# Shoulder design for packaging machines

# SHOULDER DESIGN FOR PACKAGING MACHINES

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## 1. Introduction

In this report, we deal with the design of a part of a packaging machine called "shoulder". In these machines, the packaging material (paper or plastic sheet) is unrolled from a horizontal cylinder and folded against the inner side of a vertical, hollow cylinder. During this folding process, the sheet passes over a curved surface, the shoulder, which is attached to the vertical cylinder. In Fig. 1 an example of the geometry is drawn. After a piece of sheet has been positioned inside the vertical cylinder, it is sealed at the bottom and at the side and filled by dropping the product to be packed from above into the newly formed bag. Then, the bag is drawn downwards, sealed at the top and cut off. This technique allows for packaging at high speed (hundreds of bags per minute), but is sensitive for disturbances if the packaging sheet is not guided over the shoulder in the appropriate way. The curvature of the shoulder should be such, that the sheet is nowhere stretched or torn. In the literature a mathematical discription of possible surfaces is given by Mot [1-4] and Culpin [5]. The former author constructs the shoulder out of pieces of a plane and a cone. The reliability of this approximation is not discussed by this author and is certainly not clear from a theoretical point of view. Culpin has solved the problem exactly. The solution in the present report is not essentially different from his approach. The presentation, however, is more straightforward and also deals with practical aspects of the calculation of shoulders.

In §2 we show that the shoulder is fully determined by specification of a curve in the plane. This planar bending curve corresponds in three dimensions with the intersection of the shoulder and the vertical cylinder. In terms of differential geometry, it is natural to parametrize the shoulder by parameters $(s,u)$ with $s$ the arclength along the bending curve. The planar bending curve, however, is naturally given in terms of Cartesian coordinates. In §3 we deal with the transformation between these representations.

In practice, one needs a representation of the shoulder in the form $z(x,y)$ with $(x,y,z)$ Cartesian coordinates and $z$ the height of the shoulder above the horizontal $(x,y)$-plane. Given a pair $(s,u)$, the corresponding parameter pair $(x,y)$ is trivially found, but the map $(x,y) \rightarrow (s,u)$ is not simple to describe analytically. In §4 we give a numerical approach for this mapping, which is easily implemented. In §5 we numerically investigate various possibilities for the planar bending curve and investigate their practical implications. In the Appendices we deal with the lengthy derivation of some theoretical details.

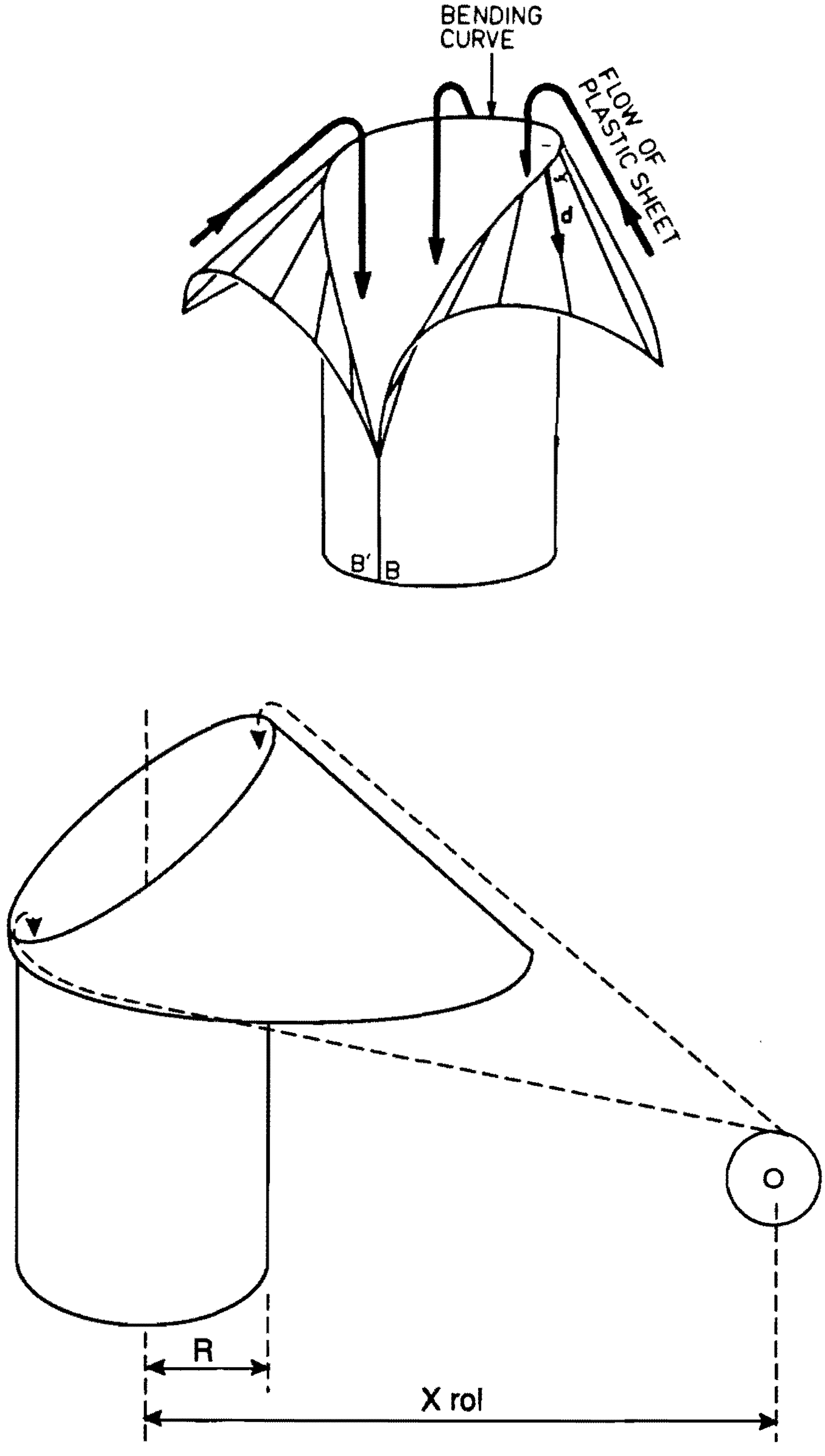


Figure 1. Sideviews of the shoulder geometry.

## 2. Determination of a shoulder representation

The shoulder must be isometric with the plane, i.e. it can be mapped unto a part of the plane such that all distances (and thus angles) are preserved. A cylinder and a cone are for example isometric with the plane. The intersection of the shoulder and the vertical cylinder is called the bending curve ($BC$) in three dimensions. Under an isometric mapping $BC$ transforms into a planar bending curve ($\overline{BC}$). In the following, we shall denote quantities referring to the plane by an overbar. Because $BC$ is obtained by wrapping $\overline{BC}$ around a given cylinder in an obvious way, the relation between $BC$ and $\overline{BC}$ is straightforward. We note that $BC$ is the intersection of two surfaces, which are both isometric with the plane. It is known from differential geometry (see e.g. Forsyth [6]), that there exist precisely two surfaces which contain $BC$ and are isometric with the plane.

In the following, we shall present an appropriate representation for the shoulder. Let a point of $BC$ be represented by a three-dimensional vector $\mathbf{r}(s)$, with parameter arclength $s$. We assume $\mathbf{r}(s)$ to be twice differentiable and choose $\mathbf{r}(0)$ as the highest point of the shoulder. We notice, that in terms of differential geometry the shoulder (and also the vertical cylinder) are so-called developable surfaces. Through each point of $BC$ a straight line passes, which is completely contained in such a surface. These lines "generate" the surface and the surface is specified by giving their directions. See e.g. Weatherburn [7], Haantjes [8] and Struik [9]. We describe the straight line contained in the shoulder and passing through the point $\mathbf{r}(s)$ of $BC$ by the unit vector $\mathbf{d}(s)$. A point $\mathbf{P}$ on the shoulder is then parametrized by

$$\mathbf{P}(s,u) = \mathbf{r}(s) + u\,\mathbf{d}(s).$$

We introduce a local, orthonormal coordinate system $(\mathbf{t},\mathbf{n},\mathbf{b})$ at $\mathbf{r}(s)$. The unit tangent vector $\mathbf{t}$ is defined as

$$\mathbf{t}(s) = \mathbf{r}_s(s)\,,$$

the unit normal vector $\mathbf{n}$ as

$$\mathbf{n}(s) = \frac{-1}{\kappa(s)}\,\mathbf{t}_s(s)$$

with the curvature $\kappa$ given by $\kappa(s) = |\mathbf{t}_s(s)|$, and the unit binormal $\mathbf{b}$ as

$$\mathbf{b}(s) = \mathbf{t}(s) \times \mathbf{n}(s).$$

Note, that we take for convenience the curvature $\kappa \geq 0$. With this definition the normal vector $\mathbf{n}$ points outwards. The derivatives of these basis vectors are given by the Serret-Frenet formulae

$$\mathbf{t}_s(s) = -\kappa(s)\,\mathbf{n}(s)$$

$$\mathbf{n}_s(s) = \kappa(s)\,\mathbf{t}(s) + \tau(s)\,\mathbf{b}(s)$$

$$\mathbf{b}_s(s) = -\tau(s)\,\mathbf{n}(s)$$

with $\tau(s)$ the torsion. We may now write quite generally

$$\mathbf{d}(s) = \cos\alpha(s)\,\mathbf{t}(s) + \sin\alpha(s)\,(\cos\phi(s)\,\mathbf{n}(s) + \sin\phi(s)\,\mathbf{b}(s)).$$

In the sequel, it will become clear why the introduction of the angles $\alpha$ and $\phi$ is useful.

We represent a point of $\overline{BC}$ by a two-dimensional vector $\bar{\mathbf{r}}(s)$. Note that both $\mathbf{r}(s)$ and $\bar{\mathbf{r}}(s)$ have the arclength $s$ as parameter, because $s$ is preserved under an isometric mapping. Analogous to the definitions above, we have

$$\bar{\mathbf{t}}(s) = \bar{\mathbf{r}}_s(s)$$

$$\bar{\mathbf{n}}(s) = \frac{-1}{\bar{\kappa}(s)}\,\bar{\mathbf{t}}_s(s)\,, \quad \bar{\kappa}(s) = |\bar{\mathbf{t}}_s(s)|$$

$$\bar{\mathbf{n}}_s(s) = \bar{\kappa}(s)\,\bar{\mathbf{t}}(s)$$

$$\bar{\mathbf{d}}(s) = \cos\alpha(s)\,\bar{\mathbf{t}}(s) + \sin\alpha(s)\,\bar{\mathbf{n}}(s).$$

The last line expresses that the angle $\alpha$ is preserved under the isometric mapping. A point $\bar{\mathbf{P}}$ in the plane is parametrized by

$$\bar{\mathbf{P}}(s,u) = \bar{\mathbf{r}}(s) + u\,\bar{\mathbf{d}}(s).$$

For each pair $(s,u)$, $\mathbf{P}_s$ and $\mathbf{P}_u$ are linearly independent tangent vectors which form a basis for the tangent plane in $\mathbf{P}(s,u)$. The fact that under the isometric mapping all distances (and angles) are preserved can be expressed by the following three conditions:

(i) $\quad |\mathbf{P}_u| = |\bar{\mathbf{P}}_u|$

(ii) $\quad \mathbf{P}_s \cdot \mathbf{P}_u = \bar{\mathbf{P}}_s \cdot \bar{\mathbf{P}}_u$

(iii) $\quad |\mathbf{P}_s| = |\bar{\mathbf{P}}_s|.$

ad i) This condition is automatically fulfilled because $\mathbf{P}_u = \mathbf{d}$, $\bar{\mathbf{P}}_u = \bar{\mathbf{d}}$ and $|\mathbf{d}| = |\bar{\mathbf{d}}| = 1$.

ad ii) We have

$$\mathbf{P}_s = \mathbf{r}_s + u\,\mathbf{d}_s = \mathbf{t} + u\,\mathbf{d}_s$$

and

$$\bar{\mathbf{P}}_s = \bar{\mathbf{r}}_s + u\,\bar{\mathbf{d}}_s = \bar{\mathbf{t}} + u\,\bar{\mathbf{d}}_s.$$

It immediately follows that

$$\mathbf{P}_s \cdot \mathbf{P}_u = \cos\alpha = \bar{\mathbf{P}}_s \cdot \bar{\mathbf{P}}_u.$$

We already anticipated this condition by writing $\mathbf{d}(s)$ and $\bar{\mathbf{d}}(s)$ in the forms given above.

ad iii) This condition states that

$$|\mathbf{t}(s) + u\,\mathbf{d}_s(s)| = |\bar{\mathbf{t}}(s) + u\,\bar{\mathbf{d}}_s(s)|$$

should hold for every $u$. This is equivalent with the conditions $|\mathbf{t}| = |\bar{\mathbf{t}}|$, which is trivially fulfilled, and

$$|\mathbf{d}_s(s)| = |\bar{\mathbf{d}}_s(s)|.$$

From lengthy but straightforward algebra we find

$$|\bar{\mathbf{d}}_s|^2 = \alpha_s^2 - 2\alpha_s\,\bar{\kappa} + \bar{\kappa}^2$$

$$|\mathbf{d}_s|^2 = \alpha_s^2 - 2\alpha_s\,\kappa\cos\phi + [\kappa^2(\sin^2\alpha\cos^2\phi + \cos^2\alpha)$$
$$+ \sin^2\alpha(\phi_s + \tau)^2 + 2\kappa\cos\alpha\sin\alpha\sin\phi(\phi_s + \tau)].$$

Equating terms with $\alpha_s$ yields

$$\cos\phi = \frac{\bar{\kappa}}{\kappa}.$$

If we use this relation and equate the remaining parts of the expressions, we find

$$\tan\alpha = \frac{-\kappa\sin\phi}{\phi_s + \tau}, \quad 0 \le \alpha < \pi.$$

The latter two equations determine $\phi$ and $\alpha$ if the $BC$ (or $\overline{BC}$) is given. Note, that in general two values $\phi_1$ and $\phi_2$ for $\phi$ are obtained with $0 \le \phi_1 \le \pi/2$ and $\phi_2 = 2\pi - \phi_1$, and two corresponding values for $\alpha$. In the following section, we shall outline how $\alpha$ and $\phi$ (and thus **d**) can be calculated from a given planar curve $\overline{BC}$. In Appendix A we show that $\phi_2$ corresponds with the cylinder and $\phi_1$ with the shoulder.

### 3. Calculation of the shoulder from a given planar bending curve

In practice, one prescribes the planar bending curve $\overline{BC}$ and wants to calculate points on the shoulder. In this section, we present formulae to calculate the vector $\mathbf{d}(s)$, introduced in the preceding section, in a numerically appropriate way. For given values of the radius $R$ of the cylinder and height $h$ of the shoulder, $\overline{BC}$ is given as a curve $z(v)$ in the $(v,z)$-plane, with $v$ and $z$ Cartesian coordinates. This curve has the properties $z(0)=h$, $z(-v)=z(+v)$ and $z(\pm\pi R)=0$ as depicted in Fig. 2. We assume $z(v)$ to be three times differentiable. Furthermore, it is assumed that $z_{vv}<0$, i.e. $\overline{BC}$ is concave downwards.

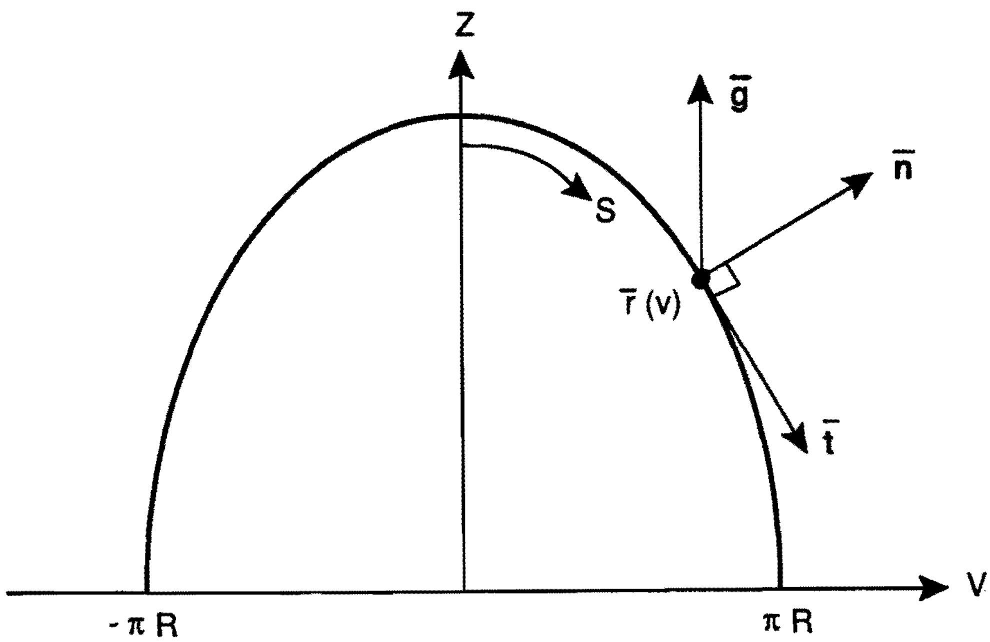


Figure 2. The planar bending curve $\overline{BC}$, given as a function $z(v)$ in the $(v,z)$-plane. The parameter $s$ denotes arclength.

We may also represent a point of $\overline{BC}$ by a two-dimensional vector $\bar{\mathbf{r}}(v)$:

$$\bar{\mathbf{r}}(v)=\begin{bmatrix} v \\ z(v) \end{bmatrix}.$$

If $\overline{BC}$ is wrapped around the cylinder, as shown in Fig. 3, the resulting $BC$ has points represented by a three-dimensional vector

$$\mathbf{r}(v) = \begin{bmatrix} R\cos(v/R) \\ R\sin(v/R) \\ z(v) \end{bmatrix}$$

with respect to Cartesian coordinates $(x, y, z)$.

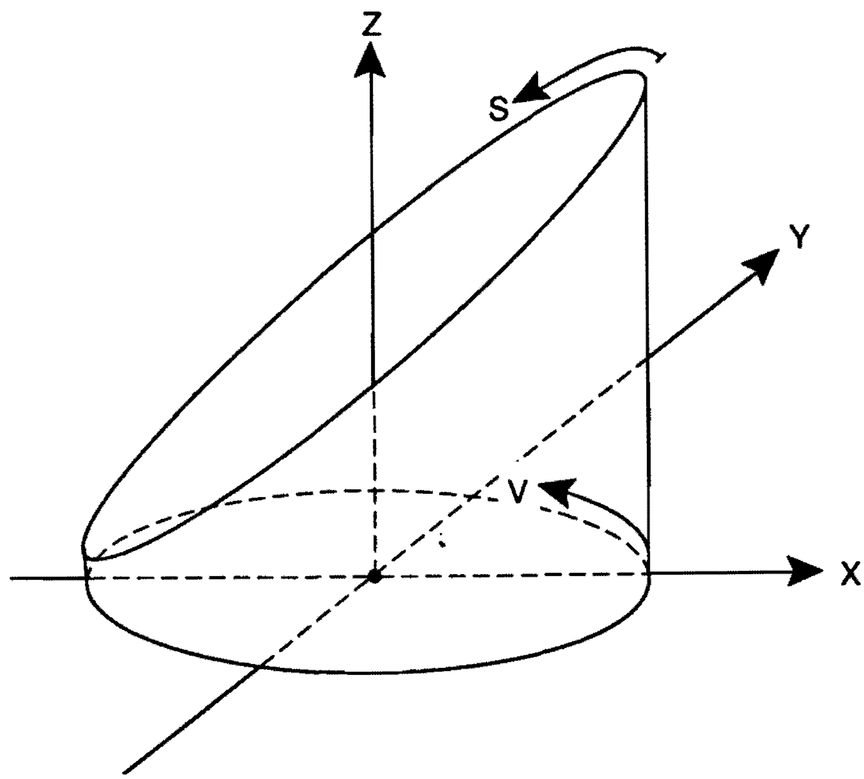


Figure 3. Choice of the Cartesian coordinates $(x, y, z)$.

In the preceding section, we used arclength $s$ as parameter instead of $v$. The relation between the two is given by

$$s = \int_0^v (1 + z_v^2(v'))^{1/2}\, dv'.$$

Only in a few cases this relationship can be brought into a simpler form by evaluating the integral analytically. For example, if

$$z(v) = a \cosh\left[\frac{v}{a}\right]$$

with $a$ some constant, we find that

$$s = a \sinh\left[\frac{v}{a}\right].$$

In the following, we assume that such a reduction is not known and point out how the quantities in §2, which are given as functions of $s$, can be calculated as functions of $v$. We need the factors

$$v_s = (1 + z_v^2)^{-½}$$

$$v_{ss} = -z_v\, z_{vv}\, v_s^4$$

$$v_{sss} = -z_{vv}^2\, v_s^5 - z_v\, z_{vvv}\, v_s^5 - 4\, z_v\, z_{vv}\, v_s^3\, v_{ss}$$

$$= -v_s^5(z_{vv}^2 + z_v\, z_{vvv} - 4\, z_v^2\, z_{vv}^2\, v_s^2)$$

and the vectors $\mathbf{r}_v$, $\mathbf{r}_{vv}$ and $\mathbf{r}_{vvv}$, which directly follow from the explicit expression for $\mathbf{r}(v)$. Then, we may write

$$\mathbf{t}(v) = \mathbf{r}_v\, v_s$$

$$\mathbf{t}_s(v) = \mathbf{r}_{vv}\, v_s^2 + \mathbf{r}_v\, v_{ss}$$

$$\mathbf{t}_{ss}(v) = \mathbf{r}_{vvv}\, v_s^3 + 3\,\mathbf{r}_{vv}\, v_s\, v_{ss} + \mathbf{r}_v\, v_{sss}.$$

From $\mathbf{t}$, $\mathbf{t}_s$ and $\mathbf{t}_{ss}$ we may calculate the other relevant quantities. The curvature $\kappa$ and its derivative are given by

$$\kappa = |\mathbf{t}_s|$$

$$\kappa_s = \frac{1}{\kappa}\,(\mathbf{t}_s \cdot \mathbf{t}_{ss}).$$

The normal $\mathbf{n}$ and binormal $\mathbf{b}$ follow from

$$\mathbf{n} = \frac{-1}{\kappa}\,\mathbf{t}_s$$

$$\mathbf{b} = \mathbf{t} \times \mathbf{n} = \frac{-1}{\kappa}\,(\mathbf{t} \times \mathbf{t}_s).$$

The torsion $\tau$ can be found from the Serret-Frenet formula

$$\mathbf{n}_s = \kappa\,\mathbf{t} + \tau\,\mathbf{b}\,,$$

which implies that

$$\tau = (\mathbf{n}_s \cdot \mathbf{b}).$$

From

$$\mathbf{n}_s = \frac{-1}{\kappa^2}\,(\kappa\,\mathbf{t}_{ss} - \kappa_s\,\mathbf{t}_s)\,,$$

we conclude that

$$\tau = \frac{1}{\kappa^2}\,(\mathbf{t}_{ss}\cdot(\mathbf{t}\times\mathbf{t}_s)).$$

In the plane, similar relations hold. We give the relevant ones:

$$\bar{\mathbf{t}} = \bar{\mathbf{r}}_v\,v_s$$

$$\bar{\mathbf{t}}_s = \bar{\mathbf{r}}_{vv}\,v_s^2 + \bar{\mathbf{r}}_v\,v_{ss}$$

$$\bar{\mathbf{t}}_{ss} = \bar{\mathbf{r}}_{vvv}\,v_s^3 + 3\,\bar{\mathbf{r}}_{vv}\,v_s\,v_{ss} + \bar{\mathbf{r}}_v\,v_{sss}$$

$$\bar{\kappa} = |\bar{\mathbf{t}}_s|$$

$$\bar{\kappa}_s = \frac{1}{\bar{\kappa}}\,(\bar{\mathbf{t}}_s\cdot\bar{\mathbf{t}}_{ss}).$$

If we compare the expressions for $\kappa$ and $\bar{\kappa}$, we obtain the relation

$$\kappa^2 = \bar{\kappa}^2 + \frac{v_s^4}{R^2}\,,$$

so that in practice it is unnecessary to calculate $\bar{\kappa}$ and $\kappa$ separately. From $\bar{\kappa}$ and $\kappa$, we can find values for $\phi$ via

$$\cos\phi = \frac{\bar{\kappa}}{\kappa}.$$

To find $\alpha$ from

$$\tan\alpha = \frac{-\kappa\sin\phi}{\phi_s + \tau}\,,$$

one may appropriately use the relations

$$\sin\phi = \pm\frac{v_s^2}{\kappa R}$$

and

$$\phi_s = \frac{-1}{\sin\phi}\,\frac{1}{\kappa^2}\,(\kappa\,\bar{\kappa}_s - \bar{\kappa}\,\kappa_s).$$

The plus sign refers to $\phi_1$, i.e. the shoulder, the minus sign to $\phi_2$, i.e. the vertical cylinder. In the following we shall use $\phi_1$.

An alternative expression for $\phi_s$ reads as

$$\phi_s = \frac{-1}{\kappa^2 R \cos\phi} (\kappa_s v_s - 2\kappa v_{ss}),$$

which avoids the evaluation of $\overline{t}_s$ and $\overline{t}_{ss}$.

## 4. The inversion problem

One often needs a representation of the shoulder in the form $z(x,y)$, i.e. its height $z$ above the horizontal $(x,y)$-plane. In §2,3 we have shown, that the shoulder is commonly parametrized by the pair $(u,v)$. While the map $(u,v) \rightarrow (x,y)$ is trivial, the inverse map $(x,y) \rightarrow (u,v)$ is not easily cast into an explicit form. Here, we follow a numerical approach. A point **P** on the shoulder is given by

$$\mathbf{P}(u,v) = \mathbf{r}(v) + u\,\mathbf{d}(v) = \begin{bmatrix} r_1(v) \\ r_2(v) \\ r_3(v) \end{bmatrix} + u \begin{bmatrix} d_1(v) \\ d_2(v) \\ d_3(v) \end{bmatrix}.$$

Given $(x,y)$, we search for the pair $(u,v)$ such that

$$x = r_1(v) + u\,d_1(v)$$

$$y = r_2(v) + u\,d_2(v).$$

This leads to the one-parameter equation

$$y = r_2(v) + \left[ \frac{x - r_1(v)}{d_1(v)} \right] d_2(v).$$

For numerical reasons, the parameter $v$ should rather be determined from the equation

$$(y - r_2(v))\,d_1(v) - (x - r_1(v))\,d_2(v) = 0.$$

If $v$ has been calculated, $u$ is obtained from either

$$u = (x - r_1(v)) \,/\, d_1(v)$$

or

$$u = (y - r_2(v)) \,/\, d_2(v).$$

## 5. Applications

From the viewpoint of the designer, information about the following properties of the shoulder may be of importance:

- The mathematical angle $\theta$ at $\mathbf{r} \in BC$. It is the angle between the two planes tangent in $\mathbf{r}$ to the shoulder and the vertical cylinder.
- The angle $\chi$ at $\mathbf{r} \in BC$. This angle is measured in the plane through $\mathbf{r}$ and the axis of the vertical cylinder (i.e. the $z$-axis). This plane intersects the shoulder along some curve. $\chi$ is the angle between the tangent in $\mathbf{r}$ to this curve and the downward vertical.
- The angle $\psi$ at $\mathbf{r} \in BC$. It is obtained by following the path of a specific point of the sheet when sliding over the shoulder and reaching $BC$ in $\mathbf{r}$. $\psi$ is defined to be the angle between the tangent in $\mathbf{r}$ to this path and the downward vertical.
- The paper or plastic sheet is unrolled from a horizontal cylinder mounted perpendicularly to the $(x,z)$-plane (see Fig. 3). For several reasons, it is advantageous if this cylinder is placed as near to the highest point of the shoulder as possible. However, this cylinder is straight and does not exactly fit to the shoulder. We introduce a measure Curv for the curvature of the shoulder in the vicinity of the horizontal cylinder:

$$\text{Curv}(x_{\text{rol}}) = z(x_{\text{rol}}, 0) - z(x_{\text{rol}}, \pi R).$$

In this formula, the shoulder is assumed to be presented as a known function $z(x,y)$ with $x,y,z$ as denoted in Fig. 3. The $x$-coordinate of the horizontal cylinder is denoted by $x_{\text{rol}}$. The length of this cylinder should be at least $2\pi R$, i.e. the circumference of the vertical cylinder. We see that Curv = 0 if the shoulder would contain the horizontal cylinder. This is never the case, because the only straight lines contained in the shoulder are the ones through the generating vectors $\mathbf{d}$.

Expressions for the calculation of the angles $\theta, \chi$ and $\psi$ are derived in Appendix B. We have calculated $\theta, \chi, \psi$ and Curv for three (families of) $\overline{BC}$'s. The radius of the vertical cylinder and the height of the shoulder are denoted by $R$ and $h$, respectively.

1. A $\overline{BC}$ given by a parabola:

$$z(v) = h\left[\frac{-v^2}{(\pi R)^2} + 1\right].$$

Results for various values of $h$ are given in Tables 1a, 1b and 1c.

2. A $\overline{BC}$ given by a catenary:

$$z(v) = h\left[-\frac{\cosh(v/b) - 1}{\cosh(\pi R/b) - 1} + 1\right].$$

The shape of this $\overline{BC}$ can be adjusted via the parameter $b$. Results for various values of $h$

and $b$ are given in Tables 2a-2f.

3. A special case of 2) is obtained if we choose for $b$ the solution $b_0$ of the equation

$$b(\cosh(\pi R/b) - 1) = h.$$

The corresponding $\overline{BC}$ is given by

$$z(v) = +b_0\,(-\cosh(v/b_0) + \cosh(\pi R/b_0)).$$

For this $\overline{BC}$ the angle $\theta$ turns out to be constant along $BC$. In Table 3 its value is given as a function of $b_0$ and $h$.

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**Tables**

| $\nu$ | $\chi$ | $\theta$ | $\psi$ |
|---|---|---|---|
| 0 | 60.02 | 60.02 | 60.02 |
| 15 | 59.17 | 59.46 | 61.67 |
| 30 | 56.73 | 57.87 | 66.20 |
| 45 | 52.94 | 55.48 | 72.58 |
| 60 | 48.21 | 52.60 | 79.81 |
| 75 | 43.02 | 49.47 | 87.14 |
| 90 | 37.79 | 46.32 | 94.16 |
| 105 | 32.88 | 43.27 | 100.66 |
| 120 | 28.48 | 40.41 | 106.55 |
| 135 | 24.64 | 37.76 | 111.85 |
| 150 | 21.37 | 35.34 | 116.58 |
| 165 | 18.60 | 33.15 | 120.81 |
| 180 | 16.27 | 31.15 | 124.58 |

*Table 1a.* Values for $\theta, \chi$ and $\psi$ in degrees for $\overline{BC}$ number 1 with $h = 2.850$ and $R = 1$. In the first column the position on $BC$ is given by specification of the parameter $\nu$ (in degrees) as introduced in §3 and Fig. 2.

| $\nu$ | $\chi$ | $\theta$ | $\psi$ |
|---|---|---|---|
| 0 | 45.00 | 45.00 | 45.00 |
| 15 | 44.60 | 44.76 | 46.36 |
| 30 | 43.42 | 44.08 | 50.12 |
| 45 | 41.56 | 43.00 | 55.55 |
| 60 | 39.18 | 41.61 | 61.91 |
| 75 | 36.43 | 40.02 | 68.61 |
| 90 | 33.50 | 38.29 | 75.28 |
| 105 | 30.53 | 36.52 | 81.70 |
| 120 | 27.65 | 34.75 | 87.74 |
| 135 | 24.95 | 33.02 | 93.35 |
| 150 | 22.46 | 31.37 | 98.51 |
| 165 | 20.20 | 29.80 | 103.24 |
| 180 | 18.19 | 28.33 | 107.57 |

*Table 1b.* Data as in Table 1a for $\overline{BC}$ number 1 with $h = 2.044$ and $R = 1$.

| $x_{rol}$ | a. | b. |
|---|---|---|
| 1.00 | 2.84 | 3.39 |
| 2.00 | 2.08 | 2.50 |
| 3.00 | 1.60 | 1.93 |
| 4.00 | 1.28 | 1.55 |
| 5.00 | 1.06 | 1.29 |
| 6.00 | 0.90 | 1.10 |
| 7.00 | 0.78 | 0.96 |
| 8.00 | 0.69 | 0.84 |
| 9.00 | 0.62 | 0.76 |
| 10.00 | 0.56 | 0.68 |

*Table 1c.* Values for Curv as a function of $x_{rol}$ for $\overline{BC}$ number 1 with $h = 2.850$ (column a.) and $h = 2.044$ (column b.), and $R = 1$.

| $\nu$ | $\chi$ | $\theta$ | $\psi$ |
|---|---|---|---|
| 0 | 44.99 | 44.99 | 44.99 |
| 15 | 44.93 | 45.10 | 46.69 |
| 30 | 44.74 | 45.41 | 51.40 |
| 45 | 44.40 | 45.91 | 58.27 |
| 60 | 43.86 | 46.54 | 66.44 |
| 75 | 43.08 | 47.25 | 75.26 |
| 90 | 42.04 | 48.01 | 84.27 |
| 105 | 40.71 | 48.75 | 93.17 |
| 120 | 39.09 | 49.46 | 101.72 |
| 135 | 37.19 | 50.09 | 109.81 |
| 150 | 35.06 | 50.66 | 117.33 |
| 165 | 32.75 | 51.14 | 124.24 |
| 180 | 30.32 | 51.54 | 130.54 |

*Table 2a.* Values for $\theta$, $\chi$ and $\theta$ in degrees for $\overline{BC}$ number 2 with $b = 2.0$, $h = 2.50$ and $R = 1$. In the first column the position on $BC$ is given by specification of the parameter $\nu$ (in degrees) as introduced in §3 and Fig. 2.

| $\nu$ | $\chi$ | $\theta$ | $\psi$ |
|---|---|---|---|
| 0 | 54.96 | 54.96 | 54.96 |
| 15 | 54.68 | 54.93 | 56.92 |
| 30 | 53.84 | 54.84 | 62.31 |
| 45 | 52.48 | 54.70 | 69.99 |
| 60 | 50.64 | 54.52 | 78.88 |
| 75 | 48.37 | 54.34 | 88.16 |
| 90 | 45.76 | 54.15 | 97.31 |
| 105 | 42.88 | 53.98 | 106.02 |
| 120 | 39.82 | 53.82 | 114.12 |
| 135 | 36.68 | 53.69 | 121.55 |
| 150 | 33.52 | 53.58 | 128.27 |
| 165 | 30.42 | 53.48 | 134.32 |
| 180 | 27.44 | 53.41 | 139.72 |

*Table 2b.* Data as in Table 2a for $\overline{BC}$ number 2 with $b = 2.0$, $h = 3.14$ and $R = 1$.

| $\nu$ | $\chi$ | $\theta$ | $\psi$ |
|---|---|---|---|
| 0 | 60.21 | 60.21 | 60.21 |
| 15 | 59.78 | 60.06 | 62.28 |
| 30 | 58.52 | 59.66 | 67.94 |
| 45 | 56.50 | 59.05 | 75.93 |
| 60 | 53.83 | 58.32 | 85.03 |
| 75 | 50.67 | 57.57 | 94.37 |
| 90 | 47.15 | 56.83 | 103.42 |
| 105 | 43.44 | 56.17 | 111.89 |
| 120 | 39.66 | 55.58 | 119.66 |
| 135 | 35.94 | 55.09 | 126.69 |
| 150 | 32.37 | 54.68 | 132.99 |
| 165 | 28.99 | 54.35 | 138.60 |
| 180 | 25.85 | 54.09 | 143.58 |

*Table 2c.* Data as in Table 2a for $\overline{BC}$ number 2 with $b = 2.0$, $h = 3.50$ and $R = 1$.

| $\nu$ | $\chi$ | $\theta$ | $\psi$ |
|---|---|---|---|
| 0 | 47.97 | 47.97 | 47.97 |
| 15 | 47.72 | 47.91 | 49.62 |
| 30 | 46.99 | 47.75 | 54.19 |
| 45 | 45.81 | 47.51 | 60.80 |
| 60 | 44.23 | 47.20 | 68.57 |
| 75 | 42.33 | 46.84 | 76.82 |
| 90 | 40.17 | 46.48 | 85.11 |
| 105 | 37.83 | 46.11 | 93.18 |
| 120 | 35.38 | 45.77 | 100.85 |
| 135 | 32.90 | 45.45 | 108.05 |
| 150 | 30.43 | 45.16 | 114.72 |
| 165 | 28.02 | 44.91 | 120.88 |
| 180 | 25.70 | 44.69 | 126.51 |

*Table 2d.* Data as in Table 2a for $\overline{BC}$ number 2 with $b = 2.50$, $h = 2.50$ and $R = 1$.

| $\nu$ | $\chi$ | $\theta$ | $\psi$ |
|---|---|---|---|
| 0 | 58.39 | 58.39 | 58.39 |
| 15 | 57.87 | 53.14 | 60.28 |
| 30 | 56.33 | 57.43 | 65.45 |
| 45 | 53.92 | 56.36 | 72.78 |
| 60 | 50.80 | 55.06 | 81.15 |
| 75 | 47.21 | 53.68 | 89.76 |
| 90 | 43.36 | 52.30 | 98.14 |
| 105 | 39.47 | 51.01 | 106.03 |
| 120 | 35.70 | 49.85 | 113.31 |
| 135 | 32.14 | 48.32 | 119.95 |
| 150 | 28.85 | 47.94 | 125.96 |
| 165 | 25.86 | 47.18 | 131.39 |
| 180 | 23.16 | 46.55 | 136.28 |

*Table 2e.* Data as in Table 2a for $\overline{BC}$ number 2 with $b = 2.50$, $h = 3.14$ and $R = 1$.

| $\nu$ | $\chi$ | $\theta$ | $\psi$ |
|---|---|---|---|
| 0 | 63.83 | 63.83 | 63.83 |
| 15 | 63.14 | 63.44 | 65.82 |
| 30 | 61.10 | 62.33 | 71.23 |
| 45 | 57.92 | 60.69 | 78.80 |
| 60 | 53.86 | 58.77 | 87.33 |
| 75 | 49.26 | 56.77 | 95.97 |
| 90 | 44.47 | 54.83 | 104.24 |
| 105 | 39.77 | 53.06 | 111.90 |
| 120 | 35.36 | 51.50 | 118.89 |
| 135 | 31.33 | 50.14 | 125.18 |
| 150 | 27.74 | 49.00 | 130.83 |
| 165 | 24.56 | 48.03 | 135.89 |
| 180 | 21.78 | 47.23 | 140.41 |

*Table 2f.* Data as in Table 2a for $\overline{BC}$ number 2 with $b = 2.50$, $h = 3.50$ and $R = 1$.

| $x_{rol}$ | a. | b. | c. | d. | e. | f. |
|---|---|---|---|---|---|---|
| 1.00 | 2.52 | 2.87 | 2.40 | 2.89 | 2.57 | 2.47 |
| 2.00 | 2.11 | 1.84 | 1.75 | 2.13 | 1.88 | 1.79 |
| 3.00 | 1.62 | 1.40 | 1.33 | 1.63 | 1.44 | 1.37 |
| 4.00 | 1.29 | 1.12 | 1.06 | 1.31 | 1.15 | 1.09 |
| 5.00 | 1.07 | 0.92 | 0.87 | 1.08 | 0.95 | 0.90 |
| 6.00 | 0.90 | 0.78 | 0.74 | 0.92 | 0.80 | 0.76 |
| 7.00 | 0.78 | 0.68 | 0.64 | 0.80 | 0.70 | 0.66 |
| 8.00 | 0.69 | 0.60 | 0.56 | 0.70 | 0.61 | 0.58 |
| 9.00 | 0.62 | 0.53 | 0.50 | 0.63 | 0.55 | 0.52 |
| 10.00 | 0.56 | 0.48 | 0.45 | 0.57 | 0.50 | 0.47 |

*Table 2g.* Values for Curv as a function of $x_{rol}$ for $\overline{BC}$ number 2.
The columns a.-f. correspond to the values of $b$, $h$ and $R$, as used in Tables 2a-2f, respectively.

| $b_0$ | $h$ | $\theta$ |
|---|---|---|
| 1.0 | 10.59 | 90.0 |
| 2.0 | 3.02 | 53.1 |
| 3.0 | 1.80 | 36.9 |
| 4.0 | 1.30 | 28.1 |
| 5.0 | 1.02 | 22.6 |

*Table 3.* Values of $\theta$ in degrees for various values of $b_0$ (and thus $h$) in $\overline{BC}$ number 3 with $R = 1$.

**Appendix A**

In this appendix we explicitly show that, indeed, one of the surfaces constructed in §2 coincides with the vertical cylinder. At the same time we find out which of the solutions $\phi_1$ and $\phi_2$ with $0 \le \phi_1 < \pi$ and $\pi \le \phi_2 < 2\pi$ of the equation $\cos\phi = \bar{\kappa}/\kappa$ corresponds with the shoulder. With an eye on the expression for $\mathbf{r}(v)$ in Cartesian coordinates given in §3, we introduce the following orthonormal basis:

$$\mathbf{e}_1(v) = \begin{bmatrix} \cos(v/R) \\ \sin(v/R) \\ 0 \end{bmatrix}$$

$$\mathbf{e}_2(v) = \begin{bmatrix} -\sin(v/R) \\ \cos(v/R) \\ 0 \end{bmatrix}$$

$$\mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

For these basisvector *s* we have the properties (with a prime denoting differentiating with respect to $v$):

$$\mathbf{e}_1' = \mathbf{e}_2 / R$$

$$\mathbf{e}_2' = -\mathbf{e}_1 / R$$

$$\mathbf{e}_3' = 0$$

$$\mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{e}_3$$

$$\mathbf{e}_2 \times \mathbf{e}_3 = \mathbf{e}_1$$

$$\mathbf{e}_3 \times \mathbf{e}_1 = \mathbf{e}_2 .$$

In terms of this basis the *BC* is given by

$$\mathbf{r}(v) = R\,\mathbf{e}_1 + z(v)\,\mathbf{e}_3 .$$

Its derivatives read as

$$\mathbf{r}_v = \mathbf{e}_2 + z_v\,\mathbf{e}_3$$

$$\mathbf{r}_{vv} = -\mathbf{e}_1 / R + z_{vv}\,\mathbf{e}_3$$

$$\mathbf{r}_{vvv} = -\mathbf{e}_2 / R^2 + z_{vvv}\,\mathbf{e}_3 .$$

The vectors $\mathbf{t}, \mathbf{t}_s$ and $\mathbf{t}_{ss}$ are represented by

$$\mathbf{t} = \mathbf{r}_s = \mathbf{r}_v\, v_s = \mathbf{e}_2\, v_s + \mathbf{e}_3\, z_v\, v_s$$

$$\mathbf{t}_s = -\mathbf{e}_1\, v_s^2 / R - \mathbf{e}_2\, z_v\, z_{vv}\, v_s^4 + \mathbf{e}_3\, z_{vv}\, v_s^4$$

$$\mathbf{t}_{ss} = \mathbf{e}_1 (z_v\, z_{vv}\, v_s^5)\,(\frac{3}{R})$$

$$-\mathbf{e}_2 (1/R^2 + z_{vv}^2\, v_s^2 + z_v\, z_{vvv}\, v_s^2 - 4\, z_v^2\, z_{vv}^2\, v_s^4)\, v_s^3 +$$

$$\mathbf{e}_3 (z_{vvv} - 4\, z_v\, z_{vv}^2\, v_s^2)\, v_s^5 .$$

Using the properties of the basisvectors given above, we find

$$\mathbf{t} \times \mathbf{t}_s = \mathbf{e}_1\, z_{vv}\, v_s^3 - \mathbf{e}_2\, z_v\, v_s^3 / R + \mathbf{e}_3\, v_s^3 / R.$$

Via these expressions, one can find **n** and **b** in terms of the basisvectors via the relations

$$\mathbf{n} = \frac{-1}{\kappa}\, \mathbf{t}_s$$

$$\mathbf{b} = \mathbf{t} \times \mathbf{n} = \frac{-1}{\kappa}\, (\mathbf{t} \times \mathbf{t}_s).$$

So, the representation

$$\mathbf{d} = \cos\alpha\, \mathbf{t} + \sin\alpha (\cos\phi\, \mathbf{n} + \sin\phi\, \mathbf{b})$$

can be rewritten in the form

$$\mathbf{d} = \mathbf{e}_1\, \frac{v_s^2 \sin\alpha}{\kappa}\, [\cos\phi / R - z_{vv}\, v_s \sin\phi] +$$

$$\mathbf{e}_2\, v_s\, [\cos\alpha + \sin\alpha (z_{vv}\, v_s \cos\phi + \sin\phi / R) \frac{z_v\, v_s^2}{\kappa}] +$$

$$\mathbf{e}_3\, v_s\, [z_v \cos\alpha - \sin\alpha (z_{vv}\, v_s \cos\phi + \sin\phi / R) \frac{v_s^2}{\kappa}].$$

To evaluate the coefficients, it is useful to have explicit expressions at hand for $\bar{\kappa}$, $\bar{\kappa}_s$ and $\kappa$. They are obtained from the general equations given in §3. We arrive at

$$\bar{\kappa} = -z_{vv}\, v_s^3$$

$$\bar{\kappa}_s = -v_s^4 (z_{vvv} - 3\, z_{vv}^2\, z_v\, v_s^2)$$

$$\kappa^2 = \bar{\kappa}^2 + \frac{v_s^4}{R^2}.$$

If we differentiate the relation between $\kappa^2$ and $\bar{\kappa}^2$, we find

$$\kappa_s = \frac{1}{\kappa}\, (\bar{\kappa}\bar{\kappa}_s - 2\, z_{vv}\, z_v\, v_s^7 / R^2).$$

Further, we shall make use of the relations

$$\cos\phi = \frac{\bar{\kappa}}{\kappa}$$

$$\sin\phi = \pm\frac{v_s^2}{\kappa R}$$

$$\tan\alpha = \frac{-\kappa\sin\phi}{\phi_s+\tau}.$$

The plus sign refers to $\phi_1$, the minus sign to $\phi_2$. We find by substitution, that the coefficient of $\mathbf{e}_1$ in $\mathbf{d}$ vanishes if $\phi_2$ is used:

$$\cos\phi/R - z_{vv}\,v_s\sin\phi = \frac{1}{\kappa R}\,(\bar{\kappa}-z_{vv}\,v_s^3) = 0.$$

To work out the coefficient of $\mathbf{e}_2$ in $\mathbf{d}$ for $\phi = \phi_2$, we write it in the form

$$\frac{v_s\cos\alpha}{\phi_s+\tau}\,[\phi_s+\tau-\sin\phi(z_{vv}\,v_s\cos\phi+\sin\phi/R)\,z_v\,v_s^2] =$$

$$\frac{v_s\cos\alpha}{\phi_s+\tau}\,[\phi_s+\tau-\frac{v_s^6\,z_v}{\kappa^2 R}\,(z_{vv}^2\,v_s^2+1/R^2)].$$

To determine the expression between parentheses, the quantities $\phi_s$ and $\tau$ have to be expressed in terms of $\kappa, v_s, z_v, z_{vv}$ etc. By differentiating the relation $\cos\phi = \bar{\kappa}/\kappa$ and using expressions given above, we find

$$\phi_s = \frac{-1}{\sin\phi}\,\frac{1}{\kappa^2}\,(\kappa\bar{\kappa}_s-\bar{\kappa}\kappa_s)$$

$$= \frac{\pm v_s^6}{\kappa^2 R}\,(z_{vvv}-z_{vv}^2\,z_v\,v_s^2).$$

From the equation $\tau = ((\mathbf{t}\times\mathbf{t}_s)\cdot\mathbf{t}_{ss})/\kappa^2$ we obtain that $\tau$ can be written as

$$\tau = \frac{v_s^6}{\kappa^2 R}\,(z_{vvv}+z_v/R^2).$$

Substitution of these representations of $\phi_s$ and $\tau$ into the coefficient of $\mathbf{e}_2$ yields, that the latter vanishes if $\phi_2$ is used. Thus the solution $\phi_2$ of §2 corresponds with the cylinder, and consequently the solution $\phi_1$ with the shoulder.

**Appendix B**

Here, we derive expressions for the angles $\theta, \chi$ and $\psi$, defined in §5.

Along BC the tangent planes of the surfaces which correspond to $\phi_1$ and $\phi_2$ (and thus to $\mathbf{d}_1$ and $\mathbf{d}_2$, say), have the line through $\mathbf{t}$ as line of intersection. These planes are spanned by the pairs of vectors $(\mathbf{t}, \mathbf{d}_1)$ and $(\mathbf{t}, \mathbf{d}_2)$. The angle $\theta$ between these planes is equal to the angle between the components of $\mathbf{d}_1$ and $\mathbf{d}_2$ perpendicular to $\mathbf{t}$. These components are given by $\sin\alpha\,(\cos\phi_1\,\mathbf{n}+\sin\phi_1\,\mathbf{b})$ and $\sin\alpha\,(\cos\phi_2\,\mathbf{n}+\sin\phi_2\,\mathbf{b}) = \sin\alpha\,(\cos\phi_1\,\mathbf{n}-\sin\phi_1\,\mathbf{b})$. Taking the inner product, we find

$$\cos\theta = 1 - 2\left[\frac{\bar{\kappa}}{\kappa}\right]^2 ,$$

where we determined the sign of the right-hand side from the requirement $\theta = 0$ if $\bar{\kappa} = 0$.

To obtain an expression for $\chi$ we introduce a vector $\mathbf{m}$ which is tangent to the shoulder at $\mathbf{r} \in BC$ and lies in the plane through $\mathbf{r}$ and the vertical axis. So, $\mathbf{m}$ lies along the line of intersection of the tangent plane and the plane spanned by the basis vectors $\mathbf{e}_1$ and $\mathbf{e}_3$ as defined in Appendix A. The tangent plane is spanned by $\mathbf{t}$ and $\mathbf{d}$ and we may write

$$\mathbf{m} = \lambda\mathbf{t} + \mathbf{d}.$$

The components of the vectors $\mathbf{t}$ and $\mathbf{d}$ with respect to the basis $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ are given in Appendix A and denoted as $\mathbf{t} = (0, t_2, t_3)$ and $\mathbf{d} = (d_1, d_2, d_3)$. Then, $\mathbf{m}$ can be written as

$$\mathbf{m} = d_1\,\mathbf{e}_1 + (\lambda t_2 + d_2)\,\mathbf{e}_2 + (\lambda t_3 + d_3)\,\mathbf{e}_3 ,$$

from which we find

$$\tan\chi = -d_1 / (\lambda t_3 + d_3).$$

From the property $(\mathbf{m}\cdot\mathbf{e}_2) = 0$ it follows that

$$\lambda = -d_2 / t_2.$$

Thus we arrive at

$$\tan\chi = -d_1 t_2 / (d_3 t_2 - d_2 t_3)$$
$$= v_s(\cos\phi / R - z_{vv}\, v_s \sin\phi) / (\sin\phi / R + z_{vv}\, v_s \cos\phi).$$

If $\phi_2$ is used, this expression should yield the solution $\chi = 0$. This is easily checked by substituting the relations $\cos\phi = \bar{\kappa}/\kappa$, $\sin\phi = -v_s^2/\kappa R$ and $\bar{\kappa} = -z_{vv}\, v_s^3$. If we use $\phi_1$, we obtain the alternative expression

$$\tan\chi = 2\, v_s^3\, \bar{\kappa} / R(\kappa^2 - 2\bar{\kappa}^2).$$

To obtain an expression for $\psi$, we remark that the path of a specific point on the sheet in three dimensions corresponds with a vertical line in two dimensions if the shoulder is isometrically mapped unto the $(v,z)$ plane depicted in Fig. 2. The vector **g** which is tangent at $\mathbf{r}(v) \in BC$ to the path of the points passing through $\mathbf{r}(v)$ corresponds with a vertical vector $\bar{\mathbf{g}}$ passing through $\bar{\mathbf{r}}(v)$ in Fig. 2. We may write

$$\mathbf{g} = \cos\gamma\,\mathbf{t} + \sin\gamma\,(\cos\phi\,\mathbf{n} + \sin\phi\,\mathbf{b})\,,$$

where the angle $\gamma$ is still to be determined.
If we take $\phi = 0$, we obtain a representation for $\bar{\mathbf{g}}$:

$$\bar{\mathbf{g}} = \cos\gamma\,\bar{\mathbf{t}} + \sin\gamma\,\bar{\mathbf{n}}.$$

Because $\bar{\mathbf{g}}$ is a unit vector in the positive $z$-direction, the following relations between $\gamma$ and the $\overline{BC}$ curve $z(v)$ hold:

$$\tan\gamma = z_v^{-1}$$

$$\sin\gamma = v_s$$

$$\cos\gamma = z_v\,v_s.$$

So, we have the relation

$$\mathbf{g} = z_v\,v_s\,\mathbf{t} + v_s(\cos\phi\,\mathbf{n} + \sin\phi\,\mathbf{b}).$$

The angle $\psi$ is obtained from the $\mathbf{e}_3$-component of **g**. The $\mathbf{e}_3$-components of **t**, **n** and **b** are given in Appendix A. If we substitute them, we find

$$(\mathbf{g}\cdot\mathbf{e}_3) = z_v^2\,v_s^2 + v_s(\cos\phi\,\frac{\bar{\kappa}}{\kappa}\,v_s - \sin\phi\,\frac{v_s^3}{\kappa R}).$$

From the relations $\cos\psi = -(\mathbf{g}\cdot\mathbf{e}_3)$, $\cos\phi = \bar{\kappa}/\kappa$ and $\sin\phi = v_s^2/\kappa R$ (using $\phi_1$) we obtain:

$$\cos\psi = -1 + 2\,\frac{v_s^6}{\kappa^2 R^2}.$$