frac{2^l}{2^l - 1}\right) (-2)^j \prod_{l=1}^{j} \frac{1}{2^l - 1}, \qquad j = 0, 1, \ldots.$$

To invert $\varphi(s)$ we first expand it in rational fractions of s:

$$\varphi(s) = \sum_{j=0}^{\infty} \frac{r_j}{s - s_j}. \tag{3.33}$$

To validate (3.33) we follow the approach in Whittaker and Watson [82], Section 7.4. This approach requires that $|\varphi(s)|$ is uniformly bounded on a sequence of circles C_j, with centre at 0 and radius R_j, not passing through any poles, and such that $R_j \to \infty$ as $j \to \infty$. In this case we can take $R_j = -(s_j + s_{j+1})/2 = 2^{j-1} + 2^j$ and it is straightforward to show that for all $s \in C_j$,

$$|\varphi(s)| \leq |\varphi(-R_j)| \leq 2 \cdot \prod_{l=0}^{\infty} \frac{2^{2+l}}{2^{2+l} - 3}.$$

Since this upper bound does not depend on j, the function $|\varphi(s)|$ is indeed uniformly bounded on the sequence of circles C_j. Now we can conclude from Section 7.4 in [82] that

$$\varphi(s) = \varphi(0) + \sum_{j=0}^{\infty} r_j \left[\frac{1}{s - s_j} + \frac{1}{s_j}\right].$$

It can be shown by combinatorial arguments (cf. Section 3.7, Remark 3.7) that

$$\sum_{j=0}^{\infty} \frac{r_j}{s_j} = \left(\prod_{l=1}^{\infty} \frac{2^l}{2^l - 1}\right) \sum_{j=0}^{\infty} (-1)^{j+1} \prod_{l=1}^{j} \frac{1}{2^l - 1} = -1 = -\varphi(0),$$

which implies (3.33). Inversion of (3.33) finally yields (3.32). □

In Figure 3.11 we demonstrate the rate at which the distribution of $(n+1)(1-T_n^{NI})$ converges to its limiting distribution.

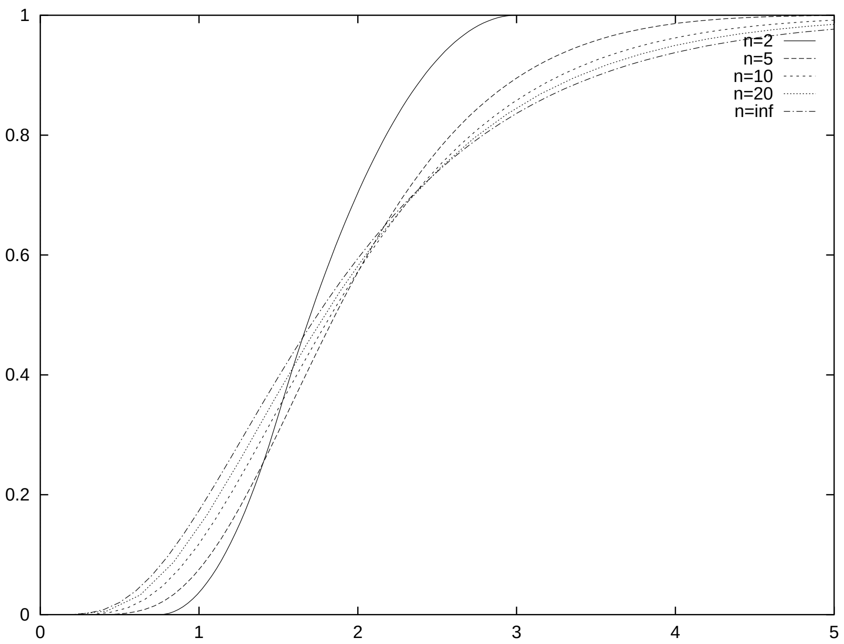

Figure 3.11: The distribution of $(n+1)(1-T_n^{NI})$ for $n = 2, 5, 10, 20$ and the limiting distribution as $n \to \infty$.

In our case convergence in distribution implies convergence in moments. Indeed, using (3.29) and (2.33) for any $k = 1, 2, \dots$ we have

$$\mathrm{E}\left(\left[(n+1)(1-T_n^{NI})\right]^k\right) = (n+1)^k \binom{n+k}{k}^{-1} \sum_{\substack{k_1,k_2,\dots,k_{n+1}\geq 0 \\ k_1+k_2+\cdots+k_{n+1}=k}} \prod_{j=1}^{n+1} \left(\frac{1}{2^{j-1}}\right)^{k_j}$$

$$\leq \frac{k!(n+1)^k}{(n+1)(n+2)\cdots(n+k)} \left(1+\frac{1}{2}+\cdots+\frac{1}{2^n}\right)^k \leq k!2^k.$$

Hence (see e.g. Chung [14], Section 4.5) for any $k = 1, 2, \dots$ it holds that

$$\lim_{n\to\infty} \mathrm{E}\left(\left[(n+1)(1-T_n^{NI})\right]^k\right) = \mathrm{E}\left(\xi^k\right) < \infty.$$

To find $\mathrm{E}\left(\xi^k\right)$ we use (3.32) and then change the order of integration and summation. This yields

$$\mathrm{E}\left(\xi^k\right) = k!\left(\prod_{\nu=1}^{\infty}\frac{2^\nu}{2^\nu-1}\right)\sum_{j=0}^{\infty}(-1)^j\frac{1}{2^{kj}}\prod_{l=1}^{j}\frac{1}{2^l-1}. \tag{3.34}$$

Changing the order of integration and summation is allowed, since the sum above is absolutely convergent. Expression (3.34) can be simplified by using the equality

$$\sum_{j=0}^{\infty}(-1)^j\frac{1}{2^{kj}}\prod_{l=1}^{j}\frac{1}{2^l-1} = \prod_{j=k+1}^{\infty}\left(1-\frac{1}{2^j}\right), \tag{3.35}$$

which holds for $k = 0, 1, \ldots$ (see Remark 3.7). Substitution of (3.35) into (3.34) gives the simple expression

$$\mathrm{E}\left(\xi^k\right) = k!\prod_{j=1}^{k}\frac{2^j}{2^j-1}.$$

3.7 Alternative way to derive the distribution of the number of turns

In this section we will determine the distribution of the number of turns under the NI heuristic by using an approach from Chapter 2. Define

$$Q_i = \mathbf{1}_{[S_{i-1}<X_i]}, \qquad i = 1, 2, \ldots .$$

Since X_i can be interpreted as the i-th non-normalized exponential spacing faced by the picker (see Section 3.5, formula (3.28)), it is clear that for $i \geq 3$ the random variable Q_i indicates whether or not the picker turns after picking the $(i-2)$-th item. So we may write

$$K_n = \sum_{i=3}^{n+1} Q_i.$$

Since $\mathrm{E}(Q_i) = \mathrm{P}\left(S_{i-1} < X_i\right) = 1/2^{i-1}$ we immediately retrieve (3.21):

$$\mathrm{E}(K_n) = \frac{1}{2} - \frac{1}{2^n}, \qquad n = 1, 2, \ldots .$$

To find the variance, and in fact, the complete distribution of the number of turns, we need the following result.

Lemma 3.3 *The random variables $Q_1, Q_2, \ldots$ are independent.*

The assertion of Lemma 3.3 follows because for any $1 \leq i < j$, the random variables X_j and S_{j-1} are independent of the event $[X_i > S_{i-1}]$. This issue was discussed in detail in Chapter 2 (see e.g. Remark 2.2).

Remark 3.6 Note that the number of turns K_n and the travel time T_n^{NI} are independent! This implies, for example, that the distribution of the travel time remains the same even given that we have to turn 9 times while collecting 10 items. Indeed, K_n is defined by the random events $[X_i > S_{i-1}]$, $i = 3, 4, \ldots, n+1$. On the other hand, T_n^{NI} is independent of such events, because (3.28) is always distributed as (3.29) under any event $[X_i > S_{i-1}]$. Actually, this result follows from the proof of formula (2.2) in Theorem 2.1. However, we did not mention this issue before because so far we did not have an interpretation for the events $[X_i > bS_{i-1}]$. Note that in Corollary 2.1 we formulated a similar result arising from the proof of (2.3). In this corollary, (i) is analogous to Lemma 3.3 and (ii) is analogous to this remark.

Now from Lemma 3.3 we obtain for the second moment that

$$\begin{aligned} \mathrm{E}(Q_n^2) &= \sum_{i=3}^{n+1} \mathrm{E}(Q_i^2) + \sum_{3 \le i < j \le n+1} 2\mathrm{E}(Q_i Q_j) \\ &= \sum_{i=3}^{n+1} \mathrm{E}(Q_i) + \sum_{3 \le i < j \le n+1} 2\mathrm{E}(Q_i)\mathrm{E}(Q_j) \\ &= \frac{1}{2} - \frac{1}{2^n} + \sum_{3 \le i < j \le n+1} \frac{1}{2^{i+j-3}} \\ &= \frac{2}{3} - \frac{1}{2^{n-1}} + \frac{1}{3 \cdot 4^{n-1}}. \end{aligned}$$

This again yields formula (3.22) for the variance of the number of turns.

Of course, from Lemma 3.3 we can also obtain the distribution of the number of turns:

$$\mathrm{P}(K_n = k) = \sum_{\substack{0 \le k_3, \ldots, k_{n+1} \le 1 \\ k_3 + \cdots + k_{n+1} = k}} \mathrm{P}(Q_3 = k_3) \cdots \mathrm{P}(Q_{n+1} = k_{n+1}), \tag{3.36}$$

where

$$\mathrm{P}(Q_i = 1) = 1 - \mathrm{P}(Q_i = 0) = \frac{1}{2^{i-1}}, \qquad i = 3, \ldots, n+1.$$

For computational purposes we mention that the probability distribution for K_n can be determined recursively. Let $p_{n,k}$ denote $\mathrm{P}(K_n = k)$. From (3.36) we then obtain, by conditioning on Q_{n+1}, the following recursion:

$$p_{n,k} = \frac{1}{2^n}\, p_{n-1,k-1} + \left(1 - \frac{1}{2^n}\right) p_{n-1,k}, \qquad 0 \le k < n,$$

with initial condition $p_{1,0} = 1$ and boundary conditions $p_{n,-1} = p_{n,n} = 0$ for $n > 1$.

From (3.36) we directly obtain

$$\begin{aligned}
\mathrm{P}(K_\infty = 0) &= \prod_{i=2}^{\infty}\left(1-\frac{1}{2^i}\right),\\
\mathrm{P}(K_\infty = 1) &= \prod_{i=2}^{\infty}\left(1-\frac{1}{2^i}\right)\sum_{k=2}^{\infty}\frac{1}{2^k-1},\\
\mathrm{P}(K_\infty = 2) &= \prod_{i=2}^{\infty}\left(1-\frac{1}{2^i}\right)\sum_{2\le k<l}^{\infty}\frac{1}{2^k-1}\cdot\frac{1}{2^l-1}.
\end{aligned} \tag{3.37}$$

Remark 3.7 Formulas (3.27) and (3.37) are of course the same, which can be proved by combinatorial arguments. Let us show, for example, that the formulas for $\mathrm{P}(K_\infty = 0)$ from (3.27) and (3.37) indeed coincide. In other words, we are going to prove the equality

$$\prod_{i=1}^{\infty}\left(1-\frac{1}{2^i}\right) = \frac{1}{3} - \frac{1}{3\cdot 7} + \frac{1}{3\cdot 7\cdot 15} - \cdots . \tag{3.38}$$

A similar type of formula is given on p. 185 of [76]:

$$\prod_{i=1}^{\infty}\left(1-\frac{1}{2^i}\right) = \sum_{k=-\infty}^{\infty}(-1)^k\frac{1}{2^{k(3k+1)/2}}.$$

In order to prove (3.38) we open braces in the left-hand side and we arrange the terms as follows:

$$\prod_{i=1}^{\infty}\left(1-\frac{1}{2^i}\right) = 1 - \sum_{i=1}^{\infty}\frac{1}{2^i} + \sum_{i=1}^{\infty}\sum_{j=i+1}^{\infty}\frac{1}{2^i}\cdot\frac{1}{2^j} - \sum_{i=1}^{\infty}\sum_{j=i+1}^{\infty}\sum_{k=j+1}^{\infty}\frac{1}{2^i}\cdot\frac{1}{2^j}\cdot\frac{1}{2^k} + \cdots .$$

Further, note that

$$\begin{aligned}
\sum_{i=1}^{\infty}\sum_{j=i+1}^{\infty}\frac{1}{2^i}\cdot\frac{1}{2^j} &= \frac{1}{4}\left(\frac{1}{2}+\frac{1}{4}+\cdots\right) + \frac{1}{16}\left(\frac{1}{2}+\frac{1}{4}+\cdots\right) + \cdots\\
&= \left(\frac{1}{2}+\frac{1}{4}+\cdots\right)\left(\frac{1}{4}+\frac{1}{16}+\cdots\right) = \frac{1}{3}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\sum_{i=1}^{\infty}\sum_{j=i+1}^{\infty}\sum_{k=j+1}^{\infty}\frac{1}{2^i}\cdot\frac{1}{2^j}\cdot\frac{1}{2^k}\\
&\quad = \left(\frac{1}{2}+\frac{1}{4}+\cdots\right)\left(\frac{1}{4}+\frac{1}{16}+\cdots\right)\left(\frac{1}{8}+\frac{1}{64}+\cdots\right) = \frac{1}{3\cdot 7},
\end{aligned}$$

and so on. Thus, the left-hand side of (3.38) becomes

$$1 - 1 + \frac{1}{3} - \frac{1}{3 \cdot 7} + \frac{1}{3 \cdot 7 \cdot 15} - \cdots ,$$

which is the required result. Along the same lines one can prove formula (3.35) which was used in the previous section to derive the moments for the limiting distribution of $(n+1)(1 - T_n^{NI})$.

3.8 Concluding remarks

In this chapter we presented an exhaustive analysis of the probabilistic properties of the travel time under the NI heuristic. The NI heuristic means that the next item to be picked is always the nearest one. Due to a reasonably good performance and the ease of implementation this algorithm is very often used in practice.

We provided a tight upper bound for the travel time. In Section 3.3 we developed a procedure to derive the conditional mean and variance of the travel time and also the conditional distribution for the number of turns given that there is an empty space of a certain size at one side of the picker's position. Further, we presented the travel time in such a way that we could use the results from Chapter 2 to obtain its distribution, moments and asymptotic properties. We also showed that similar arguments could be used to find the distribution for the number of turns. Moreover, we proved that the number of turns is independent of the travel time.

Chapter 4

Travel time under the m-step strategies

4.1 Introduction

In this chapter we study strategies whose properties are similar to the properties of the optimal route. In Section 4.2 we explore the One-Side Optimal (OSO) strategy where the picker chooses the best of the routes ending in a given direction. As well as the optimal strategy, the OSO strategy implies at most one turn. We derive the distribution of the travel time under the OSO strategy and the distribution of the number of items collected before a turn. Then we show that the travel time under the OSO strategy is stochastically larger than the travel time under the NI heuristic. Hence, it is not very wise to use the OSO strategy in practice.

Further in this chapter we present the material from paper [66] on the m-step strategies. Under such strategies, the picker chooses the shortest route among the ones that change direction at most once, and only do so after collecting no more than m items. Since it is never optimal to turn more than once, the optimal strategy is an $(n-1)$-step strategy. For randomly distributed pick positions, Rouwenhorst *et al.* [77] analyzed the m-step strategy for $m \leq 2$. Their results indicate that these strategies perform very well. In this chapter we completely analyze the travel time under the m-step strategies, provided $2m < n$. The analysis is based on the probabilistic arguments and the results obtained in Chapter 2.

In Section 4.3 we prove that the travel time under the m-step strategy can be expressed as the maximum of two sums of spacings, provided $2m < n$. This representation is exploited in Section 4.4 to show that the travel time is distributed as a probabilistic mixture of sums of spacings. In Section 4.5 we derive closed-form expressions for the moments of the travel time. Then, in Section 4.6 we compare the performance of the m-step strategies with the performance of the optimal strategy. Numerical results show that, already for small values of m, the performance of the m-step strategy is very close to optimal. In fact, with high probability, the optimal strategy coincides with the 2-step strategy. The m-step strategies are also compared with the NI heuristic. It appears that, already for $m = 2$, the m-step strategy performs better than the NI heuristic. Finally, Section 4.7 is devoted to comments and conclusions.

4.2 One-Side Optimal (OSO) strategy

In this section we analyse the One-Side Optimal (OSO) strategy defined as follows:

One-Side Optimal (OSO) strategy: The picker chooses the shortest route ending in the clockwise direction.

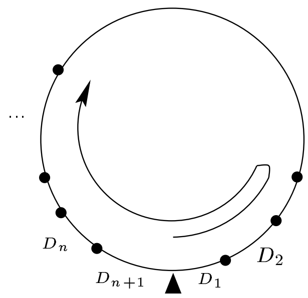

Figure 4.1: Route under the OSO strategy.

As we already pointed out above, it is never good to cover the same interval more than twice. Hence, under the OSO strategy, the picker chooses a route with at most one turn. Therefore, he either picks all the items clockwise or he makes several steps counterclockwise and then turns like in Figure 4.1. Since the last step must be performed clockwise, the number of steps performed counterclockwise can not exceed $n - 1$. Thus, the possible routes are:

$$
\begin{aligned}
&D_2 + D_3 + \cdots + D_{n+1}, \\
&2D_1 + D_3 + D_4 + \cdots + D_{n+1}, \\
&\ldots, \\
&2D_1 + 2D_2 + \cdots + 2D_{n-1} + D_{n+1}.
\end{aligned}
$$

Hence, the travel time T_n^{OSO} under the OSO strategy can be expressed as

$$\begin{aligned} T_n^{OSO} &= D_{n+1} + \min_{1\le j\le n}\left\{\sum_{l=1}^{j-1} 2D_l + \sum_{l=j+1}^{n} D_l\right\} \\ &\stackrel{d}{=} 1 - \max_{1\le j\le n}\left\{D_j - \sum_{l=1}^{j-1} D_l\right\} \\ &\stackrel{d}{=} 1 - \frac{1}{S_{n+1}}\max_{1\le j\le n}\left\{X_j - S_{j-1}\right\}, \qquad (4.1) \end{aligned}$$

where, as before, $D_1, D_2, \ldots, D_{n+1}$ are the uniform spacings defined by (2.17); $X_1, X_2, \ldots,$ are independent exponential random variables with mean 1; $S_0 = 0$; $S_i = X_1 + X_2 + \cdots + X_i$, $i \ge 1$. The second expression suggests an alternative interpretation for the OSO strategy. Clearly, $D_j - \sum_{l=1}^{j-1} D_l$ is the gain in travel time (compared to one full rotation) obtained by skipping the spacing D_j and going back instead. Under the OSO strategy the picker skips the spacing that provides the largest possible gain.

Now we apply Theorem 2.2 to derive a simple representation for T_n^{OSO}. Using (2.20) with $N = n$ and $b = 1$, we obtain

$$T_n^{OSO} \stackrel{d}{=} 1 - \sum_{j=1}^{n}\frac{1}{2^j - 1}D_j = \sum_{j=2}^{n}\left(1 - \frac{1}{2^j - 1}\right)D_j + D_{n+1}. \qquad (4.2)$$

Let K_n^{OSO} denote the number of items collected under the OSO strategy before the turn. Note that the event $[K_n^{OSO} = k]$, where $k = 0, 1, \ldots, n-1$, is equivalent to the event $[\arg\max_{1\le j\le n}\{X_j - bS_{j-1}\} = k+1]$. Then it follows from (2.14) that

$$\mathrm{P}\left(K_n^{OSO} = k\right) = \mathrm{P}\left(\arg\max_{1\le j\le n}\{X_j - bS_{j-1}\} = k+1\right) = \frac{2^{n-k-1}}{2^n - 1}, \quad 0 \le k \le n-1.$$

Let us now compare the distributions of the travel times under the OSO strategy and the NI heuristic. Both the T_n^{NI} and T_n^{OSO} are distributed as a sum of n spacings which are multiplied by certain coefficients. According to (3.29) and (4.2), the coefficients are

$$\begin{array}{lllllll} T_n^{NI}: & \frac{1}{2}, & \frac{3}{4}, & \ldots, & 1 - \frac{1}{2^{n-1}}, & 1 - \frac{1}{2^n}; \\ T_n^{OSO}: & \frac{2}{3}, & \frac{6}{7}, & \ldots, & 1 - \frac{1}{2^n - 1}, & 1. \end{array}$$

It is clearly seen that the travel time under the NI heuristic is stochastically smaller than the travel time under the OSO route, i.e.

$$\mathrm{P}\left(T_n^{NI} < t\right) \ge \mathrm{P}\left(T_n^{OSO} < t\right), \quad 0 \le t \le 1, \quad n \ge 1. \qquad (4.3)$$

Remark 4.1 Note that in the discussion above we compare the NI heuristic and the OSO strategy only in distribution. It does not at all follow from our arguments that the NI heuristic always performs better than the OSO strategy. For example, in Figure 3.3 the travel time under the NI heuristic is $31/32 - \varepsilon$, whereas the OSO strategy provides an optimal route with the travel time $17/32 + 2\varepsilon$ (see Example 3.1 and Remark 3.3).

Remark 4.2 One can also prove (4.3) directly from (3.28) and (4.1). Consider a sum

$$D_2 + D_3 + \cdots + D_{n+1}. \tag{4.4}$$

Suppose that a term D_i for any $i = 2, 3, \ldots, n+1$ can be replaced by

$$T_{i-1} = D_1 + D_2 + \cdots + D_{i-1}.$$

Let us say that $\beta_{i-1} = 1$, if such a replacement took place. Otherwise, we put $\beta_{i-1} = 0$. Denote

$$\beta = (\beta_1, \beta_2, \ldots, \beta_n) \in \{0,1\}^n.$$

After the replacements, sum (4.4) becomes

$$T_n(\beta) = \sum_{i=2}^{n+1} \left[(1 - \beta_{i-1})D_i + \beta_{i-1}T_{i-1}\right].$$

It is easy to see that

$$T_n^{NI} \stackrel{d}{=} \sum_{i=2}^{n+1} \min\{D_i, T_{i-1}\} = \min_{\beta \in \{0,1\}^n} \{T_n(\beta)\}.$$

On the other hand,

$$T_n^{OSO} \stackrel{d}{=} D_1 + \min_{2 \le i \le n+1} \left\{ \sum_{j=2}^{i-1} 2D_j + \sum_{j=i+1}^{n+1} D_j \right\} = \min_{\beta \in \mathbf{B}} \{T_n(\beta)\},$$

where $\mathbf{B} \subset \{0,1\}^n$ is a set of n vectors such that one of their coordinates is 1 and the other coordinates are 0, i.e. exactly one of D_i's from (4.4) is replaced by T_{i-1}. Clearly, $\min_{\beta \in \{0,1\}^n} \{T_n(\beta)\} \le \min_{\beta \in \mathbf{B}} \{T_n(\beta)\}$.

4.3 Representation of the travel time under the m-step strategies

Further in this chapter we will study so-called m-step strategies:

The m-step strategy: The picker chooses the shortest route among the ones that change direction at most once, and only do so after collecting no more than m items.

These strategies are closely related to the optimal strategy, i.e., the one minimizing the travel time. Since it is never optimal to turn more than once, the optimal strategy is an $(n-1)$-step strategy.

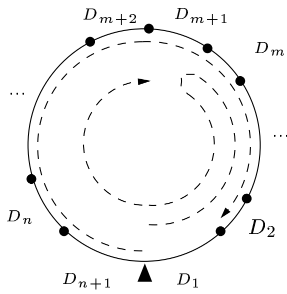

Figure 4.2: Possible routes under the m-step strategy.

As before, we assume that the locations of the items are independent and uniformly distributed. The picker operates under the m-step strategy: he chooses the shortest route among the routes that turn only once, and only turn when no more than m steps have been done. Clearly, there are $2(m+1)$ possible routes (see Figure 4.2); the ones that end in clockwise direction, i.e.,

$$\begin{aligned}
&D_2 + D_3 + \cdots + D_{n+1}, \\
&2D_1 + D_3 + D_4 + \cdots + D_{n+1}, \\
&\ldots, \\
&2D_1 + 2D_2 + \cdots + 2D_{m-1} + D_{m+1} + D_{m+2} + \cdots + D_{n+1}, \\
&2D_1 + 2D_2 + \cdots + 2D_{m-1} + 2D_m + D_{m+2} + D_{m+3} + \cdots + D_{n+1},
\end{aligned} \tag{4.5}$$

and, symmetrically, the other ones that end in counterclockwise direction,

$$\begin{aligned}
&D_n + D_{n-1} + \cdots + D_1, \\
&2D_{n+1} + D_{n-1} + D_{n-2} + \cdots + D_1, \\
&\ldots, \\
&2D_{n+1} + 2D_n + \cdots + 2D_{n-m+3} + D_{n-m+1} + D_{n-m} + \cdots + D_1, \\
&2D_{n+1} + 2D_n + \cdots + 2D_{n-m+3} + 2D_{n-m+2} + D_{n-m} + D_{n-m-1} + \cdots + D_1.
\end{aligned} \tag{4.6}$$

Under the m-step strategy, the picker chooses the shortest of the $2(m+1)$ routes (4.5), (4.6). Let the random variable $T_n^{(m)}$ denote the travel time under the m-step strategy, needed to pick n items. Then, by definition,

$$\begin{aligned} T_n^{(m)} = \min\Bigg\{ & \min_{1\le j\le m+1}\left\{\sum_{l=1}^{j-1} 2D_l + \sum_{l=j+1}^{n+1} D_l\right\}, \\ & \min_{1\le j\le m+1}\left\{\sum_{l=1}^{j-1} 2D_{n+2-l} + \sum_{l=j+1}^{n+1} D_{n+2-l}\right\}\Bigg\} \\ = 1 - \max\Bigg\{ & \max_{1\le j\le m+1}\left\{D_j - \sum_{l=1}^{j-1} D_l\right\}, \max_{1\le j\le m+1}\left\{D_{n+2-j} - \sum_{l=1}^{j-1} D_{n+2-l}\right\}\Bigg\}, \end{aligned} \tag{4.7}$$

where in the last expression we take $\sum_{l=1}^{n+1} D_l = 1$ outside the external minimum. Similarly to the OSO strategy, the m-step strategy has an alternative interpretation provided by the last expression in (4.7). The term $D_j - \sum_{l=1}^{j-1} D_l$ is the gain in travel time (compared to one full rotation) obtained by skipping the spacing D_j and going back instead, ending in a clockwise direction. On the other hand, $D_{n+2-j} - \sum_{l=1}^{j-1} D_{n+2-l}$ is the gain obtained by skipping the spacing D_{n+2-j} and going back ending counterclockwise. Under the m-step strategy the picker skips the spacing that provides the largest possible gain.

Bartholdi and Platzman [6] proved that the optimal route never allows more than one turn, and thus it is an m-step strategy with $m = n - 1$. However, we only consider the case $2m < n$. In the analysis of $T_n^{(m)}$ it appears to be crucial that the spacings $D_1, D_2, \ldots, D_{m+1}$, whose coefficients vary (-1, 0 or 1) in the first internal maximum of the last expression in (4.7), do not participate in the second internal maximum. This implies that $n - m + 1 > m + 1$, or $2m < n$.

Below we establish an elegant representation of the travel time. This representation will be used in the next section to derive the distribution of the travel time. Let us rewrite (4.7) using (2.18):

$$\begin{aligned} T_n^{(m)} \stackrel{d}{=} 1 - \frac{1}{S_{n+1}} \max\Bigg\{ & \max_{1\le j\le m+1}\{X_j - S_{j-1}\}, \\ & \max_{1\le j\le m+1}\{X_{n+2-j} - (S_{n+1} - S_{n+2-j})\}\Bigg\}. \end{aligned} \tag{4.8}$$

By exploiting properties of exponentials, we will reduce the two internal maxima in (4.8) to two sums of exponentials. First, for the term

$$\max_{1\le j\le m+1}\{X_j - S_{j-1}\} \tag{4.9}$$

it follows from Theorem 2.1 that for $2m < n$,

$$\max_{1\le j\le m+1}\{X_j - S_{j-1}\} \stackrel{d}{=} \sum_{j=1}^{m+1} \frac{1}{2^j - 1} X_j.$$

We now proceed with (4.8). Remember that in the induction step in the proof of Theorem 2.1 we only affect the random variables $X_1, \ldots, X_{m+1}$ by conditioning on the random events $E_{i,k}(\mathbf{c})$ (see (2.6)). Their sum (see (2.8)) as well as the other random variables $X_{m+2}, \ldots, X_{n+1}$ remain unaltered. Hence, during the induction, we only change (4.9) and do not affect the 'structure' of the remaining terms in (4.8). Note that we will loose this property as soon as $m+1 > n-m$. In this case replacements (2.6) will change not only (4.9), but also the other internal maximum in (4.8).

Once we have reduced the first internal maximum to a sum of exponentials, we can use the same arguments to also reduce the second internal maximum in (4.8), finally yielding the following theorem.

Theorem 4.1 *For any $m = 0, 1, \ldots; 2m < n$,*

$$T_n^{(m)} \stackrel{d}{=} 1 - \frac{1}{S_{n+1}} \max \left\{ \sum_{j=1}^{m+1} a_{m+2-j}^{-1} X_j, \sum_{j=1}^{m+1} a_{m+2-j}^{-1} X_{n+2-j} \right\}, \tag{4.10}$$

where

$$a_j = 2^j - 1, \quad j \geq 0.$$

In the remainder of this section we derive the distribution of the random variable $K_n^{(m)}$ defined as the number of steps before the picker turns, when collecting n items under the m-step strategy. By symmetry, the probability that the route under the m-step strategy ends in clockwise direction is equal to 1/2. Further, from (4.8) we see that the event

$$\left[\arg \max_{1 \leq j \leq m+1} \{X_j - bS_{j-1}\} = i \right],$$

means that, among the routes ending in clockwise direction, the shortest route has $i-1$ steps before a turn (i.e., this route skips the spacing D_i). By Corollary 2.1 such events do not provide information on the distribution of the two internal maxima in the last expression of (4.8), and thus they are independent of the event that the route under the m-step strategy ends in clockwise direction. Hence, from (2.14) we obtain

$$\begin{aligned} \mathrm{P}\left(K_n^{(m)} = k\right) &= 2 \cdot \frac{1}{2} \cdot \mathrm{P}\left(\arg \max_{1 \leq j \leq m+1} \{X_j - bS_{j-1}\} = k+1\right) \qquad (4.11) \\ &= \frac{2^{m-k}}{2^{m+1}-1} = \frac{1}{2^{k+1} - 2^{k-m}}, \quad 0 \leq k \leq m, \end{aligned}$$

where the factor 2 in (4.11) takes into account the completely symmetrical event that the route under the m-step strategy ends in counterclockwise direction. Our findings are summarized in the following theorem.

Theorem 4.2 *For any $m = 0, 1, \ldots; 2m < n$,*

$$\mathrm{P}(K_n^{(m)} = k) = \frac{1}{2^{k+1} - 2^{k-m}}, \quad 0 \leq k \leq m.$$

4.4 Distribution of the travel time

We will now use Theorem 4.1 to prove that $T_n^{(m)}$ can be expressed as a probabilistic mixture of spacings. Let us first consider

$$\max\left\{\sum_{j=1}^{m+1} a_{m+2-j}^{-1} X_j, \sum_{j=1}^{m+1} a_{m+2-j}^{-1} X_{n+2-j}\right\}. \tag{4.12}$$

Below we argue that this random variable is distributed as a probabilistic mixture of sums of $2(m+1)$ exponentials. This will be explained via the paths of the Markov chain in Figure 4.3. The states are the grid points (x, y) where $0 \le y \le x \le m+1$. From a state (x, y) with $0 < y < x$ it is possible to make a transition to $(x-1, y)$ with probability $a_x/(a_x + a_y)$, and to $(x, y-1)$ with probability $a_y/(a_x + a_y)$. From states on the horizontal axis and the diagonal, only one transition is possible. State $(0, 0)$ is absorbing. For this Markov chain we consider the paths, that start in $(m+1, m+1)$ and eventually end in $(0, 0)$, and show that these paths generate sums of exponentials representing maximum (4.12).

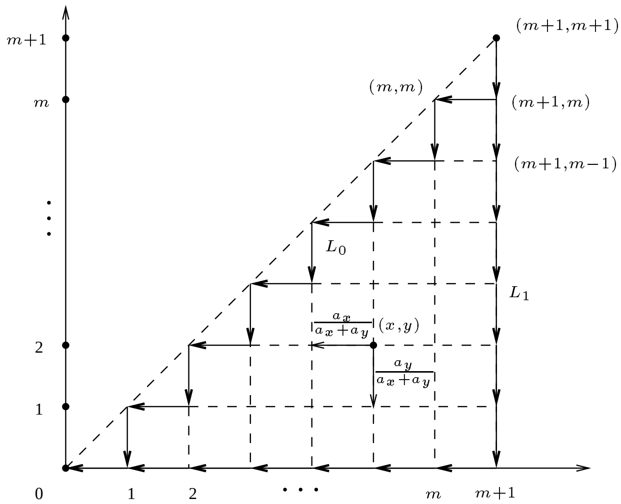

Figure 4.3: Markov chain interpretation of the maximum of two sums of exponentials.

Starting from state $(m+1, m+1)$, we compare the terms $a_{m+1}^{-1}X_1$ and $a_{m+1}^{-1}X_{n+1}$ in (4.12). Without loss of generality we assume $[a_{m+1}^{-1}X_1 > a_{m+1}^{-1}X_{n+1}]$. Then $a_{m+1}^{-1}X_{n+1}$ is distributed as $(2a_{m+1})^{-1}X_{n+1}$, and we can take it outside the maximum, reducing the second sum by one term. Further, due to the memory-less property of exponentials, the overshoot of $a_{m+1}^{-1}X_1$ is independent of X_{n+1}, and it has the same distribution as $a_{m+1}^{-1}X_1$. Hence, the first sum in (4.12) remains the same, and therefore expression (4.12) is distributed as

$$(2a_{m+1})^{-1}X_{n+1} + \max\left\{\sum_{j=1}^{m+1} a_{m+2-j}^{-1}X_j, \sum_{j=2}^{m+1} a_{m+2-j}^{-1}X_{n+2-j}\right\}. \tag{4.13}$$

So, the transition from $(m+1, m+1)$ to $(m+1, m)$ can be interpreted as a transition from (4.12) to (4.13). By leaving state $(m+1, m+1)$ we have taken the term $(2a_{m+1})^{-1}X_{n+1}$ outside the maximum. Now we are at $(m+1, m)$, and we compare $a_{m+1}^{-1}X_1$ and $a_m^{-1}X_n$. If the event $[a_{m+1}^{-1}X_1 > a_m^{-1}X_n]$ takes place, then we take $(a_{m+1}+a_m)^{-1}X_n$ outside the maximum, and thus we reduce the second sum by one term again. Given $[a_{m+1}^{-1}X_1 < a_m^{-1}X_n]$ (the probability of this event is $a_{m+1}/(a_{m+1}+a_m)$), we take the term $(a_{m+1}+a_m)^{-1}X_1$ outside the maximum, and we reduce the first sum by one term. Hence, leaving $(m+1, m)$ we always get an exponential with mean $(a_{m+1}+a_m)^{-1}$ outside the maximum. With probability $a_m/(a_{m+1}+a_m)$ we make a transition to $(m+1, m-1)$, where the terms $a_{m+1}^{-1}X_1$ and $a_{m-1}^{-1}X_{n-1}$ are to be compared, and otherwise we move to (m, m), where we have to compare the terms $a_m^{-1}X_2$ and $a_m^{-1}X_n$.

We proceed in this way traveling from $(m+1, m+1)$ to $(0, 0)$, without crossing the diagonal. Every time we leave a state (x, y), we get an exponential with mean $(a_x + a_y)^{-1}$ outside the maximum. A transition to $(x, y-1)$ means that the second sum in the maximum has been reduced by the first term; a transition to $(x-1, y)$ means the same for the first sum.

Let L denote the set of states visited along a path from $(m+1, m+1)$ to $(1, 0)$, and let $\mathrm{P}(L)$ be the probability of this path, i.e., the product of the probabilities of each transition in path L. Then it is clear, from the exposition above, that along path L, maximum (4.12) becomes a linear combination of exponentials with coefficients $(a_x + a_y)^{-1}$, $(x, y) \in L$. For example, path L_1 in Figure 4.3 generates the sum

$$\sum_{j=1}^{m+1} a_{m+2-j}^{-1}X_j + \sum_{j=1}^{m+1}(a_{m+1}+a_{m+2-j})^{-1}X_{n+2-j}.$$

The probability that maximum (4.12) is distributed as the sum above is the product of transition probabilities

$$\frac{a_m}{a_{m+1}+a_m}\cdot\frac{a_{m-1}}{a_{m+1}+a_{m-1}}\cdots\frac{1}{a_{m+1}+1}.$$

One can also say that path L_1 corresponds to the event

$$\left[a_{m+1}^{-1}X_1 > \sum_{j=1}^{m+1} a_{m+2-j}^{-1}X_{n+2-j}\right].$$

We can conclude that, with probability $\mathrm{P}(L)$, maximum (4.12) is distributed as the sum

$$S(L) = \sum_{(x,y)\in L} (a_x + a_y)^{-1} X_{x+y}.$$

Note that each path goes through the states $(m+1, m+1)$, $(m+1, m)$ and $(1, 0)$. Hence, $S(L)$ always includes exponentials with coefficients $(2a_{m+1})^{-1}$, $(a_{m+1} + a_m)^{-1}$ and 1.

It is readily verified that, just as in the derivation of Theorem 4.1, conditioning on path L does not alter the sum S_{n+1}. For example, after the first transition from $(m+1, m+1)$ to $(m+1, m)$, thus under event $[a_{m+1}^{-1} X_1 > a_{m+1}^{-1} X_{n+1}]$, we replace the X_j's by Y_j's as follows:

$$a_{m+1}^{-1} X_{n+1} = (2a_{m+1})^{-1} Y_{n+1}; \qquad a_{m+1}^{-1} X_1 = (2a_{m+1})^{-1} Y_{n+1} + a_{m+1}^{-1} Y_1;$$
$$X_j = Y_j, \quad j \neq 1, n+1,$$

where $Y_1, Y_2, \ldots$ are i.i.d. exponentials with mean 1. Since $X_1 + X_{n+1} = Y_1 + Y_{n+1}$ (cf. (2.8)), the sum S_{n+1} remains $Y_1 + Y_2 + \cdots + Y_{n+1}$. Renaming again the Y_j's by X_j's, we can repeat this procedure in the second transition, and so on. Hence, from Theorem 4.1 and (2.18), we obtain that, with probability $\mathrm{P}(L)$, the random variable $T_n^{(m)}$ is distributed as

$$T(L) = 1 - \sum_{(x,y)\in L} (a_x + a_y)^{-1} D_{x+y}.$$

This is summarized in the following theorem, where $\mathcal{L}(m)$ is the set of all paths from $(m+1, m+1)$ to $(1, 0)$.

Theorem 4.3 *For any $m = 0, 1, \ldots; 2m < n$,*

$$\mathrm{P}(T_n^{(m)} < t) = \sum_{L\in\mathcal{L}(m)} \mathrm{P}(L)\mathrm{P}(T(L) < t). \tag{4.14}$$

Remark 4.3 It is well-known (see, for example, Yaglom and Yaglom [84], problem 83a) that the cardinality of $\mathcal{L}(m)$ is a Catalan number, i.e.,

$$|\mathcal{L}(m)| = \frac{1}{m+2}\binom{2m+2}{m+1}.$$

For more detail on lattice path counting and various applications we refer to the book of Mohanty [70].

Remark 4.4 The probability $\mathrm{P}(L)$ is maximal for the path passing through all states on the diagonal, i.e., path L_0 in Figure 4.3. To prove this, we consider two possible ways to reach state (x, y) from $(x+1, y+1)$; see Figure 4.4. For $x = y$ route (a) is not possible. For $x > y$ the probability of the (a)-route is

$$\frac{2^{x+1}-1}{2^{x+1}+2^{y+1}-2} \cdot \frac{2^{y+1}-1}{2^x+2^{y+1}-2}, \; x > y+1; \qquad \frac{2^{x+1}-1}{2^{x+1}+2^{y+1}-2} \cdot 1, \; x = y+1,$$

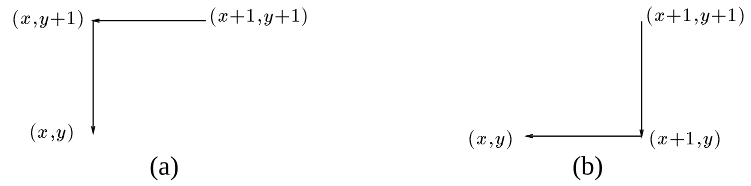

Figure 4.4: Two possible ways from $(x+1, y+1)$ to (x, y).

which is obviously larger than the probability of the (b)-route, given by

$$\frac{2^{y+1}-1}{2^{x+1}+2^{y+1}-2} \cdot \frac{2^{x+1}-1}{2^{x+1}+2^{y}-2}.$$

Hence, for $x > y$, replacing the (b)-route by the (a)-route always gives a more likely path. Thus, the probability of path L_1 in Figure 4.3 is the smallest, and the probability of the path L_0 is the biggest.

To obtain a tractable expression for (4.14), first note that

$$T(L) = D_1 + D_2 + \cdots + D_{n+1} - \sum_{(x,y)\in L} a_{(x,y)}^{-1} D_{x+y},$$

where $a_{(x,y)} = a_x + a_y$. So, $T(L)$ is a linear combination of $n+1$ spacings; actually, only of n spacings, because $a_{(1,0)} = 1$, and thus D_1 vanishes. Then a closed-form expression for the right-hand side of (4.14) straightforwardly follows from (2.26). This yields, for any $L \in \mathcal{L}(m)$,

$$\begin{aligned} \mathrm{P}\left(T(L) < t\right) &= \sum_{(x,y)\in L} \left(a_{(x,y)}t - a_{(x,y)} + 1\right)_+^n \prod_{\substack{(x',y')\in L \\ (x',y')\neq(x,y)}} \frac{a_{(x',y')}}{a_{(x',y')} - a_{(x,y)}}, \\ & \quad t < 1; \\ \mathrm{P}\left(T(L) < t\right) &= 1, \quad t \geq 1. \end{aligned}$$

Example 4.1 We will derive the distributions of the travel time for the 0-, 1- and 2-step strategies using Figure 4.5, where we display $a_{(x,y)}$ at every state (x, y), $x, y = 0, 1, 2, 3$, and the transition probabilities at the arrows.

Let us first consider the 0-step strategy, also known as the Shorter Direction (SD) heuristic described on p. 33. Under the SD heuristic the picker is not allowed to turn; he chooses the shortest of two possible routes. In Figure 4.5 there is only one possible path from $(1, 1)$ to $(0, 0)$. Hence, the travel time under the SD heuristic satisfies

$$T_n^{(0)} \stackrel{d}{=} 1 - \max\{D_1, D_{n+1}\} \stackrel{d}{=} 1 - D_1 - \frac{1}{2}D_2$$

with

$$\mathrm{P}\left(T_n^{(0)} < t\right) = 2t^n - (2t-1)_+^n, \quad 0 \leq t \leq 1.$$

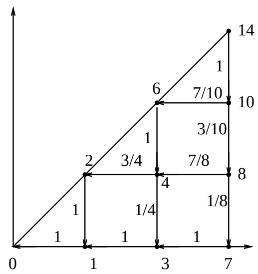

Figure 4.5: Illustration for the analysis of 0-, 1- and 2-step strategies.

Under the 1-step strategy the picker chooses the best of 4 routes. As we can see in Figure 4.5, there are two possible paths from $(2,2)$ to $(0,0)$, thus for $n \geq 3$ the travel time $T_n^{(1)}$ is distributed as:

$$\begin{array}{ll} 1-D_1-\frac{1}{2}D_2-\frac{1}{4}D_3-\frac{1}{6}D_4 & \text{with probability (w.p.) } 3/4, \\ 1-D_1-\frac{1}{3}D_2-\frac{1}{4}D_3-\frac{1}{6}D_4 & \text{w.p. } 1/4. \end{array}$$

Then for $0 \leq t \leq 1$,

$$\mathrm{P}\left(T_n^{(1)} < t\right) = 3t_+^n - \frac{9}{4}(2t-1)_+^n - (3t-2)_+^n + \frac{3}{2}(4t-3)_+^n - \frac{1}{4}(6t-5)_+^n.$$

Finally, for $n \geq 5$, the travel time under the 2-step strategy is distributed as a mixture of 5 sums of spacings, corresponding to the 5 paths from $(3,3)$ to $(0,0)$. From Figure 4.5 it is clear that $T_n^{(2)}$ is distributed as:

$$\begin{array}{ll} 1-D_1-\frac{1}{2}D_2-\frac{1}{4}D_3-\frac{1}{6}D_4-\frac{1}{10}D_5-\frac{1}{14}D_6 & \text{w.p. } \frac{3}{4}\cdot\frac{7}{10}, \\ 1-D_1-\frac{1}{3}D_2-\frac{1}{4}D_3-\frac{1}{6}D_4-\frac{1}{10}D_5-\frac{1}{14}D_6 & \text{w.p. } \frac{1}{4}\cdot\frac{7}{10}, \\ 1-D_1-\frac{1}{2}D_2-\frac{1}{4}D_3-\frac{1}{8}D_4-\frac{1}{10}D_5-\frac{1}{14}D_6 & \text{w.p. } \frac{3}{4}\cdot\frac{7}{8}\cdot\frac{3}{10}, \\ 1-D_1-\frac{1}{3}D_2-\frac{1}{4}D_3-\frac{1}{8}D_4-\frac{1}{10}D_5-\frac{1}{14}D_6 & \text{w.p. } \frac{1}{4}\cdot\frac{7}{8}\cdot\frac{3}{10}, \\ 1-D_1-\frac{1}{3}D_2-\frac{1}{7}D_3-\frac{1}{8}D_4-\frac{1}{10}D_5-\frac{1}{14}D_6 & \text{w.p. } \frac{1}{8}\cdot\frac{3}{10}. \end{array}$$

It then follows that, for $0 \leq t \leq 1$,

$$\begin{array}{rcl} \mathrm{P}(T_n^{(2)} < t) & = & \frac{7}{2}t_+^n - \frac{49}{16}(2t-1)_+^n - \frac{7}{4}(3t-2)_+^n + \frac{49}{16}(4t-3)_+^n - \frac{49}{64}(6t-5)_+^n \\ & + & \frac{1}{4}(7t-6)_+^n - \frac{7}{16}(8t-7)_+^n + \frac{7}{32}(10t-9)_+^n - \frac{1}{64}(14t-13)_+^n. \end{array}$$

4.5 Moments of the travel time

In this section we shall calculate, for any path L, the moments of $1 - T(L)$. From these moments we can obtain, by virtue of Theorem 4.3, the corresponding moments for the travel time $T_n^{(m)}$. For the kth moment of $1-T(L)$ we obtain from (2.33):

$$\begin{aligned}\mathrm{E}\left([1-T(L)]^k\right) &= \mathrm{E}\left(\left(\sum_{(x,y)\in L} a_{(x,y)}^{-1} D_{x+y}\right)^k\right)\\ &= \binom{n+k}{k}^{-1} \sum_{\substack{k_1,k_2,\dots,k_{2m+2}\geq 0\\ k_1+k_2+\cdots+k_{2m+2}=k}} \prod_{(x,y)\in L} a_{(x,y)}^{-k_{x+y}}.\end{aligned}$$

It follows from (2.34)–(2.36) that the first two moments of $1 - T(L)$ and the variance are given by

$$\begin{aligned}\mathrm{E}\,(1-T(L)) &= \frac{1}{n+1}\sum_{(x,y)\in L} a_{(x,y)}^{-1},\\ \mathrm{E}\left([1-T(L)]^2\right) &= \frac{1}{(n+1)(n+2)}\left(\sum_{(x,y)\in L} a_{(x,y)}^{-2} + \left(\sum_{(x,y)\in L} a_{(x,y)}^{-1}\right)^2\right),\\ \mathrm{Var}\,(1-T(L)) &= \frac{1}{(n+1)(n+2)}\left(\sum_{(x,y)\in L} a_{(x,y)}^{-2} - \frac{1}{n+1}\left(\sum_{(x,y)\in L} a_{(x,y)}^{-1}\right)^2\right).\end{aligned}$$

Example 4.2 The mean and variance of the travel time for the 0-, 1- and 2-step strategies can readily be derived from Example 4.1. For the 0-step strategy we obtain

$$\begin{aligned}\mathrm{E}\left(T_n^{(0)}\right) &= 1-\frac{3}{2(n+1)}, \qquad \mathrm{E}\left(\left[T_n^{(0)}\right]^2\right) = \frac{n^2-1/2}{(n+1)(n+2)},\\ \mathrm{Var}\left(T_n^{(0)}\right) &= \frac{1}{4}\cdot\frac{5n-4}{(n+1)^2(n+2)},\end{aligned}$$

and the 1-step strategy gives

$$\begin{aligned}\mathrm{E}\left(T_n^{(1)}\right) &= 1-\frac{15}{8(n+1)}, \qquad \mathrm{E}\left(\left[T_n^{(1)}\right]^2\right) = \frac{144n^2-108n-97}{144(n+1)(n+2)},\\ \mathrm{Var}\left(T_n^{(1)}\right) &= \frac{5}{576}\cdot\frac{151n-254}{(n+1)^2(n+2)},\end{aligned}$$

which is valid for $n \geq 3$. For the 2-step strategy we confine ourselves to the mean travel time only, yielding

$$\mathrm{E}\left(T_n^{(2)}\right) = 1 - \frac{9073}{4480(n+1)}, \qquad n \geq 5. \tag{4.15}$$

Of course, it holds that $\mathrm{E}\left(T_n^{(0)}\right) > \mathrm{E}\left(T_n^{(1)}\right) > \mathrm{E}\left(T_n^{(2)}\right)$, $n \geq 5$.

4.6 Performance evaluation

In this section we present numerical results on the performance of the m-step strategy, and we compare it with the performance of the optimal pick strategy and the NI heuristic.

n	m	$\mathrm{E}(T_n^{(m)})$	$\sigma(T_n^{(m)})$	$\mathrm{E}(T_n^{Opt})$	$\sigma(T_n^{Opt})$	$\mathrm{E}(T_n^{NI})$	$\sigma(T_n^{NI})$
5	0	0.750	0.144	0.659	0.123	0.672	0.128
	1	0.688	0.131				
	2	0.663	0.123				
10	0	0.864	0.089	0.805	0.083	0.818	0.086
	1	0.830	0.087				
	2	0.816	0.085				
	3	0.810	0.084				
	4	0.807	0.083				
20	0	0.929	0.050	0.897	0.049	0.905	0.050
	1	0.911	0.050				
	2	0.904	0.049				
	3	0.900	0.049				
	4	0.899	0.049				
	5	0.898	0.049				
	6	0.898	0.049				

Table 4.1: Mean and standard deviation of the travel time under the m-step strategy, the optimal strategy and the NI heuristic, respectively.

In Table 4.1 we list the mean and standard deviation of the travel time under the m-step strategy for various values of m and n, and we compare them with the ones for the optimal pick strategy and the NI heuristic. The random variables T_n^{Opt} and T_n^{NI} denote the travel time under the optimal strategy and the NI heuristic, respectively. For each n, the results for the optimal strategy have been obtained from a simulation of 10^6 trials; for the NI heuristic we use (3.11) and (3.17):

$$\begin{aligned} \mathrm{E}\left(T_n^{NI}\right) &= 1 - \frac{2}{n+1} + \frac{1}{(n+1)2^n}, \\ \mathrm{Var}\left(T_n^{NI}\right) &= \frac{1}{(n+1)^2(n+2)}\left(\frac{4n}{3} - \frac{8}{3} + \frac{1}{2^{n-2}} - \frac{n}{3\cdot 4^n} - \frac{1}{3\cdot 4^{n-1}}\right). \end{aligned}$$

Hence, from (4.15), we can immediately conclude that

$$\mathrm{E}\left(T_n^{(2)}\right) < \mathrm{E}\left(T_n^{NI}\right), \qquad n \geq 5.$$

Moreover, one can easily check that $\sigma(T_n^{(2)}) < \sigma(T_n^{NI})$, $n \geq 5$. Thus, already for $m = 2$, the m-step strategy outperforms the NI heuristic.

The results in Table 4.1 show that, indeed, already for small values of m the performance of the m-step strategy is very close to optimal. This is not only valid for the mean and standard deviation of the travel time; it is also true for the distribution. This is demonstrated in Figure 4.6, where we display for $n = 10$ the complementary

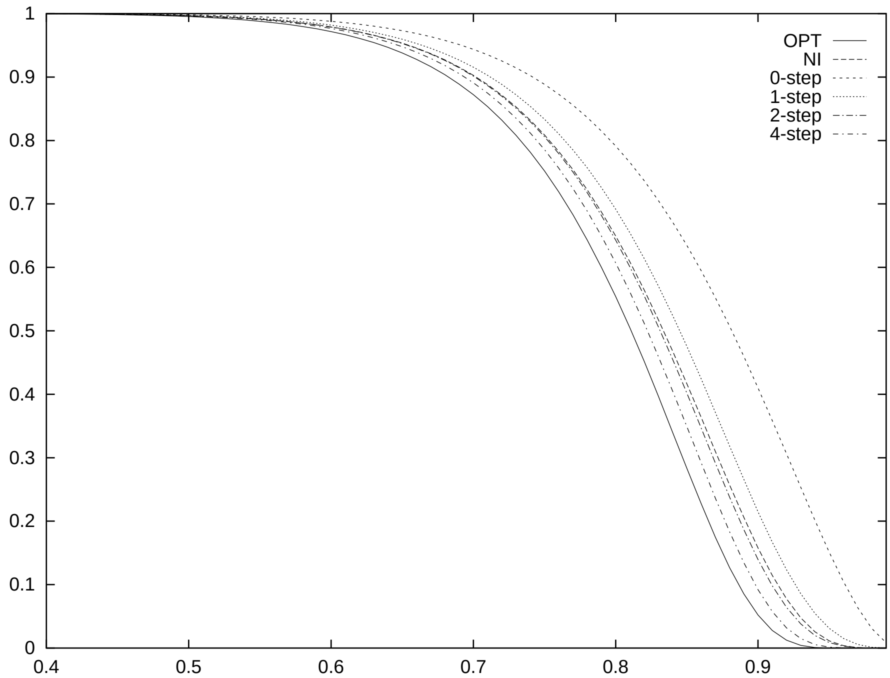

Figure 4.6: The complementary distribution function of the travel time.

distribution function of the travel time for the optimal and NI strategy, and the 0-, 1-, 2- and 4-step strategy. The distribution function for the optimal strategy has been obtained from a simulation of 10^6 trials; the one for the NI strategy has been calculated exactly (see formula (3.30)).

The results suggest that, if the picker turns under the optimal strategy, then it is very likely that he does so after collecting a small number of items. In other words, already for small values of m, the optimal strategy will coincide with the m-step strategy with high probability. This is also confirmed by the results listed in Table 4.2. For various values of n, we estimated from a simulation of 10^6 trials, the probability that the picker, operating under the optimal strategy, will turn after collecting m items, $m = 0, 1, \ldots, 5$. Here $m = 0$ means that the picker does not turn.

From Table 4.2 one can see that the probability that the optimal route turns after k steps converges to $1/2^{k+1}$ when n goes to infinity. This will be proved in Section 5.2 (see Theorem 5.1).

n	m					
	0	1	2	3	4	5
3	0.646	0.291	0.062			
5	0.558	0.277	0.124	0.037	0.004	
8	0.516	0.259	0.129	0.062	0.026	0.008
10	0.506	0.254	0.127	0.063	0.030	0.013
15	0.501	0.251	0.126	0.062	0.031	0.016
20	0.499	0.250	0.125	0.062	0.031	0.016

Table 4.2: Probability that the picker, collecting a list of n items under the optimal strategy, will turn after m steps.

4.7 Concluding remarks

In this chapter we studied strategies related to the optimal route. First, we analyzed the One-Side Optimal (OSO) strategy, where the picker chooses the best route ending in a given direction. We derived the distribution of the travel time and of the number of items collected before the turn. We have figured out that the OSO strategy does not perform very well: its travel time is stochastically larger than the travel time under the NI heuristic.

The major part of this chapter is devoted to the so-called m-step strategies. For uniformly distributed pick positions, we used arguments from Chapter 2 to find the distribution and the moments of the travel time needed to pick n items. Our method is only applied to the case $2m < n$. In principle the method also works for larger values of m, but then the resulting expressions will become essentially more complicated. We have seen that, already for small values of m, the performance of m-step strategies is very close to optimal. Also, our analysis showed that the 2-step strategy on average performs better than the NI heuristic, and it may be even easier to implement.

Chapter 5

On the analysis of the optimal route

5.1 Introduction

This chapter is devoted to properties of the optimal route of the picker who has to collect n items independently and uniformly distributed on a circle:

Optimal strategy: The picker chooses the shortest possible route.

We explore the minimal travel time in terms of the uniform $(n+1)$-spacings $D_1, D_2, \ldots, D_{n+1}$. For $n = 1$, the problem is trivial. The picker just chooses the shorter distance from the starting point to the item providing that the travel time T_1^{Opt} is distributed as $1/2D_1$. For $n = 2$, one can easily verify that the optimal route is guaranteed by the Nearest Item (NI) heuristic where the next item to be picked is always the nearest one. Therefore, it follows from formula (3.29) that the travel time T_2^{Opt} is distributed as $1/2D_1 + 3/4D_2$. For $n \geq 3$, the problem becomes much more difficult.

The crucial observation made by many authors (see [6, 42, 79]) is that the optimal route admits at most one turn. This follows because it is never optimal to cover the same segment of the circle more than twice (see also p. 4). Then there are only $2n$ candidate sequences, and thus the optimal sequence can be always found in linear time. In the previous chapter we proposed so-called m-steps strategies. Under the m-step strategies the picker chooses the shortest route among the ones that change direction at most once (as does the optimal route), and only do so after collecting no more than m items. In fact, as we already noticed in Sections 4.1, 4.3 the optimal strategy is the m-step strategy with $m = n - 1$. Therefore, the minimal travel time

T_n^{Opt} can be expressed by using formulas (4.7), (4.8):

$$\begin{aligned} T_n^{Opt} &= \min\left\{ \min_{1\le j\le n}\left\{\sum_{l=1}^{j-1} 2D_l + \sum_{l=j+1}^{n+1} D_l\right\},\right. \\ &\qquad \left.\min_{1\le j\le n}\left\{\sum_{l=1}^{j-1} 2D_{n+2-l} + \sum_{l=j+1}^{n+1} D_{n+2-l}\right\}\right\} \\ &= 1 - \max\left\{\max_{1\le j\le n}\left\{D_j - \sum_{l=1}^{j-1} D_l\right\}, \max_{1\le j\le n}\left\{D_{n+2-j} - \sum_{l=1}^{j-1} D_{n+2-l}\right\}\right\} \\ &\stackrel{d}{=} 1 - \frac{1}{S_{n+1}} \max\left\{\max_{1\le j\le n}\{X_j - S_{j-1}\}, \max_{1\le j\le n}\{X_{n+2-j} - (S_{n+1} - S_{n+2-j})\}\right\}. \end{aligned} \tag{5.1}$$

Here the second expression for T_n^{Opt} could be derived directly from the alternative (and, maybe, more intuitive) interpretation that we have already mentioned in Chapter 4 for the OSO and the m-step strategies. The term $D_j - \sum_{l=1}^{j-1} D_l$, where $j = 1, 2, \ldots, n$, is the gain in travel time (compared to one full rotation) obtained by skipping the spacing D_j and going back to the starting point instead. The same can be said about $D_j - \sum_{l=j+1}^{n+1} D_l$, where $j = 2, 3, \ldots, n+1$, but here the picker goes back in the other direction. Under the optimal strategy, the picker chooses the largest possible gain.

Although the last expression in (5.1) looks very similar to the right-hand side of (4.8), the latter could be analyzed only for $2m < n$. Thus, the optimal route requires special studies which are partially introduced in this chapter. In Section 5.2 we prove the conjecture drawn from Table 4.2. Specifically, we show that for fixed $k = 0, 1, \ldots$, the probability that the picker collects exactly k items before a turn tends to $1/2^{k+1}$ when $n \to \infty$. Further, in Section 5.3 we consider the Nearly-Optimal (NO) strategy. Roughly speaking, this is just the m-step strategy with the largest possible m, provided $2m < n$. The slight difference is that for even n, the NO strategy allows one more step before a turn in one of the directions. The NO strategy provides the route which differs from the optimal one with a probability of the order $2^{-n/2}$. While considering the NO strategy, we draw some conclusions about the minimal travel time. First, we provide a tight upper bound of the travel time under the NO and the optimal strategy for any realization of the items' locations. Then we present a stochastic lower bound as a conjecture. Further, we derive a simple and precise upper estimate for the mean travel time under the NO strategy. This upper bound also gives a good approximation for the expectation of the minimal travel time.

In Section 4.3 above we explained why the condition $2m < n$ was important to derive the distribution of the travel time under the m-step strategy. Below in Section 5.4 we consider this issue in detail. We propose some possibilities to derive the distribution of the minimal travel time, and we discover difficulties arising in the analysis. As an example, we derive the minimal travel time distribution for $n = 3$.

Then we propose a conjecture on the form of the distribution of the travel time under the optimal strategy for arbitrary n. Section 5.5 is devoted to conclusions.

5.2 The number of items collected before a turn

Let K_n^{Opt} be the number of steps before the picker turns while picking n items under the optimal strategy. Note that we define K_n^{Opt} to be zero in case of no turn (in other words, no items are collected before the turn). Then the following assertion holds.

Theorem 5.1 *For any fixed $k = 0, 1, \ldots,$*

$$\lim_{n\to\infty} \mathrm{P}\left(K_n^{Opt} = k\right) = \frac{1}{2^{k+1}}.$$

Proof. First note that under event $[K_n^{Opt} \leq m]$ the optimal strategy and the m-step strategy prescribe the same picking sequence. Hence, for any fixed $k \leq m$, event $[K_n^{Opt} = k]$ occurs if and only if (i) the optimal route turns after at most m steps, and (ii) the route under the m-step strategy turns after exactly k steps. Hence, for $0 \leq k \leq m$; $2m < n$,

$$\mathrm{P}\left(K_n^{(m)} = k\right) - \mathrm{P}\left(K_n^{Opt} > m\right) \leq \mathrm{P}\left(K_n^{Opt} = k\right) \leq \mathrm{P}\left(K_n^{(m)} = k\right). \tag{5.2}$$

By letting both m and n go to infinity such that the inequality $2m < n$ is always satisfied, we obtain from Theorem 4.2 that

$$\lim_{\substack{m,n\to\infty \\ 2m<n}} \mathrm{P}\left(K_n^{(m)} = k\right) = \frac{1}{2^{k+1}}. \tag{5.3}$$

Further, from (5.1) by using (2.14) with $b = 1$, we get

$$\begin{aligned} \mathrm{P}\left(K_n^{Opt} > m\right) &< 2 \cdot \mathrm{P}\left(\left[\arg\max_{1\leq j\leq n} \{X_j - S_{j-1}\} > m+1\right]\right) \\ &= 2 \cdot \sum_{i=m+2}^{n} \frac{2^{n-i}}{2^n - 1} < \frac{1}{2^m}, \end{aligned}$$

yielding

$$\lim_{\substack{m,n\to\infty \\ 2m<n}} \mathrm{P}\left(K_n^{Opt} > m\right) = 0. \tag{5.4}$$

Now the statement of the theorem directly follows from (5.2)-(5.4). □

5.3 Bounds for the travel time

In Chapter 4 we used formula (4.10) in order to derive the distribution of the travel time under the m-step strategies, provided $2m < n$. However, for the travel time under the optimal strategy such a formula is not available. Essentially, this can be put as follows. Remember that both the m-step strategy and the optimal strategy provide a route which skips one of the spacings going back instead. Under the optimal strategy, each spacing can be skipped in two ways: going back either clockwise or counterclockwise. Under the m-step strategy with $2m < n$, only one of the ways is possible, and in fact, this is the feature that guarantees the availability of a representation like (4.10).

Now let us think of a strategy which is as close as possible to the optimum, but whose travel time can be still expressed via a maximum of two sums of normalized exponentials like in (4.10). This would be the Nearly-Optimal (NO) strategy defined as follows:

Nearly-Optimal (NO) strategy: If $n = 2m + 1$, then the picker follows the m-step strategy (see p. 63). If $n = 2m$, then the strategy is similar, but asymmetric: the picker chooses the shortest route among the ones that change direction only once and only do so after collecting no more than m items ending clockwise and no more than $m - 1$ items ending counterclockwise.

The illustration of this strategy is given in Figure 5.1. Here we use the following notations: 1 – the area where the picker may turn ending clockwise; 2 – the area where the picker may turn ending counterclockwise.

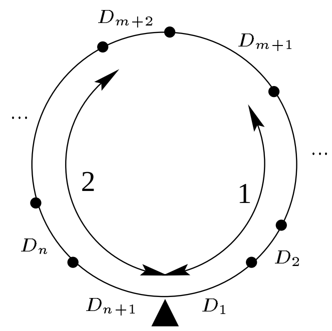

Figure 5.1: The NO strategy for $n = 2m$; $n = 2m + 1$.

Roughly speaking, under the NO strategy, the picker is not allowed to turn after collecting more than $n/2$ items. Very often such a strategy would be optimal, but not always. For example, in Figure 5.2 we show a pathological case in which it is optimal to collect $n - 1$ items clockwise and then turn ending counterclockwise. However, it follows from Theorem 5.1 that the route under the NO strategy differs from the optimal route with a very small probability which decreases with n as $2^{-n/2}$.

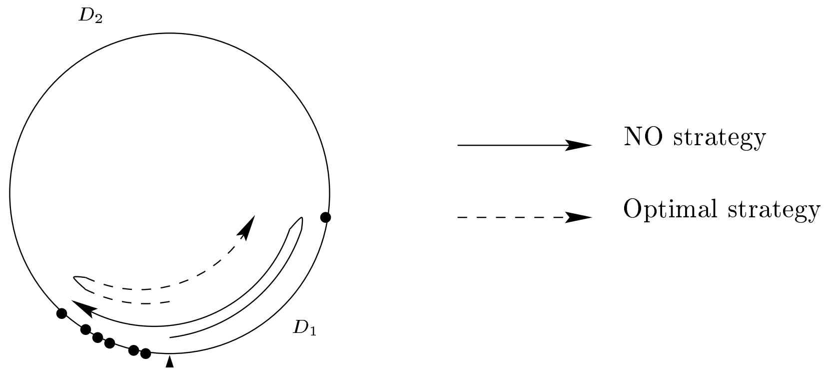

Figure 5.2: Optimal route with $n-1$ items collected before the turn; the NO route is far from optimal in this case.

Analogously to (4.10), the travel time T_n^{NO} under the NO strategy can be expressed as

$$T_n^{NO} \stackrel{d}{=} 1 - \frac{1}{S_{n+1}} \max\left\{ \sum_{j=1}^{m+1} a_{m+2-j}^{-1} X_j, \sum_{j=1}^{m} a_{m+1-j}^{-1} X_{n+2-j} \right\}, \ n = 2m;$$

$$T_n^{NO} \stackrel{d}{=} 1 - \frac{1}{S_{n+1}} \max\left\{ \sum_{j=1}^{m+1} a_{m+2-j}^{-1} X_j, \sum_{j=1}^{m+1} a_{m+2-j}^{-1} X_{n+2-j} \right\}, \ n = 2m+1,$$

where $a_j = 2^j - 1$, $j \geq 0$. The distribution of T_n^{NO} can be found as in Section 4.4 using the Markov chain interpretation described on p. 66 and illustrated in Figure 4.3. The resulting expression is similar to (4.14):

$$\mathrm{P}\left(T_n^{NO} < t\right) = \sum_{L \in \mathcal{L}_n^{NO}} \mathrm{P}(L)\mathrm{P}(T(L) < t), \tag{5.5}$$

where for $n = 2m+1$ and $n = 2m$, $\mathcal{L}_n^{NO}$ is the set of all paths from, respectively, $(m+1, m+1)$ and $(m+1, m)$ to $(1, 0)$. Note that for any $L \in \mathcal{L}_n^{NO}$, the random variable $T(L)$ is a linear combination of $D_1, D_2, \ldots, D_{n+1}$ where all the coefficients are smaller than 1:

$$T(L) = \sum_{\substack{x+y=1 \\ (x,y)\in L}}^{n+1} \left(1 - a_{(x,y)}^{-1}\right) D_{x+y}.$$

Here $a_{(x,y)} = a_x + a_y$. Since any path L arrives to the point $(1, 0)$, and $a_{(1,0)} = 1$, the coefficient of D_1 in $T(L)$ is always 0. Thus, $T(L)$ is in fact a linear combination of n spacings $D_2, D_3, \ldots, D_{n+1}$. Furthermore, for every path L the only possible point with $x+y=n+1$ is the starting point. The value of $a_{(x,y)}$ at this point is the largest. Hence, the spacing D_{n+1} has the largest coefficient in $T(L)$. This coefficient equals $(1-a^{-1}_{(m+1,m)})$ if $n=2m$, and it equals $(1-a^{-1}_{(m+1,m+1)})$ if $n=2m+1$. The probability that the travel time T_n^{NO} exceeds this value is, of course, zero. In fact, we can even prove a stronger assertion concerning not only T_n^{NO}, but also the minimal travel time T_n^{Opt}.

Lemma 5.1 *For any $n \geq 1$, the travel times T_n^{NO} and T_n^{Opt} never exceed $1-\alpha_{n+1}$, where*

$$\alpha_{n+1} = \frac{1}{2^{m+1}+2^m-2},\ n=2m; \qquad \alpha_{n+1} = \frac{1}{2\cdot 2^{m+1}-2},\ n=2m+1.$$

This upper bound is tight.

Proof. Without loss of generality assume that $n=2m+1$. For $n=2m$ the proof is the same. The positions of the items plus the picker's starting point partition the circle into $n+1$ spacings with lengths $d_1, d_2, \ldots, d_{n+1}$. Note that for any collection $d_1, d_2, \ldots, d_{n+1} \geq 0$ there exists a number $j=1,2,\ldots,m+1$ such that either (i) $d_j \geq 2^{j-1}\alpha_{n+1}$; $d_l < 2^{j-1}\alpha_{n+1}$, $l=1,2,\ldots,j-1$ or (ii) $d_{n+2-j} \geq 2^{j-1}\alpha_{n+1}$; $d_{n+2-l} < 2^{j-1}\alpha_{n+1}$, $l=1,2,\ldots,j-1$. This follows since

$$2\sum_{j=1}^{m+1} 2^{j-1}\alpha_{n+1} = d_1+d_2+\cdots+d_{n+1} = 1.$$

Without loss of generality assume (i). Then the route that skips the spacing d_j and goes back instead has length

$$1-d_j+d_1+d_2+\cdots+d_{j-1} \leq 1-\alpha_{n+1}.$$

Moreover, such a route is admissible under the NO strategy. Thus, its length is greater or equal than both T_n^{NO} and T_n^{Opt}. This proves the upper bound.

To show the tightness we just put $d_j = d_{n+2-j} = 2^{j-1}\alpha_{n+1}$, $j=1,2,\ldots,m+1$. In this case the travel time under the NO strategy as well as under the optimal strategy equals $1-\alpha_{n+1}$. The example where T_7^{NO} and T_7^{Opt} achieve the value $1-\alpha_8 = 29/30$ is shown in Figure 5.3. Obviously, a similar example can be constructed for any n. □

Remark 5.1 The values α_{n+1} play an important role in the analysis of the optimal route. For convenience of the reader, we write down the first 11 of them:

$$\alpha_1=1,\ \alpha_2=1/2,\ \alpha_3=1/4,\ \alpha_4=1/6,\ \alpha_5=1/10,\ \alpha_6=1/14,$$
$$\alpha_7=1/22,\ \alpha_8=1/30,\ \alpha_9=1/46,\ \alpha_{10}=1/62,\ \alpha_{11}=1/94.$$

By definition, the value α_{n+1} decreases with n as $2^{-n/2}$.

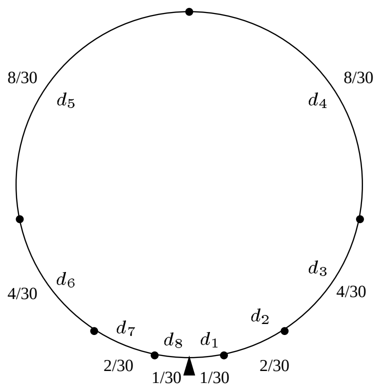

Figure 5.3: Example for which T_7^{NO} and T_7^{Opt} achieve their common upper bound.

Now we are coming back to (5.5), and we are going to derive a simple stochastic lower bound for T_n^{NO}. It is easy to see that for given $x+y$, the value $a_{(x,y)}$ is the smallest, if $(x,y) \in L_0$, where L_0 is a path attached to the diagonal (see Figure 4.3). Moreover, in this case, $a_{(x,y)} = \alpha_{x+y}^{-1}$. Hence, it follows from (5.5) that the travel time T_n^{NO} is stochastically larger than

$$T_n^* = \sum_{j=2}^{n+1} (1-\alpha_j) D_j.$$

Note that the distribution of T_n^{NO} considerably differs from the distribution of T_n^* already for $n=3$. On the other hand, the distributions of T_n^{NO} and T_n^{Opt} are very close. Therefore, we propose a natural conjecture which agrees with numerical results, although the analytical proof has not been found so far.

Conjecture 5.2 *T_n^{Opt} is stochastically larger than T_n^* for any $n \geq 3$.*

Note that $T_1^{Opt} \stackrel{d}{=} 1/2D_2 = T_1^*$ and $T_2^{Opt} \stackrel{d}{=} 1/2D_2 + 3/4D_3 = T_2^*$. For $n=3$, we will verify Conjecture 5.2 in Section 5.4 where we derive the distribution of T_3^{Opt}.

Now our objective is to estimate the mean travel time under the NO strategy. Looking at (5.5), one may say that this formula is actually difficult to use, because the number of the paths in $\mathcal{L}_n^{NO}$ is very large:

$$\left|\mathcal{L}_n^{NO}\right| = \frac{1}{m+2}\binom{2m+2}{m+1}, \quad n=2m;\ n=2m+1.$$

However, it is easy to see that if a point (x,y) is far away from the diagonal, then the corresponding coefficient $(1-a_{(x,y)}^{-1})$ has a very small probability to appear in $T(L)$. Consider, for example, some point $(x, x-2)$, where $x = 2, 3, \ldots, m+1$. To achieve this point, a path L must make at least one step down from some point

$(x_1, x_1 - 1)$ to a point $(x_1, x_1 - 2)$, where $x \le x_1$. According to the Markov chain interpretation on p. 66, the probability of such an event is

$$\frac{2^{x_1-1} - 1}{2^{x_1} + 2^{x_1-1} - 2} < \frac{1}{3}.$$

Analogously, to achieve a point $(x, x-k)$, a path necessarily makes steps down at some points $(x_1, x_1 - 1)$, $(x_2, x_2 - 2)$,..., $(x_{k-1}, x_{k-1} - k + 1)$, where $k \le x \le x_{k-1} \le \cdots \le x_1 \le m+1$. The probability of all these steps on the same path is

$$\frac{2^{x_1-1} - 1}{2^{x_1} + 2^{x_1-1} - 2} \cdot \frac{2^{x_2-2} - 1}{2^{x_2} + 2^{x_2-2} - 2} \cdots \frac{2^{x_{k-1}-k+1} - 1}{2^{x_{k-1}} + 2^{x_{k-1}-k+1} - 2} < \prod_{j=1}^{k-1} \frac{1}{2^j + 1}.$$

Hence, the probability to travel through the point $(x, x-k)$ is

$$\sum_{\substack{L \in \mathcal{L}_n^{NO} \\ (x,x-k) \in L}} \mathrm{P}(L) = \sum_{L \in \mathcal{L}_n^{NO}} \mathrm{P}\left((x, x-k) \in L\right) \mathrm{P}\left(L | (x, x-k) \in L\right) < \prod_{\nu=1}^{k-1} \frac{1}{2^\nu + 1}. \quad (5.6)$$

It follows that a term with a coefficient $(1 - 1/a_{(x,x-4)})$ appears in the probabilistic mixture (5.5) with probability smaller than $1/135$. The probability of terms with coefficients $1/a_{(x,x-k)}$, where $k \ge 5$, is negligible, and thus, these terms play almost no role in $\mathrm{E}\left(T_n^{NO}\right)$. Now we take into account that for given $j = 2, 3, \ldots, n+1$, the smallest possible coefficient of D_j in $T(L)$ is $(1 - \alpha_j)$, and we estimate the difference between this coefficient and the other possible coefficients of D_j. For any $l = 1, 2, \ldots, \lfloor j/2 \rfloor$, where $\lfloor j/2 \rfloor$ is the integer part of $j/2$, the difference $\delta_j(l)$ is given by:

$$\delta_j(l) = \alpha_j - a^{-1}_{(j/2+l, j/2-l)} = \alpha_j \cdot \frac{1 + 2^{-2l} - 2^{-l+1}}{1 + 2^{-2l} - 2^{-j/2-l+1}}, \quad j \text{ is even;}$$

$$\delta_j(l) = \alpha_j - a^{-1}_{(\lfloor j/2 \rfloor + l + 1, \lfloor j/2 \rfloor - l)} = \alpha_j \cdot \frac{1 + 2^{-2l-1} - 3 \cdot 2^{-l-1}}{1 + 2^{-2l-1} - 2^{-\lfloor j/2 \rfloor - l}}, \quad j \text{ is odd.}$$

The probability that a coefficient $(1 - \alpha_j + \delta_j(l))$ appears in $T(L)$ can be estimated by (5.6) with $k = 2l$ ($k = 2l + 1$), if j is even (odd). Now we derive the following upper estimate for $\mathrm{E}\left(T_n^{NO}\right)$:

$$\begin{aligned} \mathrm{E}\left(T_n^{NO}\right) &= \frac{1}{n+1} \sum_{j=2}^{n+1} \left(1 - \alpha_j + \sum_{l=1}^{\lfloor j/2 \rfloor} \delta_j(l) \left(\prod_{\nu=1}^{2l-1} \frac{1}{2^\nu + 1} \right) \frac{1}{(2^{2l}+1)^{\mathbf{1}_{[j \text{ is odd}]}}} \right) \\ &< \frac{1}{n+1} \sum_{j=2}^{n+1} (1 - \alpha_j) + \frac{0.09}{n+1}. \end{aligned} \quad (5.7)$$

Obviously, the upper estimate (5.7) can be also used for evaluation of the mean travel time under the optimal strategy. In Table 5.1 we compare the mean travel

n	3	5	10	15	20	30
$\mathrm{E}\left(T_n^*\right)$	.5208	.6520	.7986	.8602	.8933	.9277
$\mathrm{E}\left(T_n^{Opt}\right)$	.5261	.6592	.8053	.8652	.8973	.9304
upper estimate (5.7)	.5433	.6670	.8068	.8658	.8976	.9306

Table 5.1: Estimation of the mean travel time under the optimal strategy.

time $\mathrm{E}\left(T_n^{Opt}\right)$ obtained by simulation with the lower estimate $\mathrm{E}\left(T_n^*\right)$ and the upper estimate (5.7). It follows from the table that both the lower and the upper estimates for $\mathrm{E}\left(T_n^{Opt}\right)$ are rather tight, but the upper estimate becomes closer to the actual value of $\mathrm{E}\left(T_n^{Opt}\right)$, when n increases. This is because for sufficiently large n, the optimal strategy and the NO strategy provide the same route with a very high probability, while (5.7) gives a precise estimate for $\mathrm{E}\left(T_n^{NO}\right)$.

5.4 Distribution of the travel time: possibilities and difficulties

In this section we investigate the possibilities to derive the distribution of the minimal travel time (5.1). As we already saw in the earlier chapters (see formula (2.8), Corollary 2.2, the derivation of Theorems 2.2, 4.1, 4.3 and also Remarks 2.2, 2.4), it suffices to obtain a nice representation of the term

$$S_n^{Opt} = \max\left\{\max_{1\le j\le n}\{X_j - S_{j-1}\}, \max_{1\le j\le n}\{X_{n+2-j} - (S_{n+1} - S_{n+2-j})\}\right\}, \quad (5.8)$$

and then just plug in the new representation of S_n^{Opt} into the formula

$$T_n^{Opt} \overset{d}{=} 1 - S_n^{Opt}/S_{n+1}. \quad (5.9)$$

One may try to expand the internal maxima in (5.8) using formula (2.3) and its proof, as we did for the m-step strategies deriving (4.10) from (4.8). However, we already mentioned in Chapter 4 that there would be certain difficulties.

According to this procedure, we have to first consider $\max\{X_1, X_2 - X_1\}$. Under event $E_1 = [2X_1 < X_2]$ which has a probability 1/3, we use (2.6) to replace X_1 by $1/3Y_1$ and X_2 by $2/3Y_1 + Y_2$, where Y_1 and Y_2 are independent exponentials with mean 1. Similarly, under event $E_2 = [2X_1 > X_2]$ whose probability is 2/3, we replace X_1 by $1/3Y_1 + Y_2$, and X_2 by $2/3Y_1$. In both cases (as we already saw in the proof of (2.3)), $\max\{X_1, X_2 - X_1\}$ becomes $1/3Y_1 + Y_2$, and $X_1 + X_2$ becomes $Y_1 + Y_2$. Then we rename Y_j's back to X_j's. As a result, $\max\{X_1, X_2 - X_1\}$ in (5.8) is replaced by $1/3X_1 + X_2$ without affecting the terms where X_1 and X_2 are present only as $X_1 + X_2$. The latter holds, for example, for all terms in the first internal maximum in (5.8). However, the last term of the second internal maximum

$$X_2 - X_3 - X_4 - \cdots - X_{n+1},$$

will be affected. In the new expression, this term becomes

$$2/3X_1 + X_2 - X_3 - X_4 - \cdots - X_{n+1} \tag{5.10}$$

with probability 1/3. On the other hand, with probability 2/3, it becomes

$$2/3X_1 - X_3 - X_4 - \cdots - X_{n+1}. \tag{5.11}$$

So, already after the first step we observe some *'branching'*. Here by branching we mean that the original expression splits into several cases that later on have to be considered separately.

Nevertheless, we proceed with the first internal maximum. At the next step we consider $\max\{1/3X_1 + X_2; X_3 - S_2\}$. As we saw in the proof of (2.3), after applying transformation (2.6) to X_1, X_2, X_3, this maximum may be replaced by $1/7X_1 + 1/3X_2 + X_3$ without affecting the structure of $X_1 + X_2 + X_3$. However, as follows from Remark 2.3, the latter sum is the only linear combination of X_1, X_2, X_3 which would stay invariant under transformation (2.6). Hence, the terms (5.10), (5.11) will be affected, and thus both the corresponding branches will split again into different cases. Moreover, in the second internal maximum, the term

$$X_3 - X_4 - X_5 - \cdots - X_{n+1}$$

will be affected, too, providing even more branching. Therefore, already after the second step, it will be very difficult to get hold of all branches. Thus, the procedure does not lead to a closed-form expression for the travel time distribution.

When straightforward methods do not work, one can also think about a recursive procedure. Consider the terms X_1 and X_{n+1} arising in the internal maxima in (5.8) when $j = 1$. Without loss of generality assume the event $[X_1 > X_{n+1}]$. Then in the right-hand side of (5.8) we can replace X_{n+1} by $1/2X_{n+1}$ and X_1 by $1/2X_{n+1} + X_1$. The terms in the first internal maximum then become

$$X_j - S_{j-1} - 1/2X_{n+1}, \quad 2 \le j \le n, \tag{5.12}$$

and the terms in the second internal maximum become

$$X_{n+2-j} - (S_n - S_{n+2-j}) - 1/2X_{n+1}, \quad 2 \le j \le n. \tag{5.13}$$

Further, note that the term with $j = 2$ in (5.13) is bigger than the term with $j = n$ in (5.12):

$$X_n - 1/2X_{n+1} > X_n - 1/2X_{n+1} - S_{n-1}.$$

Hence, the latter term may be dropped out of S_n^{Opt}. Then we have

$$
\begin{aligned}
S_n^{Opt} &\stackrel{d}{=} \max\Big\{X_1 + 1/2X_{n+1}, \max_{2\le j\le n-1}\{X_j - S_{j-1}\} - 1/2X_{n+1}, \\
&\qquad \max_{1\le j\le n-1}\{X_{n+1-j} - (S_n - S_{n+1-j})\} - 1/2X_{n+1}\Big\} \\
&= \max\Big\{X_1 + 1/2X_{n+1}, \max_{1\le j\le n-1}\{X_j - S_{j-1}\} - 1/2X_{n+1}, \\
&\qquad \max_{1\le j\le n-1}\{X_{n+1-j} - (S_n - S_{n+1-j})\} - 1/2X_{n+1}\Big\},
\end{aligned}
$$

providing the recursion:

$$
S_1^{Opt} = \max\{X_1, X_2\}; \qquad S_n^{Opt} \stackrel{d}{=} \max\Big\{X_1 + X_{n+1}, S_{n-1}^{Opt}\Big\} - 1/2X_{n+1},\ n \ge 2. \tag{5.14}
$$

The problem with (5.14) is that X_1 participates in S_{n-1}^{Opt}. Thus, while expanding S_{n-1}^{Opt}, we affect the term $X_1 + X_{n+1}$, and we have to keep track of the changes of X_1. For example, let us proceed with S_{n-1}^{Opt} by comparing X_1 and X_n. Given $[X_1 > X_n]$, the term X_1 becomes $1/2X_n + X_1$; otherwise, X_1 becomes $1/2X_1$. So, in recursive procedure (5.14) we again observe branching. It seems that this kind of branching is easier to get hold of. However, it is not so, because there will be more branching while comparing the two terms in the maximum in (5.14). Therefore, the recursive procedure does not enable us to find the distribution of the travel time.

Nevertheless, one can derive the distribution of S_3^{Opt} by directly expanding the external maximum in (5.8). We first use the recursive formula (5.14):

$$
\begin{aligned}
S_3^{Opt} &\stackrel{d}{=} \max\{1/2X_4 + X_1, S_2^{Opt} - 1/2X_4\} \\
&= \max\{1/2X_4 + X_1, X_2 - X_1 - 1/2X_4, X_3 - 1/2X_4, X_2 - X_3 - 1/2X_4\}.
\end{aligned}
$$

At the next step we compare X_1 and X_3. After conditioning on all possible cases, substituting the new representation of S_3^{Opt} into (5.9) and using (2.18) we find out that the travel time T_3^{Opt} is distributed as:

$$\begin{array}{rcl}
\frac{1}{2}D_1 + \frac{1}{2}D_2 + \frac{3}{4}D_3 & \text{w.p.} & \frac{3}{8}, \\
\frac{1}{2}D_1 + \frac{3}{4}D_2 + \frac{3}{4}D_3 & \text{w.p.} & \frac{1}{4}, \\
\frac{1}{2}D_1 + \frac{1}{2}D_2 + \frac{3}{4}D_3 + \frac{3}{4}D_4 & \text{w.p.} & \frac{3}{16}, \\
\frac{1}{2}D_1 + \frac{3}{4}D_2 + \frac{3}{4}D_3 + \frac{3}{4}D_4 & \text{w.p.} & \frac{1}{16}, \\
\frac{3}{4}D_1 + \frac{3}{4}D_2 + \frac{3}{4}D_3 + \frac{5}{6}D_4 & \text{w.p.} & \frac{1}{48}, \\
\frac{3}{4}D_1 + \frac{3}{4}D_2 + \frac{5}{6}D_3 & \text{w.p.} & \frac{1}{24}, \\
\frac{1}{2}D_1 + \frac{3}{4}D_2 + \frac{3}{4}D_3 + \frac{5}{6}D_4 & \text{w.p.} & \frac{1}{48}, \\
\frac{1}{2}D_1 + \frac{3}{4}D_2 + \frac{5}{6}D_3 & \text{w.p.} & \frac{1}{24}.
\end{array}$$

Note that this distribution can be written in many other forms. The final expression depends on the order in which we expand S_3^{Opt}. In order to find the distribution of the term with distinct coefficients (the last term above), we use formula (2.26). For the terms with coinciding coefficients we use formula (2.29). Finally, for any $0 \le t < 1$ we obtain:

$$\mathrm{P}\left(T_3^{Opt} < t\right) = \frac{28}{9}t_+^3 - \frac{11}{4}(2t-1)_+^3 + \frac{41}{36}(4t-3)_+^3 - \frac{1}{4}(4t-3)_+^2 - \frac{1}{4}(6t-5)_+^3.$$

For the mean travel time, we have

$$\mathrm{E}\left(T_3^{Opt}\right) = \frac{101}{192} \approx 0.52604.$$

This is slightly bigger than $\mathrm{E}\,(T_3^*) = (1/2 + 3/4 + 5/6)/4 = 100/192$. Furthermore, one can directly verify that T_3^{Opt} is stochastically bigger than T_3^*. This confirms Conjecture 5.2 for $n = 3$.

Our further attempts to expand (5.14) for arbitrary n were unsuccessful. However, as a result of these calculations, we propose the following conjecture.

Conjecture 5.3 *For any $n \ge 1$, the travel time T_n^{Opt} is distributed as a mixture of linear combinations of uniform $(n+1)$-spacings where all the coefficients are of the form $1 - \alpha_j$, $1 \le j \le n+1$, and at most one coefficient equals to $1 - \alpha_1 = 0$.*

Note that this conjecture is of course true for $T_1^{Opt} \stackrel{d}{=} 1/2D_1$, for $T_2^{Opt} \stackrel{d}{=} 1/2D_1 + 3/4D_2$ and for T_3^{Opt} which is a mixture of the linear combinations of the spacings with coefficients 0, 1/2, 3/4 or 5/6.

5.5 Concluding remarks

In this chapter we explored the properties of the optimal picking sequence while collecting n items independently and uniformly distributed on a circle. The major feature of the optimal route is that it admits no more than one turn. When $n \to \infty$, we derived the limit probability that the picker turns after collecting exactly $k \geq 0$ items. Further, we considered the Nearly-Optimal (NO) strategy whose performance is very close to optimal. In fact, this strategy guarantees a picking sequence which differs from the optimal one only in pathological cases. On the other hand, the distribution of the travel time under the NO strategy can be found in the same way as it was done for the m-step strategies in Chapter 4. Therefore, the study of the NO strategy was very helpful to discover some properties of the optimal route. We have derived a tight upper bound for the travel time under the NO strategy and the optimal strategy. Further, we proposed a natural conjecture on a stochastic lower bound of the minimal travel time.

We investigated the possibilities to derive the distribution of the minimal travel time T_n^{Opt} for arbitrary n. As well as the travel time under the m-step strategies, the minimal travel time is distributed as a mixture of linear combinations of the uniform $(n+1)$-spacings. The problem is that the number of possible linear combinations is very large, and we could not find any regular structure to get hold of all the cases. However, we proposed a conjecture on the form of the coefficients. As an example, we derived the distribution of the minimal travel time for $n = 3$. The complete analysis of the travel time under the optimal strategy for larger n remains a challenging open problem.

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Summary

In this thesis, we study the travel time needed to collect n items randomly located on a circle. Chapter 1 is devoted to the motivation of this research and the literature survey on relevant works. The problem arises in performance analysis of carousel systems. A carousel is a widely used automated warehousing system consisting of a large number of shelves or drawers rotating in a closed loop in either direction. A picker has a fixed position in front of the carousel that rotates the required items to the picker. In this thesis, a carousel is modeled as a circle of length 1. The object is to characterize the travel time needed to pick one order represented by a list of n items whose positions are independent and uniformly distributed on the circle. The methods are based on the properties of uniform spacings and their relations with exponential random variables.

In Chapter 2 we derive theoretical results needed in the further analysis. We consider the sum and the maximum of the terms $(X_j - bS_{j-1})_+$, $j = 1, 2, \ldots, N$, where $b \geq 0$; $X_1, X_2, \ldots$ are independent exponential random variables with mean 1; $S_0 = 0$; $S_i = X_1 + X_2 + \cdots + X_i$, $i \geq 1$. Using an approach based on the memoryless property we prove that both the sum and the maximum are distributed as linear combinations of exponential random variables. This is a generalization of well-know properties of the sum and the maximum of the X_j's. We also prove some auxiliary assertions and corollaries. Then we show that similar results hold for uniform spacings. Finally, we explore the distribution and the moments of linear combinations of the spacings.

In the further chapters we study various strategies to collect n items on a circle. Chapter 3 contains the results on the Nearest Item (NI) heuristic where the next item to be picked is always the nearest one. This algorithm is often used in practice. For any realization of the items' locations, we derive a tight upper bound for the travel time. Then we develop a recursive procedure to obtain a closed-form expression for the mean and the variance of the travel time conditioned on the size of an empty space at one side of the picker's position. Further, we use a similar procedure to find the conditional, the unconditional and the limiting distribution for the number of turns. Then we express the travel time as a function of uniform spacings, and we use the results from Chapter 2 to prove that the travel time under the NI heuristic

is distributed as a linear combination of uniform spacings. That enables us to derive the closed-form expression for the distribution and the moments of the travel time. Then we give an exhaustive analysis of the limiting behavior of the travel time distribution. Finally, we show an alternative way to derive the distribution of the number of turns. Moreover, we prove that the travel time and the number of turns are independent random variables.

In Chapter 4 we address the strategies related to the optimal route. This chapter starts with the analysis of the One-Side Optimal (OSO) strategy. This algorithm chooses the best picking sequence providing that the picker ends in a given direction (say, clockwise). Applying the results of Chapter 2 we derive the distribution of the travel time under the OSO strategy. We show that this travel time is stochastically bigger than the travel time under the NI heuristic. The major part of Chapter 4 is devoted to so-called m-step strategies: the picker chooses the shortest route among the ones that change direction only once, and only do so after collecting no more than m items. Since it is never optimal to turn more than once, then the optimal strategy is in fact an $(n-1)$-step strategy. We derive, for any $m \geq 0$, explicit expressions for the distribution and all moments of the travel time under the m-step strategies, provided $2m < n$. The analysis is based on the arguments from Chapter 2. The performance of the m-step strategies is compared with the performance of the optimal pick strategy. Numerical results show that, already for small values of m, the performance of the m-step strategy is very close to optimal. In fact, with high probability, the optimal strategy coincides with the 2-step strategy. Furthermore, m-step strategies are compared with the NI heuristic. It appears that, on average, the m-step strategy performs better than the NI heuristic already for $m = 2$.

Chapter 5 is devoted to the optimal pick sequence. We first study the probability that under the optimal strategy the picker would collect exactly k items before the turn. We prove that, for fixed $k = 0, 1, \ldots$, this probability tends to $1/2^{k+1}$, when the number of items tends to infinity. Further, we find a tight upper bound of the travel time for any realization of the items' locations. Also, we obtain a stochastic upper bound and make a conjecture about a stochastic lower bound for the minimal travel time. Further, we discuss possibilities to find the distribution of the travel time under the optimal strategy. Although the methods from Chapter 2 can be applied again, we discover difficulties that don't allow us to solve the problem in general. Therefore, we can only provide some conjectures. As an example, we derive the distribution of the minimal travel time for $n = 3$. We complete the last chapter with conclusions and discussion.

Samenvatting

In dit proefschrift bestuderen we de reistijd die nodig is om n artikelen te verzamelen die stochastisch verspreid liggen op een cirkel. In hoofdstuk 1 wordt het onderzoek gemotiveerd en het geeft tevens een overzicht van relevante literatuur. Het probleem komt voort uit de prestatieanalyse van carousel systemen. Een carousel is een veel gebruikt automatisch opslag- en uitslagsysteem. Het bestaat uit een groot aantal vakken die in een gesloten ring in beide richtingen kunnen draaien. De carousel roteert de gewenste artikelen naar de verzamelaar, die een vaste positie heeft voor de carousel. In dit proefschrift modelleren we de carousel als een cirkel met omtrek 1. Het doel is om de reistijd te karakteriseren die nodig is voor het verzamelen van een order. Dit is een lijst van n artikelen, die onafhankelijk en uniform verdeeld liggen op de cirkel. De methoden die worden gebruikt voor de analyse van de reistijd zijn gebaseerd op eigenschappen van uniforme spacings en hun relatie met exponentieel verdeelde stochastische variabelen.

In hoofdstuk 2 leggen we de theoretische basis voor de rest van het proefschrift. We beschouwen de som en het maximum van termen van de vorm $(X_j - bS_{j-1})_+$, $j = 1, 2, \ldots, N$, waarin $b \geq 0$; $X_1, X_2, \ldots$ onafhankelijke exponentieel verdeelde stochastische variabelen zijn met gemiddelde 1; $S_0 = 0$; $S_i = X_1 + X_2 + \cdots + X_i$, $i \geq 1$. Door gebruik te maken van een aanpak die gebaseerd is op de geheugenloosheid eigenschap tonen we aan dat zowel de som als het maximum verdeeld zijn als lineaire combinaties van exponentieel verdeelde stochastische variabelen. Dit is een generalisatie van bekende eigenschappen van de som en het maximum van de X_j's. Daarna laten we zien dat soortgelijke resultaten gelden voor uniforme spacings. Tot slot onderzoeken we in hoofdstuk 2 de verdeling en de momenten van lineaire combinaties van spacings.

In de volgende hoofdstukken bestuderen we verschillende strategieën voor het verzamelen van n artikelen. Hoofdstuk 3 is gewijd aan de Nearest Item (NI) heuristiek, waarbij het dichtstbijliggende artikel altijd het volgende artikel is om verzameld te worden. Deze orderverzamel heuristiek wordt veel gebruikt in de praktijk. We leiden een scherpe bovengrens af voor de reistijd, die geldig is voor elke realisatie van posities van de te verzamelen artikelen. Daarna ontwikkelen we een recursieve procedure, waarmee we een gesloten uitdrukking kunnen afleiden voor

het gemiddelde en de variantie van de reistijd, gegeven de afmeting van een lege ruimte aan één kant van de positie van de verzamelaar. Een soortgelijke procedure wordt gebruikt voor het bepalen van de voorwaardelijke, onvoorwaardelijke en de limietverdeling van het aantal keren dat de carousel van draairichting verandert. Daarna laten we zien dat de reistijd kan worden uitgedrukt als een functie van uniforme spacings, zodat we de resultaten uit hoofdstuk 2 kunnen toepassen om te bewijzen dat de reistijd voor de NI heuristiek hetzelfde verdeeld is als een lineaire combinatie van uniforme spacings. Hierdoor is het mogelijk om gesloten uitdrukkingen af te leiden voor zowel de verdeling als de momenten van de reistijd. Ook analyseren we het limietgedrag van de verdeling van de reistijd als het aantal te verzamelen artikelen naar oneindig gaat. Tot slot bestuderen we het aantal keer dat de carousel van richting verandert. We tonen aan dat dit onafhankelijk is van de reistijd, en we presenteren een alternatieve afleiding voor de kansverdeling.

In hoofdstuk 4 bestuderen we strategieën die nauw verwant zijn aan de optimale strategie, i.e., de strategie waarvoor de reistijd minimaal is. We beginnen in dit hoofdstuk met de One-Side Optimal (OSO) strategie. Deze strategie kiest de beste verzamelvolgorde, onder de voorwaarde dat de carousel eindigt in een voorgeschreven draairichting (zeg, met de klok mee). Gebruikmakend van de resultaten uit hoofdstuk 2 leiden we de verdeling van de reistijd af. We laten zien dat de reistijd voor de OSO strategie stochastisch groter is dan de reistijd voor de NI heuristiek. Het grootste deel van hoofdstuk 4 is gewijd aan de bestudering van de zogenaamde m-stap strategieën: de verzamelaar kiest de kortste route onder de routes die hooguit één maal van richting veranderen, en dat doen na het verzamelen van hooguit m artikelen. De optimale strategie is een $(n-1)$-stap strategie, want het niet optimaal om meer dan één keer van richting te veranderen. Onder de voorwaarde dat $2m < n$ leiden we gesloten uitdrukkingen af voor de verdeling en de momenten van de reistijd voor de m-stap strategie. De analyse maakt weer gebruik van resultaten uit hoofdstuk 2. De prestaties van m-stap strategieën worden vergeleken met de prestaties van de optimale verzamel strategie. Numerieke resultaten laten dat reeds voor kleine waarden van m de prestaties van de m-stap strategie bijna optimaal zijn. De optimale strategie is in feite met grote waarschijnlijkheid hetzelfde als de 2-stap strategie. Tevens worden de m-stap strategieën vergeleken met de NI heuristiek. Het blijkt dat al voor $m = 2$ de gemiddelde prestaties van de m-stap strategie beter zijn dan de prestaties van de NI heuristiek.

Hoofdstuk 5 is gewijd aan de optimale verzamelstrategie. We beginnen met het bestuderen van de kans dat de carousel van draairichting verandert na het verzamelen van precies k artikelen. We bewijzen dat voor vaste $k = 0, 1, \ldots,$ deze kans convergeert naar $1/2^{k+1}$ als het aantal te verzamelen artikelen naar oneindig convergeert. Verder leiden we een scherpe bovengrens af voor de reistijd, die geldig is voor elke realisatie van posities van de te verzamelen artikelen. We presenteren ook een stochastische bovengrens, and formuleren een vermoeden over een stochastische ondergrens voor de reistijd. Tot slot bespreken we mogelijke methoden voor het vinden van de verdeling van de reistijd voor de optimale strategie. Hoewel de methoden uit hoofdstuk 2 in principe toepasbaar zijn, ontdekken we moeilijkheden in de analyse, waardoor het probleem niet in zijn algemeenheid kan worden oplost. Daarom formuleren we slechts vermoedens over de structuur van de verdeling van de reistijd. Ter illustratie leiden we de verdeling van de reistijd af voor het speciale geval $n = 3$. Het hoofdstuk wordt beëindigd met conclusies en een discussie.

Curriculum Vitae

Nelly Litvak was born in Gorky, Soviet Union (nowadays, Nizhny Novgorod, Russia) on January 27, 1972. In June 1995, she graduated (with honor) from Nizhny Novgorod State University. In August 1998, she completed her post-graduate studies at the same university. In March 1999, she defended a dissertation *Adaptive Control of Conflict Flows* and received her Candidate of Science degree. From September 1998 to June 1999, she worked as a teaching assistant at the department of Applied Probability at Nizhny Novgorod State University.

In June 1999, she started a Ph.D. project at EURANDOM (European Institute for Statistics, Probability, Stochastic Operations Research and Their Applications, Eindhoven). After the defense which is to take place on January 22, 2002, she is planning to proceed as a postdoc at EURANDOM.