# Symbolic computation and exact distributions of nonparametric test statistics

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**Abstract**

We show how to use computer algebra for computing exact distributions on nonparametric statistics. We give several examples of nonparametric statistics with explicit probability generating functions that can be handled this way. In particular, we give a new table of critical values of the Jonckheere-Terpstra test that extends tables known in the literature.

**Keywords:** Computer algebra; generating function; Jonckheere-Terpstra test.

## 1 Introduction

Nonparametric statistics is a valuable tool of applied statistics. Thus it is important to have correct and extensive tables (on paper or in a digital form) of critical values of nonparametric tests. Many nonparametric tables were computed in the fifties and sixties using recurrences. However, computations with recursions tend to be very time-consuming. Therefore, other ways of computing were developed. The most important contributions in this respect (often for the broader class of permutation tests) are the fast Fourier methods of Pagano and Tritchler (1983), various shift-algorithms (see e.g. Streitberg and Röhmel (1986) and Edgington (1995; pp. 393-398)), and the network algorithms developed by Mehta and co-workers (see Good (1994, chap. 13) for an overview). Baglivo, Pagano and Spino (1993) remark that all these methods can be described as efficient methods to calculate generating functions. It is thus not surprising that the recent availability of computer algebra systems offer new possibilities (see e.g. Baglivo et al. (1993) and Kendall (1993)). It is the purpose of this paper to show that critical values of many nonparametric tests can be computed easily within a computer algebra system at high speed, avoiding the sophisticated approaches mentioned above. The crux is to find expressions for the probability generating function of the test statistic at hand. Since many nonparametric test statistics are of a combinatorial nature (especially those based on ranks), these generating functions can be found in the literature (David and Barton (1962) is a rich source of generating functions, many of which are important for statistics). It is interesting to note that in the statistical literature generating functions of nonparametric statistics are hardly mentioned, or used for other purposes such as deriving recursions (see e.g. Pollicello and Hettmansperger (1976)).

A major advantage of using generating functions and computer algebra systems over other approaches is that one can work directly with mathematical objects like polynomials the way we are used to do as humans, as opposed to representations of these objects in arrays etc., which are suitable for computers only. Another advantage is that computer algebra systems use infinite precision, so that rounding errors during computations do not occur. Examples of computations in Mathematica (a computer algebra system of Wolfram Research) can be found in Section 4. Furthermore, we extend the existing tables for the Jonckheere-Terpstra test. A few words on asymptotics is in order here. We want to show with this paper that with computer algebra, one can compute exact distributions of many nonparametric statistics within reasonable time. Our strategy is to compute exact distributions whenever possible. We found in all cases that when computing exact distributions becomes time-consuming, asymptotic results are sufficiently accurate. We therefore see asymptotic distributions as a useful addendum to exact computations. Also note that now we can compute exact distributions, it is possible to investigate more precisely the convergence of distributions.

This paper is organised as follows. In Section 2 we present generating functions of some rank statistics, in Section 3 we give generating functions for two goodness-of-fit tests. Section 4 contains examples of the use of a generating function in Mathematica. In Section 5 we give a new extended table of critical values of the Jonckheere-Terpstra test.
For more details about the presented tests we refer to Gibbons and Chakraborti (1992). An overview of nonparametric techniques which stresses the analogies with the parametric counterparts can be found in Van der Laan and Verdooren (1987).

We assume, unless stated otherwise, that all distributions function are continuous and that hence, ties do not occur almost surely.

# 2 Generating functions of rank statistics

In this section we present examples of rank statistics the null distribution of which can be easily computed using generating functions.

## 2.1 The Wilcoxon-Mann-Whitney test

Let $X_1, \ldots, X_m$ and $Y_1, \ldots, Y_n$ be independent random samples from continuous distributions with finite expectations $\mu_X$ and $\mu_Y$, respectively, and with distribution functions $F(x)$ and $G(y) = F(y - \Delta\mu)$, respectively, where $\Delta\mu$ is an unknown shift parameter. In order to test the null hypothesis

$$H_0 : \mu_X = \mu_Y$$

against the alternative hypothesis

$$H_1 : \mu_X \neq \mu_Y,$$

Wilcoxon (1945) introduced the test statistic

$$W_{m,n} = \sum_{i=1}^{m} R_i, \tag{1}$$

where $R_i$ is the rank of $X_i$ in the combined sample $X_1, \ldots, X_m, Y_1, \ldots, Y_n$. Mann and Whitney (1947) introduced the statistically equivalent test statistic

$$M_{m,n} = \sum_{i=1}^{m} \#\{j : Y_j < X_i\}. \tag{2}$$

The generating function of $M_{m,n}$ was already known to Gauss (see, e.g., Andrews (1976), p. 51). A complete overview of recurrences and generating functions for $M_{m,n}$ can be found in Di Bucchianico (1996).

**Theorem 2.1** *Under $H_0$, the probability generating function of the Mann-Whitney test statistic $M_{m,n}$ is given by*

$$\sum_{k=0}^{mn} \Pr(M_{m,n} = k)\, x^k = \frac{1}{\binom{m+n}{m}} \frac{\prod_{i=m+1}^{m+n} (1 - x^i)}{\prod_{i=1}^{n} (1 - x^i)}. \tag{3}$$

Proof: For a proof based on recurrences we refer to Andrews (1976; Chapter 3), for a proof based on inversions we refer to David and Barton (1962; pp. 203-204). □

## 2.2 The Jonckheere-Terpstra test

A multi-sample analogue of the Mann-Whitney test is the Jonckheere-Terpstra test. Assume that random samples of size $n_1, \dots, n_k$, respectively, are given from $k$ populations. Denote by $X_{ij}$ the $j$th observation in the sample from the $i$th population, $1 \le i \le k, 1 \le j \le n_i$. Denote by $F_i$ the continuous cumulative distribution function of $X_{ij}$. Define $\phi(X_{ij})$ to be the number of observations from the first $i-1$ populations that are smaller than $X_{ij}$. Let, for $i = 2, \dots, k$,

$$S_i = \sum_{j=1}^{n_i} \phi(X_{ij})$$

and let

$$S = \sum_{j=2}^{k} S_i.$$

We wish to test the null hypothesis

$$H_0 : F_1(x) = \dots = F_k(x) \text{ for all } x$$

against the alternative hypothesis

$$H_1 : F_1(x) \le \dots \le F_k(x) \text{ for all } x,$$

with at least one strict inequality. For this testing problem Terpstra (1952) and Jonckheere (1954) proposed the following test statistic $J$ (nowadays known as the Jockheere-Terpstra statistic):

$$J = 2S - M, \tag{4}$$

where $M$ is the maximum possible value of $S$, i.e. $M = \sum_{i=2}^{k} \sum_{j=1}^{i-1} n_i n_j$. Therefore, if we know the distribution of $S$ then we also know the distribution of $J$.

**Theorem 2.2** *Let for $i = 2, \dots, k, N_i = \sum_{j=1}^{i-1} n_j$ and $M = \sum_{i=2}^{k} n_i N_i$. The probability generating function of $S$ under $H_0$ is given by*

$$\sum_{\ell=0}^{M} \Pr(S = \ell)\, x^\ell = \prod_{i=2}^{k} \frac{1}{\binom{n_i+N_i}{n_i}} \frac{\prod_{\ell=N_i+1}^{n_i+N_i} (1 - x^\ell)}{\prod_{\ell=1}^{n_i} (1 - x^\ell)} \tag{5}$$

Proof: It follows from Theorem 1 of Terpstra (1952) or Theorem 3 of Streitberg and Röhmel (1988) that under $H_0$ the random variables $S_i$ are independent. Further, note that $\Pr(S_i = t) = \Pr(M_{n_i,N_i} = t)$, with $M_{m,n}$ the Mann-Whitney statistic defined by (2). Hence, the probability generating function of $S$ is a product of the probability generating functions of the form (3). □

The trick to reduce the probability generating function of the Jonckheere-Terpstra test to a product of Mann-Whitney type generating functions can also be applied to other tests for partial orders (e.g. the Mack-Wolfe test for umbrella alternatives). See Streitberg and Röhmel (1988) for examples and a characterization of those alternatives for which the corresponding generalization of the Mann-Whitney test can be treated along the same lines as the Jonckheere-Terpstra test.

### 2.3 The Kendall rank correlation test

Let $(X_1, Y_1), \dots, (X_n, Y_n)$ be a sample of $n$ pairs of observations. A nonparametric correlation test is the Kendall rank correlation test. The rank correlation coefficient $\tau$ of Kendall is defined as

$$\tau = 1 - \frac{2I}{\binom{n}{2}}, \qquad (6)$$

where $I$ is the number of inversions, i.e. the number of pairs $\{(X_i, Y_i), (X_j, Y_j)\}$ such that $X_i < X_j$ and $Y_i > Y_j$ for $i < j, i = 1, \dots n-1$ and $j = 2, \dots, n$. The probability generating function of $I$ has the following simple form:

**Theorem 2.3** *The probability generating function of the number of inversions $I$ is*

$$\sum_{k=0}^{\binom{n}{2}} \Pr(I = k)\, x^k = \frac{1}{n!} \prod_{k=1}^{n} \frac{x^k - 1}{x - 1}. \qquad (7)$$

Proof: See Kendall and Stuart (1977; pp. 505-506). □

Recently, generating functions for the null distribution of Kendall's rank correlation statistic when ties are present in both ranks have been derived (see Valz et al. (1995) for details).

### 2.4 The Wilcoxon signed rank test

The Wilcoxon signed-rank test is used to test whether the median of a random sample $X_1, \dots, X_m$ from a symmetric distribution equals $m_0$. Under the null hypothesis the differences $D_i = X_i - m_0, i = 1, \dots, m$, are symmetrically distributed around zero. Ranks $\{1, \dots, m\}$ are assigned to the absolute values of the $D_i$'s from small to large and the rank of $|D_i|$ is denoted by $R_i$. The test statistic is

$$T_m = \sum_{i=1}^{m} R_i\, Z_i, \qquad (8)$$

where

$$Z_i = \begin{cases} 1 & \text{if } D_i > 0 \\ 0 & \text{otherwise.} \end{cases} \qquad (9)$$

**Theorem 2.4** *Under $H_0$, the probability generating function of $T_m$ is*

$$\sum_{i=0}^{\binom{m+1}{2}} \Pr(T_m = i)\, x^i = \frac{1}{2^m} \prod_{i=1}^{m} (1 + x^i). \tag{10}$$

Proof: The generating function is an easy consequence of the fact that under $H_0$, $T_m$ has the same distribution as $U = \sum_{i=1}^{m} U_i$, where $U_i = 0$ or $i$, both with probability $\frac{1}{2}$. □

Computations with this generating function are so fast, that existing algorithms as in Castagliola (1996) become obsolete. Mitic (1996) reports that existing tables contain many errors.

### 2.5 Other one-sample rank tests

Instead of assigning ranks $\{1, \ldots, m\}$ to the $|D_i|$'s as in the Wilcoxon signed-rank test, one can also assign rank scores $a(i)$ to the $|D_i|$'s, where $a : \{1, \ldots, m\} \to \mathbb{R}$. We can now define the following test statistic:

$$T_m^* = \sum_{i=1}^{m} a(i)\, Z_i, \tag{11}$$

with $Z_i$ as in (9). A similar argument as for $T_m$ yields the generating function of $T_m^*$ under $H_0$:

$$\sum_{i=1}^{M_a} \Pr(T_m^* = i)\, x^i = \frac{1}{2^m} \prod_{i=1}^{m} (1 + x^{a(i)}), \tag{12}$$

where $M_a = \sum_{i=1}^{m} a(i)$.

Examples of such scores include

- $a(i) = \max[0, i - \frac{m+1}{2}], i = 1, \ldots, m$. These are the scores proposed in Randles and Hogg (1973) for light-tailed distributions.
- $a(i) = \min[2i, m + 1], i = 1, \ldots, m$. These are the scores proposed in Pollicello and Hettmansperger (1976) for heavy-tailed distributions.
- $a(i) = \Phi^{-1}\left(\frac{1}{2} + \frac{i}{2(m+1)}\right), i = 1, \ldots m$, where $\Phi^{-1}$ is the inverse of the standard normal cumulative distribution function. These are the inverse normal scores. Note that the scores in this case are not rational and that exact computations are not possible unless we approximate the scores by rational numbers.

## 3 Generating functions for goodness-of-fit tests

### 3.1 The Kolmogorov one-sample test

The Kolmogorov one-sample test is used to test whether the sample $X_1, \ldots, X_m$ comes from a certain distribution function. The null hypothesis is

$$H_0 : F(x) = F_0(x) \quad \text{for all } x,$$

where $F(x)$ is the continuous distribution function of the observations and $F_0(x)$ is a given continuous distribution function. The two-sided alternative is

$$H_1 : F(x) \neq F_0(x) \quad \text{for at least one } x\,.$$

The test statistic is

$$D_m = \sup_x |F_m(x) - F_0(x)|\,,$$

where $F_m(x)$ denotes the empirical distribution function defined by $F_m(x) := \frac{1}{m}\,\#\{\ell : X_\ell \leq x, \ell = 1, \ldots, m\}$.
For the one-sided alternative hypotheses $H_1 : F(x) \leq F_0(x)$ and $H_1 : F(x) \geq F_0(x)$, for all $x$ and with strict inequality for at least one $x$, the test statistics are

$$D_m^+ = \sup_x\{F_0(x) - F_m(x)\} \quad \text{and} \quad D_m^- = \sup_x\{F_m(x) - F_0(x)\}, \tag{13}$$

respectively.
Kemperman (1957) (see also Niederhausen (1981)) gives the following implicit generating function, which holds under $H_0$:

$$\sum_{k=0}^{\infty} \Pr\left(-r < k\,D_k^- < s\right)\,\frac{(kx)^k}{k!} = \frac{Q_r(x)\,Q_s(x)}{Q_{r+s}(x)}, \tag{14}$$

for $x < 1/e$, where $Q_t(x) = \sum_{i=0}^{\lfloor t \rfloor}(i-t)^i\,x^i/i!$, and $\lfloor t \rfloor$ denotes the largest integer not exceeding $t$. Under $H_0$, $D_m^+$ is distributed as $-D_m^-$. We also know that $D_m = \max(D_m^+, D_m^-)$. Therefore,

$$\Pr(D_m \geq \frac{s}{k}) = 1 - \Pr(\max(-D_m^-, D_m^-) < \frac{s}{k}) = 1 - \Pr\left(-s < k\,D_m^- < s\right) \tag{15}$$

So in order to compute the exact distribution of $D_m$ we need the coefficient of $x^m$ of the right-hand side of (14). This can be done by expanding the right-hand side of (14) by hand, which yields an explicit expression for the null distribution of $D_m$ (cf. Kemperman (1957)). Alternatively, we may ask Mathematica to compute the coefficient of $x^m$ of the right-hand side of (14). Critical values can be computed using a numerical procedure for root finding. We refer to Section 4 for further details. Tail probabilities for $D_m^+$ or $D_m^-$ can be obtained by choosing $r = m$ or $s = m$.

For the corresponding two-sample Smirnov test analogous generating functions exist for sample sizes that are not relatively prime (Kemperman (1957) and Niederhausen (1981)). For a combinatorial explanation of the influence of relative primeness of the sample sizes on this statistic, see Di Bucchianico and Loeb (1997).

## 3.2 The Kuiper test

Kuiper (1960) suggested

$$K_m = D_m^+ + D_m^-,$$

where $D_m+$ and $D_m^-$ are defined in the previous subsection, as a Kolmogorov-type test statistic on a circle. It has the property that if the observations are circular data, its value does not depend on the choice of the origin for measuring $x$. From Niederhausen (1981) we obtain that the generating function of $K_m$ has the same form as (14):

$$\sum_{k=0}^{\infty} \Pr\left(K_k < \frac{s}{k}\right) \frac{(kx)^k}{k!\,k} = x\,\frac{Q_{s-1}(x)Q_1(x)}{Q_s(x)}, \tag{16}$$

with $Q_s(x)$ as in (14). Thus we can compute tail probabilities in the same way as for the Kolmogorov test.

# 4 Generating functions in Mathematica

In this section we show how we use the generating funtions to obtain tail probabilities using Mathematica. We give examples for the Mann-Whitney test and the Kolmogorov test. One can deal with the other tests in the same way.

## 4.1 Implementation of the Mann-Whitney test

```
MannWhitneyGf[m_,n_,x_:x]:=
Module[{i,mini = Min[m,n],maxi=Max[m,n]}, Expand[Factor[Product[
1-x^i,{i,maxi+1,maxi+mini}]/Product[1-x^i,i,1,mini]]]]

MannWhitneyFrequencies[m_,n_]:=
Module[{x,i,mini = Min[m,n],maxi=Max[m,n]}, Drop[FoldList[
Plus,0,CoefficientList[ MannWhitneyGf[m,n],x]],1]]

MannWhitneyRightTail[m_,n_,k_]:=
N[1-(Part[MannWhitneyFrequencies[m,n],k]/Binomial[m+n,n])]

MannWhitneyRightCriticalValue[m_,n_,alpha_]:=
Module[help= Length[Select[ MannWhitneyFrequencies[m,n],# < (1-1/2 * alpha)*
Binomial[m+n,n]&]]+1; If[m * n/2 <= help && help <= m*n,help,"*"]]

MannWhitneyGf[2,3]
```

$1 + x^1 + 2\,x^2 + 2\,x^3 + 2\,x^4 + x^5 + x^6$

```
MannWhitneyFrequencies[2,3]
```

$\{1, 2, 4, 6, 8, 9, 10\}$

```
MannWhitneyRightTail[2,3,5]
0.2

MannWhitneyRightCriticalValue[3,2,0.2]
6

Timing[MannWhitneyRightCriticalValue[25,20,0.05]]
{13.46 Second,337}
```

**Explanation**
The MannWhitneyGf[m_,n_] function factors (**Factor**) and expands (**Expand**) formula (3) without the constant $1/\binom{m+n}{m}$. With the aid of the local variables $q, i, mini, maxi$ (**Module**) the function MannWhitneyFrequencies[m_,n_] generates a list $\{c_1, \ldots, c_{mn}\}$ of coefficients (**CoefficientList**) and transforms it into $\{0, c_1, c_1 + c_2, \ldots, \sum_{i=1}^{mn} c_i\}$ (**FoldList**). Finally, it drops the '0' in this list (**Drop**). Thus, the function MannWhitneyFrequencies[m_,n_] generates up to a factor $1/\binom{m+n}{m}$ the cumulative distribution of the statistic $M_{m,n}$. The function MannWhitneyRightTail[m_,n_,k_] takes the $k$th part of MannWhitneyFrequencies[m_,n_] and divides it by $\binom{m+n}{m}$. This quotient is subtracted from 1 to obtain the right-tail probability of $k$.

The function MannWhitneyRightCriticalValue[m_,n_,alpha_] computes the right critical value corresponding to the two-sided confidence level $\alpha$. By using **Select** it selects all frequencies with right-tail probability larger or equal to $\frac{1}{2}\alpha$. We note that for every $k, k = 0, \ldots, mn$ the probability that the Mann-Whitney statistic equals $k$ is positive. Therefore, the length (**Length**) of the list that results after applying **Select** equals the largest value for which the right-tail probability is larger or equal to $\frac{1}{2}\alpha$. We add one to this number; the result (**help**) is the right critical value if (**If**) $\frac{1}{2}mn \leq$ help $\leq mn$.

The example shows that the right-tail probability in the case $m = 2, n = 3$ equals 0.2 for $k = 5$. The right critical value for this case with $\alpha = 0.2$ equals 6. To give an indication of the speed of this method we use the **Timing** function for obtaining the time needed for computing the right critical value for the case $m = 20, n = 25, \alpha = 0.05$. We see that the right critical value equals 337 and that the computing time is 13.46 CPU seconds on a SunSPARC10.

**Remark:**
In order to obtain right-tail probabilities one can also use the formula:

$$\sum_{k=0}^{\infty} \Pr(T > k)x^k = \frac{1 - \sum_{k=0}^{\infty} \Pr(T = k)\, x^k}{1 - x}, \tag{17}$$

where $T$ is an arbitrary rank statistic.
The advantage of using (17) is that we immediately know the right-tail probability of all $k$. However, expanding (17) is more time consuming than our method since it involves division.

## 4.2 Implementation of the Kolmogorov test

```
Q[s_]:=
Module[{i}, Sum[(i-s)^i*(x^i)/i!,{i,0,Floor[s]}]]

F[s_,m_]:=
Normal[Series[Simplify[(Q[s]^2)/Q[2*s]],{x,0,m}]]

F[3.5,10]
```

$1 + x + 2\,x^2 + 4.5\,x^3 + 10.6615\,x^4 + 25.8969\,x^5 + 63.6287\,x^6 + 157.128\,x^7 + 388.858\,x^8 + 963.186\,x^9 + 2386.57\,x^{10}$

```
Kolmogorov[d_,m_]:=
N[1-Coefficient[F[m*d,m],x,m]*m!/(m^m)]
```

```
Kolmogorov[0.35,10]
0.866039

Timing[Kolmogorov[0.23,40]]
{0.96 Second, 0.035}
```

**Explanation**

Floor[s] represents $\lfloor s \rfloor$. The function F[s_,m_] simplifies (Simplify) the right-hand side of (14) and then expands it into a Taylor polynomial (Series) of degree $m$, including an order term which is removed by applying Normal. The function Kolmogorov [d_,m_] first computes the coefficient of $x^m$ (Coefficient) in F[m*d,m] and multiplies this by $\frac{m!}{m^m}$ for obtaining $\Pr(-d < D_m^- < d)$, where $D_m^-$ is the Kolmogorov statistic as in (13). From equality (15) we know that subtracting this result from one gives us $\Pr(D_m \geq d)$. The function N provides a numerical result instead of a fraction. The example shows F[m*d,m] for $d = 0.35$ and $m = 10$ and it gives the right-tail probability for these values of $d$ and $m$. We use the Timing function to show that this method is very fast (computation on a SunSPARC10), even for $m = 40$.

## 5 Table for the Jonckheere-Terpstra test

With the generating function (5) we extended the existing tables for the Jonckheere-Terpstra test statistic. In Odeh (1971) the following cases were tabulated: $k = 3, 2 \leq n_1 \leq n_2 \leq n_3 \leq 8; k = 4, 5, 6, n_1 = \ldots = n_k = 2(1)6$. We tabulated the cases $n_i = n_j, i \neq j, i, j = 1, \ldots k$. Our tables are 2 to 5 times larger than the existing ones. Tabulation of the cases $n_i \neq n_j$ requires a lot of space, because there are so many different cases. For these cases we recommend to use our Mathematica packages for computing tail probabilities. A star denotes that a critical value does not exist for this case.

| $k$ | $n$ | 0.2 | 0.1 | 0.05 | 0.025 | 0.01 | 0.005 |
|---|---|---|---|---|---|---|---|
| 3 | 2 | 9 | 10 | 11 | 12 | * | * |
| | 3 | 18 | 20 | 22 | 23 | 25 | 25 |
| | 4 | 31 | 34 | 36 | 38 | 40 | 42 |
| | 5 | 47 | 51 | 54 | 57 | 60 | 62 |
| | 6 | 66 | 71 | 75 | 79 | 83 | 86 |
| | 7 | 88 | 95 | 100 | 105 | 110 | 114 |
| | 8 | 113 | 121 | 128 | 134 | 140 | 145 |
| | 9 | 142 | 152 | 160 | 166 | 174 | 180 |
| | 10 | 173 | 185 | 194 | 202 | 212 | 218 |
| | 11 | 208 | 222 | 232 | 242 | 252 | 260 |
| | 12 | 246 | 261 | 274 | 284 | 297 | 305 |
| | 13 | 287 | 304 | 318 | 330 | 344 | 353 |
| | 14 | 332 | 351 | 366 | 380 | 395 | 406 |
| | 15 | 379 | 400 | 418 | 432 | 450 | 461 |
| | 16 | 430 | 453 | 472 | 488 | 507 | 520 |
| | 17 | 483 | 509 | 530 | 548 | 568 | 582 |
| | 18 | 540 | 568 | 591 | 610 | 633 | 648 |
| | 19 | 600 | 630 | 655 | 676 | 701 | 717 |
| | 20 | 663 | 696 | 722 | 745 | 772 | 790 |
| | 21 | 729 | 764 | 793 | 818 | 846 | 865 |
| | 22 | 799 | 836 | 867 | 893 | 924 | 944 |
| | 23 | 871 | 911 | 944 | 972 | 1005 | 1027 |
| | 24 | 947 | 989 | 1024 | 1054 | 1089 | 1113 |
| | 25 | 1025 | 1071 | 1108 | 1140 | 1177 | 1202 |
| | 26 | 1107 | 1155 | 1194 | 1228 | 1268 | 1294 |
| | 27 | 1192 | 1243 | 1284 | 1320 | 1362 | 1390 |
| | 28 | 1280 | 1333 | 1377 | 1415 | 1459 | 1489 |
| | 29 | 1371 | 1427 | 1474 | 1514 | 1560 | 1591 |
| | 30 | 1465 | 1524 | 1573 | 1615 | 1664 | 1697 |
| | 31 | 1562 | 1624 | 1676 | 1720 | 1771 | 1806 |
| | 32 | 1662 | 1728 | 1782 | 1828 | 1882 | 1918 |
| | 33 | 1766 | 1834 | 1891 | 1939 | 1996 | 2034 |
| | 34 | 1872 | 1944 | 2003 | 2054 | 2113 | 2153 |
| | 35 | 1982 | 2057 | 2118 | 2171 | 2233 | 2275 |
| | 36 | 2095 | 2173 | 2237 | 2292 | 2356 | 2400 |
| | 37 | 2210 | 2292 | 2358 | 2416 | 2483 | 2528 |
| | 38 | 2329 | 2414 | 2483 | 2543 | 2613 | 2660 |
| | 39 | 2451 | 2539 | 2611 | 2674 | 2746 | 2795 |
| | 40 | 2576 | 2667 | 2742 | 2807 | 2883 | 2934 |
| 4 | 2 | 16 | 18 | 19 | 21 | 22 | 23 |
| | 3 | 34 | 37 | 40 | 42 | 44 | 45 |
| | 4 | 58 | 63 | 67 | 70 | 73 | 76 |
| 4 | 5 | 89 | 95 | 100 | 105 | 110 | 114 |
| | 6 | 126 | 134 | 141 | 147 | 154 | 158 |
| | 7 | 169 | 179 | 188 | 196 | 204 | 210 |
| | 8 | 218 | 231 | 242 | 251 | 262 | 269 |
| | 9 | 274 | 290 | 302 | 313 | 326 | 334 |
| | 10 | 336 | 354 | 369 | 382 | 397 | 407 |
| | 11 | 404 | 425 | 443 | 457 | 474 | 486 |
| | 12 | 479 | 503 | 522 | 539 | 559 | 572 |
| | 13 | 560 | 587 | 609 | 628 | 650 | 665 |
| | 14 | 647 | 677 | 701 | 723 | 747 | 764 |
| | 15 | 740 | 773 | 801 | 824 | 851 | 870 |
| | 16 | 839 | 876 | 906 | 932 | 962 | 983 |
| | 17 | 945 | 985 | 1018 | 1047 | 1080 | 1102 |
| | 18 | 1057 | 1101 | 1137 | 1168 | 1203 | 1228 |
| | 19 | 1175 | 1222 | 1261 | 1295 | 1334 | 1360 |
| | 20 | 1299 | 1350 | 1392 | 1429 | 1471 | 1499 |
| 5 | 2 | 26 | 28 | 30 | 32 | 33 | 35 |
| | 3 | 54 | 59 | 62 | 65 | 69 | 71 |
| | 4 | 94 | 100 | 106 | 110 | 116 | 119 |
| | 5 | 144 | 153 | 160 | 167 | 174 | 179 |
| | 6 | 204 | 216 | 226 | 235 | 244 | 251 |
| | 7 | 275 | 290 | 303 | 313 | 325 | 334 |
| | 8 | 357 | 375 | 390 | 403 | 418 | 428 |
| | 9 | 448 | 470 | 488 | 504 | 522 | 534 |
| | 10 | 550 | 576 | 597 | 615 | 636 | 650 |
| | 11 | 663 | 693 | 717 | 738 | 762 | 778 |
| | 12 | 786 | 819 | 847 | 871 | 899 | 917 |
| | 13 | 919 | 957 | 988 | 1015 | 1046 | 1067 |
| | 14 | 1062 | 1105 | 1140 | 1170 | 1205 | 1228 |
| | 15 | 1216 | 1263 | 1302 | 1335 | 1374 | 1400 |
| 6 | 2 | 37 | 40 | 43 | 45 | 47 | 49 |
| | 3 | 80 | 85 | 90 | 94 | 98 | 101 |
| | 4 | 138 | 147 | 154 | 160 | 167 | 171 |
| | 5 | 212 | 224 | 234 | 242 | 252 | 259 |
| | 6 | 302 | 317 | 330 | 342 | 354 | 363 |
| | 7 | 407 | 427 | 443 | 457 | 474 | 484 |
| | 8 | 528 | 552 | 572 | 589 | 609 | 623 |
| | 9 | 664 | 693 | 717 | 738 | 761 | 777 |
| | 10 | 816 | 850 | 878 | 902 | 930 | 949 |
| | 11 | 984 | 1023 | 1055 | 1083 | 1115 | 1136 |
| | 12 | 1167 | 1211 | 1248 | 1279 | 1316 | 1341 |

Table 1: Right critical values for the Jonckheere-Terpstra test, $n_i = n_j = n$

| k | n | 0.2 | 0.1 | 0.05 | 0.025 | 0.01 | 0.005 |
|---|---|---|---|---|---|---|---|
| 7 | 2 | 51 | 55 | 58 | 61 | 64 | 66 |
| | 3 | 109 | 117 | 122 | 127 | 133 | 137 |
| | 4 | 190 | 201 | 210 | 218 | 227 | 233 |
| | 5 | 293 | 308 | 321 | 332 | 344 | 352 |
| | 6 | 418 | 438 | 454 | 468 | 484 | 495 |
| | 7 | 564 | 589 | 610 | 628 | 648 | 662 |
| | 8 | 732 | 763 | 788 | 810 | 835 | 852 |
| | 9 | 922 | 959 | 989 | 1015 | 1045 | 1065 |
| | 10 | 1133 | 1176 | 1211 | 1242 | 1277 | 1301 |
| 8 | 2 | 66 | 71 | 75 | 78 | 82 | 85 |
| | 3 | 144 | 153 | 160 | 166 | 173 | 178 |
| | 4 | 251 | 264 | 275 | 285 | 296 | 303 |
| | 5 | 387 | 406 | 421 | 434 | 449 | 460 |
| | 6 | 552 | 577 | 597 | 614 | 634 | 648 |
| | 7 | 746 | 777 | 802 | 824 | 849 | 867 |
| | 8 | 969 | 1007 | 1038 | 1064 | 1095 | 1116 |
| | 9 | 1221 | 1266 | 1303 | 1335 | 1371 | 1396 |
| 9 | 2 | 84 | 90 | 95 | 99 | 103 | 106 |
| | 3 | 183 | 194 | 202 | 210 | 218 | 224 |
| | 4 | 320 | 336 | 349 | 360 | 373 | 382 |
| 9 | 5 | 494 | 516 | 535 | 550 | 568 | 581 |
| | 6 | 705 | 734 | 759 | 779 | 803 | 819 |
| | 7 | 954 | 990 | 1021 | 1047 | 1077 | 1097 |
| | 8 | 1239 | 1284 | 1321 | 1353 | 1390 | 1415 |
| 10 | 2 | 104 | 111 | 116 | 121 | 126 | 130 |
| | 3 | 227 | 239 | 249 | 258 | 268 | 274 |
| | 4 | 397 | 416 | 431 | 444 | 460 | 470 |
| | 5 | 614 | 640 | 661 | 680 | 701 | 716 |
| | 6 | 877 | 911 | 939 | 964 | 992 | 1011 |
| | 7 | 1186 | 1229 | 1265 | 1295 | 1331 | 1355 |
| 11 | 2 | 126 | 134 | 140 | 145 | 152 | 156 |
| | 3 | 276 | 290 | 301 | 311 | 323 | 330 |
| | 4 | 483 | 504 | 522 | 537 | 555 | 567 |
| | 5 | 746 | 777 | 801 | 823 | 847 | 864 |
| | 6 | 1067 | 1106 | 1139 | 1167 | 1199 | 1221 |
| 12 | 2 | 150 | 159 | 166 | 172 | 179 | 184 |
| | 3 | 329 | 345 | 358 | 369 | 383 | 391 |
| | 4 | 576 | 601 | 621 | 639 | 659 | 672 |
| | 5 | 892 | 926 | 954 | 979 | 1007 | 1026 |

Table 2: Right critical values for the Jonckheere-Terpstra test, $n_i = n_j = n$

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