Inductive and projective limits of weighted unconditional Schauder systems

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By
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INDUCTIVE AND PROJECTIVE LIMITS OF WEIGHTED UNCONDITIONAL SCHAUDER SYSTEMS

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Summary

For an unconditional Schauder system of Banach spaces X_m, $m \in \mathbf{I}$ and a Köthe power set ρ of nonnegative sequences on $\mathbf{I}$ an inductive limit of Banach spaces $X_{\mathbf{ind}}(\rho)$ and a projective limit of seminormed spaces $X_{\mathbf{proj}}(\rho)$ are constructed. Topological properties of $X_{\mathbf{ind}}(\rho)$ and $X_{\mathbf{proj}}(\rho)$ are discussed and put in correspondence with properties of ρ.
The dual spaces $X_{\mathbf{ind}}(\rho)'$ and $X_{\mathbf{proj}}(\rho)'$ turn out to be of the same type. An interesting feature is the symmetry condition on ρ ensuring the existence of a Köthe set σ such that $X_{\mathbf{ind}}(\rho) = X_{\mathbf{proj}}(\sigma)$ and $X_{\mathbf{proj}}(\rho) = X_{\mathbf{ind}}(\sigma)$.
Thus locally convex spaces which are both inductive limits and projective limits of Banach spaces, can be constructed.

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Introduction

Starting point of our discussions is a Schauder system $(X_m)_{m\in\mathbf{I}}$ of closed subspaces of a Banach space $\mathcal{X}$. Examples of such a system are Schauder bases and block Schauder bases. In the carthesian product $C = \times_{m\in\mathbf{I}} X_m$ a Banach space X is identified which is the building block in our construction. For each nonnegative sequence $\mathbf{a}$ on $\mathbf{I}$, i.e. each function $\mathbf{a}$ from $\mathbf{I}$ to $[0,\infty)$, the mapping Λ_a from C into C is defined by

$$(\Lambda_a \mathbf{u})\ (m) = \mathbf{a}(m)\ \mathbf{u}(m) \quad , \quad \mathbf{u} \in C\ .$$

To the sequence $\mathbf{a}$ we link a Banach space $X_{\mathbf{ind}}(\mathbf{a})$ and a seminormed space $X_{\mathbf{proj}}(\mathbf{a})$,

$$X_{\mathbf{ind}}(\mathbf{a}) = \Lambda_a(X) \quad , \quad X_{\mathbf{proj}}(\mathbf{a}) = \{\mathbf{u} \in C \mid \Lambda_a \mathbf{u} \in X\}\ .$$

For a pointwisely directed set ρ of nonnegative sequences on $\mathbf{I}$, this leads to an inductive system $\{X_{\mathbf{ind}}(\mathbf{a}) \mid \mathbf{a} \in \rho\}$ and a projective system $\{X_{\mathbf{proj}}(\mathbf{a}) \mid \mathbf{a} \in \rho\}$.
The present paper contains a detailed discussion of the corresponding inductive limit

$$X_{\mathbf{ind}}(\rho) = \bigcup_{\mathbf{a}\in\rho} X_{\mathbf{ind}}(\mathbf{a})$$

and projective limit

$$X_{\mathbf{proj}}(\rho) = \bigcap_{\mathbf{a}\in\rho} X_{\mathbf{proj}}(\mathbf{a})\ .$$

As such it is a revision and adaption of the theory presented in [Ma], Chapter 3, where the X_m are closed orthogonal subspaces of a Hilbert space H.
The plan of the paper is as follows.
In Section 1 we summarize some relevant notions of Banach space theory such as unconditional convergence, basic sequence, Schauder basis, Schauder decomposition and, in this connection, we recall some useful results. Let $\omega^+(\mathbf{I})$ denote the set of all nonnegative sequences on $\mathbf{I}$. In Section 2 we study the structure of the collection $P(\omega^+(\mathbf{I}))$ of all subsets of $\omega^+(\mathbf{I})$. In $P(\omega^+(\mathbf{I}))$, a quasi-ordering $\preceq$, an equivalence relation, a classification and the so called hash-operation, $\rho \to \rho^\sharp$, are defined. The third and fourth section treats the spaces $X_{\mathbf{ind}}(\rho)$ and $X_{\mathbf{proj}}(\rho)$ and in the fifth section several topological properties of these spaces are linked with the $\sharp$-symmetry property of the sequence set ρ. In the last section, we show that the dual spaces $X_{\mathbf{ind}}(\rho)'$ and $X_{\mathbf{proj}}(\rho)'$ have the same structure as the spaces $X_{\mathbf{ind}}(\rho)$ and $X_{\mathbf{proj}}(\rho)$, i.e. they can be described as projective/inductive limits of subspaces of the product space $\times_{m\in\mathbf{I}} X'_m$.
The underlying paper is almost completely self-contained and only some knowledge of the fundamentals of functional analysis is required for reading it. The theory yields a wealth of examples of inductive and projective limits of Banach spaces. Moreover, it can be applied in the description of different types of distribution theories for suitable choices of the Schauder system $(X_m)_{m\in\mathbf{I}}$ and the sequence set ρ. For instance if ρ consists of bounded sequences, only, we get the triple

$$X_{\mathbf{ind}}(\rho) \hookrightarrow X \hookrightarrow X_{\mathbf{proj}}(\rho)\ .$$

§1. Schauder systems

Throughout we shall let $\mathcal{X}$ be a Banach space with norm $\|\cdot\|$. For a countable set $\mathbf{I}$ let $\mathcal{F}(\mathbf{I})$ be the collection of all finite subsets of $\mathbf{I}$.

Definition 1.1.
A sequence $\mathbf{x} : \mathbf{I} \to \mathcal{X}$ is said to be unconditionally summable if the net

$$\Big(\sum_{i \in \boldsymbol{F}} \mathbf{x}(i)\Big)_{\boldsymbol{F} \in \mathcal{F}(\mathbf{I})}$$

is convergent in $\mathcal{X}$. If $\mathbf{x}$ is unconditionally summable, then by

$$\sum_{i \in \mathbf{I}} \mathbf{x}(i)$$

its finite-sum limit is denoted.

The following observations are rather straight forward.

Proposition 1.2.
Let the sequence $\mathbf{x} : \mathbf{I} \to \mathcal{X}$ be unconditionally summable. Then

(a) For any subset $\mathbf{J} \subset \mathbf{I}$ the subsequence $\mathbf{y} : j \mapsto \mathbf{x}(j)$, $j \in \mathbf{J}$ is unconditionally summable.

(b) For any choice of signs $\Theta : \mathbf{I} \to \{-1, 1\}$ the sequence $i \mapsto \Theta(i)\ \mathbf{x}(i)$, $i \in \mathbf{I}$, is unconditionally summable.

(c) For any permutation π of $\mathbf{I}$ the sequence $i \mapsto \mathbf{x}(\pi(i))$ is unconditionally summable.

Because of its relevance for the rest of the paper we also mention the following proposition.

Proposition 1.3.
Let $\mathbf{x} : \mathbf{I} \to \mathcal{X}$ be unconditionally summable and let $\lambda : \mathbf{I} \to \mathbb{C}$ be a bounded sequence. Then the sequence $i \mapsto \lambda(i)\ \mathbf{x}(i)$, $i \in \mathbf{I}$, is unconditionally summable.

Proof.
We may as well assume that the sequence λ is real valued by considering the real and imaginary parts separately.
Since $\mathbf{x}$ is unconditionally summable there exists $\mathbb{F}_\varepsilon \in \mathcal{F}(\mathbf{I})$ such that for all $\mathbb{F} \in \mathcal{F}(\mathbf{I})$ with $\mathbb{F} \cap \mathbb{F}_\varepsilon = \emptyset$

$$\|\sum_{i \in \boldsymbol{F}} \mathbf{x}(i)\| < \varepsilon$$

for any $\varepsilon > 0$.
Now let $\varepsilon > 0$ and let $\mathbb{F} \in \mathcal{F}(\mathbf{I})$ such that $\mathbb{F} \cap \mathbb{F}_\varepsilon = \emptyset$.
By Hahn-Banach there exists a real linear function $l : X \to \mathbb{R}$ such that $\|l\| = 1$ and

$$l\Big(\sum_{i\in \boldsymbol{F}} \lambda(i)\,\mathbf{x}(i)\Big) = \|\sum_{i\in \boldsymbol{F}} \lambda(i)\,\mathbf{x}(i)\| \,.$$

Now define $\boldsymbol{\Theta} : \mathbf{I} \to \{-1, 1\}$ by

$$\Theta(i) = \begin{cases} 1 & \text{if} \quad l(x(i)) > 0 \\ -1 & \text{if} \quad l(x(i)) < 0 \end{cases}$$

Then

$$\begin{aligned} \|\sum_{i\in \boldsymbol{F}} \lambda(i)\,\mathbf{x}(i)\| &= \sum_{i\in \boldsymbol{F}} \lambda(i)\, l(\mathbf{x}(i)) \\ &\le \sup_{i\in \mathbf{I}} |\lambda(i)| \sum_{i\in \boldsymbol{F}} l(\Theta(i)\,\mathbf{x}(i)) \\ &\le \|l\| \sup_{i\in \mathbf{I}} |\lambda(i)|\, \|\sum_{i\in \boldsymbol{F}} \Theta(i)\,\mathbf{x}(i)\| \\ &\le 2\varepsilon\, \|l\| \sup_{i\in \mathbf{I}} |\lambda(i)| \,. \end{aligned}$$

Hence $\Big(\sum_{i\in \boldsymbol{F}} \lambda(i)\,\mathbf{x}(i)\Big)_{\boldsymbol{F}\in\mathcal{F}(\mathbf{I})}$ is a Cauchy net in $\mathcal{X}$. □

Let $\{X_i\}_{i\in \mathbf{I}}$ be a countable collection of closed subspaces of $\mathcal{X}$ indexed by $\mathbf{I}$. Each X_i is a Banach space under the induced topology and we regard it as such.

Definition 1.4.
Let $\{X_i\}_{i\in \mathbf{I}}$ be as above. The collection $\{X_i\}_{i\in \mathbf{I}}$ is said to be a minimal system if for each $m \in \mathbf{I}$

$$X_m \cap \overline{< \{X_i \mid i \in \mathbf{I},\ i \neq m\} >} = \{0\} \,.$$

The collection $\{X_i\}_{i\in \mathbf{I}}$ is said to be an unconditional Schauder system if there exists a closed subspace X of $\mathcal{X}$ such that for each $x \in X$ there exists a unique unconditionally summable sequence $\{x_i\}_{i\in \mathbf{I}}$ with $x_i \in X_i$ and $x = \sum_{i\in \mathbf{I}} x_i$.

As important examples of unconditional Schauder systems we mention the unconditional Schauder bases and unconditional basic sequences.

Definition 1.5.
Let $\{e_m \mid m \in \mathbf{I}\}$ be a countable collection in a Banach space $\mathcal{X}$. Then $\{e_m \mid m \in \mathbf{I}\}$ is an unconditional Schauder basis in $\mathcal{X}$ if to each $x \in \mathcal{X}$ there exists a unique sequence of scalars $a = (a(m))_{m\in \mathbf{I}}$ such that the sequence $(a(m)\, e_m)_{m\in \mathbf{I}}$ is unconditionally summable and

$$x = \sum_{m \in \mathbf{I}} a(m)\, e_m \ .$$

The collection is an unconditional basic sequence if it is an unconditional Schauder basis in its closed linear span $\overline{< \ \{e_m \mid m \in \mathbf{I}\} \ >}$.

For example the standard basis $\{e_n\}_{n \in \mathbb{N}}$ is an unconditional basis in every l_p- space, $1 \leq p < \infty$, and the Haar system an unconditional basis in L_p, $1 < p < \infty$. For more on unconditional bases and basic sequences we refer the reader to [LT].

Definition 1.6.
For a minimal system $\{X_i\}_{i \in \mathbf{I}}$ let $\times_{i \in \mathbf{I}} X_i$ denote its carthesian product with the product topology. We may identify $x_m \in X_m$ with $(x_m \, \delta_{mj})_{j \in \mathbf{I}}$ so that X_m is also regarded as a subspace of the carthesian product. We shall indicate this by writing $\mathbf{X}_m$ in stead of X_m. The elements of $\times_{i \in \mathbf{I}} X_i$ will be written as $\mathbf{x}, \mathbf{y}$, etc., and their components by $\mathbf{x}(m)$, $\mathbf{y}(m)$ or $x(m)$, $y(m)$ when regarded as elements of $\mathbf{X}_m$ or X_m, respectively.

Throughout $\{X_i\}_{i \in \mathbf{I}}$ will denote a minimal system in $\mathcal{X}$.

Definition 1.7.
For each $m \in \mathbf{I}$, we define

(a) The projections $P_m : \times_{i \in \mathbf{I}} X_i \to \mathbf{X}_m$ by

$$P_m \, \mathbf{x} = \mathbf{x}(m) \ .$$

(b) The evaluations $E_m : \times_{i \in \mathbf{I}} X_i \to X_m$ by

$$E_m \, \mathbf{x} = x(m) \ .$$

Next we introduce a Banach space $\mathbf{X} \subset \times_{i \in \mathbf{I}} X_i$ such that the collection $\{\mathbf{X}_i\}_{i \in \mathbf{I}}$ is an unconditional Schauder system in $\mathbf{X}$.

Definition 1.8.
Define the space $\mathbf{X} \subset \times_{i \in \mathbf{I}} X_i$ by

$$\mathbf{X} = \{\mathbf{x} \in \times_{i \in \mathbf{I}} X_i \mid \text{The sequence } (\mathbf{x}(m))_{m \in \mathbf{I}} \text{ is unconditionally summable in } \mathcal{X}\}$$

and impose on $\mathbf{X}$ the topology induced by the unconditional seminorm

$$\|\mathbf{x}\|_u = \sup_{\boldsymbol{F} \in \mathcal{F}(\mathbf{I})} \ \| \sum_{m \in \boldsymbol{F}} \mathbf{x}(m) \| \ .$$

Proposition 1.9.

$\| \cdot \|_u$ is a norm .

Proof.
It is clear that $\|\cdot\|_u$ is a seminorm. So all we have to prove is $\|\mathbf{x}\|_u = 0 \Rightarrow \mathbf{x} = \mathbf{0}$.
Suppose $\|\mathbf{x}\|_u = 0$. Then $\sum_{i\in\mathbf{I}} x(i) = 0$. So for fixed $m \in \mathbf{I}$, $x(m) + \sum_{i\neq m} x(i) = 0$ from which it follows that $x(m) = 0$ because the collection $\{X_i\}_{i\in\mathbf{I}}$ is minimal. □

Proposition 1.10.
The operators E_m, $m \in \mathbf{I}$, are continuous from $\mathbf{X}$ onto X_m.

Proof.

$$\|E_m\mathbf{x}\| \leq \|x(m)\| \leq \sup_{\boldsymbol{F}\in\mathcal{F}(\mathbf{I})} \|\sum_{i\in\boldsymbol{F}} x(i)\| = \|x\|_u .$$

□

Corollary 1.11.
The canonical embedding from $\mathbf{X}$ into $\times_{i\in\mathbf{I}} X_i$ is continuous.

Theorem 1.12.
$\mathbf{X}$ is a Banach space.

Proof
Let $\{\mathbf{x}_k\}_{k\in\boldsymbol{N}}$ be a Cauchy sequence in $\mathbf{X}$. Since $\mathbf{X}$ is continuously embedded in $\times_{i\in\mathbf{I}} X_i$, and since this product space is complete, there exists $\mathbf{x} \in \times_{i\in\mathbf{I}} X_i$ such that $\lim_{k\to\infty} \|x_k(m) - x(m)\| = 0,\ m \in \mathbf{I}$.
Since $\{\mathbf{x}_k\}_{k\in\boldsymbol{N}}$ is a Cauchy sequence in $\mathbf{X}$ we have

$$\forall_{\varepsilon>0}\ \exists_{N\in\boldsymbol{N}}\ \forall_{k,l>N}\ \forall_{\boldsymbol{F}\in\mathcal{F}(\mathbf{I})}$$

$$\|\sum_{i\in\boldsymbol{F}} (x_k(i) - x_l(i))\| < \varepsilon .$$

Letting $l \to \infty$ we obtain

$$\forall_{\varepsilon>0}\ \exists_{N\in\boldsymbol{N}}\ \forall_{k>N}\ \forall_{\boldsymbol{F}\in\mathcal{F}(\mathbf{I})}$$

$$\|\sum_{i\in\boldsymbol{F}} (x_k(i) - x(i))\| < \varepsilon .$$

From this it follows that $\mathbf{x} \in \mathbf{X}$ and that $\lim_{k\to\infty} \|\mathbf{x}_k - \mathbf{x}\|_u = 0$.

Remark: The sequence of Banach spaces $\{\mathbf{X}_i\}_{i\in\mathbf{I}}$ is an unconditional Schauder decomposition of the Banach space $\mathbf{X}$. If we define $X \subset \mathcal{X}$ by

$$X = \{\sum_{i\in\mathbf{I}} x(i) \mid \mathbf{x} \in \mathbf{X}\}$$

then with its natural unconditional norm

$$\|\sum_{i\in\mathbf{I}} x(i)\|_1 := \|\mathbf{x}\|_u$$

X is a Banach space.
Working with the unconditional norm $\| \quad \|_1$ (and so with $\| \quad \|_u$) gives us the convergence properties we require. However we would much prefer to work with the original topology of $\mathcal{X}$ and so avoid to renorm the space X. Clearly this happens whenever X equals the closed linear span $\overline{< \{X_i \mid i \in \mathbf{I}\} >}$, where the closure is taken with respect to the norm of the Banach space X because for all $x \in X$

$$\|x\| \leq \|x\|_1$$

so, in this case, X is complete both with respect to $\|\cdot\|$ and $\|\cdot\|_1$, and $\|\cdot\|$ and $\|\cdot\|_1$ are equivalent.

Next we associate a so called unconditional constant to the collection $\{X_i\}_{i\in\mathbf{I}}$.

Definition 1.13.
Let $M_\theta : \times_{i\in\mathbf{I}} X_i \to \times_{i\in\mathbf{I}} X_i$ be the multiplication operator defined by

$$(M_\theta \mathbf{x})\ (m) = \theta(m)\ \mathbf{x}(m) \quad , \quad m \in \mathbf{I}$$

where $\theta : \mathbf{I} \to \{-1, 1\}$ is some choice of signs.

Proposition 1.14.
For every choice of signs θ, the operator M_θ maps $\mathbf{X}$ boundedly into $\mathbf{X}$, and there exists a constant $K \leq 2$ such that $\|M_\theta\| \leq K$ for all θ.

Proof.
Let $\Theta : \mathbf{I} \to \{-1, 1\}$ be some choice of signs and let $\mathbf{x} \in \mathbf{X}$. Let $\mathbb{F} \subset \mathbf{I}$ be a finite set. Define

$$\mathbb{F}_+ = \{m \in \mathbf{I} \mid \Theta(m) = 1\}, \quad \mathbb{F}_- = \{m \in \mathbf{I} \mid \Theta(m) = -1\}\ .$$

Then

$$\begin{aligned}\|\sum_{m\in\mathbb{F}} \Theta(m)\ \mathbf{x}(m)\| &\leq \|\sum_{m\in\mathbb{F}_+} \mathbf{x}(m)\| + \|\sum_{m\in\mathbb{F}_-} x(m)\| \\ &\leq 2\ \|\mathbf{x}\|_u\ .\end{aligned}$$

Hence $\|M_\theta\, \mathbf{x}\|_u \leq 2\ \|x\|_u$. □.

We define unconditional constant of the system $\{X_i\}_{i\in\mathbf{I}}$ to be $2\ \sup_\theta\ \|M_\theta\|$ and we denote this constant by C.

(The reason for the multiplication by 2 will become clear in the next proposition.) For the remainder of the paper we shall always take C to be the unconditional constant of $\{X_i\}_{i\in\mathbf{I}}$.

Proposition 1.15.
Let $\mathbf{x} \in \mathbf{X}$ and let $\mathbf{y} \in \times_{i\in\mathbf{I}} X_i$ be defined by $\mathbf{y}(m) = \lambda(m)\,\mathbf{x}(m)\,,\quad m \in \mathbf{I}$, where $\lambda \in l^\infty(\mathbf{I})$. Then $\mathbf{y} \in \mathbf{X}$ and

$$\|\mathbf{y}\|_u \leq C \sup_{m\in\mathbf{I}} |\lambda(m)| \quad \|\mathbf{x}\|_u\,.$$

Proof.
Assume the scalars $\lambda(m)$ are real. By Proposition 1.3, $\mathbf{y} \in \mathbf{X}$. So there exists a continuous linear functional $l : \mathbf{X} \to I\!R$ with $\|l\| = 1$ and $\sum_{m\in\mathbf{I}} \lambda(m)\, l(\mathbf{x}(m)) = \|\sum_{m\in\mathbf{I}} \lambda(m)\,\mathbf{x}(m)\|_u$. Define $\theta : \mathbf{I} \to \{-1, 1\}$ by

$$\theta(m) = \text{sign } l(\mathbf{x}(m))\,.$$

Then

$$\begin{aligned}\|\mathbf{y}\|_u &\leq \sum_{m\in\mathbf{I}} |\lambda(m)\,|\,|l(\mathbf{x}(m))| \\ &\leq \sup_{m\in\mathbf{I}} |\lambda(m)| \quad \|\sum_{m\in\mathbf{I}} \theta(m)\,\mathbf{x}(m)\|_u \\ &\leq \tfrac{1}{2}C \sup_{m\in\mathbf{I}} |\lambda(m)| \quad \|\mathbf{x}\|_u\,.\end{aligned}$$

If the scalars are complex then we consider separately the real and imaginary parts to obtain

$$\|\mathbf{y}\|_u \leq C \sup_{m\in\mathbf{I}} |\lambda(m)| \quad \|\mathbf{x}\|_u\,.$$

□

Corollary 1.16.
For each $\lambda \in l^\infty(\mathbf{I})$ the operator $\Lambda_\lambda : \mathbf{X} \to \mathbf{X}$ defined by

$$(\Lambda_\lambda \mathbf{x})\,(m) = \lambda(m)\,\mathbf{x}(m) \quad , \quad m \in \mathbf{I}$$

is bounded with $\|\Lambda_\lambda\| \leq C\,\|\lambda\|_\infty$.

Remark 1.
If $X = \overline{<\{X_i \mid i \in \mathbf{I}\}>}$, there exists $K_1 > 0$ such that for all $x \in X$

$$\|x\| \leq \|x\|_1 \leq K_1\,\|x\|\,.$$

So each M_θ can be seen as a bounded operator from $(X, \|\cdot\|)$ into $(X, \|\cdot\|)$ and

$$\|M_\theta\, x\| \leq \|M_\theta\, x\|_1 \leq K K_1 \, \|x\| \ .$$

Define $C_1 = 2\ \sup_\theta\ \sup_{x\in X}\ \|M_\theta\, x\|$.
Then we see that for each bounded sequence $\lambda \in l^\infty(\mathbf{I})$ the operator $\Lambda_\lambda\ :\ (X, \|\cdot\|) \to (X, \|\cdot\|)$ is bounded with norm smaller than $C_1\ \sup_{m\in\mathbf{I}}\ |\lambda(m)|$.

Remark 2.
In first instance we do not want to identify X and $\mathbf{X}$. The reason for this is the following. For each sequence $\mathbf{a}$ on $\mathbf{I}$ the mapping $\Lambda_{\mathbf{a}} : \times_{i\in\mathbf{I}} X_m \to \times_{m\in\mathbf{I}} X_m$ is defined by

$$(\Lambda_{\mathbf{a}}\, \mathbf{x})\ (m) = \mathbf{a}(m)\ \mathbf{x}(m)\ .$$

Now for a bounded sequence $\mathbf{a}$, $\Lambda_{\mathbf{a}}$ can also be defined as a bounded operator on X as

$$\Lambda_{\mathbf{a}}\, \mathbf{x} = \sum_{m\in\mathbf{I}}\ \mathbf{a}(m)\ \mathbf{x}(m)$$

and $\Lambda_{\mathbf{a}}(X)$ is a well-defined subspace of X. However for an arbitrary sequence $\mathbf{a}$, $\Lambda_{\mathbf{a}}$ cannot be that simply defined and we need completions to describe the space $\Lambda_{\mathbf{a}}(X)$.
However, $\Lambda_{\mathbf{a}}(\mathbf{X})$ is always properly defined.

§2. Sequence sets

Let $\mathbf{I}$ denote a countable set, as usual, and let $\omega^+(\mathbf{I})$ be the set of all nonnegative sequences (= functions) on $\mathbf{I}$.
In $\omega^+(\mathbf{I})$ we introduce the usual pointwise operations: addition $(\mathbf{a}+\mathbf{b})\ (m) = \mathbf{a}(m)+\mathbf{b}(m)$, multiplication $(\mathbf{a}\cdot\mathbf{b})\ (m) = \mathbf{a}(m)\ \mathbf{b}(m)$ scalar multiplication $(\lambda\,\mathbf{a})\ (m) = \lambda\,\mathbf{a}(m)$ and exponentiation $(\mathbf{a}^\lambda)\ (m) = \mathbf{a}(m)^\lambda$. By $\mathbf{1}$ we denote the sequence $\mathbf{1}(m) = 1$ and by δ_j the sequence defined by $\delta_j(m) = 1$ if $m = j$ and $\delta_j(m) = 0$ else.

Definition 2.1.
Let $\mathbf{a} \in \omega^+(\mathbf{I})$. Then the sequence $\mathbf{a}^-$ is defined by $\mathbf{a}^-(m) = \mathbf{a}(m)^{-1}$ if $a(m) > 0$ and $\mathbf{a}^-(m) = 0$ else.
Further, we set $\chi_\mathbf{a} = \mathbf{a}^-\,\mathbf{a}$.

Next we define a partial ordering $\leq$ and a quasi-ordering $\lesssim$ in $\omega^+(\mathbf{I})$.

Definition 2.2.
Let $\mathbf{a},\,\mathbf{b} \in \omega^+(\mathbf{I})$. We write $\mathbf{a} \leq \mathbf{b}$ if $\forall_{m\in\mathbf{I}} : \mathbf{a}(m) \leq \mathbf{b}(m)$, $\mathbf{a} \lesssim \mathbf{b}$ if $\exists_{\lambda>0} : \mathbf{a} \leq \lambda\,\mathbf{b}$, and $\mathbf{a} \sim \mathbf{b}$ if $\mathbf{a} \lesssim \mathbf{b}$ and $\mathbf{b} \lesssim \mathbf{a}$.
The relation $\sim$ is an equivalence relation.

The quasi-ordering in $\omega^+(\mathbf{I})$ induces a quasi-ordering in the collection of all subsets of $\omega^+(\mathbf{I})$.

Definition 2.3.
Let $\rho,\,\sigma \subset \omega^+(\mathbf{I})$. We write $\rho \lesssim \sigma$ if $\forall_{\mathbf{a}\in\rho}\ \exists_{\mathbf{b}\in\sigma} : a \lesssim b$. We write $\rho \approx \sigma$ if $\rho \lesssim \sigma$ and $\sigma \lesssim \rho$.
We also introduce some terminology.

Let $\rho \subset \omega^+(\mathbf{I})$. Then ρ is said to be separating if $\{\delta_m \mid m \in \mathbf{I}\} \lesssim \rho$, and quasi-directed if $\rho+\rho \lesssim \rho$, i.e. if $\forall_{\mathbf{a},\mathbf{b}\in\rho}\ \exists_{\mathbf{c}\in\rho} : \mathbf{a} \lesssim \mathbf{c}$ and $\mathbf{b} \lesssim \mathbf{c}$.

Two equivalent sequence sets are both separating (quasi-directed) or both not.

The subsets of $\omega^+(\mathbf{I})$ can be classified in three types.

Definition 2.3.
A set $\rho \subset \omega^+(\mathbf{I})$ is said to be type 1 if ρ is equivalent with a finite subset of $\omega^+(\mathbf{I})$, type 2 if ρ is not type 1 and ρ is equivalent with a countable subset of $\omega^+(\mathbf{I})$, and type 3 if ρ is not type 1 and not type 2.

Example.
The set $l^{\infty,+}(\mathbf{I})$ is type 1, $l^{\infty,+}(\mathbf{I}) \approx \{\mathbf{1}\}$. The set $\varphi^+(\mathbf{I})$ of all sequences $\mathbf{a} \in \omega^+(\mathbf{I})$ with a finite support is type 2. With a diagonal argument it follows that $\omega^+(\mathbf{I})$ is a type 3 set itself.

Quasi-directed type 1 and type 2 sets have a standard form.

Proposition 2.4.
For a quasi-directed subset of $\omega^+(\mathbf{I})$,

(a) ρ is type 1 iff $\rho \approx \{\mathbf{a}\}$ for some $\mathbf{a} \in \omega^+(\mathbf{I})$

(b) ρ is type 2 iff $\rho \approx \{\mathbf{a}_k \mid k \in \mathbb{N}\}$ with for all $k \in \mathbb{N}$, $\mathbf{a}_k \leq \mathbf{a}_{k+1}$ and $\neg(\mathbf{a}_{k+1} \lesssim \mathbf{a}_k)$.

Proof.

(a) If ρ is type 1 and quasi-directed, then there are $\mathbf{b}_1, ..., \mathbf{b}_r$ such that $\rho \approx \{\mathbf{b}_1, ..., \mathbf{b}_r\}$. Now observe that $\{\mathbf{b}_1, ..., \mathbf{b}_r\} \approx \{\mathbf{b}_1 + ... + \mathbf{b}_r\}$. The converse is trivial.

(b) If ρ is type 2 and quasi-directed, then there are $(\mathbf{b}_k)_{k\in\mathbb{N}}$ such that $\rho \approx \{\mathbf{b}_k \mid k \in \mathbb{N}\}$. Put $\mathbf{a}_k = \mathbf{b}_1 + ... + \mathbf{b}_k$. Then $\mathbf{a}_{k+1} \geq \mathbf{a}_k$ and $\rho \approx \{\mathbf{a}_k \mid k \in \mathbb{N}\}$. Similarly, the condition $\neg(\mathbf{a}_{k+1} \lesssim \mathbf{a}_k)$ can be taken care of. □

We mention the following lemma.

Lemma 2.5.
Let $\rho \subset \omega^+(\mathbf{I})$ be totally ordered. Then ρ is type 1 or type 2.

Proof.
Let $\rho = \{\mathbf{a}_\alpha \mid \alpha \in A\}$ with A a totally ordered set such that $\mathbf{a}_\alpha \leq a_\beta \Longleftrightarrow \alpha \leq \beta$. We have

$$(*) \qquad (\exists_{m\in\mathbf{I}} \;:\; \mathbf{a}_\alpha(m) < \mathbf{a}_\beta(m)) \Longrightarrow \alpha \leq \beta \;.$$

We consider two cases

$$\text{a.} \qquad \exists_{m_0\in\mathbf{I}} \;:\; \sup\{\mathbf{a}_\alpha(m_0) \mid \alpha \in A\} = \infty \;.$$

For every $k \in \mathbb{N}$ there exists $\alpha_k \in A$ with $\mathbf{a}_{\alpha_k}(m_0) > k$. Put $\mathbf{b}_k = \mathbf{a}_{\alpha_k}$. Then by (1) it simply follows that $\rho \approx \{\mathbf{b}_k \mid k \in \mathbb{N}\}$.

$$\text{b.} \qquad \forall_{m\in\mathbf{I}} \;:\; \sup\{\mathbf{a}_\alpha(m) \mid \alpha \in A\} < \infty \;.$$

Define $s \in \omega^+(\mathbf{I})$ by $\mathbf{s}(m) = \sup\{\mathbf{a}_\alpha(m) \mid \alpha \in A\}$. We may as well assume that $\mathbf{s}(m) > 0$ for all $m \in \mathbf{I}$.
Then $\rho \lesssim \{\mathbf{s}\}$. If $\rho \approx \{\mathbf{s}\}$, ρ is type 1. If not, then $\neg(\mathbf{s} \lesssim \mathbf{a}_\alpha)$ for all $\alpha \in A$. So we have

$$(**) \qquad \forall_{\alpha\in A}\, \exists_{m_\alpha\in\mathbf{I}} \;:\; 2\,a_\alpha(m_\alpha) \leq s(m_\alpha) \;.$$

There exists a countable set $\{b_j \mid j \in \mathbb{N}\} \subset \rho$ with $b_m(m) > \frac{1}{2}\,s(m)$, $m \in \mathbf{I}$. We prove that $\{b_j \mid j \in \mathbb{N}\} \gtrsim \rho$. So let $\alpha \in A$, then by $(**)$

$$a_\alpha(m_\alpha) \leq \frac{1}{2}\,s(m_\alpha) < b_{m_\alpha}(m_\alpha)$$

and by (*) $a_\alpha \leq b_{m_\alpha}$. □

Next we introduce the operation $\rho \mapsto \rho^\sharp$ on the collection of all subsets of $\omega^+(\mathbf{I})$.

Definition 2.6.
Let $\rho \subset \omega^+(\mathbf{I})$. The set $\rho^\sharp \subset \omega^+(\mathbf{I})$, called ρ hash, is defined by

$$\rho^\sharp = \{\mathbf{u} \in \omega^+(\mathbf{I}) \mid \forall_{\mathbf{a}\in\rho} \ : \ \mathbf{a}\cdot\mathbf{u} \in l^\infty(\mathbf{I})\} \,.$$

For each $\rho \subset \omega^+(\mathbf{I})$, $\rho^\sharp$ is separating and quasi-directed. We have $\rho \subset \rho^{\sharp\sharp}$ and $\rho^\sharp = \rho^{\sharp\sharp\sharp}$. Also, $\rho \approx \sigma$ implies $\rho^\sharp = \sigma^\sharp$. Quite naturally the notion of symmetric sequence set comes up.

Definition 2.7.
Let $\rho \subset \omega^+(\mathbf{I})$. The set ρ is called symmetric if $\rho \approx \rho^{\sharp\sharp}$.

Observe that symmetric sequence sets are separating and quasi-directed. We have the following result for type 1 sets.

Theorem 2.8.
Let $\rho \subset \omega^+(\mathbf{I})$ be type 1. Then ρ is symmetric iff ρ is separating and quasi-directed. If $\rho \approx \{\mathbf{a}\}$ is symmetric then $\rho^\sharp \approx \{\mathbf{a}^{-1}\}$.

It is a remarkable fact that the same result holds for type 2 sets.

Theorem 2.9.
Let $\rho \subset \omega^+(\mathbf{I})$ be type 2. Then ρ is symmetric iff ρ is separating and quasi-directed.

Proof.
By Proposition 2.4 we may assume that $\rho = \{\mathbf{a}_j \mid j \in \mathbb{N}\}$ with $\forall_m \, \exists_j \ : \ \mathbf{a}_j(m) > 0$ and $\forall_j \ : \ \mathbf{a}_j \leq \mathbf{a}_{j+1} \wedge \neg(\mathbf{a}_{j+1} \lesssim \mathbf{a}_j)$.
Suppose ρ is not symmetric. Then there is $\mathbf{b} \in \rho^{\sharp\sharp}$ such that $\forall_j \ : \ \neg(\mathbf{b} \leq j\,\mathbf{a}_j)$. Define $\mathbf{I}_j = \{m \in \mathbf{I} \mid \mathbf{b}(m) > j\,\mathbf{a}_j(m)\}$. Then $\mathbf{I}_j \supset \mathbf{I}_{j+1}$, $\mathbf{I}_j \neq \emptyset$ and $\cap_j \mathbf{I}_j = \emptyset$. So each $\mathbf{I}_j$ is infinite and there exists a sequence $(m_j)_{j\in\mathbb{N}}$ in $\mathbf{I}$ such that $m_j \in \mathbf{I}_j$ and $m_j \neq m_{j'}$, $j \neq j'$.
Define $\mathbf{c} \in \omega^+(\mathbf{I})$ by

$$\mathbf{c}(m) = \begin{cases} 0 & \text{if} \quad m \notin \{m_j \mid j \in \mathbb{N}\} \\ \mathbf{a}_j(m)^{-1} & \text{if} \quad m = m_j \wedge \mathbf{a}_j(m) > 0 \\ j\,\mathbf{b}(m)^{-1} & \text{if} \quad m = mj \wedge a_j(m) = 0 \,. \end{cases}$$

Then $\mathbf{b}\cdot\mathbf{c} \notin l^\infty(\mathbf{I})$, and for each fixed $j_0 \in \mathbb{N}$ and $j \geq j_0$, $j \in \mathbb{N}$, with $a_{j_0}(m_j) > 0$

$$(\mathbf{c}\cdot\mathbf{a}_{j_0})\,(m_j) = (\mathbf{a}_j^- \cdot \mathbf{a}_{j_0})\,(m_j) \leq 1$$

so that $\mathbf{c} \in \rho^\sharp$. We arrive at a contradiction. □

Remark: If follows from the preceding theorem that all sets $l^{p,+}(\mathbf{I})$, $1 \le p < \infty$, are type 3.

Lemma 2.10.
Let ρ and σ denote two type 2 sequence sets with ρ quasi-directed and σ symmetric. Assume $\rho \subset \sigma^\sharp$.
Then there exists $\mathbf{c} \in \omega^+(\mathbf{I})$ such that

$$\rho \lesssim \{\mathbf{c}\} \subset \sigma^\sharp .$$

Proof.
We may assume that $\rho = \{\mathbf{a}_j \mid j \in \mathbb{N}\}$ and $\sigma = \{\mathbf{b}_j \mid j \in \mathbb{N}\}$ with $\mathbf{a}_j \le \mathbf{a}_{j+1}$, $\mathbf{b}_j \le \mathbf{b}_{j+1}$ and $\mathbf{a}_1 \cdot \mathbf{b}_1 \neq 0$. We define $\mathbf{c} \in \sigma^\sharp$ as follows

$$\mathbf{c}(m) = \inf\{|\mathbf{a}_j \cdot \mathbf{b}_j|_\infty (b_j(m))^{-1} \mid j \ \text{ with } \ b_j(m) > 0\}$$

(Here $|\cdot|_\infty$ is the norm of $l^\infty(\mathbf{I})$.)
It remains to prove that $\rho \lesssim \{\mathbf{c}\}$.
Take a fixed $j_0 \in \mathbb{N}$. Then for all $j \in \mathbb{N}$ and $m \in \mathbf{I}$ with $a_{j_0}(m) > 0$ and $b_j(m) > 0$,

$$\frac{\mathbf{a}_{j_0}(m)\ \mathbf{b}_j(m)}{|\mathbf{a}_{j_0} \cdot \mathbf{b}_j|_\infty} < 1$$

and so

$$\mathbf{c}(m) > \inf\Big\{\frac{|a_j \cdot b_j|_\infty}{|\mathbf{a}_{j_0} \cdot b_j|_\infty} \mid j \ \text{ with } \ b_j(m) > 0\Big\}\ \mathbf{a}_{j_0}(m) .$$

Let $\gamma_{j_0} = \min\Big\{\dfrac{|\mathbf{a}_j \cdot \mathbf{b}_j|}{|\mathbf{a}_{j_0} \cdot \mathbf{b}_j|} \mid 1 \le j \le j_0\Big\}$.
Then $\mathbf{c} \ge \gamma_{j_0}\ \mathbf{a}_{j_0}$. □

The preceding result has the following two important consequences.

Corollary 2.11.
Let ρ be a symmetric type 2 set. Then $\rho^\sharp$ is type 3.

Proof.
If $\rho^\sharp$ were type 1, then $\rho \approx \rho^{\sharp\sharp}$ would be type 1 also by Theorem 2.8. If $\rho^\sharp$ were type 2, then Theorem 2.10 states that there would be $\mathbf{c} \in \rho^\sharp$ with $\rho^\sharp \lesssim \{\mathbf{c}\}$ taking $\sigma = \rho^\sharp$ and so $\rho^\sharp$ would be type 1, a contradiction. □

Corollary 2.12.
Let ρ and σ be symmetric type 2 sets. Then $\rho \cdot \sigma$ is a symmetric type 2 set with $(\rho \cdot \sigma)^\sharp = \rho^\sharp \cdot \sigma^\sharp$.

Proof.
Clearly $\rho \cdot \sigma$ is separating quasi-directed and type 2 whence $\rho \cdot \sigma$ is symmetric. Further it is clear that $\rho^\sharp \cdot \sigma^\sharp \subset (\rho \cdot \sigma)^\sharp$. For the converse, we note that

$$\mathbf{u} \in (\rho \cdot \sigma)^{\sharp} \iff \{\mathbf{u}\} \cdot \rho \subset \sigma^{\sharp} .$$

Now $\{\mathbf{u}\} \cdot \rho$ is quasi-directed and σ is symmetric. Hence there exists $\mathbf{c} \in \sigma^{\sharp}$ such that $\{\mathbf{u}\} \cdot \rho \lesssim \{\mathbf{c}\}$. It means that $\mathbf{u} \in \{\mathbf{c}\} \cdot \rho^{\sharp} \subset \sigma^{\sharp} \cdot \rho^{\sharp}$. □

The results presented in this section have been firstly presented by Kuylaars in his master's thesis [Ku]. The proofs as presented here are taken from the PhD thesis [Ma].

§3. The inductive limits $\mathbf{X}_{\text{ind}}(\rho)$

Let $\{X_i\}_{i\in\mathbf{I}}$ be a minimal system of closed subspaces of a Banach space $\mathcal{X}$. Let $\mathbf{X}$ denote the associated Banach subspace of the carthesian product $\times_{i\in\mathbf{I}} X_i$ as described in Section 1. Further, for each $m \in \mathbf{I}$ let $P_m : \times_{i\in\mathbf{I}} X_i \to \mathbf{X}_m$ denote the canonical projection as defined in Definition 1.7.
For each $\mathbf{a} \in \omega^+(\mathbf{I})$ we define the vector spaces $\mathbf{X}_{\text{ind}}(\mathbf{a})$ and $\mathbf{X}_{\text{proj}}(\mathbf{a})$. Then for a separating quasi-directed sequence set $\rho \subset \omega^+(\mathbf{I})$ we obtain the inductive limit $\mathbf{X}_{\text{ind}}(\rho)$ and the projective limit $X_{\text{proj}}(\rho)$. Topological properties of the inductive limit will be discussed in this section. In the next section we consider the projective limit. The theory is an adaption and refinement of the theory in [Ma].

Definition 3.1.
Let $\mathbf{a} \in \omega^+(\mathbf{I})$. The linear operator $\Lambda_{\mathbf{a}} : \times_{i\in\mathbf{I}} X_i \to \times_{i\in\mathbf{I}} X_i$ is defined by

$$\Lambda_{\mathbf{a}} = \sum_{m\in\mathbf{I}} \mathbf{a}(m)\; P_m \; .$$

Definition 3.2.
Let $a \in \omega^+(\mathbf{I})$. The space $\mathbf{X}_{\text{ind}}(\mathbf{a})$ is defined to be the space $\Lambda_{\mathbf{a}}(\mathbf{X})$ endowed with the norm $\|\cdot\|_{\mathbf{a}}$,

$$\|\mathbf{x}\|_{\mathbf{a}} = \|\Lambda_{\mathbf{a}^-}\mathbf{x}\|_u \quad , \quad x \in \Lambda_{\mathbf{a}}(\mathbf{X}) \; .$$

To see that $\mathbf{X}_{\text{ind}}(\mathbf{a})$ is a Banach space observe that $\Lambda_{\mathbf{a}}$ maps $\Lambda_{\mathbf{a}-\mathbf{a}}(\mathbf{X})$ isometrically onto $X_{\text{ind}}(\mathbf{a})$ and $\Lambda_{\mathbf{a}-\mathbf{a}}(\mathbf{X})$ is a closed subspace of $\mathbf{X}$ since $P_m : \mathbf{X} \to \mathbf{X}_m$ is continuous for each $m \in \mathbf{I}$. It is clear that the collection of Banach spaces $\{\mathbf{X}_m \mid m \in \text{supp}\ \mathbf{a}\}$ is an unconditional Schauder decomposition of $\mathbf{X}_{\text{ind}}(\mathbf{a})$.

Lemma 3.3.
Let $\mathbf{a}, \mathbf{b} \in \omega^+(\mathbf{I})$. The following are equivalent

(i) $\quad \mathbf{a} \lesssim \mathbf{b}$

(ii) $\quad \Lambda_{\mathbf{a}}(\mathbf{X}) \subset \Lambda_{\mathbf{b}}(\mathbf{X})$

(iii) $\quad \mathbf{X}_{\text{ind}}(\mathbf{a}) \hookrightarrow \mathbf{X}_{\text{ind}}(\mathbf{b})$

Proof.

(iii) $\Rightarrow$ (ii) Trivial

(ii) $\Rightarrow$ (i) Suppose $\Lambda_{\mathbf{a}}(\mathbf{X}) \subset \Lambda_{\mathbf{X}}(\mathbf{b})$. Then $\Lambda_{\mathbf{b}-\mathbf{a}}(\mathbf{X}) \subset \Lambda_{\mathbf{b}-\mathbf{b}}(\mathbf{X}) \subset \mathbf{X}$. From the closed graph theorem it follows that $\Lambda_{\mathbf{b}-\mathbf{a}}$ is a bounded operator on $\mathbf{X}$. So there exists $K > 0$ such that

$$\|\Lambda_{\mathbf{b}-\mathbf{a}}\mathbf{x}\|_u \leq K\ \|x\|_u\ .$$

Consequently,

$$(\mathbf{b}^- \mathbf{a})\ (m)\ \|P_m \mathbf{x}\| \leq K\ \|P_m \mathbf{x}\|_u\ .$$

Since, also $\mathbf{a}(m)\ P_m(\mathbf{X}) \subset \mathbf{b}(m)\ P_m(\mathbf{X})$, we get

$$\mathbf{a}(m) \leq K\ \mathbf{b}(m)$$

or equivalently $\mathbf{a} \lesssim \mathbf{b}$.

(i) $\Rightarrow$ (iii) Suppose $\mathbf{a} \lesssim \mathbf{b}$. Then $\text{supp}(\mathbf{a}) \subseteq \text{supp}(\mathbf{b})$ and for all $\mathbf{x} \in X$,

$$\Lambda_a \mathbf{x} = \Lambda_{\mathbf{b}}(\Lambda_{\mathbf{b}-\mathbf{a}}\mathbf{x})\ .$$

Since $\mathbf{b}^-\mathbf{a} \in l^\infty(\mathbf{I})$, $\Lambda_{\mathbf{b}-\mathbf{a}}$ is a bounded linear operator from $\mathbf{X}$ to $\mathbf{X}$ by Corollary 1.16 so that $\Lambda_{b-a}\mathbf{x} \in \mathbf{X}$ and $\Lambda_{\mathbf{a}}\mathbf{x} \in \Lambda_{\mathbf{b}}(\mathbf{X})$.
Moreover for all $\mathbf{z} \in \Lambda_{\mathbf{a}}(\mathbf{X})$

$$\begin{aligned} \|\mathbf{z}\|_{\mathbf{b}} &= \|\Lambda_{\mathbf{b}^-}\mathbf{z}\|_u \\ &\leq C\ \ \|\Lambda_{b-a}\|\ \ \|\mathbf{z}\|_{\mathbf{a}} \end{aligned}$$

where C is the unconditional constant.
Hence $\mathbf{X}_{\text{ind}}(\mathbf{a}) \hookrightarrow \mathbf{X}_{\text{ind}}(\mathbf{b})$. □

Proposition 3.4.
Let $\mathbf{a} \in \omega^+(\mathbf{I})$. Then $\mathbf{X}_{\text{ind}}(\mathbf{a}) \hookrightarrow \times_{i\in\mathbf{I}}\, \mathbf{X}_i$ and $\mathbf{X}_m \hookrightarrow X_{\text{ind}}(\mathbf{a})$ for all $m \in \text{supp}(\mathbf{a})$.

Proof.
For all $j \in \mathbf{I}$ and $\mathbf{z} \in \mathbf{X}_{\text{ind}}(\mathbf{a})$,

$$\|P_j\,\mathbf{z}\|_u \leq \mathbf{a}(j)\ C\ \|\mathbf{z}\|_{\mathbf{a}}$$

and so the canonical injection from $\mathbf{X}_{\text{ind}}(\mathbf{a})$ into $\times_{i\in\mathbf{I}}\, \mathbf{X}_i$ is continuous.
The second assertion is a consequence of Lemma 3.3 because for all $m \in \text{supp}(\mathbf{a})$, $\delta_m \lesssim \mathbf{a}$. □

It follows from Lemma 3.3 that each separating and quasi-directed sequence set $\rho \subset \omega^+(\mathbf{I})$, henceforth called a Köthe set, yields the inductive system of Banach spaces $\{X_{\text{ind}}(\mathbf{a}) \mid \mathbf{a} \in \rho\}$.

Definition 3.5.
Let $\rho \subset \omega^+(\mathbf{I})$ be a Köthe set. The inductive limit $\mathbf{X}_{\text{ind}}(\rho)$ is defined by

$$\mathbf{X}_{\text{ind}}(\rho) = \text{limind}_{\mathbf{a}\in\rho}\, \mathbf{X}_{\text{ind}}(\mathbf{a})\ .$$

Corollary 3.6.
Let $\rho \subset \omega^+(\mathbf{I})$ be a Köthe set. Then $X_{\mathbf{ind}}(\rho)$ is a Hausdorff space. In particular, $\mathbf{X}_{\mathbf{ind}}(\rho) \hookrightarrow \times_{i\in\mathbf{I}} X_i$ and $\mathbf{X}_m \hookrightarrow \mathbf{X}_{\mathbf{ind}}(\rho)$.

Proof.
Continuity of the canonical injections follows from Proposition 3.4. Since $\times_{i\in\mathbf{I}} X_i$ with product topology is a Hausdorff space, so is the space $X_{\mathbf{ind}}(\rho)$. □

Theorem 3.7.
Let $\rho, \sigma \subset \omega^+(\mathbf{I})$ be Köthe sets

(i) If $\rho \lesssim \sigma$ then $X_{\mathbf{ind}}(\rho) \hookrightarrow X_{\mathbf{ind}}(\sigma)$.

(ii) If $\rho \approx \sigma$ then $X_{\mathbf{ind}}(\rho) = X_{\mathbf{ind}}(\sigma)$.

Proof.
Let $\rho \lesssim \sigma$. Then for all $\mathbf{a} \in \rho$ there is $\mathbf{b} \in \sigma$ such that $\mathbf{a} \lesssim \mathbf{b}$. So by the lemma above for all $\mathbf{a} \in \rho$ there exists $\mathbf{b} \in \sigma$ such that $\mathbf{X}_{\mathbf{ind}}(\mathbf{a}) \hookrightarrow \mathbf{X}_{\mathbf{ind}}(\mathbf{b})$. Consequently, $\cup_{\mathbf{a}\in\rho} \mathbf{X}_{\mathbf{ind}}(\mathbf{a}) \subset \cup_{\mathbf{b}\in\sigma} \mathbf{X}_{\mathbf{ind}}(\mathbf{b})$. If j denotes the canonical injection from $\mathbf{X}_{\mathbf{ind}}(\rho)$ into $X_{\mathbf{ind}}(\sigma)$ then, by definition of the inductive limit topology, whence j is continuous. □

Remark
If the Köthe set ρ is type 1 then $\mathbf{X}_{\mathbf{ind}}(\rho)$ is a Banach space, since in this case $\rho \approx \{\mathbf{a}\}$ and so $\mathbf{X}_{\mathbf{ind}}(\rho) = \mathbf{X}_{\mathbf{ind}}(\mathbf{a})$. If ρ is type 2, $X_{\mathbf{ind}}(\rho)$ is a countable inductive limit of Banach spaces.

Theorem 3.8.
Let $\rho \subset \omega^+(\mathbf{I})$ be a Köthe set

(i) The space $\mathbf{X}_{\mathbf{ind}}(\rho)$ is barreled.

(ii) The space $\mathbf{X}_{\mathbf{ind}}(\rho)$ is bornological.

Proof.
Each Banach space is barreled and bornological. Being an inductive limit of barreled and bornological spaces $X_{\mathbf{ind}}(\rho)$ is barreled and bornological (cf. [Sch], ch. II). □

Remark.
The inductive limit $\mathbf{X}_{\mathbf{ind}}(\rho)$ is, in general, not strict. In Section 5 we present a necessary and sufficient condition on a Köthe set ρ such that $\mathbf{X}_{\mathbf{ind}}(\rho)$ is a regular inductive limit.

A crucial point in our setup is that we can describe the topology on the inductive limit $\mathbf{X}_{\mathbf{ind}}(\rho)$ in terms of well specified seminorms when ρ is a so called moulding set.

Definition 3.9.
A Köthe set $\rho \subset \omega^+(\mathbf{I})$ is called moulding if there exists a sequence $\mathbf{r} \in l^{1,+}(\mathbf{I})$ such that $\{\mathbf{r}\} \cdot \rho \approx \rho$.

Remark.
If $\{r\} \cdot \rho \approx \rho$, then $\text{supp}(\mathbf{r}) = \mathbf{I}$, $\{\mathbf{r}^{-1}\} \cdot \rho \approx \rho$ and $\{\mathbf{r}\} \cdot \rho^\sharp = \rho^\sharp = \{\mathbf{r}^{-1}\} \cdot \rho^\sharp$.

If $\rho \subset \omega^+(\mathbf{I})$ is a moulding set, then for all $\mathbf{a} \in \rho$ there exist $\mathbf{b} \in \rho$ and $\mathbf{s} \in l_1^+(\mathbf{I})$ such that $\mathbf{a} = \mathbf{b} \cdot \mathbf{s}$.

Before we state the main theorem of this section, we prove the following lemma.

For moulding sets, $\mathbf{X}_{\text{ind}}(\rho)$ admits the following characterization.

Lemma 3.10
$\mathbf{x} \in \mathbf{X}_{\text{ind}}(\rho)$ iff there exists a sequence $\tilde{\mathbf{x}} \in \times_{i\in\mathbf{I}} X_m$ and $\mathbf{a} \in \rho$ such that $\sup_{m\in\mathbf{I}} \|\tilde{\mathbf{x}}(m)\|_u < \infty$ and $\mathbf{x}(m) = \mathbf{a}(m)\ \tilde{\mathbf{x}}(m)\ ,\quad m \in \mathbf{I}$.

Proof.

$\Rightarrow$ Let $\mathbf{x} \in \mathbf{X}_{\text{ind}}(\rho)$. Then $\mathbf{x} = \Lambda_{\mathbf{a}}\,\tilde{\mathbf{x}}$ for some $\tilde{\mathbf{x}} \in \mathbf{X}$ and $\mathbf{a} \in \rho$. Now $\|\tilde{\mathbf{x}}(m)\|_u \leq C\ \|\tilde{\mathbf{x}}\|_u$.

$\Leftarrow$ Let $\zeta \in l_1^+(\mathbf{I})$ be such that $\{\zeta\} \cdot \rho \approx \rho$. Then $\mathbf{a}(m)\ \tilde{\mathbf{x}}(m) = (\zeta^{-1} \cdot \mathbf{a})\ (m)\ \zeta(m)\ \tilde{\mathbf{x}}(m)$ and $\sum_{m\in\mathbf{I}} \zeta(m)\ \|\tilde{\mathbf{x}}(m)\|_u < \infty$.

Lemma 3.11.
Let $\mathbf{u} \in \omega^+(\mathbf{I})$ and let $\Lambda_{\mathbf{u}}\,\mathbf{x} \in \mathbf{X}$. Then $\|\Lambda_{\mathbf{u}}\,\mathbf{x}\|_u < 1/C$ implies $\mathbf{u}(m)\ \|\mathbf{x}(m)\|_u < 1\ ,\quad m \in \mathbf{I}$.

Proof.

$$\begin{aligned} \mathbf{u}(m)\ \|\mathbf{x}(m)\|_u &= \|\mathbf{u}(m)\ \mathbf{x}(m)\|_u \\ &= \|P_m \sum_{i\in\mathbf{I}} \mathbf{u}(i)\ \mathbf{x}(i)\|_u \\ &\leq C\ \ \|\Lambda_{\mathbf{u}}\,\mathbf{x}\|_u < 1 \end{aligned}$$

where for the last inequality Corollary 1.16 is used. □

Theorem 3.12.
Let $\rho \subset \omega^+(\mathbf{I})$ be a moulding set. The inductive limit topology of $\mathbf{X}_{\text{ind}}(\rho)$ is generated by the collection of seminorms $\{p_{\mathbf{u}} \mid \mathbf{u} \in \rho^\sharp\}$, where

$$p_{\mathbf{u}}(\mathbf{z}) = \|\Lambda_{\mathbf{u}}\,\mathbf{z}\|_u\ ,\quad \mathbf{z} \in X_{\text{ind}}(\rho),\ \mathbf{u} \in \rho^\sharp\ .$$

Proof.
First, observe that $p_{\mathbf{u}}$ is well-defined on $\mathbf{X}_{\text{ind}}(\rho)$ because for $\mathbf{z} \in \mathbf{X}_{\text{ind}}(\rho)$, $\mathbf{z} = \Lambda_{\mathbf{a}}\,\mathbf{x}$, we have

$$\Lambda_{\mathbf{u}}\,\mathbf{z} = \Lambda_{\mathbf{u}}(\Lambda_{\mathbf{a}}\,\mathbf{x}) = \Lambda_{\mathbf{u}\cdot\mathbf{a}}\,\mathbf{x} \in \mathbf{X}\ ,$$

by Corollary 1.16. And by the same corollary for all $\mathbf{a} \in \rho$ and $\mathbf{x} \in \mathbf{X}$

$$p_{\mathbf{u}}(\Lambda_{\mathbf{a}}\,\mathbf{x}) = \|\Lambda_{\mathbf{u}\cdot\mathbf{a}}\,\mathbf{x}\|_u \leq C\ |\mathbf{u}\cdot\mathbf{a}|_\infty\ \|\Lambda_{\mathbf{a}}\,\mathbf{x}\|_{\mathbf{a}}\ .$$

Hence $p_{\mathbf{u}}$ is a continuous seminorm on $\mathbf{X}_{\mathrm{ind}}(\rho)$ and the inductive limit topology is stronger than the topology generated by the collection $\{p_{\mathbf{u}} \mid \mathbf{u} \in \rho^\sharp\}$.

Now let Ω be a convex balanced subset of $\mathbf{X}_{\mathrm{ind}}(\rho)$ such that for each $\mathbf{a} \in \rho$, $\Omega \cap \mathbf{X}_{\mathrm{ind}}(\mathbf{a})$ is a neighbourhood of zero in $\mathbf{X}_{\mathrm{ind}}(\mathbf{a})$. Let k_Ω denote the gauge of Ω. Then for each $\mathbf{a} \in \rho$ there exists $\varepsilon_{\mathbf{a}} > 0$ such that

$$\forall_{\mathbf{z}\in X_{\mathrm{ind}}(\mathbf{a})}\quad :\quad k_\Omega(\mathbf{z}) \leq \varepsilon_{\mathbf{a}}\ \|\mathbf{z}\|_{\mathbf{a}}\ .$$

Also, since ρ is separating, for each $m \in \mathbf{I}$, there exists $\mathbf{a} \in \rho$ such that $\mathbf{a}(m) > 0$, whence for $\mathbf{z} \in \mathbf{X}_m \subset \mathbf{X}_{\mathrm{ind}}(\mathbf{a})$

$$k_\Omega(\mathbf{z}) \leq \varepsilon_a\ \|\mathbf{z}\|_a = \mathbf{a}(m)^{-1}\,\varepsilon_a\ \|\mathbf{z}\|_u\ .$$

So for all $m \in \mathbf{I}$, $k_\Omega\Big|_{\mathbf{X}_m}$ is continuous and we can define $\mathbf{w} \in \omega^+(\mathbf{I})$ by

$$\mathbf{w}(m) = \sup\ \{k_\Omega(\mathbf{x}) \mid \mathbf{x} \in \mathbf{X}_m\ ,\ \ \|\mathbf{x}\|_u = 1\}\ .$$

Then for all $\mathbf{a} \in \rho$ and all $m \in \mathbf{I}$,

$$\begin{aligned}\mathbf{a}(m)\ \mathbf{w}(m)\ &= \sup\ \{a(m)\ k_\Omega(\mathbf{x}\,/\,\|\mathbf{x}\|_u) \mid \mathbf{x} \in \mathbf{X}_m\}\\ &\leq \sup\ \{k_\Omega(\Lambda_{\mathbf{a}}\,\mathbf{x})\ /\ \|\Lambda_{\mathbf{a}}\,\mathbf{x}\|_{\mathbf{a}} \mid \mathbf{x} \in \mathbf{X}\}\\ &\leq \varepsilon_a\end{aligned}$$

which shows $\mathbf{w} \in \rho^\sharp$.
Now since ρ is a moulding set there exists $\mathbf{r} \in l^{1,+}(\mathbf{I})$ with $\sum_{i\in\mathbf{I}} \mathbf{r}(i) = 1$ such that $\{\mathbf{r}\}\cdot\rho^\sharp = \rho^\sharp$.
Let $\mathbf{u} = \mathbf{r}^{-1}\cdot\mathbf{w}$ and consider those $\mathbf{z} \in \mathbf{X}_{\mathrm{ind}}(\rho)$ for which $p_{\mathbf{u}}(\mathbf{z}) < 1/C$. Then $\mathbf{u} \in \rho^\sharp$ and

$$\begin{aligned}k_\Omega(\mathbf{z})\ &= k_\Omega\Big(\sum_{m\in\mathbf{I}} \mathbf{z}(m)\Big) \leq \sum_{m\in\mathbf{I}} k_\Omega(\mathbf{z}(m))\\ &\leq \sum_{m\in\mathbf{I}} \mathbf{w}(m)\ \|\mathbf{z}(m)\|_u = \sum_{m\in\mathbf{I}} \mathbf{r}(m)\ \mathbf{u}(m)\ \|\mathbf{z}(m)\|_u\\ &< \sum_{m\in\mathbf{I}} \mathbf{r}(m) = 1\ .\end{aligned}$$

Hence $\mathbf{z} \in \Omega$.
It follows that the topology generated by the seminorms $p_{\mathbf{u}}$, $\mathbf{u} \in \rho^\sharp$ is stronger than the inductive limit topology. □

§4. The projective limit $\mathbf{X}_{\mathrm{proj}}(\rho)$

In the preceding section we showed how to associate to each Köthe set $\rho \subset \omega^+(\mathbf{I})$ and each minimal system $\{X_i\}_{i\in\mathbf{I}}$ in a Banach space $\mathcal{X}$ an inductive limit $\mathbf{X}_{\mathrm{ind}}(\rho)$.
Here we associate a projective limit $\mathbf{X}_{\mathrm{proj}}(\rho)$ to ρ and $\{X_i\}_{i\in\mathbf{I}}$.

Definition 4.1.
Let $\mathbf{a} \in \omega^+(\mathbf{I})$. The topological vector space $\mathbf{X}_{\mathrm{proj}}(\mathbf{a})$ is defined to be the vector space

$$\Lambda_{\mathbf{a}}^{\leftarrow}(\mathbf{X}) = \{\mathbf{x} \in \times_{i\in\mathbf{I}} X_i \mid \Lambda_{\mathbf{a}}(\mathbf{x}) \in \mathbf{X}\}$$

endowed with the seminorm

$$p_{\mathbf{a}}(\mathbf{x}) = \|\Lambda_{\mathbf{a}}\,\mathbf{x}\|_u \;.$$

The seminorm $p_{\mathbf{a}}$ is a norm if and only if $\mathrm{supp}(\mathbf{a}) = \mathbf{I}$. In that case, $X_{\mathrm{proj}}(\mathbf{a})$ is a Banach space.

Lemma 4.2.
Let $\mathbf{a}, \mathbf{b} \in \omega^+(\mathbf{I})$. Then $\mathbf{a} \lesssim \mathbf{b}$ iff $\mathbf{X}_{\mathrm{proj}}(\mathbf{b}) \hookrightarrow \mathbf{X}_{\mathrm{proj}}(\mathbf{a})$.

Proof.

$\Leftarrow$) Let $\mathbf{x} \in \mathbf{X}_{\mathrm{proj}}(\mathbf{b})$. Then $\Lambda_{\mathbf{a}}\,\mathbf{x} = \Lambda_{\mathbf{a}\mathbf{b}^-}(\Lambda_{\mathbf{b}}\,\mathbf{x})$. So $\Lambda_{\mathbf{a}}\,\mathbf{x} \in \mathbf{X}$ and

$$\|\Lambda_{\mathbf{a}}\,\mathbf{x}\|_u \le C \;\; |\mathbf{a}\,\mathbf{b}^-|_\infty \; \|\Lambda_{\mathbf{b}}\,\mathbf{x}\|_u$$

i.e.

$$\mathbf{X}_{\mathrm{proj}}(\mathbf{b}) \hookrightarrow X_{\mathrm{proj}}(\mathbf{a}) \;.$$

$\Rightarrow$) There exists $\gamma > 0$ such that for all $\mathbf{x} \in \mathbf{X}_{\mathrm{proj}}(\mathbf{b})$

$$\|\Lambda_{\mathbf{a}}\,\mathbf{x}\|_u \le \gamma \;\; \|\Lambda_{\mathbf{b}}\,\mathbf{x}\|_u \;.$$

Since $\mathbf{X}_m \subset X_{\mathrm{proj}}(\mathbf{b})$ we find that $\mathbf{a}(m) \le \gamma\; \mathbf{b}(m)$ for all $m \in \mathbf{I}$. Hence $\mathbf{a} \le \mathbf{b}$. □

So for a Köthe set $\rho \subset \omega^+(\mathbf{I})$ the set $\{\mathbf{X}_{\mathrm{proj}}(\mathbf{a}) \mid \mathbf{a} \in \rho\}$ is a projective system.

Definition 4.3.
Let $\rho \subset \omega^+(\mathbf{I})$ be a Köthe set. The projective limit $\mathbf{X}_{\mathrm{proj}}(\rho)$ is defined by

$$\mathbf{X}_{\mathrm{proj}}(\rho) = \mathrm{limproj}_{\mathbf{a}\in\rho}\, \mathbf{X}_{\mathrm{proj}}(\mathbf{a}) \;.$$

$\mathbf{X}_{\mathrm{proj}}(\rho)$ is the space $\bigcap_{\mathbf{a}\in\rho} \mathbf{X}_{\mathrm{proj}}(\mathbf{a})$ together with the topology generated by the separating system of seminorms $\{p_{\mathbf{a}} \mid \mathbf{a} \in \rho\}$.

We have the following reformulation of Theorem 3.11.

Corollary 4.4.
Let $\rho \subset \omega^+(\mathbf{I})$ be a moulding set. Then

$$\mathbf{X}_{\mathrm{ind}}(\rho) \hookrightarrow \mathbf{X}_{\mathrm{proj}}(\rho^{\sharp}) \; .$$

Theorem 4.5.
Let $\rho, \sigma \subset \omega^{+}$ be Köthe sets. Then

(i) $\rho \lesssim \sigma$ iff $X_{\mathbf{proj}}(\sigma) \hookrightarrow X_{\mathbf{proj}}(\rho)$

(ii) $\rho \approx \sigma$ iff $X_{\mathbf{proj}}(\rho) = X_{\mathbf{proj}}(\sigma)$.

Proof.
The proof is an immediate consequence of Lemma 4.2 and is omitted. □

Remark.
If ρ is type 1, $\rho \approx \{\mathbf{a}\}$, then $\mathbf{X}_{\mathbf{proj}}(\rho)$ is a Banach space, $\mathbf{X}_{\mathbf{proj}}(\rho) = \mathbf{X}_{\mathbf{proj}}(\mathbf{a}^{-1})$. If ρ is type 2, then $\mathbf{X}_{\mathbf{proj}}(\rho)$ is a metrizable locally convex space.

Theorem 4.6.
Let $\rho \subset \omega^{+}(\mathbf{I})$ be a Köthe set. The space $X_{\mathbf{proj}}(\rho)$ is complete.

Proof.
Let $(\mathbf{x}_{\alpha})_{\alpha \in A}$ be a Cauchy net in $X_{\mathbf{proj}}(\rho)$. It means that $(\Lambda_{\mathbf{a}}\,\mathbf{x}_{\alpha})_{\alpha \in A}$ is a Cauchy net in the Banach space $\mathbf{X}$. Further for each $m \in \mathbf{I}$ the net $(P_m\,\mathbf{x}_{\alpha})_{\alpha \in A}$ is a Cauchy net in $\mathbf{X}_m$. So there exists $\mathbf{x}$ in $\times_{i \in \mathbf{I}} X_i$ such that

$$P_m\,\mathbf{x} = \lim_{\alpha}\; P_m\,\mathbf{x}_{\alpha} \; .$$

Let $\mathbf{a} \in \rho$. Then

$$\begin{aligned} \mathbf{a}(m)\; P_m\,\mathbf{x} \; &= \lim_{\alpha}\; \mathbf{a}(m)\; P_m\,\mathbf{x}_{\alpha} = \\ &= P_m(\lim_{\alpha}\; \Lambda_{\mathbf{a}}\,\mathbf{x}_{\alpha}) \; . \end{aligned}$$

Hence $\Lambda_{\mathbf{a}}\,\mathbf{x} \in \mathbf{X}$. □

Theorem 4.6'.
Let $\rho \subset \omega^{+}(\mathbf{I})$ be a moulding set and let $\mathbf{x} \in \times_{i \in \mathbf{I}} X_m$. Then $\mathbf{x} \in \mathbf{X}_{\mathbf{proj}}(\rho)$ iff for all $\mathbf{a} \in \rho$

$$\sup_{m \in \mathbf{I}}\; \mathbf{a}(m)\; \|\mathbf{x}(m)\|_u < \infty \; .$$

Proof.

$\Rightarrow$ Let $\mathbf{x} \in \mathbf{X}_{\mathbf{proj}}(\rho)$. Then $\Lambda_{\mathbf{a}}\,\mathbf{x} \in \mathbf{X}$ for all $\mathbf{a} \in \rho$, whence $\mathbf{a}(m)\; \|\mathbf{x}(m)\|_u = \|(\Lambda_{\mathbf{a}}\,\mathbf{x})\;(m)\|_u \leq C\;\; \|\Lambda_{\mathbf{a}}\,\mathbf{x}\|_u$.

$\Leftarrow$ Let for all $\mathbf{a} \in \rho$, $\sup_{m \in \mathbf{I}} \mathbf{a}(m) \, \|\mathbf{x}(m)\|_u < \infty$. Then for $\mathbf{b} \in \rho$, there exists $\mathbf{s} \in l_1^+(\mathbf{I})$ such that $\mathbf{b}\,\mathbf{s}^{-1} \in \rho$, whence $\sum_{m \in \mathbf{I}} \|(\Lambda_{\mathbf{b}}\,\mathbf{x})\,(m)\| \le \sup_{m \in \mathbf{I}} (\mathbf{b}(m)\,\mathbf{s}(m)^{-1}\,\|\mathbf{x}(m)\|_u) \sum_{j \in \mathbf{I}} \mathbf{s}(j) < \infty$.

We come to the main theorem of this section in which the bounded sets of $X_{\mathbf{proj}}(\rho)$ are characterized in case ρ is a moulding set.

Theorem 4.7.
Let $\rho \subset \omega^+(\mathbf{I})$ be a moulding set. Then

(i) For all $\mathbf{u} \in \rho^\sharp$ and all bounded subset B of $\mathbf{X}$ the set $\Lambda_{\mathbf{u}}\,B$ is bounded in $\mathbf{X}_{\mathbf{proj}}(\rho)$.

(ii) For a bounded subset W in $\mathbf{X}_{\mathbf{proj}}(\rho)$ there exists $\mathbf{u} \in \rho^\sharp$ and a bounded subset B of $\mathbf{X}$ such that $\Lambda_{\mathbf{u}}$ maps B homeomorphically onto W with respect to the relative topologies.

Proof.

(i) For all $\mathbf{x} \in \mathbf{X}$, $\mathbf{a} \in \rho$ and $\mathbf{u} \in \rho^\sharp$, $\Lambda_a(\Lambda_u\,\mathbf{x}) = \Lambda_{a \cdot u}\,\mathbf{x} \in \mathbf{X}$. So $\Lambda_u\,\mathbf{x} \in \mathbf{X}_{\mathbf{proj}}(\rho)$ and

$$(*) \qquad p_{\mathbf{a}}(\Lambda_{\mathbf{u}}\,\mathbf{x}) = \|\Lambda_{\mathbf{a}\cdot\mathbf{u}}\,\mathbf{x}\| \le C \;\; |\mathbf{a}\cdot\mathbf{u}|_\infty \, \|\mathbf{x}\|_u \,.$$

It follows that for each $B \subset \mathbf{X}$ bounded, the set $\Lambda_{\mathbf{u}}(B)$ is bounded in $\mathbf{X}_{\mathbf{proj}}(\rho)$.

(ii) Let W be a bounded subset of $\mathbf{X}_{\mathbf{proj}}(\rho)$. Let $\mathbf{r} \in l_1^+$ such that $\{\mathbf{r}\} \cdot \rho \approx \rho$. For $m \in \mathbf{I}$ put

$$\mathbf{u}(m) = \mathbf{r}(m)^{-1} \; \sup \, \{\|P_m\,\mathbf{x}\|_u \mid \mathbf{x} \in W\} \,.$$

Since ρ is separating, W bounded and P_m continuous on $\mathbf{X}_{\mathbf{proj}}(\rho)$ we have $\mathbf{u}(m) < \infty$ for all $m \in \mathbf{I}$. Let $\mathbf{a} \in \rho$. Since $\mathbf{a}\cdot\mathbf{r}^{-1} \lesssim \mathbf{b}$, there exist $M > 0$ such that $\|\Lambda_{a\cdot r^{-1}}\,\mathbf{x}\| \le M$ for all $\mathbf{x} \in W$, which by unconditionality gives $\mathbf{r}(m)^{-1}\,\mathbf{a}(m)\,\|P_m\,\mathbf{x}\|_u \le MC$ for all $m \in \mathbf{I}$ and $x \in W$. Hence $|\mathbf{a}\cdot\mathbf{u}|_\infty < \infty$, and so $u \in \rho^\sharp$.
Further, for all $\mathbf{x} \in W$ we have $\Lambda_{\mathbf{u}^-\mathbf{u}}\,\mathbf{x} = \mathbf{x}$ and $\|P_m(\Lambda_{\mathbf{u}^-}\,\mathbf{x})\|_u \le r(m)$, $\;m \in \mathrm{supp}(\mathbf{u})$.
Put $B = \{\Lambda_{\mathbf{u}^-}\,\mathbf{x} \mid \mathbf{x} \in W\}$. Then B is a bounded set in $\mathbf{X}$, the set W equals $\Lambda_{\mathbf{u}}(B)$ and $\Lambda_{\mathbf{u}}$ is a bijection from B onto W.
Now we prove that Λ_u is a homeomorphism.
By $(*)$ it follows that Λ_u maps B continuously onto W, and it remains to prove that Λ_{u^-} is continuous from W onto B. For any finite subset $\mathbb{F}$ of $\mathbf{I}$ we put $w_{\mathbb{F}} = u^-\,\chi_{\mathbb{F}}$ where $\chi_{\mathbb{F}}$ denotes the characteristic sequence of $\mathbb{F}$. Then $\{\mathbf{w}_{\mathbb{F}}\} \lesssim \rho$ and there exists $\mathbf{a}_{\mathbb{F}} \in \rho$ and $\lambda_{\mathbb{F}} > 0$ such that $\mathbf{w}_{\mathbb{F}} \le \lambda_{\mathbb{F}}\,\mathbf{a}_{\mathbb{F}}$. Let $\mathbf{x} \in W$ and let $\varepsilon > 0$. Take $\mathbb{F}$ such that $\sum_{i \in \mathbf{I}\setminus\mathbb{F}} r(i) < \varepsilon/2$. Then for all $\mathbf{y} \in W$ with $\|\Lambda_{\mathbf{a}_{\mathbb{F}}}(\mathbf{x}-\mathbf{y})\| < \varepsilon/2C\lambda_{\mathbb{F}}$ we have

$$
\begin{aligned}
\|\Lambda_{u^-}(\mathbf{x}-\mathbf{y})\| &\le \|\Lambda_{\mathbf{w}_F}(x-y)\| + \sum_{m\in \operatorname{supp}\mathbf{u}\setminus F} u(m)^{-1}\,\|P_m(\mathbf{x}-\mathbf{y})\|_u \\
&\le C\,\lambda_F\,\|\Lambda_{\mathbf{a}_F}(\mathbf{x}-\mathbf{y})\| + \sum_{m\in \mathbf{I}\setminus F} \mathbf{r}(m) < \varepsilon\ .
\end{aligned}
$$

□

Corollary 4.8.
Let $\rho \subset \omega^+(\mathbf{I})$ be a moulding set and let $\mathbf{x} \in \times_{i\in\mathbf{I}} X_i$. Then $\mathbf{x} \in \mathbf{X}_{\mathbf{proj}}(\rho)$ iff there exists $\mathbf{u} \in \rho^\sharp$ and $\mathbf{y} \in \mathbf{X}$ such that $\mathbf{x} = \Lambda_{\mathbf{u}}\, y$.
Put differently, $\mathbf{X}_{\mathbf{proj}}(\rho) = \mathbf{X}_{\mathbf{ind}}(\rho^\sharp)$ as sets!

Corollary 4.9.
Let $\rho \subset \omega^+(\mathbf{I})$ be a moulding set and $(\mathbf{x}_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbf{X}_{\mathbf{proj}}(\rho)$. Then the sequence converges to zero in $\mathbf{X}_{\mathbf{proj}}(\rho)$ iff there exist $u \in \rho^\sharp$ and a null sequence $(\mathbf{y}_n)_{n\in\mathbb{N}}$ in $\mathbf{X}$ such that $\mathbf{x}_n = \Lambda_u\, \mathbf{y}_n\ ,\quad n \in \mathbb{N}$.

Corollary 4.10.
Let $\rho \subset \omega^+(\mathbf{I})$ be a moulding set and let W denote a subset of $\mathbf{X}_{\mathbf{proj}}(\rho)$. Then W is compact iff there exists $u \in \rho^\sharp$ and a compact subset K of $\mathbf{X}$ such that $W = \Lambda_{\mathbf{u}}(K)$.

Theorem 4.11.
Let $\rho \subset \omega^+(\mathbf{I})$ be a moulding set. Then the space $\mathbf{X}_{\mathbf{proj}}(\rho)$ is semi-Montel iff $\forall_{m\in\mathbf{I}}\ :\ \dim(X_m) < \infty$, i.e. each P_m is of finite rank.

Proof.

$\Leftarrow$) Let B_m denote the closed unit ball in $\mathbf{X}_m$. Then B_m is bounded and closed in $X_{\mathbf{proj}}(\rho)$ and therefore a compact subset of $X_{\mathbf{proj}}(\rho)$. Hence B_m is compact in $\mathbf{X}_m$ and so $\dim(\mathbf{X}_m) < \infty$ (since $\mathbf{X}_m$ is a Banach space).

$\Rightarrow$) Let W denote a closed and bounded subset of $\mathbf{X}_{\mathbf{proj}}(\rho)$. Let $\mathbf{r} \in l_1^+$ be such that $\{\mathbf{r}\}\cdot\rho \approx \rho$. There exists a bounded and closed subset B of $\mathbf{X}$ such that $\Lambda_u\ :\ B \to W$ is a homeomorphism. Put $K = \Lambda_{\mathbf{r}}(B)$. Then K is compact since $\Lambda_{\mathbf{r}}$ is a compact operator from $\mathbf{X}$ into $\mathbf{X}$,

$$
\Lambda_{\mathbf{r}} = \sum_{m\in\mathbf{I}} \mathbf{r}(m)\ P_m \quad , \quad P_m \text{ finite rank .}
$$

Let $\tilde{\mathbf{u}} = \mathbf{u}\cdot\mathbf{r}^{-1}$. Then $\tilde{\mathbf{u}} \in \rho^\sharp$ and $W = \Lambda_{\tilde{\mathbf{u}}}(K)$. Since $\Lambda_{\tilde{\mathbf{u}}}\ :\ \mathbf{X} \to \mathbf{X}_{\mathbf{proj}}(\rho)$ is continuous, W is compact. □

§5. Symmetric sequence sets

We recall that a Köthe set $\rho \subset \omega^+(\mathbf{I})$ is said to be symmetric if $\rho \approx \rho^{\sharp\sharp}$. In this section, we describe topological properties of $\mathbf{X}_{\mathbf{ind}}(\rho)$ and $\mathbf{X}_{\mathbf{proj}}(\rho)$ for symmetric moulding sets ρ additional to the properties presented in the preceding sections. In fact for moulding sets ρ the symmetry condition turns out equivalent with a number of topological conditions on the spaces $\mathbf{X}_{\mathbf{ind}}(\rho)$ or $\mathbf{X}_{\mathbf{proj}}(\rho)$. First we present an auxiliary result.

Lemma 5.1.
Let $\rho \subset \omega^+(\mathbf{I})$ be a moulding set. Then $\mathbf{X}_{\mathbf{proj}}(\rho^\sharp) = \mathbf{X}_{\mathbf{ind}}(\rho^{\sharp\sharp})$ and $\mathbf{X}_{\mathbf{ind}}(\rho^\sharp) = \mathbf{X}_{\mathbf{proj}}(\rho^{\sharp\sharp})$ as topological vector spaces.

Proof.
It follows from Corollary 4. that $\mathbf{X}_{\mathbf{proj}}(\rho^\sharp) = \mathbf{X}_{\mathbf{ind}}(\rho^{\sharp\sharp})$ as sets. The topology of $\mathbf{X}_{\mathbf{proj}}(\rho^\sharp)$ is brought about by the seminorm $\mathbf{x} \mapsto \|\Lambda_{\mathbf{u}}\,\mathbf{x}\|$, $u \in \rho^\sharp$. Since $\rho^{\sharp\sharp\sharp} = \rho^\sharp$ the result follows from Theorem 3.12. Further, observe that $\mathbf{X}_{\mathbf{ind}}(\rho^\sharp) = \mathbf{X}_{\mathbf{ind}}(\rho^{\sharp\sharp\sharp}) = \mathbf{X}_{\mathbf{proj}}(\rho^{\sharp\sharp})$. □

This lemma has the following immediate consequence.

Theorem 5.2.
Let $\rho \subset \omega^+(\mathbf{I})$ be a moulding. Then the following statements are equivalent

(i) ρ is symmetric.

(ii) $\mathbf{X}_{\mathbf{ind}}(\rho) = \mathbf{X}_{\mathbf{ind}}(\rho^{\sharp\sharp})$ as topological vector spaces.

(iii) $\mathbf{X}_{\mathbf{proj}}(\rho) = \mathbf{X}_{\mathbf{proj}}(\rho^{\sharp\sharp})$ as topological vector spaces.

(iv) $\mathbf{X}_{\mathbf{ind}}(\rho) = \mathbf{X}_{\mathbf{proj}}(\rho^\sharp)$ as topological vector spaces.

(v) $\mathbf{X}_{\mathbf{proj}}(\rho) = \mathbf{X}_{\mathbf{ind}}(\rho^\sharp)$ as topological vector spaces.

Proof.
The implications (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iv) and (i) $\Rightarrow$ (iii) $\Leftrightarrow$ (v) follow from Theorem (3.7) and Lemma (5.1).
For (ii) $\Rightarrow$ (i) let $\mathbf{b} \in \rho^{\sharp\sharp}$ and let $\mathbf{x} \in \mathbf{X}$ with $\|\mathbf{x}(m)\| = \mathbf{r}(m)$, $m \in \mathbf{I}$, where $\mathbf{r} \in l_1^+(\mathbf{I})$ is such that $\{\mathbf{r}\} \cdot \rho \approx \rho$. (We may assume that the Banach spaces X_m are non trivial.)
There exists $\mathbf{x_0} \in \mathbf{X}$ and $\mathbf{a} \in \rho$ such that $\Lambda_{\mathbf{b}}\,\mathbf{x} = \Lambda_{\mathbf{a}}\,\mathbf{x_0}$. Hence for all $m \in \mathbf{I}$, $(\Lambda_{\mathbf{b}}\,\mathbf{x})\ (m) = (\Lambda_{\mathbf{a}}\,\mathbf{x_0})\ (m)$ so that

$$\mathbf{b}(m)\ \mathbf{r}(m) \leq C\ \mathbf{a}(m)\ \ \|\mathbf{x_0}\|_u\ ,$$

i.e. $\{\mathbf{b}\} \lesssim \rho$.
For the implication (iii) $\Rightarrow$ (i), let $\mathbf{b} \in \rho^{\sharp\sharp}$. Then the seminorm $\mathbf{x} \mapsto \|\Lambda_{\mathbf{b}}\,\mathbf{x}\|_u$ is continuous on $X_{\mathbf{proj}}(\rho)$. It follows that there exists $A > 0$ and $\mathbf{a} \in \rho$ such that for all $\mathbf{x} \in X_{\mathbf{proj}}(\rho)$

$$\|\Lambda_b\,\mathbf{x}\|_u \leq A\ \ \|\Lambda_a\,\mathbf{x}\|_u\ ,$$

or, equivalently $\mathbf{b} \lesssim \mathbf{a}$. □

Theorem 5.3.
Let $\rho \subset \omega^+(\mathbf{I})$ be a moulding set. Then the following are equivalent

(i) ρ is symmetric.

(ii) $\mathbf{X}_{\mathbf{ind}}(\rho)$ is complete.

(iii) $\mathbf{X}_{\mathbf{proj}}(\rho)$ is barreled.

(iv) $\mathbf{X}_{\mathbf{proj}}(\rho)$ is bornological.

Proof.
The implications (i) $\Rightarrow$ (ii), (i) $\Rightarrow$ (iii) and (i) $\Rightarrow$ (iv) follow from Theorem (3.8), (4.6) and (5.2).

(ii) $\Rightarrow$ (i) Suppose $\mathbf{X}_{\mathbf{ind}}(\rho)$ is complete. Let $\mathbf{b} \in \rho^{\sharp\sharp}$ and let $\mathbf{x} \in \mathbf{X}$. The set $\mathcal{F}(\mathbf{I})$ of all finite subsets of $\mathbf{I}$ is a directed set under the ordering by inclusion. For each $\mathbb{F} \in \mathcal{F}(\mathbf{I})$ define $\mathbf{x}_{\mathbb{F}} \in X$ by

$$\mathbf{x}_{\mathbb{F}} = \sum_{m \in \mathbb{F}} P_m \, \mathbf{x} \, .$$

Then $(\Lambda_{\mathbf{b}} \, \mathbf{x}_{\mathbb{F}})_{\mathbb{F} \in \mathcal{F}(\mathbf{I})}$ is a Cauchy net in $\mathbf{X}_{\mathbf{ind}}(\rho)$, because for all $\mathbf{u} \in \rho^{\sharp}$ the net $(\Lambda_{\mathbf{u}}(\Lambda_{\mathbf{b}} \, \mathbf{x}_{\mathbb{F}}))_{\mathbb{F} \in \mathcal{F}(\mathbf{I})}$ is a Cauchy net in $\mathbf{X}$ and hence its limit $\Lambda_{\mathbf{b}} \, \mathbf{x}$ belongs to $X_{\mathbf{ind}}(\rho)$. We conclude that $X_{\mathbf{ind}}(\rho^{\sharp\sharp}) \subset X_{\mathbf{ind}}(\rho)$ and so $\rho^{\sharp\sharp} \lesssim \rho$.

(iii) $\Rightarrow$ (i) Let $b \in \rho^{\sharp\sharp}$. Then the set

$$W = \{\mathbf{x} \in \mathbf{X}_{\mathbf{proj}}(\rho) \mid \sup_{m \in \mathbf{I}} [\mathbf{b}(m) \, \|P_m \, \mathbf{x}\|] \leq 1\}$$

is a barrel, i.e. W is a closed, convex, absorbing and balanced subset of $X_{\mathbf{proj}}(\rho)$. So there exists $\mathbf{a} \in \rho$ such that

$$W \supset \{\mathbf{x} \in \mathbf{X}_{\mathbf{proj}}(\rho) \mid \|\Lambda_{\mathbf{a}} \, \mathbf{x}\| \leq 1\}$$

whence

$$\exists_{M>0} \; \forall_{\mathbf{x} \in X_{\mathbf{proj}}(\rho)} \; \forall_{m \in \mathbf{I}} \; : \; \mathbf{b}(m) \, \|P_m \, \mathbf{x}\| \leq M \|\Lambda_{\mathbf{a}} \, \mathbf{x}\|$$

or $\mathbf{b} \lesssim \mathbf{a}$.

(iv) $\Rightarrow$ (i) Let $\mathbf{b} \in \rho^{\sharp\sharp}$ and consider the convex and balanced set

$$W = \{\mathbf{x} \in \mathbf{X}_{\mathbf{proj}}(\rho) \mid \|\Lambda_{\mathbf{b}} \, \mathbf{x}\| \leq 1\}$$

which absorbs every bounded subset of $X_{\mathbf{proj}}(\rho)$. Hence W is a neighbourhood of $\mathbf{a}$ and similarly as in the proof of (iii) $\Rightarrow$ (i) we obtain $\mathbf{a} \in \rho$ with $\mathbf{b} \lesssim \mathbf{a}$. □

As a further consequence we mention

Corollary 5.4.
Let ρ be a symmetric moulding set. Then the following are equivalent

(i) Each Banach space X_m is finite dimensional.

(ii) $\mathbf{X}_{\mathbf{ind}}(\rho)$ is semi-Montel.

(iii) $\mathbf{X}_{\mathbf{proj}}(\rho)$ is semi-Montel.

Corollary 5.5.
Let ρ be a symmetric moulding set.

(i) A sequence $(\mathbf{x}_n)_{n\in\mathbb{N}}$ in $\mathbf{X}_{\mathbf{ind}}(\rho)$ converges to zero iff there is $\mathbf{a} \in \rho$ such that $(\mathbf{x}_n)_{n\in\mathbb{N}}$ converges to zero in the Banach space $\mathbf{X}_{\mathbf{ind}}(a)$.

(ii) A set W in $\mathbf{X}_{\mathbf{ind}}(\rho)$ is bounded iff there is $\mathbf{a} \in \rho$ such that W is a bounded subset of the Banach space $\mathbf{X}_{\mathbf{ind}}(\mathbf{a})$.

(iii) A set K in $\mathbf{X}_{\mathbf{ind}}(\rho)$ is compact iff there is $\mathbf{a} \in \rho$ such that K is a compact subset of the Banach space $\mathbf{X}_{\mathbf{ind}}(\mathbf{a})$.

Remark.
Although $\mathbf{X}_{\mathbf{ind}}(\rho)$ is not a strict inductive limit, in general, it very much behaves like one if ρ is a symmetric moulding set. The inductive limit is said to be regular.
If ρ is type 2, then $\mathbf{X}_{\mathbf{proj}}(\rho)$ is a Frechet space and $\mathbf{X}_{\mathbf{ind}}(\rho)$ is a so called DF-space, cf. [Sch], p.88 DF-spaces are regular inductive limits. This result has already been obtained by Grothendiek, [Gr]. By Theorem 2.9 and Theorem 5.2 the same result has been derived.
Also by Theorem 2.11 if ρ is type 2 then $\rho^\sharp$ is type 3 and so the inductive limit topology of $\mathbf{X}_{\mathbf{ind}}(\rho)$ is not metrizable. The aforementioned statement reflects the classical result that a countable inductive limit of Banach spaces is not a Frechet space unless it is a Banach space.

§6. The dual of $\mathbf{X}_{\mathbf{ind}}(\rho)$ and $\mathbf{X}_{\mathbf{proj}}(\rho)$

In this section we develop a description of the duals of the inductive limits $\mathbf{X}_{\mathbf{ind}}(\rho)$ and the projective limits $\mathbf{X}_{\mathbf{proj}}(\rho)$ in the terminology of the preceding sections. When ρ is a moulding set the fact that the topology of both cases is induced by seminorms allows us to obtain a satisfactory representation of the dual.
Recall that we have a minimal collection of closed subspaces X_i, $i \in \mathbf{I}$, of a Banach space $\mathcal{X}$ and from these we construct an unconditional Schauder system $(\mathbf{X}, \|\ \|_u)$ where $\mathbf{X} = \{\mathbf{x} \in \times_{i\in\mathbf{I}} X_i \mid \sum_{m\in\mathbf{I}} E_m \mathbf{x}$ converges unconditionally$\}$. We now construct an unconditional Schauder system on the dual space $\mathcal{X}'$ using the duals of the spaces $\{X_i\}$.

Definition 6.1.
Let X'_m denote the dual of X_m, $m \in \mathbf{I}$. On X'_m we impose the norm topology induced from X_m given by the norm

$$\|f\|'_m = \sup\ \{f(x) \mid x \in X_m,\ \|x\| = 1\}\ ,\quad f \in X'_m\ .$$

When we give X'_m this topology we write $(X'_m,\ \|\ \|'_m)$.

Definition 6.2.
Let $\mathbf{X}'$ denote the dual of $\mathbf{X}$ and impose on $\mathbf{X}'$ the usual norm topology induced by $\mathbf{X}$, i.e.

$$\|f\|'_u = \sup\ \{|f(x)| \mid \mathbf{x} \in \mathbf{X},\ \|\mathbf{x}\|_u = 1\}\ .$$

We can identify the elements of X'_m with a subspace of $\mathbf{X}'$ as follows: define $i_m\ :\ X'_m \to \mathbf{X}'$ by

$$i_m(f) = f \cdot E_m \quad ,\quad f \in X'_m\ .$$

It is clear that the identification is unique.

Considered as a subspace of $\mathbf{X}'$ we can impose on X'_m the induced topology and when we do this we write $(X'_m,\ \|\ \|'_{u,m})$.
We have the following proposition

Proposition 6.3.

$$(X'_m,\ \|\ \|'_m)\ \text{ is isomorphic to }\ (X'_m,\ \|\cdot\|'_{u,m})\ .$$

Proof.
The mapping i_m defined above is $1-1$, linear and onto. So all we have to show is that the norms are equivalent.

– Let $f \in (X'_m,\ \|\cdot\|'_m)$. Then

$$\|i_m(f)\|'_{u,m} = \|f \circ E_m\|'_{u,m} \le \|f\|'_m\ \|E_m\| \le C\ \|f\|'_m\ .$$

– Let $f \in (X'_m, \|\ \|'_{u,m})$. Then $f = i_m(\tilde{f})$ for some $\tilde{f} \in (X'_m, \|\ \|'_m)$ and

$$\begin{aligned} \|\tilde{f}\|'_m &= \sup \{|\tilde{f}(x)| \mid x \in X_m, \ \|x\| = 1\} = \\ &= \sup \{|\tilde{f}(E_m \mathbf{x})| \mid \mathbf{x} \in \mathbf{X}, \ \|E_m \mathbf{x}\| = 1\} \\ &\leq \sup \{|\tilde{f}(E_m \mathbf{x})| \mid \mathbf{x} \in \mathbf{X}, \ \|\mathbf{x}\|_u = 1\} \ . \end{aligned}$$

Hence (with a slight abuse of notation)

$$\|f\|'_m \leq \|f\|'_{u,m} \leq C \ \|f\|'_m \ .$$

So X'_m is a closed subspace of X' and $(X'_m, \|\ \|'_{u,m})$ is a Banach space. □

We are in danger here of drowning in our own notation, since there are so many identifications going on. However the preceding proposition allows us to simplify and henceforth we shall regard X'_m as the closed subspace of $\mathbf{X}'$ consisting of all linear functionals of the form

$$f \circ P_m \quad , \quad f \in \mathbf{X}'$$

where P_m is the projection of $\mathbf{X}$ onto $\mathbf{X}_m$.

Lemma 6.4.
Let $f \in X'$. Then there exists a unique sequence $\{f_m\}_{m \in \mathbf{I}}$, $f_m \in X'_m$ such that for all $\mathbf{x} \in \mathbf{X}$

$$f(x) = \sum_{m \in \mathbf{I}} f_m(x) \ .$$

Proof.
Let $f \in \mathbf{X}'$ and let $\mathbf{x} \in \mathbf{X}$. Then

$$f(\mathbf{x}) = \sum_{m \in \mathbf{I}} (f \circ P_m)\ (\mathbf{x}) \ .$$

So with $f_m = f \circ P_m$ existence of the sequence has been proved.
Now suppose

$$f(\mathbf{x}) = \sum_{m \in \mathbf{I}} g_m(\mathbf{x}) \ , \quad \mathbf{x} \in \mathbf{X} \ ,$$

where $g_m \in X'_m$. Then for $m' \in \mathbf{I}$

$$(f \circ P_{m'})\ (x) = f(P_{m'}\, x) = g_{m'}(x) \ .$$

Hence the result. □

Corollary 6.5.
The collection of Banach spaces $\{X'_m\}_{m\in\mathbf{I}}$ establishes a minimal collection in the Banach space $\mathbf{X}'$.

Proof.
We have to show that for each $m \in \mathbf{I}$

$$X'_m \cap \overline{< \{\mathbf{X}'_i \mid i \neq m\} >} = \{0\} \; .$$

Since the norm closure of a subspace of a Banach space equals its weak $*$-closure the result is a straightforward consequence of Lemma 4.4. □

Having a minimal collection we can introduce projective and inductive limits. For this we introduce some notation.

Definition 6.6.
We define the space $\mathbf{X}^+ \subset \mathbf{X}'$ by

$$\mathbf{X}^+ = \{\mathbf{f} \in \times_{i\in\mathbf{I}} X'_i \mid \mathbf{f} \text{ is unconditionally summable in } \mathbf{X}'\}$$

and we impose on $\mathbf{X}^+$ the topology given by the unconditional norm $\|\ \|_u^+$,

$$\|\mathbf{f}\|_u^+ = \sup_{\boldsymbol{F}\in\mathcal{F}(\mathbf{I})} \|\sum_{m\in\boldsymbol{F}} \mathbf{f}(m)\|'_u \; .$$

Remark.
Since the elements of $\mathbf{X}$ are sums of unconditionally summable sequences in $\times_{i\in\mathbf{I}} \mathbf{X}_i$, the norm $\|\ \|_u^+$ on $\mathbf{X}^+$ is equivalent to the dual norm $\|\cdot\|'_u$.
So we are in the same position as in the beginning of Section 2 and for each Köthe set $\rho \subset \omega^+(\mathbf{I})$ we can introduce the inductive limit $\mathbf{X}^+_{\text{ind}}(\rho)$ and the projective limit $\mathbf{X}^+_{\text{proj}}(\rho)$. Herefor we must replace X_m by X'_m and $\mathbf{X}$ by $\mathbf{X}^+$.

We want to prove that for moulding sets ρ, $\mathbf{X}^+_{\text{ind}}(\rho)$ and $\mathbf{X}^+_{\text{proj}}(\rho)$ represent the duals of $\mathbf{X}_{\text{proj}}(\rho)$ and $X_{\text{ind}}(\rho)$, respectively. By $< \cdot, \cdot >_m$ we denote the duality pairing of X_m of X'_m.

Proposition 6.7.

(i) Let $\mathbf{x} \in \mathbf{X}_{\text{ind}}(\rho)$ and $\mathbf{f} \in \mathbf{X}^+_{\text{proj}}(\rho)$. Then the sequence $(< \mathbf{x}(m), \mathbf{f}(m) >_m)_{m\in\mathbf{I}}$ is absolutely summable.

(ii) Let $\mathbf{x} \in \mathbf{X}_{\text{proj}}(\rho)$ and $\mathbf{f} \in \mathbf{X}^+_{\text{ind}}(\rho)$. Then the sequence $(< \mathbf{x}(m), \mathbf{f}(m) >_m)_{m\in\mathbf{I}}$ is absolutely summable.

Proof.

(i) There exists $\mathbf{a} \in \rho$ and $\tilde{\mathbf{x}} \in \mathbf{X}$ such that

$$\mathbf{x}(m) = \mathbf{a}(m)\, \tilde{\mathbf{x}}(m) .$$

Now by definition of $\mathbf{X}^+_{\mathbf{proj}}(\rho)$, the sequence $\{\mathbf{a}(m)\ \mathbf{f}(m)\}_{m\in\mathbf{I}}$ belongs to $\mathbf{X}^+$. Hence

$$< \mathbf{x}(m),\ \mathbf{f}(m) >_m = < \tilde{\mathbf{x}}(m),\ \mathbf{a}(m)\ \mathbf{f}(m) >_m , \quad m \in \mathbf{I} ,$$

and the result follows.

(ii) Similarly. □

We have the following definition.

Definition 6.8.
Let ρ be a Köthe set

(i) On the product $\mathbf{X}_{\mathbf{ind}}(\rho) \times \mathbf{X}^+_{\mathbf{proj}}(\rho)$ we define the pairing $< \cdot, \cdot >_{ip}$,

$$< \mathbf{x}, \mathbf{f} >_{ip} = \sum_{m\in\mathbf{I}} < \mathbf{x}(m),\ \mathbf{f}(m) >_m .$$

(ii) On the product $X_{\mathbf{proj}}(\rho) \times X_{\mathbf{ind}}(\rho)$ we define the pairing $< \cdot, \cdot >_{pi}$,

$$(\mathbf{x}, \mathbf{f}) = \sum_{m\in\mathbf{I}} < \mathbf{x}(m),\ \mathbf{f}(m) >_m .$$

Theorem 6.9.
Let ρ be a Köthe set.

(i) For each $\mathbf{f} \in \mathbf{X}^+_{\mathbf{proj}}(\rho)$ the linear functional

$$\mathbf{x} \mapsto < \mathbf{x}, \mathbf{f} >_{ip} \quad , \quad x \in \mathbf{X}_{\mathbf{ind}}(\rho)$$

is continuous.

(ii) For each $\mathbf{f} \in \mathbf{X}^+_{\mathbf{ind}}(\rho)$ the linear functional

$$\mathbf{x} \mapsto < \mathbf{x}, \mathbf{f} >_{pi} \quad , \quad \mathbf{x} \in \mathbf{X}_{\mathbf{proj}}(\rho)$$

is continuous.

Proof.

(i) Let $f \in \mathbf{X}^+_{\mathbf{proj}}(\rho)$ and let l denote the linear functional

$$l(\mathbf{x}) = <\mathbf{x}, f>_{pi} \quad , \quad x \in \mathbf{X}_{\mathbf{ind}}(\rho) \, .$$

We have to prove that $l\Big|_{\mathbf{X}_{\mathbf{ind}}(\mathbf{a})}$ is continuous for each $\mathbf{a} \in \rho$. But this follows from the inequality

$$|<\Lambda_{\mathbf{a}}\tilde{\mathbf{x}}, \ \mathbf{f}>_{ip}| \le \|\tilde{\mathbf{x}}\|_u \ \|\Lambda'_{\mathbf{a}}\mathbf{f}\|'_u \, , \quad \tilde{\mathbf{x}} \in \mathbf{X} \, .$$

(ii) Let $f \in \mathbf{X}^+_{\mathbf{ind}}(\rho)$. Then there is $\mathbf{a} \in \rho$ and $\tilde{\mathbf{f}} \in \mathbf{X}^+$ such that

$$f(m) = \mathbf{a}(m) \ \tilde{f}(m) \quad , \quad m \in \mathbf{I} \, .$$

So

$$|<\mathbf{x}, \mathbf{f}>| \le \|\Lambda_a \mathbf{x}\|_u \ \|\tilde{f}\|'_u \, , \quad \mathbf{x} \in \mathbf{X}_{\mathbf{proj}}(\rho)$$

and the result follows. □

Theorem 6.10.
Let ρ be a moulding set in $\omega^+(\mathbf{I})$.
A linear functional l on $\mathbf{X}_{\mathbf{ind}}(\rho)$ is continuous iff there is $f \in \mathbf{X}^+_{\mathbf{proj}}(\rho)$ such that

$$l(\mathbf{x}) = <\mathbf{x}, f>_{ip} \ .$$

Proof.
One side of the equivalence is a consequence of the preceding theorem.
So let $l \in \mathbf{X}_{\mathbf{ind}}(\rho)'$. Then for each $\mathbf{a} \in \rho$ there is a sequence $\mathbf{f}_a \in \times_{m\in\mathbf{I}} X'_m$ such that

$$(*) \qquad l(\Lambda_{\mathbf{a}}\tilde{\mathbf{x}}) = \sum_{m\in\mathbf{I}} <\tilde{\mathbf{x}}(m), \ \mathbf{f}_a(m)>_m \, , \quad \tilde{\mathbf{x}} \in \mathbf{X}$$

(cf. Lemma 4.4).
Further, for each $m \in \mathbf{I}$ there is $\tilde{\mathbf{f}}(m) \in \mathbf{X}_m$ such that

$$(**) \qquad l(P_m\mathbf{x}) = <\mathbf{x}(m), \ \tilde{\mathbf{f}}(m)>_m \, , \quad \mathbf{x} \in \mathbf{X}_{\mathbf{ind}}(\rho)$$

since $\mathbf{X}_m \hookrightarrow \mathbf{X}_{\mathbf{ind}}(\rho)$.
It follows from $(*)$ and $(**)$ that $\mathbf{f}_a(m) = \mathbf{a}(m) \ \tilde{\mathbf{f}}(m)$.
Now $\tilde{\mathbf{f}} \in X^+_{\mathbf{proj}}(\rho)$ since

$$\sup_{m\in\mathbf{I}} \ \mathbf{a}(m) \ \|\tilde{\mathbf{f}}(m)\|'_u =$$

$$= \sup_{m\in\mathbf{I}} \ \|l \cdot \Lambda_a \circ P_m\|'_u \le C \ \|l \cdot \Lambda_a\|$$

applying Theorem (4.5'). □

Theorem 6.11.
Let ρ be a moulding set in $\omega^+(\mathbf{I})$.
A linear functional m on $\mathbf{X}_{\mathbf{proj}}(\rho)$ is continuous iff there is $g \in \mathbf{X}^+_{\mathbf{ind}}(\rho)$ such that

$$k(\mathbf{x}) = < \mathbf{x}, g >_{pi} \quad .$$

Proof.
One side of the equivalence being a consequence of Theorem 5.9 we only have to show that each $k \in \mathbf{X}_{\mathbf{proj}}(\rho)'$ corresponds to a $g \in \mathbf{X}^+_{\mathbf{ind}}(\rho)$ as indicated.
So let $k \in \mathbf{X}_{\mathbf{ind}}(\rho)'$. Then there are $D > 0$ and $a \in \rho$ such that

$$|k(x)| \leq D \; \|\Lambda_a(x)\|_{uc} \, .$$

The space $\mathbf{X}_a = \{\Lambda_a(\mathbf{x}) \mid \mathbf{x} \in \mathbf{X}_{\mathbf{proj}}(\rho)\}$ is a subspace of $\mathbf{X}$. Define the linear functional $\mathbf{a}$ on $\mathbf{X}_a$ by

$$k_a(\Lambda_a(\mathbf{x})) = k(\mathbf{x}) \quad , \quad \mathbf{x} \in \mathbf{X}_{\mathbf{a}} \, .$$

Then for all $\mathbf{y} \in X_a$ we have

$$k_a(h) \leq C \; \|\mathbf{y}\|_u \, .$$

By Hahn-Banach, k_a extends to a continuous linear functional on X and so there exists a sequence $\mathbf{f} \in \times_{m\in\mathbf{I}} X'_m$ such that for all $\mathbf{y} \in \mathbf{X}_a$

$$k_a(\mathbf{y}) = \sum_{m\in\mathbf{I}} < \mathbf{y}(m), \; \mathbf{f}(m) >_m$$

or, equivalently, for all $x \in X_{\mathbf{proj}}(\rho)$

$$k(x) = \sum_{m\in\mathbf{I}} \mathbf{a}(m) \; < \mathbf{x}(m), \; \mathbf{f}(m) >_m \quad .$$

We show that $\Lambda_a \mathbf{f} \in \times_{m\in\mathbf{I}} X'_m$ belongs to $\mathbf{X}^+_{\mathbf{ind}}(\rho)$.
First observe that $\|f(m)\| \leq C \; \|k_a\|$. Since ρ is moulding there exists $\mathbf{r} \in l^+_1(\mathbf{I})$ and $\mathbf{b} \in \rho$ such that $\mathbf{a} = \mathbf{b} \cdot \mathbf{r}$. It follows that

$$\Lambda_a \; = \Lambda_b(\Lambda_r \mathbf{f})$$

and $\Lambda_r f \in X^+$ because

$$\sum_{m\in\mathbf{I}} \mathbf{r}(m) \; \|\mathbf{f}(m)\| \leq C \; \|k_a\| \sum_{m\in\mathbf{I}} \mathbf{r}(m) \, .$$

So $g = \Lambda_a \mathbf{f} \in \mathbf{X}^+_{\text{ind}}(\rho)$ and

$$k(\mathbf{x}) = < \mathbf{x}, g >_{pi} \ .$$

□

For a moulding set we have characterized the bounded set of $\mathbf{X}_{\text{proj}}(\rho)$ and hence of $\mathbf{X}^+_{\text{proj}}(\rho)$. This characterization yield the following result.

Theorem 6.12.
Let ρ be a moulding set. The strong topology $\beta(\mathbf{X}_{\text{ind}}(\rho),\ \mathbf{X}^+_{\text{proj}}(\rho))$ is equivalent to the inductive limit topology on $\mathbf{X}_{\text{ind}}(\rho)$.

Proof.
As we have shown in Theorem 4.7 a set $V \subset \mathbf{X}^+_{\text{proj}}(\rho)$ is bounded if and only if there exists a bounded subset V_0 of X^+ and $u \in \rho^\sharp$ such that $V = \{\Lambda_u(f) \mid f \in V_0\}$. It follows that

$$\begin{aligned} \sup_{g \in V} | < x, g >_{ip} | &= \sup_{f \in V_0} | < \Lambda_u x, f > | \\ &\leq (\sup_{f \in V_0} \|f\|_{uc}) \ \|\Lambda_u x\|_{uc} \ . \end{aligned}$$

Hence the seminorm $x \mapsto \sup_{g \in V} \ | < x, g >_{ip} |$ is continuous with respect to the inductive limit topology.
Conversely, denoting by E^+ the unit ball in X^+ we get

$$\begin{aligned} \|\Lambda_u x\|_{uc} &= \sup_{f \in E^+} | < \Lambda_u , f > | \\ &= \sup_{g \in \Lambda_u(E^+)} | < x, g > | \ . \end{aligned}$$

□

Corollary 6.13.
Let ρ be a symmetric moulding set. Then the strong topology $\beta(\mathbf{X}_{\text{proj}}(\rho),\ \mathbf{X}^+_{\text{ind}}(\rho))$ is equivalent to the projective limit topology on $\mathbf{X}_{\text{proj}}(\rho)$.

Proof.
For symmetric moulding set $\mathbf{X}_{\text{proj}}(\rho) = \mathbf{X}_{\text{ind}}(\rho^\sharp)$ and $\mathbf{X}^+_{\text{ind}}(\rho) = \mathbf{X}^+_{\text{proj}}(\rho^\sharp)$. Cf. Theorem 5.2.
□

Theorem 6.14.
Let ρ be a symmetric moulding set. Then both $\mathbf{X}_{\text{ind}}(\rho)$ and $\mathbf{X}_{\text{proj}}(\rho)$ are reflexive locally convex spaces if each Banach space X_m is reflexive.
In particular, if each X_m is finite dimensional.

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