On interpolating periodic quintic spline functions with equally spaced nodes

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by

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1. Let C denote the Banach space (with supremum norm) of all real-valued, continuous, periodic functions with period 1. To each division of the interval $[0,1]$ into n subintervals $\{0 = x_0 < x_1 < \dots < x_{n-1} < x_n = 1\}$, there corresponds an n-dimensional subspace $S(x_0,x_1,\dots,x_n)$ in C whose members are the periodic quintic spline functions with nodes x_i. Thus, $s \in S(x_0,x_1,\dots,x_n)$ if and only if

1) $s \in C^4[0,1]$ and $s^{(i)}(0) = s^{(i)}(1)$, $(i = 0,1,2,3,4)$;
2) s reduces to a polynomial of degree at most five on each subinterval $[x_{i-1},x_i]$.

Throughout the paper we assume that the nodes are uniformly distributed on $[0,1]$, i.e. $x_i = i/n$ $(i = 0,1,\dots,n)$. We begin by proving two theorems which strengthen some results communicated in [2]. Then an exact expression is given for the norm of the interpolating periodic quintic spline operator. The last part of the paper contains estimates (which are best possible or nearly so) for the difference of the function f to be approximated and the associated spline in terms of the modulus of continuity of f. The contents of these sections (6 and 7) should be compared with reference [3], where similar results for the interpolating periodic cubic splines are derived by an analogous method.

2. It is known ([1], p. 135; [2]) that to each $f \in C$ there corresponds a uniquely determined element $s \in S(x_0,x_1,\dots,x_n)$ with the interpolating property, $s(x_i) = f(x_i)$ for $i = 1,\dots,n$. If we write

$$f_i = f(x_i)\,,\quad \bar{\lambda}_i = s'(x_i)\,,\quad \bar{\mu}_i = s''(x_i)\,,\quad \bar{m}_i = s'''(x_i)\,,$$

then using Hermite interpolation the quintic spline function s can be given explicitly in the following form on the interval $[x_{i-1},x_i]$:

$$(1)\quad s(x) = f_{i-1}A_i(x) + f_i B_i(x) + \bar{\lambda}_{i-1}C_i(x) + \bar{\lambda}_i D_i(x) + \bar{m}_{i-1}E_i(x) + \bar{m}_i F_i(x)\,.$$

Here $A_i(x),\dots,F_i(x)$ are certain quintic polynomials. If we denote these polynomials by $A(t),\dots,F(t)$ when $[x_{i-1},x_i]$ is replaced by $[0,1]$, we have

$$(2)\qquad A(t) = \tfrac{1}{2}(1-t)^2(-2t^3+t^2+4t+2)\ , \qquad B(t) = A(1-t)\ ,$$

$$(3)\qquad C(t) = \tfrac{1}{4}t(1-t)^2(-2t^2+t+4)\ , \qquad D(t) = -C(1-t)\ ,$$

$$(4)\qquad E(t) = \frac{1}{48}t^2(1-t)^2(2t-3)\ , \qquad F(t) = -E(1-t)\ .$$

The expressions for $A_i(x),\dots,F_i(x)$ are now obtained by setting $t=n(x-x_{i-1})$, multiplying $C_i(x)$, $D_i(x)$ by n^{-1} and $E_i(x)$, $F_i(x)$ by n^{-3}.
On $[x_{i-1},x_i]$ we have $A_i(x),B_i(x),C_i(x),F_i(x) \geq 0$, whereas $D_i(x),E_i(x) \leq 0$ on this interval. Moreover,

$$(5)\qquad A_i(x) + B_i(x) = 1\ ,$$

$$(6)\qquad C_i(x) - D_i(x) = (x-x_{i-1})(1-n(x-x_{i-1})) \leq \frac{1}{4n}\ ,$$

$$(7)\qquad F_i(x) - E_i(x) = \frac{1}{12n}(x-x_{i-1})^2\{1-n(x-x_{i-1})\}^2 \leq \frac{1}{192n^3}\ .$$

As a consequence of the fact that $s \in C^4[0,1]$ and $s^{(i)}(0) = s^{(i)}(1)$, $(i = 0,1,2,3,4)$, the parameters $\overline{\lambda}_i$, $\overline{\mu}_i$ and $\overline{m}_i$ have to satisfy some particular relations for $i = 1,2,\dots,n$, which were derived in [2]. Assuming that all indices which occur are interpreted modulo n, we have

$$(8)\quad \overline{\lambda}_{i-2} + 26\overline{\lambda}_{i-1} + 66\overline{\lambda}_i + 26\overline{\lambda}_{i+1} + \overline{\lambda}_{i+2} = 5n(f_{i+2}+10f_{i+1}-10f_{i-1}-f_{i-2})\ ,$$

$$(9)\quad \overline{\mu}_{i-2} + 26\overline{\mu}_{i-1} + 66\overline{\mu}_i + 26\overline{\mu}_{i+1} + \overline{\mu}_{i+2} = 20n^2(f_{i+2}+2f_{i+1}-6f_i+2f_{i-1}+f_{i-2})$$

$$(10)\quad \overline{m}_{i-2} + 26\overline{m}_{i-1} + 66\overline{m}_i + 26\overline{m}_{i+1} + \overline{m}_{i+2} = 60n^3(f_{i+2}-2f_{i+1}+2f_{i-1}-f_{i-2})\ ,$$

$$(11)\quad \overline{\mu}_i = \frac{5}{2}n^2(f_{i+1}-2f_i+f_{i-1}) + \frac{3}{4}n(\overline{\lambda}_{i-1}-\overline{\lambda}_{i+1}) + \frac{1}{48n}(\overline{m}_{i+1}-\overline{m}_{i-1})\ .$$

Due to the fact that the matrix associated with the two systems of equations (8) and (10) is diagonally dominant, it follows by a standard procedure (cf. [2]) that

$$(12)\qquad \max_i|\overline{\lambda}_i| \leq \frac{25}{6}n\ \omega(f;1/n)\ ,$$

$$(13)\qquad \max_i|\overline{m}_i| \leq 20n^3\ \omega(f;1/n)\ .$$

If $s^i(x) \in S(x_0,x_1,\ldots,x_n)$ denotes the i-th cardinal spline - this function is defined by the equation $s^i(x_j) = \delta^i_j$ for $i,j = 1,2,\ldots,n$ -, then in terms of these functions we have

$$(14) \qquad s \equiv L_n f = \sum_{i=1}^{n} f(x_i)s^i(x) \, .$$

Accordingly, the norm of the interpolating periodic quintic spline operator

$$\|L_n\| = \sup\{\|L_n f\| : f \in C, \ \|f\| = 1\}$$

is given by

$$(15) \qquad \|L_n\| = \Big\| \sum_{i=1}^{n} |s^i(x)| \Big\| \, .$$

3. We will now prove two theorems, which improve similar results given in [2] (cf. [2], theorems 3 and 4).

Theorem 1

Let f belong to C and let s be the interpolating periodic quintic spline function associated with f. Then

$$\|s - f\| \leq 2\,\tfrac{7}{48}\, \omega(f;1/n) \, .$$

Proof

Let x be an arbitrary point of $[0,1]$ and assume $x \in [x_{i-1},x_i]$. Using (1) and (5) we get

$$s(x) - f(x) = (f_{i-1}-f(x))A_i(x) + (f_i-f(x))B_i(x) + \bar{\lambda}_{i-1}C_i(x) + \bar{\lambda}_i D_i(x) + \bar{m}_{i-1}E_i(x) + \bar{m}_i F_i$$

We recall that on the interval $[x_{i-1},x_i]$ the functions $A_i(x)$, $B_i(x)$, $C_i(x)$ and $F_i(x)$ are nonnegative, while $D_i(x)$, $E_i(x) \leq 0$ there. Consequently one has

$$|s(x) - f(x)| \leq \omega(f;1/n) + \max_j|\bar{\lambda}_j|(C_i(x) - D_i(x)) + \max_j|\bar{m}_j|(F_i(x) - E_i(x)) \, .$$

From this we obtain the result of theorem 1 by a simple calculation using inequalities (6), (7), (12) and (13). ■

Theorem 2

A uniform upper bound for the norm of the interpolating periodic quintic spline operator L_n, as defined in (14), is given by

$$\|L_n\| \leq 3\,\frac{7}{24}\ .$$

Proof

As we already noted in (15), the norm of the quintic spline operator is equal to the Chebyshev norm of the function $\sum_{i=1}^{n} |s^i(x)|$. Select ξ such that $\|\sum_{i=1}^{n} |s^i(x)|\| = \sum_{i=1}^{n} |s^i(\xi)|$, and let f be a continuous function of norm 1 which satisfies the equations $f_i = \operatorname{sgn} s^i(\xi)$ and is linear in each interval $[x_{i-1}, x_i]$. Then $\|L_n\| = \|L_n f\| = \|s\|$. To determine an upper bound for $\|L_n\|$ it is sufficient to consider the spline function s on $[x_{i-1}, x_i]$. In view of (1) we have on this interval

$$s(x) = f_{i-1}\,A_i(x) + f_i\,B_i(x) + \bar{\lambda}_{i-1}\,C_i(x) + \bar{\lambda}_i\,D_i(x) + \bar{m}_{i-1}\,E_i(x) + \bar{m}_i\,F_i(x)\ ,$$

whence

$$|s(x)| \leq \|f\|(A_i(x)+B_i(x)) + \max_j|\bar{\lambda}_j|\,(C_i(x)-D_i(x)) + \max_j|\bar{m}_j|\,(F_i(x)-E_i(x)) \leq$$

$$\leq \|f\| + \frac{25}{6}\,n\,\omega(f;1/n)\,\frac{1}{4n} + 20n^3\,\omega(f;1/n)\,\frac{1}{192n^3} = 1 + \frac{55}{48}\,\omega(f;1/n)\ .$$

Here we have made use of the formulae (5), (6), (7), (12) and (13). Theorem 2 now follows by observing that $\omega(f;1/n) \leq 2$. ∎

4. In view of formulae (14), (15), it will be obvious that knowledge about the cardinal spline functions would be useful. Proceeding in a similar way as when investigating the interpolating periodic cubic splines (cf. [3]), the ultimate aim of this paper is to derive an exact expression for the norm of L_n ($n = 1,2,3,\ldots$). Moreover, an improved version of theorem 1 will be deduced.

The information about the cardinal spline functions which is needed to arrive at these results is given mostly in the form of lemmas and assertions. Because the calculations which are involved to prove these statements are often quite long and tedious, most of the details of their proofs will be omitted.

The purpose of the first two lemmas is to show how in an appropriate case the computation of the numbers $(s^i)'(x_j)$ and $(s^i)'''(x_j)$ is connected with a particular solution of a difference equation of order four. If $n = 2k$ or $n = 2k+1$, we put $\lambda_i = (s^k)'(x_i)$ and $m_i = (s^k)'''(x_i)$. Since the functions

s^i are periodic and the nodes are equally spaced, we have $s^i(x) = s^k(x - x_{i-k})$. Thus it is only necessary to compute one cardinal quintic spline function, and we choose s^k. Then

$$(s^i)'(x_j) = (s^k)'(x_j - x_{i-k}) = (s^k)'(x_{j-i+k}) = \lambda_{j-i+k} \,.$$

Consequently, on the interval $[x_{j-1}, x_j]$ we get

$$(16)\quad \begin{cases} s^i(x) = \delta^i_{j-1} A_j(x) + \delta^i_j B_j(x) + (s^i)'(x_{j-1})C_j(x) + \\ \quad + (s^i)'(x_j)D_j(x) + (s^i)'''(x_{j-1})E_j(x) + (s^i)'''(x_j)F_j(x) = \\ \quad = \delta^i_{j-1} A_j(x) + \delta^i_j B_j(x) + \lambda_{j-i+k-1} C_j(x) + \lambda_{j-i+k} D_j(x) + \\ \quad + m_{j-i+k-1} E_j(x) + m_{j-i+k} F_j(x) \,. \end{cases}$$

In order to compute $s^k(x)$, we first have to rewrite equations (8) and (10) in their appropriate form. We get

$$(17)\quad \lambda_{i-2} + 26\lambda_{i-1} + 66\lambda_i + 26\lambda_{i+1} + \lambda_{i+2} = 5n(\delta^k_{i+2} + 10\delta^k_{i+1} - 10\delta^k_{i-1} - \delta^k_{i-2}) \,, \quad (i=1,2,\ldots$$

$$(18)\quad m_{i-2} + 26m_{i-1} + 66m_i + 26m_{i+1} + m_{i+2} = 60n^3(\delta^k_{i+2} - 2\delta^k_{i+1} + 2\delta^k_{i-1} - \delta^k_{i-2}) \,.$$

<u>Lemma 1</u>

Let $n = 2k$ $(k = 1,2,\ldots)$ and let $\{\rho_{-1}, \rho_0, \rho_1, \ldots, \rho_{k+1}\} = \{a_{-1} = -a_1, a_0 = 0, a_1, \ldots, a_{k+1}\}$ be a non-trivial solution of the difference equation

$$(19)\quad \rho_{i+1} - 26\rho_i + 66\rho_{i-1} - 26\rho_{i-2} + \rho_{i-3} = 0 \,, \quad (i = 2,3,\ldots,k) \,,$$

which satisfies the end condition

$$(20)\quad 16\rho_k - \rho_{k-1} - \rho_{k+1} = 0 \,.$$

If we put

$$(21)\quad \begin{cases} \lambda_i = (-1)^{k+i+1}\, 5n\, a_k^{-1} a_i \,, & (i = 0,1,\ldots,k-1) \,, \\ \lambda_i = -\lambda_{2k-i} \,, & (i = k,k+1,\ldots,2k) \,, \end{cases}$$

then $\{\lambda_0, \lambda_1, \ldots, \lambda_{2k}\}$ is the solution of (17).

When dealing with the third derivatives m_i of the cardinal spline s^k, condition (20) has to be replaced by

(22) $$28\rho_k - \rho_{k-1} - \rho_{k+1} = 0 .$$

Let the set $\{\rho_{-1},\rho_0,\rho_1,\ldots,\rho_{k+1}\} = \{a^*_{-1} = -a^*_1, a^*_0 = 0, a^*_1,\ldots,a^*_{k+1}\}$ be a non-trivial solution of (19), which satisfies relation (22). If we define

(23) $$\begin{cases} m_i = (-1)^{k+i+1}\, 60n^3\, a_k^{*-1} a_i^* , & (i = 0,1,\ldots,k-1) , \\ m_i = -\, m_{2k-i} , & (i = k,k+1,\ldots,2k) , \end{cases}$$

then (23) is the solution of system (18).

Lemma 2

Let $n = 2k+1$ $(k = 1,2,\ldots)$ and let $\{\rho_{-2},\rho_{-1},\rho_0,\rho_1,\ldots,\rho_{k+1}\} =$ $= \{b_{-2} = b_1, b_{-1} = b_0, b_0, b_1,\ldots,b_{k+1}\}$ be a non-trivial solution of the difference equation (19) for $i = 1,2,\ldots,k$, which satisfies (20). Then $\{\lambda_0,\lambda_1,\ldots,\lambda_{2k}\}$ with

(24) $$\begin{cases} \lambda_i = (-1)^{k+i+1}\, 5n\, b_k^{-1} b_i , & (i = 0,1,\ldots,k-1) , \\ \lambda_i = -\, \lambda_{2k-i} , & (i = k,k+1,\ldots,2k) \end{cases}$$

is the solution of (17).

Assume now that $\{\rho_{-2},\rho_{-1},\rho_0,\rho_1,\ldots,\rho_{k+1}\} = \{b^*_{-2} = b^*_1, b^*_{-1} = b^*_0, b^*_0, b^*_1,\ldots,b^*_{k+1}\}$ is a non-trivial solution of (19) for $i = 1,2,\ldots,k$, which has property (22). Then

(25) $$\begin{cases} m_i = (-1)^{k+i+1}\, 60n^3\, b_k^{*-1} b_i^* , & (i = 0,1,\ldots,k-1) , \\ m_i = -\, m_{2k-i} , & (i = k,k+1,\ldots,2k) \end{cases}$$

solves the set of equations (18).

Proof

There are apparently four different cases to be considered, all of which can be dealt with in a similar way. Therefore we only prove the first part of lemma 1.

If $i = 1,2,\ldots,k-3$, then using (21) we have

$$\lambda_{i-2} + 26\lambda_{i-1} + 66\lambda_i + 26\lambda_{i+1} + \lambda_{i+2} =$$

$$= (-1)^{k+i-1}\, 5n\, a_k^{-1} \{a_{i-2} - 26a_{i-1} + 66a_i - 26a_{i+1} + a_{i+2}\} = 0 =$$

$$= 5n(\delta_{i+2}^k + 10\delta_{i+1}^k - 10\delta_{i-1}^k - \delta_{i-2}^k) .$$

All indices of the parameter λ have to be taken modulo n (= 2k). As a consequence of (21), $\lambda_{-1} = \lambda_{2k-1} = -\lambda_1$; this gives rise to the definition $a_{-1} = -a_1$.

In case $i = k-2$ and noting that $\lambda_k = 0$, we have

$$\lambda_{k-4} + 26\lambda_{k-3} + 66\lambda_{k-2} + 26\lambda_{k-1} + \lambda_k = -5n\, a_k^{-1}(a_{k-4} - 26a_{k-3} + 66a_{k-2} - 26a_{k-1}) =$$

$$= -5n\, a_k^{-1}(-a_k) = 5n = 5n(\delta_k^k + 10_{k-1}^k - 10_{k-3}^k - \delta_{k-4}^k) .$$

Assume now that $i = k-1$. In view of (21) and (20) one gets

$$\lambda_{k-3} + 26\lambda_{k-2} + 66\lambda_{k-1} + 26\lambda_k + \lambda_{k+1} = 5n\ a_k^{-1}(a_{k-3} - 26a_{k-2} + 66a_{k-1} - a_{k-1}) =$$

$$= 5n\ a_k^{-1}(26a_k - a_{k+1} - a_{k-1}) = 50n = 5n(\delta_{k+1}^k + 10\delta_k^k - 10\delta_{k-2}^k - \delta_{k-3}^k) .$$

For the case $i = k$ we have

$$\lambda_{k-2} + 26\lambda_{k-1} + 66\lambda_k + 26\lambda_{k+1} + \lambda_{k+2} = 0 = 5n(\delta_{k+2}^k + 10\delta_{k+1}^k - 10\delta_{k-1}^k - \delta_{k-2}^k) .$$

Taking into account the symmetry-relation of formula (21), the cases $i = k+1,\dots,2k$ can be dealt with in an analysis which is the same as the one just given. ■

As a consequence of lemmas 1 and 2, it becomes necessary to solve the fourth-order difference equation (19) with end conditions (20) and (22) respectively, in order to get explicit expressions for the first and third quintic spline derivatives at the nodes. It is obvious that for the determination of λ_i in case $n = 2k$ the ratio of a_i and a_k is the only important thing. Without any restriction one can therefore assume $a_1 = 1$, $a_2 = \alpha$ in case $n = 2k$ and $b_0 = 1$, $b_1 = \beta$ in case $n = 2k+1$, when dealing with the first derivatives. The numbers $a_3, a_4, \dots, a_{k+1}$ and $b_2, b_3, \dots, b_{k+1}$ then can be calculated successively from recurrence relation (19); the quantities α and β are determined by means of (20). In the same way we put $a_1^* = 1$, $a_2^* = \alpha^*$ when $n = 2k$ and $b_0^* = 1$, $b_1^* = \beta^*$ in case $n = 2k+1$; after generating

$a_3^*,\dots,a_{k+1}^*$ and $b_2^*,\dots,b_{k+1}^*$, the unknown numbers α^* and β^* follow from condition (22). We remark that the values of α, β, α^*, β^* are dependent on k and have to be determined each time anew. To fix ideas, let us write down the first elements of the number sequence $a_0, a_1, a_2, \dots$. We get

(26) $a_0 = 0,\ a_1 = 1,\ a_2 = \alpha,\ a_3 = 26\alpha - 65,\ a_4 = 610\alpha - 1664,\ a_5 = 14170\alpha - 38975, \dots$.

Because the difference equation (19) is homogeneous, we can form two number sequences $\{\rho_i^{(1)}\}$, $\{\rho_i^{(2)}\}$ $(i = 0,1,\dots)$ out of (26), which are again solutions of (19). Moreover, using (26), condition (20) gives rise to another sequence, which can be split up in two sequences, all of which are solutions of (19). For all these sequences there can be written down an explicit formula, which makes it possible to determine the value of α for each fixed number k. The other three sequences $\{b_0, b_1, \dots\}$, $\{a_0^*, a_1^*, \dots\}$, $\{b_0^*, b_1^*, \dots\}$ can be treated in a similar way. When dealing with the first derivatives λ_i of the cardinal spline function s^k, a simple calculation gives the following results.
In case $n = 2k$ $(k = 1,2,\dots)$ we have

(27) $\{\rho_i^{(1)}\}\ (i = 0,1,\dots) = \{0, 0, 1, 26, 610, 14170, \dots\}$,

(28) $\{\rho_i^{(2)}\}\ (i = 0,1,\dots) = \{0, 1, 0, -65, -1664, -38975, \dots\}$,

(29) $\{16\rho_k^{(1)} - \rho_{k-1}^{(1)} - \rho_{k+1}^{(1)}\}\ (k = 1,2,\dots) = \{-1, -10, -195, -4436, -102725, \dots\}$,

(30) $\{16\rho_k^{(2)} - \rho_{k-1}^{(2)} - \rho_{k+1}^{(2)}\}\ (k = 1,2,\dots) = \{16, 64, 624, 12416, 283280, \dots\}$.

If $n = 2k+1$ $(k = 1,2,\dots)$, then

(31) $\{\rho_i^{(3)}\}\ (i = 0,1,\dots) = \{0, 1, 25, 584, 13560, 314665, \dots\}$,

(32) $\{\rho_i^{(4)}\}\ (i = 0,1,\dots) = \{1, 0, -40, -1015, -23751, -551576, \dots\}$,

(33) $\{16\rho_k^{(3)} - \rho_{k-1}^{(3)} - \rho_{k+1}^{(3)}\}\ (k = 1,2,\dots) = \{-9, -185, -4241, -98289, \dots\}$,

(34) $\{16\rho_k^{(4)} - \rho_{k-1}^{(4)} - \rho_{k+1}^{(4)}\}\ (k = 1,2,\dots) = \{39, 375, 7551, 172575, \dots\}$.

We recall that in case of the third derivatives m_i condition (20) has to be replaced by (22). As a consequence of this, the sequences (29), (30), (33) and (34) are transformed as follows.

If $n = 2k$ $(k = 1,2,\ldots)$, then

$$(35) \qquad \{28\rho_k^{(1)} - \rho_{k-1}^{(1)} - \rho_{k+1}^{(1)}\} \; (k = 1,2,\ldots) = \{-1, 2, 117, 2884, 67315, \ldots\} ,$$

$$(36) \qquad \{28\rho_k^{(2)} - \rho_{k-1}^{(2)} - \rho_{k+1}^{(2)}\} \; (k = 1,2,\ldots) = \{28, 64, -156, -7552, -184420, \ldots\} .$$

In case $n = 2k+1$ $(k = 1,2,\ldots)$ we get

$$(37) \qquad \{28\rho_k^{(3)} - \rho_{k-1}^{(3)} - \rho_{k+1}^{(3)}\} \; (k = 1,2,\ldots) = \{3, 115, 2767, 64431, \ldots\} ,$$

$$(38) \qquad \{28\rho_k^{(4)} - \rho_{k-1}^{(4)} - \rho_{k+1}^{(4)}\} \; (k = 1,2,\ldots) = \{39, -105, -4629, -112437, \ldots\} .$$

In order to give explicit formulae for the sequences (27), (28), ..., (38), we need the general solution of the difference equation (19). This turns out to be

$$(39) \qquad \rho_i = C_1 z_1^i + C_2 z_2^i + C_3 z_2^{-i} + C_4 z_1^{-i} ,$$

where

$$(40) \qquad z_1 = \tfrac{1}{2}(13 + \sqrt{105} - \sqrt{270 + 26\sqrt{105}}) = 0.04309\ldots ,$$

$$(41) \qquad z_2 = \tfrac{1}{2}(13 - \sqrt{105} - \sqrt{270 - 26\sqrt{105}}) = 0.43057\ldots .$$

For each of the sequences (27), (28), (31), (32), the constants $C_1,\ldots,C_4$ have to be determined from the initial elements of these sequences. This can be done by means of generating functions in the following way. Put

$$(42) \qquad G_j(z) = \sum_{i=0}^{\infty} \rho_i^{(j)} z^i , \quad (j = 1,2,3,4) ,$$

where the numbers $\rho_i^{(j)}$ satisfy the difference equation (19). An elementary calculation shows that

$$(43) \quad G_j(z) = \frac{(\rho_3^{(j)} - 26\rho_2^{(j)} + 66\rho_1^{(j)} - 26\rho_0^{(j)})z^3 + (\rho_2^{(j)} - 26\rho_1^{(j)} + 66\rho_0^{(j)})z^2 + (\rho_1^{(j)} - 26\rho_0^{(j)})z + \rho_0^{(j)}}{z^4 - 26z^3 + 66z^2 - 26z + 1}$$

If we put

$$\begin{cases} P(z) = z^4 - 26z^3 + 66z^2 - 26z + 1 , \\ Q(z) = z(z^2 - 26z + 1) , \\ R(z) = - z^2 + z , \\ S(z) = - z^3 + 26z^2 - 26z + 1 , \end{cases}$$

and apply (43) to the sequences (27), (28), (31), (32), then we get the following results:

$$(\rho_0^{(1)},\rho_1^{(1)},\rho_2^{(1)},\rho_3^{(1)}) = (0,0,1,26) \Rightarrow G_1(z) = \frac{z^2}{P(z)} ,$$

$$(\rho_0^{(2)},\rho_1^{(2)},\rho_2^{(2)},\rho_3^{(2)}) = (0,1,0,-65) \Rightarrow G_2(z) = \frac{Q(z)}{P(z)} ,$$

$$(\rho_0^{(3)},\rho_1^{(3)},\rho_2^{(3)},\rho_3^{(3)}) = (0,1,25,584) \Rightarrow G_3(z) = \frac{R(z)}{P(z)} ,$$

$$(\rho_0^{(4)},\rho_1^{(4)},\rho_2^{(4)},\rho_3^{(4)}) = (1,0,-40,-1015) \Rightarrow G_4(z) = \frac{S(z)}{P(z)} .$$

The polynomial $P(z)$ can be written in the form

$$(z-z_1)(z-z_2)(z-z_3)(z-z_4) ,$$

where z_1 and z_2 are given by (40), (41) and $z_3 = z_2^{-1}$, $z_4 = z_1^{-1}$. A computation shows that

$$z_3 = 2.32\ldots ,$$

$$z_4 = 23.2\ldots .$$

Thus

$$z_1 < z_2 < z_3 < z_4 ,$$

and the root z_4 of $P(z) = 0$ is highly dominant over the other ones. This fact will be of importance in subsequent estimations.

To determine the unknown coefficients $C_1,\ldots,C_4$ in formula (39) when dealing with the sequences (27), (28), (31), (32), we proceed as follows. We can write successively

$$(44)\qquad \left\{ \begin{aligned} G_1(z) &= \frac{C_1}{z-z_1} + \frac{C_2}{z-z_2} + \frac{C_3}{z-z_3} + \frac{C_4}{z-z_4} , & C_j &= \frac{z_j^2}{P'(z_j)} , \\ G_2(z) &= \frac{C_1^*}{z-z_1} + \frac{C_2^*}{z-z_2} + \frac{C_3^*}{z-z_3} + \frac{C_4^*}{z-z_4} , & C_j^* &= \frac{Q(z_j)}{P'(z_j)} , \\ G_3(z) &= \frac{D_1}{z-z_1} + \frac{D_2}{z-z_2} + \frac{D_3}{z-z_3} + \frac{D_4}{z-z_4} , & D_j &= \frac{R(z_j)}{P'(z_j)} , \\ G_4(z) &= \frac{D_1^*}{z-z_1} + \frac{D_2^*}{z-z_2} + \frac{D_3^*}{z-z_3} + \frac{D_4^*}{z-z_4} , & D_j^* &= \frac{S(z_j)}{P'(z_j)} . \end{aligned} \right. \qquad (j=1,2,3,4)$$

Now we have for instance

$$G_1(z) = -C_1 z_1^{-1}\left(1 + \frac{z}{z_1} + \left(\frac{z}{z_1}\right)^2 + \dots\right) - C_2 z_2^{-1}\left(1 + \frac{z}{z_2} + \left(\frac{z}{z_2}\right)^2 + \dots\right) +$$

$$- C_3 z_3^{-1}\left(1 + \frac{z}{z_3} + \left(\frac{z}{z_3}\right)^2 + \dots\right) - C_4 z_4^{-1}\left(1 + \frac{z}{z_4} + \left(\frac{z}{z_4}\right)^2 + \dots\right) .$$

In view of this and (42), (39), there hold the following explicit formulae for the elements of the sequences (27), (28), (31), (32), respectively:

$$\rho_i^{(1)} = -C_1(z_1)^{-i-1} - C_2(z_2)^{-i-1} - C_3(z_3)^{-i-1} - C_4(z_4)^{-i-1} , \tag{45}$$

$$\rho_i^{(2)} = -C_1^*(z_1)^{-i-1} - C_2^*(z_2)^{-i-1} - C_3^*(z_3)^{-i-1} - C_4^*(z_4)^{-i-1} , \tag{46}$$

$$(i = 0,1,2,\dots$$

$$\rho_i^{(3)} = -D_1(z_1)^{-i-1} - D_2(z_2)^{-i-1} - D_3(z_3)^{-i-1} - D_4(z_4)^{-i-1} , \tag{47}$$

$$\rho_i^{(4)} = -D_1^*(z_1)^{-i-1} - D_2^*(z_2)^{-i-1} - D_3^*(z_3)^{-i-1} - D_4^*(z_4)^{-i-1} . \tag{48}$$

In this set of formulae the coefficients $C_j,\dots,D_j^*$ $(j = 1,2,3,4)$ are given by (44). Moreover, because $z_4 \sim 540z_1$ and $z_3 \sim 5z_2$, it is obvious that already for rather small values of i the first term in each of the formulae (45),...,(48) is by far the largest one, and the contribution of the last two terms is very small.

Now we are ready to state lemma 3.

<u>Lemma 3</u>

The first and third derivatives λ_i and m_i $(i = 0,1,\dots,k-1;\ k = 1,2,\dots)$ of the cardinal quintic spline function s^k are given by the following formulae:

$$\lambda_i = (-1)^{k+i+1}\, 5n\, a_k^{-1} a_i , \qquad (n = 2k,\ k = 1,2,\dots) ,$$

$$\lambda_i = (-1)^{k+i+1}\, 5n\, b_k^{-1} b_i , \qquad (n = 2k+1,\ k = 1,2,\dots) ,$$

$$m_i = (-1)^{k+i+1}\, 60n^3\, a_k^{*-1} a_i^* , \qquad (n = 2k,\ k = 1,2,\dots) ,$$

$$m_i = (-1)^{k+i+1}\, 60n^3\, b_k^{*-1} b_i^* , \qquad (n = 2k+1,\ k = 1,2,\dots) ,$$

where

$$(49)\qquad a_i = \rho_i^{(1)} \frac{(16\rho_k^{(2)} - \rho_{k-1}^{(2)} - \rho_{k+1}^{(2)})}{(-16\rho_k^{(1)} + \rho_{k-1}^{(1)} + \rho_{k+1}^{(1)})} + \rho_i^{(2)} ,$$

$$(50)\qquad b_i = \rho_i^{(3)} \frac{(16\rho_k^{(4)} - \rho_{k-1}^{(4)} - \rho_{k+1}^{(4)})}{(-16\rho_k^{(3)} + \rho_{k-1}^{(3)} + \rho_{k+1}^{(3)})} + \rho_i^{(4)} ,$$

$$(i = 0,1,\ldots,k-1)$$

$$(51)\qquad a_i^* = \rho_i^{(1)} \frac{(28\rho_k^{(2)} - \rho_{k-1}^{(2)} - \rho_{k+1}^{(2)})}{(-28\rho_k^{(1)} + \rho_{k-1}^{(1)} + \rho_{k+1}^{(1)})} + \rho_i^{(2)} ,$$

$$(52)\qquad b_i^* = \rho_i^{(3)} \frac{(28\rho_k^{(4)} - \rho_{k-1}^{(4)} - \rho_{k+1}^{(4)})}{(-28\rho_k^{(3)} + \rho_{k-1}^{(3)} + \rho_{k+1}^{(3)})} + \rho_i^{(4)} .$$

Proof

The first part of the lemma is a partial restatement of the contents of lemmas 1 and 2. Without lack of generality we only examine formula (49). It gives an expression for a_i $(i = 0,1,\ldots,k-1)$, where $\{a_i\}$ is given by (26). We recall that the sequence (26) was split up into two sequences (27) and (28), the elements of which we denoted by $\rho_i^{(1)}$ and $\rho_i^{(2)}$. An explicit formula for them is given in (45) and (46). Finally, the quantity α has to be determined from (20); in our terminology we have

$$\alpha = \frac{16\rho_k^{(2)} - \rho_{k-1}^{(2)} - \rho_{k+1}^{(2)}}{-16\rho_k^{(1)} + \rho_{k-1}^{(1)} + \rho_{k+1}^{(1)}} .$$

This proves (49); the other expressions are derived in quite a similar way. ■

5. Using lemmas 1, 2, 3 and formula (16) all cardinal spline functions can be completely determined. But the expressions with which we have to deal with are not so tractable. Indeed, the set of formulae (21), (23), (24), (25), (44), (45), (46), (47), (48), (49), (50), (51), (52) are all needed to describe the behaviour of the cardinal splines. In this section we give additional information about the first and third derivatives of the spline s^k. The assertions are based upon the contents of lemma 3 and can be proved by elementary, yet tedious, calculations. These calculations are too lengthy to be given here.

We first state the relations between the coefficients C_j, C_j^*, D_j, D_j^*, which are used for the proof of assertions 3, 4, 5 and which we will need again in the sequel.

If we write

$$c_{12} = C_1^*C_2 - C_1C_2^* , \quad d_{12} = D_1^*D_2 - D_1D_2^* ,$$

and use similar abbreviations for other expressions of this kind, then we have

<u>Assertion 1</u>

(53) $\quad c_{12}, c_{24} < 0,\ c_{14} = c_{23} = 0,\ c_{13}, c_{34} > 0,$

(54) $\quad d_{12}, d_{13} < 0,\ d_{14} = d_{23} = 0,\ d_{24}, d_{34} > 0.$

<u>Assertion 2</u>

(55) $\quad c_{12} - c_{24}z_1^2 = 0,\ c_{13} - c_{34}z_1^2 = 0,\ c_{12} + c_{13}z_2^2 = 0,\ c_{24} + c_{34}z_2^2 = 0 ,$

(56) $\quad d_{12} + d_{24}z_1 = 0,\ d_{13} + d_{34}z_1 = 0,\ d_{12} - d_{13}z_2 = 0,\ d_{24} - d_{34}z_2 = 0 ,$

where z_1 and z_2 are given by (40), (41), respectively.

<u>Proof</u>

The results of these two assertions can be verified by simple calculations based upon formula (44). ■

The form of the expressions for λ_i and m_i $(i = 0,1,\ldots,k-1)$, as stated in lemmas 1, 2, 3, suggest that the first and third derivatives of the cardinal spline function s^k alternate in sign. This indeed is true, as follows from

<u>Assertion 3</u>

Let k be an arbitrary, but fixed, positive integer. Then

(57) $\quad a_i, b_i > 0 , \quad (i = 1,2,\ldots,k) ,$

(58) $\quad a_i^*, b_i^* > 0 , \quad (i = 1,2,\ldots,k-1) ,$

(59) $\quad a_k^*, b_k^* < 0 ,$

where a_i, b_i, a_i^*, b_i^* are given by (49), (50), (51), (52).

Proof

In order to verify that $a_i > 0$, $(i = 1,2,\dots,k)$, k being a fixed positive integer, a brief examination indicates that it is sufficient to show that

$$\text{(i)} \qquad \frac{-\rho_i^{(2)}}{\rho_i^{(1)}} < \frac{-\rho_{i+1}^{(2)}}{\rho_{i+1}^{(1)}}\ , \quad (i = 2,3,\dots)\ ,$$

$$\text{(ii)} \qquad \frac{16\rho_i^{(2)} - \rho_{i-1}^{(2)} - \rho_{i+1}^{(2)}}{-16\rho_i^{(1)} + \rho_{i-1}^{(1)} + \rho_{i+1}^{(1)}} > \frac{16\rho_{i+1}^{(2)} - \rho_i^{(2)} - \rho_{i+2}^{(2)}}{-16\rho_{i+1}^{(1)} + \rho_i^{(1)} + \rho_{i+2}^{(1)}}\ , \quad (i = 2,3,\dots)\ ;$$

this in view of the fact that both sequences $\left\{\frac{-\rho_i^{(2)}}{\rho_i^{(1)}}\right\}$ $(i = 2,3,\dots)$ and $\left\{\frac{16\rho_i^{(2)} - \rho_{i-1}^{(2)} - \rho_{i+1}^{(2)}}{-16\rho_i^{(1)} + \rho_{i-1}^{(1)} + \rho_{i+1}^{(1)}}\right\}$ $(i = 1,2,\dots)$ have the same limit, viz. $\left|\frac{C_1^*}{C_1}\right| = 13 - \sqrt{105}$.

Inequality (i) can be established by using assertion 1. Its proof rests heavily upon the fact that there is a considerable difference in magnitude between the numbers z_1, z_2, z_3, z_4 and, moreover, that $c_{12} < 0$. Verification of the second part of formula (57) can be done in exactly the same way. Formulae (58) and (59) may be proved in the same fashion. For instance, in case $b_i^* > 0$, $(i = 1,2,\dots,k-1)$ and $b_k^* < 0$, one has to establish that

$$\text{(iii)} \qquad \frac{-\rho_i^{(4)}}{\rho_i^{(3)}} < \frac{-\rho_{i+1}^{(4)}}{\rho_{i+1}^{(3)}}\ , \quad (i = 1,2,3,\dots)\ ,$$

$$\text{(iv)} \qquad \frac{-28\rho_i^{(4)} + \rho_{i-1}^{(4)} + \rho_{i+1}^{(4)}}{28\rho_i^{(3)} - \rho_{i-1}^{(3)} - \rho_{i+1}^{(3)}} < \frac{-28\rho_{i+1}^{(4)} + \rho_i^{(4)} + \rho_{i+2}^{(4)}}{28\rho_{i+1}^{(3)} - \rho_i^{(3)} - \rho_{i+2}^{(3)}}\ , \quad (i = 2,3,\dots)\ ,$$

$$\text{(v)} \qquad \frac{-\rho_{i-1}^{(4)}}{\rho_{i-1}^{(3)}} < \frac{-28\rho_i^{(4)} + \rho_{i-1}^{(4)} + \rho_{i+1}^{(4)}}{28\rho_i^{(3)} - \rho_{i-1}^{(3)} - \rho_{i+1}^{(3)}} < \frac{-\rho_i^{(4)}}{\rho_i^{(3)}}\ , \quad (i = 2,3,\dots)\ .$$

To prove these three inequalities we have to use the already mentioned information about the roots z_1, z_2, z_3, z_4 of $P(z) = 0$ and (54) of assertion 1. We omit the details. ■

The next assertion states that the absolute values of the first and third derivatives λ_i and m_i $(i = 0,1,\dots,k-1)$ are decreasing if we move from the node x_k to the left. There is only one exception to this rule and this occurs for the third derivative in case $n = 5$; indeed, $|m_0| > |m_1|$, as can be verified by a simple calculation. Moreover, $\lambda_0 = m_0 = 0$ in case n is even; this follows from lemma 3.

Assertion 4

Let n be an arbitrary but fixed positive integer and put $n = 2k$ when n is even, respectively $n = 2k+1$ when n is odd. Then

$$|\lambda_0| < |\lambda_1| < \dots < |\lambda_{k-1}|$$

and

$$|m_0| < |m_1| < \dots < |m_{k-1}| , \quad (n \neq 5) .$$

Proof

In case n is small, say $n \leq 6$, the asserted inequalities can be verified by calculating the values of λ_i and m_i by using lemma 3 or directly from equations (17), (18). The establishment of the remaining values of n involves quite large formulae; we only give a sketch of the proof. Starting out with the expressions for the λ_i and m_i as given in lemma 3, the calculations make use of assertions 1 and 2. Moreover, we exploit the fact that $z_1 < z_2 < z_3 < z_4$ and use some more additional information about these numbers. We omit all further details. ■

Now define $\sigma_{-1} = 0$, $\sigma_i = \lambda_0 + \lambda_1 + \dots + \lambda_i$ for $i = 0,1,\dots,k$ and in the same way $\pi_{-1} = 0$, $\pi_i = m_0 + m_1 + \dots + m_i$ $(i = 0,1,\dots,k)$. Using assertions 3 and 4, it follows by mathematical induction that $|\sigma_i| < |\lambda_{i+1}|$ $(i = 0,1,\dots,k-2)$ and $|\pi_i| < |m_{i+1}|$ $(i = 0,1,\dots,k-2;\ n \neq 5)$. Moreover, $\operatorname{sgn} \sigma_i = \operatorname{sgn}(\lambda_i + \sigma_{i-1}) = \operatorname{sgn} \lambda_i = (-1)^{k+1+i}$ for $i \in \{0,1,\dots,k-1\}$. Also we have $\operatorname{sgn} \pi_i = (-1)^{k+i}$, $i = 0,\dots,k-1$; $n \neq 5$.

Assertion 5

If $n \neq 3$, then

(60) $$|\sigma_k| < n ,$$

(61) $$|\pi_k| < 4n^3 .$$

Proof

As a consequence of the remarks preceding assertion 5 we have

$$\sigma_k = \sigma_{k-1} = \lambda_{k-1} + \sigma_{k-2} = \lambda_{k-1} - |\sigma_{k-2}| < \lambda_{k-1} ,$$

$$\pi_k = \pi_{k-1} = m_{k-1} + \pi_{k-2} = m_{k-1} + |\pi_{k-2}| > m_{k-1} .$$

It is thus sufficient to prove that for all values of $n \neq 3$

(62) $|\lambda_{k-1}| < n$,

(63) $|m_{k-1}| < 4n^3$;

it can be verified by a separate calculation that inequalities (60), (61) do not hold in case $n = 3$. No further details of the proof will be given here. Just as when dealing with assertion 4, the whole analysis is rather tedious. The calculations involved are based upon lemma 3, the formulae (53), (54), (55), (56) of assertions 1 and 2 and some information about the roots z_i $(i = 1,2,3,4)$ of the polynomial $P(z)$. ■

We end this section by proving a result about the magnitude of the second cardinal spline derivatives μ_i $(i=0,1,\dots,k)$.

Assertion 6

If μ_i denotes the second derivative of the cardinal spline function s^k at the node x_i, then $|\mu_k| > |\mu_i|$, $(i \neq k)$.

Proof

In view of (9) we have

$$\mu_{i-2} + 26\mu_{i-1} + 66\mu_i + 26\mu_{i+1} + \mu_{i+2} = 20n^2(\delta^k_{i+2} + 2\delta^k_{i+1} - 6\delta^k_i + 2\delta^k_{i-1} + \delta^k_{i-2}) .$$

Denote the right-hand side of this equation by R^k_i and assume that $\max_i|\mu_i| = |\mu_s|$. Then we get

$$66|\mu_s| = |20n^2 R^k_s - \mu_{s-2} - 26\mu_{s-1} - 26\mu_{s+1} - \mu_{s+2}| \leq 20n^2|R^k_s| + 54|\mu_s| .$$

Thus $|\mu_s| \leq \frac{5}{3}n^2|R^k_s|$. Because $R^k_i = 0$ when $i \neq k-2,k-1,k,k+1,k+2$, it follows that

$$\max_i|\mu_i| = \max\{|\mu_{k-2}|,|\mu_{k-1}|,|\mu_k|,|\mu_{k+1}|,|\mu_{k+2}|\} .$$

Now take into account formula (11). Using assertions 3, 4 and inequalities (62), (63), it is for our purpose sufficient to observe that $|\mu_k| > 3n^2$ and $|\mu_i| < \frac{5}{2}n^2$ $(i \neq k)$. These inequalities also hold in case $n = 3$. This proves the assertion. ■

Remark. Taking into account the contents of assertions 3 and 4, it is a consequence of formula (11) that the second derivatives of the cardinal spline function s^k also alternate in sign, with in particular $\mu_k < 0$.

6. In this section we will first deduce some expressions for the norm of the interpolating periodic quintic spline operator, which involve the values of the various cardinal spline derivatives at the nodes. Then an intricate formula will be derived by which it is possible to compute the exact value of $\|L_n\|$ for each positive integer n. A few conclusions will be drawn from this formula (theorem 4). We close this section by giving some numerical results.

Lemma 4

If the numbers λ_i and m_i $(i = 0,1,\ldots,n)$ are defined as in lemma 3, then the norm of the interpolating periodic quintic spline operator L_n is given by

$$\|L_n\| = 1 + \frac{1}{4n} \sum_{i=1}^{n} |\lambda_i| + \frac{1}{192n^3} \sum_{i=1}^{n} |m_i| . \tag{64}$$

Proof

In view of (15) we know that the norm of L_n is equal to the Chebyshev norm of the function $\sum_{i=1}^{n} |s^i|$. Select x so that $\|L_n\| = \sum_{i=1}^{n} |s^i(x)|$ and select j such that $x_{j-1} \leq x \leq x_j$. By equation (16) we have

$$\sum_{i=1}^{n} |s^i(x)| =$$

$$= \sum_{i=1}^{n} |\delta^i_{j-1} A_j(x) + \delta^i_j B_j(x) + \lambda_{j-i+k-1} C_j(x) + \lambda_{j-i+k} D_j(x) + m_{j-i+k-1} E_j(x) + m_{j-i+k} F_j(x)|$$

$$= |A_j(x) + \lambda_k C_j(x) + \lambda_{k+1} D_j(x) + m_k E_j(x) + m_{k+1} F_j(x)| +$$

$$+ |B_j(x) + \lambda_{k-1} C_j(x) + \lambda_k D_j(x) + m_{k-1} E_j(x) + m_k F_j(x)| +$$

$$+ \left(\sum_{i=1}^{k-1} + \sum_{i=k+2}^{n} \right) |\lambda_{i-1} C_j(x) + \lambda_i D_j(x) + m_{i-1} E_j(x) + m_i F_j(x)| .$$

By assertion 3 the coefficients λ_i alternate in sign as the index i runs through the sets $\{0,1,\ldots,k-1\}$ and $\{k+1,\ldots,n\}$. Assertion 3 also establishes the alternation of the parameters m_i when i runs through these sets.

Furthermore, $\lambda_k = m_k = 0$, $\lambda_{k-1} > 0$, $m_{k-1} < 0$ and $\lambda_{k+1} < 0$, $m_{k+1} > 0$. These facts together with the properties of the functions $A_j(x),\dots,F_j(x)$ (viz. formulae (2), (3), (4) and (5)), imply that

$$\sum_{i=1}^{n} |s^i(x)| = 1 + \{C_j(x) - D_j(x)\} \sum_{i=1}^{n} |\lambda_i| + \{F_j(x) - E_j(x)\} \sum_{i=1}^{n} |m_i| .$$

Since x was chosen to make $\sum_{i=1}^{n} |s^i|$ a maximum, it is apparent from formulae (6) and (7) that we must take $x = \frac{1}{2}(x_j + x_{j-1})$. Then $C_j(x) - D_j(x)$ and $F_j(x) - E_j(x)$ both attain their maximal value. In view of (6), (7) we obtain

$$\| \sum_{i=1}^{n} |s^i| \| = 1 + \frac{1}{4n} \sum_{i=1}^{n} |\lambda_i| + \frac{1}{192n^3} \sum_{i=1}^{n} |m_i| ,$$

which is equivalent to the lemma. ■

The next lemma gives an expression for $\|L_n\|$ in which the first and second cardinal spline derivatives of s^k are involved. It will be used in the sequel to find an upper and a lower bound for $\|L_n\|$, in which only the first derivatives λ_i are present.

Lemma 5

Let μ_j denote the second derivative of the cardinal spline function s^k at the node x_j. Then we have the following formulae for the norm of the interpolating periodic quintic spline operator:

(65) $$\|L_n\| = 1 + \frac{5}{16n} \sum_{i=1}^{n} |\lambda_i| + \frac{\mu_k}{32n^2} , \qquad (n = 2k+1) ,$$

(66) $$\|L_n\| = 1 + \frac{5}{16n} \sum_{i=1}^{n} |\lambda_i| + \frac{(\mu_k + |\mu_0|)}{32n^2} , \qquad (n = 2k) .$$

Proof

We give only an outline of the proof. If we express the cardinal spline function s^i on the interval $[x_{j-1}, x_j]$ in terms of the first and second derivatives of the cardinal function s^k, then we have (cf. [2])

(67) $$s^i(x) = \delta^i_{j-1} A^*_j(x) + \delta^i_j B^*_j(x) + \lambda_{j-i+k-1} C^*_j(x) + \lambda_{j-i+k} D^*_j(x) + \\ + \mu_{j-i+k-1} E^*_j(x) + \mu_{j-i+k} F^*_j(x) .$$

Here $A_j^*(x),\dots,F_j^*(x)$ are quintic polynomials, the formulae of which are given in [2]. In fact, we have: if $A^*(t),\dots,F^*(t)$ denote these polynomials when the interval $[x_{j-1},x_j]$ is replaced by $[0,1]$, then

$$A^*(t) = (1-t)^3(6t^2+3t+1) \;, \qquad B^*(t) = A^*(1-t) \;,$$

$$C^*(t) = t(1-t)^3(1+3t) \;, \qquad D^*(t) = -\,C^*(1-t) \;,$$

$$E^*(t) = \tfrac{1}{2}t^2(1-t)^3 \;, \qquad F^*(t) = E^*(1-t) \;.$$

The expressions for $A_j^*(x),\dots,F_j^*(x)$ are obtained by setting $t = n(x-x_{j-1})$, multiplying $C_j^*(x)$, $D_j^*(x)$ by n^{-1} and $E_j^*(x)$, $F_j^*(x)$ by n^{-2}.
One easily computes that

(68) $$C_j^*(x) - D_j^*(x) \le \frac{5}{16n}$$

and

(69) $$E_j^*(x) + F_j^*(x) \le \frac{1}{32n^2} \;.$$

The way in which the numbers λ_i and m_i alternate, together with the properties of the polynomials $A_j(x),\dots,F_j(x)$, completely determine the shape of the cardinal spline functions. Between two adjacent nodes the cardinal functions do not have zeros and the sign of the function changes when a node is passed. Moreover, $s^k > 0$ on (x_{k-1},x_{k+1}) and this function is symmetric with respect to x_k. If we evaluate the sum $\sum_{i=1}^{n} |s^i(x)|$ on $[x_{j-1},x_j]$ using (67), then we obtain as a consequence of these remarks that

$$\sum_{i=1}^{n} |s^i(x)| = 1 + (C_j^*(x) - D_j^*(x)) \sum_{i=1}^{n} |\lambda_i| + (E_j^*(x) + F_j^*(x))\mu_k \;, \quad (n = 2k+1)$$

and

$$\sum_{i=1}^{n} |s^i(x)| = 1 + (C_j^*(x) - D_j^*(x)) \sum_{i=1}^{n} |\lambda_i| + (E_j^*(x) + F_j^*(x))(\mu_k + |\mu_0|) \;, \quad (n = 2k$$

In view of lemma 4 we know that the maximum value of $\sum_{i=1}^{n} |s^i(x)|$ is attained when $x = \frac{1}{2}(x_{j-1}+x_j)$. However, in formulae (68) and (69) equality holds for this choice of x. This establishes the identities (65), (66) of lemma 5. ∎

Lemmas 4 and 5 imply the following simple corollary.

Corollary

$$1 + \frac{1}{4n} \sum_{i=1}^{n} |\lambda_i| < \|L_n\| < 1 + \frac{5}{16n} \sum_{i=1}^{n} |\lambda_i| \ .$$

Proof

The left-hand side inequality is an immediate consequence of lemma 4. The upper bound for $\|L_n\|$ follows from lemma 5, assertion 6 and taking into account that $\mu_k < 0$. ∎

We will now state a formula which enables us to compute the exact value of $\|L_n\|$; however, the expression of lemma 4, together with lemma 3, is much better suited for numerical purposes. Because of the intricateness of the expression involved we have to introduce a number of abbreviations. If the numbers $C_1, \ldots, D_4^*$ and z_1, z_2 are defined as in (44), respectively (40), (41), then we put

$$C_1 z_1^{-k-1} + C_4 z_1^{k+1} = u(c) \ ,$$

$$C_2 z_2^{-k-1} + C_3 z_2^{k+1} = v(c) \ ,$$

$$C_1 z_1^{-k} + C_4 z_1 = \overline{u}(c) \ ,$$

$$C_2 z_2^{-k} + C_3 z_2 = \overline{v}(c) \ .$$

Corresponding expressions are denoted similarly, for instance

$$D_2^* z_2^{-k-1} + D_3^* z_2^{k+1} = v(d^*) \ , \quad D_1 z_1^{-k} + D_4 z_1 = \overline{u}(d) \ .$$

If additionally we set

$$1 - 16 z_1^{-1} + z_1^{-2} = \alpha(z_1)$$

$$1 - 28 z_1^{-1} + z_1^{-2} = \beta(z_1) \ ,$$

then the theorem takes the following form.

Theorem 3

Assume $n = 2k$ $(k = 1, 2, \ldots)$. If the nodes x_i $(i = 0, 1, \ldots, n)$ are equally spaced on the interval $[0,1]$, then the norm of the interpolating periodic quintic spline operator L_n is given by

(70) $\quad \|L_n\| - 1 =$

$$= \frac{5}{2}\,\frac{U(c^*)W(c) - U(c)W(c^*)}{U(c)(\bar{u}(c^*) + \bar{v}(c^*)) - U(c^*)(u(c) + v(c))} - \frac{5}{8}\,\frac{V(c^*)W(c) - V(c)W(c^*)}{V(c)(\bar{u}(c^*) + \bar{v}(c^*)) - V(c^*)(u(c) + v(c))}$$

where

$$U(c) = z_1 u(c)\alpha(z_1) + z_2 v(c)\alpha(z_2)\ ,$$

$$V(c) = z_1 u(c)\beta(z_1) + z_2 v(c)\beta(z_2)\ ,$$

$$W(c) = \bar{u}(c)\left(\frac{1 - z_1^k}{z_1 - 1}\right) + \bar{v}(c)\left(\frac{1 - z_2^k}{z_2 - 1}\right)\ .$$

The norm of the operator L_n in case $n = 2k+1$ is given by the same formula if it is adjusted in the following way: the arguments of $u, v, \bar{u}, \bar{v}, U, V, W$, i.e. c, c^*, have to be replaced everywhere by d, d^*, respectively.

Proof

This can be given by using the contents of lemma 3, together with the formulae (45), (46), (47), (48), and (64) of lemma 4. We have to delete all further details because the calculations involved are much too lengthy to be reproduced here. ■

The next theorem is proved by direct calculations based upon theorem 3; the upper bound given here for $\|L_n\|$ is best possible.

Theorem 4

The norms $\|L_n\|$ are ordered as follows:

i) $\quad \|L_2\| < \|L_4\| < \|L_6\| < \ldots < 1 + \frac{15}{8}\,\frac{(z_1 + z_2 - 13 z_1 z_2 + (z_1 z_2)^2)}{(1 - z_1)(1 - z_2)(1 - z_1 z_2)} = 1.8161\ldots\ ,$

ii) $\quad \|L_3\| < \|L_5\| < \|L_7\| < \ldots < 1 + \frac{15}{8}\,\frac{(z_1 + z_2 - 13 z_1 z_2 + (z_1 z_2)^2)}{(1 - z_1)(1 - z_2)(1 - z_1 z_2)} = 1.8161\ldots\ ,$

iii) $\|L_3\| = \|L_6\|,\ \|L_5\| = \|L_{10}\|,\ \|L_7\| = \|L_{14}\|, \ldots$.

Proof

The proof that the sequences $\{\|L_2\|, \|L_4\|, \ldots\}$ and $\{\|L_3\|, \|L_5\|, \ldots\}$ are increasing can be based upon (70), using assertions 1 and 2. By a careful examination of formula (70) the best possible upper bound for $\|L_n\|$ is then obtained. Statement iii) can be derived from (70) by making use of assertions 1 and 2; all further details have to be omitted. ■

We remark that similar results hold for the norm of the interpolating periodic cubic spline operator, the nodes being equally spaced (cf. theorem 2 in [3]).

In the following table we have collected some numerical results. The values of $\|L_n\|$ $(n = 2,3,\ldots,11)$ were obtained by applying formula (64) of lemma 4, together with lemma 3. They clearly show that already for small values of n the norm of L_n is very close to the upper bound 1.8161... .

Table 1

$\|L_2\| = 1$	$\|L_3\| = 1\frac{5}{8} = 1.625$
$\|L_4\| = 1\frac{105}{256} = 1.4101\ldots$	$\|L_5\| = 1\frac{123}{158} = 1.7784\ldots$
$\|L_6\| = 1\frac{5}{8} = 1.625$	$\|L_7\| = 1\frac{680745}{841352} = 1.8091\ldots$
$\|L_8\| = 1\frac{8775}{12016} = 1.7302\ldots$	$\|L_9\| = 1\frac{39385}{48333} = 1.8148\ldots$
$\|L_{10}\| = 1\frac{123}{158} = 1.7784\ldots$	$\|L_{11}\| = 1\frac{1988418655}{2436972728} = 1.8158\ldots$

7. We recall that theorem 1 of the third section gives an error estimate for the difference between the function f and the corresponding interpolating spline function in terms of the modulus of continuity of f with argument 1/n. Once we know the values of the first and third derivatives of the function s^k, this information can be used to improve theorem 1. This is the purpose of this section. First we need a few preliminary lemmas. If from now on we denote the function $A_j(x)$ by A, etc. then one has

Lemma 6

Let $n = 2k$, respectively $n = 2k+1$. If the numbers σ_i, π_i are defined as on p. 15, then

$$\sum_{i=0}^{k-1}\{|\sigma_i D + \sigma_{i-1} C + \pi_i F + \pi_{i-1} E|\} + \sum_{i=1}^{k}\{|\sigma_{k-i-1} D + \sigma_{k-i} C + \pi_{k-i-1} F + \pi_{k-i} E|\} =$$

$$= (C - D)\sum_{i=0}^{k-1}|\lambda_i| + (F - E)\sum_{i=0}^{k-1}|m_i| .$$

Proof

We recall (p. 15) that $\operatorname{sgn} \sigma_i = (-1)^{k+i+1}$ $(i = 0,1,\dots,k-1)$ and $\operatorname{sgn} \pi_i = (-1)^{k+i}$ $(i = 0,1,\dots,k-1;\ n \neq 5)$. Furthermore it was established that $C,F \geq 0$ and $D,E \leq 0$ on $[x_{j-1},x_j]$. As a consequence we have $\operatorname{sgn}(\sigma_i D + \sigma_{i-1} C) = \operatorname{sgn} \sigma_{i-1}$ and $\operatorname{sgn}(\pi_i F + \pi_{i-1} E) = \operatorname{sgn} \pi_i$. The sum on the left side in the statement of the lemma therefore is

$$\sum_{i=0}^{k-1} \{|\sigma_i D + \sigma_{i-1} C + \pi_i F + \pi_{i-1} E| + |\sigma_i C + \sigma_{i-1} D + \pi_i E + \pi_{i-1} F|\} =$$

$$= \sum_{i=0}^{k-1} (-1)^{k+i+1} \{-\sigma_i D - \sigma_{i-1} C - \pi_i F - \pi_{i-1} E + \sigma_i C + \sigma_{i-1} D + \pi_i E + \pi_{i-1} F\}$$

$$= (C-D) \sum_{i=0}^{k-1} (-1)^{k+i+1} (\sigma_i - \sigma_{i-1}) + (F-E) \sum_{i=0}^{k-1} (-1)^{k+i} (\pi_i - \pi_{i-1}) =$$

$$= (C-D) \sum_{i=0}^{k-1} |\lambda_i| + (F-E) \sum_{i=0}^{k-1} |m_i| .$$

In the last step of this deduction we used that $\operatorname{sgn} \lambda_i = (-1)^{k+i+1}$ and $\operatorname{sgn} m_i = (-1)^{k+i}$. The case $n = 5$ can be verified by a direct calculation of the sums involved. This completely proves the lemma. ■

Lemma 7

On the interval $[x_{j-1},x_j]$ the functions $A - \sigma_k(C+D) - \pi_k(E+F)$ and $B + \sigma_k(C+D) + \pi_k(E+F)$ are nonnegative.

Proof

Put $J = A - \sigma_k(C+D) - \pi_k(E+F)$. Then $B + \sigma_k(C+D) + \pi_k(E+F) = 1 - J$. If we put $x = x_{j-1} + \theta n^{-1}$ $(0 \leq \theta \leq 1)$, then it follows from (3), (4) that

$$C + D = \frac{1}{2n} \theta(1-\theta)(1+\theta)(\theta-2)(2\theta-1) , \tag{71}$$

$$E + F = \frac{1}{24n^3} \theta^2 (1-\theta)^2 (2\theta-1) . \tag{72}$$

Because $\sigma_k > 0$ and $\pi_k < 0$ $(n \neq 5)$, it is easy to verify that for $\frac{1}{2} \leq \theta \leq 1$ we have $J \geq 0$. We will also show that $J \leq 1$ when $\theta \in [\frac{1}{2},1]$. We note that the function A is monotone decreasing on $[x_{j-1},x_j]$ with $A = \frac{1}{2}$ when $\theta = \frac{1}{2}$. Moreover, by a weakened version of assertion 5 we have $|\sigma_k| < 2n$, $|\pi_k| < 24n^3$. It is therefore sufficient to establish that

$$- \theta(1-\theta)(1+\theta)(\theta-2)(2\theta-1) + \theta^2(1-\theta)^2(2\theta-1) \leq \tfrac{1}{2} ,$$

which is an elementary calculation.

The case $n = 5$ can be handled separately, thus lemma 7 is proved when $\theta \in [\frac{1}{2}, 1]$. Similar considerations hold in case $0 \leq \theta \leq \frac{1}{2}$. ■

Lemma 8

Let n be an arbitrary but fixed positive integer. Moreover, let $x = x_{j-1} + \delta$ with $0 \leq \delta \leq (2n)^{-1}$. If p denotes the smallest integer satisfying $p \geq (n\delta)^{-1}$, then for $n \neq 3$ we have

$$(p-1)|B+\sigma_k(C+D)+\pi_k(E+F)| + |A-\sigma_k(C+D)-\pi_k(E+F)| < 2-(n\delta)^2$$

If $\delta = (2n)^{-1}$, the bound can be lowered to 1.

Proof

In view of (5) and lemma 7, the left-hand side of the asserted inequality is

$$I \equiv 1 + (p-2)\{B+\sigma_k(C+D)+\pi_k(E+F)\} .$$

If we insert the expressions for B, $C+D$ and $E+F$ from equations (2), (71), (72) into this identity and abbreviate $n\delta$ by θ in the next formula we obtain

$$I = 1 + (p-2)\left\{\theta^5 - \frac{5}{2}\theta^4 + \frac{5}{2}\theta^2 + \frac{\sigma_k}{n}(-\theta^5 + \frac{5}{2}\theta^4 - \frac{5}{2}\theta^2 + \theta) + \frac{\pi_k}{24n^3}(2\theta^5 - 5\theta^4 + 4\theta^3 - \theta^2)\right\}$$

If we make use of the inequalities for σ_k, π_k $(n \neq 3)$ as stated in assertion 5, then after some elementary estimations there follows

$$I < 1 + (p-2)n\delta(1+n\delta) .$$

Since $p < 1 + (n\delta)^{-1}$, we have $pn\delta < n\delta + 1$ so that $I < 1 + (n\delta+1)^2 - 2n\delta(n\delta+1) = 2 - (n\delta)^2$. Note that if $\delta = (2n)^{-1}$ then $p = 2$, $I = 1$, and the bound $2-(n\delta)^2$ can be lowered to 1. This proves the lemma. ■

The improvement of theorem 1 now reads as follows.

Theorem 5

Let $f \in C$ and let $L_n f$ be the associated interpolating periodic quintic spline function, the nodes x_i $(i = 0,1,\dots,n)$ being equally spaced. Then for $n \neq 3$ there holds the error-estimate

$$|(L_n f - f)(x)| \leq c_n\, \omega(f;\delta) ,$$

where $\delta = \min_i |x - x_i|$ and $\|L_n\| \leq c_n \leq 2\|L_n\|$. If $n = 3$, then $\|L_3\| \leq c_3 \leq 2.04\|L_3\|$.

Proof

Let $n = 2k$ or $n = 2k+1$ $(k \neq 1)$. Let x be any point, and select j so that $x_{j-1} \leq x \leq x_j$. From equation (16), together with the equation $A + B = 1$, we obtain

$$\varepsilon = (L_n f - f)(x) = \sum_{i=1}^{n} f_i s^i(x) - f(x)A - f(x)B =$$

$$= (f_{j-1} - f(x))A + (f_j - f(x))B +$$

$$+ \sum_{i=1}^{n} f_i \{\lambda_{j+k-i-1} C + \lambda_{j+k-i} D + m_{j+k-i-1} E + m_{j+k-i} F\} .$$

In order to simplify the notation we abbreviate f_{j+k-i} by $\tilde{f}_i$ and $f(x)$ by f_x. We also note from lemma 3 that $\lambda_{k+i} = -\lambda_{k-i}$ and $m_{k+i} = -m_{k-i}$ for $i = 0,1,\dots,k$. Furthermore, $\lambda_0 = m_0 = 0$ when n is even. Hence

$$(73)\quad \begin{cases} \varepsilon = (f_{j-1} - f_x)A + (f_j - f_x)B + \sum_{i=0}^{n-1} \tilde{f}_i(\lambda_{i-1} C + \lambda_i D + m_{i-1} E + m_i F) = \\ = (f_{j-1} - f_x)A + (f_j - f_x)B + \sum_{i=0}^{k-1} \tilde{f}_i \lambda_i D + \sum_{i=1}^{k} \tilde{f}_i \lambda_{i-1} C - \sum_{i=0}^{k-1} \tilde{f}_{2k-i} \lambda_i D + \\ - \sum_{i=0}^{k-1} \tilde{f}_{2k-i+1} \lambda_i C + \sum_{i=0}^{k-1} \tilde{f}_i m_i F + \sum_{i=1}^{k} \tilde{f}_i m_{i-1} E - \sum_{i=0}^{k-1} \tilde{f}_{2k-i} m_i F - \sum_{i=0}^{k-1} \tilde{f}_{2k-i+1} m_i \end{cases}$$

Recalling the definitions for σ_i and π_i, we apply to (73) the method of partial summation. The result is

$$(74)\quad \begin{cases} \varepsilon = \sum_{i=0}^{k-1} \{(\tilde{f}_i - \tilde{f}_{i+1})(\sigma_i D + \sigma_{i-1} C + \pi_i F + \pi_{i-1} E)\} + \\ + (f_j - f_x)\{B + \sigma_k(C + D) + \pi_k(E + F)\} + (f_x - f_{j-1})\{-A + \sigma_k(C + D) + \pi_k(E + F)\} \\ + \sum_{i=1}^{k} \{(\tilde{f}_{k+i} - \tilde{f}_{k+i+1})(\sigma_{k-i-1} D + \sigma_{k-i} C + \pi_{k-i-1} F + \pi_{k-i} E)\} . \end{cases}$$

Now let $\delta = \min_i |x - x_i| = \min\{x - x_{j-1}, x_j - x\}$. If $\delta = 0$ then x is a node and the inequality in question is trivial. We assume therefore that $\delta > 0$. If $\omega(f;\delta) = 0$, then f is a constant function and $L_n f = f$. We assume therefore that $\omega(f;\delta) > 0$. Since the inequality of the theorem is homogeneous in f, it is sufficient to give the proof for functions f such that $\omega(f;\delta) = 1$. Now let p denote the smallest integer satisfying $p \geq (n\delta)^{-1}$. Since each interval of length $1/n$ can be subdivided into p intervals of length at most δ, we have $|\tilde{f}_i - \tilde{f}_{i+1}| \leq p$. Assume now that $x - x_{j-1} = \delta$ and that $x_j - x = n^{-1} - \delta$. (The analysis of the other case when $x_j - x = \delta$, is almost exactly the same.) Then $|f_x - f_{j-1}| \leq 1$ and $|f_j - f_x| \leq p - 1$. Thus

$$\varepsilon \leq p \sum_{i=0}^{k-1} |\sigma_i D + \sigma_{i-1} C + \pi_i F + \pi_{i-1} E| + (p-1)|B + \sigma_k(C+D) + \pi_k(E+F)| +$$

$$+ |-A + \sigma_k(C+D) + \pi_k(E+F)| + p \sum_{i=1}^{k} |\sigma_{k-i-1} D + \sigma_{k-i} C + \pi_{k-i-1} F + \pi_{k-i} E| .$$

We have analysed the sum on the right in this inequality in lemmas 6 and 8. Using this results we obtain

$$\varepsilon \leq p\left\{(C-D)\sum_{i=0}^{k-1} |\lambda_i| + (F-E)\sum_{i=0}^{k-1} |m_i| \right\} + 2 - (n\delta)^2 . \tag{75}$$

If we evaluate the functions $C - D$ and $F - E$ at the point $x = x_{j-1} + \delta$, then in view of formulae (6), (7) we obtain $\delta(1 - n\delta)$, $\frac{1}{12n}\delta^2(1-n\delta)^2$, respectively. Also we note from the proof of lemma 4 that

$$\frac{1}{96n^3}\sum_{i=0}^{k-1} |m_i| = \|L_n\| - 1 - \frac{1}{2n}\sum_{i=0}^{k-1} |\lambda_i| . \tag{76}$$

Finally we use the inequality $p \leq 1 + (n\delta)^{-1}$. Consequently

(77)

$$\varepsilon \leq (1 + (n\delta)^{-1})\left\{8n^2\delta^2(1-n\delta)^2(\|L_n\|-1) + (\delta(1-n\delta) - 4n\delta^2(1-n\delta)^2)\sum_{i=0}^{k-1} |\lambda_i|\right\} + 2 - (n\delta)^2 .$$

On account of (76) we may write

$$\sum_{i=0}^{k-1} |\lambda_i| < 2n(\|L_n\| - 1) .$$

Using this inequality in (77), we obtain eventually

$$\varepsilon \leq (1 + (n\delta)^{-1})2n\delta(1 - n\delta)(\|L_n\| - 1) + 2 - (n\delta)^2$$

and the right-hand side can be proved to be smaller than $2\|L_n\|$.

In the special case that $\delta = (2n)^{-1}$, we have $p = 2$, and by lemma 8 the bound $2-(n\delta)^2$ can be lowered to 1. Hence in this case $\varepsilon \leq \|L_n\|$, because of (75) and (64).

In order to see that $c_n \geq \|L_n\|$ we construct a particular function f by specifying $\tilde{f}_{k-i} = \tilde{f}_{k+i+1} = p$ for $i = 0,2,4,\ldots$ and $\tilde{f}_{k-i} = \tilde{f}_{k+i+1} = 0$ for $i = 1,3,5,\ldots$. Also, we take $f_x = p-1$. The function f varies linearly between the specified values, is periodic, and satisfies $\omega(f;\delta) = 1$. In view of (74) and (75) we obtain for this function

$$\varepsilon = p\left\{(C-D)\sum_{i=0}^{k-1}|\lambda_i| + (F-E)\sum_{i=0}^{k-1}|m_i|\right\} + 1 .$$

Taking into account formulae (6), (7) and (76), the above expression for ε can be evaluated in terms of p, n, δ, $\|L_n\|$ and $\sum_{i=0}^{k-1}|\lambda_i|$. Also we derive from the corollary on page 20 that

$$\frac{5}{8n}\sum_{i=0}^{k-1}|\lambda_i| > \|L_n\| - 1 ;$$

moreover, $pn\delta > 1$.

Using these two facts, elementary calculations show that $\varepsilon \geq \|L_n\|$. (The example just given is satisfactory when n is odd. If n is even, it is modified by defining $\tilde{f}_0$ to be equal to $\tilde{f}_1$.)

Finally, we remark that the statement in theorem 5 about the particular case $n = 3$ has to be dealt with separately. We omit the details. This ends the proof of theorem 5. ■

<u>Corollary</u>

If $f \in C$ and $L_n f$ is the interpolating periodic quintic spline function associated with f, then the estimate

$$|(L_n f - f)(x)| \leq \|L_n\|\omega(f;\delta)$$

holds for an arbitrary point x such that $\delta = (2n)^{-1}$. It is not possible to introduce a constant factor < 1 on the right-hand side.

<u>Proof</u>

This follows from the proof of theorem 5. ■

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