

Ray-optical analysis of reflection in open-ended parallel plane waveguide. I: TM case

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RAY-OPTICAL ANALYSIS OF REFLECTION IN AN OPEN-ENDED PARALLEL-PLANE WAVEGUIDE. I: TM CASE*

J. BOERSMA†

Abstract. The reflection problem for a TM mode traveling toward the open end of a semi-infinite parallel-plane waveguide is solved by ray methods. Unlike a previous solution due to Yee, Felsen and Keller, the present ray-optical solution is a rigorous asymptotic result, i.e., it is identical with the asymptotic expansion of the exact solution when the width of the waveguide is large compared to the wavelength. Numerical results for the modal reflection coefficients are presented and are compared with calculations based on the exact solution. It is found that the agreement between ray-optical and exact values is excellent and even better than in the approach of Yee et al., especially in the vicinity of cutoff frequencies of higher order modes.

1. Introduction. The systematic application of ray methods to the solution of waveguide scattering problems was initiated by Yee, Felsen and Keller [1]. In the latter paper, the ray-optical approach was illustrated at the reflection of an incident TM or TE mode from the open end of a semi-infinite parallel-plane waveguide with perfectly conducting walls. However, as was pointed out by Bowman [2], the ray-optical solution of the reflection problem fails to agree with the asymptotic expansion (width of waveguide large compared to wavelength) of the exact solution. In this paper, and its companion [3], the reflection problem is reconsidered and the approach of [1] is critically reviewed. A corrected ray-optical solution is derived which is in complete agreement with the asymptotic form of the exact solution.

The start of our analysis is the same as in [1]. The incident mode is decomposed into two plane waves. Each plane wave hits one of the edges of the open end of the guide and generates a primary diffracted field. The latter field is a cylindrical wave centered at the diffracting edge and, as such, is determined by Keller's geometrical theory of diffraction [4]. Each primary (first order) diffracted field acts as an incident wave on the opposite half-plane and gives rise to second and higher order diffractions. The actual calculation of the successive edge-edge interaction fields is complicated by the fact that in the case of multiple diffraction the back-scattered direction coincides with the shadow boundary of the specularly reflected wave. In other words, each edge lies on the ray-optical reflection boundary of the opposite half-plane. Now, as is well known, Keller's theory is not valid along shadow boundaries.

In order to overcome this difficulty, Yee et al. [1] introduced the basic assumption that each interaction field can be approximated by the field of an equivalent set of isotropic line sources, the source strengths being such as to provide the correct interaction field in the direction toward the opposite edge. The interaction fields were then determined recursively by means of a special asymptotic formula for scattering of an isotropic cylindrical wave by a half-plane.

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The assumption on the character of the multiply diffracted fields is merely a working hypothesis that simplifies the analysis of the interaction process. It should be emphasized that the actual interaction fields do not satisfy the basic assumption, and this explains the discrepancy observed by Bowman [2].

In the present paper, the successive interaction fields are calculated by means of the uniform asymptotic theory of edge diffraction as developed by Ahluwalia, Lewis and Boersma [5]. The latter theory deals with diffraction of an arbitrary incident wave by a plane screen. The uniform asymptotic theory provides an asymptotic solution of the diffraction problem that is uniformly valid near the edge and the shadow boundaries. Away from these regions the solution reduces to an expansion for the diffracted field which contains Keller's result as its leading term. Higher order terms are obtained as well, whereas Keller's theory is incapable of determining these terms.

The interaction fields thus determined and the primary diffracted fields are regarded as being due to induced line sources located at the edges of the waveguide. The radiation of these sources back into the waveguide establishes the reflected wave. The radiation fields and their repeated reflections at the waveguide walls are converted into a sum of waveguide modes according to the procedure detailed in [1]. In so doing, the modal reflection coefficients are obtained explicitly, and turn out to agree with the asymptotic form of the exact coefficients.

In conclusion, from the viewpoint of the geometrical theory of diffraction, the present paper has two contributions. First of all, the analysis of this paper shows that interaction between parallel half-planes can be built up from a local consideration of a single half-plane. Secondly, the paper provides a concrete example for using the uniform asymptotic theory of edge diffraction as developed by Ahluwalia et al. [5].

The plan of this paper is as follows. In § 2 we formulate the reflection problem for an incident TM mode. Diffraction of the incident wave at the edges of the waveguide produces two primary diffracted fields which are determined in § 3. Multiple diffraction due to interaction between the edges is discussed in §§ 4–7. First, in § 4 we present a nonuniform high-frequency expansion for the $(n + 1)$ st order interaction field due to $(n + 1)$ -fold interaction. The expansion is expressed in terms of the n th order interaction field evaluated at the diffracting edge. The expansion is called nonuniform because it is not valid along the line connecting the edges where a shadow-boundary-type singularity appears. Secondly, in §§ 5–7 we derive uniform expansions for the first, second and n th order interaction fields which remain finite at the line connecting the edges. These uniform expansions provide the edge values required to completely specify the nonuniform expansions of the successive interaction fields. The uniform expansions are derived by means of the uniform asymptotic theory of edge diffraction [5]. A summary of the latter theory, specialized to diffraction by a half-plane, is presented in Appendix A. Some intricate mathematical details of the analysis in §§ 6, 7 are deferred to Appendices B, C, D. In § 8 we obtain our final result for the reflected wave in the waveguide. Numerical results for the reflection coefficients of the lowest order modes are presented in § 9. A comparison with calculations based on the exact solution show that the ray-optical results are in excellent agreement with the exact ones not only at high frequencies but also in

the dominant-mode regime. Our ray-optical solution is even more accurate than the previous solution of Yee et al. [1], especially in the vicinity of cutoff frequencies of higher order modes.

In the companion paper [3], the corrected ray method is applied to the reflection problem for an incident TE mode. The original ray-optical approach of [1] was extended by Felsen and his associates to various other waveguide scattering problems, viz., reflection from the open end of a flanged parallel-plane waveguide [1] or a cylindrical pipe [6]; scattering by a strip or a bifurcation in a parallel-plane waveguide [7]; electromagnetic scattering in waveguides [8]; scattering by obstacles in inhomogeneously filled waveguides [9], [10]. The same simplifying assumption on the character of the multiply diffracted fields is basic to the analysis of these papers. Therefore it is to be expected that the ray-optical solution of these problems can be improved also by utilizing the uniform asymptotic theory of edge diffraction.

A different ray-optical approach was recently proposed by Lee [11], [12]. His method employs a modified diffraction coefficient for diffraction by a half-plane in the presence of a second parallel half-plane. This modified coefficient, which automatically includes the interaction between the diffracting edge and the second half-plane, is derived from the solution of a canonical problem, viz., diffraction of a plane wave by two staggered, parallel half-planes. It seems, however, that Lee's method is only applicable to scattering in waveguide configurations with parallel plane walls.

2. Statement of the problem. Let x, y, z be rectangular coordinates. Consider a semi-infinite waveguide bounded by the perfectly conducting half-planes $y = 0, z \geq 0$, and $y = a, z \geq 0$, with edges at $y = 0, z = 0$, and $y = a, z = 0$. Inside the waveguide, the TM_{0N} mode, $N = 0, 1, 2, \dots$, with transverse magnetic field

$$(2.1) \quad H_x^i(y, z) = \cos(N\pi y/a) \exp(-i\kappa_N z),$$

travels toward the open end of the guide. The propagation constant $\kappa_n, n = 0, 1, 2, \dots$, is generally defined through

$$(2.2) \quad \kappa_n = (k^2 - n^2\pi^2/a^2)^{1/2}, \quad \text{Re } \kappa_n \geq 0, \quad \text{Im } \kappa_n \geq 0,$$

where k is the free-space wave number. A time dependence $\exp(-i\omega t)$ is implied and suppressed throughout.

At the open end of the waveguide, the incident mode is partly reflected back into the guide and partly radiated into free space. The reflected field is represented by the modal expansion

$$(2.3) \quad H_x^r(y, z) = \sum_{n=0}^{\infty} \varepsilon_n \Gamma_{Nn} \cos\left(\frac{n\pi y}{a}\right) \exp(i\kappa_n z),$$

where $\varepsilon_0 = 1, \varepsilon_n = 2$ for $n \neq 0$ (Neumann's factor). In this paper we shall determine the modal reflection coefficients Γ_{Nn} by means of a ray-optical method.

We introduce some general notations to be used throughout this paper.

The angle $\theta_n, n = 0, 1, 2, \dots$, is uniquely determined by

$$(2.4) \quad \sin \theta_n = \frac{n\pi}{ka}, \quad \cos \theta_n = \frac{\kappa_n}{k}, \quad 0 \leq \text{Re } \theta_n \leq \frac{\pi}{2}.$$

The various diffracted fields are properly described by means of polar coordinates r, φ and $r_{\pm m}, \varphi_{\pm m}, m = 1, 2, 3, \dots$, with respect to the centers $y = 0, z = 0$ and $y = \pm ma, z = 0$, respectively; cf. Fig. 1. The angle φ_m is measured in a clockwise

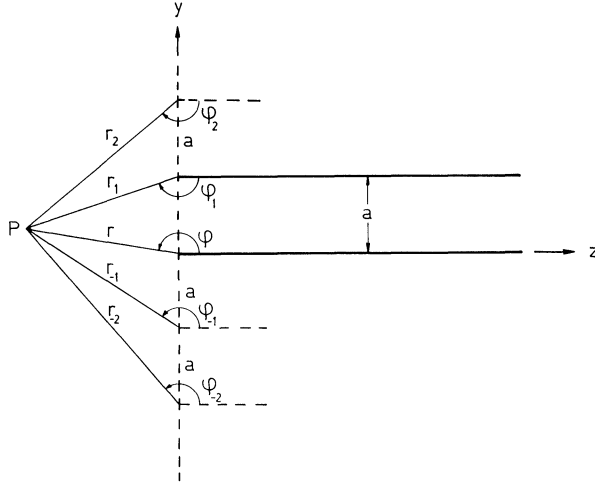


FIG. 1. Waveguide configuration and choice of coordinates

sense, whereas the angles φ, φ_{-m} are measured in a counterclockwise sense; furthermore, $0 \leq \varphi \leq 2\pi, 0 \leq \varphi_{\pm m} \leq 2\pi$. The variables $\xi, \xi_{\pm 1}$ are defined by

$$(2.5) \quad \begin{aligned} \xi &= (r + a - r_{-1})^{1/2} \operatorname{sgn} \left[\cos \frac{1}{2} \left(\varphi + \frac{\pi}{2} \right) \right] \\ &= \left(\frac{4ar}{r + a + r_{-1}} \right)^{1/2} \cos \frac{1}{2} \left(\varphi + \frac{\pi}{2} \right), \\ \xi_{\pm 1} &= (r_{\pm 1} + a - r_{\pm 2})^{1/2} \operatorname{sgn} \left[\cos \frac{1}{2} \left(\varphi_{\pm 1} + \frac{\pi}{2} \right) \right] \\ &= \left(\frac{4ar_{\pm 1}}{r_{\pm 1} + a + r_{\pm 2}} \right)^{1/2} \cos \frac{1}{2} \left(\varphi_{\pm 1} + \frac{\pi}{2} \right), \end{aligned}$$

in accordance with the law of cosines.

Finally, the functions $g(\varphi, \theta), h(\varphi)$ are defined by

$$(2.6) \quad g(\varphi, \theta) = -[\sec \frac{1}{2}(\varphi - \theta) + \sec \frac{1}{2}(\varphi + \theta)] = -\frac{4 \cos \frac{1}{2}\varphi \cos \frac{1}{2}\theta}{\cos \varphi + \cos \theta},$$

$$(2.7) \quad h(\varphi) = \frac{1}{4} \cos \varphi \left[\sec^3 \frac{1}{2} \left(\varphi - \frac{\pi}{2} \right) + \sec^3 \frac{1}{2} \left(\varphi + \frac{\pi}{2} \right) \right] = \frac{2^{1/2} \cos \frac{1}{2}\varphi (2 - \cos \varphi)}{\cos^2 \varphi}.$$

For later use we establish the relation

$$(2.8) \quad \frac{\partial g(\pi/2, \theta)}{\partial \varphi} = -h(\theta),$$

which can be verified by simple calculation.

3. Primary diffraction. Following Yee, Felsen and Keller [1], the incident mode (2.1) is decomposed into two plane waves:

$$(3.1) \quad H_x^i(y, z) = \frac{1}{2} \{ \exp [ik(y \sin \theta_N - z \cos \theta_N)] + \exp [ik(-y \sin \theta_N - z \cos \theta_N)] \},$$

traveling in the directions $\pi \pm \theta_N$, where θ_N is defined by (2.4). The wave in the direction $\pi + \theta_N$ hits the lower half-plane $y = 0$, $z \geq 0$, and produces a primary diffracted field to be denoted by $u_0(r, \varphi)$ (the subscript refers to zero order interaction). A high-frequency expansion for u_0 is provided by Keller's geometrical theory of diffraction [4], viz.,

$$(3.2) \quad u_0(r, \varphi) = \frac{\exp [ikr + \pi i/4]}{4(2\pi kr)^{1/2}} [g(\varphi, \theta_N) + O((kr)^{-1})], \quad \varphi \neq \pi \pm \theta_N,$$

where $g(\varphi, \theta_N)$ is defined by (2.6). The expansion becomes singular when $\varphi = \pi \pm \theta_N$, i.e., along the shadow boundaries of the incident and reflected primary wave. Similarly, the plane wave in the direction $\pi - \theta_N$ hits the upper half-plane and gives rise to the primary diffracted field

$$(3.3) \quad \bar{u}_0(r_1, \varphi_1) = (-1)^N \frac{\exp [ikr_1 + \pi i/4]}{4(2\pi kr_1)^{1/2}} [g(\varphi_1, \theta_N) + O((kr_1)^{-1})], \quad \varphi_1 \neq \pi \pm \theta_N.$$

So far, our results are in full agreement with [1].

4. Multiple diffraction (nonuniform expansion). Each of the primary diffracted fields u_0, \bar{u}_0 acts as an incident wave on the opposite half-plane thus leading to double diffraction, or first order interaction. Diffraction of the first order interaction fields at opposite edges gives rise to second order interaction fields, and so on. The successive interaction fields are denoted by $u_n(r, \varphi)$ arising at the lower edge $y = 0$, $z = 0$ and $\bar{u}_n(r_1, \varphi_1)$ arising at the upper edge $y = a$, $z = 0$; the subscript n , $n = 1, 2, 3, \dots$, refers to the order of interaction. The interaction field $u_n(\bar{u}_n)$ is the scattered field, i.e., total field minus incident field, that arises due to diffraction of the incident field $\bar{u}_{n-1}(u_{n-1})$ at the lower (upper) edge. For reasons of symmetry the functions u_n, \bar{u}_n are connected through

$$(4.1) \quad \bar{u}_n(r_1, \varphi_1) = (-1)^N u_n(r_1, \varphi_1).$$

The $(n + 1)$ st order interaction field u_{n+1} arises from diffraction of the n th order field \bar{u}_n at the lower edge. A high-frequency expansion for u_{n+1} is provided by Keller's theory [4] and its extension, the uniform asymptotic theory of edge diffraction [5]; see Appendix A for a summary of the latter theory. In accordance with (A.13) we find that u_{n+1} consists of the geometrical-optics scattered

field and an additional diffracted field u_{n+1}^d :

$$(4.2) \quad \begin{aligned} u_{n+1}(r, \varphi) = & -\bar{u}_n(r_1, \varphi_1)H\left(\varphi - \frac{3\pi}{2}\right) + \bar{u}_n(r_{-1}, \varphi_{-1})H\left(\frac{\pi}{2} - \varphi\right) \\ & + u_{n+1}^d(r, \varphi), \end{aligned} \quad \varphi \neq \frac{\pi}{2}, \frac{3\pi}{2},$$

where $H(x)$ is the unit step function, i.e., $H(x) = 1$ for $x > 0$ and $H(x) = 0$ for $x < 0$. According to (A.14) the diffracted field u_{n+1}^d is completely determined by the incident field $\bar{u}_n = (-1)^N u_n$ and its derivatives evaluated at the diffracting edge. We now anticipate some results on these ‘‘edge values’’, to be established in subsequent sections. Setting

$$u_n(r, \varphi) = e^{ikr} z^{(n)}(r, \varphi),$$

it will be shown that

$$(4.3) \quad \begin{aligned} z^{(n)}\left(a, \frac{\pi}{2}\right) &= O((ka)^{-1/2}), \quad a \frac{\partial z^{(n)}(a, \pi/2)}{\partial r} = O((ka)^{-1/2}), \\ \frac{\partial z^{(n)}(a, \pi/2)}{\partial \varphi} &= O(1), \quad ka \rightarrow \infty. \end{aligned}$$

Replacing z^i by $z^{(n)}$ in (A.14), we only retain the terms up to order k^{-2} . Thus we obtain the expansion

$$(4.4) \quad \begin{aligned} u_{n+1}^d(r, \varphi) = & (-1)^N \frac{\exp [ikr + \pi i/4]}{2(2\pi kr)^{1/2}} \left[u_n\left(a, \frac{\pi}{2}\right) g\left(\varphi, \frac{\pi}{2}\right) \right. \\ & - i(ka)^{-1} \frac{\partial u_n(a, \pi/2)}{\partial \varphi} h(\varphi) + O((ka)^{-3/2}) \\ & \left. + O(k^{-3/2} a^{-1/2} r^{-1}) \right], \end{aligned} \quad \varphi \neq \frac{\pi}{2}, \frac{3\pi}{2},$$

where $g(\varphi, \pi/2)$, $h(\varphi)$ are defined by (2.6), (2.7). Notice that the lower sign applies in (A.14) because of the boundary condition $\partial H_x / \partial y = 0$ on the half-plane. Strictly speaking, the expansion (A.14) pertains to diffraction of a cylindrical wave due to a line source located at the upper edge $y = a, z = 0$. It will be shown in § 7 that the interaction field \bar{u}_n can be represented by an (infinite) sum of cylindrical waves centered at the upper edge. Therefore, we feel free to apply (A.14) in the present case.

The expansion (4.4) is called nonuniform, since it is not valid at the shadow boundaries $\varphi = 3\pi/2, \varphi = \pi/2$ of the incident and reflected n th order field. Both functions $g(\varphi, \pi/2)$ and $h(\varphi)$ become singular when $\varphi = \pi/2, 3\pi/2$. The leading term in (4.4) is the same as provided by Keller’s theory. The second term which is essentially due to the uniform theory, is missing in the corresponding expansion of Yee et al. [1].

According to (4.4), the diffracted field u_{n+1}^d is completely determined by the ‘‘edge values’’ $u_n(a, \pi/2), \partial u_n(a, \pi/2) / \partial \varphi$. From (3.2) we deduce

$$(4.5) \quad u_0\left(a, \frac{\pi}{2}\right) = \frac{\exp[ika + \pi i/4]}{4(2\pi ka)^{1/2}} \left[g\left(\frac{\pi}{2}, \theta_N\right) + O((ka)^{-1}) \right],$$

$$(4.6) \quad \frac{\partial u_0(a, \pi/2)}{\partial \varphi} = O((ka)^{-1/2}).$$

For $n \geq 1$ the edge values are yet to be determined. Now each edge lies on the ray-optical reflection boundary of the opposite half-plane. Therefore the required edge values cannot be derived from (4.2), (4.4) (with $n + 1$ replaced by n), since both expansions are not valid at the shadow boundary $\varphi = \pi/2$. Instead of the nonuniform expansion, we rather need a uniform expansion for $u_n(r, \varphi)$ that remains finite when $\varphi = \pi/2$. Such a uniform expansion is provided by the uniform asymptotic theory of edge diffraction as summarized in Appendix A. In subsequent sections we shall derive uniform expansions for the first and second order interaction fields u_1, u_2 and for the n th order interaction field u_n .

5. First order interaction field (uniform expansion). The first order interaction field $u_1(r, \varphi)$ arises from diffraction of the primary diffracted field $\bar{u}_0(r_1, \varphi_1)$, as given by (3.3), at the lower edge $y = 0, z = 0$. Utilizing (A.9), (A.15) (with the lower sign in view of the current boundary condition $\partial H_x / \partial y = 0$), we derive the uniform expansion

$$(5.1) \quad u_1(r, \varphi) = (-1)^N \frac{\exp[ik(r+a) + \pi i/4]}{4(2\pi)^{1/2}} k^{-1/2} \left\{ \left[F(k^{1/2}\xi) + \frac{e^{\pi i/4}}{2\pi^{1/2}} k^{-1/2} \xi^{-1} \right] (r_{-1})^{-1/2} g(\varphi_{-1}, \theta_N) + \frac{e^{\pi i/4}}{2(2\pi)^{1/2}} k^{-1/2} a^{-1/2} g\left(\frac{\pi}{2}, \theta_N\right) r^{-1/2} g\left(\varphi, \frac{\pi}{2}\right) + O(k^{-1}) \right\}, \quad 0 \leq \varphi \leq \pi,$$

truncated at terms of relative order k^{-1} . Here the Fresnel integral F is given by (A.4); $\xi, g(\varphi, \theta)$ are defined by (2.5), (2.6). At the reflected-wave boundary $\varphi = \pi/2$ one has $\xi = 0$, and $g(\varphi, \pi/2)$ becomes infinite there. The resultant singularities in (5.1) do, however, cancel, since

$$(5.2) \quad \lim_{\varphi \rightarrow \pi/2} \left[2^{1/2} \xi^{-1} (r_{-1})^{-1/2} g(\varphi_{-1}, \theta_N) + a^{-1/2} g\left(\frac{\pi}{2}, \theta_N\right) r^{-1/2} g\left(\varphi, \frac{\pi}{2}\right) \right] = -a^{-1/2} r^{-1/2} g\left(\frac{\pi}{2}, \theta_N\right) + \frac{2r^{1/2}}{a^{1/2}(r+a)} h(\theta_N),$$

where $h(\theta_N)$ is given by (2.7), (2.8). It is now clear that the expansion (5.1) remains finite at the shadow boundary $\varphi = \pi/2$.

From (5.1), (5.2) we deduce the edge values

$$(5.3) \quad u_1\left(a, \frac{\pi}{2}\right) = (-1)^N \frac{\exp[2ika + \pi i/4]}{4(2\pi ka)^{1/2}} \left[\frac{g(\pi/2, \theta_N)}{2^{3/2}} + \frac{e^{\pi i/4}}{2(2\pi ka)^{1/2}} \left\{ h(\theta_N) - g\left(\frac{\pi}{2}, \theta_N\right) \right\} + O((ka)^{-1}) \right],$$

$$(5.4) \quad \frac{\partial u_1(a, \pi/2)}{\partial \varphi} = (-1)^{N+1} \frac{e^{2ika}}{16\pi} \left[g\left(\frac{\pi}{2}, \theta_N\right) + O((ka)^{-1/2}) \right].$$

A comparison of (5.3) with Yee et al. [1, (18b)] shows that in the latter result the term $h(\theta_N)$ is missing. In [1] the field $u_1(a, \pi/2)$ was obtained from a special formula for diffraction of an isotropic cylindrical wave by a half-plane. However, the present incident field \bar{u}_0 is not isotropic, and this explains the missing term $h(\theta_N)$. Finally, we remark that the edge values of u_1 and its derivatives do satisfy (4.3).

6. Second order interaction field (uniform expansion). The second order interaction field $u_2(r, \varphi)$ arises from diffraction of the first order field $\bar{u}_1(r_1, \varphi_1)$ at the lower edge. According to (4.1) the incident field \bar{u}_1 is given by (5.1) multiplied by $(-1)^N$, and with $r, \varphi, r_{-1}, \varphi_{-1}, \xi$ replaced by $r_1, \varphi_1, r_2, \varphi_2, \xi_1$, respectively. We shall derive a uniform expansion for the field u_2 . Now, because of the rapid variation of the Fresnel integral $F(k^{1/2}\xi_1)$ across $\xi_1 = 0$, the incident field \bar{u}_1 cannot be regarded as a cylindrical wave in the vicinity of the diffracting lower edge; in this vicinity ξ_1 (as defined by (2.5)) is close to zero. Therefore the uniform theory as summarized in Appendix A does not immediately apply. In order to overcome this difficulty, the incident field is handled in the following way. The Fresnel integral $F(k^{1/2}\xi_1)$ is expanded in a Taylor series around $\xi_1 = 0$, viz.,

$$(6.1) \quad F(k^{1/2}\xi_1) = \frac{1}{2} \sum_{q=0}^{\infty} \frac{\exp[-q\pi i/4]}{\Gamma(\frac{1}{2}q + 1)} k^{(1/2)q} \xi_1^q,$$

thus leading to

$$(6.2) \quad \begin{aligned} \bar{u}_1(r_1, \varphi_1) = & \frac{\exp[ik(r_1 + a) + \pi i/4]}{4(2\pi)^{1/2}} k^{-1/2} \\ & \cdot \left\{ \frac{1}{2} \sum_{q=0}^{\infty} \frac{\exp[-q\pi i/4]}{\Gamma(\frac{1}{2}q + 1)} k^{(1/2)q} \xi_1^q r_2^{-1/2} g(\varphi_2, \theta_N) \right. \\ & + \frac{e^{\pi i/4}}{2(2\pi)^{1/2}} k^{-1/2} \left[2^{1/2} \xi_1^{-1} r_2^{-1/2} g(\varphi_2, \theta_N) \right. \\ & \left. \left. + a^{-1/2} g\left(\frac{\pi}{2}, \theta_N\right) r_1^{-1/2} g\left(\varphi_1, \frac{\pi}{2}\right) \right] + O(k^{-1}) \right\}. \end{aligned}$$

The present representation for \bar{u}_1 comprises an infinite sum of cylindrical waves centered at the upper edge $r_1 = 0$. We now perform a term by term application of the uniform theory, i.e., to each cylindrical-wave term the uniform theory is applied and the corresponding scattered field is determined.

Omitting multiplicative powers of k , the cylindrical-wave constituents of \bar{u}_1 are shortly written as

$$(6.3) \quad \bar{u}_1^{(q)}(r_1, \varphi_1) = \exp[ikr_1] z^{(q)}(r_1, \varphi_1) \xi_1^q, \quad q = 0, 1, 2, \dots,$$

where the precise form of $z^{(q)}$ can be gathered from (6.2). Diffraction of the field $\bar{u}_1^{(q)}$ at the lower edge furnishes the constituent $u_2^{(q)}$ of the second order interaction field u_2 . An expansion for $u_2^{(q)}$ is derived by means of (A.15). The coefficients \hat{v}_m

in (A.15) are given by (A.7), (A.8). In the present case, one has

$$(6.4) \quad \lambda_m(\varphi) = 0, \quad \hat{v}_m(r, \varphi) = 0 \quad \text{for } m = 0, 1, \dots, q-1,$$

since the incident field $\bar{u}_1^{(q)}$ and its derivatives of orders $0, 1, \dots, q-1$, vanish at the diffracting edge, due to the factor ζ_1^q in (6.3). In the special case $q = 0$, one has, according to (A.9),

$$(6.5) \quad \hat{v}_0(r, \varphi) + \hat{v}_0(r, 4\pi - \varphi) = \frac{e^{\pi i/4}}{2(2\pi)^{1/2}} z^{(0)} \left(a, \frac{\pi}{2} \right) r^{-1/2} g \left(\varphi, \frac{\pi}{2} \right),$$

where $g(\varphi, \pi/2)$ is defined by (2.6). The field $\bar{u}_1^{(q)}$ is accompanied by a multiplicative factor $k^{(1/2)q}$. Therefore the expansion (A.15) for $u_2^{(q)}$ is truncated at terms of order $k^{-(1/2)q-1}$, thus yielding

$$(6.6) \quad u_2^{(q)}(r, \varphi) = e^{ik(r+a)} \left\{ \left[F(k^{1/2}\xi) + \frac{e^{\pi i/4}}{2\pi^{1/2}} k^{-1/2} \sum_{m=0}^{[\frac{1}{2}q]} \binom{1}{2}_m (ik)^{-m} \xi^{-2m-1} \right] \cdot z^{(q)}(r_{-1}, \varphi_{-1}) (\xi_{-1})^q \right. \\ \left. + \frac{e^{\pi i/4}}{2(2\pi)^{1/2}} \delta_{q0} k^{-1/2} z^{(0)} \left(a, \frac{\pi}{2} \right) r^{-1/2} g \left(\varphi, \frac{\pi}{2} \right) + O(k^{-(1/2)q-1}) \right\},$$

where $[\frac{1}{2}q]$ is the largest integer $\leq \frac{1}{2}q$, $\delta_{00} = 1$, $\delta_{q0} = 0$ for $q \neq 0$ (Kronecker's symbol), and ξ, ξ_{-1} are defined by (2.5). Upon replacing F by its Taylor series (6.1), one can easily show that

$$(6.7) \quad F(k^{1/2}\xi) + \frac{e^{\pi i/4}}{2\pi^{1/2}} k^{-1/2} \sum_{m=0}^{[\frac{1}{2}q]} \binom{1}{2}_m (ik)^{-m} \xi^{-2m-1} \\ = \frac{1}{2} e^{q\pi i/4} k^{-(1/2)q} \xi^{-q} \sum_{m=0}^{\infty} \frac{\exp[-m\pi i/4]}{\Gamma(\frac{1}{2}m - \frac{1}{2}q + 1)} k^{(1/2)m} \xi^m \\ + \frac{e^{(q+1)\pi i/4}}{2\Gamma(\frac{1}{2} - \frac{1}{2}q)} k^{-(1/2)q-1/2} \xi^{-q-1}.$$

Notice that the last term vanishes when q is odd. Upon substitution of (6.7), the expansion (6.6) passes into

$$(6.8) \quad u_2^{(q)}(r, \varphi) = e^{ik(r+a)} \left\{ \left[\frac{1}{2} e^{q\pi i/4} k^{-(1/2)q} \xi^{-q} \sum_{m=0}^{\infty} \frac{\exp[-m\pi i/4]}{\Gamma(\frac{1}{2}m - \frac{1}{2}q + 1)} k^{(1/2)m} \xi^m \right. \right. \\ \left. \left. + \frac{e^{(q+1)\pi i/4}}{2\Gamma(\frac{1}{2} - \frac{1}{2}q)} k^{-(1/2)q-1/2} \xi^{-q-1} \right] z^{(q)}(r_{-1}, \varphi_{-1}) (\xi_{-1})^q \right. \\ \left. + \frac{e^{\pi i/4}}{2(2\pi)^{1/2}} \delta_{q0} k^{-1/2} z^{(0)} \left(a, \frac{\pi}{2} \right) r^{-1/2} g \left(\varphi, \frac{\pi}{2} \right) + O(k^{-(1/2)q-1}) \right\}.$$

We now insert the actual value of $z^{(q)}$ and collect the constituents $u_2^{(q)}$. Then it is found that the second order interaction field u_2 is given by the uniform expansion

$$\begin{aligned}
 u_2(r, \varphi) = & \frac{\exp [ik(r + 2a) + \pi i/4]}{4(2\pi)^{1/2}} k^{-1/2} \left\{ \left[\frac{1}{4} \sum_{q=0}^{\infty} \frac{(\xi_{-1}/\xi)^q}{\Gamma(\frac{1}{2}q + 1)} \sum_{m=0}^{\infty} \frac{\exp [-m\pi i/4]}{\Gamma(\frac{1}{2}m - \frac{1}{2}q + 1)} \right. \right. \\
 & \cdot k^{(1/2)m} \xi^m + \frac{e^{\pi i/4}}{4} k^{-1/2} \xi^{-1} \sum_{q=0}^{\infty} \frac{(\xi_{-1}/\xi)^{2q}}{q! \Gamma(\frac{1}{2} - q)} \left. \right] (r_{-2})^{-1/2} g(\varphi_{-2}, \theta_N) \\
 (6.9) \quad & + \frac{e^{\pi i/4}}{4(2\pi)^{1/2}} k^{-1/2} (2a)^{-1/2} g\left(\frac{\pi}{2}, \theta_N\right) r^{-1/2} g\left(\varphi, \frac{\pi}{2}\right) \\
 & + \frac{e^{\pi i/4}}{2(2\pi)^{1/2}} k^{-1/2} F(k^{1/2} \xi) \left[2^{1/2} (\xi_{-1})^{-1} (r_{-2})^{-1/2} g(\varphi_{-2}, \theta_N) \right. \\
 & \left. + a^{-1/2} g\left(\frac{\pi}{2}, \theta_N\right) (r_{-1})^{-1/2} g\left(\varphi_{-1}, \frac{\pi}{2}\right) \right] + O(k^{-1}),
 \end{aligned}$$

valid for $0 \leq \varphi \leq \pi$, away from the edge $r = 0$. The latter expansion can be simplified by explicit summation of the series involved. Setting $\eta = \xi_{-1}/\xi$, it is obvious that $0 \leq \eta < 1$, and one has

$$(6.10) \quad \sum_{q=0}^{\infty} \frac{\eta^{2q}}{q! \Gamma(\frac{1}{2} - q)} = \pi^{-1/2} (1 + \eta^2)^{-1/2}.$$

The summation of the double series in (6.9) is performed in Appendix B; see (B.6) for the final result. Thus the uniform expansion (6.9) can be reduced to

$$\begin{aligned}
 u_2(r, \varphi) = & \frac{\exp [ik(r + 2a) + \pi i/4]}{4(2\pi)^{1/2}} k^{-1/2} \left\{ [F(k^{1/2} \xi) F(k^{1/2} \xi_{-1}) \right. \\
 & + \frac{1}{2} F(k^{1/2} \xi \sqrt{1 + \eta^2}) - \frac{1}{2} \exp(-ik \xi_{-1}^2) F(k^{1/2} \xi) \\
 & + G(\eta, k^{1/2} \xi)] (r_{-2})^{-1/2} g(\varphi_{-2}, \theta_N) \\
 (6.11) \quad & + \frac{e^{\pi i/4}}{4\pi^{1/2}} k^{-1/2} \left[\xi^{-1} (1 + \eta^2)^{-1/2} (r_{-2})^{-1/2} g(\varphi_{-2}, \theta_N) \right. \\
 & \left. + \frac{1}{2} a^{-1/2} g\left(\frac{\pi}{2}, \theta_N\right) r^{-1/2} g\left(\varphi, \frac{\pi}{2}\right) \right] \\
 & + \frac{e^{\pi i/4}}{2(2\pi)^{1/2}} k^{-1/2} F(k^{1/2} \xi) \left[2^{1/2} (\xi_{-1})^{-1} (r_{-2})^{-1/2} g(\varphi_{-2}, \theta_N) \right. \\
 & \left. + a^{-1/2} g\left(\frac{\pi}{2}, \theta_N\right) (r_{-1})^{-1/2} g\left(\varphi_{-1}, \frac{\pi}{2}\right) \right] + O(k^{-1}) \left. \right\}, \quad 0 \leq \varphi \leq \pi,
 \end{aligned}$$

where the following notation is employed: $\xi, \xi_{-1}, g(\varphi, \theta)$ are defined by (2.5), (2.6), $\eta = \xi_{-1}/\xi$; the Fresnel integral F is introduced in (A.4); the function G is given by

$$(6.12) \quad G(\eta, t) = \frac{\exp[-i\eta^2 t^2]}{2\pi} \int_0^\eta \frac{\exp[i\sigma^2 t^2]}{1 + \sigma^2} d\sigma.$$

At the shadow boundary $\varphi = \pi/2$, one has $\xi = \xi_{-1} = 0$, and $g(\varphi, \pi/2)$, $g(\varphi_{-1}, \pi/2)$ become infinite there. As before (compare (5.2)), it can be shown that the resultant singularities in (6.11) do cancel. Hence the expansion (6.11) remains finite at the shadow boundary $\varphi = \pi/2$. Away from the shadow boundary, both F and G can be replaced by their asymptotic expansions, and (6.11) reduces to the nonuniform expansion (4.2), (4.4) with $n = 1$.

At the opposite edge $r = a$, $\varphi = \pi/2$, one has $\xi = \xi_{-1} = 0$, $\eta = 3^{-1/2}$. From (6.11) we deduce the edge values

$$(6.13) \quad u_2\left(a, \frac{\pi}{2}\right) = \frac{\exp[3ika + \pi i/4]}{4(2\pi ka)^{1/2}} \left[\frac{g(\pi/2, \theta_N)}{3^{3/2}} + \frac{e^{\pi i/4}}{4(\pi ka)^{1/2}} \left\{ h(\theta_N) - g\left(\frac{\pi}{2}, \theta_N\right) \right\} + O((ka)^{-1}) \right],$$

$$(6.14) \quad \frac{\partial u_2(a, \pi/2)}{\partial \varphi} = -\frac{e^{3ika}}{16\pi} \left[\frac{g(\pi/2, \theta_N)}{2^{1/2}} + O((ka)^{-1/2}) \right].$$

We remark that the edge values of u_2 and its derivatives do satisfy (4.3).

We want to comment on the expansion (6.11) and its derivation. First of all, just like Keller's theory of edge diffraction, the uniform theory is a formal asymptotic method, and no general proof has yet been given that the formal asymptotic solution is identical with the asymptotic expansion of the exact solution of the problem at hand. Nevertheless, the agreement found at various special problems provides strong evidence of the validity of both Keller's theory and its uniform extension. In the present case there is still another formal aspect. The expansion (6.11) was obtained by a formal term-by-term application of the uniform theory. In order to justify this formal procedure, we compare (6.11) with a rigorous asymptotic result due to Jones [13]. The latter deals with diffraction of a plane wave by two staggered, parallel half-planes. Starting from an exact integral representation, Jones derives a uniform asymptotic expansion for, in our terminology, the first order interaction field. In [13, § 7], the special case is considered when the incident plane wave propagates in a direction parallel to the line through the edges of the half-planes. Then the second diffracting edge lies on the shadow boundary corresponding to the first edge. In this particular case, Jones' uniform expansion is of the same form as (6.11), containing Fresnel integrals and a generalization of the Fresnel integral that is closely related to our function G . In a forthcoming paper we shall present a more detailed comparison between Jones' rigorous results and the formal asymptotic solution as provided by the uniform theory.

Finally, we compare (6.13) with the corresponding result in Yee et al. [1, (22a, b)], viz.,

$$(6.15) \quad u_2\left(a, \frac{\pi}{2}\right)_{YFK} = \frac{\exp[3ika + \pi i/4]}{4(2\pi ka)^{1/2}} \left[\frac{g(\pi/2, \theta_N)}{4\sqrt{3}} - \frac{e^{\pi i/4}}{4(\pi ka)^{1/2}} g\left(\frac{\pi}{2}, \theta_N\right) + O((ka)^{-1}) \right].$$

The leading terms in (6.15), (6.13) are different, and in the second term of (6.15) the factor $h(\theta_N)$ is missing. In the approach of [1] the incident interaction field \bar{u}_1 is approximated by the field of two isotropic line sources located at $y = 2a$, $z = 0$ and $y = a$, $z = 0$. In our notation this approximation is given by

$$(6.16) \quad \bar{u}_1|_{\text{approx}} = \frac{\exp [ikr_2 + \pi i/4]}{8(2\pi)^{1/2}} k^{-1/2} r_2^{-1/2} g\left(\frac{\pi}{2}, \theta_N\right) - \frac{\exp [ik(r_1 + a) + \pi i/2]}{16\pi} k^{-1} a^{-1/2} r_1^{-1/2} g\left(\frac{\pi}{2}, \theta_N\right).$$

Diffraction of the latter field at the lower edge furnishes the second order field u_2 , and it is found that $u_2(a, \pi/2)$ is given by (6.15). The discrepancy between (6.15) and (6.13) shows that the approximation (6.16) is not permissible.

7. Higher order interaction fields (uniform expansions). In this section we present a uniform expansion for the n th order interaction field $u_n(r, \varphi)$ which remains finite at the shadow boundary $\varphi = \pi/2$. Guided by the special results (6.2), (6.9) for $n = 1, 2$, we introduce the Ansatz

$$(7.1) \quad u_n(r, \varphi) = (-1)^{Nn} \frac{\exp [ik(r + na) + \pi i/4]}{4(2\pi)^{1/2}} k^{-1/2} \left\{ \sum_{q=0}^{\infty} e^{-q\pi i/4} u_{n,q}(r, \varphi) k^{(1/2)q\xi q} + \frac{e^{\pi i/4}}{(2\pi)^{1/2}} k^{-1/2} \sum_{q=0}^{\infty} e^{-q\pi i/4} v_{n,q}(r, \varphi) k^{(1/2)q\xi q} + O(k^{-1}) \right\},$$

$0 \leq \varphi \leq \pi,$

where ξ is defined by (2.5). The Ansatz contains the first and second term of a high-frequency expansion in inverse powers of k . Each of these terms is represented by a convergent Taylor series with coefficients $u_{n,q}, v_{n,q}$, respectively, which are to be determined. A similar Ansatz holds for the interaction field $\bar{u}_n(r_1, \varphi_1)$ arising at the upper edge; it is given by (7.1) multiplied by $(-1)^N$, and with r, φ, ξ replaced by r_1, φ_1, ξ_1 , respectively. From (7.1) we deduce the edge values

$$(7.2) \quad u_n\left(a, \frac{\pi}{2}\right) = (-1)^{Nn} \frac{\exp [i(n + 1)ka + \pi i/4]}{4(2\pi)^{1/2}} k^{-1/2} \left[u_{n,0}\left(a, \frac{\pi}{2}\right) + \frac{e^{\pi i/4}}{(2\pi)^{1/2}} k^{-1/2} v_{n,0}\left(a, \frac{\pi}{2}\right) + O(k^{-1}) \right],$$

$$(7.3) \quad \frac{\partial u_n(a, \pi/2)}{\partial \varphi} = (-1)^{Nn+1} \frac{\exp [i(n + 1)ka]}{8(2\pi)^{1/2}} \left[a^{1/2} u_{n,1}\left(a, \frac{\pi}{2}\right) + O(k^{-1/2}) \right].$$

Hence our main object will be the evaluation of $u_{n,0}, u_{n,1}, v_{n,0}$ at the opposite edge $r = a, \varphi = \pi/2$.

Diffraction of the n th order interaction field $\bar{u}_n(r_1, \varphi_1)$ at the lower edge gives rise to the $(n + 1)$ st order field $u_{n+1}(r, \varphi)$. Proceeding as in § 6, the scattered field u_{n+1} is determined by a term by term application of the uniform theory. According to the Ansatz, the incident field \bar{u}_n comprises an infinite sum of

cylindrical waves centered at the upper edge $r_1 = 0$. The cylindrical-wave constituents of \bar{u}_n are of the form (6.3). Then the corresponding constituents of u_{n+1} are given by an expansion of the form (6.8), derived by means of the uniform theory. Upon collecting the latter constituents, we obtain the uniform expansion

$$\begin{aligned}
 (7.4) \quad u_{n+1}(r, \varphi) &= (-1)^{N(n+1)} \frac{\exp [ik(r + (n + 1)a) + \pi i/4]}{4(2\pi)^{1/2}} k^{-1/2} \\
 &\cdot \left\{ \frac{1}{2} \sum_{q=0}^{\infty} u_{n,q}(r_{-1}, \varphi_{-1}) \eta^q \sum_{m=0}^{\infty} \frac{\exp [-m\pi i/4]}{\Gamma(\frac{1}{2}m - \frac{1}{2}q + 1)} k^{(1/2)m} \xi^m \right. \\
 &+ \frac{e^{\pi i/4}}{2(2\pi)^{1/2}} k^{-1/2} \sum_{q=0}^{\infty} v_{n,q}(r_{-1}, \varphi_{-1}) \eta^q \sum_{m=0}^{\infty} \frac{\exp [-m\pi i/4]}{\Gamma(\frac{1}{2}m - \frac{1}{2}q + 1)} k^{(1/2)m} \xi^m \\
 &+ \frac{1}{2} e^{\pi i/4} k^{-1/2} \xi^{-1} \sum_{q=0}^{\infty} \frac{u_{n,2q}(r_{-1}, \varphi_{-1})}{\Gamma(\frac{1}{2} - q)} \eta^{2q} \\
 &\left. + \frac{e^{\pi i/4}}{2(2\pi)^{1/2}} k^{-1/2} u_{n,0} \left(a, \frac{\pi}{2} \right) r^{-1/2} g \left(\varphi, \frac{\pi}{2} \right) + O(k^{-1}) \right\}, \quad 0 \leq \varphi \leq \pi,
 \end{aligned}$$

where $\eta = \xi_{-1}/\xi$ and $\xi, \xi_{-1}, g(\varphi, \pi/2)$ are defined by (2.5), (2.6).

We examine the behavior of the expansion (7.4) at the shadow boundary $\varphi = \pi/2$. Along this line, one has $\xi = 0, \eta = r^{1/2}/(r + 2a)^{1/2}$, and $g(\varphi, \pi/2)$ becomes infinite there. The resultant singularities in the third and fourth term of (7.4) will cancel provided that the following ‘‘finiteness condition’’ is fulfilled, viz.,

$$(7.5) \quad u_{n,0} \left(a, \frac{\pi}{2} \right) = \pi^{1/2} \left(\frac{r + a}{a} \right)^{1/2} \sum_{q=0}^{\infty} \frac{u_{n,2q}(r + a, \pi/2)}{\Gamma(\frac{1}{2} - q)} \left(\frac{r}{r + 2a} \right)^q.$$

In Appendix C it is shown that the actual coefficients $u_{n,q}$ do satisfy (7.5). Hence the expansion (7.4) is indeed finite at the shadow boundary $\varphi = \pi/2$. It has been verified that the next term in (7.4), of relative order k^{-1} , remains also finite when $\varphi = \pi/2$. In that case, two additional finiteness conditions are to be imposed. It has been shown that the actual coefficients $u_{n,q}, v_{n,q}$ do satisfy these conditions.

The expansion (7.4) for the field u_{n+1} is now compared to the Ansatz (7.1) with n replaced by $n + 1$. By equating corresponding terms we are led to a set of recurrence relations for the coefficients $u_{n,q}, v_{n,q}$, viz.,

$$\begin{aligned}
 (7.6) \quad u_{n+1,m}(r, \varphi) &= \frac{1}{2} \sum_{q=0}^{\infty} \frac{u_{n,q}(r_{-1}, \varphi_{-1})}{\Gamma(\frac{1}{2}m - \frac{1}{2}q + 1)} \eta^q, \\
 (7.7) \quad v_{n+1,m}(r, \varphi) &= \frac{1}{2} \sum_{q=0}^{\infty} \frac{v_{n,q}(r_{-1}, \varphi_{-1})}{\Gamma(\frac{1}{2}m - \frac{1}{2}q + 1)} \eta^q \\
 &+ \frac{1}{2} \delta_{m0} \left[(2\pi)^{1/2} \xi^{-1} \sum_{q=0}^{\infty} \frac{u_{n,2q}(r_{-1}, \varphi_{-1})}{\Gamma(\frac{1}{2} - q)} \eta^{2q} \right. \\
 &\left. + u_{n,0} \left(a, \frac{\pi}{2} \right) r^{-1/2} g \left(\varphi, \frac{\pi}{2} \right) \right],
 \end{aligned}$$

where $m = 0, 1, 2, \dots$, and $\delta_{00} = 1, \delta_{m0} = 0$ for $m \neq 0$. The present relations are accompanied by the initial values

$$(7.8) \quad u_{0,q}(r, \varphi) = \delta_{q0}r^{-1/2}g(\varphi, \theta_N), \quad v_{0,q}(r, \varphi) = 0, \quad q = 0, 1, 2, \dots,$$

quoted from the expansion (3.2) for the primary diffracted field u_0 . Thus the coefficients $u_{n,q}, v_{n,q}$ are completely determined.

The recurrence relations (7.6), (7.7) may be solved by the methods described in Appendix C; see the remark at the end of that Appendix. However, since we are mainly interested in the edge values (7.2), (7.3), we confine our investigation to the recurrence relations along the line $\varphi = \pi/2$. Assuming that the finiteness condition (7.5) is satisfied, it is found that for $\varphi \rightarrow \pi/2$ the relations (7.6), (7.7) pass into

$$(7.9) \quad u_{n+1,m}\left(r, \frac{\pi}{2}\right) = \frac{1}{2} \sum_{q=0}^{\infty} \frac{u_{n,q}(r+a, \pi/2)}{\Gamma(\frac{1}{2}m - \frac{1}{2}q + 1)} \left(\frac{r}{r+2a}\right)^{(1/2)q},$$

$$(7.10) \quad v_{n+1,m}\left(r, \frac{\pi}{2}\right) = \frac{1}{2} \sum_{q=0}^{\infty} \frac{v_{n,q}(r+a, \pi/2)}{\Gamma(\frac{1}{2}m - \frac{1}{2}q + 1)} \left(\frac{r}{r+2a}\right)^{(1/2)q}$$

$$- \delta_{m0}\pi^{1/2} \frac{r^{1/2}}{a^{1/2}(r+a)^{1/2}} \sum_{q=0}^{\infty} \frac{1}{\Gamma(\frac{1}{2} - q)} \frac{\partial u_{n,2q}(r+a, \pi/2)}{\partial \varphi} \left(\frac{r}{r+2a}\right)^q$$

$$- \frac{1}{2} \delta_{m0} u_{n,0}\left(a, \frac{\pi}{2}\right) r^{-1/2},$$

subject to the initial conditions (cf. (7.8)),

$$(7.11) \quad u_{0,q}\left(r, \frac{\pi}{2}\right) = \delta_{q0}r^{-1/2}g\left(\frac{\pi}{2}, \theta_N\right), \quad v_{0,q}\left(r, \frac{\pi}{2}\right) = 0, \quad q = 0, 1, 2, \dots$$

The derivatives $\partial u_{n,q}/\partial \varphi$ at $\varphi = \pi/2$ are determined by the recurrence relation

$$(7.12) \quad \frac{\partial u_{n+1,m}(r, \pi/2)}{\partial \varphi} = \frac{1}{2} \frac{r}{r+a} \sum_{q=0}^{\infty} \frac{1}{\Gamma(\frac{1}{2}m - \frac{1}{2}q + 1)} \frac{\partial u_{n,q}(r+a, \pi/2)}{\partial \varphi} \left(\frac{r}{r+2a}\right)^{(1/2)q},$$

which is obtained by differentiation of (7.6). From (7.8) we deduce the initial values

$$(7.13) \quad \frac{\partial u_{0,q}(r, \pi/2)}{\partial \varphi} = -\delta_{q0}r^{-1/2}h(\theta_N),$$

where $h(\theta_N)$ is given by (2.7), (2.8). It is observed that (7.12) is of the same form as (7.9), except for an extra factor $r/(r+a)$. Hence, the solutions of these recurrence relations are simply related and it is easily seen that

$$(7.14) \quad \frac{\partial u_{n,q}(r, \pi/2)}{\partial \varphi} = -\frac{r}{r+na} \frac{h(\theta_N)}{g(\pi/2, \theta_N)} u_{n,q}\left(r, \frac{\pi}{2}\right).$$

Upon substitution of the latter result, the second sum in (7.10) can be evaluated

by means of the finiteness condition (7.5). Then the recurrence relation (7.10) takes the simplified form

$$(7.15) \quad v_{n+1,m}\left(r, \frac{\pi}{2}\right) = \frac{1}{2} \sum_{q=0}^{\infty} \frac{v_{n,q}(r+a, \pi/2)}{\Gamma(\frac{1}{2}m - \frac{1}{2}q + 1)} \left(\frac{r}{r+2a}\right)^{(1/2)q} + \frac{1}{2} \delta_{m0} u_{n,0}\left(a, \frac{\pi}{2}\right) r^{-1/2} \left[\frac{2r}{r+(n+1)a} \frac{h(\theta_N)}{g(\pi/2, \theta_N)} - 1 \right].$$

The actual solution of the recurrence relations (7.9), (7.15), subject to the initial conditions (7.11), is derived in Appendix C. Referring to (C.1), (C.2), (C.17), we present the solution

$$(7.16) \quad u_{n,q}\left(r, \frac{\pi}{2}\right) = r^{-1/2} g\left(\frac{\pi}{2}, \theta_N\right) I_{n,q}(r),$$

$$(7.17) \quad v_{n,q}\left(r, \frac{\pi}{2}\right) = \frac{a^{-1/2} r^{-1/2}}{r+na} h(\theta_N) \sum_{m=1}^n [r+(n-m)a] I_{m-1,0}(a) I_{n-m,q}(r) - \frac{1}{2} a^{-1/2} r^{-1/2} g\left(\frac{\pi}{2}, \theta_N\right) \sum_{m=1}^n I_{m-1,0}(a) I_{n-m,q}(r),$$

where $I_{n,q}(r)$ is given by the n -fold integral (see (C.13))

$$(7.18) \quad I_{n,q}(r) = \pi^{-(1/2)n} \frac{2^q}{q!} \left(\frac{r+a}{r}\right)^{(1/2)q} \int_0^{\infty} \dots \int_0^{\infty} x_1^q \exp\left[-\frac{r+a}{r} x_1^2 - 2 \sum_{m=2}^n x_m^2 + 2 \sum_{m=2}^n x_{m-1} x_m\right] dx_1 \dots dx_n.$$

Two special cases of the latter integral are evaluated in Appendix D, where we find (see (D.9), (D.12))

$$(7.19) \quad I_{n,0}(a) = \frac{1}{(n+1)^{3/2}}, \quad I_{n,1}(a) = \frac{1}{(2\pi)^{1/2}} \sum_{m=1}^n \frac{1}{m^{3/2}(n-m+1)^{3/2}}.$$

The present results are needed in order to establish the edge values (cf. (7.2), (7.3))

$$(7.20) \quad u_n\left(a, \frac{\pi}{2}\right) = (-1)^{N_n} \frac{\exp [i(n+1)ka + \pi i/4]}{4(2\pi ka)^{1/2}} \left[\frac{g(\pi/2, \theta_N)}{(n+1)^{3/2}} + \frac{e^{\pi i/4}}{2(2\pi ka)^{1/2}} \cdot \left\{ h(\theta_N) - g\left(\frac{\pi}{2}, \theta_N\right) \right\} \sum_{m=1}^n \frac{1}{m^{3/2}(n-m+1)^{3/2}} + O((ka)^{-1}) \right],$$

$$(7.21) \quad \frac{\partial u_n(a, \pi/2)}{\partial \varphi} = (-1)^{N_{n+1}} \frac{\exp [i(n+1)ka]}{16\pi} \cdot \left[g\left(\frac{\pi}{2}, \theta_N\right) \sum_{m=1}^n \frac{1}{m^{3/2}(n-m+1)^{3/2}} + O((ka)^{-1/2}) \right].$$

The diffracted field u_n^d as given by the nonuniform expansion (4.4), is now completely specified. It is remarked that the edge values of u_n and its derivatives do satisfy (4.3).

The edge value $u_n(a, \pi/2)$ as derived by Yee et al. [1], contains a leading term $g(\pi/2, \theta_N)/2^n(n + 1)^{1/2}$ versus our term $g(\pi/2, \theta_N)/(n + 1)^{3/2}$ in (7.20). In the approach of [1] the incident interaction field \bar{u}_{n-1} is approximated by the field of an equivalent set of isotropic line sources located at $y = ma, z = 0, m = 1, 2, \dots, n$, the source strengths being such as to provide the correct interaction field in the direction toward the lower edge. Then the back-scattered field $u_n(a, \pi/2)$ is determined by means of a special formula for scattering of an isotropic cylindrical wave by a half-plane. This explains the discrepancy noticed above.

8. Reflected field. The total diffracted field $u^d(r, \varphi)$ arising at the lower edge is obtained by summation of the primary and multiply diffracted fields u_0, u_n^d , as given by (3.2), (4.4), (7.20), (7.21). Thus we find

$$(8.1) \quad u^d(r, \varphi) = u_0(r, \varphi) + \sum_{n=1}^{\infty} u_n^d(r, \varphi) = \frac{\exp [ikr + \pi i/4]}{2(2\pi kr)^{1/2}} [f(\varphi) + O((kr)^{-1})],$$

valid for $\varphi \neq \pi \pm \theta_N, \varphi \neq \pi/2, 3\pi/2$, where the radiation pattern $f(\varphi)$ is given by

$$(8.2) \quad f(\varphi) = \frac{1}{2}g(\varphi, \theta_N) - \frac{e^{\pi i/4}}{4(2\pi ka)^{1/2}}S^{\pm}(ka)g\left(\frac{\pi}{2}, \theta_N\right)g\left(\varphi, \frac{\pi}{2}\right) + \frac{i}{16\pi ka}[S^{\pm}(ka)]^2 \cdot \left[h(\theta_N)g\left(\varphi, \frac{\pi}{2}\right) - g\left(\frac{\pi}{2}, \theta_N\right)g\left(\varphi, \frac{\pi}{2}\right) + g\left(\frac{\pi}{2}, \theta_N\right)h(\varphi) \right] + O((ka)^{-3/2}),$$

with $S^{\pm}(ka)$ defined by

$$(8.3) \quad S^{\pm}(ka) = \pm \sum_{m=1}^{\infty} \frac{(\mp 1)^{m-1} e^{imka}}{m^{3/2}}.$$

In (8.2), (8.3) the upper sign applies for N odd and the lower for N even. Likewise, the total diffracted field \bar{u}^d arising at the upper edge $y = a, z = 0$, is given by

$$(8.4) \quad \bar{u}^d(r_1, \varphi_1) = (-1)^N \frac{\exp [ikr_1 + \pi i/4]}{2(2\pi kr_1)^{1/2}} [f(\varphi_1) + O((kr_1)^{-1})],$$

valid for $\varphi_1 \neq \pi \pm \theta_N, \varphi_1 \neq \pi/2, 3\pi/2$, in view of the symmetry relation (4.1).

The diffracted fields u^d, \bar{u}^d are regarded as being due to induced line sources located at the edges of the waveguide. The radiation of these sources back into the waveguide establishes the reflected wave. The radiation fields and their repeated reflections at the waveguide walls are converted into modal form by utilizing the procedure detailed in [1, § 2]. Thus we find that the reflected field is given by

$$(8.5) \quad H_x^r(y, z) = \sum_{n=0}^{\infty} \varepsilon_n \Gamma_{Nn} \cos \left(\frac{n\pi y}{a} \right) \exp (i\kappa_n z),$$

where $\varepsilon_0 = 1, \varepsilon_n = 2$ for $n \neq 0$, and with reflection coefficients Γ_{Nn} that are simply

related to the radiation pattern f of the line sources, viz.,

$$(8.6) \quad \Gamma_{Nn} = [1 + (-1)^{N+n}] \frac{i}{4a\kappa_n} f(\theta_n).$$

Here the angle θ_n is defined by (2.4). From (8.6) it is clear that $\Gamma_{Nn} = 0$ when $N + n$ is odd, in agreement with the requirement of zero coupling between modes with different symmetries. When $N + n$ is even, we evaluate $f(\theta_n)$ as given by (8.2), by employing the definitions (2.4), (2.6), (2.7) of the angle θ_n and the functions g, h . As a result we find

$$(8.7) \quad \Gamma_{Nn} = -\frac{i}{2} \frac{(k + \kappa_N)^{1/2}(k + \kappa_n)^{1/2}}{a\kappa_n(\kappa_N + \kappa_n)} \left[1 + \frac{e^{\pi i/4}}{(2\pi ka)^{1/2}} \frac{k(\kappa_N + \kappa_n)}{\kappa_N \kappa_n} S^\pm(ka) \right. \\ \left. + \frac{1}{2} \left\{ \frac{e^{\pi i/4}}{(2\pi ka)^{1/2}} \frac{k(\kappa_N + \kappa_n)}{\kappa_N \kappa_n} S^\pm(ka) \right\}^2 + O((ka)^{-3/2}) \right], \quad N + n \text{ even,}$$

where $S^\pm(ka)$ is given by (8.3). The upper sign applies for both N and n odd and the lower for both N and n even. The present result (8.7) is valid even if the involved modes are nonpropagating. In that case, $(k + \kappa_j)^{1/2}$, $j = N, n$, must be understood as the principal value of the square root, i.e., $\text{Re}(k + \kappa_j)^{1/2} \geq 0$. The expansion (8.7) breaks down when $\kappa_N = 0$ or $\kappa_n = 0$, i.e., at the cutoff frequencies of the N th and n th modes. The reflection coefficients Γ_{Nn} as given by (8.7) satisfy the reciprocity relation $\kappa_n \Gamma_{Nn} = \kappa_N \Gamma_{nN}$, which is known to be exact for this class of waveguide discontinuity problems. The present solution (8.7) is to be compared with the result of Yee, Felsen and Keller [1], viz.,

$$(8.8) \quad \Gamma_{Nn}|_{YFK} = -\frac{i}{2} \frac{(k + \kappa_N)^{1/2}(k + \kappa_n)^{1/2}}{a\kappa_n(\kappa_N + \kappa_n)} \left[1 + \frac{e^{\pi i/4}}{(2\pi ka)^{1/2}} \frac{k(\kappa_N + \kappa_n)}{\kappa_N \kappa_n} A^\pm(ka) \right. \\ \left. + \frac{i}{4\pi ka} \frac{k(\kappa_N + \kappa_n)}{\kappa_N \kappa_n} \{A^\pm(ka)\}^2 + O((ka)^{-3/2}) \right], \quad N + n \text{ even,}$$

where

$$(8.9) \quad A^\pm(ka) = \pm \sum_{m=1}^{\infty} \frac{(\mp 1)^{m-1} e^{imka}}{2^{m-1} m^{1/2}}.$$

Notice that the series S^\pm and A^\pm are different except for the first and second terms. The discrepancy between the two results for Γ_{Nn} is due to a simplifying approximation that underlies the calculation of the higher order interaction fields in [1]; see the comments at the end of §§ 6, 7.

We now reduce (8.7) to the equivalent form

$$(8.10) \quad \Gamma_{Nn} = -\frac{i}{2} \frac{(k + \kappa_N)^{1/2}(k + \kappa_n)^{1/2}}{a\kappa_n(\kappa_N + \kappa_n)} \exp \left[\frac{e^{\pi i/4}}{(2\pi ka)^{1/2}} \frac{k(\kappa_N + \kappa_n)}{\kappa_N \kappa_n} S^\pm(ka) \right. \\ \left. + O((ka)^{-3/2}) \right].$$

The latter result is compared with the exact solution of the reflection problem to be derived by means of the Wiener-Hopf technique; cf. [14]–[16]. Referring to

Weinstein [15, (10.37), (10.38)], the exact value of Γ_{Nn} can be represented by

$$(8.11) \quad \Gamma_{Nn} = -\frac{i}{2} \frac{(k + \kappa_N)^{1/2}(k + \kappa_n)^{1/2}}{a\kappa_n(\kappa_N + \kappa_n)} \exp \left[U^\pm \left(\frac{\kappa_N}{k}, ka \right) + U^\pm \left(\frac{\kappa_n}{k}, ka \right) \right],$$

valid for $N + n$ even, where $U^\pm(s, ka)$ is given by

$$(8.12) \quad U^\pm(s, ka) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \log(1 \pm e^{ika - kat^2}) \frac{(1 + it^2)(1 + \frac{1}{2}it^2)^{-1/2}}{t(1 + \frac{1}{2}it^2)^{1/2} - 2^{-1/2} e^{\pi i/4} s} dt$$

and the same sign convention applies as before. In (8.12) the logarithm and the square root stand for principal values. It is remarked that U^+ , U^- correspond with the original, exact Weinstein functions V , U , as defined in [15, (10.07), (10.18)]. By means of Laplace's method we derive the asymptotic expansion

$$(8.13) \quad \begin{aligned} U^\pm(s, ka) &= -\frac{2^{1/2} e^{-\pi i/4} s^{-1}}{2\pi i} \int_{-\infty}^{\infty} \log(1 \pm e^{ika - kat^2}) dt + O(s^{-3}(ka)^{-3/2}) \\ &= \frac{e^{\pi i/4} s^{-1}}{(2\pi ka)^{1/2}} S^\pm(ka) + O(s^{-3}(ka)^{-3/2}), \end{aligned}$$

valid for large ka provided that s is not close to zero. Upon inserting (8.13) into (8.11), one can easily see that the ray-optical result (8.10) is identical with the asymptotic expansion of the exact coefficient Γ_{Nn} for large ka provided that the N th and n th modes are not close to cutoff.

The integral $U^\pm(s, ka)$ arises in the Wiener-Hopf procedure of factorization of a certain analytic function. This factorization can also be performed in terms of infinite products, cf. Noble [14, §§ 3.2, 3.3]. Omitting the details, we thus obtain the following alternative representations for the amplitude and phase of the exact reflection coefficient Γ_{Nn} :

$$(8.14a) \quad \begin{aligned} |\Gamma_{Nn}| &= \frac{a\kappa_N(k + \kappa_N)^{1/2}(k + \kappa_n)^{1/2}}{\pi N^{1/2} n^{1/2} (\kappa_N + \kappa_n)} \exp \left[-(\kappa_N + \kappa_n) \frac{a}{4} \right] \\ &\cdot \left| \prod_{\substack{m=1, \neq N \\ m \text{ odd}}}^{M^+} \frac{\kappa_N + \kappa_m}{\kappa_N - \kappa_m} \prod_{\substack{m=1, \neq n \\ m \text{ odd}}}^{M^+} \frac{\kappa_n + \kappa_m}{\kappa_n - \kappa_m} \right|^{1/2}, \quad N, n \text{ odd;} \end{aligned}$$

$$(8.14b) \quad \begin{aligned} |\Gamma_{Nn}| &= \frac{2\kappa_N}{\varepsilon_N^{1/2} e_n^{1/2} (\kappa_N + \kappa_n)} \exp \left[-(\kappa_N + \kappa_n) \frac{a}{4} \right] \\ &\cdot \left| \prod_{\substack{m=0, \neq N \\ m \text{ even}}}^{M^-} \frac{\kappa_N + \kappa_m}{\kappa_N - \kappa_m} \prod_{\substack{m=0, \neq n \\ m \text{ even}}}^{M^-} \frac{\kappa_n + \kappa_m}{\kappa_n - \kappa_m} \right|^{1/2}, \quad N, n \text{ even;} \end{aligned}$$

$$(8.15) \quad \begin{aligned} \arg \Gamma_{Nn} &= \pi - \frac{1}{2} M^\pm \pi + \frac{1}{2} N \arcsin \frac{N\pi}{ka} + \frac{1}{2} n \arcsin \frac{n\pi}{ka} \\ &+ \frac{(\kappa_N + \kappa_n)a}{2\pi} \left[\psi \left(\frac{1}{2} M^\pm + 1 \right) + 1 - \log \frac{ka}{4\pi} \right] \\ &- S_{M^\pm+2}(\kappa_N a/\pi; N, 0) - S_{M^\pm+2}(\kappa_n a/\pi; n, 0), \quad N + n \text{ even.} \end{aligned}$$

Here $M^+(M^-)$ is the largest odd-(even) integer $\leq ka/\pi$, $\psi(w)$ is the logarithmic derivative of the Γ -function, i.e., $\psi(w) = \Gamma'(w)/\Gamma(w)$, and S_M stands for the well-known sinc-function (cf. Marcuvitz [17, Appendix A]) in a slightly modified form, viz.,

$$(8.16) \quad S_M(x; \alpha, 0) = \sum_{\substack{m=M \\ m-M \text{ even}}}^{\infty} \left[\arcsin \frac{x}{\sqrt{m^2 - \alpha^2}} - \frac{x}{m} \right].$$

In (8.15) the upper sign applies for both N and n odd and the lower for both N and n even. The present results (8.14), (8.15) are valid only for propagating modes, i.e., for $N \leq ka/\pi$, $n \leq ka/\pi$. It seems that (8.14), (8.15) are new except that some special cases were presented by Weinstein [15].

Finally, we remark that the diffracted fields u^d, \bar{u}^d also radiate into the free space surrounding the waveguide. The exterior radiation field can easily be determined and it can be shown that the result does agree with the asymptotic expansion (for large ka) of the exact radiation field. The details will be presented in a forthcoming paper.

9. Numerical results. The present reflection problem for an open-ended parallel-plane waveguide is especially convenient to assess the accuracy of the ray-optical method, since the exact solution of the reflection problem is available for comparison. Numerical calculations have been performed for the reflection coefficients of some lower order modes, based on (i) the exact solution (8.11), (ii) the ray-optical solution (8.10), (iii) the previous ray-optical solution [1] as quoted in (8.8). The integral $U^\pm(s, ka)$, appearing in the exact solution, was evaluated by numerical integration. As a check, the exact reflection coefficient amplitude was also computed from (8.14a), (8.14b). Numerical data for the amplitude and phase of $\Gamma_{00}, \Gamma_{02}, \Gamma_{11}$, are plotted in Figs. 2-7 as functions of the waveguide height-to-wavelength ratio a/λ . The exact data are drawn as solid curves. The ray-optical results based on (8.10) are indicated by black dots. Data based on the solution of Yee, Felsen and Keller [1], are represented by crosses. It is observed that the ray-optical and exact values are in excellent agreement even for a/λ as small as 0.3 in the case of the coefficient Γ_{00} . The present ray-optical solution is even more accurate than the previous solution [1], especially near the cutoff frequencies of higher order modes.

These cutoff frequencies correspond to integral or half-integral values of $ka/2\pi$, dependent on N and n being even or odd, respectively. From (8.14), (8.15) we infer that the exact curves for $|\Gamma_{Nn}|, \arg \Gamma_{Nn}$ will show the following behavior at cutoff frequencies: the amplitude curve has a skew left tangent and a vertical right tangent, whereas the phase curve has a vertical left tangent and a skew right tangent. Consider now the series $S^\pm(ka)$ as defined by (8.3), and notice that $S^\pm(x)$ is periodic with period 2π . Referring to [18, § 1.11], $S^-(x)$ can be expressed in terms of Lerch's transcendent $\Phi(z, s, v)$, and we find

$$(9.1) \quad \begin{aligned} S^-(x) &= -e^{ix}\Phi(e^{ix}, \frac{3}{2}, 1) \\ &= 2\pi^{1/2} e^{-\pi i/4} x^{1/2} - \sum_{m=0}^{\infty} \zeta\left(\frac{3}{2} - m\right) \frac{(ix)^m}{m!}, \quad -2\pi < x < 2\pi, \end{aligned}$$

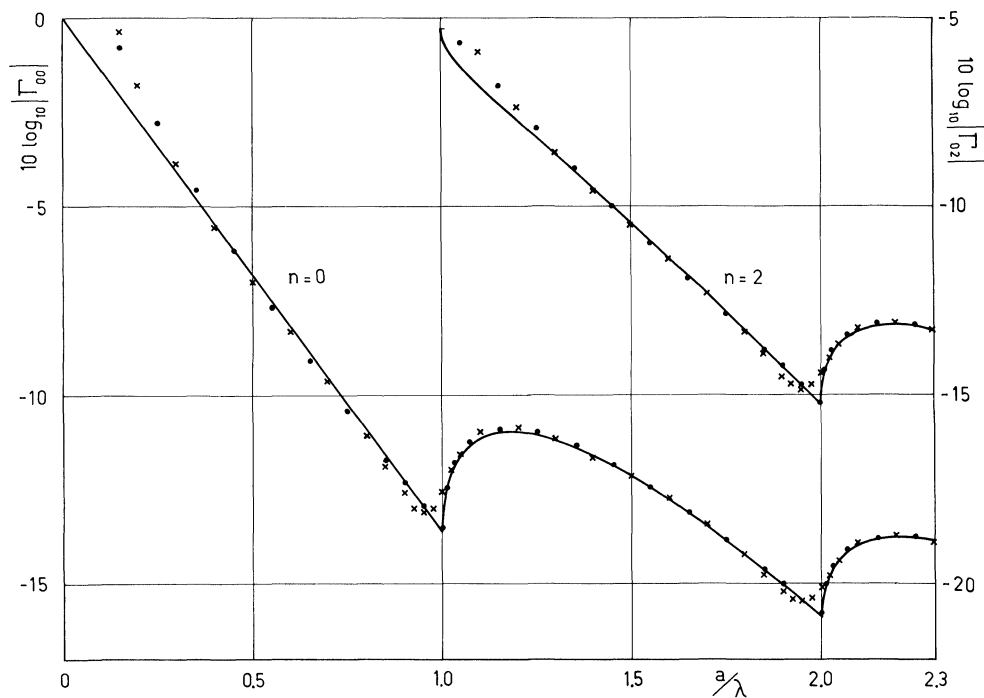


FIG. 2. Reflection coefficient amplitude for TM_{0n} mode, with TM_{00} mode incident. Legend: — : exact (8.11); ···· : ray-optics (8.10); ×××× : Yee, Felsen and Keller [1] (8.8)

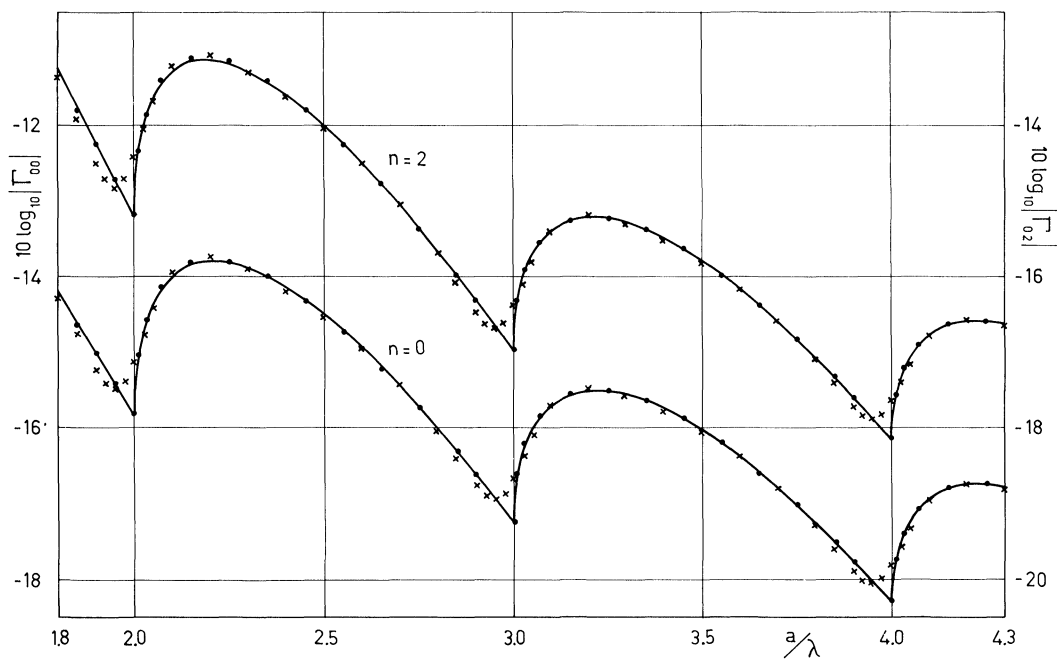


FIG. 3. Reflection coefficient amplitude for TM_{0n} mode, with TM_{00} mode incident (continuation of Fig. 2). Legend: as in Fig. 2

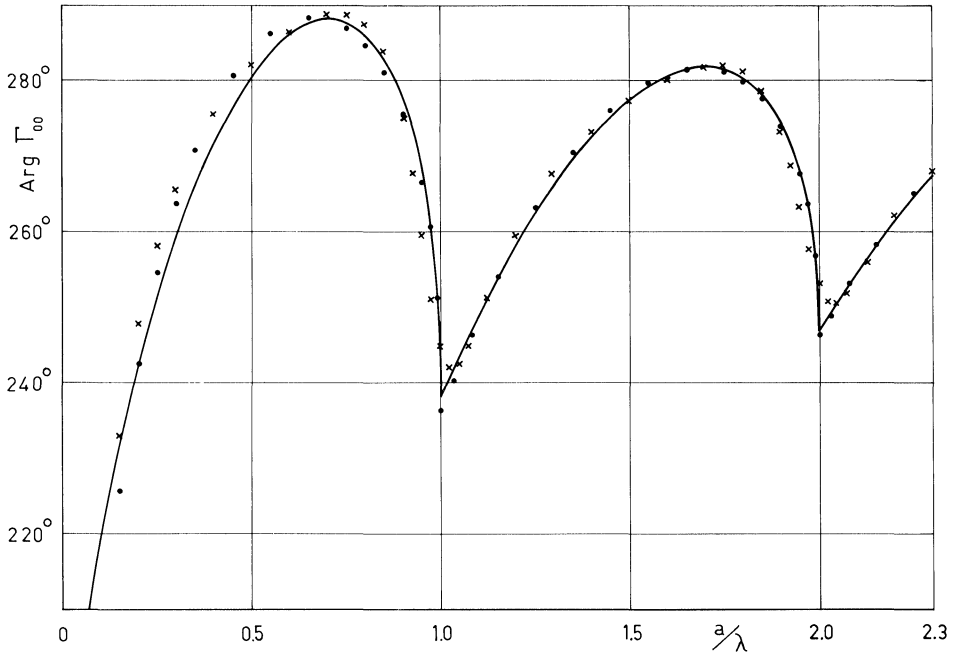


FIG. 4. Reflection coefficient phase for TM_{00} mode, with TM_{00} mode incident. Legend: as in Fig. 2

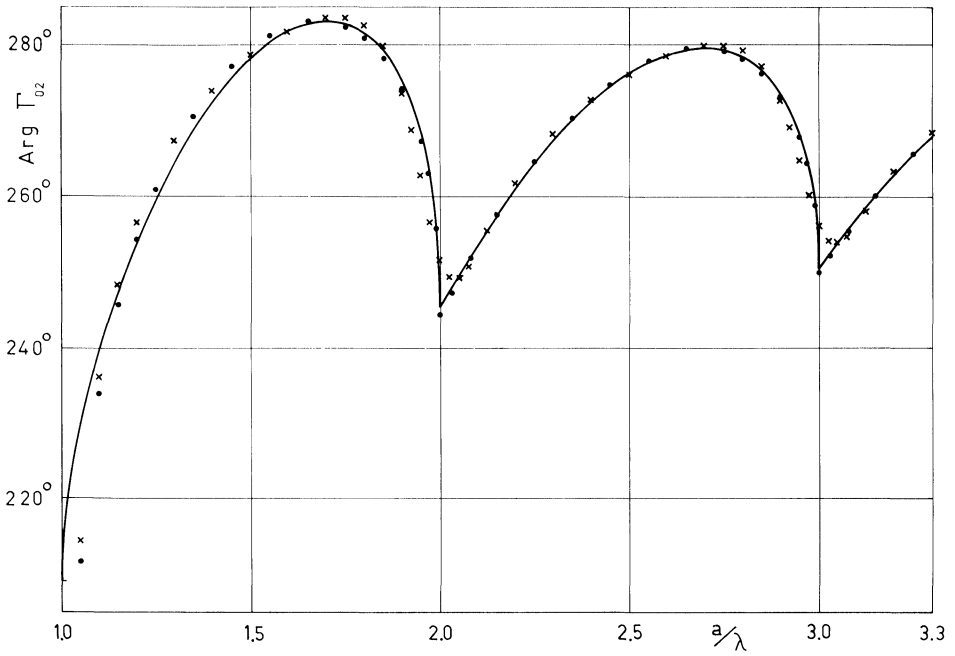


FIG. 5. Reflection coefficient phase for TM_{02} mode, with TM_{00} mode incident. Legend: as in Fig. 2

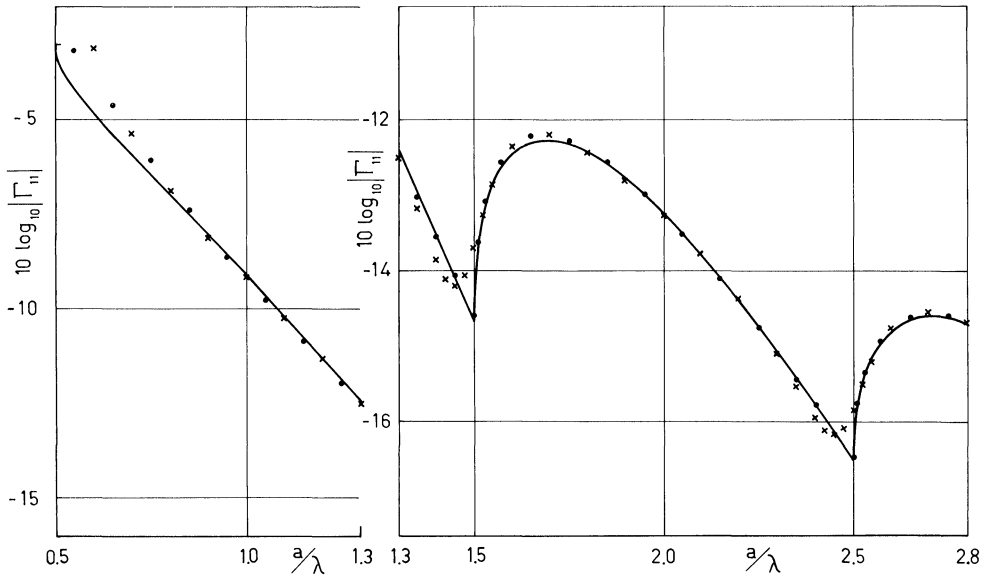


FIG. 6. Reflection coefficient amplitude for TM_{01} mode, with TM_{01} mode incident. Legend: as in Fig. 2

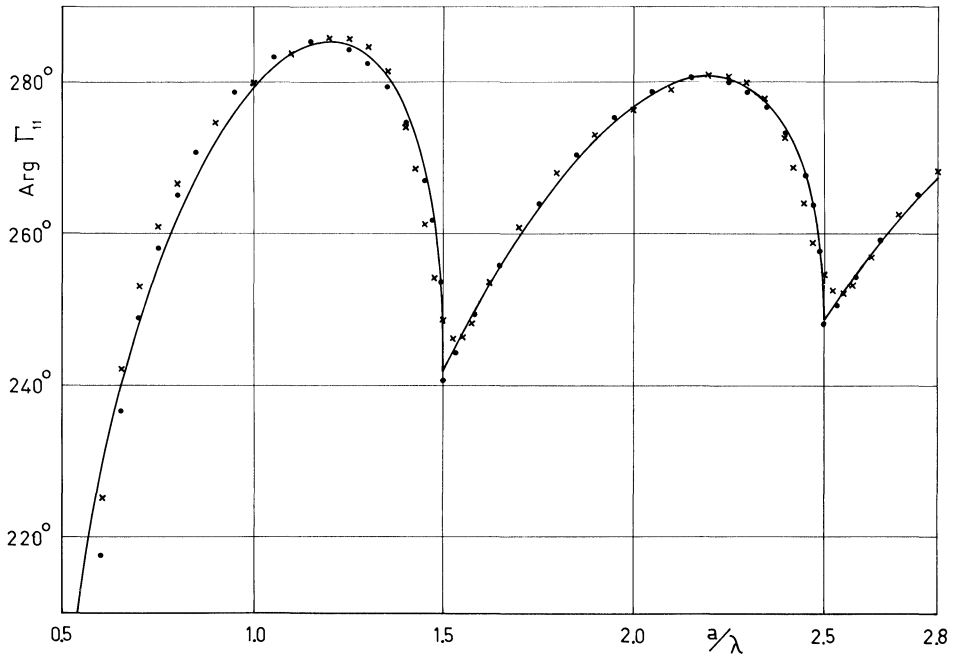


FIG. 7. Reflection coefficient phase for TM_{01} mode, with TM_{01} mode incident. Legend: as in Fig. 2

where $\zeta(s)$ stands for Riemann's zeta function and $x^{1/2} = i|x|^{1/2}$ when $-2\pi < x < 0$. A similar expansion holds for $S^+(x) = S^-(x - \pi)$. In view of the periodicity, the behavior of $S^\pm(ka)$ near integral or half-integral values of $ka/2\pi$ is now completely established. Then it is easily seen that the amplitude and phase curves based on the ray-optical solution (8.10) do have precisely the same behavior at cutoff frequencies as the corresponding exact curves. On the other hand, the previous solution [1] gives rise to smooth curves for $|\Gamma_{Nn}|, \arg \Gamma_{Nn}$. This may explain the superior accuracy of the present ray-optical solution in the vicinity of cutoff frequencies of higher order modes.

Appendix A. In this appendix we present a summary of the uniform asymptotic theory of edge diffraction as developed in [5]. The uniform theory is illustrated by means of a diffraction problem that is basic in the analysis of this paper, viz., diffraction of a cylindrical wave by a half-plane.

Let the half-plane $y = 0, z \geq 0$ be excited by a scalar cylindrical wave due to a line source located at $y = a, z = 0$. Polar coordinates r, φ and $r_{\pm 1}, \varphi_{\pm 1}$ are employed as introduced in § 2. Let the incident field be given by the asymptotic representation¹

$$(A.1) \quad u^i(r_1, \varphi_1) = \exp [ikr_1] z^i(r_1, \varphi_1) \sim \exp [ikr_1] \sum_{m=0}^{\infty} (ik)^{-m} z_m(r_1, \varphi_1).$$

Then the total field u is expressed in terms of a double-valued wave function U , viz.,

$$(A.2) \quad u(r, \varphi) = U(r, \varphi) \mp U(r, 4\pi - \varphi),$$

where the upper (lower) sign applies in the case of a boundary condition $u = 0$ ($\partial u / \partial y = 0$) on the half-plane. This sign convention is adopted throughout this appendix. According to [5], the function U is represented by the uniform asymptotic expansion

$$(A.3) \quad U(r, \varphi) \sim e^{ik(r+a)} \left\{ \left[F(k^{1/2} \zeta^*) + \frac{e^{\pi i/4}}{2\pi^{1/2}} k^{-1/2} \sum_{m=0}^{\infty} \left(\frac{1}{2} \right)_m (ik)^{-m} (\zeta^*)^{-2m-1} \right] z^i(r_1, \varphi_1) + k^{-1/2} \sum_{m=0}^{\infty} (ik)^{-m} \hat{v}_m(r, \varphi) \right\},$$

where

$$(A.4) \quad F(x) = \pi^{-1/2} e^{-\pi i/4} e^{-ix^2} \int_{-\infty}^x e^{it^2} dt$$

and

$$(A.5) \quad \left(\frac{1}{2} \right)_0 = 1, \quad \left(\frac{1}{2} \right)_m = \frac{1}{2} \left(\frac{1}{2} + 1 \right) \cdots \left(\frac{1}{2} + m - 1 \right), \quad m = 1, 2, 3, \dots$$

¹ The notation z^i, z_m for the amplitude and amplitude coefficients is copied from [5]; it should not be confused with the coordinate z .

The variable ξ^* is defined by

$$(A.6) \quad \xi^* = (r + a - r_1)^{1/2} \operatorname{sgn} \left[\cos \frac{1}{2} \left(\varphi - \frac{\pi}{2} \right) \right] = \left(\frac{4ar}{r + a + r_1} \right)^{1/2} \cos \frac{1}{2} \left(\varphi - \frac{\pi}{2} \right),$$

in accordance with the law of cosines. Notice that $\xi^* = 0$ along the shadow boundary $\varphi = 3\pi/2$ of the incident wave. In fact, the sign of the square root (A.6) is chosen in such a way that $\xi^* > 0$ ($\xi^* < 0$) in the illuminated region (shadow region) of the incident wave. The variable ξ^* has a simple physical meaning: $(\xi^*)^2$ measures the detour of the ray path from the source to the observation point via the edge of the half-plane.

The coefficients \hat{v}_m are recursively determined through

$$(A.7) \quad \hat{v}_m(r, \varphi) = \lambda_m(\varphi)r^{-1/2} - \frac{1}{2}r^{-1/2} \int_0^r \sigma^{1/2} \Delta \hat{v}_{m-1}(\sigma, \varphi) d\sigma, \\ m = 0, 1, 2, \dots; \quad \hat{v}_{-1} \equiv 0,$$

where the symbol \int indicates that the finite part of the (divergent) integral, in the sense of Hadamard, is to be calculated. The initial values λ_m are given by

$$(A.8) \quad \lambda_m = \sum_{n=0}^m \mathcal{D}_n z_{m-n} \left(a, \frac{\pi}{2} \right),$$

where $\mathcal{D}_n = \mathcal{D}_n(\varphi)$ is a linear differential operator of order n , detailed in [5]. Hence, λ_m is equal to a linear combination of the amplitude z^i and its derivatives of orders $1, 2, \dots, m$, evaluated at the diffracting edge. The expansion (A.3) is now completely determined. In [5, Appendix 2], it is proved that the expansion is finite throughout the (y, z) -plane including the shadow boundary $\varphi = 3\pi/2$. For this reason the expansion (A.3) is called uniform.

In [5] explicit results were obtained for the initial values λ_0, λ_1 . Upon substitution of these values in (A.7), it is found that the coefficients \hat{v}_0, \hat{v}_1 are given by

$$(A.9) \quad \hat{v}_0(r, \varphi) = -\frac{e^{\pi i/4}}{2(2\pi)^{1/2}} z_0 \left(a, \frac{\pi}{2} \right) r^{-1/2} s, \\ (A.10) \quad \hat{v}_1(r, \varphi) = -\frac{e^{\pi i/4}}{2(2\pi)^{1/2}} \left[z_1 \left(a, \frac{\pi}{2} \right) r^{-1/2} s + \frac{1}{4} \left\{ \frac{3}{2a} z_0 \left(a, \frac{\pi}{2} \right) \sin^2 \frac{1}{2} \left(\varphi - \frac{\pi}{2} \right) \right. \right. \\ \left. \left. - \frac{\partial z_0(a, \pi/2)}{\partial r_1} \cos \left(\varphi - \frac{\pi}{2} \right) + \frac{1}{a} \frac{\partial z_0(a, \pi/2)}{\partial \varphi_1} \sin \left(\varphi - \frac{\pi}{2} \right) \right\} r^{-1/2} s^3 \right. \\ \left. + \frac{1}{4} z_0 \left(a, \frac{\pi}{2} \right) r^{-3/2} s^3 \right],$$

where $s = \sec \frac{1}{2}(\varphi - \pi/2)$.

Away from the shadow boundary $\varphi = 3\pi/2$, one has $\xi^* \neq 0$ and the Fresnel integral $F(k^{1/2}\xi^*)$ can be replaced by its asymptotic expansion

$$(A.11) \quad F(x) \sim e^{-ix^2} H(x) - \frac{e^{\pi i/4}}{2\pi^{1/2}} x^{-1} \sum_{m=0}^{\infty} \left(\frac{1}{2} \right)_m (ix^2)^{-m}, \quad x \rightarrow \pm \infty,$$

where $H(x)$ is the unit step function, i.e., $H(x) = 1$ for $x > 0$ and $H(x) = 0$ for $x < 0$. As a result, the uniform expansion (A.3) passes into the nonuniform expansion

$$(A.12) \quad U(r, \varphi) \sim u^i(r_1, \varphi_1) H \left[\cos \frac{1}{2} \left(\varphi - \frac{\pi}{2} \right) \right] + e^{ik(r+a)} k^{-1/2} \sum_{m=0}^{\infty} (ik)^{-m} \hat{v}_m(r, \varphi),$$

$$\varphi \neq \frac{3\pi}{2}.$$

Then, according to (A.2), the total field u is given by the nonuniform expansion

$$(A.13) \quad u(r, \varphi) \sim u^i(r_1, \varphi_1) H \left[\cos \frac{1}{2} \left(\varphi - \frac{\pi}{2} \right) \right] \mp u^i(r_{-1}, \varphi_{-1}) H \left[\cos \frac{1}{2} \left(\varphi + \frac{\pi}{2} \right) \right]$$

$$+ e^{ik(r+a)} k^{-1/2} \sum_{m=0}^{\infty} (ik)^{-m} [\hat{v}_m(r, \varphi) \mp \hat{v}_m(r, 4\pi - \varphi)], \quad \varphi \neq \frac{\pi}{2}, \frac{3\pi}{2}.$$

The first and second term in (A.13) just describe the geometrical-optics incident and reflected fields; the third term corresponds to the diffracted field to be denoted by u^d . The expansion (A.13) is called nonuniform because it is not valid along the shadow boundaries $\varphi = 3\pi/2$, $\varphi = \pi/2$ of the incident and reflected waves. Upon substitution of (A.9), (A.10), the diffracted field expansion becomes

$$(A.14) \quad u^d(r, \varphi) \sim \frac{\exp [ik(r+a) + \pi i/4]}{2(2\pi)^{1/2}} \left[-k^{-1/2} z^i \left(a, \frac{\pi}{2} \right) r^{-1/2} (s_1 \mp s_2) \right.$$

$$+ \frac{i}{4} k^{-3/2} \left\{ \frac{3}{2} z^i \left(a, \frac{\pi}{2} \right) a^{-1} r^{-1/2} [(s_1^3 - s_1) \mp (s_2^3 - s_2)] \right.$$

$$- \frac{\partial z^i(a, \pi/2)}{\partial r_1} r^{-1/2} \sin \varphi (s_1^3 \pm s_2^3)$$

$$- \left. \frac{\partial z^i(a, \pi/2)}{\partial \varphi_1} a^{-1} r^{-1/2} \cos \varphi (s_1^3 \mp s_2^3) \right.$$

$$\left. + z^i \left(a, \frac{\pi}{2} \right) r^{-3/2} (s_1^3 \mp s_2^3) \right\} + \text{higher order terms} \Big], \quad \varphi \neq \frac{\pi}{2}, \frac{3\pi}{2},$$

where $s_1 = \sec \frac{1}{2}(\varphi - \pi/2)$, $s_2 = \sec \frac{1}{2}(\varphi + \pi/2)$. Here the diffracted field is expressed in terms of the incident amplitude z^i and its first order derivatives at the diffracting edge. The leading term in (A.14) agrees exactly with the result provided by Keller's geometrical theory of diffraction [4]. Notice that the expansion (A.14) has been written in a form that is independent of the particular asymptotic representation (A.1) of the incident wave.

Finally, we derive a uniform expansion for the scattered field u^s , away from the edge. Consider first the backward region $0 \leq \varphi \leq \pi$, where $U(r, \varphi)$ can be replaced by its nonuniform expansion (A.12). However, the constituent $U(r, 4\pi - \varphi)$ is to be replaced by its uniform expansion (A.3), since the reflected-wave boundary

points in the direction $\varphi = \pi/2$. Thus we obtain

$$\begin{aligned}
 u^s(r, \varphi) &= U(r, \varphi) \mp U(r, 4\pi - \varphi) - u^i(r_1, \varphi_1) \\
 &\sim e^{ik(r+a)} \left\{ \mp \left[F(k^{1/2}\xi) + \frac{e^{\pi i/4}}{2\pi^{1/2}} k^{-1/2} \sum_{m=0}^{\infty} \left(\frac{1}{2}\right)_m (ik)^{-m} \xi^{-2m-1} \right] z^i(r_{-1}, \varphi_{-1}) \right. \\
 (A.15) \quad &\left. + k^{-1/2} \sum_{m=0}^{\infty} (ik)^{-m} [\hat{v}_m(r, \varphi) \mp \hat{v}_m(r, 4\pi - \varphi)] \right\}, \quad 0 \leq \varphi \leq \pi,
 \end{aligned}$$

where ξ is defined by (2.5). The expansion (A.15) remains finite at the shadow boundary $\varphi = \pi/2$ of the reflected wave. In the same way, we may determine the scattered field in the forward directions $\pi \leq \varphi \leq 2\pi$. It is found that u^s satisfies the symmetry relation

$$(A.16) \quad u^s(r, \varphi) = \pm u^s(r, 2\pi - \varphi),$$

which can be shown to be exact for diffraction by a plane screen.

Appendix B. This appendix deals with the summation of the double series

$$(B.1) \quad F(s, t) = \sum_{q=0}^{\infty} \frac{s^q}{\Gamma(\frac{1}{2}q + 1)} \sum_{m=0}^{\infty} \frac{\exp[-m\pi i/4]}{\Gamma(\frac{1}{2}m - \frac{1}{2}q + 1)} t^m,$$

where $0 \leq s < 1$. The double series appears in (6.9). Consider first the double sum F_1 consisting of terms with q odd, viz.,

$$(B.2) \quad F_1(s, t) = \sum_{q=0}^{\infty} \frac{s^{2q+1}}{\Gamma(q + \frac{3}{2})} \sum_{m=0}^{\infty} \frac{\exp[-m\pi i/4]}{\Gamma(\frac{1}{2}m - q + \frac{1}{2})} t^m.$$

Upon differentiation of F_1 with respect to s , it is easily found that

$$(B.3) \quad F'_1 + 2ist^2F_1 = \frac{2\pi^{-1}}{1 + s^2} + 4\pi^{-1/2} e^{-\pi i/4} tF(t),$$

where F stands for the Fresnel integral as defined by (A.4), and a prime denotes differentiation with respect to s . The differential equation (B.3) is solved by variation of parameters. In view of the initial value $F_1 = 0$ at $s = 0$, we obtain

$$\begin{aligned}
 (B.4) \quad F_1(s, t) &= \frac{2}{\pi} \exp[-is^2t^2] \int_0^s \frac{\exp[i\sigma^2t^2]}{1 + \sigma^2} d\sigma \\
 &\quad + 4F(t)F(st) - 2 \exp[-is^2t^2]F(t).
 \end{aligned}$$

Secondly, consider the double sum F_2 consisting of terms with q even. By interchanging the order of summation we find

$$\begin{aligned}
 (B.5) \quad F_2(s, t) &= \sum_{m=0}^{\infty} \exp\left[\frac{-m\pi i}{4}\right] t^m \sum_{q=0}^{\infty} \frac{s^{2q}}{q! \Gamma(\frac{1}{2}m - q + 1)} \\
 &= \sum_{m=0}^{\infty} \frac{\exp[-m\pi i/4]}{\Gamma(\frac{1}{2}m + 1)} t^m (1 + s^2)^{(1/2)m} = 2F(t\sqrt{1 + s^2}),
 \end{aligned}$$

according to (6.1). From (B.4), (B.5) we deduce the final result

$$(B.6) \quad \begin{aligned} F(s, t) &= 4F(t)F(st) + 2F(t\sqrt{1+s^2}) \\ &- 2 \exp[-is^2t^2]F(t) + \frac{2}{\pi} \exp[-is^2t^2] \int_0^s \frac{\exp[i\sigma^2t^2]}{1+\sigma^2} d\sigma. \end{aligned}$$

Appendix C. This appendix deals with the solution of the recurrence relations (7.9), (7.15) for the coefficients $u_{n,q}(r, \pi/2)$, $v_{n,q}(r, \pi/2)$. In order to simplify these relations we set

$$(C.1) \quad u_{n,q}\left(r, \frac{\pi}{2}\right) = (r + na)^{-1/2} g\left(\frac{\pi}{2}, \theta_N\right) U_{n,q}(r),$$

$$(C.2) \quad v_{n,q}\left(r, \frac{\pi}{2}\right) = \frac{a^{-1/2}}{r + na} h(\theta_N) V_{n,q}^{(+)}(r) - \frac{1}{2} a^{-1/2} g\left(\frac{\pi}{2}, \theta_N\right) V_{n,q}^{(-)}(r).$$

Then the coefficients $U_{n,q}$, $V_{n,q}^{(\pm)}$ must satisfy the recurrence relations

$$(C.3) \quad U_{n+1,m}(r) = \frac{1}{2} \sum_{q=0}^{\infty} \frac{U_{n,q}(r+a)}{\Gamma(\frac{1}{2}m - \frac{1}{2}q + 1)} \left(\frac{r}{r+2a}\right)^{(1/2)q},$$

$$(C.4) \quad V_{n+1,m}^{(\pm)}(r) = \frac{1}{2} \sum_{q=0}^{\infty} \frac{V_{n,q}^{(\pm)}(r+a)}{\Gamma(\frac{1}{2}m - \frac{1}{2}q + 1)} \left(\frac{r}{r+2a}\right)^{(1/2)q} + \delta_{m0} \frac{U_{n,0}(a)}{(n+1)^{1/2}} r^{\pm 1/2},$$

where $m = 0, 1, 2, \dots$, and $\delta_{00} = 1$, $\delta_{m0} = 0$ for $m \neq 0$. The present relations are accompanied by the initial values $U_{0,q}(r) = \delta_{q0}$, $V_{0,q}^{(\pm)}(r) = 0$, for $q = 0, 1, 2, \dots$, derived from (7.11).

Consider first the recurrence relation (C.3) which is solved by a generating-function technique. We introduce the generating function

$$(C.5) \quad F_n(r; z) = \sum_{q=0}^{\infty} U_{n,q}(r)(iz)^q, \quad n = 0, 1, 2, \dots,$$

where z is a complex variable (not to be confused with the coordinate z). Then (C.3) can be reduced to a recurrence relation for F_{n+1} expressed in terms of F_n . For that purpose, the reciprocal Γ -function in (C.3) is replaced by Hankel's contour integral (cf. [18, 1.6(2)])

$$(C.6) \quad \frac{1}{\Gamma(\frac{1}{2}m - \frac{1}{2}q + 1)} = \frac{1}{2\pi i} \int_{-\infty}^{(0+)} e^{t-(1/2)m+(1/2)q-1} dt, \quad |\arg t| \leq \pi,$$

where the integration contour starts at $-\infty$, encircles the origin once counter-clockwise and returns to its starting point. By means of (C.3), (C.6) we derive

$$(C.7) \quad \begin{aligned} F_{n+1}(r; z) &= \frac{1}{2} \sum_{m=0}^{\infty} (iz)^m \sum_{q=0}^{\infty} U_{n,q}(r+a) \left(\frac{r}{r+2a}\right)^{(1/2)q} \frac{1}{2\pi i} \int_{-\infty}^{(0+)} e^{t-(1/2)m+(1/2)q-1} dt \\ &= \frac{1}{4\pi i} \int_{-\infty}^{(0+)} e^{t-1} (1 - izt^{-1/2})^{-1} F_n\left(r+a; -i\sqrt{\frac{rt}{r+2a}}\right) dt \end{aligned}$$

under a (formal) interchange of the order of summation and integration. Furthermore the Hankel contour is chosen in such a way that $|zt^{-1/2}| < 1$ along the contour. In (C.7) the variable t is replaced by $(it)^2$, yielding

$$(C.8) \quad F_{n+1}(r; z) = \frac{1}{2\pi i} \int_{-\infty+i\alpha}^{\infty+i\alpha} \frac{e^{-t^2}}{t-z} F_n\left(r+a; t\sqrt{\frac{r}{r+2a}}\right) dt,$$

where $\alpha < \text{Im } z$, i.e., the path of integration passes below the pole $t = z$. Through repeated application of (C.8), the generating function $F_n(r; z)$ can be expressed in terms of $F_0(r; z) = 1$, viz.,

$$(C.9) \quad F_n(r; z) = \frac{1}{(2\pi i)^n} \int_{-\infty+i\alpha_1}^{\infty+i\alpha_1} \cdots \int_{-\infty+i\alpha_n}^{\infty+i\alpha_n} \exp\left[-\sum_{m=1}^n t_m^2\right] (t_1 - z)^{-1} \\ \cdot \prod_{m=2}^n \left[t_m - t_{m-1} \left(\frac{r + (m-2)a}{r + ma} \right)^{1/2} \right]^{-1} dt_1 \cdots dt_n,$$

where $\alpha_1 < \text{Im } z$, $\alpha_m < \alpha_{m-1}(r + (m-2)a)^{1/2}/(r + ma)^{1/2}$, $m = 2, 3, \dots, n$. In order to simplify the latter integral, we replace the reciprocal linear factors by the following integrals:

$$(C.10) \quad -i(t_1 - z)^{-1} = \int_0^\infty \exp[-ix_1(t_1 - z)] dx_1, \\ -i \left[t_m - t_{m-1} \left(\frac{r + (m-2)a}{r + ma} \right)^{1/2} \right]^{-1} \\ = \int_0^\infty \exp \left[-ix_m \left(t_m - t_{m-1} \sqrt{\frac{r + (m-2)a}{r + ma}} \right) \right] dx_m.$$

Then, upon interchanging the order of integration in (C.9), we find that the inner integrals with respect to t_1, \dots, t_n can be explicitly determined. Thus we obtain the representation

$$(C.11) \quad F_n(r; z) = 2^{-n} \pi^{-(1/2)n} \int_0^\infty \cdots \int_0^\infty \exp \left[ix_1 z \right. \\ \left. - \frac{1}{4} \sum_{m=2}^n \left\{ x_{m-1} - x_m \left(\frac{r + (m-2)a}{r + ma} \right)^{1/2} \right\}^2 - \frac{1}{4} x_n^2 \right] dx_1 \cdots dx_n.$$

In the latter integral the variable x_m , $m = 1, 2, \dots, n$, is replaced by

$$2x_m(r + ma)^{1/2}/(r + (m-1)a)^{1/2},$$

yielding

$$(C.12) \quad F_n(r; z) = \pi^{-(1/2)n} \left(\frac{r + na}{r} \right)^{1/2} \int_0^\infty \cdots \int_0^\infty \exp \left[2ix_1 z \left(\frac{r + a}{r} \right)^{1/2} - \frac{r + a}{r} x_1^2 \right. \\ \left. - 2 \sum_{m=2}^n x_m^2 + 2 \sum_{m=2}^n x_{m-1} x_m \right] dx_1 \cdots dx_n.$$

The latter result can easily be expanded in a power series in powers of iz , comparable to (C.5). Then it is found that $U_{n,q}(r)$ is given by the n -fold integral

$$(C.13) \quad U_{n,q}(r) = \pi^{-(1/2)n} \left(\frac{r+na}{r}\right)^{1/2} \frac{2^q}{q!} \left(\frac{r+a}{r}\right)^{(1/2)q} \cdot \int_0^\infty \cdots \int_0^\infty x_1^q \exp \left[-\frac{r+a}{r} x_1^2 - 2 \sum_{m=2}^n x_m^2 + 2 \sum_{m=2}^n x_{m-1} x_m \right] dx_1 \cdots dx_n.$$

The solution (C.13) of the recurrence relation (C.3) was derived by a formal use of generating functions. We shall briefly comment on the rigorous verification of the solution. First of all, for $n = 1$ the integral (C.13) can be evaluated, viz.,

$$(C.14) \quad U_{1,q}(r) = \frac{1}{2\Gamma(\frac{1}{2}q + 1)}, \quad q = 0, 1, 2, \dots,$$

which agrees with the result obtained from (C.3) with $n = 0$. Secondly, it is to be proved that (C.13) does satisfy (C.3). For that purpose, (C.13) is substituted in the right-hand side of (C.3) and the reciprocal Γ -function in (C.3) is replaced by Hankel's contour integral (C.6). Then, upon interchanging the order of summation and integration in (C.3), the resulting $(n + 1)$ -fold integral can be reduced to the form (C.13) with n, q replaced by $n + 1, m$, respectively, i.e., the coefficient $U_{n+1,m}(r)$ is recovered. It is found that the interchange of summation and integration is permissible provided that $0 \leq r < a(n + 1)/(n - 1)$. In fact, through repeated application of the auxiliary estimate

$$(C.15) \quad \frac{1}{2}\pi^{1/2} p^{-1/2} \exp\left(\frac{s^2}{p}\right) \leq \int_0^\infty \exp[-px^2 + 2sx] dx \leq \pi^{1/2} p^{-1/2} \exp\left(\frac{s^2}{p}\right),$$

valid for $p > 0, s \geq 0$, the following inequality can be derived for $U_{n,q}(r)$ as given by (C.13):

$$(C.16) \quad \frac{1}{2^n \Gamma(\frac{1}{2}q + 1)} \left(\frac{n(r+a)}{r+na}\right)^{(1/2)q} \leq U_{n,q}(r) \leq \frac{1}{2\Gamma(\frac{1}{2}q + 1)} \left(\frac{n(r+a)}{r+na}\right)^{(1/2)q}.$$

In view of this estimate it is clear that the series in the right-hand side of (C.3) converges only for $0 \leq r < a(n + 1)/(n - 1)$. This explains why the solution (C.13) does satisfy (C.3) over the range $0 \leq r < a(n + 1)/(n - 1)$ only. However, the latter range is sufficient for our purpose, since it includes the line segment $0 \leq r \leq a$ that connects the edges. In a similar way it is verified that the solution (C.13) meets the finiteness condition (7.5) over the range $0 \leq r < a(n + 1)/(n - 1)$.

The recurrence relation (C.4) for the coefficients $V_{n,q}^{(\pm)}$ is linear and inhomogeneous. The associated homogeneous equation is just the recurrence relation (C.3). It is observed that (C.3) is independent of n . Then the coefficients $V_{n,q}^{(\pm)}$ can be expressed in terms of the solution $U_{n,q}$ of (C.3) by using an analogue of Duhamel's principle, viz.,

$$(C.17) \quad V_{n,q}^{(\pm)}(r) = \sum_{m=1}^n \frac{[r + (n - m)a]^{\pm 1/2}}{m^{1/2}} U_{m-1,0}(a) U_{n-m,q}(r).$$

It can easily be verified by substitution that (C.17) does indeed satisfy (C.4).

Remark. The original recurrence relations (7.6), (7.7) for the coefficients $u_{n,q}(r, \varphi)$, $v_{n,q}(r, \varphi)$ can be solved by the same methods as described before. Utilizing the generating-function technique, it is found that $u_{n,q}(r, \varphi)$ is given by an n -fold integral which is, although more complicated, essentially of the same type as (C.13). By means of Duhamel's principle, $v_{n,q}(r, \varphi)$ can be expressed in terms of the coefficients $u_{n,q}$. Upon inserting these results in the Ansatz (7.1), it can be shown that the infinite series appearing in (7.1) are indeed convergent for each r, φ .

Appendix D. In this appendix simple closed-form results are derived for the integrals $I_{n,0}(a)$, $I_{n,1}(a)$, as given by (7.18), viz.,

$$(D.1) \quad I_{n,0}(a) = \pi^{-(1/2)n} \int_0^\infty \cdots \int_0^\infty \exp \left[-2 \sum_{m=1}^n x_m^2 + 2 \sum_{m=2}^n x_{m-1} x_m \right] dx_1 \cdots dx_n,$$

$$(D.2) \quad I_{n,1}(a) = \frac{2^{3/2}}{\pi^{(1/2)n}} \int_0^\infty \cdots \int_0^\infty x_1 \exp \left[-2 \sum_{m=1}^n x_m^2 + 2 \sum_{m=2}^n x_{m-1} x_m \right] dx_1 \cdots dx_n.$$

Consider first (D.1), where the exponent is rewritten as

$$2 \sum_{m=1}^n x_m^2 - 2 \sum_{m=2}^n x_{m-1} x_m = x_1^2 + \sum_{m=2}^n (x_m - x_{m-1})^2 + x_n^2.$$

We introduce the new variables

$$(D.3) \quad y_1 = x_1; \quad y_m = x_m - x_{m-1}, \quad m = 2, 3, \dots, n; \quad y_{n+1} = -x_n,$$

and conversely,

$$(D.4) \quad x_m = \sum_{j=1}^m y_j, \quad m = 1, 2, \dots, n; \quad \sum_{j=1}^{n+1} y_j = 0.$$

Then (D.1) transforms into

$$(D.5) \quad I_{n,0}(a) = \pi^{-(1/2)n} \int_{G_n} \cdots \int \exp \left[- \sum_{m=1}^{n+1} y_m^2 \right] dy_1 \cdots dy_n,$$

where G_n is an n -dimensional domain given by

$$(D.6) \quad G_n: \sum_{j=1}^m y_j \geq 0, \quad m = 1, 2, \dots, n; \quad \sum_{j=1}^{n+1} y_j = 0,$$

i.e., G_n is a polyhedral cone in the hyperplane $H_n: \sum_{j=1}^{n+1} y_j = 0$, in E^{n+1} . The integral (D.5) is rewritten as a surface integral

$$(D.7) \quad I_{n,0}(a) = \frac{\pi^{-(1/2)n}}{(n+1)^{1/2}} \int_{G_n} \cdots \int \exp \left[- \sum_{m=1}^{n+1} y_m^2 \right] d\sigma,$$

where $d\sigma = (n+1)^{1/2} dy_1 \cdots dy_n$ denotes the surface area element of H_n .

We now reduce (D.7) to an integral over the entire hyperplane H_n .² We introduce the orthogonal transformation $S: H_n \rightarrow H_n$, defined by

$$Sy = (y_2, y_3, \dots, y_{n+1}, y_1),$$

where $y = (y_1, y_2, \dots, y_{n+1})$. In particular, let y be an interior point of G_n , i.e.,

$$\sum_{j=1}^m y_j > 0, \quad m = 1, 2, \dots, n; \quad \sum_{j=1}^{n+1} y_j = 0,$$

according to (D.6). Then for $m = 1, 2, \dots, n$, one has

$$S^m y = (y_{m+1}, y_{m+2}, \dots, y_{n+1}, y_1, \dots, y_m) \notin G_n,$$

since

$$\sum_{j=m+1}^{n+1} y_j = - \sum_{j=1}^m y_j < 0.$$

Similarly it can be shown that $S^m y \neq S^p y$ for $m \neq p$, $m, p = 1, 2, \dots, n$, and $\bigcup_{m=0}^n S^m(G_n) = H_n$. Thus H_n is covered by $n + 1$ disjoint domains which are equivalent to G_n . Furthermore, we observe that the integrand (D.7) is invariant under the transformation S . Then it is clear that (D.7) can be reduced to

$$(D.8) \quad I_{n,0}(a) = \frac{\pi^{-(1/2)n}}{(n+1)^{3/2}} \int_{H_n} \dots \int \exp \left[- \sum_{m=1}^{n+1} y_m^2 \right] d\sigma.$$

Since the latter integrand is invariant under a rotation around the origin, we may rotate H_n to the hyperplane $y_{n+1} = 0$, yielding

$$(D.9) \quad I_{n,0}(a) = \frac{\pi^{-(1/2)n}}{(n+1)^{3/2}} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \exp \left[- \sum_{m=1}^n y_m^2 \right] dy_1 \dots dy_n = \frac{1}{(n+1)^{3/2}}.$$

Secondly, consider the integral (D.2). For reasons of symmetry, the factor x_1 in the integrand can be replaced by

$$(D.10) \quad \frac{1}{2}(x_1 + x_n) = \frac{1}{4} \sum_{m=1}^n (-2x_{m-1} + 4x_m - 2x_{m+1}),$$

where $x_0 = x_{n+1} = 0$ by definition. Then (D.2) passes into a sum of integrals which permit explicit integration with respect to x_m . Thus we obtain

$$(D.11) \quad \begin{aligned} I_{n,1}(a) = & \frac{2^{-1/2}}{\pi^{(1/2)n}} \sum_{m=1}^n \int_0^{\infty} \dots \int_0^{\infty} \exp \left[-2 \sum_{j=1}^{m-1} x_j^2 + 2 \sum_{j=2}^{m-1} x_{j-1} x_j \right] dx_1 \dots dx_{m-1} \\ & \cdot \int_0^{\infty} \dots \int_0^{\infty} \exp \left[-2 \sum_{j=m+1}^n x_j^2 + 2 \sum_{j=m+2}^n x_{j-1} x_j \right] dx_{m+1} \dots dx_n, \end{aligned}$$

where a zero-fold integral is to be set equal to 1 by definition. Each of the multiple integrals appearing in (D.11) is of the same type as (D.1). Using (D.9) we find

$$(D.12) \quad I_{n,1}(a) = \frac{1}{(2\pi)^{1/2}} \sum_{m=1}^n \frac{1}{m^{3/2}(n-m+1)^{3/2}}.$$

² The author is indebted to N. G. de Bruijn for pointing out this reduction.

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