On Stieltjes integral transforms involving Γ-functions

TECHNISCHE HOGESCHOOL EINDHOVEN
NEDERLAND
ONDERAFDELING DER WISKUNDE

TECHNOLOGICAL UNIVERSITY EINDHOVEN
THE NETHERLANDS
DEPARTMENT OF MATHEMATICS

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V. Belevitch and J. Boersma

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Abstract

After some methodological remarks on the theory of Stieltjes transforms, a systematic classification of transforms involving Γ-functions is presented. As a consequence, many new transforms are established and much simpler proofs for a few known transforms are obtained. A shortened version of this report will appear in Mathematics of Computation.

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In circuit and system theory a real function $f(z)$, analytic in $\operatorname{Re} z \geq 0$ (hence finite at infinity), is deduced from its real part on the imaginary axis

$$u(y) = \operatorname{Re} f(iy) \tag{1}$$

by [5]

$$f(z) = \frac{2z}{\pi} \int_0^\infty \frac{u(y)\,dy}{z^2 + y^2}, \quad \operatorname{Re} z > 0. \tag{2}$$

Relation (2) is even valid when $f(z)$ has sufficiently mild singularities on the imaginary axis and at infinity [8]. In the case of logarithmic singularities or branch points, however, cuts along the imaginary axis may be necessary to define $f(z)$ as single-valued in $\operatorname{Re} z > 0$, and the real part (1) must then be replaced by the real part on the right lip of the cut, i.e. by [1]

*) Philips Research Laboratory, Brussels, Belgium.

**) Department of Mathematics, Eindhoven University of Technology, Eindhoven, The Netherlands.

$$u(y) = \lim_{\varepsilon \to +0} \operatorname{Re} f(\varepsilon + iy). \tag{3}$$

The right-hand side of (2) is odd in z whereas f(z) is not odd, else u(y) would be zero. Consequently (2) does not hold for Re $z \leq 0$; in any case, the integrand of (2) is singular for $z = \pm iy$. For $f(z) = u(y) = 1$ (2) reduces to the elementary integral

$$1 = \frac{2z}{\pi} \int_0^\infty \frac{dy}{z^2 + y^2} \quad , \quad \operatorname{Re} z > 0. \tag{4}$$

Subtracting u(s) times (4) from (2), one obtains:

$$f(z) - u(s) = \frac{2z}{\pi} \int_0^\infty \frac{u(y) - u(s)}{z^2 + y^2} dy \tag{5}$$

where the integrand is no longer singular for $z = is$. With $f(is) = u(s) + iv(s)$, (5) divided by i yields

$$v(s) = \frac{2s}{\pi} \int_0^\infty \frac{u(y) - u(s)}{y^2 - s^2} dy.$$

This is equivalent [7] to the Hilbert transform

$$v(s) = \frac{2s}{\pi} ⨍_0^\infty \frac{u(y)}{y^2 - s^2} dy$$

where the integral is a Cauchy principal value.

In (2), change z into $\sqrt{z}$ and y into $\sqrt{y}$; next change $f(\sqrt{z})/\sqrt{z}$ into f(z) and $u(\sqrt{y})/\sqrt{y}$ into u(y); this yields the Stieltjes transform

$$f(z) = \frac{1}{\pi} \int_0^\infty \frac{u(y)\,dy}{y + z} \tag{6}$$

holding in the whole z-plane cut along the negative real axis, and u(y) is now related to the discontinuity of Im f(z) across the cut through

$$u(y) = -\frac{1}{2}\,\mathrm{Im}\,[f(ye^{i\pi}) - f(ye^{-i\pi})], \tag{7}$$

a relation given in Ref. 4, p.215, eq. (5) with a sign error. Relation (7) is less conspicuous than (1) or (3); moreover, the change of variables from (2) to (6) (corresponding in circuit theory to the transformation of an LC-impedance into an RC-impedance) is responsible for the many square-roots appearing in the table [4] of Stieltjes transforms. Finally, additional transforms are deduced from (2), for $z = iy$, by

$$\mathrm{Re}\; z\,\frac{df(z)}{dz} = y\,\frac{du(y)}{dy} \tag{8}$$

which is simpler than Ref. 5, p.215, eq. (9), and by

$$\mathrm{Re}\;\frac{f(z) - f(0) - zf'(0)}{z^2} = \frac{u(0) - u(y)}{y^2}\;. \tag{9}$$

The idea that (2) is essentially simpler than (6) has been (somewhat unsystematically) exploited in two previous papers, thus generating a number of new transforms for Bessel functions [3] and for complete elliptic integrals [2]. In addition to the remarks just made, the purpose of this note is to derive some new transforms, and to present much simpler proofs for some known transforms, involving Γ-functions.

Owing to the complement relation for Γ-functions, the real or imaginary parts of some linear combinations of logarithms of Γ-functions have elementary expressions. For $z = iy$, and with the definition (3) of the real part whenever (1) is ambiguous (and similarly for the imaginary part), we have

$$\mathrm{Re}\,[\log \Gamma(z+a) + \log \Gamma(z+1-a)] = \frac{1}{2} \log \frac{2\pi^2}{\cosh(2\pi y) - \cos(2\pi a)}, \tag{10}$$

$$\mathrm{Im}\,[\log \Gamma(z+a) - \log \Gamma(z+1-a)] = -\arctan[\cot(\pi a)\tanh(\pi y)]. \tag{11}$$

A number of Stieltjes transforms corresponding to the definition (2) and resulting from (10) or (11) are given in Tables A to D. For $0 \le a \le 1$, the functions $\log \Gamma(z+a)$, $z \log \Gamma(z+a)$ and $z^{-1}[\log \Gamma(z+a) - \log \Gamma(a)]$ are analytic in $\mathrm{Re}\, z > 0$. They can be made finite at infinity by subtracting from $\log \Gamma(z+a)$ the necessary number of terms of its asymptotic expansion

$$\log \Gamma(z+a) \sim (z + a - \tfrac{1}{2})\log z - z + \tfrac{1}{2} \log 2\pi + O(z^{-1}) \tag{12}$$

and the resulting differences have at most a logarithmic singularity on the imaginary axis (at $z = 0$ for $a = 0$). Transforms I to III are established by combining the resulting functions with parameters a and 1-a. Transform IV results from I by (8). Transforms V and VI result from II and III, respectively, by (8) after adding a multiple of the original transforms. Also VI is the derivative of I with respect to a; similarly IV is the derivative of II.

Transform VII results from I by (9). Since $f(z)$ is singular for $a = 0$ and $a = 1$, the transform is only valid in the range $0 < a < 1$. Transform VIII results from VII by (8) or from III by differentiation with respect to a. By adding to $f(z)$ of VII the function $\log(1+z/a) - z/a$ whose real part is $\frac{1}{2} \log(1+y^2/a^2)$, one suppresses the singularity for $a = 0$; the result is transform IX which now holds for $0 \le a < 1$. Transform X is deduced from IX by (8) combined with a multiple of the original transform.

All the transforms of Table D except XXVII are particular cases of transforms I to X for special values of the parameter a, as indicated in the last column of the Table. Whenever a certain transform can be derived in several ways, only the most straightforward derivation is mentioned. Also, special values of the parameter yielding trivial identities are omitted.

Additional transforms can be obtained dy differentiation of transforms VII and IX with respect to a. We only present the special result XXVII obtained by differentiating VII and setting $a = 3/4$. In that result, C is Catalan's constant given by

$$C = \frac{1}{16} [\psi'(\tfrac{1}{4}) - \psi'(\tfrac{3}{4})] = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} . \tag{13}$$

Transforms XI, XVI, XX, XII and XVII are equivalent (sometimes after integration by parts) to Ref. 6, pp.181-182, eq. (10) to (14), and transforms XIII and XVIII are trivial consequences. Transform I is equivalent to a result presented in an annex to Boersma's thesis (1964). In all these known cases, our proofs are much simpler than the original ones. All the other transforms are believed to be new.

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Table A $(0 \le a \le 1)$

	$f(z)$	$u(y)$
I	$\log \Gamma(z+a) + \log \Gamma(z+1-a) - 2z \log z + 2z$	$\frac{1}{2} \log \frac{2\pi^2 e^{2\pi y}}{\cosh(2\pi y) - \cos(2\pi a)}$
II	$z[\log \Gamma(z+a) - \log \Gamma(z+1-a) - (2a-1)\log z]$	$y \arctan[\cot(\pi a) \tanh(\pi y)] + (a - \frac{1}{2})\pi y$
III	$\frac{1}{z}[\log \Gamma(z+a) - \log \Gamma(z+1-a) - \log \Gamma(a) + \log \Gamma(1-a)]$	$-\frac{1}{y} \arctan[\cot(\pi a) \tanh(\pi y)]$
IV	$z[\psi(z+a) + \psi(z+1-a) - 2 \log z]$	$-\pi y \frac{\cos(2\pi a) - e^{-2\pi y}}{\cosh(2\pi y) - \cos(2\pi a)}$
V	$z^2[\psi(z+a) - \psi(z+1-a) - \frac{2a-1}{z}]$	$\pi y^2 \frac{\sin(2\pi a)}{\cosh(2\pi y) - \cos(2\pi a)}$
VI	$\psi(z+a) - \psi(z+1-a)$	$-\frac{\pi \sin(2\pi a)}{\cosh(2\pi y) - \cos(2\pi a)}$

Table B $(0 < a < 1)$

VII	$\frac{1}{z^2}[\log \Gamma(z+a) + \log \Gamma(z+1-a) - \log \frac{\pi}{\sin(\pi a)} - z\{\psi(a) + \psi(1-a)\}]$	$\frac{1}{2y^2} \log \frac{\cosh(2\pi y) - \cos(2\pi a)}{2 \sin^2(\pi a)}$
VIII	$\frac{1}{z}[\psi(z+a) + \psi(z+1-a) - \psi(a) - \psi(1-a)]$	$\frac{\pi}{y} \frac{\sinh(2\pi y)}{\cosh(2\pi y) - \cos(2\pi a)}$

Table C $(0 \le a < 1)$

IX	$\frac{1}{z^2}[\log \Gamma(z+1+a) + \log \Gamma(z+1-a) - \log \frac{\pi a}{\sin(\pi a)} - z\{\psi(1+a) + \psi(1-a)\}]$	$\frac{1}{2y^2} \log \left[\frac{a^2}{2 \sin^2(\pi a)} \cdot \frac{\cosh(2\pi y) - \cos(2\pi a)}{y^2 + a^2} \right]$
X	$\frac{1}{z}[\psi(z+1+a) + \psi(z+1-a) - \psi(1+a) - \psi(1-a)]$	$\frac{\pi}{y} \frac{\sinh(2\pi y)}{\cosh(2\pi y) - \cos(2\pi a)} - \frac{1}{y^2+a^2}$

Table D

XI	$\log\Gamma(z) - (z-\frac{1}{2})\log z + z$	$\frac{1}{2}\log\frac{2\pi}{1-e^{-2\pi y}}$	I for $a=0$ or 1
XII	$\log\Gamma(z+\frac{1}{2}) - z\log z + z$	$\frac{1}{2}\log\frac{2\pi}{1+e^{-2\pi y}}$	I for $a=\frac{1}{2}$
XIII	$\log\Gamma(z+\frac{1}{2}) - \log\Gamma(z) - \frac{1}{2}\log z$	$\frac{1}{2}\log\tanh(\pi y)$	XII minus XI
XIV	$z[\log\Gamma(z+\frac{3}{4}) - \log\Gamma(z+\frac{1}{4}) - \frac{1}{2}\log z]$	$\frac{\pi y}{4} - y\arctan[\tanh(\pi y)]$	II for $a=\frac{1}{4}$ or $\frac{3}{4}$
XV	$\frac{1}{z}[\log\Gamma(z+\frac{3}{4}) - \log\Gamma(z+\frac{1}{4}) - \log\Gamma(\frac{3}{4}) + \log\Gamma(\frac{1}{4})]$	$\frac{1}{y}\arctan[\tanh(\pi y)]$	III for $a=\frac{1}{4}$ or $\frac{3}{4}$
XVI	$z[\psi(z) - \log z] + \frac{1}{2}$	$\frac{\pi y}{1-e^{2\pi y}}$	IV for $a=0$ or 1
XVII	$z[\psi(z+\frac{1}{2}) - \log z]$	$\frac{\pi y}{1+e^{2\pi y}}$	IV for $a=\frac{1}{2}$
XVIII	$z[\psi(z+\frac{1}{2}) - \psi(z)] - \frac{1}{2}$	$\frac{\pi y}{\sinh(2\pi y)}$	XVII minus XVI
XIX	$z^2[\psi(z+\frac{3}{4}) - \psi(z+\frac{1}{4}) - \frac{1}{2z}]$	$-\frac{\pi y^2}{\cosh(2\pi y)}$	V for $a=\frac{1}{4}$ or $\frac{3}{4}$
XX	$\psi(z+\frac{3}{4}) - \psi(z+\frac{1}{4})$	$\frac{\pi}{\cosh(2\pi y)}$	VI for $a=\frac{1}{4}$ or $\frac{3}{4}$
XXI	$\frac{1}{z^2}[\log\Gamma(z+\frac{1}{2}) - \frac{1}{2}\log\pi + z(\gamma + 2\log 2)]$	$\frac{1}{2y^2}\log\cosh(\pi y)$	VII for $a=\frac{1}{2}$
XXII	$\frac{1}{z}[\psi(z+\frac{1}{2}) + \gamma + 2\log 2)]$	$\frac{\pi}{2y}\tanh(\pi y)$	VIII for $a=\frac{1}{2}$
XXIII	$\frac{1}{z^2}[\log\Gamma(z+1) + \gamma z]$	$\frac{1}{2y^2}\log\frac{\sinh(\pi y)}{\pi y}$	IX for $a=0$
XXIV	$\frac{1}{z^2}[\log\Gamma(z+\frac{1}{2}) - \log\Gamma(z+1) - \frac{1}{2}\log\pi + 2z\log 2]$	$\frac{1}{2y^2}\log[\pi y\coth(\pi y)]$	XXI minus XXIII
XXV	$\frac{1}{z}[\psi(z+1) + \gamma]$	$\frac{\pi}{2y}[\coth(\pi y) - \frac{1}{\pi y}]$	X for $a=0$
XXVI	$\frac{1}{z}[\psi(z+1) - \psi(z+\frac{1}{2}) - 2\log 2]$	$\frac{\pi}{y}[\frac{1}{\sinh(2\pi y)} - \frac{1}{2\pi y}]$	XXV minus XXII
XXVII	$\frac{1}{z^2}[\psi(z+\frac{3}{4}) - \psi(z+\frac{1}{4}) - \pi + 16Cz]$	$\frac{\pi}{y^2}[1 - \frac{1}{\cosh(2\pi y)}]$	see (13)