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UNIFORM ASYMPTOTIC THEORY OF DIFFRACTION BY A PLANE SCREEN*

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1. Introduction. The study of diffraction phenomena requires the solution of an appropriate boundary value problem for the reduced wave equation or Maxwell's equations. With few exceptions these problems cannot be solved exactly. Often useful approximate solutions are given by geometrical optics, but these solutions fail to account for diffraction, i.e., the existence of nonzero fields in the shadow regions. It is now known that geometrical optics yields the leading term of a high-frequency asymptotic expansion of the solution of the boundary value problem, and that higher order terms account for diffraction. Keller's "geometrical theory of diffraction" [3] provides a systematic means of computing such terms.

Keller's theory has not only been of great practical value but has formed the foundation for important further developments in the asymptotic theory of diffraction. Many of these developments have been motivated by the attempt to overcome some of the defects of the geometrical theory of diffraction. These defects, such as the singularities at caustics and shadow boundaries, are listed at the end of §3.

In a recent paper [4] Lewis and Boersma presented a method of obtaining a "uniform" asymptotic solution of problems involving diffraction by thin screens. That work was largely motivated by an earlier paper of Wolfe [8], who treated special cases involving plane and spherical waves incident on a plane screen, by a somewhat different method. More recently Boersma and Kersten [1] have extended the method of [4] to the electromagnetic case, and Wolfe [9] has introduced a new method for the scalar problem based on the representation of the solution as an integral over the aperture.

In several respects the work of Lewis and Boersma [4] is incomplete. Only the first two terms of the asymptotic expansion were actually obtained, and it was conjectured that all terms could be obtained by the same method. However the calculations were prohibitively complex. It was also conjectured that all terms would be regular at the shadow boundaries, but this was proved only for the leading term. In this paper we complete

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the work of [4] for the special case of screens which are portions of planes. We begin with the same Ansatz introduced in [4], but our treatment of the Ansatz is significantly simpler. This enables us to obtain all terms of the expansion and to prove the conjectures. Except for one reference to a result obtained in [4] our work here is essentially self-contained.

In §2 we formulate the boundary value problem, and in §3 we briefly summarize Keller's solution. In §4 we reduce the boundary value problem to the determination of a certain double-valued function. This device, which was first introduced by Sommerfeld [6], simplifies the remaining work. In §5 we introduce our Ansatz and derive the consequences of inserting it into the reduced wave equation. There we state two theorems which assert the existence of the integrals that define the terms of the expansion and the regularity of the solution. These theorems are proved in Appendix 2. In §6 we present alternate forms of the solution, and in §7 we compare our results with Keller's theory. There we obtain all terms of the expansion of the "diffracted wave". Keller's theory yields only the leading term and involves a "diffraction coefficient" D. We find that our leading term agrees with Keller's and all the terms can be described simply in terms of successive diffraction coefficients $D_0 = D, D_1, D_2, \cdots$. Explicit formulas for the coefficients D_n are given. Appendix 1 contains a brief summary of a basic method for obtaining asymptotic solutions of the reduced wave equation.

2. Formulation of the problem. We consider problems of diffraction by a screen S which lies in the plane $x_3 = 0$. The screen may have one or more apertures of arbitrary shape or may consist of a collection of disjoint regions of arbitrary shape. The complications of the geometry of the screen will not concern us because our considerations will be local. We shall construct the diffracted field in a certain neighborhood[1] N of the edge of a typical portion of the screen and shall ignore contributions from other portions of the screen as well as those due to interactions between portions of the screen. Such contributions will be considered in a later paper. We shall require that the edge curve $\mathbf{x} = \mathbf{x}_0(\eta)$ be regular, i.e., have derivative of all orders.[2] The parameter η denotes arc length along the edge.

An incident field $u_0(\mathbf{x})$ which is a solution of the reduced wave equation

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[1] The neighborhood N extends up to the first caustic point along each "diffracted ray" emanating from the edge (see §5).

[2] This requirement can be weakened. We shall construct our asymptotic solution to all orders and show that the functions in every term are regular. However it can be shown that the construction can be carried out to any given finite order and the terms will have any specified number of derivatives if the edge function $\mathbf{x}_0(\eta)$ has sufficiently many derivatives. In fact the required order of differentiability of $\mathbf{x}_0(\eta)$ might be determined exactly.

(2.1) is prescribed. The total field $u(\mathbf{x})$ must then satisfy the following conditions:

$$\Delta u + k^2 u = 0; \tag{2.1}$$

$$u = 0 \quad \text{on} \quad S \tag{2.2a}$$

or

$$\partial u / \partial x_3 = 0 \quad \text{on} \quad S; \tag{2.2b}$$

$$u \quad \text{has a finite limit at the edge;} \tag{2.3}$$

$$u - u_0 \quad \text{is outgoing from} \quad S. \tag{2.4}$$

Thus we are in fact simultaneously considering two problems corresponding to the two boundary conditions (2.2a) and (2.2b). Condition (2.4) is a form of the "radiation condition" which is more convenient for our asymptotic method. The definition of the condition is given in Appendix 1. The "edge condition" (2.3) is an essential part of the problem. It is well known that without it the solution is not unique.

We assume that the incident field has an asymptotic expansion of the form

$$u_0 \sim e^{iks(\mathbf{x})} \sum_{m=0}^{\infty} (ik)^{-m} z_m(\mathbf{x}), \qquad k \to \infty. \tag{2.5}$$

Then (see Appendix 1) the phase function $s(\mathbf{x})$ satisfies the eiconal equation

$$(\nabla s)^2 = 1, \tag{2.6}$$

and the amplitude functions $z_m(\mathbf{x})$ satisfy the recursive system of transport equations

$$2 \nabla s \cdot \nabla z_m + z_m \Delta s = -\Delta z_{m-1}, \quad m = 0, 1, 2, \cdots, \quad z_{-1} \equiv 0. \tag{2.7}$$

The solutions of these equations are discussed in Appendix 1.

3. Keller's asymptotic solution. According to Keller's geometrical theory of diffraction [3], the asymptotic solution of our diffraction problem is given by

$$u \sim u_i + u_r + \hat{u}, \tag{3.1}$$

where

$$u_i(x_1, x_2, x_3) = \delta_i u_0(x_1, x_2, x_3), \tag{3.2}$$

$$u_r(x_1, x_2, x_3) = \mp \delta_r u_0(x_1, x_2, -x_3), \tag{3.3}$$

and

$$\hat{u} = k^{-1/2} e^{ik\hat{s}(\mathbf{x})} \sum_{m=0}^{\infty} (ik)^{-m} \hat{z}_m(\mathbf{x}). \tag{3.4}$$

The factor δ_i is one in the illuminated region of the incident wave and zero in the (complementary) shadow region. We assume that this wave is incident from the region $x_3 < 0$. Then the illuminated region includes the region $x_3 < 0$ and that portion of the region $x_3 > 0$ reached by incident rays. Similarly δ_r is one in the illuminated region of the reflected wave (the region reached by the reflected rays of geometrical optics) and zero in the corresponding shadow region. The upper sign in (3.3) corresponds to the boundary condition (2.2a) and the lower sign to (2.2b). From (3.1) we see that, in addition to the incident and reflected waves, there is a "diffracted wave" $\hat{u}$ given by (3.4). In order to describe this function we must first discuss the two-parameter family of "diffracted rays". These rays emanate from the edge. The diffracted rays through a point $\mathbf{x}_0(\eta)$ of the edge generate a cone of semiangle $\beta = \beta(\eta)$ with vertex at $\mathbf{x}_0(\eta)$ and axis tangent to the edge. Thus, for each fixed η, ϕ, a diffracted ray is given by

$$\mathbf{x} = \mathbf{x}(\sigma, \eta, \phi) = \mathbf{x}_0(\eta) + \sigma \mathbf{U}(\eta, \phi), \tag{3.5}$$

where $\mathbf{U}$ is the unit vector

$$\mathbf{U} = \cos\beta\, \mathbf{t}_1 + \sin\beta \cos\phi\, \mathbf{t}_2 - \sin\beta \sin\phi\, \mathbf{t}_3\,, \quad -\pi \leqq \phi \leqq \pi. \tag{3.6}$$

Here $\mathbf{t}_1 = \dot{\mathbf{x}}_0(\eta) = d\mathbf{x}_0/d\eta$ is the unit tangent vector to the edge; $\mathbf{t}_2(\eta)$ is the unit vector orthogonal to the edge, in the plane of the screen, pointing away from the screen; and $\mathbf{t}_3$ is a unit vector in the direction of the negative x_3-axis. These vectors are illustrated in Fig. 1. The positive direction of η along the edge is so chosen that $\mathbf{t}_1 = \mathbf{t}_2 \times \mathbf{t}_3$. In (6), $\beta(\eta)$ is the angle between the incident ray and the tangent to the edge at the point $\mathbf{x}_0(\eta)$. Thus, since ∇s is the unit vector in the direction of the incident ray, $\cos\beta = \nabla s \cdot \mathbf{t}_1$. In fact

$$\nabla s = \cos\beta\, \mathbf{t}_1 - \sin\beta \cos\phi_0\, \mathbf{t}_2 - \sin\beta \sin\phi_0\, \mathbf{t}_3\,. \tag{3.7}$$

This equation merely determines the angle $\phi_0(\eta)$. (See Fig. 1.)

If $\mathbf{n}$ denotes the unit normal to the edge, then $\mathbf{t}_2 = \pm\mathbf{n}$, and the upper or lower sign holds when the screen is locally concave or convex. In either case the curvature is given by $\kappa_0 = \mathbf{n} \cdot \dot{\mathbf{t}}_1 = |\kappa|$, where $\kappa = -\mathbf{t}_2 \cdot \dot{\mathbf{t}}_1 = \mp\kappa_0$ is the "signed curvature." Since $\dot{\mathbf{t}}_1 = \kappa_0\, \mathbf{n}$ and $\dot{\mathbf{n}} = -\kappa_0\, \mathbf{t}_1$, it follows that

$$\dot{\mathbf{t}}_1 = -\kappa \mathbf{t}_2\,, \qquad \dot{\mathbf{t}}_2 = \kappa \mathbf{t}_1\,, \qquad \dot{\mathbf{t}}_3 = 0. \tag{3.8}$$

Equation (3.5) defines a transformation from "ray coordinates" σ, η, ϕ to Cartesian coordinates x_1, x_2, x_3. The Jacobian

$$j = \frac{\partial(x_1\,, x_2\,, x_3)}{\partial(\sigma, \eta, \phi)} = \frac{\partial \mathbf{x}}{\partial \sigma} \cdot \frac{\partial \mathbf{x}}{\partial \eta} \times \frac{\partial \mathbf{x}}{\partial \phi} \tag{3.9}$$

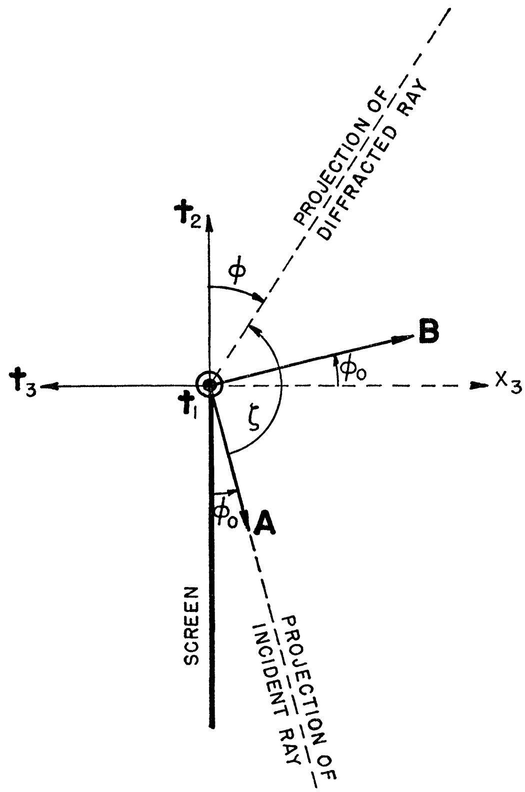

FIG. 1. *Angles and vectors at an edge of the screen. The vectors* $\mathbf{t}_1$, $\mathbf{t}_2$, $\mathbf{t}_3$, $\mathbf{A}$, *and* $\mathbf{B}$ *are of unit length:* $\mathbf{t}_1$ *is tangent to the edge of the screen and points out of the plane of the figure,* $\mathbf{t}_2$ *lies in the plane of the screen and points away from the screen, and* $\mathbf{t}_3$ *points in the direction of the negative* x_3*-axis. The projections of incident and diffracted rays into the plane of the figure are shown.* $\zeta = \pi - \phi_0 - \phi$ *is the angle between these projections. The incident wave propagates to the right, i.e.,* $0 \leqq \phi_0 \leqq \pi$.

can be obtained from (3.5), (3.6) and (3.8). A brief calculation yields

$$j = \sin^2 \beta \cdot \sigma \left(1 + \frac{\sigma}{\rho}\right), \tag{3.10}$$

where

$$\rho = \frac{\sin \beta}{\kappa \cos \phi - \dot{\beta}}. \tag{3.11}$$

In order to complete the description of Keller's solution (3.1) we must specify the functions that appear in (3.4). Along the diffracted ray (3.5),

$\hat{s}(\mathbf{x})$ is given by

$$\hat{s} = s[\mathbf{x}_0(\eta)] + \sigma, \tag{3.12}$$

where s is the phase function of the incident wave (2.5). The functions $\hat{z}_m$ are given recursively along the diffracted rays (see (A1.14) of Appendix 1) by

$$\hat{z}_m(\sigma) = \frac{\delta_m(\eta, \phi)}{y(\sigma)} - \frac{1}{2} \,⨍_0^\sigma \frac{y(\sigma')}{y(\sigma)} \Delta\hat{z}_{m-1}(\sigma')\, d\sigma', \quad m = 0, 1, 2, \cdots, \tag{3.13}$$

where

$$y = \frac{|j|^{1/2}}{\sin\beta} = \left|\sigma\left(1 + \frac{\sigma}{\rho}\right)\right|^{1/2}. \tag{3.14}$$

The finite part integral $⨍$ in (3.13) is defined in Appendix 1. Keller's method yields $\delta_m(\eta, \phi)$ only for $m = 0$, hence only the leading term $\hat{z}_0$ of (3.4). It is given by

$$\hat{z}_0 = Dz_0[\mathbf{x}_0(\eta)] \left|\sigma\left(1 + \frac{\sigma}{\rho}\right)\right|^{-1/2}, \tag{3.15}$$

where D is Keller's "diffraction coefficient",

$$D = -\frac{e^{i\pi/4}}{2\sqrt{2\pi}\sin\beta}[\sec\tfrac{1}{2}(\phi + \phi_0) \pm \sec\tfrac{1}{2}(\phi + \phi_0)]. \tag{3.16}$$

The upper or lower sign holds for the boundary condition (2.2a) or (2.2b). Since $\hat{s}$ increases with distance from the edge along the diffracted rays, the last term in (3.1) is clearly outgoing from S. The reflected wave u_r is also clearly outgoing. Then, since $u_i - u_0 = (1 - \delta_i)u_0$ is nonzero only in the *shadow* region of the incident wave, we see that (3.1) satisfies the outgoing condition (2.4).

Keller's solution has been very useful and yields excellent agreement with experimental results. It also agrees perfectly with the asymptotic expansion of the few exact solutions that are known. However it suffers from the following defects:

(a) As can be seen from (3.2) and (3.3), u_i is discontinuous across the shadow boundary of the incident wave (the surface that separates the illuminated and shadow regions). Similarly u_r is discontinuous across the shadow boundary of the reflected wave.

(b) The diffracted wave $\hat{u}$ becomes infinite at both shadow boundaries, where $\phi = \pi - \phi_0$ and $\phi = -\pi + \phi_0$, because the diffraction coefficient (3.16) becomes infinite there.

(c) From (3.15) we see that the diffracted wave becomes infinite at the edge where $\sigma = 0$; thus the edge condition is violated.

(d) The higher order terms $\hat{z}_m$, $m = 1, 2, \cdots$, in (3.4) cannot be determined.

(e) The value (3.16) of the diffraction coefficient does not arise as an integral part of Keller's method; rather it is obtained by comparison with the asymptotic expansion of the exact solution of a "canonical problem," the problem of diffraction of a plane wave by a half-plane.

(f) The solution becomes infinite at the caustic $\sigma = -\rho$ of the diffracted wave (see (3.15)) as well as at any caustics of the incident and reflected waves.

(g) A rigorous proof of the asymptotic nature of the formal solution has not been given.

Buchal and Keller [2] have overcome defects (a)–(e) by boundary layer methods. However these methods yield separate expansions in various regions and require relatively complicated computations. In the succeeding sections we shall obtain, by relatively simple means, a single (uniform) asymptotic expansion which is free of defects (a)–(e). However (f) and (g) remain. Our expansion is the same as that obtained by a more complicated method in [4]. The present method enables us to prove the conjectures made in [4].

4. The double-valued solution. The solution of our diffraction problem is facilitated by the introduction of a double-valued solution of the reduced wave equation. A similar device was used by Sommerfeld [6] for the solution of the half-plane diffraction problem. We shall attempt to construct a function U of the ray coordinates σ, η, ϕ which satisfies the conditions (corresponding to (2.1)–(2.3))

$$\Delta U + k^2 U = 0 \quad \text{for} \quad \sigma > 0, \tag{4.1}$$

$$U(\sigma, \eta, \phi + 4\pi) \equiv U(\sigma, \eta, \phi), \tag{4.2}$$

$$\lim_{\sigma \to 0} U(\sigma, \eta, \phi) = U_0(\eta) \quad \text{exists and is finite for all } \eta. \tag{4.3}$$

From the transformation (3.5), (3.6) we see that the periodicity condition (4.2) makes U a double-valued function of $\mathbf{x}$. We now define a single-valued function $u(\mathbf{x})$ by setting

$$u = U(\sigma, \eta, \phi) \mp U(\sigma, \eta, 2\pi - \phi), \qquad -\pi \leqq \phi \leqq \pi; \tag{4.4}$$

and we observe that if (4.1)–(4.3) are satisfied, then u satisfies the conditions (2.1)–(2.3) of the diffraction problem. (Condition (2.4) will be verified later.) In fact, conditions (2.1) and (2.3) are clearly satisfied and it remains to verify the boundary conditions (2.2). From (3.5) and

(3.6) we see that on S, i.e., for $\phi = \pm\pi$,

$$\frac{\partial}{\partial\phi} = \frac{\partial \mathbf{x}}{\partial\phi}\cdot\nabla = -\sigma \sin\beta \frac{\partial}{\partial x_3}. \tag{4.5}$$

We assume that $0 < \beta < \pi$. Hence (2.2a) and (2.2b) are equivalent to

$$u(\pm\pi) = 0, \tag{4.6a}$$

$$u_\phi(\pm\pi) = 0. \tag{4.6b}$$

Using the upper sign in (4.4) for the boundary condition (4.6a) we see from (4.2) that

$$u(\pi) = U(\pi) - U(\pi) = 0, \qquad u(-\pi) = U(-\pi) - U(3\pi) = 0. \tag{4.7}$$

Similarly for the other boundary condition $u_\phi(\phi) = U_\phi(\phi) - U_\phi(2\pi - \phi)$ and

$$\begin{aligned} u_\phi(\pi) &= U_\phi(\pi) - U_\phi(\pi) = 0, \\ u_\phi(-\pi) &= U_\phi(-\pi) - U_\phi(3\pi) = 0. \end{aligned} \tag{4.8}$$

Thus the boundary condition is verified in both cases.

5. The uniform asymptotic solution. We shall construct the function U (asymptotically) in a neighborhood N of the edge defined as follows:

$$N = \{\mathbf{x} = \mathbf{x}_0(\eta) + \sigma\mathbf{U},\ 0 \leqq \sigma < \sigma_1\},$$

where $\sigma = \sigma_1$ is the smallest positive value of σ such that $\mathbf{x} = \mathbf{x}_0 + \sigma_1\mathbf{U}$ is a caustic point of the incident or diffracted wave. Thus that segment of each diffracted ray (3.5), beginning at the edge and terminating at the nearest caustic point, lies in N. We shall also refer later to the neighborhood

$$N_0 = \{\mathbf{x} = \mathbf{x}_0(\eta) + \sigma\mathbf{U},\ 0 < \sigma < \sigma_1\},$$

from which the edge itself has been deleted.

In order to find the function U we introduce the Ansatz[3]

$$U \sim e^{ik\hat{s}}\left[f(k^{1/2}\theta)\sum_{m=0}^{\infty}(ik)^{-m} z_m + ck^{-1/2}\sum_{m=0}^{\infty}(ik)^{-m} v_m\right] \qquad k \to \infty, \tag{5.1}$$

where

$$f(x) = -ice^{-ix^2}\int_{-\infty}^{x} e^{it^2}\,dt, \qquad c = \pi^{-1/2} e^{i\pi/4}, \tag{5.2}$$

[3] This form was suggested by the study of the exact solution of the half-plane diffraction problem (see [4]).

and

$$\theta^2 = \hat{s} - s. \tag{5.3}$$

The functions s and z_m are the phase and amplitude functions of the incident wave (2.5), and $\hat{s}$ is the phase function of Keller's diffracted wave. It is given by (3.12). The functions v_m are to be determined. It is easy to show that $\hat{s} - s \geqq 0$ in N (see [4, Section 2, Lemma 1]). Hence θ is real and double-valued in N. We note that if we set

$$\zeta = \pi - \phi - \phi_0 \tag{5.4}$$

then (see Fig. 1) $\sin(\zeta/2) = \cos((\phi + \phi_0)/2)$ vanishes at the shadow boundary, where $\zeta = 2n\pi$, $n = 0, \pm 1, \pm 2, \cdots$. Furthermore the incident and diffracted rays coincide on the shadow boundary, and hence $\hat{s} = s$ there. It follows that θ vanishes on the shadow boundary, and we may choose

$$\operatorname{sgn} \theta = \operatorname{sgn}\left(\sin \frac{\zeta}{2}\right) = \operatorname{sgn} \cos \frac{\phi + \phi_0}{2}. \tag{5.5}$$

Then θ satisfies the periodicity condition (4.2). In fact the first term in (5.1) satisfies the same condition. This follows from the fact that $\hat{s}$ and z_m are single-valued functions of $\mathbf{x}$, hence have period 2π (therefore 4π) in ϕ.

Later we shall verify that the second term in (5.1) also satisfies (4.2). First however we insert (5.1) into the reduced wave equation, using

$$f'(x) = -ic - 2ixf(x) \tag{5.6}$$

to eliminate derivatives of f. The calculation is simplified if we set

$$g = e^{ik\hat{s}} f(k^{1/2}\theta), \qquad h = ck^{-1/2} e^{ik\hat{s}} \tag{5.7}$$

and

$$U = (ik)^{-m}[g z_m + h v_m]. \tag{5.8}$$

Here we sum over all integer values of the repeated index m, and it is understood that z_m and v_m vanish identically for $m = -1, -2, \cdots$. In computing derivatives of U we note that

$$\frac{\partial g}{\partial x_\nu} = ik\left(\frac{\partial s}{\partial x_\nu} g - \frac{\partial \theta}{\partial x_\nu} h\right), \qquad \frac{\partial h}{\partial x_\nu} = ik \frac{\partial \hat{s}}{\partial x_\nu} h. \tag{5.9}$$

Then it is easy to show that (4.1) is satisfied, provided

$$(\nabla \hat{s})^2 = 1, \tag{5.10}$$

$$\nabla\theta \cdot (\nabla s + \nabla \hat{s}) = 0 \tag{5.11}$$

and

$$2\nabla\hat{s} \cdot \nabla v_m + v_m \Delta \hat{s} = -\Delta v_{m-1} + q_m, \tag{5.12}$$

where

$$q_m = 2\nabla\theta\cdot\nabla z_m + z_m\Delta\theta. \tag{5.13}$$

In verifying (4.1) we also made use of (2.6) and (2.7).

Now (5.10) is just the eiconal equation for $\hat{s}$ and is clearly satisfied by (3.12). Furthermore (5.11) is satisfied because, from (5.3), $2\theta\,\nabla\theta = \nabla\hat{s} - \nabla s$, and

$$\begin{aligned} 2\theta\,\nabla\theta\cdot(\nabla s + \nabla\hat{s}) &= (\nabla\hat{s} - \nabla s)\cdot(\nabla\hat{s} + \nabla s) \\ &= (\nabla\hat{s})^2 - (\nabla s)^2 = 1 - 1 = 0. \end{aligned} \tag{5.14}$$

Thus we are left with (5.12) which we shall use to determine the functions v_m. We first transform (5.12) by using the identity

$$\Delta\hat{s} = \frac{d}{d\sigma}\log|j| = 2y^{-1}\frac{dy}{d\sigma}, \qquad y = \frac{|j|^{1/2}}{\sin\beta}, \tag{5.15}$$

which follows from (A1.9). Here, since $\hat{s}$ is the phase function of the diffracted wave, j is the Jacobian of the transformation defined by the diffracted rays. It is given by (3.10). Now, since $\nabla\hat{s}\cdot\nabla v_m = dv_m/d\sigma$, (5.12) becomes

$$\frac{d}{d\sigma}(yv_m) = \frac{y}{2}(\Delta v_{m-1} + q_m). \tag{5.16}$$

From (4.3) we see that v_m must be finite at $\sigma = 0$, and from (3.10) and (5.15) we see that y vanishes at $\sigma = 0$. Therefore integration of (5.16) yields

$$v_m(\sigma) = \frac{1}{2y(\sigma)}\int_0^\sigma y(-\Delta_{m-1} + q_m)\,d\sigma', \qquad m = 0, 1, 2, \cdots, \tag{5.17}$$

provided the integral exists. In (5.17) the dependence on the ray coordinates η and ϕ is not explicitly indicated.

In Appendix 2 we shall prove the following theorems. (The definitions of σ_1 and N_0 are given at the beginning of this section.)

THEOREM 1. *For every* $m = 0, 1, 2, \cdots$, *the integral* (5.17) *exists for* $0 \leqq \sigma < \sigma_1$ *and*

$$\lim_{\sigma\to 0} v_m(\sigma) = 0.$$

THEOREM 2. *U is a regular function of* $\mathbf{x}$ *in* N_0 *and it satisfies* (4.3).

Since (4.1) is satisfied by construction and Theorem 2 establishes the validity of (4.3), it remains to verify (4.2). We have already seen that $\theta(\phi + 4\pi) \equiv \theta(\phi)$; therefore to verify (4.2) we need show only that

$v_m(\phi + 4\pi) \equiv v_m(\phi)$. This can be proved by induction on m beginning with $m = -1$. (v_{-1} clearly satisfies the periodicity condition since it vanishes identically.) Since z_m is a single-valued function of $\mathbf{x}$, it is 2π-periodic in ϕ; hence it follows from (5.13) that $q_m(\phi + 4\pi) \equiv q_m(\phi)$. If now we make the induction assumption $v_{m-1}(\phi + 4\pi) \equiv v_{m-1}(\phi)$ we see from (5.17) that $v_m(\phi + 4\pi) \equiv v_m(\phi)$.

According to (4.4) our uniform asymptotic solution of the diffraction problem (2.1)–(2.4) is now given by

$$u(\mathbf{x}) = U(\sigma, \eta, \phi) \mp U(\sigma, \eta, 2\pi - \phi), \qquad -\pi \leqq \phi \leqq \pi, \tag{5.18}$$

where $U(\sigma, \eta, \phi)$ is given by (5.1), (5.2), (5.3), (5.5), (5.15), (5.17) and (5.13). The present solution (5.18) satisfies the conditions (2.1)–(2.3). It only remains to be verified that the outgoing condition (2.4) is satisfied. For that purpose we shall show that away from the shadow boundaries and from the edge the solution (5.18) reduces to (3.1). At the same time we shall verify Keller's theory and obtain the higher order terms in the expansions (3.4). We begin with the asymptotic expansion of $f(x)$, which can be obtained from (5.2) by integration by parts:

$$f(x) \sim e^{-ix^2}\eta_0(x) - \tfrac{1}{2}cx^{-1}\sum_{n=0}^{\infty} (\tfrac{1}{2})_n(ix^2)^{-n}, \qquad x \to \pm\infty. \tag{5.19}$$

Here

$$(\tfrac{1}{2})_0 = 1, \quad (\tfrac{1}{2})_n = \tfrac{1}{2}(\tfrac{1}{2} + 1) \cdots (\tfrac{1}{2} + n - 1), \quad n = 1, 2, 3, \cdots, \tag{5.20}$$

and $\eta_0(x)$ is the unit step function. Thus $\eta_0(x) = 1$ for $x > 0$ and $\eta_0(x) = 0$ for $x < 0$. Except near the shadow boundary and the edge, where $\theta = 0$, $k^{1/2}\theta$ is large, and we may use (5.19) in (5.1). This yields

$$U \sim \eta_0\left[\cos\frac{\phi + \phi_0}{2}\right]u_0 + k^{-1/2}\, e^{ik\hat{s}} \sum_{m=0}^{\infty} (ik)^{-m}\, \hat{v}_m\,, \tag{5.21}$$

where u_0 is the incident field, given by (2.5), and

$$\hat{v}_m = c\left[v_m - \frac{1}{2}\sum_{n=0}^{m}\left(\frac{1}{2}\right)_n \theta^{-2n-1} z_{m-n}\right]. \tag{5.22}$$

In the interval $-\pi \leqq \phi \leqq \pi$, $\eta_0\,[\cos((\phi + \phi_0)/2)]$ is nonzero only for $-\pi \leqq \phi < \pi - \phi_0$, which (see Fig. 1) coincides with the illuminated region of the incident wave. Similarly, in the same interval,

$$\eta_0\left[\cos\frac{2\pi - \phi + \phi_0}{2}\right] = \eta_0\left[-\cos\frac{\phi - \phi_0}{2}\right]$$

is nonzero only for $-\pi \leqq \phi < -\pi + \phi_0$, which coincides with the illuminated region of the reflected wave. Therefore

$$
\begin{aligned}
&\eta_0\left[\cos\frac{\phi+\phi_0}{2}\right] = \delta_i\,,\\
(5.23)\qquad &\eta_0\left[\cos\frac{2\pi-\phi+\phi_0}{2}\right] = \delta_r\,, \qquad -\pi \leqq \phi \leqq \pi.
\end{aligned}
$$

Thus away from the shadow boundaries and from the edge we see from (5.21) and (5.23) that (5.18) reduces to the (nonuniform) asymptotic solution (3.1)–(3.4), where

$$
(5.24)\qquad \hat{z}_m(x) = \hat{v}_m(\sigma, \eta, \phi) \mp \hat{v}_m(\sigma, \eta, 2\pi - \phi), \qquad -\pi \leqq \phi \leqq \pi,
$$

and $\hat{v}_m$ is given by (5.22). Hence the outgoing condition is satisfied.

6. Alternate forms of the uniform expansion. We first obtain a useful alternate expression for q_m which is given by (5.13). From (5.3) we see that

$$
(6.1)\qquad \nabla\theta = \frac{1}{2\theta}(\nabla\hat{s} - \nabla s), \qquad \Delta\theta = \frac{\Delta\hat{s} - \Delta s}{2\theta} - \frac{1 - \nabla\hat{s}\cdot\nabla s}{2\theta^3}.
$$

Hence

$$
(6.2)\qquad q_m = \frac{(\nabla\hat{s} - \nabla s)\cdot\nabla z_m}{\theta} + \frac{1}{2}z_m\left(\frac{\Delta\hat{s} - \Delta s}{\theta} - \frac{1 - \nabla\hat{s}\cdot\nabla s}{\theta^3}\right).
$$

But

$$
(6.3)\qquad \frac{d}{d\sigma}\left(\frac{1}{\theta}\right) = -\frac{1}{\theta^2}\nabla\theta\cdot\nabla\hat{s} = -\frac{1 - \nabla s\cdot\nabla\hat{s}}{2\theta^3}.
$$

Thus, from (5.15) and (6.3),

$$
(6.4)\qquad y^{-1}\frac{d}{d\sigma}\left(\frac{yz_m}{\theta}\right) = \frac{z_m\,\Delta\hat{s}}{2\theta} + \frac{\nabla z_m\cdot\nabla\hat{s}}{\theta} - z_m\left(\frac{1 - \nabla s\cdot\nabla\hat{s}}{2\theta^3}\right).
$$

Now from (6.2), (6.4) and (2.7) we see that

$$
(6.5)\qquad q_m = \frac{\Delta z_{m-1}}{2\theta} + y^{-1}\frac{d}{d\sigma}\left(\frac{yz_m}{\theta}\right).
$$

If we insert (6.5) in (5.17), we obtain

$$
(6.6)\qquad v_m = \frac{1}{2y}\int_0^\sigma y\left[-\Delta v_{m-1} + \frac{\Delta z_{m-1}}{2\theta}\right]d\sigma' + \frac{1}{2y}\left[\frac{yz_m}{\theta}\right]_0^\sigma.
$$

By expanding θ and y for small σ (see (7.19) and (7.20)) we find that

$$
(6.7)\qquad \lim_{\sigma\to 0}\frac{\theta}{y} = 2^{1/2}\sin\frac{\zeta}{2}\sin\beta.
$$

Hence

$$v_m = \frac{1}{2y}\int_0^{\sigma} y\left[-\Delta v_{m-1} + \frac{\Delta z_{m-1}}{2\theta}\right] d\sigma' + \frac{z_m}{2\theta} - \frac{z_m[\mathbf{x}_0(\eta)]}{2^{3/2}\, y \sin(\zeta/2)\sin\beta}. \tag{6.8}$$

For $m = 0$, (6.8) becomes

$$v_0 = \frac{z_0}{2\theta} + \frac{z_0[\mathbf{x}_0(\eta)]}{2^{3/2}\, y \sin(\zeta/2)\sin\beta}. \tag{6.9}$$

In the important special case of an incident plane wave, $z_0 \equiv 1$ and $z_m \equiv 0$ for $m = 1, 2, \cdots$. Then (6.8) simplifies to

$$v_m = -\frac{1}{2y}\int_0^{\sigma} y\Delta v_{m-1}\, d\sigma', \qquad m = 1, 2, \cdots. \tag{6.10}$$

7. The nonuniform expansion. In §5 we obtained the nonuniform expansion (5.21), (5.22) for U valid away from the shadow boundaries and from the edge. Using the results of Appendix 1 we shall now derive a simple recursive formula for the coefficients $\hat{v}_m$. According to (A1.14), (A1.15), $\hat{v}_m(\sigma)$ can be represented by

$$\hat{v}_m(\sigma) = \frac{\lambda_m}{y(\sigma)} - \frac{1}{2y(\sigma)} \not\!\!\int_0^{\sigma} y\Delta\hat{v}_{m-1}\, d\sigma', \quad m = 0, 1, 2, \cdots, \quad v_{-1} \equiv 0, \tag{7.1}$$

where

$$\lambda_m = \operatorname*{fin}_{\sigma\to 0}\, y(\sigma)\hat{v}_m(\sigma). \tag{7.2}$$

Here, $y = |j|^{1/2}/\sin\beta$ is given by (3.10). Using (5.22), the initial value λ_m can be expressed in terms of the known coefficients v_m and z_m, viz.,

$$\lambda_m = c \operatorname*{fin}_{\sigma\to 0} y\left[v_m - \frac{1}{2}\sum_{n=0}^{m}\left(\frac{1}{2}\right)_n \theta^{-2n-1} z_{m-n}\right]. \tag{7.3}$$

Since $v_m \to 0$ (Theorem 1) and $y \to 0$ as $\sigma \to 0$, the finite part (7.3) reduces to

$$\lambda_m = -\frac{c}{2}\operatorname*{fin}_{\sigma\to 0}\sum_{n=0}^{m}\left(\frac{1}{2}\right)_n \theta^{-2n-1}\, y z_{m-n} = \sum_{n=0}^{m} \mathfrak{D}_n z_{m-n}. \tag{7.4}$$

Here the $\mathfrak{D}_n = \mathfrak{D}_n(\phi)$ are linear operators defined by

$$\mathfrak{D}_n z = -\frac{c}{2}\left(\frac{1}{2}\right)_n \operatorname*{fin}_{\sigma\to 0}\, (\theta^{-2n-1} y z). \tag{7.5}$$

For example, from (6.7),

$$\begin{aligned} \mathfrak{D}_0 z &= -\frac{c}{2}\lim_{\sigma\to 0}\frac{y}{\theta} z = -\frac{c z(\mathbf{x}_0)}{2^{3/2}\sin\beta\sin(\zeta/2)} \\ &= -\frac{e^{i\pi/4}}{2\sqrt{2\pi}\sin\beta}\sec\frac{\phi+\phi_0}{2}\, z(\mathbf{x}_0). \end{aligned} \tag{7.6}$$

Thus $\mathfrak{D}_0$ is a multiplication operator. However, for $n > 0$, $\mathfrak{D}_n$ is a differential operator, as we shall see shortly.

If we now insert (7.1) into (5.24), we see that

$$\hat{z}_m = \frac{\delta_m}{y} - \frac{1}{2y} \int_0^\sigma y \Delta \hat{z}_{m-1}\, d\sigma', \tag{7.7}$$

where

$$\delta_m = \lambda_m(\phi) \mp \lambda_m(2\pi - \phi) = \sum_{n=0}^{m} D_n z_{m-n} . \tag{7.8}$$

Here the *diffraction coefficients* D_n are linear operators defined by (7.5) and

$$D_n = \mathfrak{D}_n(\phi) \mp \mathfrak{D}_n(2\pi - \phi), \qquad -\pi \leqq \phi \leqq \pi. \tag{7.9}$$

Thus from (7.6),

$$D_0 z_0 = D z_0(\mathbf{x}_0), \tag{7.10}$$

where D is Keller's diffraction coefficient (3.16), and

$$\hat{z}_0 = D z_0(x_0) y^{-1}. \tag{7.11}$$

We note that (7.7) and (7.11) agree exactly with (3.13) and (3.15). Thus we have verified Keller's theory.

The higher order terms in the expansion of the diffracted wave cannot be obtained by Keller's method. Here we see that they are given recursively by (7.7), (7.8), (7.9) and (7.5). In conclusion we may state that the uniform asymptotic solution as derived in §4 and §5 is not only of great value in itself, but it is also fundamental for the completion of Keller's nonuniform asymptotic solution. The initial value δ_m in (3.13) and (7.7), which was unknown until now (except for $m = 0$), is directly obtained from the uniform asymptotic solution.

To illustrate the application of this nonuniform asymptotic solution, we complete the correction term $\hat{z}_1$. This requires the evaluation of

$$\mathfrak{D}_1 z = -\frac{c}{4} \operatorname*{fin}_{\sigma \to 0} \theta^{-3} yz. \tag{7.12}$$

To evaluate the finite part we expand θ, y, and z for small σ. First we see from (3.5) and (3.6) that

$$z = z(\mathbf{x}_0) + \sigma \mathbf{U} \cdot \nabla z(\mathbf{x}_0) + O(\sigma^2) \tag{7.13}$$

and

$$s = s(\mathbf{x}_0) + \sigma \mathbf{U} \cdot \nabla s(\mathbf{x}_0) + b\sigma^2 + O(\sigma^3). \tag{7.14}$$

Here

$$(7.15)\qquad b = \frac{1}{2}\sum_{i,j=1}^{3} \frac{\partial^2 s}{\partial y_i\, \partial y_j}(\mathbf{x}_0)\, U_i\, U_j,$$

$$(7.16)\qquad \mathbf{U} = (U_1, U_2, U_3) = (\cos\beta, \sin\beta\cos\phi, -\sin\beta\sin\phi),$$

and the y_i are Cartesian coordinates corresponding to the base vectors $\mathbf{t}_i$, $i = 1, 2, 3$. From (3.7) we see that

$$(7.17)\qquad \mathbf{U}\cdot\nabla s(\mathbf{x}_0) = \cos^2\beta - \sin^2\beta\cos(\phi + \phi_0) = \cos^2\beta + \sin^2\beta\cos\zeta.$$

Since $\hat{s} = s(\mathbf{x}_0) + \sigma$, and $1 - \cos\zeta = \sin^2(\zeta/2)$,

$$(7.18)\qquad \hat{s} - s = 2\sigma\sin^2\beta\sin^2\frac{\zeta}{2}\left[1 - \frac{b\sigma}{2\sin^2\beta\sin^2(\zeta/2)} + O(\sigma^2)\right].$$

Now, (5.3) and (5.5) yield

$$(7.19)\qquad \theta^{-3} = (2\sigma)^{-3/2}\left(\sin\beta\sin\left(\frac{\zeta}{2}\right)\right)^{-3}\left[1 + \frac{3}{4}\frac{b\sigma}{\sin^2\beta\sin^2(\zeta/2)} + O(\sigma^2)\right].$$

Furthermore (3.10) yields

$$(7.20)\qquad y = \frac{|j|^{1/2}}{\sin\beta} = \sigma^{1/2}\left[1 + \frac{\sigma}{2\rho} + O(\sigma^2)\right].$$

We now form the product of (7.13), (7.19) and (7.20). Then we delete the singular terms (negative powers of σ) and then let $\sigma \to 0$. This yields $\mathrm{fin}_{\sigma\to 0}\, \theta^{-3} yz$, and (7.12) becomes

$$(7.21)\qquad \begin{aligned} \mathfrak{D}_1 z = {} & -2^{-7/2} c(\sin\beta\sin(\zeta/2))^{-3} \\ & \cdot\left[\left(\frac{1}{2\rho} + \frac{3b}{4\sin^2\beta\sin^2(\zeta/2)}\right) z(\mathbf{x}_0) + \mathbf{U}\cdot\nabla z(\mathbf{x}_0)\right]. \end{aligned}$$

Here $\zeta = \pi - \phi - \phi_0$, b is given by (7.15) and $\mathbf{U}$ is given by (3.6). The last term in (7.21) illustrates the fact that the $\mathfrak{D}_j$ are in general differential operators.

We shall not complete the evaluation of $\hat{z}_1$ in general, because the integral in (7.7) for $m = 1$ cannot be explicitly evaluated in general. However there are two important special cases which can be evaluated. We consider first the case in which $\phi_0 \equiv 0$ (grazing incidence toward the screen: see Fig. 1) and the second boundary condition (2.2b) holds. In this case we see from (3.16) that the diffraction coefficient $D_0 = D$ vanishes. Then $\hat{z}_0 \equiv 0$ and it is especially important to evaluate $\hat{z}_1$ because it now provides the leading term in (3.4). From (7.7), (7.8) and (7.9) we see that

$$(7.22)\qquad \hat{z}_1 = \frac{D_1 z_0}{y}, \qquad D_1 = \mathfrak{D}_1(\phi) + \mathfrak{D}_1(2\pi - \phi), \qquad -\pi \leqq \phi \leqq \pi.$$

Here $\mathfrak{D}_1 z = \mathfrak{D}_1(\phi)z$ is given by (7.21) with $\phi_0 = 0$, $\zeta = \pi - \phi$, and $\sin(\zeta/2) = \cos(\phi/2)$. Since $\cos((2\pi - \phi)/2) = -\cos(\phi/2)$ and $\rho(2\pi - \phi) = \rho(\phi)$ (see (3.11)), the first term in (7.21) contributes nothing to the sum in (7.22). Furthermore, since the incident rays are tangent to the screen, $\partial s/\partial y_3 \equiv 0$ on S and $\partial^2 s(\mathbf{x}_0)/\partial y_i \partial y_3 = 0$, $i = 1, 2, 3$. It follows from (7.15) and (7.16) that $b(2\pi - \phi) = b(\phi)$; hence the second term in (7.21) also does not contribute. Now from (3.6) we see that

$$\begin{aligned} \mathbf{U}(\phi)\cdot\nabla z - \mathbf{U}(2\pi - \phi)\cdot\nabla z &= -2\sin\beta\sin\phi\,\nabla z\cdot t_3 \\ &= 2\sin\beta\sin\phi\,\frac{\partial z}{\partial x_3}. \end{aligned} \tag{7.23}$$

It follows that

$$D_1 z_0 = \frac{-e^{i\pi/4}\sin\phi}{4\sqrt{2\pi}\sin^2\beta\cos^3(\phi/2)}\frac{\partial z_0}{\partial x_3}(\mathbf{x}_0); \tag{7.24}$$

and if we insert (7.22) and (7.24) into (3.4) we obtain, for the leading term of the diffracted wave,

$$\hat{u} \sim k^{-3/2}e^{ik\hat{s}}\left|\sigma\left(1+\frac{\sigma}{\rho}\right)\right|^{-1/2} D'\,\frac{\partial z_0}{\partial x_3}(\mathbf{x}_0), \tag{7.25}$$

where

$$D' = \frac{-e^{-i\pi/4}\sin(\phi/2)}{2\sqrt{2\pi}\sin^2\beta\cos^2(\phi/2)}. \tag{7.26}$$

This result was also obtained by Keller by expanding the exact solution of a special diffraction problem. It is easily seen that (7.25) and (7.26) agree exactly with (12) of [3]. (We must first correct an error in the last part of (12) which has the wrong sign. Then the results agree because $\phi = \theta - \pi/2$.)

The second special case occurs when $\phi_0 \equiv \pi$ (grazing incidence from the screen: see Fig. 1) and the first boundary condition (2.2a) holds. In this case we see from (3.16) that the diffraction coefficient $D_0 = D$ again vanishes. Again $\hat{z}_0 \equiv 0$ and $\hat{z}_1$ provides the leading term in (3.4). Now (7.7), (7.8) and (7.9) yield

$$z_1 = \frac{D_1 z_0}{y}, \quad D_1 = \mathfrak{D}_1(\phi) - \mathfrak{D}_1(2\pi - \phi), \qquad -\pi \leqq \phi \leqq \pi. \tag{7.27}$$

Here $\mathfrak{D}_1 z = \mathfrak{D}_1(\phi)z$ is given by (7.21) with $\phi_0 = \pi$, $\zeta = -\phi$ and $\sin(\zeta/2) = -\sin(\phi/2)$. Since $\sin((2\pi - \phi)/2) = \sin(\phi/2)$ and $\rho(2\pi - \phi) = \rho(\phi)$, the first term in (7.21) contributes nothing to the sum in (7.22). Since again $b(2\pi - \phi) = b(\phi)$, the second term in (7.21) also does not contribute. It then follows from (7.23) that

$$D_1 z_0 = \frac{e^{i\pi/4} \sin \phi}{4\sqrt{2\pi} \sin^2 \beta \sin^3 (\phi/2)} \frac{\partial z_0}{\partial x_3} (\mathbf{x}_0). \tag{7.28}$$

The leading term of the diffracted wave is now given by (7.25), with

$$D' = \frac{e^{-i\pi/4} \cos (\phi/2)}{2\sqrt{2\pi} \sin^2 \beta \sin^2 (\phi/2)}. \tag{7.29}$$

This result was also obtained by Keller. (If we set $\theta = \phi + \pi/2$, $n = 2$, and correct some errors in (19) of [3], it then agrees with (29).)

Appendix 1. Asymptotic solutions of the reduced wave equation. We consider solutions $u(\mathbf{x})$ of

$$\Delta u + k^2 u = 0 \tag{A1.1}$$

which have an asymptotic expansion of the form

$$u \sim e^{iks(\mathbf{x})} \sum_{m=0}^{\infty} (ik)^{-m} z_m(\mathbf{x}), \qquad k \to \infty. \tag{A1.2}$$

By formally substituting (A1.2) into (A1.1) we find that (A1.1) is satisfied if

$$(\nabla s)^2 = 1, \tag{A1.3}$$

and

$$2\nabla s \cdot \nabla z_m + z_m \Delta s = -\Delta z_{m-1}, \qquad m = 0, 1, 2, \cdots, \quad z_{-1} \equiv 0. \tag{A1.4}$$

The solutions of (A1.3) and (A1.4) may be described conveniently by introducing a two-parameter family of straight lines (rays)

$$\mathbf{x} = \mathbf{x}(\sigma, \sigma_2, \sigma_3) \tag{A1.5}$$

which are orthogonal to a level surface (wave front) $s(\mathbf{x}) = s_0$ of s. The labeling parameters σ_2, σ_3 are fixed on a ray and σ denotes arc length along the ray from the given wave front in the direction of increasing s. Then we see from (A1.3) that

$$s[\mathbf{x}(\sigma, \sigma_2, \sigma_3)] = s_0 + \sigma. \tag{A1.6}$$

This provides the solution of (A1.3). It is easily seen that the rays are orthogonal to every wave front $s =$ const.

An asymptotic solution of (A1.1) of the form (A1.2) is said to be *outgoing* from a manifold M if all of the rays of the family associated with the solution emanate from M and on each ray, in a neighborhood of M, the phase function s increases with distance from M along the ray.

For each m, (A1.4) is an ordinary differential equation along a ray because $\nabla s \cdot \nabla z_m = dz_m/d\sigma$. This equation can be conveniently solved by introducing the Jacobian of the "ray transformation" $\mathbf{x} = \mathbf{x}(\sigma, \sigma_2, \sigma_3)$,

$$j = \det\left(\frac{\partial x_i}{\partial \sigma_\nu}\right) = \sum_{\nu=1}^{3} \frac{\partial x_i}{\partial \sigma_\nu} \operatorname{cof} \frac{\partial x_i}{\partial \sigma_\nu}, \qquad \sigma_1 = \sigma. \tag{A1.7}$$

Here we have used the expansion of the determinant in terms of cofactors of the ith row, $i = 1, 2$ or 3. Since the determinant vanishes if two rows are identical, we have

$$\sum_{\nu=1}^{3} \frac{\partial x_k}{\partial \sigma_\nu} \operatorname{cof} \frac{\partial x_i}{\partial \sigma_\nu} = j\delta_{ik}, \tag{A1.8}$$

where δ_{ik} is the Kronecker symbol. It follows that

$$\begin{aligned} \frac{dj}{d\sigma} = \frac{\partial j}{\partial \sigma_1} &= \sum_{i,\nu} \frac{\partial^2 x_1}{\partial \sigma_\nu \, \partial \sigma_1} \operatorname{cof} \frac{\partial x_i}{\partial \sigma_\nu} = \sum_{i,\nu,k} \frac{\partial}{\partial x_k}\left(\frac{\partial x_i}{\partial \sigma_1}\right)\left[\frac{\partial x_k}{\partial \sigma_\nu} \operatorname{cof} \frac{\partial x_i}{\partial \sigma_\nu}\right] \\ &= j \sum_i \frac{\partial}{\partial x_1}\left(\frac{\partial x_i}{\partial \sigma_1}\right) = j\nabla \cdot \frac{d\mathbf{x}}{d\sigma} = j\nabla \cdot \nabla s = j\Delta s. \end{aligned} \tag{A1.9}$$

Thus, from (A1.4),

$$\begin{aligned} \frac{d}{d\sigma}(|j|^{1/2} z_m) = |j|^{1/2}\left[\frac{dz_m}{d\sigma} + \frac{z_m}{2j}\frac{dj}{d\sigma}\right] &= \frac{|j|^{1/2}}{2}[2\nabla s \cdot \nabla z_m + z_m \Delta s] \\ &= -\frac{|j|^{1/2}}{2} \Delta z_{m-1}. \end{aligned} \tag{A1.10}$$

By integration (along rays) we obtain the recursive formulas for the z_m's:

$$z_m(\sigma) = \left|\frac{j(\sigma_0)}{j(\sigma)}\right|^{1/2} z_m(\sigma_0) - \frac{1}{2}\int_{\sigma_0}^{\sigma} \left|\frac{j(\sigma')}{j(\sigma)}\right|^{1/2} \Delta z_{m-1}(\sigma')\, d\sigma', \qquad m = 0, 1, 2, \cdots. \tag{A1.11}$$

Here we have not indicated the dependence of all quantities on σ_2 and σ_3.[4]

In general we can of course take $\sigma_0 = 0$ in (A1.11). However if $j(0) = 0$, the point $\sigma = 0$ is called a *caustic point* and it can be shown that the integral in (A1.11) would then diverge at the lower endpoint $\sigma' = 0$. To avoid this difficulty we introduce a *finite part integral* defined as follows:

For $\epsilon \geqq 0$ let $f(\epsilon)$ have an asymptotic expansion in powers (perhaps fractional) of ϵ as $\epsilon \to 0$. Left $f_\infty(\epsilon)$ denote the singular terms (negative powers of ϵ) of this expansion. We define the *finite part of* $f(\epsilon)$ as $\epsilon \to 0$ by

$$\operatorname*{fin}_{\epsilon \to 0} f(\epsilon) = \lim_{\epsilon \to 0}[f(\epsilon) - f_\infty(\epsilon)]. \tag{A1.12}$$

---

[4] The present method of solution of the equations (A1.4) is different from the method used in [5] and elsewhere. The latter method led to a solution containing the expansion ratio $da(\sigma_0)/da(\sigma)$, where da stands for the cross-sectional area of a tube of rays. The solutions are equivalent because $j(\sigma_0)/j(\sigma) = da(\sigma_0)/da(\sigma)$.

Now if $\int_0^a g(x)dx$ is divergent or convergent at $x = 0$, we define the *finite part of the integral* as

$$\text{(A1.13)} \qquad \mathop{\int\!\!\!\!\!\!-}_0^a g(x)\,dx = \operatorname*{fin}_{\epsilon\to 0} \mathop{\int\!\!\!\!\!\!-}_\epsilon^a g(x)\,dx.$$

If $\sigma = 0$ is a caustic point, the solution (A1.11) is meaningful for $\sigma_0 > 0$. Let us now take the finite part of (A1.11) as $\sigma_0 \to 0$. Then

$$\text{(A1.14)} \quad z_m(\sigma) = \frac{\xi_m}{|j(\sigma)|^{1/2}} - \frac{1}{2}\mathop{\int\!\!\!\!\!\!-}_0^\sigma \left|\frac{j(\sigma')}{j(\sigma)}\right|^{1/2} \Delta z_{m-1}(\sigma')\,d\sigma',\ m = 0, 1, 2, \cdots,$$

where

$$\text{(A1.15)} \qquad \zeta_m = \zeta_m(\sigma_2, \sigma_3) = \operatorname*{fin}_{\sigma_0\to 0} |j(\sigma_0)|^{1/2} z_m(\sigma_0).$$

The initial value ζ_m may be chosen to meet the boundary conditions of the problem for (A1.1). For $m = 0$ the integral term in (A1.14) is missing. If $\sigma = 0$ is not a caustic point, the integral in (A1.14) is an ordinary integral and the finite part of (A1.15) reduces to an ordinary limit, so

$$\text{(A1.16)} \qquad \zeta_m = |j(0)|^{1/2} z_m(0).$$

It is then clear that (A1.14) reduces to (A1.11) with σ_0 replaced by zero.

Appendix 2. Proofs of theorems. In this Appendix we shall prove Theorems 1 and 2 which are stated in §5. In the body of the paper we made heavy use of the "ray coordinates" σ, η, ϕ defined by the transformation

$$\text{(A2.1)} \qquad \mathbf{x} = \mathbf{x}_0(\eta) + \sigma\mathbf{U}(\eta, \phi),$$

where $\mathbf{U}$ is a unit vector in the direction of the diffracted ray. Thus $\mathbf{U}$ is given by (3.6) or, in terms of the unit vectors $\mathbf{t}_1$, $\mathbf{A}$, $\mathbf{B}$ (illustrated in Fig. 1), by

$$\text{(A2.2)} \quad \mathbf{U} = \cos\beta\,\mathbf{t}_1 + \sin\beta\cos\zeta\,\mathbf{A} + \sin\beta\sin\zeta\,\mathbf{B}, \quad \zeta = \pi - \phi - \phi_0.$$

Here it is convenient to introduce a new set of coordinates η_1, η_2, η_3 defined by

$$\text{(A2.3)} \qquad \eta_1 = \eta, \qquad \eta_2 = (2\sigma)^{1/2}\sin(\zeta/2), \qquad \eta_3 = (2\sigma)^{1/2}\cos(\zeta/2).$$

Thus

$$\text{(A2.4)} \qquad 2\sigma = \eta_2^2 + \eta_3^2, \qquad 2\sigma\cos\zeta = \eta_3^2 - \eta_2^2, \qquad \sigma\sin\zeta = \eta_2\eta_3,$$

and, from (A1.1) and (A1.2),

$$\text{(A2.5)} \quad \mathbf{x} = \mathbf{x}_0 + \tfrac{1}{2}(\eta_2^2 + \eta_3^2)\cos\beta\,\mathbf{t}_1 + \tfrac{1}{2}(\eta_3^2 - \eta_2^2)\sin\beta\,\mathbf{A} + \eta_2\eta_3\sin\beta\,\mathbf{B}.$$

Here $\mathbf{x}_0$ and the orthogonal unit vectors $\mathbf{t}_1$, $\mathbf{A}$, $\mathbf{B}$ are functions of $\eta = \eta_1$, and (A2.5) defines a transformation $\mathbf{x} = \mathbf{x}(\eta_1, \eta_2, \eta_3)$. This transformation maps the η-space on the *doubly-sheeted* $\mathbf{x}$-space. Two points $(\eta_1, \pm\eta_2, \pm\eta_3)$ have the same image in $\mathbf{x}$-space.

In order to compute the gradient and Laplacian operators in the new coordinates we first note that

$$\text{(A2.6)} \qquad \mathbf{A} = -\cos\phi_0\,\mathbf{t}_2 - \sin\phi_0\,\mathbf{t}_3, \qquad \mathbf{B} = \sin\phi_0\,\mathbf{t}_2 - \cos\phi_0\,\mathbf{t}_3;$$

hence (3.8) yields

$$\text{(A2.7)} \qquad \dot{\mathbf{A}} = -\kappa\cos\phi_0\,\mathbf{t}_1 + \dot{\phi}_0\,\mathbf{B}, \qquad \dot{\mathbf{B}} = \kappa\sin\phi_0\,\mathbf{t}_1 - \dot{\phi}_0\,\mathbf{A}.$$

It follows that

$$\text{(A2.8)} \qquad \dot{\mathbf{x}} = \mathbf{x}_1 = \partial\mathbf{x}/\partial\eta_1 = (1 + e_1)\,\mathbf{t}_1 + e_2\,\mathbf{A} + e_3\,\mathbf{B},$$

$$\text{(A2.9)} \qquad \mathbf{x}_2 = \partial\mathbf{x}/\partial\eta_2 = \eta_2\cos\beta\,\mathbf{t}_1 - \eta_2\sin\beta\,\mathbf{A} + \eta_3\sin\beta\,\mathbf{B},$$

$$\text{(A2.10)} \qquad \mathbf{x}_3 = \partial\mathbf{x}/\partial\eta_3 = \eta_3\cos\beta\,\mathbf{t}_1 + \eta_3\sin\beta\,\mathbf{A} + \eta_2\sin\beta\,\mathbf{B},$$

where

$$\text{(A2.11)} \qquad \begin{aligned} e_1 = -\sin\beta[\tfrac{1}{2}\dot{\beta}(\eta_2^2 + \eta_3^2) + \tfrac{1}{2}\kappa\cos\phi_0(\eta_3^2 - \eta_2^2) \\ - \kappa\sin\phi_0\eta_2\eta_3], \end{aligned}$$

$$\text{(A2.12)} \qquad \begin{aligned} e_2 = \tfrac{1}{2}\dot{\beta}\cos\beta(\eta_3^2 - \eta_2^2) + \tfrac{1}{2}\kappa\cos\beta\cos\phi_0(\eta_2^2 + \eta_3^2) \\ - \dot{\phi}_0\sin\beta\eta_2\eta_3, \end{aligned}$$

$$\text{(A2.13)} \qquad \begin{aligned} e_3 = \dot{\beta}\cos\beta\eta_2\eta_3 - \tfrac{1}{2}\kappa\cos\beta\sin\phi_0(\eta_2^2 + \eta_3^2) \\ + \tfrac{1}{2}\dot{\phi}_0\sin\beta(\eta_3^2 - \eta_2^2). \end{aligned}$$

The Jacobian $J = \partial(x_1, x_2, x_3)/\partial(\eta_1, \eta_3, \eta_2)$ of the transformation (A2.5) can be computed directly from (A2.8)–(A2.13). However it is simpler to use (A2.2), (A2.3) and (3.10), which yield

$$\text{(A2.14)} \qquad \begin{aligned} \sin^2\beta\,\sigma\left(1 + \frac{\sigma}{\rho}\right) = j = \frac{\partial(x_1, x_2, x_3)}{\partial(\sigma, \eta, \phi)} = \frac{\partial(x_1, x_2, x_3)}{\partial(\eta, \sigma, \zeta)} \\ = J\,\frac{\partial(\eta_3, \eta_2)}{\partial(\sigma, \zeta)} = \frac{1}{2}J. \end{aligned}$$

The metric coefficients g_{ij} of (A2.5) are defined by

$$\text{(A2.15)} \qquad g_{ij} = \mathbf{x}_i\cdot\mathbf{x}_j = \frac{\partial x_\nu}{\partial\eta_i}\frac{\partial x_\nu}{\partial\eta_j} \quad \text{or} \quad (g_{ij}) = \left(\frac{\partial x_i}{\partial\eta_j}\right)'\left(\frac{\partial x_i}{\partial\eta_j}\right).$$

Here the accent denotes the transposed matrix. Clearly,

$$\text{(A2.16)} \qquad g = \det\,(g_{ij}) = \left[\det\left(\frac{\partial x_i}{\partial\eta_j}\right)\right]^2 = J^2.$$

The reciprocal coefficients g^{ij} are defined by

$$(g^{ij}) = (g_{ij})^{-1} \quad \text{or} \quad g^{ij} = \frac{1}{g} G^{ij}, \tag{A2.17}$$

where $G^{ij} = G^{ji}$ is the cofactor of g_{ij}. Then (see, e.g., [7]) for arbitrary functions ψ, γ,

$$\Delta\psi = \frac{1}{\sqrt{g}} \frac{\partial}{\partial \eta_i}\left(\sqrt{g}\, g^{ij} \frac{\partial \psi}{\partial \eta_j}\right) = J^{-1} \frac{\partial}{\partial \eta_i}\left(J^{-1} G^{ij} \frac{\partial \psi}{\partial \eta_j}\right) \tag{A2.18}$$

and

$$\nabla\gamma \cdot \nabla\psi = g^{ij} \frac{\partial \gamma}{\partial \eta_i} \frac{\partial \psi}{\partial \eta_j} = \frac{1}{J^2} G^{ij} \frac{\partial \gamma}{\partial \eta_i} \frac{\partial \psi}{\partial \eta_j}. \tag{A2.19}$$

We now introduce two classes of functions $g(\eta_1, \eta_2, \eta_3)$. We shall say that g is an *odd* or *even* function if it is regular in a neighborhood of the edge $\eta_2 = \eta_3 = 0$ (i.e., can be expressed as a power series in η_2 and η_3 with coefficients that are regular functions of η_1) and if

$$g(\eta_1, -\eta_2, -\eta_3) = -g(\eta_1, \eta_2, \eta_3) \quad \text{or} \quad g(\eta_1, -\eta_2, -\eta_3) = g(\eta_1, \eta_2, \eta_3), \tag{A2.20}$$

respectively. The definitions have some immediate and useful consequences: If g is odd, then $g(\eta_1, 0, 0) = 0$. The product of two odd functions is even, etc. From (A2.5) we see that $\mathbf{x}$ is even; hence if $g(\mathbf{x})$ is regular in a neighborhood of the edge, then $g[\mathbf{x}(\eta_1, \eta_2, \eta_3)]$ is even. From (A2.14), (A2.4) and (3.11) it is easy to show that

$$\sigma J^{-1} \text{ is even.} \tag{A2.21}$$

In order to prove an important lemma about the regularity of the function θ defined by (5.3) and (5.5), we introduce that segment $\mathcal{S}$ of the shadow boundary that lies in the neighborhood N which was defined at the beginning of §5. In terms of the coordinates (η_1, η_2, η_3) we see from (A2.3) that

$$\mathcal{S} = \{(\eta_1, \eta_2, \eta_3), \eta_2 = 0, 0 \leqq \eta_3 < \sqrt{2\sigma_1}\}.$$

LEMMA 1. *θ is a regular function of (η_1, η_2, η_3) in a neighborhood M of $\mathcal{S}$. Furthermore θ is odd.*

Proof. Let

$$\mathbf{U}_1 = \cos\beta\, \mathbf{t}_1 + \sin\beta\, \mathbf{A} = \cos\beta\, \mathbf{t}_1 - \sin\beta \cos\phi_0\, \mathbf{t}_2 - \sin\beta \sin\phi_0\, \mathbf{t}_3, \tag{A2.22}$$

$$\mathbf{U}_2 = \mathbf{B} = \sin\phi_0\, \mathbf{t}_2 - \cos\phi_0\, \mathbf{t}_3, \tag{A2.23}$$

$$\mathbf{U}_3 = \mathbf{U}_1 \times \mathbf{U}_2 = \sin\beta\, \mathbf{t}_1 - \cos\beta\, \mathbf{A}. \tag{A2.24}$$

Then $\mathbf{U}_1$ has the direction of the incident ray (see (3.7)), and from (A2.2) we see that in the $\mathbf{U}_1$, $\mathbf{U}_2$, $\mathbf{U}_3$ basis

$$\mathbf{U} = [\cos^2\beta + \sin^2\beta \cos\zeta,\ \sin\beta \sin\zeta,\ \cos\beta \sin\beta(1 - \cos\zeta)]' \tag{A2.25}$$

We consider an arbitrary point $\mathbf{P} = \mathbf{x}_0 + \sigma \mathbf{U}_1$ on the shadow boundary and a neighboring point $\mathbf{x} = \mathbf{x}_0 + \sigma \mathbf{U}$. The difference is

$$\begin{aligned} \mathbf{h} = \mathbf{x} - \mathbf{P} &= \sigma(\mathbf{U} - \mathbf{U}_1) \\ &= \sigma[\sin^2\beta(\cos\zeta - 1), \sin\beta \sin\zeta, \cos\beta \sin\beta(1 - \cos\zeta)]. \end{aligned} \tag{A2.26}$$

Hence, from (A2.4),

$$\mathbf{h} = (h_1, h_2, h_3) = (-\sin^2\beta\, \eta_2^2, \sin\beta\, \eta_2\eta_3, \cos\beta \sin\beta\, \eta_2^2). \tag{A2.27}$$

Now $\hat{s}(\mathbf{x}) = s(\mathbf{x}_0) + \sigma = s(\mathbf{P})$; therefore by Taylor's theorem, provided $\mathbf{P}$ is not a caustic point of the incident wave,

$$\hat{s}(\mathbf{x}) - s(\mathbf{x}) = s(\mathbf{P}) - s(\mathbf{x}) = -\sum_{n=1}^{\infty} \frac{1}{n!} s_{i_1 \cdots i_n}(\mathbf{P})\, h_{i_1} \cdots h_{i_n}. \tag{A2.28}$$

Since at $\mathbf{P}$, $(s_1, s_2, s_3) = \nabla s = (1, 0, 0)$, we see from (A2.27) and (A2.28) that

$$\hat{s}(\mathbf{x}) - s(\mathbf{x}) = (\eta_2 \sin\beta)^2 - \tfrac{1}{2} s_{22}\, (\eta_2\, \eta_3 \sin\beta)^2 + r, \tag{A2.29}$$

where every term in r contains a factor η_2^3. Thus

$$\hat{s}(\mathbf{x}) - s(\mathbf{x}) = (\eta_2 \sin\beta)^2 p(\eta_1, \eta_2, \eta_3), \tag{A2.30}$$

where $p = 1 - \frac{1}{2} s_{22}\, \eta_3^2 + \cdots$ is regular in a neighborhood of $\eta_2 = 0$ and even. Furthermore we see from (A2.4) that, on the shadow boundary where $\eta_2 = 0$, $\eta_3^2 = 2\sigma$ and

$$p = 1 - \sigma s_{22}. \tag{A2.31}$$

We now use the following identity which is given by [4, (18), Appendix 2]:

$$(\rho_2 + \sigma)(\rho_3 + \sigma)\, s_{22} = \sigma + \rho_2 + \rho_3 - \frac{\rho_2\, \rho_3}{\rho}. \tag{A2.32}$$

Here ρ_2, ρ_3 are the principal radii of curvature of the incident wavefront at $\mathbf{x}_0(\eta)$. It follows from (A2.31) that on the shadow boundary

$$p = \frac{1 + \sigma/\rho}{(1 + \sigma/\rho_2)(1 + \sigma/\rho_3)}. \tag{A2.33}$$

At the edge, $\sigma = 0$ and $p = 1$. Since p can vanish only at the caustic point $\sigma = -\rho$ and is continuous except at the caustic points $\sigma = -\rho_2$ and $\sigma = -\rho_3$, we see that p is finite and positive in $\mathcal{S}$, hence in a neighborhood M of $\mathcal{S}$. From (5.3), (5.5), (A2.3) and (A2.30) we now see that

$$\theta = \operatorname{sgn} \eta_2 \sqrt{\hat{s} - s} = \operatorname{sgn} \eta_2\, |\eta_2| \sin\beta \sqrt{p} = \eta_2 \sin\beta \sqrt{p}. \tag{A2.34}$$

Since p is regular and positive at $\eta_2 = 0$, we see that θ is a regular function of (η_1, η_2, η_3) in a neighborhood of $\eta_2 = 0$; and since p is even, θ is odd.

COROLLARY 1. *θ is a regular function of* $\mathbf{x} = (x_1, x_2, x_3)$ *in* N_0.

Proof. From (5.3) we see that θ is a regular function of $\mathbf{x}$ except at a caustic (where s or $\hat{s}$ fails to be regular) and perhaps at the shadow boundary where $\hat{s} = s$ and $\eta_2 = 0$. But from Lemma 1, in a neighborhood M of the shadow boundary segment $\mathcal{S}$, θ is a regular function of (η_1, η_2, η_3), hence of $\mathbf{x}$, except where the Jacobian J vanishes. From (A2.14) we see that J vanishes only at the caustic $\sigma = -\rho$ and at the edge $\sigma = 0$. Hence θ is a regular function of $\mathbf{x}$ in N_0.

Proof of Theorem 2. The function f defined by (5.2) is entire and the z_m and $\hat{s}$ are regular functions of $\mathbf{x}$ except at caustics. Hence from Corollary 1 the first term in (5.1) is regular in N_0. The regularity of the second term can be proved by induction: If v_{m-1} is regular in N_0, then Δv_{m-1} is regular, and from Corollary 1 and (5.13) we see that q_m is regular. Thus from (5.17), (3.10) and the formula $y = | j |^{1/2}/\sin \beta$, v_m is regular in N_0. Condition (4.3) follows from Theorem 1.

The proof of Theorem 1 is based on three more lemmas.

LEMMA 2. (i) *If* $i = 1, j = 2, 3$ *or* $j = 1, i = 2, 3$, *then* $J^{-1}G^{ij}$ *is odd.*

(ii) *If* $i = j$, *then* $J^{-1}G^{ij}$ *is even.*

(iii) *If* $i = 2, j = 3$ *or* $j = 2, i = 3$, *then* $J^{-1}G^{ij}$ *is even.*

Proof. From (A2.15), (A2.8), (A2.9) and (A2.10),

$$g_{11} = (1 + e_1)^2 + e_2^2 + e_3^2, \tag{A2.35}$$

$$g_{12} = (1 + e_1)\eta_2 \cos \beta - e_2\eta_2 \sin \beta + e_3\eta_3 \sin \beta, \tag{A2.36}$$

$$g_{13} = (1 + e_1)\eta_3 \cos \beta + e_2\eta_3 \sin \beta + e_3\eta_2 \sin \beta, \tag{A2.37}$$

$$g_{22} = \eta_2^2 + \eta_3^2 \sin^2\beta, \quad g_{23} = \eta_2\eta_3 \cos^2\beta, \quad g_{33} = \eta_3^2 + \eta_2^2 \sin^2\beta. \tag{A2.38}$$

Let P_n, Q_n, R_n, S_n denote nth degree homogeneous polynomials in η_2, η_3 with coefficients that are regular functions of $\eta = \eta_1$. From (A2.14), (A2.4) and (3.11),

$$J = \sin^2\beta(\eta_2^2 + \eta_3^2)\{1 - P_2(\eta_2, \eta_3)\}; \tag{A2.39}$$

hence

$$J^{-1} = \csc^2\beta(\eta_2^2 + \eta_3^2)^{-1}\{1 - P_2(\eta_2, \eta_3)\}^{-1}. \tag{A2.40}$$

From (A2.11)–(A2.13) we obtain by straightforward calculation

$$e_2\eta_3 + e_3\eta_2 = -(\eta_2^2 + \eta_3^2)Q_1(\eta_2, \eta_3), \tag{A2.41}$$

$$e_2\eta_3 + e_3\eta_2 = (\eta_2^2 + \eta_3^2)R_1(\eta_2, \eta_3), \tag{A2.42}$$

$$e_2^2 + e_3^2 = (\eta_2^2 + \eta_3^2)R_2(\eta_2, \eta_3). \tag{A2.43}$$

Now from (A2.35)–(A2.38) we compute G^{ij} = cofactor (g_{ij}) using (A2.41)–(A2.43). We find, e.g., that $(\eta_2^2 + \eta_3^2)^{-1}G^{11}$ is even; hence from (A2.40), $J^{-1}G^{11}$ is even, etc.

LEMMA 3. *If a is odd and b is even, then* $\sigma\nabla a \cdot \nabla b$ *is odd.*

Proof. From (19),

$$\sigma\nabla a \cdot \nabla b = (\sigma J^{-1})(J^{-1}G^{ij}) \frac{\partial a}{\partial \eta_i} \frac{\partial b}{\partial \eta_j}. \tag{A2.44}$$

By using (A2.21) and Lemma 2 we find: in case (i),

$$\frac{\partial a}{\partial \eta_i} \frac{\partial b}{\partial \eta_j} \quad \text{is even, hence} \quad (\sigma J^{-1})(J^{-1}G^{ij}) \frac{\partial a}{\partial \eta_i} \frac{\partial b}{\partial \eta_j} \quad \text{is odd;}$$

in case (ii),

$$\frac{\partial a}{\partial \eta_i} \frac{\partial b}{\partial \eta_j} \quad \text{is odd, hence} \quad (\sigma J^{-1})(J^{-1}G^{ij}) \frac{\partial a}{\partial \eta_i} \frac{\partial b}{\partial \eta_j} \quad \text{is odd;}$$

in case (iii),

$$\frac{\partial a}{\partial \eta_i} \quad \text{is even,} \quad \frac{\partial b}{\partial \eta_j} \quad \text{is odd, hence} \quad (\sigma J^{-1})(J^{-1}G^{ij}) \frac{\partial a}{\partial \eta_i} \frac{\partial b}{\partial \eta_j} \quad \text{is odd.}$$

LEMMA 4. *If a is odd, then* $\sigma\Delta a$ *is odd.*

Proof. Let $J^{-1}G^{ij}\, \partial a/\partial \eta_j = h^i$. Then it is easily seen from Lemma 2 that h^1 is odd and h^2 and h^3 are even. It follows that $\partial h^i/\partial \eta_i$ is odd and from (A2.18) that $\sigma\Delta a = (\sigma J^{-1})\partial h^i/\partial \eta_i$ is odd.

Proof of Theorem 1. From (A2.39) and (A2.4) we see that

$$(\sigma^{-1}J)^{1/2} \quad \text{is even and} \quad (\sigma^{-1}J)^{-1/2} \quad \text{is even.} \tag{A2.45}$$

Since $y \sin \beta = j^{\cdot 1/2} = j^{\cdot 1/2} = (J/2)^{1/2}$, it follows that

$$\sigma^{-1/2}\, y \quad \text{is even and} \quad \sigma^{1/2}y^{-1} \quad \text{is even.} \tag{A2.46}$$

Since z_m is regular in a neighborhood of the edge, z_m is even. From Lemma 1, θ is odd, and from Lemma 3, $\sigma\nabla\theta \cdot \nabla z_m$ is odd. Furthermore, from Lemma 4, $\sigma z_m \Delta\theta$ is odd; hence from (5.13) we see that σq_m is odd. We shall prove by induction that for each m the integral (5.17) exists and v_m is odd. The assertion is clearly true for $m = -1$ because $v_{-1} \equiv v_{-2} \equiv q_{-1} \equiv 0$. If v_{m-1} is odd, it follows from Lemma 4 that $\sigma\Delta v_{m-1}$ is odd; hence from (A2.46) we see that $a_m(\eta_1, \eta_2, \eta_3)$ is odd, where

$$a_m = \sigma^{1/2}y \cdot (-\Delta v_{m-1} + q_m) = \sigma^{-1/2}y[-\sigma\Delta v_{m-1} + \sigma q_m]. \tag{A2.47}$$

Thus

$$y \cdot (-\Delta v_{m-1} + q_m) = \sigma^{-1/2}a_m[\eta, (2\sigma)^{1/2} \sin(\zeta/2), (2\sigma)^{1/2} \cos(\zeta/2)] \tag{A2.48}$$

has an expansion in nonnegative integral powers of σ, i.e., is regular in σ

at $\sigma = 0$. Thus the integral in (5.17) exists, and $v_m = \frac{1}{2}\sigma^{1/2}y^{-1}\alpha_m$, where

$$\text{(A2.49)}\qquad \begin{aligned} \alpha_m &= \sigma^{-1/2}\int_0^\sigma y(-\Delta v_{m-1} + q_m)\, d\sigma' \\ &= \sigma^{-1/2}\int_0^\sigma (\sigma')^{-1/2} a_m[\eta, (2\sigma')^{1/2}\sin(\zeta/2), (2\sigma')^{1/2}\cos(\zeta/2)]\, d\sigma'. \end{aligned}$$

We see that α_m is odd because a_m is odd. It follows from (A2.46) that v_m is odd. This completes the induction argument. Since $v_m(\eta_1, \eta_2, \eta_3)$ is odd, we see from (A2.3) that

$$\text{(A2.50)}\qquad \lim_{\sigma\to 0} v_m = v_m(\eta_1, 0, 0) = 0.$$

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REFERENCES

[1] J. Boersma and P. H. M. Kersten, *Uniform asymptotic theory of electromagnetic diffraction by a plane screen*, to appear.

[2] R. N. Buchal and J. B. Keller, *Boundary layer problems in diffraction theory*, Comm. Pure Appl. Math., 13 (1960), pp. 85–114.

[3] J. B. Keller, *The geometric theory of diffraction*, J. Opt. Soc. Amer., 52 (1962), pp. 116–130.

[4] R. M. Lewis and J. Boersma, *Uniform asymptotic theory of edge diffraction*, J. Mathematical Phys., to appear.

[5] R. M. Lewis and J. B. Keller, *Asymptotic methods for partial differential equations: the reduced wave equation and Maxwell's equations*, Res. Rep. EM-194, New York University, New York, 1964.

[6] A. Sommerfeld, *Optics*, Lectures on Theoretical Physics, vol. IV, Academic Press, New York, 1964.

[7] J. A. Stratton, *Electromagnetic Theory*, McGraw-Hill, New York, 1964.

[8] P. Wolfe, *Diffraction of a scalar wave by a plane screen*, this Journal, 14 (1966), pp. 577–599.

[9] ———, *A new approach to edge diffraction*, this Journal, 15 (1967), pp. 1434–1469.