Facet inducing inequalities for single-machine scheduling problems

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We report new results for a time-indexed formulation of nonpreemptive single-machine scheduling problems. We give complete characterizations of all facet inducing inequalities with integral coefficients and right-hand side 1 or 2. Our results may lead to improved cutting plane algorithms for single-machine scheduling problems.

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1 Introduction

Recently developed polyhedral methods have yielded substantial progress in solving many important NP-hard combinatorial optimization problems. Some well-known examples are the traveling salesman problem [Padberg and Rinaldi 1991], and large-scale 0-1 integer programming problems [Crowder, Johnson and Padberg 1983]. We refer to Hoffman and Padberg [1985] and Nemhauser and Wolsey [1988] for general descriptions of the approach.

For machine scheduling problems, however, polyhedral methods have not been nearly so successful. The investigation and development of polyhedral methods for machine scheduling problems is important because traditional combinatorial algorithms do not perform well on difficult problem types in this class.

Relatively few papers have been written in this area. Balas [1985] pioneered the study of scheduling polyhedra with his work on the facial structure of the job shop scheduling problem. Queyranne [1986] completely characterized the polyhedron associated with the simple nonpreemptive single-machine scheduling problem. Queyranne and Wang [1988] generalized Queyranne's results to include precedence constraints. Wolsey [1989] compared different formulations for the problem with precedence constraints. Dyer and Wolsey [1990] examined several formulations for the single-machine scheduling problem with release times. Furthermore, Nemhauser and Savelsbergh [1992] developed a cutting plane algorithm for this problem. Sousa and Wolsey [1992] investigated a time-indexed formulation for several variants of the nonpreemptive single-machine scheduling problem. Crama and Spieksma [1991] studied the same formulation for problems in which the jobs have equal processing times. Lasserre and Queyranne [1992] presented a mixed integer programming formulation motivated by a control theoretic view of scheduling decisions.

In this paper, we report new results for the time-indexed formulation of nonpreemptive single-machine scheduling problems studied by Sousa and Wolsey [1992]. They introduced three classes of inequalities: a class of inequalities with right-hand side 1, and two classes of inequalities with right-hand side $k \in \{2, \ldots, n\}$. In their cutting plane algorithm, they used an exact separation method only for inequalities with right-hand side 1 and for inequalities with right-hand side 2 in one class. They used a simple heuristic to identify violated inequalities in the remaining class. Their computational experiments revealed that the bounds obtained are strong compared to bounds obtained from other mixed integer programming formulations.

These promising computational results stimulated us to study the inequalities with right-hand side 1 or 2 more thoroughly. This has resulted in complete characterizations of all facet inducing inequalities with integral coefficients and right-hand side 1 or 2. It appears that only some of the classes of inequalities used in the computational experiments by Sousa and Wolsey were facet inducing. Our results may hence lead to improved cutting plane algorithms for single-machine scheduling problems. For reasons of brevity, this paper does not discuss separation for the identified classes of facet inducing inequalities and it does not present any computational results. We are currently studying and implementing separation algorithms. The results of these activities will be reported in a sequel paper.

2 Problem formulation

The usual setting for nonpreemptive single-machine scheduling problems is as follows. A set J of n jobs has to be scheduled on a single machine. Each job $j \in J$ requires uninterrupted processing for a period of length p_j. The machine can handle no more than one job at a time.

The time-indexed formulation studied by Sousa and Wolsey [1992] is based on time-discretization. The planning horizon is denoted by T. This formulation is given by:

$$\text{minimize} \sum_{1\le j\le n} \sum_{1\le t\le T-p_j+1} c_{jt}x_{jt}$$

subject to

$$\sum_{1\le t\le T-p_j+1} x_{jt} = 1 \quad (j = 1, ..., n),$$

$$\sum_{1\le j\le n} \sum_{t-p_j<s\le t} x_{js} \le 1 \quad (t = 1, ..., T),$$

$$x_{jt} \in \{0,1\} \qquad (j = 1, ..., n;\ t = 1, ..., T),$$

where $x_{jt} = 1$ if job j is started in period t and 0 otherwise. This formulation can be used to model several variants of single-machine scheduling problems by an appropriate choice of the objective coefficients and possibly a restriction of the set of variables. For instance, if the objective is to minimize the weighted sum of the starting times, we take coefficients $c_{jt} = w_j t$, and if there are release dates r_j, we discard the variables x_{jt} for $t = 1, \ldots, r_j - 1$.

In the above formulation the convex hull of the set of feasible schedules is not full-dimensional. As it is often easier to study full-dimensional polyhedra, we consider the polyhedron P that is associated with an extended set of solutions and defined by

$$\sum_{1\le t\le T-p_j+1} x_{jt} \le 1 \quad (j = 1, ..., n), \tag{1}$$

$$\sum_{1\le j\le n} \sum_{t-p_j<s\le t} x_{js} \le 1 \quad (t = 1, ..., T), \tag{2}$$

$$x_{jt} \in \{0,1\} \qquad (j = 1,...,n;\ t = 1,...,T).$$

Although it may seem more natural to relax the equations into inequalities with sense greater-than-or-equal instead of less-than-or-equal, we have chosen for the latter option, since it has the advantage that the origin and the unit vectors are elements of the polyhedron, which is often convenient for dealing with affine independence.

Note that the collection of facet inducing inequalities for the polyhedron associated with the extended set of solutions includes the collection of facet inducing inequalities for the polyhedron associated with the original set of solutions.

Before we present our analysis of the structure of facet inducing inequalities with right-hand side 1 or 2, we introduce some notation and definitions.

The index-set of variables with nonzero coefficients in an inequality is denoted by V. The set of variables with nonzero coefficients in an inequality associated with job j defines a set of time periods $V_j = \{s|(j,s) \in V\}$. If job j is started in period $s \in V_j$, then we say that job j is started in V. With each set V_j we associate two values

$$l_j = \min\{s|s - p_j + 1 \in V_j\}$$

and

$$u_j = \max\{s|s \in V_j\}.$$

For convenience, let $l_j = \infty$ and $u_j = -\infty$ if $V_j = \emptyset$. Note that if $V_j \neq \emptyset$, then l_j is the first period in which job j can be finished if it is started in V, and that u_j is the last period in which job j can be started in V. Furthermore, let $l = \min\{l_j|j \in \{1,...,n\}\}$ and $u = \max\{u_j|j \in \{1,...,n\}\}$.

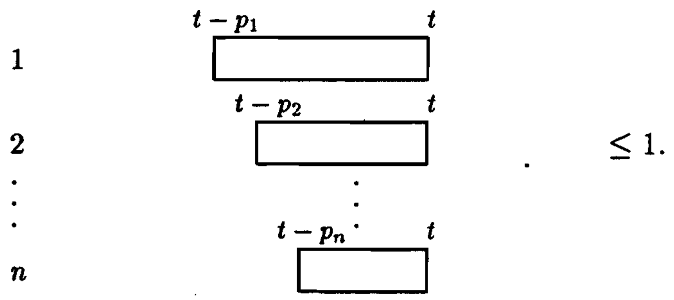
We say that period t starts at time $t-1$ and ends at time t. Now, an interval $[a,b]$ is defined as the set of periods $\{a+1, a+2, \ldots, b\}$, i.e., the set of periods between times a and b. If $a \geq b$, then $[a,b] = \emptyset$.

Lemma 1 *A facet inducing inequality with integral coefficients and integral right-hand side b has coefficients in* $\{0, 1, \ldots, b\}$.

Proof. Since the schedule x defined by $x_{js} = 1$ and $x_{j's'} = 0$ for all $(j',s') \neq (j,s)$ is feasible for every (j,s), a valid inequality with right-hand side b does not have coefficients greater than b. We now show that a facet inducing inequality does not have negative coefficients. Let $\sum_{j=1}^{n} \sum_{s=1}^{T-p_j+1} a_{js}x_{js} \leq b$ be a valid inequality with $a_{j's'} < 0$. Let x be any feasible schedule with $x_{j's'} = 1$. If $x_{j's'} = 1$ is replaced by $x_{j's'} = 0$, then the schedule remains feasible and $\sum_{j=1}^{n} \sum_{s=1}^{T-p_j+1} a_{js}x_{js}$ is increased. Therefore, any feasible schedule such that $\sum_{j=1}^{n} \sum_{s=1}^{T-p_j+1} a_{js}x_{js} = b$ satisfies $x_{j's'} = 0$. Since P is full-dimensional, it follows that the inequality $\sum_{j=1}^{n} \sum_{s=1}^{T-p_j+1} a_{js}x_{js} \leq b$ cannot be facet inducing. □

For presentational convenience, we use $x(S)$ to denote $\sum_{s \in S} x_s$. As a consequence of the previous lemma, valid inequalities with right-hand side 1 will be denoted by $x(V) \leq 1$ and valid inequalities with right-hand side 2 will be denoted by $x(V^1) + 2x(V^2) \leq 2$, where $V = V^1 \cup V^2$. Furthermore, we define $V_j^2 = \{(j,s) \mid (j,s) \in V^2\}$.

In the sequel, we shall often represent inequalities by diagrams. A diagram contains a horizontal line for each job. The line associated with job j represents the time periods s for which x_{js} occurs in the inequality. For example, an inequality of the form (2) can be represented by the following diagram:



3 Facet inducing inequalities with right-hand side 1

The purpose of this section is twofold. First, we present new results that extend and complement the work of Sousa and Wolsey [1992]. Second, we familiarize the reader with our approach in deriving complete characterizations of classes of facet inducing inequalities.

A valid inequality $x(V) \leq 1$ is called *maximal* if there does not exist a valid inequality $x(W) \leq 1$ with $V \subsetneq W$. The following lemma is frequently used in the proofs in this section.

Lemma 2 *A facet inducing inequality $x(V) \leq 1$ is maximal.*

Establishing complete characterizations of facet inducing inequalities proceeds in two phases. First, we derive necessary conditions in the form of various structural properties. Second, we show that these necessary conditions on the structure of facet inducing inequalities are also sufficient.

Property 1 *If $x(V) \leq 1$ is facet inducing, then the sets V_j are intervals, i.e., $V_j = [l_j - p_j, u_j]$.*

Proof. Let $j \in \{1,\ldots,n\}$ and assume $V_j \neq \emptyset$. Let $t_1, t_2 \in V_j$ be such that $t_1 \leq t_2$. We show that $s \in V_j$ for all s such that $t_1 \leq s \leq t_2$. Let $t_1 \leq s \leq t_2$. If $x_{jt_2} = 1$, then it is impossible to start any other job in V. Since $s \leq t_2$, this implies that, if $x_{js} = 1$, then it is impossible to start any job in V before job j. In the same way, since $s \geq t_1$, it is impossible to start any job after job j if $x_{js} = 1$. Since $x(V) \leq 1$ is maximal, it follows that $s \in V_j$. □

Property 2 *Let $x(V) \leq 1$ be facet inducing.*
(a) *Assume $l = l_1 \leq l_2 = \min\{l_j | j \in \{2,...,n\}\}$. Then $V_1 = [l - p_1, l_2]$ and $V_j = [l_j - p_j, l]$ for all $j \in \{2,...,n\}$.*
(b) *Assume $u = u_1 \geq u_2 = \max\{u_j | j \in \{2,...,n\}\}$. Then $V_1 = [u_2 - p_1, u]$ and $V_j = [u - p_j, u_j]$ for all $j \in \{2,...,n\}$.*

Proof. (a) Let $x(V) \leq 1$ be facet inducing with $l = l_1 \leq l_2 = \min\{l_j \mid j \in \{2,\ldots,n\}\}$. If $x_{1s} = 1$ for some $s > l_2$, then job 2 can be started in period $l_2 - p_2 + 1$, i.e., job 2 can be started in V. Hence $V_1 \subseteq [l - p_1, l_2]$. Now, let $x_{1s} = 1$ for some $s \in [l - p_1, l_2]$. Since by definition $l - p_1 + 1 \in V_1$ and $s \geq l - p_1 + 1$, it is impossible to start any job in V after job 1.

Since $s \leq l_2$, it is impossible to start any job in V before job 1. Since $x(V) \leq 1$ is maximal, we conclude that $s \in V_1$ and we find $V_1 = [l - p_1, l_2]$. Let $j \in \{2, \ldots, n\}$. If $x_{js} = 1$ for some $s > l$, then job 1 can be started in period $l - p_1 + 1$. Hence $V_j \subseteq [l_j - p_j, l]$. It is easy to see that if $x_{js} = 1$ for some $s \in [l_j - p_j, l]$, then it is impossible to start any other job in V. Since $x(V) \leq 1$ is maximal, we find $V_j = [l_j - p_j, l]$.

(b) Similar to (a). □

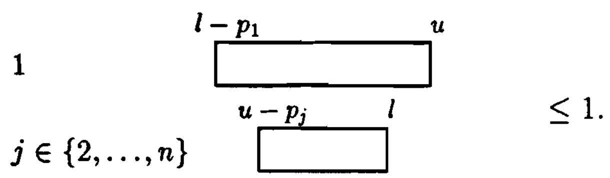
Observe that by Property 2(a) a facet inducing inequality $x(V) \leq 1$ with $l = l_1$ necessarily has $u_1 = u$. Consequently, Property 2(a) and 2(b) can be combined to give the following theorem.

Theorem 1 *A facet inducing inequality $x(V) \leq 1$ has the following structure:*

$$\begin{array}{l} V_1 = [l - p_1, u], \\ V_j = [u - p_j, l] \quad (j \in \{2, ..., n\}), \end{array} \tag{3}$$

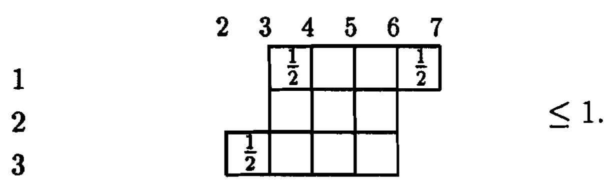
where $l = l_1 \leq u_1 = u$.

This theorem says that a facet inducing inequality with right-hand side 1 can be represented by the following diagram:



Note that if $V_j = \emptyset$ for all $j \in \{2, \ldots, n\}$, $l = p_1$ and $u = T - p_1 + 1$, the inequalities with structure (3) coincide with the inequalities (1), and if $l = u$, the inequalities with structure (3) coincide with the inequalities (2).

Example 1 Let $n = 3$, $p_1 = 3$, $p_2 = 4$ and $p_3 = 5$. The inequality with stucture (3), $l = 6$ and $u = 7$ is given by the following diagram:



The fractional solution $x_{14} = x_{17} = x_{33} = \frac{1}{2}$ violates this inequality.

Sousa and Wolsey [1992] have shown that the given necessary conditions are also sufficient.

Theorem 2 *A valid inequality $x(V) \leq 1$ with structure* (3) *that is maximal is facet inducing.*

Specific necessary and sufficient conditions for a valid inequality $x(V) \leq 1$ with structure (3) to be maximal are given by the following theorem. The proof of this theorem uses the concept of a *counterexample.* If $x(V) \leq 1$ is a valid inequality, then a counterexample for $(j, s) \notin V$ is a feasible schedule such that $x_{js} = 1$ and $x(V) = 1$.

Theorem 3 *A valid inequality $x(V) \leq 1$ with structure* (3) *is maximal if and only if $V_j \neq \emptyset$ for some $j \in \{2,\ldots,n\}$, or $x(V) \leq 1$ is one of the inequalities* (1), *i.e., $l = p_1$ and $u = T - p_1 + 1$.*

Proof. Let $x(V) \leq 1$ be a valid inequality with structure (3). Let $x(V) \leq 1$ be maximal and suppose $V_j = \emptyset$ for all $j \in \{2,\ldots,n\}$. It is not hard to see that $x(V) \leq 1$ must be one of the inequalities (1) and hence $l = p_1$ and $u = T - p_1 + 1$. On the other hand, it is easy to see that if $V_j \neq \emptyset$ for some $j \in \{2,\ldots,n\}$, then there is a counterexample for any $(j,s) \notin V$. Hence $x(V) \leq 1$ is maximal. Analogously, if $x(V) \leq 1$ is one of the inequalities (1), then it is maximal. □

4 Facet inducing inequalities with right-hand side 2

In the previous section, we have derived a complete characterization of all facet inducing inequalities with right-hand side 1. We now derive a similar characterization of all facet inducing inequalities with right-hand side 2.

First, we study the structure of valid inequalities with right-hand side 2 and coefficients 0, 1, and 2. Consider a valid inequality $x(V^1) + 2x(V^2) \leq 2$. Clearly, at most two jobs can be started in V. Let $j \in \{1,...,n\}$ and $s \in V_j$. It is easy to see that, if job j is started in period s, at least one of the following three statements is true.

(i) It is impossible to start any job in V before job j, and at most one job can be started in V after job j.

(ii) At most one job can be started in V before job j, and it is impossible to start any job in V after job j.

(iii) There exists a job i with $i \neq j$ such that job i can be started in V before or after job j and any job j' with $j' \neq j, i$ cannot be started in V.

Therefore, we can write $V = L \cup M \cup U$, where $L \subseteq V$ is the set of variables for which statement (i) holds, $M \subseteq V$ is the set of variables for which statement (ii) holds, and $U \subseteq V$ is the set of variables for which statement (iii) holds. Analogously, we can write $V_j = L_j \cup M_j \cup U_j$. Note that each of the sets L_j, M_j and U_j may be empty.

If job j is started in a period in V_j^2, then it is impossible to start any job in V before or after job j. Hence $V^2 \subseteq L \cap U$. Analogously $V_j^2 \subseteq L_j \cap U_j$. It is not hard to see that if $L_j \neq \emptyset$ and $U_j \neq \emptyset$, then the minimum of L_j is less than or equal to the minimum of U_j, and the maximum of L_j is less than or equal to the maximum of U_j. By definition $L_j \cap M_j = \emptyset$ and $M_j \cap U_j = \emptyset$. If $L_j \neq \emptyset$ and $M_j \neq \emptyset$, then the maximum of L_j is less than the minimum of M_j. Furthermore, if $M_j \neq \emptyset$ and $U_j \neq \emptyset$, then the maximum of M_j is less than the minimum of U_j. It follows that if $L_j \cap U_j \neq \emptyset$, then $M_j = \emptyset$. By definition of the sets L and U, $x(L) \leq 1$ and $x(U) \leq 1$.

We conclude that a valid inequality $x(V^1) + 2x(V^2) \leq 2$ can be represented by a collection of sets L_j, M_j and U_j. To derive necessary conditions on the structure of facet inducing inequalities with right-hand side 2, we study this LMU-structure more closely.

A valid inequality $x(V^1)+2x(V^2) \leq 2$ is called *nondecomposable* if it cannot be written as the sum of two valid inequalities $x(W) \leq 1$ and $x(W') \leq 1$. A valid inequality $x(V^1)+2x(V^2) \leq 2$ is called *maximal* if there does not exist a valid inequality $x(W^1)+2x(W^2) \leq 2$ with $V^1 \subseteq W^1$, $V^2 \subseteq W^2$, and not $W^1 = V^1$ and $W^2 = V^2$.

Lemma 3 *A facet inducing inequality $x(V^1)+2x(V^2) \leq 2$ is nondecomposable and maximal.*

The remaining part of the analysis of the LMU-structure proceeds in two phases. In the first phase, we derive conditions on the structure of the sets L and U by considering them separately from the other sets. The thus derived structural properties reveal that we have to distinguish three situations when we consider the overall LMU-structure, based on how the sets L and U can be joined. In the second phase, we investigate each of these three situations and derive conditions on the structure of the set M.

Property 3 *If $x(V^1) + 2x(V^2) \leq 2$ is facet inducing, then the sets L_j, M_j and U_j are intervals.*

Proof. Let $j \in \{1,\ldots,n\}$ and assume $L_j \neq \emptyset$. Let $t_1, t_2 \in L_j$ be such that $t_1 \leq t_2$. We show that $s \in L_j$ for all s with $t_1 \leq s \leq t_2$. Let $t_1 \leq s \leq t_2$. If $x_{jt_2} = 1$, then it is impossible to start any job in V before job j. Since $s \leq t_2$, this is also impossible if $x_{js} = 1$. Similarly, if $x_{jt_1} = 1$, then it is impossible to start more than one job after job j. Since $s \geq t_1$, this is also impossible if $x_{js} = 1$. Since $x(V^1) + 2x(V^2) \leq 2$ is maximal, it follows that $s \in L_j$.

Analogously, the sets M_j and U_j are intervals. □

Consider a facet inducing inequality $x(V^1) + 2x(V^2) \leq 2$. We have seen that $V^2 \subseteq L \cap U$. Observe that if job j is started in $L_j \cap U_j$, then it is impossible to start any job in V before or after job j. Since $x(V^1) + 2x(V^2) \leq 2$ is maximal, this implies $V_j^2 = L_j \cap U_j$ for all j, i.e., $V^2 = L \cap U$.

Property 4 *Let $x(V^1) + 2x(V^2) \leq 2$ be facet inducing.*
(a) *Assume $l = l_1 \leq l_2 \leq \min\{l_j \mid j \in \{3,\ldots,n\}\}$. Then $L_1 = [l - p_1, l_2]$ and $L_j = [l_j - p_j, l]$ for all $j \in \{2,\ldots,n\}$. Furthermore, there exists a $j \in \{2,\ldots,n\}$ such that $L_j \neq \emptyset$.*
(b) *Assume $u = u_1 \geq u_2 \geq \max\{u_j \mid j \in \{3,\ldots,n\}\}$. Then $U_1 = [u_2 - p_1, u]$ and $U_j = [u - p_j, u_j]$ for all $j \in \{2,\ldots,n\}$. Furthermore, there exists a $j \in \{2,\ldots,n\}$ such that $U_j \neq \emptyset$.*

Proof. (a) By definition $l - p_1 + 1 \in V_1$. It is easy to see that $l - p_1 + 1 \in L_1$. If $x_{1s} = 1$ for some $s > l_2$, then job 2 can be started in V before job 1. Consequently, $L_1 \subseteq [l - p_1, l_2]$. Now, let $x_{1s} = 1$ for some $s \in [l - p_1, l_2]$. Since $s \leq l_2$, it is impossible to start any job in V before job 1. From $l - p_1 + 1 \in L_1$ and $s \geq l - p_1 + 1$, it follows that at most one job can be started in V after job 1. Since $x(V^1) + 2x(V^2) \leq 2$ is maximal, we conclude that $s \in L_j$ and hence $L_1 = [l - p_1, l_2]$.

Let $j \in \{2,\ldots,n\}$. If $x_{js} = 1$ for some $s > l$, then job 1 can be started in V before job j. It follows that $L_j \subseteq [l_j - p_j, l]$. Assume $l_j - p_j < l$. By definition $l_j - p_j + 1 \in V_j$ and clearly $l_j - p_j + 1 \in L_j$. It is now easy to see that if $x_{js} = 1$ for some $s \in [l_j - p_j, l]$, then no job can be started in V before job j, and at most one job can be started in V after job j. Since $x(V^1) + 2x(V^2) \leq 2$ is maximal, we find $L_j = [l_j - p_j, l]$.

Suppose $L_j = \emptyset$ for all $j \in \{2,\ldots,n\}$. We show that $x(V^1) + 2x(V^2) \leq 2$ can be written as the sum of two valid inequalities with right-hand side 1, which contradicts the fact that $x(V^1) + 2x(V^2) \leq 2$ is facet inducing. Let $W = \{(1,s) \mid s \in L_1 \cap U_1\} \cup \{(j,s) \mid j \in \{2,\ldots,n\},\ s \in V_j\}$ and $W' = \{(1,s) \mid s \in V_1\}$. Clearly, $x(W') \leq 1$ and $x(W) + x(W') = x(V^1) + 2x(V^2)$. We still have to show $x(W) \leq 1$. Note that it suffices to show $\sum_{j\in\{2,\ldots,n\}} \sum_{s\in V_j} x_{js} \leq 1$. For $j \in \{2,\ldots,n\}$ we have $l_j - p_j \geq l$, since by assumption $L_j = \emptyset$, i.e., $s > l$ for all $s \in V_j$. Consequently, if $x_{j_1s_1} = x_{j_2s_2} = 1$ is a feasible schedule such that $\sum_{j\in\{2,\ldots,n\}} \sum_{s\in V_j} x_{js} = 2$, then $x_{1,l-p_1+1} = x_{j_1s_1} = x_{j_2s_2} = 1$ is also a feasible schedule. As this schedule violates the inequality, it follows that $\sum_{j\in\{2,\ldots,n\}} \sum_{s\in V_j} x_{js} \leq 1$.

(b) Similar to (a). □

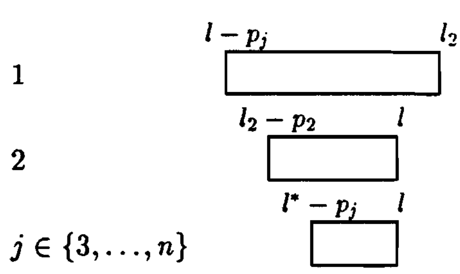
Like the proof of theorem 3, many of the proofs of the properties and theorems presented in this section use the concept of a *counterexample*. If $x(V^1) + 2x(V^2) \leq 2$ is a valid inequality, then a counterexample for $(j,s) \notin V$ is a feasible schedule such that $x_{js} = 1$ and $x(V^1) + 2x(V^2) = 2$. Observe that if $x(V^1) + 2x(V^2) \leq 2$ is facet inducing and $(j,s) \notin V$, then there exists a counterexample for (j,s), since $x(V^1) + 2x(V^2) \leq 2$ is maximal.

Property 5 *Let $x(V^1) + 2x(V^2) \leq 2$ be facet inducing.*
(a) Assume $l = l_1 \leq l_2 \leq l^*$, where $l^* = \min\{l_j \mid j \in \{3,\ldots,n\}\}$. Then for all $j \in \{3,\ldots,n\}$ such that $L_j \neq \emptyset$ we have $l_j = l^*$ and for all $j \in \{3,\ldots,n\}$ such that $L_j = \emptyset$ we have $l^* - p_j \geq l$, i.e., $L_j = [l^* - p_j, l]$ for all $j \in \{3,\ldots,n\}$.
(b) Assume $u = u_1 \geq u_2 \geq u^*$, where $u^* = \max\{u_j \mid j \in \{3,\ldots,n\}\}$. Then for all $j \in \{3,\ldots,n\}$ such that $U_j \neq \emptyset$ we have $u_j = u^*$ and for all $j \in \{3,\ldots,n\}$ such that $U_j = \emptyset$ we have $u^* \leq u - p_j$, i.e., $U_j = [u - p_j, u^*]$ for all $j \in \{3,\ldots,n\}$.

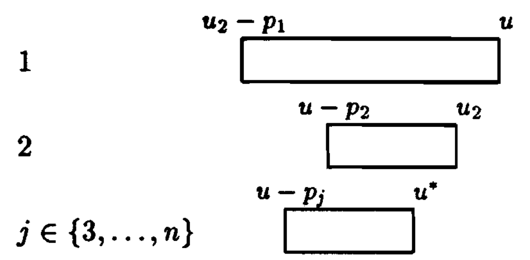
Proof. (a) By Property 4, $L_j \subseteq [l^* - p_j, l]$ for all $j \in \{3,\ldots,n\}$. Assume w.l.o.g. $l^* = l_3$. Suppose that $L_j \neq [l^* - p_j, l]$ for some $j \in \{4,\ldots,n\}$, say $L_4 \neq [l^* - p_4, l]$. Clearly, if $l^* - p_4 \geq l$, then $L_4 = \emptyset$ and hence $L_4 = [l^* - p_4, l]$. Consequently $l^* - p_4 < l$ and $l_4 > l^*$, i.e., $l^* - p_4 + 1 \notin V_4$. Since $x(V^1) + 2x(V^2) \leq 2$ is maximal, there is a counterexample for $(4, l^* - p_4 + 1)$. Let $(j_1,s_1),(j_2,s_2) \in V$ be such that $x_{4,l^*-p_4+1} = x_{j_1s_1} = x_{j_2s_2} = 1$ is a feasible schedule. Since $l^* - p_4 + 1 \leq l$, the jobs j_1 and j_2 are started after job 4. Clearly one of the jobs 1, 2 and 3 does not occur in $\{j_1, j_2\}$. Suppose job 3 does not occur. It is now easy to see that $x_{3,l^*-p_3+1} = x_{j_1s_1} = x_{j_2s_2} = 1$ is a feasible schedule. This schedule violates the inequality, which yields a contradiction. If job 1 or job 2 does not occur in $\{j_1, j_2\}$ we find a contradiction in the same way.

(b) Similar to (a). □

Properties 4 and 5 say that if $x(V^1) + 2x(V^2) \leq 2$ is facet inducing and we assume $l = l_1 \leq l_2 \leq l^*$, then the set L can be represented by the following diagram:



Similarly, if we assume $u = u_1 \geq u_2 \geq u^*$, then the set U can be represented by the following diagram:



Furthermore, a facet inducing inequality with right-hand side 2 has at most three types of intervals L_j, each characterized by the definition of the first period of the interval, and at most three types of intervals U_j, each characterized by the definition of the last period of the interval. Stated slightly differently, with the exception of two jobs the intervals L_j have the same structure for all jobs. Similarly, the intervals U_j have the same structure for all but two jobs. As a consequence, when we study the overall LMU-structure, it suffices to consider three situations, based on the jobs with the deviant intervals L_j and U_j:

(1a) $l = l_1 < l_2 \leq l^*$ and $u = u_1 > u_2 \geq u^*$, where $l^* = \min\{l_j \mid j \in \{3, \ldots, n\}\}$ and $u^* = \max\{u_j \mid j \in \{3, \ldots, n\}\}$;
(1b) $l = l_1 < l_2 < l^*$ and $u = u_1 > u_3 > u^*$, where $l^* = \min\{l_j \mid j \in \{3, \ldots, n\}\}$ and $u^* = \max\{u_j \mid j \in \{2, 4, \ldots, n\}\}$;
(2) $l = l_1$ and $u = u_2$.

Before we investigate each of the three situations, we prove a property that applies to case 1.

Property 6 *If $x(V^1) + 2x(V^2) \leq 2$ is facet inducing with $l = l_1 < l_2 = \min\{l_j \mid j \in \{2, \ldots, n\}\}$ and $u = u_1 > u_i = \max\{u_j \mid j \in \{2, \ldots, n\}\}$, then $l_2 < u_i$.*

Proof. Suppose that $l_2 \geq u_i$. We show that $x(V^1) + 2x(V^2) \leq 2$ can be written as the sum of two valid inequalities with right-hand side 1, which yields a contradiction. Let $W = \{(1, s) \mid s \in L_1 \cap U_1\} \cup \{(j, s) \mid j \in \{2, \ldots, n\},\ s \in V_j\}$ and $W' = \{(1, s) \mid s \in V_1\} \cup \{(j, s) \mid j \in \{2, \ldots, n\},\ s \in L_j \cap U_j\}$. Clearly $x(W) + x(W') = x(V^1) + 2x(V^2)$ and $x(W') \leq 1$. From $V_j \subseteq [l_j - p_j, u_j] \subseteq [l_2 - p_j, u_i]$ for all $j \in \{2, \ldots, n\}$ and $l_2 \geq u_i$, it easily follows that $\sum_{j=2}^{n} \sum_{s \in V_j} x_{js} \leq 1$ and hence $x(W) \leq 1$. □

4.1 Case 1a

Observe that the conditions on l_j and u_j and Properties 4 and 5 completely determine the sets L and U. Therefore, all that remains to be investigated is the structure of the set M.

Property 7 *If $x(V^1) + 2x(V^2) \leq 2$ is facet inducing with $l = l_1 < l_2 \leq l^*$ and $u = u_1 > u_2 \geq u^*$, then $M_1 = [u^* - p_1, l^*] \cap [l_2, u_2 - p_1]$, $M_2 = [u^* - p_2, l^*] \cap [l, u - p_2] \cap [l_2 - p_2, u_2]$ and $M_j = [u_2 - p_j, l_2] \cap [l, u - p_j]$ for $j \in \{3, \ldots, n\}$.*

Proof. Let $x(V^1) + 2x(V^2) \leq 2$ be facet inducing with $l = l_1 < l_2 \leq l^*$ and $u = u_1 > u_2 \geq u^*$. Since $L_1 \cap M_1 = \emptyset$ and $M_1 \cap U_1 = \emptyset$, we have $M_1 \subseteq [l_2, u_2 - p_1]$. If job 1 is started in M_1, then job 2 should be the only job that can be started in V before or after job 1, i.e, it should be impossible to start any job $j \in \{3, \ldots, n\}$ in V. Hence $M_1 \subseteq [u^* - p_1, l^*]$. We conclude that $M_1 \subseteq [u^* - p_1, l^*] \cap [l_2, u_2 - p_1]$. If job 1 is started in period $s \in [u^* - p_1, l^*] \cap [l_2, u_2 - p_1]$, then, since $s \in [l_2, u_2 - p_1]$, $[l_2, u_2 - p_1] \subset [l - p_1, u]$, and $L_2 \cap U_2 = [u - p_2, l]$, job 2 cannot be started in $L_2 \cap U_2$. Since $x(V^1) + 2x(V^2) \leq 2$ is maximal, it follows that $M_1 = [u^* - p_1, l^*] \cap [l_2, u_2 - p_1]$.

Since $L_2 \cap M_2 = \emptyset$ and $M_2 \cap U_2 = \emptyset$, we have $M_2 \subseteq [l, u - p_2]$. By definition $M_2 \subseteq [l_2 - p_2, u_2]$. If job 2 is started in M_2, then job 1 should be the only job that can be started before or after job 2, i.e., it should be impossible to start any job $j \in \{3, \ldots, n\}$ in V. Hence $M_2 \subseteq [u^* - p_2, l^*]$. We conclude that $M_2 \subseteq [u^* - p_2, l^*] \cap [l, u - p_2] \cap [l_2 - p_2, u_2]$. If job 2 is started in period $s \in [u^* - p_2, l^*] \cap [l, u - p_2] \cap [l_2 - p_2, u_2]$, then, since $s \in [l_2 - p_2, u_2]$ and $L_1 \cap U_1 = [u_2 - p_1, l_2]$, job 1 cannot be started in $L_1 \cap U_1$. Since $x(V^1) + 2x(V^2) \leq 2$ is maximal, it follows that $M_2 = [u^* - p_2, l^*] \cap [l, u - p_2] \cap [l_2 - p_2, u_2]$.

Let $j \in \{3, \ldots, n\}$. Since $L_j \cap M_j = \emptyset$ and $M_j \cap U_j = \emptyset$, we have $M_j \subseteq [l, u - p_j]$. If job j is started in M_j, then job 1 should be the only job that can be started in V before or after job j, i.e., it should be impossible to start any job $j' \in \{2, 3, \ldots, n\} \setminus \{j\}$ in V. Hence $M_j \subseteq [u_2 - p_j, l_2]$. We conclude that $M_j \subseteq [u_2 - p_j, l_2] \cap [l, u - p_j]$. If job j is started in period $s \in [u_2 - p_j, l_2] \cap [l, u - p_j]$, then, since $s \in [u_2 - p_j, l_2]$, $L_1 \cap U_1 = [u_2 - p_1, l_2]$, and, by Property 6, $l_2 < u_2$, job 1 cannot be started in $L_1 \cap U_1$. Since $x(V^1) + 2x(V^2) \leq 2$ is maximal, it follows that $M_j = [u_2 - p_j, l_2] \cap [l, u - p_j]$.

Observe that by definition $M_k \subseteq [l_k - p_k, u_k]$ for all $k \in \{1, \ldots, n\}$ and that for all but $k = 2$ this condition is dominated by other conditions. □

Properties 4, 5 and 7 completely determine the LMU-structure of a facet inducing inequality $x(V^1) + 2x(V^2) \leq 2$ with $l = l_1 < l_2 \leq l^*$ and $u = u_1 > u_2 \geq u^*$. However, in order to emphasize the inherent structure of the intervals M_j, we prefer to use a different representation of the set M. It is easy to show that, if $x(V^1) + 2x(V^2) \leq 2$ is facet inducing with $l = l_1 < l_2 \leq l^*$ and $u = u_1 > u_2 \geq u^*$, then for all $j \in \{3, \ldots, n\}$, $[u_2 - p_j, l] \subseteq L_j$ and $[u - p_j, l_2] \subseteq U_j$. We can use this observation to show that Properties 4, 5 and 7 can be combined to give the following theorem.

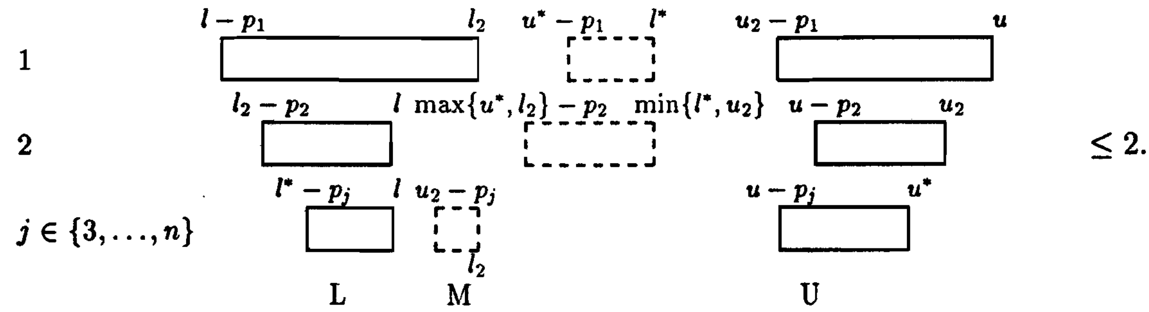
Theorem 4 *A facet inducing inequality $x(V^1) + 2x(V^2) \leq 2$ with $l = l_1 < l_2 \leq l^*$ and $u = u_1 > u_2 \geq u^*$ has the following LMU-structure:*

$$\begin{aligned} L_1 &= [l - p_1, l_2], & M_1 &= [u^* - p_1, l^*] \setminus (L_1 \cup U_1), \\ L_2 &= [l_2 - p_2, l], & M_2 &= [\max\{u^*, l_2\} - p_2, \min\{l^*, u_2\}] \setminus (L_2 \cup U_2), \\ L_j &= [l^* - p_j, l], & M_j &= [u_2 - p_j, l_2] \setminus (L_j \cup U_j), \end{aligned}$$

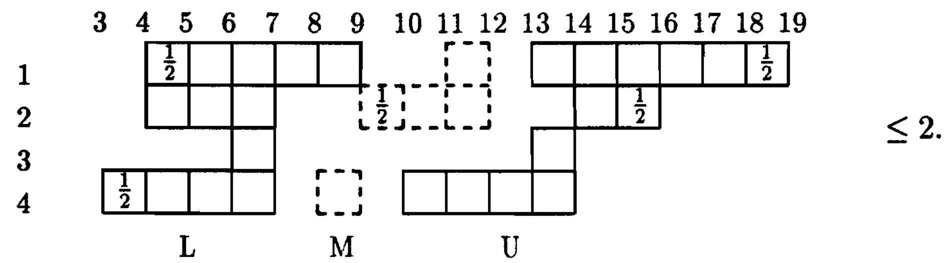
$$\begin{aligned} U_1 &= [u_2 - p_1, u], \\ U_2 &= [u - p_2, u_2], \\ U_j &= [u - p_j, u^*] \quad (j \in \{3, \ldots, n\}), \end{aligned} \tag{4}$$

where $[u_2 - p_j, l] \subseteq L_j$ and $[u - p_j, l_2] \subseteq U_j$ for all $j \in \{3, \ldots, n\}$.

This theorem says that a facet inducing inequality $x(V^1) + 2x(V^2) \leq 2$ with $l = l_1 < l_2 \leq l^*$ and $u = u_1 > u_2 \geq u^*$ can be represented by the following diagram:



Example 2 Let $n = 4$, $p_1 = 3$, $p_2 = 5$, $p_3 = 6$, and $p_4 = 9$. The inequality with LMU-structure (4) and $l = 7, l_2 = 9, l^* = 12, u^* = 14, u_2 = 16$ and $u = 19$ is given by the following diagram:



The fractional solution $x_{15} = x_{1,19} = x_{2,10} = x_{2,16} = x_{4,4} = \frac{1}{2}$ violates this inequality. It is easy to check that this solution satisfies all inequalities with structure (3).

Sufficient conditions are given by the following theorem.

Theorem 5 *A valid inequality* $x(V^1) + 2x(V^2) \leq 2$ *with* $l = l_1 < l_2 \leq l^*$ *and* $u = u_1 > u_2 \geq u^*$ *and LMU-structure* (4) *that is nondecomposable and maximal is facet inducing.*

Proof. Let $x(V^1) + 2x(V^2) \leq 2$ be a valid inequality with $l = l_1 < l_2 \leq l^*$ and $u = u_1 > u_2 \geq u^*$ and LMU-stucture (4) that is nondecomposable and maximal, and let $F = \{x \in P | x(V^1) + 2x(V^2) = 2\}$. We show that $\dim(F) = \dim(P) - 1$ by exhibiting $\dim(P) - 1$ linearly independent directions in F; a direction in F is a vector $d = x - y$ with $x, y \in F$. For notational convenience, a direction will be specified by its nonzero components. We give three sets of directions: unit vectors $d_{js} = 1$ for all $(j,s) \notin V$, $d_{js} = 1, d_{1,l-p_1+1} = d_{2u_2} = -1$ for all $(j,s) \in V^2$, and a set of $|V| - |V^2| - 1$ linearly independent directions $d_{j_1,s_1} = 1, d_{j_2,s_2} = -1$ with $(j_1,s_1),(j_2,s_2) \in V \setminus V^2$. Together these give $\dim(P) - 1$ linearly independent directions in F.

If $(j,s) \notin V$, then, since $x(V^1) + 2x(V^2) \leq 2$ is maximal, there is a counterexample for (j,s), say, defined by $x_{js} = x_{j_1s_1} = x_{j_2s_2} = 1$. Clearly this schedule is an element of F. Note that the schedule $y_{j_1s_1} = y_{j_2s_2} = 1$ also is an element of F and hence $d = x - y$ yields the direction $d_{js} = 1$.

Note that for $(j,s) \in V^2$ the schedule $x_{js} = 1$ is an element of F. Since $l < l_2$ and, by Property 6, $l_2 < u_2$, we have that $y_{1,l-p_1+1} = y_{2u_2} = 1$ is a feasible schedule. This schedule also is an element of F and hence $d_{js} = 1, d_{1,l-p_1+1} = d_{2u_2} = -1$ is a direction in F for all $(j,s) \in V^2$.

We determine the $|V| - |V^2| - 1$ directions $d_{j_1 s_1} = 1, d_{j_2 s_2} = -1$ with $(j_1, s_1), (j_2, s_2) \in V \setminus V^2$ in such a way that the undirected graph G whose vertices are the elements of $V \setminus V^2$ and whose edges are given by the pairs $\{(j_1, s_1), (j_2, s_2)\}$ corresponding to the determined directions is a spanning tree. This implies that the determined directions are linearly independent.

Observe that $d_{j_1 s_1} = 1, d_{j_2 s_2} = -1$ with $(j_1, s_1), (j_2, s_2) \in V \setminus V^2$ is a direction in F, if there exists an index $(i, t) \in V \setminus V^2$ such that $x_{j_1 s_1} = x_{it} = 1$ and $y_{j_2 s_2} = y_{it} = 1$ are both feasible schedules. In this case, we say that $d_{j_1 s_1} = 1, d_{j_2 s_2} = -1$ is a direction by (i, t).

First, we determine directions that correspond to edges in G within the sets $\{(j, s) \mid s \in (L_j \cup M_j) \setminus U_j\}$ and $\{(j, s) \mid s \in U_j \setminus L_j\}$. For $s - 1, s \in L_1 \setminus U_1$, $d_{1,s-1} = -1, d_{1s} = 1$ is a direction by $(2, u_2)$. If $M_1 \neq \emptyset$, then $d_{1l_2} = -1, d_{1m} = 1$ is a direction by $(2, u_2)$, where $m = \min\{s \mid s \in M_1\}$, and for $s - 1, s \in M_1$, $d_{1,s-1} = -1, d_{1s} = 1$ is a direction by $(2, u_2)$. Furthermore, for $s - 1, s \in U_1 \setminus L_1$, $d_{1,s-1} = -1, d_{1s} = 1$ is a direction by $(2, l_2 - p_2 + 1)$. Now, let $j \in \{2, \ldots, n\}$. For $s - 1, s \in L_j \setminus U_j$, $d_{j,s-1} = -1, d_{js} = 1$ is a direction by $(1, u)$. If $M_j \neq \emptyset$, then $d_{jl} = -1, d_{jm} = 1$ is a direction by $(1, u)$, where $m = \min\{s \mid s \in M_j\}$, and for $s - 1, s \in M_j$, $d_{j,s-1} = -1, d_{js} = 1$ is a direction by $(1, u)$. Furthermore, for $s - 1, s \in U_j \setminus L_j$, $d_{j,s-1} = -1, d_{js} = 1$ is a direction by $(1, l - p_1 + 1)$.

Second, we determine directions that correspond to edges in G between sets $\{(j, s) \mid s \in (L_j \cup M_j) \setminus U_j\}$ belonging to different jobs and between sets $\{(j, s) \mid s \in U_j \setminus L_j\}$ belonging to different jobs. We define $W = \{(1, s) \mid s \in L_1 \cap U_1\} \cup \{(j, s) \mid j \in \{2, \ldots, n\}, s \in V_j\}$ and $W' = \{(1, s) \mid s \in V_j\} \cup \{(j, s) \mid j \in \{2, \ldots, n\}, s \in L_j \cap U_j\}$. Clearly $x(W') \leq 1$. Since $x(V^1) + 2x(V^2) \leq 2$ is nondecomposable, there is a feasible schedule such that $x(W) = 2$, i.e., $\sum_{j=2}^{n} \sum_{s \in V_j} x_{js} = 2$. Let $x_{j_1 s_1} = x_{j_2 s_2} = 1$ with $s_1 < s_2$ be such a schedule. It is easy to see that job j_1 is started in L_{j_1} and job j_2 is started in U_{j_2}. Since $l = l_1$, $y_{1,l-p_1+1} = y_{j_2 s_2} = 1$ also is a feasible schedule and it follows that $d_{1,l-p_1+1} = -1, d_{j_1 s_1} = 1$ is a direction by (j_2, s_2). In the same way, since $u = u_1$, $y_{j_1 s_1} = y_{1u} = 1$ is a feasible schedule and it follows that $d_{1u} = -1, d_{j_2 s_2} = 1$ is a direction by (j_1, s_1). For $j \in \{2, \ldots, n\} \setminus \{j_1\}$ such that $L_j \cup M_j \neq \emptyset$, $d_{j_1 s_1} = -1, d_{j,l_j-p_j+1} = 1$ is a direction by $(1, u)$. Furthermore, for $j \in \{2, \ldots, n\} \setminus \{j_2\}$ such that $U_j \neq \emptyset$, $d_{j_2 s_2} = -1, d_{ju_j} = 1$ is a direction by $(1, l - p_1 + 1)$.

Finally, we determine a direction that corresponds to an edge in G between $L \cup M$ and U. Since $x(V^1) + 2x(V^2) \leq 2$ is nondecomposable and $x(U) \leq 1$, there is a feasible schedule with $x(L) + x(M) = 2$. Let $x_{j_1 s_1} = x_{j_2 s_2} = 1$ be such a schedule. Since $l_1 = l$, we may assume w.l.o.g. $j_1 = 1$. Since $s_2 \in L_{j_2} \cup M_{j_2}$, $y_{j_2 s_2} = y_{1u} = 1$ also is a feasible schedule. It follows that $d_{1s_1} = -1, d_{1u} = 1$ is a direction by (j_2, s_2).

It is easy to see that the determined directions form a spanning tree of G and hence we have determined $|V| - |V^2| - 1$ linearly independent directions.□

Specific necessary and sufficient conditions for a valid inequality $x(V^1) + 2x(V^2) \leq 2$ with $l = l_1 < l_2 \leq l^*$ and $u = u_1 > u_2 \geq u^*$ and LMU-structure (4) to be nondecomposable and maximal are given by the following two theorems.

Theorem 6 *A valid inequality $x(V^1) + 2x(V^2) \leq 2$ with $l = l_1 < l_2 \leq l^*$ and $u = u_1 > u_2 \geq u^*$ and LMU-structure (4) is nondecomposable if and only if $M_j \neq \emptyset$ for some $j \in \{1, \ldots, n\}$, and $l^* < u_2$ or $l_2 < u^*$.*

Proof. Let $x(V^1) + 2x(V^2) \leq 2$ be a valid inequality with $l = l_1 < l_2 \leq l^*$ and $u = u_1 > u_2 \geq u^*$ and LMU-structure (4).

Suppose that $x(V^1) + 2x(V^2) \leq 2$ is nondecomposable. Since $x(L) \leq 1$ and $x(U) \leq 1$, we have $M_j \neq \emptyset$ for some $j \in \{1, \ldots, n\}$. Suppose $l^* \geq u_2$ and $l_2 \geq u^*$. Let $W = \{(1,s) \mid s \in L_1 \cap U_1\} \cup \{(j,s) \mid j \in \{2, \ldots, n\}, s \in V_j\}$ and $W' = \{(1,s) \mid s \in V_1\} \cup \{(j,s) \mid j \in \{2, \ldots, n\}, s \in L_j \cap U_j\}$. Clearly $x(W) + x(W') = x(V^1) + 2x(V^2)$ and $x(W') \leq 1$. Observe that $V_2 \subseteq [l_2 - p_2, u_2]$ and for $j \in \{3, \ldots, n\}$, $V_j \subseteq [l^* - p_j, u^*] \subseteq [u_2 - p_j, l_2]$. Since, by Property 6, $l_2 < u_2$, it follows that $\sum_{j=2}^{n} \sum_{s \in V_j} x_{js} \leq 1$ and hence $x(W) \leq 1$, which yields a contradiction. It follows that $l^* < u_2$ or $l_2 < u^*$.

Suppose that $M_j \neq \emptyset$ for some $j \in \{1, \ldots, n\}$, and $l^* < u_2$ or $l_2 < u^*$. Let W and W' be such that $x(W) + x(W') = x(V^1) + 2x(V^2)$ and $x(W) \leq 1$ and $x(W') \leq 1$. We assume w.l.o.g. $(1, l - p_1 + 1) \in W$. Note that $x_{1l-p_1+1} = x_{2u_2} = 1$ is a feasible schedule. As $x(W) \leq 1$ and $(1, l - p_1 + 1) \in W$, it follows that $(2, u_2) \in W'$. We conclude that since $l < u_2$ and $(1, l - p_1 + 1) \in W$, we have $(2, u_2) \in W'$. We show that $(1, u) \in W'$ and $(2, l_2 - p_2 + 1) \in W$. By assumption either $l^* < u_2$ or $l_2 < u^*$. Suppose $l^* < u_2$. Note that $U_2 \neq \emptyset$ and $L_j \neq \emptyset$ for some $j \in \{3, \ldots, n\}$. Since $l^* < u_2$ and $(2, u_2) \in W'$, we have $(j, l^* - p_j + 1) \in W$ for all $j \in \{3, \ldots, n\}$ such that $L_j \neq \emptyset$. Furthermore, since $l^* < u_2 < u$, it follows that $(1, u) \in W'$, and since $l_2 < u_2 < u$, it follows that $(2, l_2 - p_2 + 1) \in W$. Analogously, $(2, l_2 - p_2 + 1) \in W$ and $(1, u) \in W'$ if $l_2 < u^*$.

By assumption $M_j \neq \emptyset$ for some $j \in \{1, \ldots, n\}$. Suppose that $M_j \neq \emptyset$ for some $j \in \{2, \ldots, n\}$. If job j is started in M_j, then job 1 can be started in V before or after job j. If $s \in M_j$, then, since $x_{1,l-p_1+1} = x_{js} = 1$ is a feasible schedule and $(1, l - p_1 + 1) \in W$, it follows that $(j, s) \in W'$. We find that $x_{js} = x_{1u} = 1$ is a feasible schedule such that $x(W') = 2$, which yields a contradiction. We conclude that from $(1, l - p_1 + 1) \in W$, $(1, u) \in W'$ and $M_j \neq \emptyset$, it follows that $x(V^1) + 2x(V^2) \leq 2$ is nondecomposable.

Suppose that $M_1 \neq \emptyset$. If job 1 is started in M_1 then job 2 can be started in V before or after job 1. As in the previous case, from $(2, l_2 - p_2 + 1) \in W$, $(2, u_2) \in W'$ and $M_1 \neq \emptyset$, it follows that $x(V^1) + 2x(V^2) \leq 2$ is nondecomposable. □

Theorem 7 *A valid inequality* $x(V^1) + 2x(V^2) \leq 2$ *with* $l = l_1 < l_2 \leq l^*$ *and* $u = u_1 > u_2 \geq u^*$*, and LMU-structure* (4) *is maximal if and only if*

(1) *If* $u^* < l_2$ *and* $l < l_2 - p_2$*, then* $L_1 \cap U_1 \neq \emptyset$;

(2) *If* $u_2 < l^*$ *and* $u_2 < u - p_2$*, then* $L_1 \cap U_1 \neq \emptyset$;

One of the following holds:
(3a) $L_j \cap U_j \neq \emptyset$ *for some* $j \in \{2, \ldots, n\}$;
(3b) $l \leq u^* - p_2$ *and* $L_2 \neq \emptyset$ *and* $U_j \neq \emptyset$ *for some* $j \in \{3, \ldots, n\}$;
(3c) $l \leq u_2 - \min\{p_j \mid j \in \{3, \ldots, n\}, L_j \neq \emptyset\}$ *and* $U_2 \neq \emptyset$ *and* $L_j \neq \emptyset$ *for some* $j \in \{3, \ldots, n\}$;

One of the following holds:
(4a) $L_j \cap U_j \neq \emptyset$ *for some* $j \in \{2, \ldots, n\}$;
(4b) $l^* \leq u - p_2$ *and* $U_2 \neq \emptyset$ *and* $L_j \neq \emptyset$ *for some* $j \in \{3, \ldots, n\}$;
(4c) $l_2 \leq u - \min\{p_j \mid j \in \{3, \ldots, n\}, U_j \neq \emptyset\}$ *and* $L_2 \neq \emptyset$ *and* $U_j \neq \emptyset$ *for some* $j \in \{3, \ldots, n\}$;

One of the following holds:
(5a) $L_1 \cap U_1 \neq \emptyset$;

(5b) $\min\{l_2, l+p_2\} \leq u^* - p_1$ *and* $U_j \neq \emptyset$ *for some* $j \in \{3,\ldots,n\}$;
(5c) $\min\{l_2, l+p_2\} \leq u - \min\{p_j \mid j \in \{3,\ldots,n\},\ M_j \neq \emptyset\}$ *and* $M_j \neq \emptyset$ *for some* $j \in \{3,\ldots,n\}$;

One of the following holds:
(6a) $L_1 \cap U_1 \neq \emptyset$;
(6b) $l^* \leq \max\{u_2, u-p_2\} - p_1$ *and* $L_j \neq \emptyset$ *for some* $j \in \{3,\ldots,n\}$;
(6c) $l \leq \max\{u_2, u-p_2\} - \min\{p_j \mid j \in \{3,\ldots,n\},\ M_j \neq \emptyset\}$ *and* $M_j \neq \emptyset$ *for some* $j \in \{3,\ldots,n\}$;

For all $j \in \{3,\ldots,n\}$, *one of the following holds:*
(7a) $\min\{l^*, l+p_j\} \leq u_2 - p_1$ *and* $M_1 \neq \emptyset$;
(7b) $\min\{l^*, l+p_j\} \leq u - p_2$ *and* $M_2 \neq \emptyset$;
(7c) $\min\{l^*, l+p_j\} \leq l_2$;

For all $j \in \{3,\ldots,n\}$, *one of the following holds:*
(8a) $l_2 \leq \max\{u^*, u-p_j\} - p_1$ *and* $M_1 \neq \emptyset$;
(8b) $l \leq \max\{u^*, u-p_j\} - p_2$ *and* $M_2 \neq \emptyset$;
(8c) $u_2 \leq \max\{u^*, u-p_j\}$.

Proof. Let $x(V^1) + 2x(V^2) \leq 2$ be a valid inequality with $l = l_1 < l_2 \leq l^*$ and $u = u_1 > u_2 \geq u^*$ and LMU-structure (4). Observe that $x(V^1) + 2x(V^2) \leq 2$ is maximal if and only if it is impossible to extend any of the intervals L_j, M_j and U_j.

First, we show that M_1 cannot be extended. If $l_2 < u^* - p_1$, i.e., $[l_2, u^* - p_1] \neq \emptyset$, then $x_{2,l_2-p_2+1} = x_{1s} = x_{ju^*} = 1$ defines a counterexample for $(1,s)$ for all $s \in [l_2, u^* - p_1]$, where $j \in \{3,\ldots,n\}$ is such that $u_j = u^*$. If $l^* < u_2 - p_1$, i.e., $[l^*, u_2 - p_1] \neq \emptyset$, then $x_{j,l^*-p_j+1} = x_{1s} = x_{2u_2} = 1$ defines a counterexample for $(1,s)$ for $s \in [l^*, u_2 - p_1]$, where $j \in \{3,\ldots,n\}$ is such that $l_j = l^*$. Hence M_1 cannot be extended. In the same way we can show that the intervals M_j with $j \in \{3,\ldots,n\}$ cannot be extended.

We show that M_2 cannot be extended if and only if (1) and (2) hold. Suppose that (1) does not hold, i.e., $u^* < l_2$, $l < l_2 - p_2$, and $L_1 \cap U_1 = \emptyset$. Clearly $(2, l_2 - p_2) \notin V$. Let job 2 be started in period $l_2 - p_2$. Since $l^* > l_2$, it impossible to start a job $j \in \{3,\ldots,n\}$ in V before job 2. Since $u^* < l_2$, it is impossible to start a job $j \in \{3,\ldots,n\}$ in V after job 2. Clearly, job 1 cannot be started in $L_1 \cap U_1$ and we find that M_2 can be extended by $l_2 - p_2$. In the same way, if $u_2 < l^*$, $u_2 < u - p_2$ and $L_1 \cap U_1 = \emptyset$, then M_2 can be extended by $u_2 + 1$.

Now, suppose that (1) and (2) hold. Suppose $u^* \geq l_2$, i.e., $[l, \max\{u^*, l_2\} - p_2] = [l, u^* - p_2]$. If $l < u^* - p_2$, then $x_{1,l-p_1+1} = x_{2s} = x_{ju^*} = 1$ defines a counterexample for $(2,s)$ for all $s \in [l, u^* - p_2]$, where $j \in \{3,\ldots,n\}$ is such that $u_j = u^*$. Suppose that $u^* < l_2$, i.e., $[l, \max\{u^*, l_2\} - p_2] = [l, l_2 - p_2]$. If $l < l_2 - p_2$, then, by (1), $L_1 \cap U_1 \neq \emptyset$ and $x_{2s} = x_{1l_2} = 1$ defines a counterexample for $(2,s)$ for all $s \in [l, l_2 - p_2]$. Hence there is a counterexample for $(2,s)$ for all $s \in [l, \max\{u^*, l_2\} - p_2]$. In the same way, we can show that there is a counterexample for $(2,s)$ for all $s \in [\min\{l^*, u_2\}, u - p_2]$. We conclude that M_2 cannot be extended.

Clearly, the upper bound of L_1 cannot be increased. The lower bound of L_1 cannot be decreased if and only if there is a counterexample for $(1, l - p_1)$. Such a counterexample is given by a feasible schedule $x_{1,l-p_1} = x_{js} = 1$ with $(j,s) \in V^2$ or by a feasible schedule

$x_{1,l-p_1} = x_{j_1 s_1} = x_{j_2 s_2} = 1$ with $(j_1, s_1), (j_2, s_2) \in V \setminus V^2$. Observe that if $j \in \{2, \ldots, n\}$ is such that $L_j \cap U_j \neq \emptyset$, then $x_{1,l-p_1} = x_{jl} = 1$ defines a counterexample for $(1, l - p_1)$. We find that there is a counterexample of the first type if and only if (3a) holds. Consider a counterexample of the second type. Since $u_2 \geq \max\{u_j \mid j \in \{3, \ldots, n\}\}$, we may assume that job 2 occurs in this counterexample, i.e., the counterexample contains job 2 and a job $j_1 \in \{3, \ldots, n\}$. Suppose that job 2 is started before job j_1. It is easy to see that job 2 is started in L_2. So $L_2 \neq \emptyset$ and job 2 is started in period l. Furthermore, job j_1 is started in U_{j_1}. Hence $l \leq u^* - p_2$ and $U_{j_1} \neq \emptyset$. We find that there is a counterexample of the second type such that job 2 is started before job j_1 if and only if (3b) holds. If job 2 is started after job j_1, then job j_1 is started in period l and job 2 in U_2. Job j_1 may be choosen such that $p_{j_1} = \min\{p_j \mid j \in \{3, \ldots, n\},\ L_j \neq \emptyset\}$. We find that there is a counterexample of the second type such that job 2 is started after job j_1 if and only if (3c) holds. We conclude that the lower bound of L_1 cannot be decreased if and only if (3) holds. Analogously, U_1 cannot be extended if and only if (4) holds.

Clearly, L_2 cannot be extended if and only if there is a counterexample for $(2, l_2 - p_2)$, if $L_2 \neq \emptyset$, and a counterexample for $(2, l)$, if $L_2 = \emptyset$, i.e., if and only if there is a counterexample for $(2, y - p_2)$, where $y = \min\{l_2, l + p_2\}$. The proof that there is such a counterexample if and only if (5) holds, is similar to the proof that there is a counterexample for $(1, l - p_1)$ if and only if (3) holds. Analogously, U_2 cannot be extended if and only if (6) holds.

Let $j \in \{3, \ldots, n\}$. It is easy to see that L_j cannot be extended if and only if there is a counterexample for $(j, y - p_j)$, where $y = \min\{l^*, l + p_j\}$. Suppose $y \leq l_2$. If $L_1 \cap U_1 \neq \emptyset$, then $x_{j,y-p_j} = x_{1l_2} = 1$ defines a counterexample for $(j, y - p_j)$. If $L_1 \cap U_1 = \emptyset$, i.e., $l_2 \leq u_2 - p_1$, then $x_{j,y-p_j} = x_{1l_2} = x_{2u_2}$ defines such a counterexample. Hence, if $y \leq l_2$, then there is a counterexample for $(j, y - p_j)$. Now, suppose $y > l_2$. Since $x(U) \leq 1$, in any counterexample, at least one job is started in $L \cup M$. If $x_{j,y-p_j} = 1$, then job 1 and 2 are the only jobs that can be started in $L \cup M$. It is now easy to see that a counterexample for $(j, y - p_j)$ contains job 1 and job 2 and we find that there is such a counterexample if and only if (7a) or (7b) holds. Analogously, the intervals U_j with $j \in \{3, \ldots, n\}$, cannot be extended if and only if (8) holds. □

4.2 Case 1b

As in case 1a, the conditions on l_j and u_j and Properties 4 and 5 completely determine the sets L and U. All that remains to be investigated is the structure of the set M.

Property 8 *If $x(V^1) + 2x(V^2) \leq 2$ is facet inducing with $l = l_1 < l_2 < l^*$ and $u = u_1 > u_3 > u^*$, then $M_1 = \emptyset$, $M_2 = [u_3 - p_2, l^*] \cap [l, u - p_2] \cap [l_2 - p_2, u^*]$, $M_3 = [u^* - p_3, l_2] \cap [l, u - p_3] \cap [l^* - p_3, u_3]$ and $M_j = [u_3 - p_j, l_2] \cap [l, u - p_j] \cap [l^* - p_j, u^*]$ for $j \in \{4, \ldots, n\}$.*

Proof. Let $x(V^1) + 2x(V^2) \leq 2$ be facet inducing with $l = l_1 < l_2 < l^*$ and $u = u_1 > u_3 > u^*$. If job 1 is started in V and it is possible to start a job in V before job 1, then, since $l_2 = \min\{l_j \mid j \in \{2, \ldots, n\}\}$, job 2 can be started in V before job 1. If it is possible to start a job in V after job 1, then, since $u_3 = \max\{u_j \mid j \in \{2, \ldots, n\}\}$, job 3 can be started after job 1. It follows that $M_1 = \emptyset$.

Since $L_2 \cap M_2 = \emptyset$ and $M_2 \cap U_2 = \emptyset$, we have $M_2 \subseteq [l, u - p_2]$. By definition $M_2 \subseteq [l_2 - p_2, u^*]$. If job 2 is started in M_2, then job 1 should be the only job that can be started before or after job 2, i.e., it should be impossible to start any job j $j \in \{3, \ldots, n\}$ in V. Hence

$M_2 \subseteq [u_3 - p_2, l^*]$. We conclude that $M_2 \subseteq [u_3 - p_2, l^*] \cap [l, u - p_2] \cap [l_2 - p_2, u^*]$. If job 2 is started in period $s \in [u_3 - p_2, l^*] \cap [l, u - p_2] \cap [l_2 - p_2, u^*]$, then, since $s \in [l_2 - p_2, u^*]$, $L_1 \cap U_1 = [u_3 - p_1, l_2]$, and $u_3 > u^*$, job 1 cannot be started in $L_1 \cap U_1$. Since $x(V^1) + 2x(V^2) \leq 2$ is maximal, it follows that $M_2 = [u_3 - p_2, l^*] \cap [l, u - p_2] \cap [l_2 - p_2, u^*]$. Analogously, $M_3 = [u^* - p_3, l_2] \cap [l, u - p_3] \cap [l^* - p_3, u_3]$ and $M_j = [u_3 - p_j, l_2] \cap [l, u - p_j] \cap [l^* - p_j, u^*]$ for $j \in \{4, \ldots, n\}$. □

Properties 4, 5 and 8 determine the LMU-structure of a facet inducing inequality $x(V^1) + 2x(V^2) \leq 2$ with $l = l_1 < l_2 < l^*$ and $u = u_1 > u_3 > u^*$. As in case 1a, we prefer to use a different representation of the set M, in order to emphasize the inherent structure of the intervals M_j. It turns out that a facet inducing inequality $x(V^1) + 2x(V^2) \leq 2$ with $l = l_1 < l_2 < l^*$ and $u = u_1 > u_3 > u^*$ has the following property, which restricts the class of inequalities determined by Properties 4, 5 and 8 and leads to a simpler form of the intervals M_j.

Property 9 *If $x(V^1) + 2x(V^2) \leq 2$ is facet inducing with $l = l_1 < l_2 < l^*$ and $u = u_1 > u_3 > u^*$, then $l^* \leq u^*$.*

Proof. Let $x(V^1) + 2x(V^2) \leq 2$ be facet inducing with $l = l_1 < l_2 < l^*$ and $u = u_1 > u_3 > u^*$. To be able to prove that $l^* \leq u^*$, we first show that $l^* < u_3$ and $l_2 < u^*$. Suppose $l^* \geq u_3$. We show that $[u_3 - p_2, u_3] \subseteq V_2$, which contradicts $u_3 > u^*$. Let job 2 be started in $[u_3 - p_2, u_3]$. Since by assumption $l^* \geq u_3$, it is impossible to start any job $j \in \{3, \ldots, n\}$ before job 2. Clearly, it is impossible to start any job $j \in \{3, \ldots, n\}$ after job 2. So job 1 is the only job that may be started in V. Since $L_1 \cap U_1 = [u_3 - p_2, l_2]$ and, by Property 6, $l_2 < u_3$, job 1 cannot be started in $L_1 \cap U_1$. Since $x(V^1) + 2x(V^2) \leq 2$ is maximal, it follows that $[u_3 - p_2, u_3] \subseteq V_2$. If $l_2 \geq u^*$, then analogously $[l_2 - p_3, l_2] \subseteq V_3$, which contradicts $l_2 > l^*$. Hence $l^* < u_3$ and $l_2 < u^*$.

To prove that $l^* \leq u^*$ we consider two cases: $l^* > u - p_j$ for some $j \in \{2, 4, \ldots, n\}$, and $l^* \leq u - p_j$ for all $j \in \{2, 4, \ldots, n\}$. Suppose that $j_1 \in \{2, 4, \ldots, n\}$ is such that $l^* > u - p_{j_1}$ and let job j_1 be started in $[u - p_{j_1}, l^*]$. Clearly, any job $j \in \{3, \ldots, n\} \setminus \{j_1\}$ cannot be started before job j_1. If job 2 is started before job j_1, then, since $M_2 \subseteq [u_3 - p_2, l^*]$ and $l^* < u_3$, job 2 is not started in M_2 and hence job 2 is started in L_2. If job 1 is started before job j_1, then, since $M_1 = \emptyset$, job 1 is started in L_1. It follows that at most one job can be started in V before job j_1. Since $L_1 \cap U_1 = [u_3 - p_1, l_2]$ and $l_2 < l^* < u_3$, job 1 cannot be started in $L_1 \cap U_1$. Since $x(V^1) + 2x(V^2) \leq 2$ is maximal, it follows that $[u - p_{j_1}, l^*] \subseteq U_{j_1}$. Hence $l^* \leq u^*$. Now, suppose that $l^* \leq u - p_j$ for all $j \in \{2, 4, \ldots, n\}$. Observe that from $l_2 < u^*$ and Property 8, it follows that $U_j \neq \emptyset$ for some $j \in \{2, 4, \ldots, n\}$ or $M_2 \neq \emptyset$. If $U_j \neq \emptyset$ for some $j \in \{2, 4, \ldots, n\}$, then clearly $l^* \leq u^*$. Suppose $U_j = \emptyset$ for all $j \in \{2, 4, \ldots, n\}$. Clearly, $M_2 \neq \emptyset$ and, since by Property 8, $M_2 = [u_3 - p_2, l^*] \cap [l, u - p_2] \cap [l_2 - p_2, u^*]$, we must have $u_3 - p_2 < l^*$. It is easy to see that if job 2 is started in $[u_3 - p_2, l^*] \cap [l, u - p_2]$, then job 1 is the only job that can be started before or after job 2 and job 1 cannot be started in $L_1 \cap U_1$. Since $x(V^1) + 2x(V^2) \leq 2$ is maximal, this implies $M_2 = [u_3 - p_2, l^*] \cap [l, u - p_2]$ and it follows that $l^* \leq u^*$. □

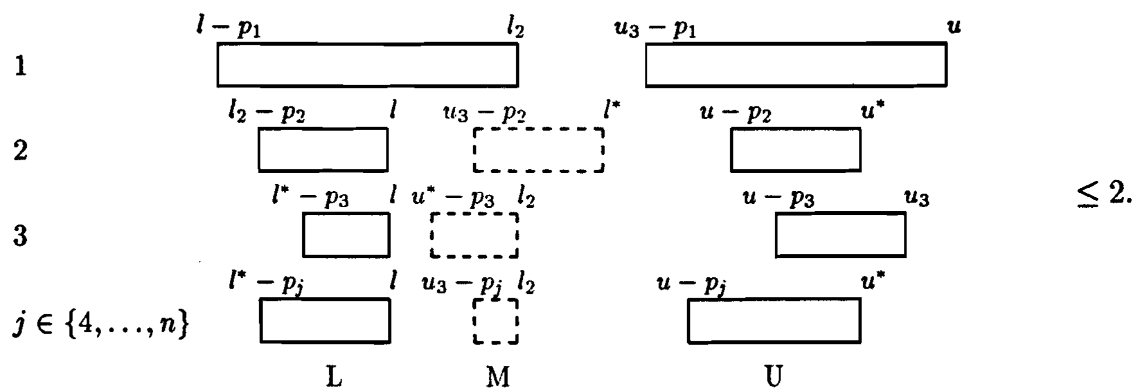
It is not hard to see that Properties 4, 5, 8, and 9 can be combined to give the following theorem.

Theorem 8 *A facet inducing inequality* $x(V^1) + 2x(V^2) \leq 2$ *with* $l = l_1 < l_2 < l^*$ *and* $u = u_1 > u_3 > u^*$ *has the following LMU-structure:*

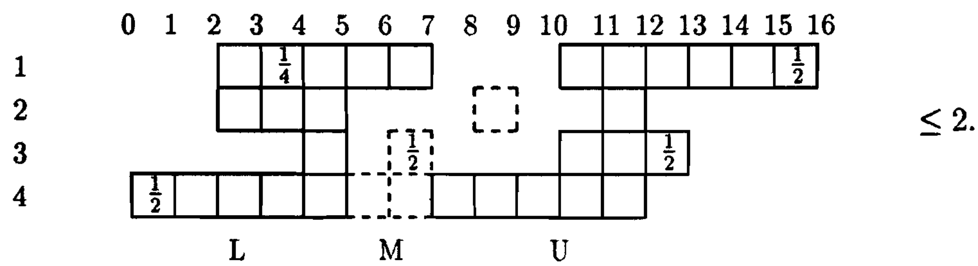
$$\begin{array}{llll} L_1 = [l - p_1, l_2], & M_1 = \emptyset, & U_1 = [u_3 - p_1, u], & \\ L_2 = [l_2 - p_2, l], & M_2 = [u_3 - p_2, l^*] \setminus (L_2 \cup U_2), & U_2 = [u - p_2, u^*], & \\ L_3 = [l^* - p_3, l], & M_3 = [u^* - p_3, l_2] \setminus (L_3 \cup U_3), & U_3 = [u - p_3, u_3], & \\ L_j = [l^* - p_j, l], & M_j = [u_3 - p_j, l_2] \setminus (L_j \cup U_j), & U_j = [u - p_j, u^*] & (j \in \{4, \ldots, n\}), \end{array} \quad (5)$$

where $l^* \leq u^*$.

This theorem says that a facet inducing inequality $x(V^1) + 2x(V^2) \leq 2$ with $l = l_1 < l_2 < l^*$ and $u = u_1 > u_3 > u^*$ can be represented by the following diagram:



Example 3 Let $n = 4$, $p_1 = 3$, $p_2 = 5$, $p_3 = 6$, and $p_4 = 9$. The inequality with LMU-structure (5) and $l = 5$, $l_2 = 7$, $l^* = 9$, $u^* = 12$, $u_3 = 13$ and $u = 16$ is given by the following diagram:



The fractional solution $x_{1,16} = x_{37} = x_{3,13} = x_{41} = \frac{1}{2}$ and $x_{14} = \frac{1}{4}$ violates this inequality. It is easy to check that this solution satisfies all inequalities with structure (3).

Sufficient conditions are given by the following theorem.

Theorem 9 *A valid inequality* $x(V^1) + 2x(V^2) \leq 2$ *with* $l = l_1 < l_2 < l^*$ *and* $u = u_1 > u_3 > u^*$ *and LMU-structure* (5) *that is nondecomposable and maximal is facet inducing.*

The proof of this theorem proceeds along the same lines as that of Theorem 5. Specific necessary and sufficient conditions for a valid inequality $x(V^1) + 2x(V^2) \leq 2$ with $l = l_1 < l_2 < l^*$ and $u = u_1 > u_3 > u^*$ and LMU-structure (5) to be nondecomposable and maximal are given by the following two theorems.

Theorem 10 *A valid inequality $x(V^1)+2x(V^2) \leq 2$ with $l = l_1 < l_2 < l^*$ and $u = u_1 > u_3 > u^*$ and LMU-structure* (5) *is nondecomposable if and only if $M_j \neq \emptyset$ for some $j \in \{2, \ldots, n\}$.*

Proof. Let $x(V^1) + 2x(V^2) \leq 2$ be a valid inequality with $l = l_1 < l_2 < l^*$ and $u = u_1 > u_3 > u^*$ and LMU-structure (5). If $x(V^1) + 2x(V^2) \leq 2$ is nondecomposable, then, since $x(L) \leq 1$ and $x(U) \leq 1$, $M_j \neq \emptyset$ for some $j \in \{2, \ldots, n\}$.

Suppose $M_j \neq \emptyset$ for some $j \in \{2, \ldots, n\}$. Let W and W' be such that $x(W) + x(W') = x(V^1) + 2x(V^2)$, and $x(W) \leq 1$ and $x(W') \leq 1$. We assume w.l.o.g. that $(1, l - p_1 + 1) \in W$. Since $l < l_2 < u_3$, $x_{1,l-p_1+1} = x_{3u_3} = 1$ is a feasible schedule. From $(1, l - p_1 + 1) \in W$ and $x(W) \leq 1$ it follows that $(3, u_3) \in W'$. In the same way, since $l_2 < u_3$ and $(3, u_3) \in W'$, it follows that $(2, l_2 - p_2 + 1) \in W$ and, since $l_2 < u$, it follows that $(1, u) \in W'$. Now, let $j \in \{2, \ldots, n\}$ be such that $M_j \neq \emptyset$. If job j is started in M_j, then job 1 can be started in V before or after job j. If $s \in M_j$, then, since $x_{1,l-p_1+1} = x_{js} = 1$ is a feasible schedule and $(1, l - p_1 + 1) \in W$, we have $(j, s) \in W'$. We find that $x_{js} = x_{1u} = 1$ is a feasible schedule such that $x(W') = 2$, which yields a contradiction. We conclude that from $(1, l - p_1 + 1) \in W$, $(1, u) \in W'$ and $M_j \neq \emptyset$, it follows that $x(V^1) + 2x(V^2) \leq 2$ is nondecomposable. □

Theorem 11 *A valid inequality $x(V^1) + 2x(V^2) \leq 2$ with $l = l_1 < l_2 < l^*$ and $u = u_1 > u_3 > u^*$ and LMU-structure* (5) *is maximal if and only if*

One of the following holds:
(1a) *$L_j \cap U_j \neq \emptyset$ for some $j \in \{2, \ldots, n\}$;*
(1b) *$l \leq u^* - p_3$ and $L_3 \neq \emptyset$ and $U_j \neq \emptyset$ for some $j \in \{2, 4, \ldots, n\}$;*
(1c) *$l \leq u_3 - \min\{p_j \mid j \in \{2, 4, \ldots, n\}, L_j \neq \emptyset\}$;*

One of the following holds:
(2a) *$L_j \cap U_j \neq \emptyset$ for some $j \in \{2, \ldots, n\}$;*
(2b) *$l^* \leq u - p_2$ and $U_2 \neq \emptyset$ and $L_j \neq \emptyset$ for some $j \in \{3, \ldots, n\}$;*
(2c) *$l_2 \leq u - \min\{p_j \mid j \in \{3, \ldots, n\}, U_j \neq \emptyset\}$;*

One of the following holds:
(3a) *$\min\{l^*, l + p_3\} \leq u - p_2$ and $M_2 \neq \emptyset$;*
(3b) *$\min\{l^*, l + p_3\} \leq l_2$ and $L_1 \cap U_1 \neq \emptyset$;*
(3c) *$\min\{l^*, l + p_3\} \leq l_2$ and $\min\{l^*, l + p_3\} \leq u^* - p_1$;*
(3d) *$\min\{l^*, l + p_3\} \leq l_2$ and $\min\{l^*, l + p_3\} \leq u - \min\{p_j \mid j \in \{4, \ldots, n\}, M_j \neq \emptyset\}$ and $M_j \neq \emptyset$ for some $j \in \{4, \ldots, n\}$;*

One of the following holds:
(4a) *$l \leq \max\{u^*, u - p_2\} - p_3$ and $M_3 \neq \emptyset$;*
(4b) *$u_3 \leq \max\{u^*, u - p_2\}$ and $L_1 \cap U_1 \neq \emptyset$;*
(4c) *$u_3 \leq \max\{u^*, u - p_2\}$ and $l^* \leq \max\{u^*, u - p_2\} - p_1$;*
(4d) *$u_3 \leq \max\{u^*, u - p_2\}$ and $l \leq \max\{u^*, u - p_2\} - \min\{p_j \mid j \in \{4, \ldots, n\}, M_j \neq \emptyset\}$ and $M_j \neq \emptyset$ for some $j \in \{4, \ldots, n\}$;*

For all $j \in \{4, \ldots, n\}$, one of the following holds:
(5a) *$\min\{l^*, l + p_j\} \leq u - p_2$ and $M_2 \neq \emptyset$;*

(5b) $\min\{l^*, l+p_j\} \leq l_2$;

For all $j \in \{4,\ldots,n\}$, one of the following holds:
(6a) $l \leq \max\{u^*, u-p_j\} - p_3$ *and* $M_3 \neq \emptyset$;
(6b) $u_3 \leq \max\{u^*, u-p_j\}$.

Proof. Let $x(V^1) + 2x(V^2) \leq 2$ be a valid inequality with $l = l_1 < l_2 < l^*$ and $u = u_1 > u_3 > u^*$ and LMU-structure (5). Observe that $x(V^1) + 2x(V^2) \leq 2$ is maximal if and only if it is impossible to extend any of the intervals L_j, M_j and U_j.

In the same way as in the proof of Theorem 7 it can be shown that the intervals M_j cannot be extended.

We show that L_2 cannot be extended. Note that since $l_2 < u_3$, we have $L_2 \neq \emptyset$. It is now easy to see that it suffices to show that there is a counterexample for $(2, l_2 - p_2)$. If $L_1 \cap U_1 \neq \emptyset$, then $x_{2,l_2-p_2} = x_{1l_2} = 1$ defines a counterexample for $(2, l_2 - p_2)$. If $L_1 \cap U_1 = \emptyset$, i.e., $l_2 \leq u_3 - p_1$, then $x_{2,l_2-p_2} = x_{1l_2} = x_{3u_3} = 1$ defines a counterexample for $(2, l_2 - p_2)$. Hence L_2 cannot be extended. Analogously, U_3 cannot be extended.

Clearly, L_1 cannot be extended if and only if there is a counterexample for $(1, l - p_1)$. If such a counterexample is given by a feasible schedule $x_{1,l-p_1} = x_{j_1 s_1} = x_{j_2 s_2} = 1$ with $(j_1, s_1), (j_2, s_2) \in V \setminus V^2$, then, since $u_3 = \max\{u_j \mid j \in \{2,\ldots,n\}\}$, we may assume that job 3 occurs in this counterexample. We can use this observation to show that L_1 cannot be extended if and only if (1) holds. Analogously, U_1 cannot be extended if and only if (2) holds.

It is easy to see that L_3 cannot be extended if and only if there is a counterexample for $(3, y-p_3)$, where $y = \min\{l^*, l+p_3\}$. Since $u_1 = u$, we may assume that such a counterexample contains job 1. Suppose $y > l_2$. Since $x(U) \leq 1$, in any counterexample at least one job is started in $L \cup M$. If $x_{3,y-p_3} = 1$, then job 2 is the only job that can be started in $L \cup M$. Hence in a counterexample for $(3, y - p_3)$ job 2 is started in M_2 and job 1 is started in U_1. Such a counterexample exists if and only if (3a) holds. If $y \leq l_2$ and $x_{3,y-p_3} = 1$, then job 1 and any job $j \in \{2, 4, \ldots, n\}$ and $M_j \neq \emptyset$ may be started in $L \cup M$. It is now not hard to see that there is a counterexample for $(3, y - p_3)$ if and only if (3) holds. Analogously, U_2 cannot be extended if and only if (4) holds.

Let $j \in \{4,\ldots,n\}$. It is easy to see L_j cannot be extended if and only if there is a counterexample for $(j, y-p_j)$, where $y = \min\{l^*, l+p_j\}$. Suppose $y \leq l_2$. If $L_1 \cap U_1 \neq \emptyset$, then $x_{j,y-p_j} = x_{1l_2} = 1$ defines a counterexample for $(j, y - p_j)$. If $L_1 \cap U_1 = \emptyset$, i.e., $l_2 \leq u_3 - p_1$, then $x_{j,y-p_j} = x_{1l_2} = x_{3u_3} = 1$ defines such a counterexample. Hence, if $y \leq l_2$, then there is a counter example for $(j, y-p_j)$. Now, suppose $y > l_2$. Since $x(U) \leq 1$, in any counterexample at least one job is started in $L \cup M$. If $x_{j,y-p_j} = 1$, then job 2 is the only job that can be started in $L \cup M$. It is now easy to see that a counterexample for $(j, y - p_j)$ contains job 2 and job 1 and we find that there is such a counterexample if and only if (5a) holds. We conclude that the intervals L_j with $j \in \{4,\ldots,n\}$ cannot be extended if and only if (5) holds. Analogously, the intervals U_j with $j \in \{4,\ldots,n\}$ cannot be extended if and only if (6) holds. □

4.3 Case 2

Observe that in this case the conditions on l_j and u_j and Properties 4 and 5 do not completely determine the sets L and U. It turns out to be beneficial to introduce a notion slightly different from that of l^* and u^*, namely $l' = \min\{l_j \mid j \in \{3,\ldots,n\}\}$ and $u' = \max\{u_j \mid j \in \{3,\ldots,n\}\}$. Note that it is possible that $l_2 > l'$ or $u_1 < u'$, i.e., l' and u' do not necessarily coincide with l^* and u^* as defined in Property 5. We can however prove a property that is similar to Property 5.

Property 10 *Let $x(V^1) + 2x(V^2) \leq 2$ be facet inducing with $l = l_1$ and $u = u_2$.*
(a) *For all $j \in \{3,\ldots,n\}$ such that $L_j \neq \emptyset$, we have $l_j = l'$ and for all $j \in \{3,\ldots,n\}$ such that $L_j = \emptyset$, we have $l' - p_j \geq l$, i.e., $L_j = [l' - p_j, l]$ for all $j \in \{3,\ldots,n\}$.*
(b) *For all $j \in \{3,\ldots,n\}$ such that $U_j \neq \emptyset$, we have $u_j = u'$ and for all $j \in \{3,\ldots,n\}$ such that $U_j = \emptyset$, we have $u' \leq u - p_j$, i.e., $U_j = [u - p_j, u']$ for all $j \in \{3,\ldots,n\}$.*

Proof. (a) By Property 4, $L_j \subseteq [l' - p_j, l]$ for all $j \in \{3,\ldots,n\}$. Assume w.l.o.g. $l' = l_3$. Suppose that $L_j \neq [l' - p_j, l]$ for some $j \in \{4,\ldots,n\}$, say $L_4 \neq [l' - p_4, l]$. Clearly, if $l' - p_4 \geq l$, then $L_4 = \emptyset$ and hence $L_4 = [l' - p_4, l]$. Consequently, $l' - p_4 < l$ and $l_4 > l'$, i.e., $l' - p_4 + 1 \notin V_4$. Since $x(V^1) + 2x(V^2) \leq 2$ is maximal, there is a counterexample for $(4, l' - p_4 + 1)$. Let $(j_1, s_1), (j_2, s_2) \in V$ be such that $x_{4,l'-p_4+1} = x_{j_1 s_1} = x_{j_2 s_2} = 1$ is a feasible schedule. Since $l' - p_4 + 1 \leq l$, the jobs j_1 and j_2 are started after job 4. Assume that job j_1 is started before job j_2. If job 2 does not occur in $\{j_1, j_2\}$, then, since $u_2 = u$, job j_2 may be replaced by job 2. So we may assume that job 2 occurs in $\{j_1, j_2\}$. It follows that one of the jobs 1 and 3 does not occur in $\{j_1, j_2\}$. Suppose that job 3 does not occur. It is now easy to see that $x_{3,l'-p_3+1} = x_{j_1 s_1} = x_{j_2 s_2} = 1$ is a feasible schedule. This schedule violates the inequality, which yields a contradiction. If job 1 does not occur in $\{j_1, j_2\}$, then we find a contradiction in the same way.

(b) Similar to (a). □

We next investigate the structure of the set M.

Property 11 *If $x(V^1) + 2x(V^2) \leq 2$ is facet inducing with $l = l_1$ and $u = u_2$, then $M_1 = [u' - p_1, l'] \cap [\min\{l_2, l'\}, u - p_1] \cap [l - p_1, u_1]$, $M_2 = [u' - p_2, l'] \cap [l, \max\{u_1, u'\} - p_2] \cap [l_2 - p_2, u]$ and $M_j = \emptyset$ for $j \in \{3,\ldots,n\}$.*

Proof. Let $x(V^1) + 2x(V^2) \leq 2$ be facet inducing with $l = l_1$ and $u = u_2$. To determine M_1 we consider two cases: $l_2 \geq l'$ and $l_2 < l'$. Suppose $l_2 \geq l'$. Let job 1 be started in V. If it is possible to start a job in V after job 1, then, since $u_2 = u$, job 2 can be started in V after job 1. If it is possible to start a job in V before job 1, then, since $l' \leq l_2$, there exists a $j \in \{3,\ldots,n\}$ such that job j can be started in V before job 1. It easily follows that $M_1 = \emptyset$. Suppose $l_2 < l'$. Since $L_1 \cap M_1 = \emptyset$ and $M_1 \cap U_1 = \emptyset$, we have $M_1 \subseteq [\min\{l_2, l'\}, u - p_1]$. By definition $M_1 \subseteq [l - p_1, u_1]$. If job 1 is started in M_1, then job 2 should be the only job that can be started in V before or after job 1, i.e., it should be impossible to start any job $j \in \{3,\ldots,n\}$ in V. Hence $M_1 \subseteq [u' - p_1, l']$. We conclude that $M_1 \subseteq [u' - p_1, l'] \cap [\min\{l_2, l'\}, u - p_1] \cap [l - p_1, u_1]$. If job 1 is started in period $s \in [u' - p_1, l'] \cap [\min\{l_2, l'\}, u - p_1] \cap [l - p_1, u_1]$, then, since $s \in [l - p_1, u_1]$ and $L_2 \cap U_2 = [\max\{u_1, u'\} - p_2, l]$, job 2 cannot be started in $L_2 \cap U_2$. Hence, if $l_2 < l'$, then $M_1 = [u' - p_1, l'] \cap [\min\{l_2, l'\}, u - p_1] \cap [l - p_1, u_1]$. Note that the intersection of these three intervals is empty if $l_2 \geq l'$. We conclude that $M_1 = [u' - p_1, l'] \cap [\min\{l_2, l'\}, u - p_1] \cap [l - p_1, u_1]$. Analogously $M_2 = [u' - p_2, l'] \cap [l, \max\{u_1, u'\} - p_2] \cap [l_2 - p_2, u]$.

Let $j \in \{3,\ldots,n\}$. If job j is started in V and it is possible to start a job in V before job j, then, since $l_1 = l$, job 1 can be started before job j. If it is possible to start a job in V after job j, then, since $u_2 = u$, job 2 can be started after job j. It follows that $M_j = \emptyset$. □

Properties 4, 10 and 11 completely determine the LMU-structure of a facet inducing inequality $x(V^1) + 2x(V^2) \leq 2$ with $l = l_1$ and $u = u_2$. As in the previous two cases, we prefer to use a different representation of the set M, in order to emphasize the inherent structure of the intervals M_j. It is easy to show that if $x(V^1) + 2x(V^2) \leq 2$ is facet inducing with $l = l_1$ and $u = u_2$, then $[l' - p_2, l] \subseteq L_2$ and $[u - p_1, u'] \subseteq U_1$. It is now not hard to see that Properties 4, 10 and 11 can be combined to give the following theorem.

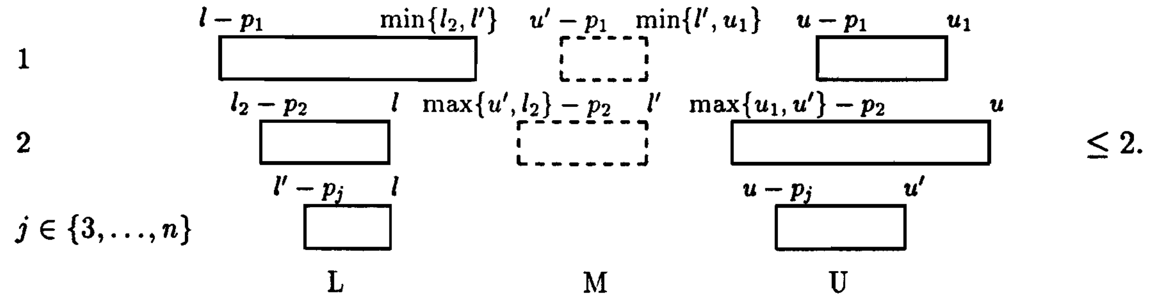
Theorem 12 *A facet inducing inequality $x(V^1) + 2x(V^2) \leq 2$ with $l = l_1$ and $u = u_2$ has the following LMU-structure:*

$$\begin{array}{ll} L_1 = [l - p_1, \min\{l_2, l'\}], & M_1 = [u' - p_1, \min\{l', u_1\}] \setminus (L_1 \cup U_1), \\ L_2 = [l_2 - p_2, l], & M_2 = [\max\{u', l_2\} - p_2, l'] \setminus (L_2 \cup U_2), \\ L_j = [l' - p_j, l], & M_j = \emptyset, \end{array}$$

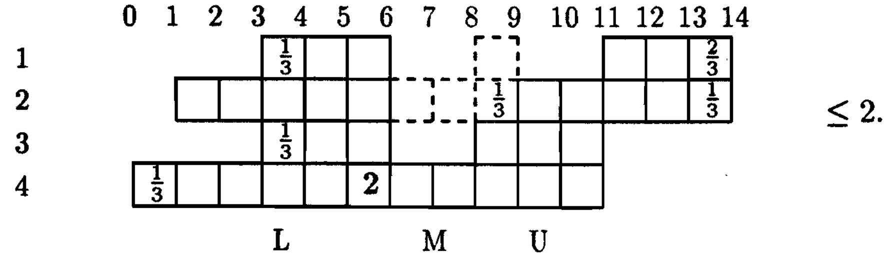
$$\begin{array}{ll} U_1 = [u - p_1, u_1], & \\ U_2 = [\max\{u_1, u'\} - p_2, u], & \\ U_j = [u - p_j, u'] & (j \in \{3,\ldots,n\}), \end{array} \tag{6}$$

where $[l' - p_2, l] \subseteq L_2$ *and* $[u - p_1, u'] \subseteq U_1$.

This theorem says that a facet inducing inequality $x(V^1)+2x(V^2) \leq 2$ with $l = l_1$ and $u = u_2$ can be represented by the following diagram:



Example 4 Let $n = 4$, $p_1 = 3$, $p_2 = 5$, $p_3 = 6$, and $p_4 = 9$. The inequality with LMU-structure (6) and $l = l_2 = 6$, $l^* = 9$, $u^* = 11$, and $u_1 = u = 14$ is given by the following diagram:



Note that $(4,6) \in L \cap U$, i.e., x_{46} has coefficient 2. The fractional solution $x_{14} = x_{29} = x_{2,14} = x_{34} = x_{41} = \frac{1}{3}$ and $x_{1,14} = \frac{2}{3}$ violates this inequality. It is easy to check that this solution satisfies all inequalities with structure (3).

Sufficient conditions are given by the following theorem.

Theorem 13 *A valid inequality $x(V^1) + 2x(V^2) \leq 2$ with $l = l_1$ and $u = u_2$ and LMU-structure (6) that is nondecomposable and maximal is facet inducing.*

Proof. Let $x(V^1) + 2x(V^2) \leq 2$ be a valid inequality with $l = l_1$ and $u = u_2$ and LMU-structure (6) that is nondecomposable and maximal, and let $F = \{x \in P | x(V^1) + 2x(V^2) = 2\}$. As in the proof of Theorem 5, we show that $\dim(F) = \dim(P) - 1$ by exhibiting $\dim(P) - 1$ linearly independent directions in F. Again, we give three sets of directions: unit vectors $d_{js} = 1$ for all $(j,s) \notin V$, directions $d_{js} = 1, d_{1,l-p_1+1} = d_{2u} = -1$ for all $(j,s) \in V^2$, and a set of $|V| - |V^2| - 1$ linearly independent directions $d_{j_1 s_1} = 1, d_{j_2, s_2} = -1$ with $(j_1, s_1), (j_2, s_2) \in V \setminus V^2$. Together these give $\dim(P) - 1$ linearly independent directions in F. The first and second set of directions can be determined as in the proof of Theorem 5.

Again, we determine the $|V| - |V^2| - 1$ directions $d_{j_1 s_1} = 1, d_{j_2 s_2} = -1$ with $(j_1, s_1), (j_2, s_2) \in V \setminus V^2$ in such that the undirected graph G whose vertices are the elements of $V \setminus V^2$ and whose edges are given by the pairs $\{(j_1, s_1), (j_2, s_2)\}$ corresponding to the determined directions is a spanning tree. It is not hard to see that this implies that the determined directions are linearly independent.

Furthermore, we say that $d_{j_1 s_1} = 1, d_{j_2 s_2} = -1$ with $(j_1, s_1), (j_2, s_2) \in V \setminus V^2$ is a direction by (i,t), if there exists an index $(i,t) \in V \setminus V^2$ such that $x_{j_1 s_1} = x_{it} = 1$ and $y_{j_2 s_2} = y_{it} = 1$ are both feasible schedules.

First, we determine directions that correspond to edges in G within the sets $\{(j,s) \mid s \in (L_j \cup M_j) \setminus U_j\}$ and $\{(j,s) \mid s \in U_j \setminus L_j\}$. For $s-1, s \in L_1 \setminus U_1$, $d_{1,s-1} = -1, d_{1s} = 1$ is a direction by $(2,u)$. If $M_1 \neq \emptyset$, then $d_{1,\min\{l_2,l'\}} = -1, d_{1m} = 1$ is a direction by $(2,u)$, where $m = \min\{s \mid s \in M_1\}$, and for $s-1, s \in M_1$, $d_{1,s-1} = -1, d_{1s} = 1$ also is a direction by $(2,u)$. Let $s-1, s \in U_1 \setminus L_1$. Note that $s - 1 > \min\{l_2, l'\}$. If $l_2 \leq l'$, then $d_{1,s-1} = -1, d_{1s} = 1$ is a direction by $(2, l_2 - p_2 + 1)$. If $l_2 > l'$, then $d_{1,s-1} = -1, d_{1s} = 1$ is a direction by $(j, l' - p_j + 1)$, where $j \in \{3, \ldots, n\}$ is such that $l_j = l'$. In the same way we find that for $s-1, s \in L_2 \setminus U_2$, $d_{2,s-1} = -1, d_{2s} = 1$ is a direction by $(1, u_1)$, if $u_1 \geq u'$, and by (j, u'), if $u_1 < u'$, where $j \in \{3, \ldots, n\}$ is such that $u_j = u'$. Observe that if job 2 is started in M_2, then job 1 is the only job that can be started before or after job 2. We find that if $L_2 \neq \emptyset$ and $M_2 \neq \emptyset$, then $d_{2l} = -1, d_{2m} = 1$ is a direction by $(1, u_1)$, where $m = \min\{s \mid s \in M_2\}$. For $s-1, s \in M_2$, $d_{2,s-1} = -1, d_{2s} = 1$ also is a direction by $(1, u_1)$. Furthermore, for $s-1, s \in U_2 \setminus L_2$, $d_{2,s-1} = -1, d_{2s} = 1$ is a direction by $(1, l - p_1 + 1)$. Now, let $j \in \{3, \ldots, n\}$. Note that $M_j = \emptyset$. Clearly, for $s-1, s \in L_j \setminus U_j$, $d_{j,s-1} = -1, d_{js} = 1$ is a direction by $(2,u)$ and for $s-1, s \in U_j \setminus L_j$, $d_{j,s-1} = -1, d_{js} = 1$ is a direction by $(1, l - p_1 + 1)$.

Second, we determine directions that correspond to edges in G between the sets $\{(j,s) \mid s \in (L_j \cup M_j) \setminus U_j\}$ belonging to different jobs and between sets $\{(j,s) \mid s \in U_j \setminus L_j\}$ belonging to different jobs. It is easy to see that for $j \in \{3, \ldots, n\}$ such that $L_j \neq \emptyset$, $d_{1,l-p_1+1} = -1, d_{l'-p_j+1} = 1$ is a direction by $(2,u)$. For $j \in \{3, \ldots, n\}$ such that $U_j \neq \emptyset$, $d_{2u} = -1, d_{ju'} = 1$ is a direction by $(1, l - p_1 + 1)$. We still have to determine a direction that corresponds to an edge in G between $\{(2,s) \mid s \in (L_2 \cup M_2) \setminus U_2\}$ and one of the sets $\{(j,s) \mid s \in (L_j \cup M_j) \setminus U_j\}$ with $j \in \{1, 3, \ldots, n\}$, and a direction that corresponds to an edge in G between $\{(1,s) \mid s \in U_1 \setminus L_1\}$ and one of the sets $\{(j,s) \mid s \in U_j \setminus L_j\}$ with $j \in \{2,\ldots,n\}$. Note that, since $x(V^1) + 2x(V^2) \leq 2$ is nondecomposable, we have $(L_2 \cup M_2) \setminus U_2 \neq \emptyset$. We define $W = \{(1,s) \mid s \in V_1\} \cup \{(2,s) \mid s \in L_2 \cap U_2\} \cup \{(j,s) \mid j \in \{3,\ldots,n\},\ s \in L_j\}$ and $W' = \{(1,s) \mid s \in L_1 \cap U_1\} \cup \{(2,s) \mid s \in V_2\} \cup \{(j,s) \mid j \in \{3,\ldots,n\},\ s \in U_j\}$. Note that $x(W) + x(W') = x(V^1) + 2x(V^2)$. Since $x(V^1) + 2x(V^2) \leq 2$ is nondecomposable, there exists a feasible schedule such that $x(W) = 2$ or there exists a feasible schedule such that $x(W') = 2$. Suppose that there exists a feasible schedule such that $x(W) = 2$. It is easy to see that in such a schedule some job $j \in \{3,\ldots,n\}$ is started in L_j and that job 1 is started after job j. It easily follows that $x_{j,l'-p_j+1} = x_{1u_1} = 1$ is a feasible schedule for all $j \in \{3,\ldots,n\}$ with $L_j \neq \emptyset$. Let $j_1 \in \{3,\ldots,n\}$ be such that $L_{j_1} \neq \emptyset$. If $L_2 \neq \emptyset$, then, since $[l'-p_2, l] \subseteq L_2$, we have $l_2 \leq l'$. It follows that $y_{2,l_2-p_2+1} = y_{1u_1} = 1$ is a feasible schedule. If job 2 is started in M_2, then job 1 can be started after job 2. Hence, if $M_2 \neq \emptyset$, then $y_{2,l_2-p_2+1} = y_{1u_1} = 1$ also is a feasible schedule. We conclude that $d_{2,l_2-p_2+1} = -1, d_{j_1,l'-p_{j_1}+1} = 1$ is a direction by $(1,u_1)$. Since $x_{j_1,l'-p_{j_1}+1} = x_{1u_1} = 1$ is a feasible schedule, it follows that $y_{j_1,l'-p_{j_1}+1} = y_{2u}$ is a feasible schedule. We find that $d_{1u_1} = -1, d_{2u} = 1$ is a direction by $(j_1, l'-p_{j_1}+1)$.

Suppose that there is a feasible schedule such that $x(W') = 2$. It is now not hard to see that $x_{2,l_2-p_2+1} = x_{ju'} = 1$ is a feasible schedule for some $j \in \{3,\ldots,n\}$ such that $U_j \neq \emptyset$. Let $j_1 \in \{3,\ldots,n\}$ be such that $U_{j_1} \neq \emptyset$. Clearly, $y_{1,l-p_1+1} = y_{j_1u'} = 1$ also is a feasible schedule and we find that $d_{2,l_2-p_2+1} = 1, d_{1,l-p_1+1} = -1$ is a direction by (j_1, u'). If $U_1 \neq \emptyset$, then, since $[u-p_1, u'] \subseteq U_1$, we have $u_1 \geq u'$. Since $x_{2,l_2-p_2+1} = x_{j_1u'} = 1$ is a feasible schedule, it follows that $y_{2,l_2-p_2+1} = y_{1u_1} = 1$ is a feasible schedule. We find that $d_{j_1,u'} = 1, d_{1u_1} = -1$ is a direction by $(2, l_2-p_2+1)$.

Finally, we determine a direction that corresponds to an edge in G between $L \cup M$ and U. Since $x(V^1)+2x(V^2) \leq 2$ is nondecomposable and $x(U) \leq 1$, there is a feasible schedule such that $x(L) + x(M) = 2$. It is easy to see that in such a schedule job 1 and job 2 are started in $L \cup M$. Let $x_{1s_1} = x_{2s_2} = 1$ be such a schedule. Since $s_1 \in L_1 \cup M_1$, $y_{1s_1} = y_{2u} = 1$ also is a feasible schedule. It follows that $d_{2s_2} = 1, d_{2u} = -1$ is a direction by $(1, s_1)$.

It is easy to see that the determined directions form a spanning tree of G and hence we have determined $|V| - |V^2| - 1$ linearly independent directions. □

Specific necessary and sufficient conditions for a valid inequality $x(V^1) + 2x(V^2) \leq 2$ with $l = l_1$ and $u = u_2$ and LMU-structure (6) to be nondecomposable and maximal are given by the following two theorems.

Theorem 14 *A valid inequality $x(V^1) + 2x(V^2) \leq 2$ with $l = l_1$ and $u = u_2$ and LMU-structure (6) is nondecomposable if and only if $M_1 \neq \emptyset$ or $M_2 \neq \emptyset$, and $l' < u_1$ or $l_2 < u'$.*

Proof. Let $x(V^1) + 2x(V^2) \leq 2$ be a valid inequality with $l = l_1$ and $u = u_2$ and LMU-structure (6).

Suppose that $x(V^1) + 2x(V^2) \leq 2$ is nondecomposable. Since $x(L) \leq 1$ and $x(U) \leq 1$, $M_j \neq \emptyset$ for some $j \in \{1,\ldots,n\}$ and, since by definition of LMU-structure (6), $M_j = \emptyset$ for all $j \in \{3,\ldots,n\}$, it follows that $M_1 \neq \emptyset$ or $M_2 \neq \emptyset$. Suppose that $l' \geq u_1$ and $l_2 \geq u'$. We define $W = \{(1,s) \mid s \in V_1\} \cup \{(2,s) \mid s \in L_2 \cap U_2\} \cup \{(j,s) \mid j \in \{3,\ldots,n\},\ s \in L_j\}$ and $W' = \{(1,s) \mid s \in L_1 \cap U_1\} \cup \{(2,s) \mid s \in V_2\} \cup \{(j,s) \mid j \in \{3,\ldots,n\},\ s \in U_j\}$. Clearly $x(W)+x(W') = x(V^1)+2x(V^2)$. Since $l' \geq u_1$, it follows that $\sum_{s\in V_1} x_{1s} + \sum_{j=3}^{n} \sum_{s\in L_j} x_{js} \leq 1$ and hence $x(W) \leq 1$. In the same way, it follows from $l_2 \geq u'$ that $x(W') \leq 1$, which yields a contradiction. Hence $l' < u_1$ or $l_2 < u'$.

Suppose that $M_1 \neq \emptyset$ or $M_2 \neq \emptyset$, and $l' < u_1$ or $l_2 < u'$. Let W and W' be such that $x(W) + x(W') = x(V^1) + 2x(V^2)$ and $x(W) \leq 1$ and $x(W') \leq 1$. We assume w.l.o.g. that $(1, l - p_1 + 1) \in W$. Since $l < u$, $x_{1,l-p_1+1} = x_{2u} = 1$ is a feasible schedule. As $x(W) \leq 1$ and $(1, l - p_1 + 1) \in W$, it follows that $(2, u) \in W'$. We conclude that, since $l < u$ and $(1, l - p_1 + 1) \in W$, we have $(2, u) \in W'$. If $L_j \neq \emptyset$ for some $j \in \{3, \ldots, n\}$, then $l' < u$. Since $l' < u$ and $(2, u) \in W'$, $(j, l' - p_j + 1) \in W$ for all $j \in \{3, \ldots, n\}$ such that $L_j \neq \emptyset$. In the same way, if $U_j \neq \emptyset$ for some $j \in \{3, \ldots, n\}$, then $l < u'$. Since $l < u'$ and $(1, l - p_1 + 1) \in W$, $(j, u') \in W'$ for all $j \in \{3, \ldots, n\}$ such that $U_j \neq \emptyset$. By assumption $M_1 \neq \emptyset$ or $M_2 \neq \emptyset$. We consider each of these two cases.

Suppose $M_1 \neq \emptyset$. To prove that $x(V^1) + 2x(V^2) \leq 2$ is nondecomposable, we first show that $(2, l_2 - p_2 + 1) \in W$. By assumption $l' < u_1$ or $l_2 < u'$. Suppose $l' < u_1$. Note that $U_1 \neq \emptyset$ and $L_j \neq \emptyset$ for some $j \in \{3, \ldots, n\}$. Since $l' < u_1$ and $(j, l' - p_j + 1) \in W$ for all $j \in \{3, \ldots, n\}$ such that $L_j \neq \emptyset$, it follows that $(1, u_1) \in W'$. Observe that if job 1 is started in M_1, then job 2 can be started in V before job 1. It follows that $x_{2,l_2-p_2+1} = x_{1s} = 1$ is a feasible schedule for all $s \in M_1$ and hence $x_{2,l_2-p_2+1} = x_{1u_1} = 1$ is a feasible schedule. Since $(1, u_1) \in W'$, we find that $(2, l_2 - p_2 + 1) \in W$. Now, suppose $l_2 < u'$. Note that $L_2 \neq \emptyset$ and $U_j \neq \emptyset$ for some $j \in \{3, \ldots, n\}$. Since $l_2 < u'$ and $(j, u') \in W'$ for all $j \in \{3, \ldots, n\}$ such that $U_j \neq \emptyset$, it follows that $(2, l_2 - p_2 + 1) \in W$. We conclude that $(2, l_2 - p_2 + 1) \in W$ in case $l' < u_1$ and in case $l_2 < u'$, i.e., $(2, l_2 - p_2 + 1) \in W$. If $s \in M_1$, then, as $x_{2,l_2-p_2+1} = x_{1s} = 1$ is a feasible schedule and $(2, l_2 - p_2 + 1) \in W$, we have $(1, s) \in W'$. We find that $x_{1s} = x_{2u} = 1$ is a feasible schedule such that $x(W') = 2$, which yields a contradiction. We conclude that from $(2, l_2 - p_2 + 1) \in W$, $(2, u) \in W'$ and $M_1 \neq \emptyset$, it follows that $x(V^1) + 2x(V^2) \leq 2$ is nondecomposable.

Analogously, if $M_2 \neq \emptyset$, then $(1, u_1) \in W'$ and from $(1, l - p_1 + 1) \in W$, $(1, u_1) \in W'$, and $M_2 \neq \emptyset$, it follows that $x(V^1) + 2x(V^2) \leq 2$ is nondecomposable. □

Theorem 15 *A valid inequality $x(V^1) + 2x(V^2) \leq 2$ with $l = l_1$ and $u = u_2$ and LMU-structure* (6) *is maximal if and only if*

(1) *If $u_1 < l'$ and $u_1 < u - p_1$, then $u_1 \geq u'$ and $L_2 \cap U_2 \neq \emptyset$;*

(2) *If $l_2 > u'$ and $l_2 - p_2 > l$, then $l_2 \leq l'$ and $L_1 \cap U_1 \neq \emptyset$;*

One of the following holds:
(3a) *$L_j \cap U_j \neq \emptyset$ for some $j \in \{2, \ldots, n\}$;*
(3b) *$l \leq u' - p_2$ and $L_2 \neq \emptyset$ and $U_j \neq \emptyset$ for some $j \in \{3, \ldots, n\}$;*
(3c) *$l \leq u - \min\{p_j \mid j \in \{3, \ldots, n\}, L_j \neq \emptyset\}$ and $L_j \neq \emptyset$ for some $j \in \{3, \ldots, n\}$;*

One of the following holds:
(4a) *$L_j \cap U_j \neq \emptyset$ for some $j \in \{1, 3, \ldots, n\}$;*
(4b) *$l' \leq u - p_1$ and $L_j \neq \emptyset$ for some $j \in \{3, \ldots, n\}$;*
(4c) *$l \leq u - \min\{p_j \mid j \in \{3, \ldots, n\}, U_j \neq \emptyset\}$ and $U_j \neq \emptyset$ for some $j \in \{3, \ldots, n\}$;*

If $\min\{l_2, l + p_2\} > l$, then one of the following holds:
(5a) *$L_1 \cap U_1 \neq \emptyset$;*
(5b) *$\min\{l_2, l + p_2\} \leq u' - p_1$ and $U_j \neq \emptyset$ for some $j \in \{3, \ldots, n\}$;*

If $l_2 = l$, *then one of the following holds:*
(6a) $L_j \cap U_j \neq \emptyset$ *for some* $j \in \{1, 3, \ldots, n\}$;
(6b) $l \leq u_1 - \min\{p_j \mid j \in \{3, \ldots, n\}, L_j \neq \emptyset\}$ *and* $U_1 \neq \emptyset$ *and* $L_j \neq \emptyset$ *for some* $j \in \{3, \ldots, n\}$;
(6c) $l \leq u' - \min\{p_j \mid j \in \{1, 3, \ldots, n\}, L_j \neq \emptyset\}$ *and* $U_j \neq \emptyset$ *for some* $j \in \{3, \ldots, n\}$;

If $\max\{u_1, u - p_1\} < u$, *then one of the following holds:*
(7a) $L_2 \cap U_2 \neq \emptyset$;
(7b) $l' \leq \max\{u_1, u - p_1\} - p_2$ *and* $U_j \neq \emptyset$ *for some* $j \in \{3, \ldots, n\}$;

If $u_1 = u$, *then one of the following holds:*
(8a) $L_j \cap U_j \neq \emptyset$ *for some* $j \in \{2, \ldots, n\}$;
(8b) $l_2 \leq u - \min\{p_j \mid j \in \{3, \ldots, n\}, U_j \neq \emptyset\}$ *and* $L_2 \neq \emptyset$ *and* $U_j \neq \emptyset$ *for some* $j \in \{3, \ldots, n\}$;
(8c) $l' \leq u - \min\{p_j \mid j \in \{2, \ldots, n\}, U_j \neq \emptyset\}$ *and* $U_j \neq \emptyset$ *for some* $j \in \{3, \ldots, n\}$;

For all $j \in \{3, \ldots, n\}$, *one of the following holds:*
(9a) $\min\{l', l + p_j\} \leq l_2$;
(9b) $\min\{l', l + p_j\} \leq u - p_1$ *and* $M_1 \neq \emptyset$;
(9c) $\min\{l', l + p_j\} \leq u_1 - p_2$ *and* $M_2 \neq \emptyset$;

For all $j \in \{3, \ldots, n\}$, *one of the following holds:*
(10a) $u_1 \leq \max\{u', u - p_j\}$;
(10b) $l_2 \leq \max\{u', u - p_j\} - p_1$ *and* $M_1 \neq \emptyset$;
(10c) $l \leq \max\{u', u - p_j\} - p_2$ *and* $M_2 \neq \emptyset$.

Proof. Let $x(V^1) + 2x(V^2) \leq 2$ be a valid inequality with $l = l_1$ and $u = u_2$ and LMU-structure (6).

We show that M_1 cannot be extended if and only if (1) holds. Suppose that (1) holds. Observe that if $l_2 \geq l'$, then $M_1 = \emptyset$ and we have to show that there is a counterexample for all $(1, s)$ with $s \in [l', u - p_1]$. If $l_2 < l'$, then we have to show that there is a counterexample for all $(1, s)$ with $s \in [l_2, u' - p_1]$ or $s \in [\min\{l', u_1\}, u - p_1]$. If $l' < u - p_1$, then $x_{j,l'-p_j+1} = x_{1s} = x_{2u} = 1$ defines a counterexample for all $(1, s)$ with $s \in [l', u - p_1]$, where $j \in \{3, \ldots, n\}$ is such that $l_j = l'$. If $l_2 \geq l'$, this implies that M_1 cannot be extended. Now, suppose $l_2 < l$. If $u_1 < l'$ and $u_1 < u - p_1$, then by (1) $u_1 - p_2 + 1 \in L_2 \cap U_2$ and we find that $x_{2,u_1-p_2+1} = x_{1s} = 1$ defines a counterexample for all $(1, s)$ with $s \in [u_1, u - p_1]$. It follows that there is a counterexample for all $(1, s)$ with $s \in [\min\{l', u_1\}, u - p_1]$. Clearly, if $l_2 < u' - p_1$, then $x_{l_2-p_2+1} = x_{1s} = x_{ju'} = 1$ defines a counterexample for all $(1, s)$ with $s \in [l_2, u' - p_1]$, where $j \in \{3, \ldots, n\}$ is such that $u_j = u'$. We conclude that M_1 cannot be extended. It is not hard to see that, if (1) does not hold, then M_1 can be extended by $\max\{u' - p_1, u_1\} + 1$. Hence M_1 cannot be extended if and only if (1) holds. Analogously, M_2 cannot be extended if and only if (2) holds.

In the same way as in the proof of Theorem 7, it can be shown that L_1 cannot be extended if and only if (3) holds. Analogously, U_2 cannot be extended if and only if (4) holds.

Clearly, L_2 cannot be extended if and only if there is a counterexample for $(2, y - p_2)$, where $y = \min\{l_2, l + p_2\}$. It is not hard to see that if $y > l$, then any counterexample for

$(2, y-p_2)$ contains job 1. Such a counterexample exists if and only if (5) holds. Now, suppose $y = l$, i.e., $l_2 = l$. We assume w.l.o.g. $p_3 \geq p_4 \geq \ldots \geq p_n$. Note that by this assumption we have that if $L_j \neq \emptyset$ for some $j \in \{3, \ldots, n\}$, then $L_3 \neq \emptyset$, and if $U_j \neq \emptyset$ for some $j \in \{3, \ldots, n\}$, then $U_3 \neq \emptyset$. Clearly, there is a counterexample for $(2, l-p_2)$ defined by $x_{2,l-p_2} = x_{js} = 1$ with $(j, s) \in V^2$ if and only if (6a) holds. We show that there is a counterexample for $(2, l-p_2)$ defined by $x_{2,l-p_2} = x_{j_1 s_1} = x_{j_2 s_2} = 1$ with $(j_1, s_1), (j_2, s_2) \in V \setminus V^2$ if and only if (6b) and (6c) hold. If (6b) holds, then $x_{2,l-p_2} = x_{j_1 l} = x_{1u_1} = 1$ defines a counterexample for $(2, l-p_2)$, where $j_1 \in \{3, \ldots n\}$ is such that $L_{j_1} \neq \emptyset$ and $p_{j_1} = \min\{p_j \mid j \in \{3, \ldots, n\}, L_j \neq \emptyset\}$. Now, suppose (6c) holds. Note that $U_3 \neq \emptyset$. Let $j_1 \in \{1, 3, \ldots, n\}$ be such that $p_{j_1} = \min\{p_j \mid j \in \{1, 3, \ldots, n\}, L_j \neq \emptyset\}$. If $j_1 \neq 3$, then $x_{2,l-p_2} = x_{j_1 l} = x_{3u'} = 1$ defines a counterexample for $(2, l-p_2)$. If $j_1 = 3$, then, since $L_1 \neq \emptyset$ and $p_{j_1} = \min\{p_j \mid j \in \{1, 3, \ldots, n\}, L_j \neq \emptyset\}$, we have $p_1 \geq p_3$. Note that since $x(V^1) + 2x(V^2) \leq 2$ has LMU-structure (6), we have $[u-p_1, u'] \subseteq U_1$. Since $U_3 \neq \emptyset$ and $[u-p_1, u'] \subseteq U_1$, it follows that $U_1 \neq \emptyset$ and hence $u_1 \geq u'$. It follows that (6b) holds and hence there is a counter example for $(2, l-p_2)$. Now, let $x_{2,l-p_2} = x_{j_1 s_1} = x_{j_2 s_2} = 1$ with $(j_1, s_1), (j_2, s_2) \in V \setminus V^2$ and $s_1 < s_2$, define a counterexample for $(2, l-p_2)$. It is easy to see that if $j_2 = 1$, then (6b) holds, and if $j_2 \in \{3, \ldots, n\}$, then (6c) holds. We find that if $y = l$, then L_2 cannot be extended if and only if (6) holds. We conclude that L_2 cannot be extended if and only if (5) and (6) hold. Analogously, U_1 cannot be extended if and only if (7) and (8) hold.

In the same way as in the proof of Theorem 7, it can be shown that the intervals L_j with $j \in \{3, \ldots, n\}$ cannot be extended if and only if (9) holds. Analogously, the intervals U_j with $j \in \{3, \ldots, n\}$ cannot be extended if and only if (10) holds. □

5 Related research

As mentioned in the introduction, Sousa and Wolsey [1992] and Crama and Spieksma [1991] have also studied the time-indexed formulation of single machine scheduling problems. In this section, we briefly indicate the relation between their research and our research.

Sousa and Wolsey present three classes of valid inequalities. The first class consists of inequalities with right-hand side 1, and the second and third class consist of inequalities with right-hand side $k \in \{2, \ldots, n-1\}$. Each class of inequalities is derived by considering a set of jobs and a certain time period. The right-hand side of the resulting inequality is equal to the cardinality of the considered set of jobs.

They show that the inequalities in the first class, which is exactly the class of inequalities with structure (3), are all facet inducing. In Section 3, we have complemented this result by showing that all facet inducing inequalities with right-hand side 1 are in this class. With respect to the other two classes of valid inequalities we make the following observations. Any inequality in the second class that has right-hand side 2 can be lifted to an inequality with LMU-structure (4) if $p_{k_1} \neq p_{k_2}$, and to an inequality with LMU-structure (6) if $p_{k_1} = p_{k_2}$, where $\{k_1, k_2\}$ is the set of jobs considered. Any inequality in the third class that has right-hand side 2 can be written as the sum of two valid inequalities with right-hand side 1. For either of the two classes, Sousa and Wolsey give an example of a fractional solution that violates one of the inequalities in the class and for which they claim that it does not violate any valid inequality with right-hand side 1. We found that in both cases the latter claim is false.

Crama and Spieksma investigate the special case of equal processing times. They completely characterize all facet inducing inequalities with right-hand side 1 and present two other classes of facet inducing inequalities with right-hand side $k \in \{2, \ldots, n-1\}$.

Our characterization of all facet inducing inequalties with right-hand side 1 was found independently and generalizes their result. The inequalities in their second class that have right-hand side 2 are special cases of the inequalities with LMU-structure (6), and the inequalities in their third class that have right-hand side 2 are special cases of the inequalities with LMU-structure (4). In addition to the facet inducing inequalities reported in their paper, they have identified other classes of facet inducing inequalities with right-hand side 2 [Spieksma 1991].

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