Series expansions with respect to polynomial bases

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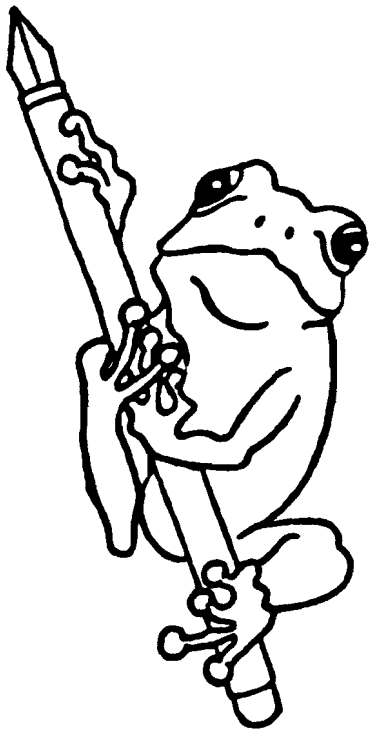

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Summary

Given a polynomial basis $(P_n)_{n\in\mathbb{N}_0}$ and a sequence $(\mu_n)_{n\in\mathbb{N}_0}$ of positive numbers spaces, $F_t[(P_n),(\mu_n)]$ are discussed which consist of functions ϕ with a series expansion $\phi(x)=\sum_{n=0}^{\infty} a_n P_n(x)$ where $a_n = O(\exp(-\mu_n s))$ for all s, $0<s<t$. For two such bases (P_n) and (Q_n) the connection matrices (S_{nm}) and (T_{nm}) are defined by $Q_m=\sum S_{nm}P_n$, $P_m=\sum T_{nm}Q_n$. Conditions on the connection matrices are presented which quarantee that $F_t[(P_n),(\mu_n)]=F_t[(Q_n),(\nu_n)]$. These classification results are applied to bases of Hermite, Laguerre and Jacobi polynomials.

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Introduction

For $\alpha > -1$, $\beta > -1$ let $P_n^{(\alpha,\beta)}$ denote the Jacobi polynomial

$$P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n n!} (1-x)^{-\alpha} (1+x)^{-\beta} (\frac{d}{dx})^n (1-x)^{n+\alpha} (1+x)^{n+\beta}.$$

It is a classical result of Szegö (see [Sz], Ch. IX) that a function f is analytic inside the ellipse E_t,

$$\frac{x^2}{\cosh^2 t} + \frac{y^2}{\sinh^2 t} = 1, \quad t > 0,$$

if and only if f admits a Jacobi series

$$f(z) = \sum_{n=0}^{\infty} a_n(f)\, P_n^{(\alpha,\beta)}(z)$$

where for all s, $0 < s < t$, $\sup_{n \in \mathbb{N}_0} | a_n(f) |\, e^{ns} < \infty$. It follows from Szegö's result that the order of decay of the coefficients $a_n(f)$ does not depend on α and β. If we introduce the space $F_t^{(\alpha,\beta)}$ as the space of functions f on [-1, 1],

$$f(x) = \sum_{n=0}^{\infty} a_n(f)\, P_n^{(\alpha,\beta)}(x),$$

where $a_n(f) = O(e^{-ns})$, for all s, $0 < s < t$. Then Szegö's result is twofold: It gives the *classification*

$$\forall_{\alpha,\beta>-1}\ \forall_{\gamma,\delta>-1} : F_t^{(\alpha,\beta)} = F_t^{(\gamma,\delta)}$$

and also the *characterization*

$f \in F_t(\alpha,\beta)$ if and only if f extends to an analytic function inside the ellipse E_t.

From the paper [SY] of Szasz and Yeardley a similar result follows for the Laguerre polynomials $L_n^{(\alpha)}$, defined by

$$L_n^{(\alpha)}(x) = \frac{1}{n!} x^{-\alpha} e^x \frac{d}{dx} (e^{-\alpha} x^{n+\alpha}).$$

Indeed, an even function f is analytic on the strip $| \operatorname{Im} z | < t$ and satisfies the growth order estimate

$$\forall_{s,\,0<s<t}\ \forall_{y,\,|y|\le s} : f(x+iy) = O(\exp[-|x|\,(s^2-y^2)^{1/2}])$$

if and only if f can be expanded into a "Laguerre series"

$$f(z) = \sum_{n=0}^{\infty} a_n(f)\, e^{-\frac{1}{2}z^2} L_n(\alpha)(z^2)$$

where $a_n(f) = O(\exp(-s\sqrt{n}))$ for all s, $0 < s < t$. Here any $\alpha > -1$ can be taken.

In this paper we present a general approach to handle this kind of classification results.

We start with a polynomial basis (P_n) and a sequence of positive numbers (μ_n). The Hilbert space $X_t[(P_n), (\mu_n)]$ consists of all $f = \sum_{n=0}^{\infty} a_n P_n$ with

$$\sum_{n=0}^{\infty} |a_n|^2 \exp(2\mu_n t) < \infty.$$

Besides we introduce the space

$$F_t[(P_n), (\mu_n)] := \bigcap_{0<s<t} X_s[(P_n), (\mu_n)].$$

Let (Q_n) denote another polynomial basis. Then the connection matrices (S_{nm}) and (T_{nm}) are defined by

$$Q_m = \sum S_{nm} P_n \quad \text{and} \quad P_m = \sum T_{nm} Q_n.$$

We shall show that there exist conditions on these connection matrices such that

$$X_t[(P_n), (\mu_n)] = X_t[(Q_n), (\mu_n)]$$

and / or

$$F_t[(P_n), (\mu_n)] = F_t[(Q_n), (\mu_n)].$$

In case of the Laguerre polynomials and in case of the Jacobi polynomials the connection matrices are known. It turns out that the classification results are applicable to the spaces $F_t[(P_n^{\alpha,\beta}), (\mu_n)]$ and $F_t[(L_n^{(\alpha)}), (\mu_n)]$ whenever

$$\frac{\mu_n}{\log n} \to \infty \quad \text{as } n \to \infty.$$

We also present characterization results. E.g. for each $t > 0$ the spaces $F_t[(P_n^{\alpha,\beta}), (n^{\nu})]$ with $\nu > 1$ consists of entire analytic functions of slow growth. The spaces $F_t[(L_n^{(\alpha)}), (n^{\nu})]$, $0 < \nu < 1$, $t > 0$, are related to the Gelfand Shilov spaces $S_{1/\nu}^{1/\nu}$.

1. General theory

Let **P** denote the vectorspace of all polynomials on $\mathbb{R}$. Consider a linear basis $(P_n)_{n\in\mathbb{N}_0}$ in **P** where each P_n denotes a polynomial of degree n. In **P** we introduce the inner product $(.\,,.)_p$ by

$$(\sum_n \alpha_n P_n, \sum_m \beta_m P_m)_P = \sum_n \alpha_n \bar{\beta}_n.$$

By $X[(P_n)]$ we denote a completion of the pre-Hilbert space **P** with inner product $(\,,\,)_p$.
We introduce the following subspaces of $X[(P_n)]$.

Definition 1.1.

Let (μ_n) denote a sequence of nonnegative real numbers and let $t > 0$.
The space $X_t[(P_n), (\mu_n)]$ is defined by

$$X_t[(P_n), (\mu_n)] = \{f \in X[(P_n)] \mid \sum_{n=0}^{\infty} e^{2\mu_n t} \mid (f, P_n) \mid^2 < \infty\}.$$

With the inner product

$$(f, g)_{p,t} = \sum_{n=0}^{\infty} e^{2\mu_n t} (f, P_n)_p (P_n, g)_p$$

$X_t[(P_n), (\mu_n)]$ is a Hilbert space.
The space $F_t[(P_n), (\mu_n)]$ is defined by

$$F_t[(P_n), (\mu_n)] = \bigcap_{0<s<t} X_s[(P_n), (\mu_n)].$$

The topology in $F_t[(P_n), (\mu_n)]$ is generated by the norms

$$P_s(f) = \sqrt{(f, f)_{p,s}}, \quad 0 < s < t.$$

Thus $F_t[(P_n), (\mu_n)]$ is a Frechet space. In the definition of $F_t[.]$ also $t = \infty$ is a permissible value.

Sometimes the formal spaces $X_t[(P_n), (\mu_n)]$ and $F_t[(P_n), (\mu_n)]$ can be considered as functions spaces. To this end we introduce the following condition

(A) There exists an interval I and $T > 0$ such that

$$\sum_{n=0}^{\infty} \mid P_n(x) \mid^2 \exp(-2\mu_n T) < \infty, \quad x \in I.$$

Lemma 1.2.

Let the sequence (μ_n) satisfy condition (A) and let $(\alpha_n)_{n\in\mathbb{N}}$ denote an l_2-sequence. Then for all $x \in I$ and all $t \geq T$ the series $\sum_{n=0}^{\infty} \alpha_n e^{-\mu_n t} P_n(x)$ converges absolutely and

$$| \sum_{n=0}^{\infty} \alpha_n e^{-\mu_n t} P_n(x) |^2 \le (\sum_{n=0}^{\infty} |\alpha_n|^2)(\sum_{n=0}^{\infty} e^{-2\mu_n t} |P_n(x)|^2).$$

Proof.
Use Cauchy-Schwartz' inequality. []

Definition 1.3.

Let the sequence (μ_n) satisfy condition (A) and let $s \ge T$. To each $f \in X_s[(P_n), (\mu_n)]$ we link the function

$$\hat{f}: x \mapsto \sum_{n=0}^{\infty} (f, P_n)_p P_n(x), \quad x \in I$$

Thus for $t \ge T$ the (formally defined) spaces $X_t[(P_n), \mu_n)]$ and $F_t[(P_n), (\mu_n)]$ will be considered as function spaces in which point evaluation is continuous. Indeed, for each $s \ge T$, $x \in I$ and $f \in X_s[(P_n), (\mu_n)]$,

$$|\hat{f}(x)| \le \|f\|_{p,s} (\sum_{n=0}^{\infty} e^{-2\mu_n s} |P_n(x)|^2)^{1/2}.$$

Remark: If condition (A) is satisfied $X_t[(P_n), (\mu_n)]$ is a functional Hilbert space with reproducing kernel $K_{p,t}(w, y) = \sum e^{-2\mu_n t} P_n(x) \overline{P_n}(v)$, $x, y \in I$.

Clearly, the polynomials (P_n) establish an orthogonal basis in $X_t[(P_n), (\mu_n)]$. We have $\|P_n\|_{p,t} = e^{\mu_n t}$. They establish a Schauder basis in $F_t[(P_n), (\mu_n)]$. Each $f \in F_t[(P_n), (\mu_n)]$ equals the series

$$\sum_{n=0}^{\infty} (f, P_n)_p P_n$$

with convergence in the topology of $F_t[(P_n), (\mu_n)]$.
We proceed by introducing another linear basis (Q_n) in the vector space **P**, where each Q_n is of the order n. Let (ν_n) denote a sequence of nonnegative numbers. Then by Definition 1.1 for each $t > 0$ the spaces $X_t[(Q_n), (\nu_n)]$ and $F_t[(Q_n), (\nu_n)]$ are well defined. Let $(.,.)_q$ denote the corresponding inner product of the Hilbert space $X[(Q_n)]$.

Definition 1.4.

Let L denote a linear mapping from $X_t[(P_n), (\nu_n)]$ into $X_\tau[(Q_n), (\nu_n)]$ or from $F_t[(P_n), (\mu_n)]$ into $F_\tau[(Q_n), (\nu_n)]$. Then the matrix of L is defined by

$$L_{nm} = (L P_m, Q_n)_q.$$

Lemma 1.5.

a. A matrix (L_{nm}) is the matrix of a continuous linear mapping L from $X_t[(P_n),(\mu_n)]$ into $X_\tau[(Q_n),(\nu_n)]$ if the matrix $(\exp[-\mu_m t+\nu_n \tau] L_{nm})$ represents a bounded linear operator on l_2.

b. A matrix (L_{nm}) is the matrix of a continuous linear mapping L from $F_t[(P_n),(\mu_n)]$ into $F_\tau[(Q_n),(\nu_n)]$ iff $\forall_{\sigma,0<\sigma<\tau}\ \exists_{s,0<s<t}$:

the matrix $(\exp[-\mu_m s+\nu_n \sigma] L_{nm})$ represents a bounded linear operator on l_2.

In both cases we have

$$Lf=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} L_{nm}\,(f,P_m)_p\,Q_n.$$

Proof.

a. We set $P_{m,r}=\exp(-\mu_m r)\,P_m$, $Q_{m,r}=\exp(-\nu_m r)\,Q_m$. Then $(P_{m,r})_{m=0}^{\infty}$ and $(Q_{m,r})_{m=0}^{\infty}$ are orthonormal bases in $X_r[(P_m),(\mu_m)]$ and $X_r[(Q_m),(\nu_m)]$, respectively
Now $L: X_t[(P_n),(\mu_n)]\to X_\tau[(Q_n),(\nu_n)]$ is continuous if and only if

$$((L\,P_{m,t}\,,\,Q_{n,\tau})_{q,\tau})$$

is the matrix of a bounded linear operator on l_2. By a simple computation we obtain

$$(L\,P_{m,t}\,,\,Q_{n,\tau})_{q,\tau}=\exp[-\mu_n t+\nu_n \tau]\,(L\,P_m\,,\,Q_n)_q.$$

b. L is continuous from $F_t[(P_n),(\mu_n)]$ into $F_t[(Q_n),(\nu_n)]$ if and only if $\forall_{\sigma,0<\sigma<\tau}\ \exists_{s,0<s<t}$:
L extends to a continuous linear mapping from $X_s[(P_n),(\mu_n)]E$ into $X_\sigma[(Q_n),(\nu_n)]$.
Now apply a. □

Consider the following conditions on a sequence (μ_n)

$$(B_0):\ \forall_{s>0}:\ \sum_{n=0}^{\infty}\exp(-\mu_n s)<\infty$$

$$(B_\infty):\ \exists_{T>0}:\ \sum_{n=0}^{\infty}\exp(-\mu_n T)<\infty.$$

Condition (B_0) yields a simple characterization of the matrices of the continuous linear mappings from $F_t[(P_n),(\mu_n)]$ into $F_\tau[(Q_n),(\nu_n)]$. Condition (B_∞) yields a simple characterization for the continuous linear mappings from $F_\infty[(P_n),(\mu_n)]$ into $F_\infty[(Q_n),(\nu_n)]$.

Lemma 1.6.

a. Let the sequences (μ_n) and (ν_n) satisfy Condition (B_0). A matrix (L_{nm}) represents a continuous linear mapping L from $F_t[(P_n),(\mu_n)]$ into $F_\tau[(Q_n),(\nu_n)]$ iff

$$\forall_{\sigma,0<\sigma<\tau}\ \exists_{s,0<s<t} : \sup_{n,m} | L_{nm} | \exp[-\mu_m s + \nu_n \sigma] < \infty.$$

b. Let the sequences (μ_n) and (ν_n) satisfy condition (B_∞). A matrix (L_{nm}) is the matrix of a continuous linear mapping L from $F_\infty[(P_n), (\mu_n)]$ into $F_\infty[(Q_n), (\nu_n)]$ iff

$$\forall_{\sigma,0<\sigma<\infty}\ \exists_{s,0<s<\infty} : \sup_{n,m} | L_{nm} | \exp[-\mu_m s + \nu_n \sigma] < \infty.$$

Proof.

a. If $L : F_t[(P_n), (\mu_n)] \to F_\tau[(Q_n), (\nu_n)]$ is continuous then it can be easily deduced from the preceding lemma that its matrix $((L P_m, Q_n)_q)$ satisfies the requirements. Conversely, let the matrix (L_{nm}) satisfy the stated conditions. Consider the equality

(*) $$\exp[-s\mu_m + \sigma\nu_n] | L_{mn} | = \exp[-\varepsilon(\mu_m + \nu_n)] \exp[-(s-\varepsilon)\mu_m + (\sigma+\varepsilon)\nu_n] | L_{mn} |.$$

Let σ, $0 < \sigma < \tau$. Choose ε_1, $0 < \varepsilon_1 < \tau - \sigma$. There exists $\hat{s}$, $0 < \hat{s} < t$ such that

$$\sup_{m,n} \exp[-\hat{s}\mu_m + (\sigma+\varepsilon_1)\nu_n] | L_{mn} | < \varepsilon.$$

Now take $0 < \varepsilon < \min\{\varepsilon_1, t - \hat{s}\}$ and set $s = \hat{s} + \varepsilon$.
Then it follows from (*) that

$$\sum_{m,n} \exp[-2s\mu_m + 2\sigma\nu_m] | L_{mn} |^2 < \infty.$$

Finally, apply the preceding lemma.

b. The proof of b. runs similar to the proof of a. and therefore is omitted. □

Consider the following infinite matrices.

Definition 1.7.

The upper triangular matrices (S_{nm}) and (T_{nm}) are defined by

$$S_{nm} = (Q_m, P_n)_p, \quad T_{nm} = (P_m, Q_n)_q.$$

Observe that $Q_m = \sum_{n=0}^{m} S_{nm} P_n$ and $P_m = \sum_{n=0}^{m} T_{nm} Q_n$.

Furthermore,

$$\delta_{nm} = \sum_{j=n}^{m} S_{nj} T_{jm} = \sum_{j=n}^{m} T_{nj} S_{jm}.$$

On the basis of these transition matrices we derive the following classification results. (Similar ideas appear in [EG3], Section 2)

Theorem 1.8.

Suppose for some $t > 0$ the matrices $(\delta_{nm} \exp(\nu_n - \mu_m)\, t)$, $(S_{nm} \exp(\mu_n - \mu_m)\, t)$, $(\delta_{nm} \exp(\mu_n - \nu_m)\, t)$ and $(T_{nm} \exp(\nu_n - \nu_m)\, t)$ represent bounded linear operators on l_2. Then there exists a continuous linear bijection j from $X_t\,[(P_n), (\mu_n)]$ onto $X_t\,[(Q_n), (\nu_n)]$ with the property that $j(p) = p$ for each polynomial p.

Proof.

Due to the conditions on the matrices (S_{nm}) and (T_{nm}) we can properly define the continuous linear mappings

$$\mathbf{S}_p^q : X_t\,[(P_n), (\mu_n)] \to X_t\,[(Q_n), (\nu_n)]$$

$$\mathbf{T}_q^q : X_t\,[(Q_n), (\nu_n)] \to X_t\,[(Q_n), (\nu_n)]$$

$$\mathbf{S}_p^p : X_t\,[(P_n), (\mu_n)] \to X_t\,[(P_n), (\mu_n)]$$

$$\mathbf{T}_q^p : X_t\,[(Q_n), (\nu_n)] \to X_t\,[(P_n), (\mu_n)]$$

by

$$\mathbf{S}_p^q f = \sum_{n=0}^{\infty} (f, P_n)_p\, Q_n = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \delta_{nm} (f, P_m)_p\, Q_n$$

$$\mathbf{T}_q^q g = \sum_{n=0}^{\infty} \sum_{m=n}^{\infty} T_{nm} (g, Q_m)_q\, Q_n$$

$$\mathbf{S}_p^p f = \sum_{n=0}^{\infty} \sum_{m=n}^{\infty} S_{nm} (f, P_m)_p\, P_n$$

$$\mathbf{T}_q^p f = \sum_{n=0}^{\infty} (g, Q_n)_q\, P_n = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \delta_{nm} (g, Q_m)_q\, P_n.$$

For all $n = 0, 1, 2, \cdots$ we have

$$\mathbf{S}_p^q P_n = S_p^p P_n = Q_n$$

$$\mathbf{T}_q^p Q_n = T_q^q Q_n = P_n.$$

So for all $p \in \mathbf{P}$,

$$(T_q^q \circ S_p^q)\, p = p = (S_p^p \circ T_q^p)\, (p).$$

Now set $j = T_q^q \circ S_p^q$. Then j is a continuous bijection from $X_t\,[(P_n), (\mu_n)]$ onto $X_t\,[(Q_n), (\nu_n)]$ with $j^{-1} = S_p^p \circ T_q^p$. We have $j(p) = j^{-1}(p) = p$ for all $p \in \mathbf{P}$. □

The homeomorphism j of the preceding theorem yields an *identification* between the elements of $X_t\,[(P_n), (\mu_n)]$ and $X_t\,[(Q_n), (\nu_n)]$ with the property that $j \restriction \mathbf{P} : \mathbf{P} \to \mathbf{P}$ is the identity. In the case that both $X_t\,[(P_n), (\mu_n)]$ and $X_t\,[(Q_n), (\nu_n)]$ are functional Hilbert spaces on some interval $I \subset \mathbb{R}$ we

have $(jf)(x) = f(x)$ and so it makes sense to write

$$X_t[(P_n), (\mu_n)] = X_t[(Q_n), (\nu_n)]$$

These assertions are contained in the following result.

Corollary 1.9.

Let the sequence (μ_n) satisfy condition (A), viz. there exists an interval $I \subset \mathbb{R}$ and $T > 0$ such that

$$\sum_{n=0}^{\infty} |P_n(x)|^2 \exp(-2\mu_n T) < \infty.$$

In addition, assume the conditions of Theorem 1.8. are valid for some $t \geq T$. Then for all $x \in I$

$$\sum_{n=0}^{\infty} \exp(-2\nu_n t) \, |Q_n(x)|^2 < \infty$$

(i.e. $X_t[(Q_n), (\nu_n)]$ is a functional Hilbert space) and

$$X_t[(P_n), (\mu_n)] = X_t[(Q_n), (\nu_n)]$$

as Hilbert spaces with equivalent norms.

Proof.

Since $\sup_{m \in \mathbb{N}_0} \exp(\mu_m - \nu_m) t < \infty$, it follows that also the matrix $(S_{nm} \exp(-\nu_m + \mu_n) t)$ represents a bounded linear operator on l_2. Hence for each $x \in I$ the sequence

$$\exp(-\nu_m t) \, Q_m(x) = \sum_{m=0}^{n} \exp(-\nu_m t) \, S_{nm} \, P_n(x) =$$

$$= \sum_{m=0}^{n} \{\exp((-\nu_m + \mu_n) t) \, S_{nm}\} \, e^{-\mu_n t} \, P_n(x) \;, \quad m \in \mathbb{N} \cup \{0\} \,,$$

belongs to l_2. In both Hilbert spaces $X_t[(P_n), (\mu_n)]$ and $X_t[(Q_n), (\nu_n)]$ point evaluation is continuous. Now let $f \in X_t[(P_n), (\nu_n)]$. Then there exists a sequence $(f_n)_{n \in \mathbb{N}}$ of polynomials such that $\|f - f_n\|_{p,t} \to 0$. It follows that $\|j(f - f_n)\|_{q,t} \to 0$, whence

$$f(x) = \lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} (jf_n)(x) = (jf)(x).$$

□

Corresponding to Theorem (1.9) we have the following result.

Theorem 1.11.

Suppose for some $t > 0$ the identity matrix (δ_{nm}) and the matrices (S_{nm}) and (T_{nm}) satisfy the following conditions:

$\forall_{\sigma,0<\sigma<t}\ \exists_{s,0<s<t}$: the following matrices

$$(S_{nm}\exp(-\mu_m s+\mu_n\sigma)),\quad (\delta_{nm}\exp(-\mu_m s+\nu_n\sigma)),$$

$$(T_{nm}\exp(-\nu_m s+\nu_n\sigma)),\quad (\delta_{nm}\exp(-\nu_m s+\mu_n\sigma))$$

represent a bounded linear operator on l_2.

Then there exists a continuous linear bijection j from $F_t[(P_n),(\mu_n)]$ onto $F_t[(Q_n),(\nu_n)]$ such that $j(p)=p$ for all $p\in P$.

Proof.

The proof is only a minor modification of the proof of the preceding theorem. We observe that the matrices (S_{nm}) and (T_{nm}) generate continuous linear mappings on $F_t[(P_n),(\mu_n)]$ and on $F_t[(Q_n),(\nu_n)]$, respectively, which map the P_n's onto the Q_n's and conversely. The identity matrix (δ_{nm}) generates a continuous linear mapping from $F_t[(P_n),(\mu_n)]$ onto $F_t[(Q_n),(\nu_n)]$ and a continuous linear mapping from $F_t[(Q_n),(\nu_n)]$ onto $F_t[(P_n),(\mu_n)]$. □

Remark 1.12.

- Suppose the sequences (μ_n) and (ν_n) satisfy condition (B_0). Then the conditions of the previous theorem may be replaced by the following ones:

$$\forall_{\sigma,0<\sigma<t}\ \exists_{s,0<s<t}:$$

$$\sup_{n,m}|S_{nm}|\exp(-\mu_m s+\mu_n\sigma)<\infty,\quad \sup_m(-\mu_m s+\nu_m\sigma)<\infty$$

$$\sup_{n,m}|T_{nm}|\exp(-\nu_m s+\nu_n\sigma)<\infty,\quad \sup_m(-\nu_m s+\mu_m\sigma)<\infty$$

- Suppose the sequences (μ_n) and (ν_n) satisfy condition (B_∞). Then for $t=\infty$ the conditions on the matrices (S_{nm}), (T_{nm}) and (δ_{nm}) can also be replaced by the above boundedness conditions.

Corollary 1.13.

Let (μ_n) satisfy the condition (A) for some $T>0$ and $I\subset\mathbb{R}$. In addition, assume that the conditions of Theorem (1.11) are valid for some $t>T$. Then there exists $s>T$ such that

$$\sum_{m=0}^{\infty}\exp(-2s\nu_m)\,|Q_m(x)|^2<\infty.$$

Furthermore,

$$F_t[(P_n),(\mu_n)]=F_t[(Q_n),(\nu_n)]$$

as Frechet function spaces with equivalent metrics.

2. Application to Laguerre polynomials

In this section we apply the results of the preceding section to bases of Laguerre polynomials. Before we proceed, we present some elementary estimates which are consequences of Stirling's formula.

Lemma 2.1.

(i) $$\forall_{a>0}\,\forall_{b>0}\,\exists_{K>0}: \frac{\Gamma(n+a)}{\Gamma(n+b)} \le K(n+1)^{a-b}, \quad n=0,1,2,\cdots$$

(ii) $$\forall_{a\in \mathbb{R}}\,\forall_{b>0}\,\exists_{K>0}: \left|\frac{(a)_n}{(b)_n}\right| \le K(n+1)^{|a|-b}.$$

Proof.

Statement (i) follows simply from Stirling's formula.

$$\Gamma(x) = \sqrt{2\pi}\,\exp[-x+(x-\tfrac{1}{2})\log x]\,(1+O(\frac{1}{x})), \quad x\to\infty.$$

Moreover for each $x \in \mathbb{R}$ we have

$$(x)_n = x(x+1)\cdots(x+n-1), \quad (x)_0 = 1.$$

It follows that

$$|(x)_n| \le |x|_n = \frac{\Gamma(|x|+n)}{\Gamma(|x|)}.$$

Thus (ii) follows applying (i). □

Let $\alpha \in \mathbb{R}$, $\alpha > -1$. For $n = 0, 1, 2, \cdots$ the polynomial $L_n^{(\alpha)}$ defined by

$$L_n^{(\alpha)}(x) = \sum_{m=0}^{n} \frac{(-1)^m}{m!} \begin{bmatrix} n+\alpha \\ n-m \end{bmatrix} x^m$$

is called the n-the Laguerre polynomial of order α. Here we use the standard notation

$$\begin{bmatrix} a \\ b \end{bmatrix} = \frac{\Gamma(a+1)}{\Gamma(b+1)\Gamma(a-b+1)}.$$

For fixed α the polynomials $L_n^{(\alpha)}$ satisfy the following orthogonality relations

$$(L_n^{(\alpha)}, L_m^{(\alpha)})_\alpha = \frac{\Gamma(n+\alpha+1)}{2\Gamma(n+1)}\,\delta_{nm}$$

where

$$(p\,,\,q)_\alpha = \int_0^\infty p(x)\,\overline{q(x)}\,e^{-x}\,x^\alpha\,dx\,,\quad p\,,q \in P.$$

The Hilbert space $X_\alpha = L_2((0\,,\,\infty)\,,\,x^\alpha\,e^{-x}\,dx)$ is the natural completion of the pre-Hilbert space $(\mathbf{P}\,,\,(.\,,\,.)_\alpha)$.

From [MOS], p. 249 we obtain the relations

$$L_m^{(\alpha)} = \sum_{n=0}^{m} \frac{(\alpha-\beta)_{m-n}}{(m-n)\,!}\,L_n^{(\beta)}.$$

In order to arrive at an orthonormal basis we introduce the normalized polynomials $\Lambda_n^{(\alpha)}$

$$\Lambda_n^{(\alpha)} = \left[\frac{2\,\Gamma(n+1)}{\Gamma(n+\alpha+1)}\right]^{1/2} L_n^{(\alpha)}.$$

Then we have with the above formula

$$\Lambda_m^{(\alpha)} = \sum_{n=0}^{m} S_{nm}^{\alpha,\beta}\,\Lambda_n^{(\beta)}$$

where

$$S_{nm}^{\alpha,\beta} = \frac{(\alpha-\beta)_{m-n}}{(m-n)\,!}\left\{\frac{\Gamma(m+1)}{\Gamma(m+\alpha+1)}\;\frac{\Gamma(n+\beta+1)}{\Gamma(n+1)}\right\}.$$

Definition 2.2.

For each $\alpha > -1$ and each $t > 0$ we write

$$X_t^{(\alpha)}\,[(\mu_n)] = X_t\,[(\Lambda_n^{(\alpha)})\,,\,(\mu_n)]$$

and

$$F_t^{(\alpha)}\,[(\mu_n)] = F_t\,[(\Lambda_n^{(\alpha)})\,,\,(\mu_n)].$$

From [MOS], p. 248, we obtain that for any fixed $x \in (0,\,\infty)$

$$\Lambda_n^{(\alpha)}\,(x) = O\,(n^{-\frac{1}{4}}).$$

First, let us consider sequences (μ_n) satisfying condition (B_0), viz.

$$\forall_{t>0} : \sum_{n=0}^{\infty} e^{-t\mu_n}\,,\,\infty.$$

This condition is equivalent with

$$\forall_{t>0} : \sup_{n\in\mathbb{N}}\, n\,\exp\,[-t\,\mu_n] < \infty.$$

It follows that condition (A) is automatically fulfilled for all $t > 0$; we mean that

$$\forall_{\alpha>-1}\ \forall_{t>0}\ \forall_{x\geq 0} : \sum_{n=0}^{\infty} | \Lambda_n^{(\alpha)}(x) |^2 e^{-t\mu_n} < \infty.$$

Lemma 2.3.

Let the sequence (μ_n) satisfy condition (B_0). Then for each $\alpha > -1$ and $t > 0$ the spaces $X_t^{(\alpha)}[(\mu_n)]$ and $F_t^{(\alpha)}[(\mu_n)]$ are genuine function spaces. Also, $F_t^{(\alpha)}[(\mu_n)]$ consists of all functions on $(0, \infty)$ which admit a Laguerre series expansion $\sum_{n=0}^{\infty} a_n \Lambda_n^{\alpha}$ where $a_n = O(\exp(-\mu_n s))$ for all s, $0 < s < t$.

We have the following classification theorem.

Theorem 2.3.

Let (μ_n) denote a monotoneously increasing sequence satisfying (B_0), let $t > 0$. Then for all $\alpha > -1$ and $\beta > -1$

$$F_t^{(\alpha)}[(\mu_n)] = F_t^{(\beta)}[(\mu_n)].$$

Proof.

By Theorem 1.11 and Remark 1.12 we have to prove that for all $\alpha, \beta > -1$,

$$\forall_{\sigma, 0<\sigma<t}\ \exists_{s, 0<s<t} :$$

$$\sup_{n,m} | S_{nm}^{\alpha,\beta} | \exp[-s\mu_m + \sigma\mu_n] < \infty.$$

A straightforward estimation based on Lemma 2.1 yields

$$| S_{nm}^{\alpha,\beta} | \leq (m-n+1)^{|\alpha-\beta|-1} \left[\frac{(n+1)^{\beta}}{(m+1)^{\alpha}} \right]^{1/2}.$$

Now with $\sigma < s < t$ we get

$$| S_{nm}^{\alpha\beta} | \exp[-s\nu_m + \sigma\mu_n] \leq$$

$$\leq (m-n+1)^{|\alpha-\beta|-1} \left\{ \frac{(n+1)^{\alpha}}{(m+1)^{\beta}} \right\}^{\frac{1}{2}} \exp[-(s-\sigma)\mu_m]$$

$$\leq (m+1)^{k_\omega} \exp(-(s-\sigma)\mu_m)$$

where

$$k_{\alpha,\beta} = \begin{cases} \max\{\alpha, \frac{3}{2}\alpha\} - \frac{3}{2}\beta - 1, & \alpha > \beta \\ \frac{1}{2}\beta - \min\{\frac{1}{2}\alpha, \alpha\} - 1, & \alpha \leq \beta. \end{cases}$$

Thus we see that

$$\forall_{\alpha>-1}\ \forall_{\beta>-1}\ \forall_{\sigma,0<\sigma<t}\ \forall_{s,\sigma<s<t}:$$

$$\sup_{n,m} | S_{nm}^{\alpha,\beta} | \exp(-s\mu_m + \sigma\mu_n) < \infty.$$

Remark.

By Theorem 1.11, the condition that the sequence (μ_n) is monotoneously increasing can be weakened in the following sense: there exists a monotoneously increasing sequence $(\tilde{\mu}_n)$ such that for all $\varepsilon > 0$

$$\limsup_{n\to\infty} [\tilde{\mu}_n - (1+\varepsilon)\mu_n] = -\infty$$

and

$$\limsup_{n\to\infty} [\mu_n - (1+\varepsilon)\tilde{\mu}_n] = -\infty.$$

It then follows that

$$F_t^{(\alpha)}[(\mu_n)] = F_t^{(\alpha)}[(\tilde{\mu}_n)] = F_t^{\beta}[(\tilde{\mu}_n)] = F_t^{(\beta)}[(\mu_n)].$$

Next, we impose condition (B_∞) on the sequence (μ_n), viz. $\exists_{T>0} : \sum_{n=0}^{\infty} \exp(-\mu_n T) < \infty$.

Then for all $\alpha > -1$,

$$\sum_{n=0}^{\infty} | \Lambda_n^{(\alpha)} |^2 \exp(-\mu_n T) < \infty.$$

So we get

Lemma 2.4.

Let (μ_n) satisfy condition (B_∞). Then for each $\alpha > -1$ the space $F_\infty^{(\alpha)}[(\mu_n)]$ is a genuine function space over $(0, \infty)$ and can be characterized as follows: A function f on $(0, \infty)$ belongs to $F_\infty^{(\alpha)}[(\mu_n)]$ iff f admits a Laguerre series expansion

$$f = \sum a_n \Lambda_n^{(\alpha)}$$

with $a_n = O(\exp(-\mu_n t))$ for all $t > 0$.

The following statement is valid

Theorem 2.5.

Let (μ_n) denote a monotoneously increasing sequence satisfying (B_∞). Then for all $\alpha > -1$ and $\beta > -1$

$$F_\infty^{(\alpha)}\,[(\mu_n)] = F_\infty^{(\beta)}\,[(\mu_n)].$$

Proof.

According to Theorem 1.11 and the remark proceding it we have to prove that for all $\alpha, \beta > -1$

$$\forall_{\sigma>0}\,\exists_{s>0} : \sup_{n,m} \mid S_{nm}^{\alpha,\beta} \mid \exp\left[-s\,\mu_m + \sigma\,\mu_n\right] < \infty.$$

With the aid of the estimate on the matrix entries we see that the above condition is satisfied for the sequence (μ_n). □

Remark: In the above theorem the monotoneously increasing sequence (μ_n) satisfying (B_∞) can be replaced by any sequence (μ_n) satisfying (B_∞) for which there exists a monotoneously increasing sequence $(\tilde{\mu}_n)$ with

$$\exists_{\varepsilon>0} : \limsup_{n\to\infty} (\mu_n - (1+\varepsilon)\,\tilde{\mu}_n) = -\infty$$

and

$$\exists_{\varepsilon>0} : \limsup_{n\to\infty} (\tilde{\mu}_n - (1+\varepsilon)\,\mu_n) = -\infty.$$

It then follows that

$$F_\infty^{(\alpha)}\,[(\mu_n)] = F_\infty^{(\alpha)}\,[(\tilde{\mu}_n)] = F_\infty^{(\beta)}\,[(\tilde{\mu}_n)] = F_\infty^{(\beta)}\,[(\mu_n)].$$

Closely related to the Laguerre polynomials are the Laguerre functions defined by

$$\mathbf{L}_n^{(\alpha)}(x) = e^{-\frac{1}{2}x^2}\,\Lambda_n^{(\alpha)}(x^2), \quad n = 0, 1, 2, \cdots$$

The functions $\mathbf{L}_n^{(\alpha)}$ establish an orthonormal basis in the Hilbert space $L_2((0,\infty), x^{2\alpha+1}\,dx)$. Correspondingly we introduce the following spaces.

Definition 2.6.

- The subspace $Y_t^{(\alpha)}\,[(\mu_n)]$ of $L_2((0,\infty), x^{2\alpha+1}\,dx)$ consists of all $\phi \in L_2((0,\infty), x^{2\alpha+1}\,dx)$ for which

$$\sum_{n=0}^{\infty} \exp(2t\,\mu_n) \mid (\phi, \mathbf{L}_n^{(\alpha)}) \mid^2 < \infty.$$

- The subspace $G_t^{(\alpha)}[(\mu_n)]$ is given by

$$G_t^{(\alpha)}[(\mu_n)] = \bigcup_{0<s<t} Y_t^{(\alpha)}[(\mu_n)].$$

The functions $L_n^{(\alpha)}$ establish an orthogonal basis in $Y_t^{(\alpha)}[(\mu_n)]$ and a Schauder basis in $G_t^{(\alpha)}[(\mu_n)]$. Now the Hankel transformation $\mathcal{H}_\alpha$ is defined by

$$(\mathcal{H}_\alpha \phi)(x) = \int_0^\infty (xy)^{-\alpha} J_\alpha(xy)\, \phi(y)\, y^{2\alpha+1}\, dy$$

Then from [MOS], p. 244, we obtain the relations

$$\mathcal{H}_\alpha L_n^{(\alpha)} = (-1)^n L_n^{(\alpha)}$$

It follows immediately that the spaces $Y_t^{(\alpha)}[(\mu_n)]$ and $G_t^{(\alpha)}[(\mu_n)]$ remain invariant with respect to $\mathcal{H}_\alpha$. The following stronger result is valid.

Theorem 2.7.

Let (μ_n), (ν_n) denote monotoneously increasing sequences satisfying condition (B_0) and (B_∞), respectively, and let $t > 0$. Then for all $\alpha > -1$ and $\beta > -1$

$$G_t^{(\alpha)}[(\mu_n)] = G_t^{(\beta)}[(\mu_n)] := G_t[(\mu_n)]$$

and

$$G_\infty^{(\alpha)}[(\nu_n)] = G_\infty^{(\beta)}[(\nu_n)] := G_\infty[(\nu_n)].$$

Moreover, for each $\gamma > -1$ the functions $L_n^{(\gamma)}$, $n = 0, 1, 2, \cdots$ establish a Schauder basis in $G_t[(\mu_n)]$ and $G_\infty[(\nu_n)]$.
The (function) spaces $G_t[(\mu_n)]$ and $G_\infty[(\nu_n)]$ remain invariant under each Hankel transformation $\mathcal{H}_\gamma$.

The remaining part of this section is devoted to analytic characterizations of certain spaces $G_t[(\mu_n)]$ and $G_\infty[(\nu_n)]$. Therefore we introduce the Hermite functions. Namely, the functions $L_n^{(-1/2)}$ are equal to the even Hermite functions ψ_{2n}, where

$$\psi_n = \frac{(-1)^n}{(\pi^{1/2} 2^n n!)^{1/2}} e^{\frac{1}{2}x^2} \left(\frac{d}{dx}\right)^n \{e^{-x^2}\}.$$

The functions ψ_n establish an orthonormal basis in $L_2(\mathbb{R})$, and satisfy

$$\mathcal{F}\psi_n = i^n \psi_n$$

where $\mathcal{F}$ denotes the Fourier transformation. So on the basis of the functions ψ_n there arise a great lot of Fourier invariant function spaces. We mention the following.

- The Schwartz space S.
The space S consists of all C^∞-functions ϕ with the following growth behaviour

$$\forall_{k,l\in\mathbb{N}} : \sup_{x\in\mathbb{R}} | x^k \phi^{(l)}(x) | < \infty.$$

Now Simon has proved the following characterization of S in terms of Hermite expansions:
A square integrable function ϕ belongs to S if and only if

$$\forall_{k\in\mathbb{N}} : \quad (\phi, \psi_n)_{L_2(\mathbb{R})} = O(n^{-k})$$

cf. [Si]. Let S_{even} denote the subspace of all even functions in S. Then we have the following characterizations of S_{even}.

Theorem 2.8.

For each $\alpha > -1$, S_{even} equals $G^{(\alpha)}[(\log n + 1)]$ as a Frechet space. So an even square integrable function ϕ on $\mathbb{R}$ belongs to S_{even} if and only if

$$\forall_{k\in\mathbb{N}} : (\phi, L_n^{(\alpha)})_\alpha = O(n^{-k}).$$

Proof.

From Simon's result we get

$$S_{even} = G^{(-1/2)}[(\log n + 1)].$$

Next apply Theorem 2.7 with $\nu_n = \log(n+1)$. □

Remark: The above result has also been obtained in [EG1] by a different method based on complex analysis.

- The Gelfand-Shilov spaces S_ω^ω
For $\omega > 0$, S_ω^ω denotes the subspace of S consisting of all $\phi \in S$ with the following growth behaviour

$$\exists_{A,B,C>0}\ \forall_{k,l} : \sup_{x\in\mathbb{R}} | x^k \phi^{(l)}(x) | \le C A^k B^l (k!)^\omega (l!)^\omega.$$

For $0 < \omega < \frac{1}{2}$ the space S_ω^ω is trivial, and for $\frac{1}{2} \le \omega < 1$ it consists of entire functions with the following growth behaviour in the complex plane

$$\exists_{C,a,b>0} : | \phi(x+iy) | \le C \exp[-a | x |^{1/\omega} + b | y |^{1/1-\omega}].$$

For details on these spaces we refer to [GS2].

Zhang has proved the following characterization (cf. [Zh]).

A square integrable function ϕ belongs to S^{ω}_{ω}, $\omega \geq \frac{1}{2}$ if and only if $\exists_{t>0} : (\phi, \psi_n)_{L_2(\mathbb{R})} = O(\exp(-t\, n^{\frac{1}{2\omega}}))$.

Zhang's result obviously implies the following.

Corollary 2.9.

Let $\omega \geq \frac{1}{2}$. Then we have

$$S^{\omega}_{\omega,\text{even}} = \bigcup_{t>0} G_t^{(-\frac{1}{2})}[(n^{\frac{1}{2\omega}})],$$

whence for each $\alpha > -1$

$$S^{\omega}_{\omega,\text{even}} = \bigcup_{t>0} G_t^{(\alpha)}[(n^{\frac{1}{2\omega}})].$$

□

It follows that for each $\alpha > -1$, $S^{\omega}_{\omega,\text{even}}$ remains invariant under the Hankel transformation $\mathcal{H}_{\alpha}$. In addition, to Zhang's characterization we have

A even square integrable function ϕ belongs to S^{ω}_{ω}, $\omega \geq \frac{1}{2}$, if and only if $\exists_{t>0} : (\phi, L_n^{(\alpha)})_{\alpha} = O(\exp(-t\, n^{\frac{1}{2\omega}}))$.

Remarks.

- From De Bruijn's paper [Br], Theorem 6.4 it follows that $G_t^{(-\frac{1}{2})}[(n)]$ consists of all even entire functions ϕ with the following property

$$\forall_{A,\, 0<A<\frac{1}{2}\tanh\frac{1}{2}t}\ \exists_{C>0} :$$

$$|\phi(x+iy)| \leq C \exp[-A x^2 + \frac{1}{A} y^2].$$

So Theorem 2.7 yields the same characterization for the spaces $G_t^{(\alpha)}[(n)]$ $\alpha > -1$. In [Hi], Hille has proved the following result A square integrable function ϕ on $\mathbb{R}$ can be extended to an analytic function $\hat{\phi}$ on a strip $|Im\, z| < t$ on which it satisfies the growth condition

(*) $$|\hat{\phi}(x+iy)| \leq C \exp[-|x|(t^2-y^2)^{\frac{1}{2}}]$$

if and only if

$$(\phi, \psi_n)_{L_2} = O(\exp(-t\, n^{\frac{1}{2}})).$$

It follows that for each $\alpha > -1$, the Frechet space $G_t^{(\alpha)}[(n^{\frac{1}{2}})]$ consists of even functions on $\mathbb{R}$ which admit an analytic extension to a strip $|Im\, z| < t$ where it satisfies the estimate (*).

-The Gelfand-Shilov spaces $W_M^{M^*}$.

Let m denote a monotoneously increasing differentiable function on $[0, \infty]$ with $m(0) = 0$. We write

$$M(x) = \int_0^x m(t)\, dt\,, \quad x \geq 0$$

and

$$M^*(y) = \int_0^y m^{\leftarrow}(t)\, dt\,, \quad y \geq 0.$$

The pair (M, M^*) satisfies Young's inequality

$$xy \leq M(x) + M^*(y)$$

with equality if and only if $y = m(x)$. M is called an Orlicz function.

In [GS3] the space $W_M^{M^*}$ is introduced as follows.

An entire function ϕ belongs to $W_M^{M^*}$ if and only if

$$|\phi(x+iy)| \leq C \exp\left[-M(a|x|) + M^*(b|y|)\right]$$

where a, b and C are suitable constants.

Under the following mild conditions on the function m also the space $W_M^{M^*}$ admits a characterization in terms of Hermite expansion coefficients, viz.

- m is concave and $m(t) \to \infty$ $(t \to \infty)$
- $\dfrac{m(t)}{t}$ decreases strictly to zero as $t \to \infty$.

Now the characterization is as follows

A square integrable function ϕ belongs to $W_M^{M^*}$ if and only if $\exists_{t>0} : (\phi, \psi_n)_{L_2(\mathbb{R})} = V(\exp(-t\,M(n^{1/2})))$.

For a proof of this result we refer to [JE]. So consequently as for the spaces S_∞^∞ we have

Corollary 2.10.

For all $\alpha > -1$,

$$W_{M,\text{even}}^{M^*} = \bigcup_{t>0} G_t^{(\alpha)}\left[(M(n^{1/2}))\right].$$

In particular, for each $\alpha > -1$, $W_{M,\text{even}}^{M^*}$ is $\mathbb{H}_v$-invariant.

3. Jacobi polynomials.

In [MOS], p. 201, the Jacobi polynomials $P_n^{(\alpha,\beta)}$ are defined by

$$P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{n!\,2^n}(1-x)^{-\alpha}(1+x)^{-\beta}\left(\frac{d}{dx}\right)^n[(1-x)^{\alpha+n}(1+x)^{\beta+n}].$$

They satisfy the following orthogonality relations

$$\int_{-1}^{1} P_n^{(\alpha,\beta)}(x)\,P_m^{(\alpha,\beta)}(x)\,(1-x)^{\alpha}(1+x)^{\beta}\,dx =$$

$$= \frac{2^{\alpha+\beta+1}}{2n+\alpha+\beta+1}\,\frac{\Gamma(n+\alpha+1)\,\Gamma(n+\beta+1)}{\Gamma(n+1)\,\Gamma(n+\alpha+\beta+1)}\,\delta_{nm}.$$

Here we consider the normalized Jacobi polynomials $R_n^{(\alpha,\beta)}$

$$R_n^{(\alpha,\beta)} = \kappa_n^{(\alpha,\beta)}\,P_n^{(\alpha,\beta)}$$

with

$$\kappa_n^{(\alpha,\beta)} = \left\{\frac{2n+\alpha+\beta+1}{2^{\alpha+\beta+1}}\,\frac{\Gamma(n+1)\,\Gamma(n+\alpha+\beta+1)}{\Gamma(n+\alpha+1)\,\Gamma(n+\beta+1)}\right\}^{1/2}.$$

The polynomials $R_n^{(\alpha,\beta)}$ establish an orthonormal basis in the Hilbert space $X_{\alpha,\beta} = L_2([-1,1],(1-x)^{\alpha}(1+x)^{\beta}\,dx)$. Hence $X_{\alpha,\beta}$ is the natural completion of the vector space P with respect to the inner product

$$(p,q)_{\alpha,\beta} = \int_{-1}^{1} p(x)\,\overline{q(x)}\,(1-x)^{\alpha}(1+x)^{\beta}\,dx.$$

We want to estimate the matrix entries $S_{nm}^{(\alpha\beta,\gamma\delta)}$ where

$$R_m^{(\gamma,\delta)} = \sum_{n=0}^{m} S_{nm}^{(\alpha\beta,\gamma\delta)}\,R_n^{(\alpha,\beta)}$$

To this end, we apply the following formula, derived in [As], p. 63

$$P_m^{(\alpha,\delta)} = \frac{(\alpha+1)_m}{(\alpha+\beta+2)_m}\sum_{n=0}^{m}(-1)^{m-n}\,\frac{(\delta-\beta)_{m-n}}{(m-n)!}\,\frac{(\alpha+\beta+1)_n}{(\alpha+1)_n}\cdot$$

$$\cdot\,\frac{(\alpha+\beta+2)_{2n}}{(\alpha+\beta+1)_{2n}}\,\frac{(m+\alpha+\delta+1)_n}{(m+\alpha+\beta+2)_n}\,P_n^{(\alpha,\beta)}.$$

It follows that $S_{nm}^{(\alpha\beta,\alpha\delta)} =$

$$(-1)^{m-n}\left\{\frac{K_m^{(\alpha,\delta)}}{K_n^{(\alpha,\beta)}}\right\}\cdot\left\{\frac{(\alpha+1)_m}{(\alpha+\beta+2)_m}\,\frac{(\delta-\beta)_{m-n}}{(m-n)!}\,\frac{(\alpha+\beta+1)_n}{(\alpha+1)_n}\,\frac{(\alpha+\beta+2)_{2n}}{(\alpha+\beta+1)_{2n}}\,\frac{(m+\alpha+\delta+1)_n}{(m+\alpha+\beta+2)_n}\right\}.$$

Employing the inequalities of Lemma 2.1 the first factor between braces { } is estimated by $K_1\left(\frac{m+1}{n+1}\right)^{1/2}$ and the second factor by

$$K_2 \left(\frac{n+1}{m+1}\right)^{\beta+1} (m-n+1)^{|\delta-\beta|-1}$$

for certain K_1 and $K_2 > 0$.
Observe that

$$\frac{(m+\alpha+\delta+1)_n}{(m+\alpha+\beta+2)_n} = \frac{\Gamma(m+\alpha+\beta+2)}{\Gamma(m+\alpha+\delta+1)} \frac{\Gamma(m+n+\alpha+\delta)}{\Gamma(m+n+\alpha+\beta+1)} \leq$$

$$\leq K_3 (m+1)^{\beta+1-\delta} (m+n+1)^{\delta-\beta-1}.$$

We arrive at the following estimate

$$| S_{nm}^{(\alpha\beta,\alpha\delta)} | \leq K \left(\frac{n+1}{m+1}\right)^{\beta+\frac{1}{2}} (m-n+1)^{|\delta-\beta|-1}.$$

Further, since $P_n^{(\alpha,\beta)}(x) = (-1)^n P_n^{(\beta,\alpha)}(-x)$ and $\kappa_n^{(\alpha,\beta)} = \kappa_n^{(\beta,\alpha)}$ we have

$$S_{nm}^{(\alpha\beta,\gamma\beta)} = (-1)^{m-n} S_{nm}^{(\beta\alpha,\beta\gamma)}.$$

Hence

$$| S_{nm}^{(\alpha\beta,\gamma\beta)} | \leq K \left(\frac{n+1}{m+1}\right)^{\alpha+\frac{1}{2}} (m-n+1)^{|\gamma-\alpha|-1}.$$

Lemma 3.1.

Let $\alpha, \beta, \gamma, \delta > -1$. Then the following estimation is valid for all $m, n \in \mathbb{N}_0$, $m \geq n$,

$$| S_{nm}^{(\alpha\beta,\gamma\delta)} | \leq K (m-n+1)^{|\alpha-\gamma|+|\beta-\delta|} \left(\frac{n+1}{m+1}\right)^{\frac{1}{2}+\min(\alpha,\beta)}.$$

Proof.

We have

$$S_{nm}^{(\alpha\beta,\gamma\delta)} = \sum_{j=n}^{m} S_{nj}^{(\gamma\beta,\gamma\delta)} S_{jm}^{(\alpha\beta,\gamma\beta)}$$

so that

$$| S_{nm}^{(\alpha\beta,\gamma\delta)} | \leq$$

$$\leq \sum_{j=n}^{m} \left(\frac{n+1}{j+1}\right)^{\beta+\frac{1}{2}} \left(\frac{j+1}{m+1}\right)^{\alpha+\frac{1}{2}} (j-n+1)^{|\delta-\beta|-1} (m-j+1)^{|\gamma-\alpha|-1}$$

$$\leq (m-n+1)^{|\alpha-\gamma|+|\beta-\delta|-1} \frac{(n+1)^{\beta+\frac{1}{2}}}{(m+1)^{\alpha+\frac{1}{2}}} \sum_{j=n}^{m} (j+1)^{\alpha-\beta} \frac{m-n+1}{(m-j+1)(j-n+1)}$$

Finally, observe that $\frac{m-n+1}{(m-j+1)(j-n+1)} \leq 1$ and that $\sum_{j=n}^{m} (j+1)^{\alpha-\beta} \leq (m-n+1) \begin{cases} (n+1)^{\alpha-\beta} \text{ if } \alpha < \beta \\ (m+1)^{\alpha-\beta} \text{ if } \alpha \geq \beta. \end{cases}$

□

For convenience, we set

$$X_t^{\alpha,\beta}[(\mu_n)] = X_t[(R_n^{(\alpha,\beta)}), (\mu_n)]$$

and

$$F_t^{\alpha,\beta}[(\mu_n)] = \bigcap_{0<s<t} X_s^{\alpha,\beta}[(\mu_n)].$$

From [MOS], p. 216, we derive that

$$R_n^{(\alpha,\beta)}(x) = O(n^q), \quad x \in [-1, 1]$$

where $q = \max\{\alpha+\frac{1}{2}, \beta+\frac{1}{2}, 0\}$. So we can deal with the same class of sequences (μ_n) as used in the case of the Laguerre polynomials, viz. we consider sequences (μ_n) satisfying condition (B_0) or condition (B_∞).

Lemma 3.2.

Let (μ_n) denote a sequence satisfying condition (B_0) and let $t > 0$. Then for all $\alpha, \beta > -1$ the spaces $X_t^{\alpha,\beta}[(\mu_n)]$ and $F_t^{\alpha,\beta}[(\mu_n)]$ are function spaces. We have

$$\phi \in F_t^{\alpha,\beta}[(\mu_n)] \text{ iff } \phi(x) = \sum_{n=0}^{\infty} a_n R_n^{(\alpha,\beta)}(x), \quad x \in [-1, 1],$$

$$\text{with } \forall_{s,0<s<t} : a_n = O(\exp(-\mu_n s)).$$

Proof.

We observe that for all $s > 0$ and $x \in [-1, 1]$

$$\sum_{n=0}^{\infty} e^{-\mu_n s} |R_n^{(\alpha,\beta)}(x)|^2 < \infty.$$

□

Theorem 3.3.

Let (μ_n) denote a monotoneously increasing sequence satisfying condition (B_0). Then for all $t > 0$ and $\alpha, \beta, \gamma, \delta > -1$

$$F_t^{\alpha,\beta}[(\mu_n)] = F_t^{\gamma,\delta}[(\mu_n)].$$

Proof.

Due to the estimate on the matrix entries $S_{nm}^{(\alpha\beta,\gamma\delta)}$ the proof contains precisely the same arguments as the proof of Theorem 2.3. We leave it to the reader. □

Remark: As in Theorem 2.3, the condition that (μ_n) is monotoneously increasing can be replaced by a weaker condition.

The statements corresponding Lemma 2.4 and Theorem 2.5 are the following

Lemma 3.4.

Let (μ_n) denote a sequence satisfying condition (B_∞). Then for all $\alpha, \beta > -1$, the space $F_\infty^{\alpha,\beta}[(\mu_n)]$ is a function space:

$$\phi \in F_\infty^{\alpha,\beta}[(\mu_n)] \text{ iff } \phi(x) = \sum a_n R_n^{(\alpha,\beta)}(x), \quad x \in [-1, 1]$$

$$\text{with } \forall_{t>0} : a_n = O(\exp(-\mu_n t)).$$

□

Theorem 3.5.

Let (μ_n) denote a monotoneously increasing sequence satisfying condition (B_∞). Then for all $\alpha, \beta, \gamma, \delta > -1$

$$F_\infty^{\alpha,\beta}[(\mu_n)] = F_\infty^{\gamma,\delta}[(\mu_n)].$$

□

The polynomials $R_n^{(-\frac{1}{2},-\frac{1}{2})}$ are called Chebysev polynomials. They satisfy the following useful relation.

$$R_n^{-\frac{1}{2},-\frac{1}{2}}(\cos w) = \begin{cases} \sqrt{\frac{1}{2\pi}} & , \ n = 0 \\ \sqrt{\frac{1}{\pi}} \cos nw & , \ n = 1, 2, \cdots \end{cases}$$

With this relation a number of space of type $F_t^{\alpha,\beta}[(\mu_n)]$ can be completely characterized. We start with a derivation of a classical result of Szegö, see [Sz].

Theorem 3.6.

Let $\alpha, \beta > -1$ and let $t > 0$. The space $F_t^{\alpha,\beta}[(n)]$ consists of all functions ϕ which are analytic within the ellipse E_t,

$$\frac{x^2}{\cosh^2 t} + \frac{y^2}{\sinh^2 t} = 1.$$

Proof.

The following statement can be readily checked:

A function ψ is 2π-periodic and analytic on the strip $|Im\, w| < t$ iff there exists a sequence

$(b_n)_{n=-\infty}^{\infty}$ satisfying $\forall_{s,\,0<s<t} : \sup_n |b_n| \exp(|n|s) < \infty$ such that $\psi(w) = \sum_{n=-\infty}^{\infty} b_n e^{inw}$.

Now let $\phi \in F_t^{\alpha,\beta}[(n)] = F_t^{-\frac{1}{2},-\frac{1}{2}}[(n)]$. Then $\phi(\cosh w) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos nw$ where

$$\forall_{s,\,0<s<t} \sup_n |a_n| \exp(ns) < \infty.$$

So $w \mapsto \phi(\cos w)$ is a 2π-periodic even function which is analytic on the strip $|Im\, w| < t$. The conformal mapping $z = \cos w$ sends the rectangle $\{w \mid |Im\, w| < t \wedge -\pi \le \mathrm{Re}\, w \le \pi\}$ onto the interior of the ellipse $\frac{x^2}{\cosh^2 t} + \frac{y^2}{\sinh^2 t} = 1$, $z = x + iy$. Hence ϕ is analytic within E_t.
Conversely, if ϕ is analytic within E_t, then the function $w \mapsto \phi(\cos w)$ is 2π-periodic, even and analytic on the strip $|Im\, w| < t$. Hence

$$\phi(\cos w) = \tfrac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos nw, \quad |Im w| < t,$$

with $a_n = O(\exp(-ns))$, $0 < s < t$, which yields

$$\phi \in F_t^{-\frac{1}{2},-\frac{1}{2}}[(n)] = F_t^{\alpha,\beta}[(n)].$$

□

Next, we present a characterization of the spaces $F_t^{\alpha,\beta}[(M(n))]$, where $\alpha, \beta > -1$, $t > 0$ and, where M denotes an Orlicz function,

$$M(x) = \int_0^x m(t)\, dt$$

with m monotoneously increasing, $m(0) = 0$ and $m(\infty) = \infty$.

Theorem 3.7.

The space $F_t^{\alpha,\beta}[(M(n))]$ consists of all entire analytic functions ϕ with the following growth behaviour in the complex plane

$$\forall_{s,\,0<s<t}\, \exists_{C_s} : |\phi(z)| \le C_s \exp[s M^*(\tfrac{1}{s} \log |z|)].$$

Proof.

Let $(b_n)_{n=-\infty}^{\infty}$ denote a bounded sequence. Then for each $\sigma > 0$ the function

$$\chi(w) = \sum_{n=-\infty}^{\infty} b_n \exp[-\sigma M(|n|)]\, e^{inw}$$

is 2π-periodic and holomorphic. Further, a simple application of Cauchy-Schwartz' and of Young's inequality yields the following estimate

(*) $$|\chi(w)| \le C_{s,\chi} \exp[s M^*(\frac{|v|}{s})], \quad w = u + iv$$

where $0 < s < \sigma$.

Conversely, if a 2π-periodic holomorphic function θ admits the asymptotic behaviour as given in (*) for each s, $0 < s < t$, then we have

$$\theta(w) = \sum_{n=-\infty}^{\infty} b_n e^{inw}$$

where for each $v \in \mathbb{R}$,

$$b_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} \theta(u) e^{inu}\, du = \frac{1}{2\pi} \int_{-\pi}^{\pi} \theta(u+iv) e^{-in(u+iv)}\, du.$$

So for each s, $0 < s < t$

(**) $$|b_n| \le C_{s,\theta} \inf_{v \in \mathbb{R}} \exp[s M^*(\frac{|v|}{s}) + nv] = C_{s,\theta} \exp[-s M(|n|)].$$

Let $\phi \in F_t^{-\frac{1}{2},-\frac{1}{2}}[(M(n))]$. Then

$$\chi(w) := \phi(\cos w) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos nw$$

with for each $0 < \sigma < t$, $\sup_{n \in \mathbb{N}_0} (|a_n| \exp[\sigma M(|n|)]) < \infty$.

It follows that χ is an even 2π-periodic holomorphic function with

$$|\chi(w)| \le C_{s,\chi} \exp[s M^*\left[\frac{|v|}{s}\right]], \quad 0 < s < \sigma.$$

Conversely, an even, 2π-periodic holomorphic function θ can be written as $\theta(w) = \psi(\cos w)$ where ψ is holomorphic. If θ satisfies

$$\forall_{s, 0<s<t}\, \exists_{C_s>0} : |\theta(w)| \le C_s \exp[s M^*(\frac{|v|}{s})]$$

then by (**) we see that $\psi \in F_t^{(-\frac{1}{2},-\frac{1}{2})}[(M(n))]$.

Finally, the wanted characterization is obtained by applying the conformal mapping $w = \cos z$, viz. $z = \log(w + i\sqrt{1-w^2})$ where we observe that

$$4|w|^2 - 3 \le |w + i\sqrt{1-w^2}|^2 \le 4|w|^2 + 1.$$

□

In the next theorem we present a condition on the sequence (μ_n) which yields a classification of the Hilbert spaces $X_t^{\alpha,\beta}[(\mu_n)]$.

Theorem 3.8.

Let (μ_n) denote a sequence of nonnegative numbers. Suppose there exists a sequence (ν_j) with the following properties

$$- \quad \forall_{t>0} : \sum_{j=1}^{\infty} e^{-\nu_j t} < \infty$$

$$- \quad \forall_{n \in \mathbb{N}_0} \forall_{j \in \mathbb{N}} : \mu_{n+j} - \mu_n \geq \nu_j .$$

Then for all $t > 0$ and all $\alpha, \beta, \gamma, \delta > -1$

$$X_t^{\alpha,\beta}[(\mu_n)] = X_t^{\gamma,\delta}[(\mu_n)]$$

as function Hilbert spaces.

Proof.

It is clear that $\sum e^{-\mu_n s} < \infty$ for all $s > 0$. Hence the spaces $X_t^{\alpha,\beta}[(\mu_n)]$ can be regarded as functional Hilbert spaces. According to Theorem 1.9 we have to prove that $\alpha, \beta, \gamma, \delta > -1$ and all $t > 0$ the matrix

$$\sigma_{nm,t}^{(\alpha\beta,\gamma\delta)} = S_{nm}^{\alpha\beta,\gamma\delta} \exp(\mu_n - \mu_m) t$$

represent a bounded linear operator $\hat{\sigma}_t^{(\alpha\beta,\gamma\delta)}$ from l_2 into l_2.
Therefore, we proceed as follows. Fix $\alpha, \beta, \gamma, \delta > -1$, and put $\hat{\sigma}_t = \hat{\sigma}_t^{(\alpha\beta,\gamma\delta)}$. Then we write

$$\hat{\sigma}_t = \sum_{j=0}^{\infty} \Delta_{t,j} U^j$$

where U denotes the unilateral shift

$$U(\xi_0, \xi_1, \xi_2, \cdots) = (\xi_1, \xi_2, \ldots)$$

and $\Delta_{t,j}$ the diagonal operator on l_2 with entries

$$(\Delta_{t,j})_{kk} = \sigma_{k,k+j,t}^{(\alpha\beta,\gamma\delta)}.$$

From Lemma 3.1 we obtain

$$\| \Delta_{t,j} \|_{l_2 \to l_2} = \sup_k | \sigma_{k,k+j,t}^{(\alpha\beta,\gamma\delta)} | \leq$$

$$\leq K (j+1)^r \sup_k \left[\frac{k+1}{k+j+1} \right]^p \exp[-(\mu_{k+j} - \mu_k) t] \leq$$

$$\leq K (j+1)^{\tilde{q}} \exp[-\nu_j t].$$

Here $r = |\alpha - \gamma| + |\beta - \delta|$, $p = \frac{1}{2}(1 + \min\{\alpha, \beta\})$ and $\tilde{q} = q - \min\{0, p\}$. So we get

$$\| \hat{\sigma}_t \|_{l_2 \to l_2} \leq \sum_{j=0}^{\infty} \| \Delta_{t,j} U^j \|_{l_2 \to l_2}$$

$$\leq K \sum_{j=0}^{\infty} (j+1)^{\bar{q}} \exp(-\nu_j t) < \infty.$$

It follows that $\hat{\sigma}_t$ is a bounded linear operator from l_2 into l_2. □

For each $\nu \geq 1$ the sequence (n^ν) satisfies the conditions stated in the preceding theorem. Hence for all $\nu \geq 1, \alpha, \beta, \gamma, \delta > -1$ and all $t > 0$ we have

$$X_t^{\alpha,\beta} [(n^\nu)] = X_t^{(\gamma,\delta)} [(n^\nu)].$$

In the paper [EG3] there is given a characterization of the spaces $X_t^{\alpha,\beta} [(n^\nu)]$ for $\nu > 1$ and $t > 0$. Indeed,

$\phi \in X_t^{\alpha,\beta} [(n^\nu)]$ if and only if ϕ is a holomorphic function satisfying

$$\iint_{\mathbb{R}^2} | \phi(x+iy) |^2 g_{t,\nu} (x, y) \, dx \, dy < \infty$$

where

$$g_{t,\nu} (x, y) = 1 \text{ for } x^2 + y^2 \leq 1$$

and

$$g_{t,\nu} (x, y) = (x^2+y^2)^{-1} (\log(x^2+y^2)^{2\kappa} \exp [-\frac{2}{\lambda} (\log(x^2+y^2))^{1/\mu}]$$

$$\kappa = \frac{1}{4} \frac{2-\nu}{\nu-1}, \quad \mu = \frac{\nu-1}{\nu}, \quad \lambda = \frac{1}{\mu} \left[\frac{t}{1-\mu} \right]^{1-\mu/\mu}.$$

Finally, we devote some attention to the standard example of a sequence satisfying condition (B_∞): we consider the sequence $\mu_n = \log n + 1$. Following Lemma 3.4 the spaces $F_\infty^{\alpha,\beta} [(\log n + 1)]$, $\alpha, \beta > -1$, are genuine function spaces and according to Theorem 3.5

$$F_\infty^{\alpha,\beta} [\log(n+1)] = F_\infty^{\gamma,\delta} [\log(n+1)]$$

for all $\alpha, \beta, \gamma, \delta > -1$.

Theorem 3.9.

For all $\alpha, \beta > -1$, $F_\infty^{\alpha,\beta} [(\log(n+1)]$ consists of all C^∞-functions on [-1, 1].

Proof.

It can be readily checked that each even, 2π-periodic C^∞-function χ on $\mathbb{R}$ can be expanded into a Fourier cosine series

$$\chi(u) = \tfrac{1}{2}\, a_0 + \sum_{n=1}^{\infty} a_n \cos nu$$

where $a_n = V(n^{-k})$ for all $k \in \mathbb{N}$. Conversely, each such series represents an even 2π-periodic C^∞-function. It follows that $\phi \in F_\infty^{-\frac{1}{2},-\frac{1}{2}}[(\log(n+1))]$ if and only if ϕ is a function on [-1, 1] such that $u \mapsto \phi(\cos u)$ is infinitely differentiable. Thus the result follows. □

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