Optimal control of one-warehouse multi-retailer systems : an assessment of the balance assumption

Optimal Control of One-Warehouse Multi-Retailer Systems:

An Assessment of the Balance Assumption

Mustafa Kemal Doğru

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Optimal control of one-warehouse multi-retailer systems: an assessment of the balance assumption / by Mustafa Kemal Doğru. – Eindhoven : Technische Universiteit Eindhoven, 2006. – Proefschrift.

ISBN 90-386-0616-85-X

NUR 804

Keywords: Multi-echelon / One-warehouse multi-retailer systems / Distribution systems / Balance assumption / Optimal control / Serial systems / Newsboy characterizations / Stochastic demand / Replenishment policies

Printed by Printpartners Ipskamp, Enschede, The Netherlands

Cover designed by Paul Verspaget

Optimal Control of One-Warehouse Multi-Retailer Systems:

An Assessment of the Balance Assumption

PROEFSCHRIFT

ter verkrijging van de graad van doctor aan de
Technische Universiteit Eindhoven, op gezag van de
Rector Magnificus, prof.dr.ir. C.J. van Duijn voor een
commissie aangewezen door het College voor
Promoties in het openbaar te verdedigen
op woensdag 1 februari 2006 om 16.00 uur

door

Mustafa Kemal Doğru

geboren te Besni, Turkije

Dit proefschrift is goedgekeurd door de promotor:

prof.dr. A.G. de Kok

Copromotor:
dr.ir. G.J.J.A.N. van Houtum

This research is supported by NWO (National Science Foundation of the Netherlands) grant 425-10-004.

Acknowledgements

This dissertation is an outcome of valuable contributions of many people even though only my name is on the cover. First and foremost, I would like to thank my supervisors prof. dr. A.G. de Kok and dr. ir. G.J.J.A.N. van Houtum. Professor A.G. de Kok's enthusiasm and numerous ideas for research have always been motivating and helpful. I have benefitted much from his knowledge and experience. I highly appreciate his continuous support in academic and nonacademic matters, especially during the first two years of my PhD study. Although he got the most loaded schedule in the world, he always had created the time for listening to me. Whenever I knocked his door, I was welcomed by "I always have time for research!".

I would like to express my gratitude to dr. ir. G.J.J.A.N. van Houtum. This dissertation would not exist without his support, guidance, careful reading and prompt feedbacks. He was an indispensable resource during the four years of my PhD education and I learned very much from the discussions during our research meetings. I am indebted to the energy, time and care he invested in our joint research.

I would like to thank prof. dr. O.J. Boxma, prof. dr. N. Erkip, prof. dr. W.H.M. Zijm and dr. M.C. van der Heijden who kindly accepted to be on my dissertation committee. Their helpful comments and involvement improved both the content and the presentation of the manuscript.

Moreover, I want to thank all my colleagues at OPAC for creating such an enjoyable and motivating atmosphere. I am grateful to Will Bertrand, Darek Chodyniecki, Bogdana Drăguţ, Jan Fransoo, Geert-Jan van Houtum, Cristina Ivănescu, Gudrun Kiesmuller, Bram Kranenburg, Pieter van Nyen, Pim Ouwehand, Ulaş Özen, Barış Selçuk, Marco Slikker, Judith Spitter, Tarkan Tan, Ineke Verbakel, Erik Winands, and many others whom I fail to mention here.

Chapter 4 of this dissertation benefitted from the discussions with dr. Lerzan Örmeci and prof. dr. Ger Koole. I highly appreciate them spending time and helping me.

The four years I spent during my PhD education was not solely an academic activity. I have met very pleasant people and made many good friends. The following people have contributed either their faith, time, energy, vision, passion, support or friendship in an important and appreciated way. I want to thank Darek Chodyniecki, Bogdana Drăguţ, Cristina Ivănescu, Alex Norta, Judith Spitter; it would have been very hard without your friendship. I am grateful to all members of the Turkish community in the Netherlands; your support was vital: Sandra Balkestein Tan, Ayşe Başdemir, Pelin Bayındır, İlker Birbil, Özgü Bulut, Seyhan Bulut, Kanat Çamlıbel, Feyzan Erkip, Nesim Erkip, Gül Gürkan, Okan Oyman, Özge Özdemir, Ulaş Özen, Aslı Selçuk, Barış Selçuk, Tarkan Tan, Altuğ Yalçıntaş. I greatly acknowledge my Dutch parents Jan and Toon for their hospitality; I always felt at home.

Finally, I want to express my sincere appreciation to my family who has offered their unconditional love and understanding. Last but not the least, I thank Canan for changing my entire life in the Netherlands. This work would not have had any sense without her wholeheartedly given support and love.

Mustafa Kemal Doğru,

November 2005

Contents

List of Figures . . . xii

List of Tables . . . xiv

1 Introduction **1**

1.1 System under Study . . . 1

1.2 Motivation of the Research . . . 4

1.3 Problem Statement and Research Questions . . . 13

1.4 Outline of the Dissertation . . . 14

2 Distribution Systems Facing Discrete Demands **17**

2.1 Introduction . . . 17

2.2 Model . . . 19

2.3 Analysis . . . 20

2.3.1 Dynamics of the System . . . 21

2.3.2 Analysis of the Allocation Decision . . . 24

2.3.3 Analysis of a Single Cycle . . . 25

2.3.4 Analysis of the Infinite Horizon Problem . . . 30

2.3.5 Newsboy Inequalities . . . 31

2.3.6 Computational Issues . . . 33

2.4 Appendix: Proofs . . . 34

3 Relative Gap between the Upper and Lower Bound **41**

3.1 Introduction . . . 41

3.2 Literature on the Balance Assumption 46
3.3 Model and Analysis 48
3.3.1 Preliminaries and Notation 48
3.3.2 Analysis of the System 50
3.3.3 Lower Bound Model 51
3.3.4 Constructing an Upper Bound 53
3.3.5 The Gap between LB and UB 53
3.4 Numerical Study and the Results 54
3.4.1 Identical Retailers 54
3.4.2 Nonidentical Retailers 62
3.4.3 Summary and Insights 72
3.5 Conclusion 74
3.6 Appendix: Demand Data from Practice 75

4 Relative Gap between the Optimal Cost and the Bounds 77

4.1 Introduction 77
4.2 Model 82
4.2.1 Stochastic Dynamic Programming Formulation 86
4.3 Analysis 87
4.3.1 Discounted Cost Criterion 87
4.3.1.1 Properties of an Optimal Policy 89
4.3.1.2 Bounding the State and Action Spaces 91
4.3.2 Lower Bound Model 93
4.3.3 Upper Bound 93
4.3.4 Quantifying the Effect of the Balance Assumption . . . 94
4.4 Computational Results 95
4.4.1 Computational Procedure 95
4.4.2 Identical Retailers 96
4.4.3 Nonidentical Retailers 100
4.4.4 Optimal Policy Behavior 109

4.4.5 Summary . . . 119
4.5 Conclusion and Further Research . . . 122
4.6 Appendix: Proofs . . . 123

5 Newsboy Characterizations for Serial Inventory Systems 125
5.1 Introduction . . . 125
5.2 Model . . . 127
5.3 Analysis . . . 129
5.3.1 Preliminaries . . . 130
5.3.2 Newsboy Characterizations . . . 132
5.3.3 Continuous Demand . . . 143
5.4 Concluding Remarks . . . 145

6 Conclusions and Further Research 147
6.1 Balance Assumption . . . 148
6.2 Newsboy Characterizations . . . 150

References 151

Summary 158

Samenvatting 163

Curriculum Vitae 168

List of Figures

1.1 One-warehouse multi-retailer system. 2

1.2 N-echelon serial system. 6

2.1 The consequences of an order given to the supplier in period t. 22

3.1 Probability of imbalance vs. relative gap for scenarios with identical retailers. 59

3.2 Probability of imbalance vs. relative gap for scenarios with nonidentical retailers. 66

4.1 Relative gaps for scenarios 1-36. 101

4.2 Relative gap vs. penalty cost for various h_0 with (a): $(l_0, l_i) = (1, 0)$, (b): $(l_0, l_i) = (2, 0)$ for scenarios 1-36. (i=1,2) 102

4.3 Relative gap vs. penalty cost for various h_0 with (a): $(l_0, l_i) = (1, 1)$, (b): $(l_0, l_i) = (1, 3)$ for scenarios 1-36. (i=1,2) 103

4.4 Relative gap vs. coefficient of variation-mean demand combination for the scenarios of set 6. 110

4.5 Behavior of optimal and LB heuristic policies in scenario 1: (a) Amount of stock retained at the warehouse after the shipment decision vs. echelon stock of the warehouse just before the shipment, (b) Order size vs. echelon stock of the warehouse just before the shipment. 113

4.6 Behavior of optimal and LB heuristic policies in scenario 2: (a) Amount of stock retained at the warehouse after the shipment decision vs. echelon stock of the warehouse just before the shipment, (b) Order size vs. echelon stock of the warehouse just before the shipment. 115

4.7 Behavior of optimal and LB heuristic policies in scenario 13: (a) Amount of stock retained at the warehouse after the shipment decision vs. echelon stock of the warehouse just before the shipment, (b) Order size vs. echelon stock of the warehouse just before the shipment. 116

4.8 Behavior of optimal and LB heuristic policies in scenario 39: (a) Amount of stock retained at the warehouse after the shipment decision vs. echelon stock of the warehouse just before the shipment, (b) Order size vs. echelon stock of the warehouse just before the shipment. 118

4.9 Behavior of optimal and LB heuristic policies in scenario 55: (a) Amount of stock retained at the warehouse after the shipment decision vs. echelon stock of the warehouse just before the shipment, (b) Order size vs. echelon stock of the warehouse just before the shipment. 120

5.1 The representation of the system considered in Example 5.1. . 129

5.2 The relationship between the decisions and their cost-wise consequences in the system of Example 5.1. 131

5.3 The representation of the 1-stage subsystem in Example 5.1. . 135

List of Tables

3.1 The summary of the results from scenarios with identical retailers. 57

3.2 The summary of the results for various values of h_i with identical retailers. 60

3.3 The summary of the results from scenarios with nonidentical retailers, $N = 2$. 64

3.4 Forwarding-to-the-small-retailer effect. 68

3.5 The effect of coefficient of variation on the balance of inventories when retailer sizes (mean demands) are equal. 71

3.6 The effect of retailer size on the balance of inventories when coefficient of variations are fixed. 72

3.7 Weekly demand data for a product family from a consumer electronics company. 76

4.1 The probability distribution functions used in the numerical study for one-period retailer demand. 95

4.2 The results of the computational procedure for various values of the lower bounds of the state variables from a scenario with $l_0 = 1$, $l_1 = l_2 = 3$, $h_0 = 0.90$, $h_1 = h_2 = 0.10$, $p_1 = p_2 = 4$ and $cv_1 = cv_2 = 2$. 97

4.3 The results for scenarios 1-27 with $cv_i = 2$. (i=1,2) 98

4.4 The results for scenarios 28-36. $(i = 1, 2)$ 99

4.5 The composition of 101 scenarios having (h_0, h_1, h_2) =(0.5,0.5,0.5) or (0.9,0.1,0.1) with respect to coefficient of variation. 104

4.6 The parameters of scenarios 37-73. 105

4.7 The results of scenarios 37-73. 106

Chapter 1

Introduction

In this dissertation, we consider the inventory control of a multi-echelon divergent system. Divergent structures are characterized by the property that each stock point has a *single predecessor*. The system under study is composed of a central inventory facility (referred to as the *warehouse*) serving several downstream stock points (referred to as the *retailers*). In the literature, this system is known as *one-warehouse multi-retailer* or *distribution* system; we use these terms interchangeably.

1.1 System under Study

The warehouse orders a single item from an external supplier with ample stock, and the retailers are replenished by the shipments from the warehouse (see Figure 1.1). There are deterministic replenishment leadtimes between the supplier and the warehouse, and in between the warehouse and the retailers. The stochastic demand of the customers occur at the retailers. Any unfulfilled demand is backlogged and satisfied as soon as possible. We assume that the system has a single decision maker; hence, there is centralized control.

The one-warehouse multi-retailer system can be observed in inventory, manufacturing and hierarchial production planning contexts; an example within each context is given below:

- *Inventory:* Consider a retail chain with a central warehouse that replenishes multiple retailer stores, which are geographically dispersed over a wide area. A planner responsible for the inventory control of a certain good in the whole supply chain would be facing the problem of determining when and

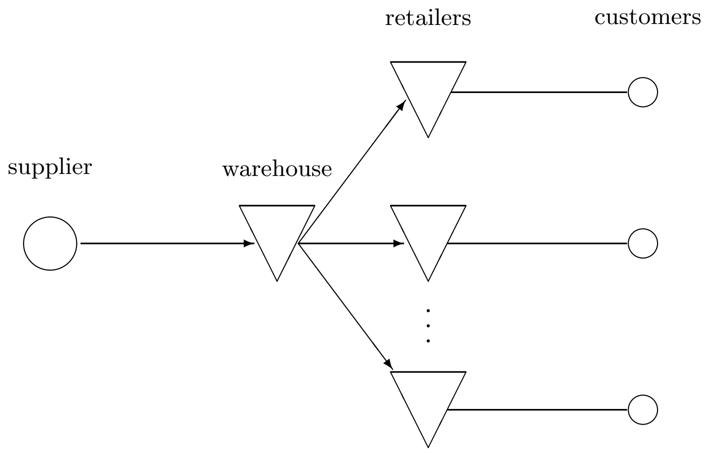

Figure 1.1: One-warehouse multi-retailer system.

how much to order from the supplier (e.g., the manufacturer or the national distributor of the good), and when and how much to ship to the retailers. It is common practice that an order has a leadtime to become available at the warehouse, and there are additional time lags to get the goods from the warehouse to the retailers.

- *Manufacturing:* Suppose an intermediate product/subassembly is stocked at a central location, which is used in the manufacturing/assembly of multiple different end products. The production manager has to decide how much to produce/order of the intermediate product/subassembly, and how much to manufacture of each end product. For example, in a company that produces photocopiers, the planner makes the production decision for a base model and end products that are manufactured by adding various modules to this base model. The leadtimes are the assembly times of the base model and the extra modules. See Rogers and Tsubakitani (1991), for example.

- *Hierarchial production planning:* Consider a production facility that manufactures a product family consisting of several product types. The production planner periodically decides on how much to manufacture at the aggregate product family level and passes this information to the shop floor. This aggregate volume decision is fixed over a time lag (orientation leadtime) during which the planner is flexible in determining the product mix. In other words, once the aggregate volume decision is made for the product family, disaggregation among the individual product types (i.e., allocating the aggregate volume between the product types) is carried out after an orientation leadtime. After a further time lag, known as the frozen leadtime, the manufacturing is completed and the items from each product type become available to satisfy customer demand. (See Wijngaard (1982) and de Kok (1990) for detailed discussions.)

Generally, the warehouse is a stock-keeping facility in the inventory and manufacturing contexts. However, the warehouse might also be a cross-docking point where no physical stock is carried. The functions of such a warehouse are to benefit from quantity discounts through consolidation and to exploit risk pooling by carrying a single inventory during the warehouse leadtime[1]. Eppen and Schrage (1981) refer to the former function as economies of scale and the latter as economies of statistics. Moreover, in both contexts, the aforementioned decisions of ordering, shipment or manufacturing give insights about the amount and positioning (where to keep stock) of the safety stocks in the system. On the other hand, the warehouse in the hierarchial production planning context is a virtual entity. It should be interpreted as a stock point that does not hold stock.

One-warehouse multi-retailer models are also used to study *delayed product differentiation* in the field of operations management. In this setting, an intermediate product is manufactured through a common process (possibly consisting of multiple production stages) and used in the production of several distinct finished goods. The stage after which the products obtain their unique features is called the point of differentiation. In one-warehouse multi-retailer system, while the warehouse may be viewed as the point of differentiation, each retailer can be regarded as a particular finished good. For more detailed information on this topic, see Lee and Tang (1997) and Aviv and Federgruen (2001).

The one-warehouse multi-retailer system under study can be analyzed in continuous or periodic review setting. While the opportunities to review the system information and make decisions exist continuously over time in the former, they are restricted to discrete and equidistant points in time[2] in the latter. The decision maker may have several objectives while controlling the system. The more common objectives one may find in the literature and practice are minimizing the expected total costs (these costs may consist of holding costs due to inventory carrying, penalty costs due to backlogging and fixed cost of ordering/shipment) solely, or minimizing expected total costs (these costs may consist of inventory holding costs and fixed cost of ordering/shipment) subject

[1]Note that these benefits are also available for a stock-keeping warehouse.

[2]Time in between these points are called periods. An hour, a day, a week, etc. can be a period.

to service level constraints like target no-stockout probabilities or target fill rates at the retailers. In this dissertation, we consider the minimization of the expected inventory holding and penalty costs. For more information on service measure models, see Diks *et al.* (1996).

We close this section by giving the definitions of some conventional terms in the inventory literature. *On-hand stock* is the physical stock at a stock point; it is nonnegative. *Echelon stock* of a stock point is the stock at that stock point plus in transit to or on hand at any stock point downstream minus the backorders at the end stock points. *Echelon inventory position* of a stock point is the echelon stock plus all orders that are in transit to this stock point. For example, in our distribution model, echelon stock of the warehouse is the on-hand stock at the warehouse plus pipeline inventories between the warehouse and the retailers (the shipments that have already left the warehouse, but have not reached their final destinations) plus the on-hand stock at the retailers minus the backorders. Echelon inventory position of the warehouse is the echelon stock of the warehouse plus the orders that have already been placed but not yet received by the warehouse.

1.2 Motivation of the Research

The literature on one-warehouse multi-retailer systems can be divided into two streams depending on the review type employed. In the first stream, *continuous review* is used. Here, the structure of the optimal policy is unknown. Generally, a class of policies for the control of the system is assumed like base stock policies, and the policy parameters are optimized (optimization within the class). The seminal work in this stream is by Sherbrooke (1968) who built a mathematical model for the analysis of base stock policies (coined as METRIC model), and developed an approximate evaluation scheme based on the first moment of the resupply time of a retailer order. An exact evaluation for the METRIC model was provided by Simon (1971) and an approximation based on two-moment fits by Graves (1985). Moreover, Graves developed an exact optimization scheme to find the optimal base stock levels, which holds for general distribution structures. The extension of the METRIC approach to installation stock (R, Q) policies[3] was carried out by Deuermeyer and Schwarz (1981). Later, Chen and Zheng (1997) developed an exact evaluation scheme for echelon stock (R, Q) policies[4] under centralized control. Axsäter (1990)

[3]When the inventory position drops to or below R, one or more batches of size Q are ordered to bring the inventory position above R.

[4]When the echelon inventory position drops to or below R, one or more batches of size Q are ordered to bring the echelon inventory position above R.

proposed a new approach for the exact evaluation of the METRIC model. In this analysis, a unit is followed from the moment it enters the system until it exits by fulfilling a demand for the purpose of determining the holding and penalty costs associated with this unit. This approach led to further extensions, see Axsäter (2000). For an extensive review on continuous review models we refer to Axsäter (2003) and the references therein.

A common assumption in the stream of continuous review papers is that the backlogged retailer orders are fulfilled at the warehouse on a *first-come-first-served (FCFS)* discipline. This is a restrictive assumption because the allocation of the warehouse stock is based on the arrival times of the retailer requests only. Thus, the current inventory levels of the retailers are not taken into account, and no differentiation between the retailer attributes (e.g., cost parameters) is made. There is recent research that investigates ways to relax the FCFS rule, see Axsäter and Marklund (2004), and Marklund (2004).

The second stream assumes *periodic review.* In this stream, a main analysis technique emanates from the early work by Clark and Scarf (1960). The authors considered an N-echelon serial inventory system[5], see Figure 1.2 for a representation of the system. Stock point 1 is facing the stochastic demand of the customers and replenished by shipments from its predecessor, stock point 2. Stock point 2 is replenished by shipments from 3, ..., $N-1$ by shipments from N, and stock point N orders from an external supplier with ample stock. There are fixed replenishment leadtimes for each stock point. Clark and Scarf developed a dynamic programming formulation for the centralized control of this system in a finite horizon of n periods. The objective is to minimize the expected total discounted holding and penalty cost of the system. They showed that the resulting N-dimensional dynamic program (DP) can be optimized by solving N single-dimensional DPs sequentially. Their approach works as follows. First, stock point 1 is considered in isolation assuming there is ample stock at stock point 2 (this structure is referred to as subsystem 1). An n-period DP for subsystem 1 is formulated and the optimal policy is a base stock policy, i.e., in each period the inventory position of stock point 1 is raised to a period specific base stock level. Next, a subsystem that consists of stock point 1 and 2 is considered in isolation assuming there is ample stock at stock point 3 (this structure is referred to as subsystem 2). An n-period DP is constructed for subsystem 2. Next, an induced-penalty cost function is introduced for each period, which shows the cost increase at stock point 1 in case the optimal base stock level (that is found by solving the DP for subsystem 1) is not attained. Clark and Scarf showed that the induced-penalty cost

[5]A serial system is a special divergent structure where each stock point has a single successor and a single predecessor.

function is a function of the echelon inventory of stock point 2 and the solution of the DP for subsystem 1. Further, they proved that the DP for subsystem 2 can be decomposed into two distinct DPs where one is the DP for subsystem 1, and the other is a DP that minimizes the sum of the expected holding costs of stock point 2 and the expected induced penalty costs. Although the second DP uses the solution of the DP for subsystem 1, it is a single-dimensional DP, and the optimal policy is an echelon base stock policy. Continuing in this manner, the N-dimensional DP formulated for the entire system can be worked out by solving single-dimensional DPs recursively starting from subsystem 1. We refer to this result as the *decomposition property*, and the analysis carried out by Clark and Scarf as the *decomposition technique*. The decomposition property allowed Clark and Scarf to characterize the optimal policy for an N-echelon serial system where each stock point follows an echelon base stock policy.

The decomposition property and the optimality of echelon base stock policies were extended to infinite horizon case under the average and discounted cost criteria by Federgruen and Zipkin (1984c). In an infinite horizon problem, the optimal base stock levels become stationary and these base stock levels can be computed by solving N nested single-dimensional convex cost functions sequentially (See Langenhof and Zijm (1990), van Houtum and Zijm (1991), Chen and Zheng (1994b)).

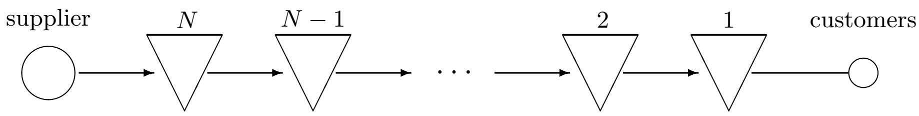

Figure 1.2: N-echelon serial system.

Clark and Scarf (1960) also tried to apply their decomposition technique to divergent structures, but it is not possible to show the decomposition property due to the allocation (rationing) problem. (In a one-warehouse multi-retailer setting, in each period the decision maker decides on the amount of stock to keep at the warehouse and the shipment sizes for the retailers, which is called the allocation problem.) Thus, the optimization requires the analysis of a multi-dimensional DP, which is intricate. Like in the continuous review stream, the structure of the optimal policy for the one-warehouse multi-retailer model is unknown. However, a presupposition, known as the *balance assump-*

tion, leads to the decomposition property and the full characterization of the optimal policy. This key assumption is the relaxation of the physical constraint that the inventory positions of the retailers just after the shipment decisions are greater than or equal to the inventory positions prior to these decisions. Other interpretations of the balance assumption are

- allowing negative quantities to be shipped to the retailers,
- permitting immediate return (with no leadtime) of stock at any retailer to the warehouse at no cost,
- allowing the lateral transhipment (shipments between the retailers) of stock[6] with the leadtime of the receiving retailer at no cost[7].

Under the balance assumption, the retailers can be aggregated into a single stock point and the decomposition technique can be applied for the resulting two-echelon serial system.

The balance assumption was introduced by Eppen and Schrage (1981)[8]. They considered a one-warehouse multi-retailer system with a stockless warehouse (cross-docking point) and identical retailers (in terms of cost and leadtime parameters). By optimizing within the class of base stock and (T, S) policies[9], they were able to derive closed-form expressions for the optimal inventory control parameters under the balance assumption. The optimal policies under the balance assumption have been characterized for finite and infinite horizon problems by Federgruen and Zipkin (1984b,c). Federgruen and Zipkin's analysis is based on dynamic programming and they were the first to make the connection between the decomposition technique and the balance assumption.

The three aforementioned papers have pioneered in the development of the theory on the analysis of divergent systems. Here are some papers in the literature that make the balance assumption and lead to various extensions in the one-warehouse multi-retailer system: Jönsson and Silver (1987), Jackson (1988), Schwarz (1989), Erkip *et al.* (1990), Chen and Zheng (1994b), Kumar *et al.* (1995), Bollapragada *et al.* (1998), Diks and de Kok (1998), Cachon and Fisher (2000), Rappold and Muckstadt (2000), Aviv and Federgruen (2001),

[6] In this setting, lateral transhipment has a broader meaning. It does not only imply the shipment of on-hand stock from one retailer to the other, but also includes shipment of stock from one retailer's pipeline inventory to the other.

[7] The lateral transhipment interpretation was first addressed by Clark and Scarf (1960).

[8] Eppen and Schrage refer to the balance assumption as the allocation assumption.

[9] Ordering every T periods such that the echelon inventory position of the warehouse is increased to S.

and Özer (2003). See also the review papers by van Houtum *et al.* (1996), and Axsäter (2003).

Although the balance assumption was proposed by Eppen and Schrage (1981), Clark and Scarf (1960) were the first to address the issue of *balanced retailer inventories*. The balance assumption allows the shipment decisions to be just based on the echelon stock of the warehouse. The individual inventory positions of the retailers become irrelevant since negative shipments are permitted. The inventory positions of the retailers under such a shipment scheme represent an *ideal* state where the retailer inventories are called to be in *balance*. However, in a real setting where the balance assumption is not made, there may be departures from the ideal state and this contributes to the *imbalance* of the system.

As demonstrated, the balance assumption is utilized extensively in the literature, but there are only a few studies that analyze the quality of this assumption. Zipkin (1984) proposed the first analytical approach, which uses dynamic programming. In a relatively simple setting (zero leadtimes for orders and shipments, warehouse as a cross-docking point), he developed a DP that accounts for the imbalance in the system. In the spirit of the induced penalty cost approach of Clark and Scarf (1960), the system cost is modelled consisting of two components: cost obtained under the balance assumption, and an additional cost that is a consequence of the imbalance in the system. A numerical study shows the accuracy of the approximation for the limited number of scenarios considered. It is concluded that imbalance can be significant when demand variances are large. The studies by van Donselaar and Wijngaard (1987), and van Donselaar (1990) investigate the effect of imbalance on P_1 service level, i.e., probability of stockout. In the former study, the results of a numerical study conclude the little impact of imbalance on the system service level. We believe that the numerical study conducted is rather restricted to come to such a conclusion. The latter paper, incorporates the effect of batch sizes on the imbalance of retailer inventories.

Federgruen and Zipkin (1984a), Kumar and Jacobson (1998), and Axsäter *et al.* (2002) developed heuristics for the control of one-warehouse multi-retailer systems. Since the optimal policy and the associated cost is unknown, instead of comparing the cost of their heuristics to the optimal cost, they all used the relative gap between the cost of the system under the balance assumption and the cost of the heuristics (found by simulation) as the performance measure[10]. The numerical results of these papers give ideas on the effect of

[10]Note that the balance assumption leads to a relaxation of the original optimization problem. Thus, the system-wide cost calculated analytically under the balance assumption is a lower bound for the true optimal cost. An estimate for the cost of a given policy can be

the balance assumption on system-wide cost. A small relative gap implies that the heuristic used is a good alternative for the optimal policy, and the lower bound value (analytical cost obtained under the balance assumption) is an accurate approximation for the true optimal cost.

The numerical results provided by Federgruen and Zipkin (1984a) indicate that the relative gaps are small (maximum relative gap of 6.42%) when the retailers are identical in terms of holding and penalty cost parameters, lead-times and demand distributions, and there is a fixed cost of ordering. For scenarios with no fixed cost of ordering, the gaps are still small (mostly less than 2% and a maximum relative gap of 4.14%) when the holding and penalty costs and demand distributions are not identical across the retailers.

Kumar and Jacobson (1998) developed a new heuristic coined as the hybrid heuristic. They also used the relative gap between the cost of the hybrid heuristic obtained by simulation and the cost of the system obtained under the balance assumption in order to test the impact of the balance assumption on system cost. The numerical results from 64 scenarios for identical retailers (equal holding and penalty cost parameters, leadtimes, and demand distributions) show that the maximum relative gap among (i) 28 moderate variance scenarios (coefficient of variation 0.8 or 1) is 0.78% (ii) 36 low variance scenarios (coefficient of variation 0.2, 0.4 or 0.6) is 0.15%.

In a recent study, Axsäter *et al.* (2002) studied a system composed of a stock keeping warehouse and multiple (possibly nonidentical) retailers. While the retailers follow base stock policies, the warehouse applies an echelon (R, Q) policy. Considering two heuristics for the warehouse replenishment policy and two heuristics for the allocation problem, the average expected holding and penalty costs of four different combinations of these heuristics are determined via simulation under various scenarios. The relative gap between the simulation results and the analytical optimal expected cost under the balance assumption is used to assess the performance of the heuristics. The results for 68 scenarios show that even the best heuristic may result in a relative gap of 93.9%. Although they conclude that the balance assumption is less appropriate in situations with long order cycles and large differences between the retailers in terms of service requirements and demand characteristics, we believe that they cannot come to such strong conclusions without knowing the true optimal cost.

The numerical results of Aviv and Federgruen (2001) and Rappold and Muck-

obtained by simulation, so the cost of a feasible policy constitutes an upper bound for the true optimal cost of the system. Using the relative gap between the upper and the lower bounds as a performance measure is common in the analysis of one-warehouse multi-retailer systems; see also Cachon and Fisher (2000), Özer (2003).

stadt (2000) also advocate the use of the balance assumption. Although they consider other issues like capacity and seasonality, the topology of the systems considered is distribution type.

The balance assumption is a key step in the analysis of one-warehouse multi-retailer systems under periodic review. Although it has been used extensively, and there is evidence that it *may not* be an accurate approximation, there is no clear-cut study in the literature that investigates the appropriateness of the balance assumption. The numerical results from studies that develop heuristics are limited and the focus of these studies is to test the performance of the heuristics developed.

In this research, we analyze the impact of the balance assumption on the expected average system-wide cost in an infinite horizon in one-warehouse multi-retailer systems under periodic review. The details of the model are discussed in §1.1; notice that we consider neither capacity nor lot sizing issues. Our methodology is explained in the subsequent paragraph.

The balance assumption allows one to apply the decomposition technique and characterize the optimal policy. Under the balance assumption, the optimal ordering policy is echelon base stock policy (the echelon inventory position of the warehouse is raised to a certain base stock level in each period), and the optimal shipment policy is *myopic allocation* (echelon stock of the warehouse is allocated among all stock points such that the sum of the expected holding and penalty costs of the retailers in the periods the shipped quantities reach their destinations is minimized). The average expected cost of this policy can be determined by solving analytical cost functions. Since the balance assumption is the relaxation of some constraints in the original optimization problem, the average expected system-wide cost calculated is a lower bound (LB) for the true optimal cost (cf. Geoffrion (1970)). Even though the optimal policy is fully characterized under the balance assumption, the optimal shipment decisions can be negative, which makes the implementation of this policy infeasible. In such a case, the shipment decisions can be modified as the retailers with negative shipments are given nothing and the warehouse on-hand stock is rationed among the rest of the retailers. Since the resulting policy (we refer to as *LB heuristic policy*[11] throughout the dissertation) is feasible, the simulation of it gives an upper bound (UB) for the true optimal cost.

We set up two numerical studies for the assessment of the balance assump-

[11] This policy is indeed optimal if the system is controlled myopically, see Axsäter *et al.* (2002, p.79) or Zipkin (2000, pp. 340-342). The LB heuristic policy was also used by Federgruen and Zipkin (1984a) and Axsäter *et al.* (2002).

tion. In the first one, we use the relative gap between the optimal average cost obtained analytically under the balance assumption (LB) and the cost estimated by simulating the LB heuristic policy (UB). Two test beds consisting of 2000 and 3888 scenarios are developed for identical (in terms of cost parameters, leadtimes and demand distributions) and nonidentical retailers, respectively. The results direct us to a clear and complete overview on when the relative gap is *small* and when *not*. The scenarios with small gaps are the cases in which the use of balance assumption is justified. For these scenarios, it can be concluded that (i) LB is a good proxy for the true optimal average expected cost, and (ii) LB heuristic policy performs well. We also look at the relationship between the various scenario parameters (holding and penalty costs, demand parameters, leadtimes, number of retailers) and the relative gap. Many practically relevant cases with nonidentical retailers exhibit moderate or large relative gaps. In order to be able to analyze the effect of the balance assumption in these cases, it is necessary to compute the true optimal cost of the system, which is the main objective of the second numerical study.

As done in studies by Clark and Scarf (1960), and Federgruen and Zipkin (1984b,c), a DP, which is inevitably multi-dimensional, can be developed for the control of the system under study. Due to the curse of dimensionality, it is unrealistic to solve the resulting DP numerically in a practical setting. The need for a numerical solution method (e.g., value iteration) and the curse of dimensionality compel us to use discrete demand distributions and limited values for input parameters for the second study. However, the decomposition and optimality results under the balance assumption only exist for continuous demand distributions. Therefore, we extend the optimality of the base stock policies to the discrete demand case as a preliminary study.

In the second numerical study, assuming discrete and bounded demand distributions, we calculate the optimal system-wide cost by value iteration (successive approximation) for the settings with moderate or large gaps that are identified in the first study. Due to the curse of dimensionality, we are forced to work with a limited number of retailers and demands distributed over a restricted number of points. The results allow us to evaluate the true impact of the balance assumption by determining the relative gap between the optimal cost (found by solving the DP) and the cost under the balance assumption (LB). While the results demonstrate the impact of the balance assumption and the performance of the LB heuristic policy, they also provide valuable information about the optimal policy behavior.

As mentioned before, we prove that the decomposition property and the optimality of base stock policy is still valid in one-warehouse multi-retailer systems facing discrete customer demands under the balance assumption. In addition, we show that the optimal base stock levels satisfy *newsboy characterizations*. The newsboy model was introduced by Arrow *et al.* (1951). The name emanates from a typical problem faced by a newsboy who has to decide how many newspapers to purchase each day to maximize his profit, where demand is stochastic. For each copy that is left at the end of a day there is a loss associated and any copy that is sold brings profit. The newsboy model of Arrow *et al.* (1951) uses this analogy for modelling the main trade-off faced by an inventory manager under demand uncertainty: keeping stock vs. backlogging demand. Assigning a holding cost for each item that is kept in stock at the end of a period and a penalty cost for each demand unit that is backlogged, the optimal policy that minimizes the expected cost is characterized as base stock policy. The optimal base stock level results in a probability of no-stockout that is equal to a ratio of the holding and penalty costs. We refer to such conditions for optimal policy parameters as newsboy characterizations. Newsboy characterizations show a direct relation between the probability of no-stockout (at a stock point facing customer demand) as a result of an optimal policy parameter (e.g., base stock level, reorder level) and the cost parameters (e.g., holding and penalty costs) in the from of equalities or inequalities. Newsboy characterizations are appealing because they

- are easy to explain to nonmathematical oriented people like managers and MBA students,
- contribute to the understanding of optimal control,
- help intuition development by providing a direct relation between cost and optimal policy parameters.

Newsboy characterizations for multi-echelon inventory systems were first developed by van Houtum and Zijm (1991) who derived newsboy equalities for the optimal base stock levels in a serial system. Similar characterizations were identified for serial systems with fixed replenishment intervals by van Houtum *et al.* (2003). Diks and de Kok (1998) derived newsboy equalities for optimal base stock levels of general divergent structures under the balance assumption. In this dissertation, we show newsboy characterizations for (i) one-warehouse multi-retailer systems facing *discrete* customer demands, and (ii) serial systems with fixed batch quantities. Our newsboy inequalities for one-warehouse multi-retailer systems with discrete demands extend the newsboy equalities of Diks and de Kok (1998) for the continuous demand case. The newsboy inequalities that we have derived for one-warehouse multi-retailer systems encouraged us to seek for similar expressions in other multi-echelon structures. Hence, we also study an N-echelon serial system where materials flow from one stock point to another in fixed batches[12] for which Chen (2000) has proved that echelon (R, Q) policies are optimal. Based on the work of Chen (1998), we develope a new cost formulation for the system and show that the optimal reorder levels satisfy newsboy equalities (inequalities) when the demand distribution is continuous (discrete). These results generalize the newsboy equalities of van Houtum and Zijm (1991) to systems with fixed batch sizes and discrete demand distributions.

There is a line of research that is closely related to the studies on newsboy characterizations. Shang and Song (2003) studied an N-echelon serial inventory system under the average cost criterion in an infinite horizon. As mentioned before, N nested cost functions have to be solved sequentially for the purpose of computing the optimal base stock levels. Shang and Song rewrote these cost functions in terms of installation holding costs, which allowed them to bound the aforementioned cost functions from above and below by single-stage newsboy-type functions. They showed that the base stock levels minimizing the newsboy-type bound functions envelope the optimal base stock levels. Moreover, they developed a heuristic which uses the average of the bounds as a proxy for the optimal base stock level, and the numerical results show that the heuristic performs successfully. The results are extended to serial systems with fixed batch sizes in Shang and Song (2005). Following the approach of Shang and Song (2003), Lystad and Ferguson (2005) developed a similar newsboy-type heuristic for general divergent structures. The main difference between our newsboy characterizations and Shang and Song's studies is that while we derive newsboy characterizations for the optimal policy parameters (base stock and reorder levels), they bound the cost functions to be solved by single-stage newsboy-type functions. Moreover, our approach leads to the lower bound cost function, and the corresponding lower bound reorder level (for the optimal reorder level) developed by Shang and Song (2005).

1.3 Problem Statement and Research Questions

The research presented in this dissertation aims to investigate the appropriateness of an assumption, the balance assumption, which is widely used in the analysis of one-warehouse multi-retailer systems. In order to achieve this goal, we try to find answers to the following questions:

1. Does the balance assumption lead to an accurate approximation?

[12] This model is a generalization of Clark and Scarf model where fixed batch quantities equal to one at each stock point.

2. In which system settings (scenarios) is the use of the balance assumption justified (unjustified)? In other words, under which conditions does the balance assumption lead to an accurate (inaccurate) approximation?

3. What is the optimal policy behavior in the scenarios where the balance assumption leads to a mediocre approximation?

4. How is the performance of the LB heuristic policy?

5. Does the decomposition property and the optimality of the base stock policy still hold under the balance assumption when the demand distributions are discrete? Under the balance assumption, Diks and de Kok (1998) have shown that the optimal base stock levels satisfy newsboy equations in N-echelon divergent inventory systems with continuous demand distributions. If the extension of the decomposition property and the optimality of the base stock policies is possible, does a similar newsboy characterization also exist for the optimal base stock levels (under the balance assumption)?

1.4 Outline of the Dissertation

The remainder of the dissertation is organized as follows. In Chapter 2, we analyze the one-warehouse multi-retailer system facing discrete customer demands under the balance assumption. The decomposition property and the optimality of the base stock policies are shown to hold for the discrete demand case, too. Further, we provide newsboy characterizations for the optimal base stock levels.

Chapter 3 is dedicated to the first numerical study where the relative gap between the optimal average cost under the balance assumption (obtained analytically) and the cost obtained by simulating the LB heuristic policy is used. We explicitly identify the scenarios resulting in small (moderate or large) relative gaps.

In Chapter 4, we develop a DP for the control of the system under study for the purpose of computing the true optimal cost. We assume discrete demand distributions with finite supports (specifically, demand is assumed to be distributed over a small set of points) in order to be able to solve the resulting DP. Using value iteration algorithm, we compute the true optimal cost for various settings that are found to lead to moderate or large relative gaps in Chapter 3. As a second numerical study, the optimal cost under the balance assumption is compared to the true optimal cost. The results allow us to determine the impact of the balance assumption precisely. In addition, we are able to evaluate the performance of the *LB* heuristic policy. Moreover, we get new and interesting insights into the optimal policy behavior.

The newsboy characterizations presented in Chapter 2 encouraged us to seek for similar results in other multi-echelon systems. In Chapter 5, we analyze an N-echelon serial system with a fixed batch size at each echelon. Chen (2000) has shown that echelon stock (R, Q) policies are optimal for such systems. We show that the optimal reorder levels satisfy newsboy inequalities (equalities) when the demand distribution is discrete (continuous).

Finally, we summarize the main results of the research and discuss directions for further research in Chapter 6.

This dissertation resulted in three Beta Research School working papers. We are currently in the process of getting them published in scholarly journals. The studies in Chapter 2, 3 and 5 led to Doğru *et al.* (2004), Doğru *et al.* (2005a) and Doğru *et al.* (2005b), respectively. At present, we are working on turning the study of Chapter 4 into a paper.

Chapter 2

Distribution Systems Facing Discrete Demands

Abstract: *In this chapter, we consider a two-echelon distribution system, which consists of a single warehouse serving N (possibly nonidentical) retailers that face discrete stochastic demand of the customers. We assume periodic review and centralized control, and the objective is to minimize the average expected inventory holding and penalty costs. Under the balance assumption, we show that base stock policies are optimal. Actually, we extend the optimality of base stock policies for continuous demand models under the balance assumption to the discrete demand case. Further, we derive newsboy inequalities for the optimal base stock levels and develop an efficient algorithm for the computations of an optimal policy.*

2.1 Introduction

This chapter treats a two-echelon distribution system under periodic review and centralized control. There are several retailers supplied by shipments from a warehouse, which in return orders from an exogenous supplier with ample stock. There are fixed leadtimes between the supplier and the warehouse, and between the warehouse and the retailers. The retailers face *discrete* stochastic demand of the customers. Excess demand is backlogged and linear penalty costs are incurred. There are no fixed costs and the objective is to minimize the average inventory holding and penalty costs of the system in the long-run.

Clark and Scarf (1960) were the first to consider the inventory control problem in a distribution system facing continuous demands. They were not able to apply the decomposition technique to these systems due to the so-called *allocation (rationing) problem*. There is an allocation decision of how to distribute the physical stock at the warehouse among all the stock points, which has to be made in each period.

Although the optimal policy for the inventory control of a distribution system is unknown, there are some approximate approaches. The key assumption (in these approaches) is that the inventories of the retailers are balanced, i.e., the allocation policy may apportion negative quantities to the retailers. The balance assumption allows one to apply Clark and Scarf's decomposition technique to distribution systems.

The distribution model under consideration with continuous demands is well studied in the literature; see Clark and Scarf (1960), Eppen and Schrage (1981), Federgruen and Zipkin (1984a,b,c), Diks and de Kok (1998). See also van Houtum *et al.* (1996), and Axsäter (2003) for extensive reviews. However, the discrete demand case has not been studied up to now. Discrete demand processes are important since it makes it possible to handle positive probability mass at any point in the demand distribution, particularly at zero. This is highly important in case of intermittent demand.

Our contribution in this study is twofold. First, under the balance assumption, we extend the optimality of base stock policies to two-echelon distribution systems facing discrete demands. The proof is not a trivial extension because the proof in the continuous demand case heavily depends on identifying equalities that have to be satisfied by the optimal control parameters (base stock levels and allocation functions). In contrast, in the discrete demand case, optimal control parameters have to satisfy certain inequalities. Second, we show that the optimal base-stock levels satisfy *newsboy inequalities*. Under continuous demand, newsboy equalities have been derived for multi-echelon serial and distribution systems by van Houtum and Zijm (1991), and Diks and de Kok (1998), respectively. Our newsboy inequalities extend the newsboy equalities of Diks and de Kok, and to the best of our knowledge are the first newsboy characterization for multi-echelon systems with discrete demand. In addition, we develop an efficient algorithm for the computation of an optimal policy.

This chapter is organized as follows. In §2.2, we introduce the model. The complete analysis is presented in §2.3; §2.3.1 and §2.3.2 set the stage for the proof, which is conducted in §2.3.3 and §2.3.4. Newsboy inequalities and the algorithm for the computation of an optimal policy are discussed in §2.3.5 and §2.3.6, respectively.

2.2 Model

Consider a two-stage distribution system that consists of a single warehouse and N retailers. The warehouse (indexed as stock point 0) *orders* from an exogenous supplier with ample stock and the retailers are supplied by *shipments* from the warehouse. Retailers face stochastic and independent demands of the customers. Demands in different periods are i.i.d., discrete nonnegative random variables. Any unfulfilled demand at a retailer is backlogged. Time is divided into periods of equal length and the following sequence of events takes place during a period:

- inventory levels are observed and the current period's ordering/shipment decisions are made considering the arrival of the orders/shipments given before (at the beginning of the period),
- orders/shipments arrive following their respective leadtimes (at the beginning of the period),
- demand occurs,
- holding and penalty costs are assessed on the period ending inventory and backorder levels (at the end of the period).

Leadtimes of orders (between the supplier and the warehouse), and shipments (between the warehouse and the retailers) are fixed. Costs consist of linear holding and penalty costs. Finally, we assume that the system is centrally controlled and the objective is to minimize the expected holding and penalty costs of the system in the long-run.

Here is the basic notation for this study:

$\mathbb{Z}$	=	set of integer numbers. $\mathbb{Z}^+ = \{1, 2, ...\}$, and $\mathbb{Z}_0^+ = \mathbb{Z}^+ \cup \{0\}$
t	=	index for time. Period t is defined as the time interval between epochs t and $t+1$ for $t \in \mathbb{Z}_0^+$.
N	=	number of retailers, $N \in \mathbb{Z}^+$.
i	=	index for stock points, $i = 0$ is the warehouse, and $i = 1, 2, ..., N$ are the retailers.
J	=	set of retailers, i.e., $J = \{1, 2, ..., N\}$.
l_i	=	leadtime parameter for stock point i. $l_i \in \mathbb{Z}_0^+ \ \forall i \in J$ and $l_0 \in \mathbb{Z}^+$
p_i	=	penalty cost parameter for retailer i. A cost p_i is charged for each unit of backlog at the end of a period at retailer i. $p_i > 0 \ \forall i \in J$

h_i = additional inventory holding cost parameter for stock point i. At the end of a period:
(i) cost h_0 is charged for each unit on stock at the warehouse or in transit to any retailers, $h_0 \geq 0$,
(ii) cost $h_0 + h_i$ is charged for each unit on stock at retailer i, $h_i \geq 0 \ \forall i \in J$.

μ_i = mean of one-period demand faced by retailer i.

μ_0 = mean of one-period demand faced by the system, i.e., $\mu_0 = \sum_{i \in J} \mu_i$.

$D_i(t, t+s)$ = discrete random variable denoting the demand faced by retailer i during the periods $t, t+1, ..., t+s$ for $t, s \in \mathbb{Z}_0^+$.

$D_0(t, t+s)$ = discrete random variable denoting the aggregate demand faced by the system during the periods $t, t+1, ..., t+s$, i.e., $D_0(t, t+s) = \sum_{i \in J} D_i(t, t+s)$ for $t, s \in \mathbb{Z}_0^+$.

$D_i^{(l)}$ = discrete random variable denoting l-period demand faced by retailer i, $l \in \mathbb{Z}_0^+$.

$D_0^{(l)}$ = discrete random variable denoting l-period aggregate demand faced by the system, $l \in \mathbb{Z}_0^+$.

$F_i^{(l)}$ = cumulative distribution function of l-period demand of retailer i defined over $\mathbb{Z}_0^+$.

$F_0^{(l)}$ = cumulative distribution function of l-period demand faced by the system defined over $\mathbb{Z}_0^+$, i.e., $F_0^{(l)} = F_1^{(l)} * F_2^{(l)} * ... * F_N^{(l)}$.

$I_i(t)$ = echelon stock of stock point i at the beginning of period t just after the receipt of the incoming order/shipment.

$\hat{I}_i(t)$ = echelon stock of stock point i at the end of period t.

$\hat{IP}_i(t)$ = echelon inventory position of stock point i at the end of period $t-1$, $t \in \mathbb{Z}^+$ = echelon inventory position of stock point i at the beginning of period t just before ordering (if $i = 0$) or shipment (if $i \in J$).

$IP_i(t)$ = echelon inventory position of stock point i at the beginning of period t just after ordering (if $i = 0$) or shipment (if $i \in J$).

2.3 Analysis

We discuss the ordering and allocation decisions with their impacts on the costs, and introduce the optimization problem under study in §2.3.1. The allocation decision is analyzed and the balance assumption is introduced in

§2.3.2. This constitutes the basis for the analysis of a single ordering cycle, and the derivation of an average cost optimal policy, in §2.3.3 and §2.3.4, respectively. Finally, we discuss the newsboy inequalities in §2.3.5 and conclude with an algorithm for computations in §2.3.6.

2.3.1 Dynamics of the System

The total cost of the system at the end of an arbitrary period t is equal to

$$h_0\left(\hat{I}_0(t) - \sum_{i\in J}\hat{I}_i(t)\right) + \sum_{i\in J}(h_0+h_i)\hat{I}_i^+(t) + \sum_{i\in J}p_i\hat{I}_i^-(t)$$

where $a^+ = \max\{0,a\}$ and $a^- = -\min\{0,a\}$ for $a\in\mathbb{R}$. Substituting $\hat{I}_i(t) = \hat{I}_i^+(t) - \hat{I}_i^-(t)$ first, rearranging the terms, and then using the identity $\hat{I}_i^+(t) = \hat{I}_i(t) + \hat{I}_i^-(t)$ leads to the following result:

$$\begin{aligned}
&h_0\left(\hat{I}_0(t) - \sum_{i\in J}\hat{I}_i(t)\right) + \sum_{i\in J}(h_0+h_i)\hat{I}_i^+(t) + \sum_{i\in J}p_i\hat{I}_i^-(t)\\
&\qquad = h_0\hat{I}_0(t) + \sum_{i\in J}h_i\hat{I}_i^+(t) + \sum_{i\in J}(h_0+p_i)\hat{I}_i^-(t)\\
&\qquad = h_0\hat{I}_0(t) + \sum_{i\in J}h_i\hat{I}_i(t) + \sum_{i\in J}(h_0+h_i+p_i)\hat{I}_i^-(t). \qquad (2.1)
\end{aligned}$$

We define $h_0\hat{I}_0(t)$ as the *cost attached to the echelon of the warehouse* (echelon of stock point 0) at the end of period t. This cost is denoted by $C_0(t)$. We define $h_i\hat{I}_i(t)+(h_0+h_i+p_i)\hat{I}_i^-(t)$ as the *cost attached to the echelon of retailer* i at the end of period t, and denote it by $C_i(t)$.

Consider the following two connected decisions and the resulting effects on the expected costs, which start with an order given to the supplier in period t, $t\in\mathbb{Z}_0^+$. Figure 2.1 illustrates the dependence among these decisions and the resulting cost consequences.

Ordering Decision: Assume that at the beginning of period t the warehouse gives an order that raises the inventory position of the system to some level y_0, i.e., $IP_0(t) = y_0$. The order materializes at the beginning of period $t+l_0$ and the echelon stock of the warehouse at that epoch is $y_0 - D_0(t, t+l_0-1)$. There are two consequences of the ordering decision:

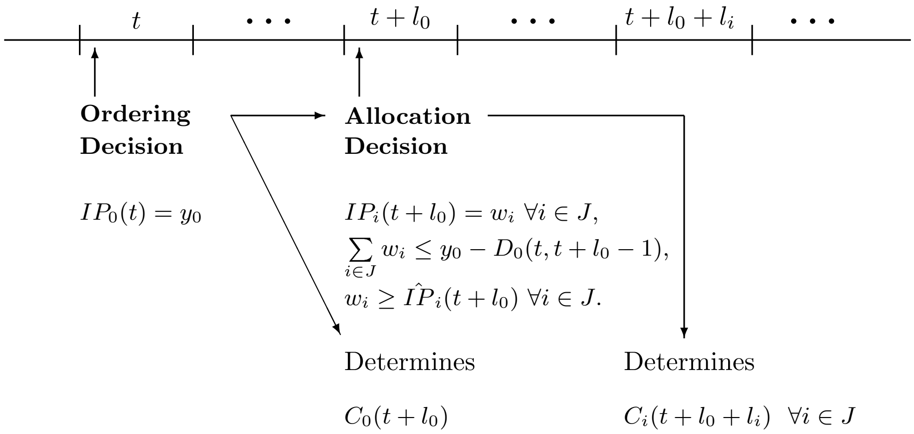

Figure 2.1: The consequences of an order given to the supplier in period t.

- It directly determines the expected value of the cost attached to the echelon of the warehouse at the end of period $t + l_0$,

$$\begin{aligned}\mathbf{E}\left[C_0(t+l_0)|IP_0(t)=y_0\right] &= \mathbf{E}[h_0(y_0 - D_0(t,t+l_0))] \\ &= h_0(y_0-(l_0+1)\mu_0).\end{aligned}$$

- It limits the shipment quantities to the retailers. In other words, it puts an upper bound on the level to which one can increase the aggregate echelon inventory positions of the retailers in period $t + l_0$,

$$\sum_{i\in J} IP_i(t+l_0) \leq y_0 - D_0(t, t+l_0-1).$$

Allocation Decision: At the beginning of period $t+l_0$, the system-wide stock is rationed among *all* stock points. In other words, the shipment quantities to the retailers are determined; as a result, the decision of how much stock to retain at the warehouse is made. At epoch $t + l_0$, the inventory position of retailer i is increased to some level w_i such that $\sum_{i\in J} w_i \leq y_0 - D_0(t, t+l_0-1)$ and $w_i \geq \hat{IP}_i(t + l_0)$ for all $i \in J$. These decisions directly affect the cost of echelon i at the end of period $t + l_0 + l_i$, for all $i \in J$. The expected value of the cost attached to echelon i is

$$
\begin{aligned}
&\mathbf{E}\left[C_i(t+l_0+l_i)|IP_i(t+l_0)=w_i\right] \\
&= \mathbf{E}\Big[h_i(w_i - D_i(t+l_0, t+l_0+l_i)) \\
&\qquad\qquad + (h_0+h_i+p_i)\left(w_i - D_i(t+l_0, t+l_0+l_i)\right)^-\Big] \\
&= h_i\left(w_i - (l_i+1)\mu_i\right) + (h_0+h_i+p_i)\mathbf{E}\left[D_i(t+l_0, t+l_0+l_i) - w_i\right]^+ .
\end{aligned}
$$

We define the expected costs as a consequence of the ordering and allocation decisions that begin with the warehouse's order given at epoch t as the *cycle cost of period* t and denote it by $C_{cyc}(t)$.

$$C_{cyc}(t) = C_0(t+l_0) + \sum_{i \in J} C_i(t+l_0+l_i)$$

Let Π and $g(\pi)$ denote the set of all ordering/allocation policies and the average expected cost of policy π, respectively. The expected long-run average cost of any policy $\pi \in \Pi$ is simply the average of the expected value of the sum of costs over all cycles:

$$
\begin{aligned}
g(\pi) &\stackrel{\text{def}}{=} \lim_{T \longrightarrow \infty} \frac{1}{T}\mathbf{E}\left[\sum_{t=0}^{T-1}\sum_{i=0}^{N} C_i(t)\right] \\
&= \lim_{T \longrightarrow \infty} \frac{1}{T}\mathbf{E}\Bigg[\sum_{t=0}^{l_0-1} C_0(t) + \sum_{t=0}^{l_0+l_i-1}\sum_{i \in J} C_i(t) + \sum_{t=0}^{T-1} C_{cyc}(t) \\
&\qquad\qquad - \sum_{t=T}^{T+l_0-1} C_0(t) - \sum_{t=T}^{T+l_0+l_i-1}\sum_{i \in J} C_i(t)\Bigg] \\
&= \lim_{T \longrightarrow \infty} \frac{1}{T}\sum_{t=0}^{T-1}\mathbf{E}[C_{cyc}(t)].
\end{aligned}
$$

The expression above requires the existence and the finiteness of the expectations. Although this may not be the case for any given policy, especially for the ones that do not order enough to satisfy demand, any policy with an underlying Markov process that is unichain meets this requirement. We are interested in such policies. (In the subsequent sections, we show the optimality of base stock policies. The class of base stock policies are well known to satisfy these necessities.) Thus, the optimization problem that we consider is

$$\min_{\pi \in \Pi} \lim_{T \longrightarrow \infty} \frac{1}{T}\sum_{t=0}^{T-1}\mathbf{E}[C_{cyc}(t)]. \tag{2.2}$$

The minimization problem given above is intricate since the decisions are highly interdependent. In the next subsection, we introduce the myopic allocation problem and discuss the balance assumption.

2.3.2 Analysis of the Allocation Decision

In this subsection, we discuss the Allocation Decision described in §2.3.1. Consider the sequence of decisions and the resulting costs as a result of increasing the inventory position of the system up to y_0 at the beginning of period $t,\ t \in \mathbb{Z}_0^+$. Suppose the echelon stock of the warehouse at the beginning of period $t+l_0$ (i.e., $y_0 - D_0(t, t+l_0-1)$) is distributed among all stock points such that the sum of the expected holding and penalty costs of the retailers in the periods the allocated quantities reach their destinations (i.e., period $t+l_0+l_i$ for retailer i) is minimized. This way of rationing is called *myopic allocation* because the effect of the allocation decisions on the subsequent periods is not considered. The mathematical formulation of the problem is as follows:

$$\min_{w_i,\ i\in J} \quad \sum_{i\in J} \mathbf{E}\left[C_i(t+l_0+l_i) | IP_i(t+l_0) = w_i\right] \tag{2.3}$$

$$s.t. \quad \sum_{i\in J} w_i \leq y_0 - D_0(t, t+l_0-1) \tag{2.4}$$

$$\hat{IP}_i(t+l_0) \leq w_i \quad \forall\ i \in J \tag{2.5}$$

Both constraints serve for the physical balance of the stocks. While (2.5) assures that no negative quantity is allocated to the retailers, (2.4) compels that the sum of the allocated quantities cannot exceed the available stock in the system.

Although myopic allocation allows the allocation decisions to be made independent of the future allocation and ordering decisions, it still depends on previous periods' decisions due to (2.5). Consider a relaxed version of the myopic allocation problem where (2.5) is omitted. This is equivalent to assuming that the quantities allocated to the retailers may be negative. We refer to this assumption as the *balance assumption.*

In the absence of (2.5), $C_{cyc}(t)$ depends only on the ordering and allocation decisions that start with the order given by the warehouse in period t, not on decisions of other periods. Next, we focus on how to minimize the cycle cost of period t.

2.3.3 Analysis of a Single Cycle

First of all, we give the definition of convexity for functions defined over $\mathbb{Z}$. Let $\Delta f(x)$ and $\Delta^2 f(x)$ denote first and second order difference equations for a function f where $\Delta f(x) = f(x+1) - f(x)$ and $\Delta^2 f(x) = \Delta f(x+1) - \Delta f(x) = f(x+2) - 2f(x+1) + f(x)$. A discrete function is convex over $\mathbb{Z}$ if $\Delta^2 f(x) \geq 0$ for all $x \in \mathbb{Z}$.

For retailer i, define $G_i(y_i)$ as the expected cost attached to echelon i at the end of period $t + l_i$ when the inventory position at the beginning of period t is increased to y_i for $y_i \in \mathbb{Z}$, and $t \in \mathbb{Z}_0^+$, i.e., $G_i(y_i) = \mathbf{E}[C_i(t + l_i)|IP_i(t) = y_i]$. Now, we analyze the function $G_i(\cdot)$ in Lemma 2.1.

Lemma 2.1 *For all $i \in J$:*

(i) $G_i(y_i) = h_i\,(y_i - (l_i + 1)\mu_i) + (h_0 + h_i + p_i)\mathbf{E}[D_i(t, t + l_i) - y_i]^+, \; y_i \in \mathbb{Z},$

(ii) $\Delta G_i(y_i) = (h_0 + h_i + p_i)F_i^{(l_i+1)}(y_i) - (h_0 + p_i), \; y_i \in \mathbb{Z},$

(iii) $G_i(y_i)$ *is convex over* $\mathbb{Z}$*,*

(iv) $G_i(y_i)$ *is minimized at all* $y_i \in Y_i^* = \{\underline{y}_i^*, \underline{y}_i^* + 1, ..., \overline{y}_i^*\}$ *where*

$$\begin{aligned} \underline{y}_i^* &= \min\left\{ y_i | F_i^{(l_i+1)}(y_i) \geq \tfrac{h_0+p_i}{h_0+h_i+p_i} \right\} \text{ and} \\ \overline{y}_i^* &= \min\left\{ y_i | F_i^{(l_i+1)}(y_i) > \tfrac{h_0+p_i}{h_0+h_i+p_i} \right\}. \end{aligned}$$

If $\left\{ y_i | F_i^{(l_i+1)}(y_i) > \tfrac{h_0+p_i}{h_0+h_i+p_i} \right\} = \emptyset$ *then* $\overline{y}_i^* = \infty$.
(Note that $\overline{y}_i^* = \infty$ *if* $h_i = 0$*. Moreover,* $\underline{y}_i^* = \infty$ *if* $h_i = 0$ *and* $F_i^{(1)}$ *has an infinite support.)*

Proof : The proof is straightforward and omitted.

Notice that if $\underline{y}_i^*$ satisfies $F_i^{(l_i+1)}(\underline{y}_i^*) = \frac{h_0+p_i}{h_0+h_i+p_i}$, then there are multiple optimal values. The results of Lemma 2.1 are utilized for the solution of the relaxed myopic allocation problem.

Let $z_i : \mathbb{Z} \to \mathbb{Z}$, $i \in J$ be an allocation function such that $z_i(a)$ is the portion of a allocated to retailer i for $a \in \mathbb{Z}$. Assume that the system-wide stock at time $t \in \mathbb{Z}_0^+$ is $x \in \mathbb{Z}$, i.e., $I_0(t) = x$. The myopic allocation problem of period

t under the balance assumption may be rewritten as:

$$\min_{z_i(x)\ \forall i \in J} \quad \sum_{i \in J} G_i(z_i(x)) \tag{2.6}$$

$$s.t. \quad \sum_{i \in J} z_i(x) \leq x, \tag{2.7}$$

where a solution is denoted by $\{z_i(x)\}_{i \in J}$. Define $\{z_i^*(x)\}_{i \in J}$ and $H^*(x)$ as an optimal solution and the optimal objective function value of (2.6)-(2.7) for a given x, respectively; $H^*(x) = \sum_{i \in J} G_i(z_i^*(x))$.

Next, we discuss how to characterize an optimal solution for the myopic allocation problem under the balance assumption, i.e., for (2.6)-(2.7).

Lemma 2.2 *Let $x \in \mathbb{Z}$.*

(i) If $x \geq \sum_{i \in J} \underline{y}_i^*$, then $z_i^*(x) \in Y_i^*$ for all $i \in J$ such that $\sum_{i \in J} z_i^*(x) \leq x$,

(ii) If $x \leq \sum_{i \in J} \underline{y}_i^*$, then (2.7) is binding, i.e., $\sum_{i \in J} z_i^*(x) = x$.

Proof : See §2.4.

Notice that the optimal solution of (2.6)-(2.7) for $x \geq \sum_{i \in J} \underline{y}_i^*$ is fully characterized by Lemma 2.2. The following lemma identifies optimal solutions for $x < \sum_{i \in J} \underline{y}_i^*$.

Lemma 2.3 *Let $x < \sum_{i \in J} \underline{y}_i^*$, and $x \in \mathbb{Z}$.*

(i) A given solution $\{z_i^*(x)\}_{i \in J}$ is optimal if and only if:

$$\Delta G_i(z_i^*(x)) \geq \Delta G_j(z_j^*(x) - 1) \qquad \forall i, j \in J,\ i \neq j.$$

(ii) Given an optimal solution $\{z_i^*(x)\}_{i \in J}$ for x, an optimal solution $\{z_i^*(x+1)\}_{i \in J}$ for $x+1$ is given by

$$\begin{gathered} z_k^*(x+1) = z_k^*(x) + 1, \text{ where} \\ k \in \left\{ i \in J | \Delta G_i(z_i^*(x)) = \min_{j \in J} \Delta G_j(z_j^*(x)) \right\}, \text{ and} \\ z_j^*(x+1) = z_j^*(x) \quad \forall j \in J \setminus \{k\}. \end{gathered}$$

(iii) Given an optimal solution $\{z_i^*(x)\}_{i \in J}$ for x, an optimal solution $\{z_i^*(x-1)\}_{i \in J}$ for $x-1$ is given by

$$z_k^*(x-1) = z_k^*(x) - 1, \; where$$
$$k \in \Big\{ i \in J | \Delta G_i(z_i^*(x)-1) = \max_{j \in J} \Delta G_j(z_j^*(x)-1) \Big\}, \; and$$
$$z_j^*(x-1) = z_j^*(x) \quad \forall j \in J \setminus \{k\}.$$

Proof : See §2.4.

Define

$\mathbf{z}$	$=$	set of allocation functions, i.e., $\{z_i\}_{i\in J}$.
$\mathbf{z}^*$	$=$	set of optimal allocation functions, i.e., $\{z_i^*\}_{i\in J}$ such that $z_i^*(x)$ is optimal for all $x \in \mathbb{Z}$ and for all $i \in J$.
$\hat{\mathbf{z}}^*$	$=$	set of nondecreasing optimal allocation functions (i.e., $\{\hat{z_i}^*\}_{i\in J}$ such that $\hat{z_i}^*(x)$ is optimal and $\Delta\hat{z_i}^*(x) = \hat{z_i}^*(x+1) - \hat{z_i}^*(x) \geq 0$ for all $x \in \mathbb{Z}$ and for all $i \in J$).
$\tilde{\mathbf{z}}^*$	$=$	set of nondecreasing optimal allocation functions with the additional property that for all $i \in J$ with $\lvert Y_i^*\rvert > 1$, $\tilde{z}_i^*(x) \in Y_i^* \setminus \{\overline{y}_i^*\}$ for $x > \sum_{i\in J} \underline{y}_i^*$.

Lemma 2.3 simply says that given an optimal solution for x, an optimal allocation of $x+1$ $(x-1)$ units can be carried out by taking the optimal solution for x and giving a retailer k with the smallest $\Delta G_k(z_k(x))$ (largest $\Delta G_k(z_k(x)-1)$) value one unit more (less). This greedy procedure is also known as the *marginal allocation*. An important implication of this lemma is that if an optimal solution of the myopic allocation problem is known for some $x \in \mathbb{Z}$, starting from this optimal solution, one can find optimal allocation functions ($\mathbf{z}^*$) by following the aforementioned greedy procedure.

We are prepared for the main result in regard to optimal allocation functions of the retailers.

Theorem 2.4 *There exist nondecreasing optimal allocation functions* $\hat{\mathbf{z}}^*$.

Proof : Distinguish two cases: (i) $\sum_{i\in J} \underline{y}_i^*$ is finite, (ii) $\sum_{i\in J} \underline{y}_i^*$ is infinite. In case (i), $z_i^*(\sum_{i\in J} \underline{y}_i^*) = \underline{y}_i^*$ $\forall i \in J$ due to Lemma 2.2. Starting from this optimal solution, optimal solutions for $x < \sum_{i\in J} \underline{y}_i^*$ can be obtained using part (iii) of Lemma 2.3, which leads to $z_i^*(x+1) - z_i^*(x) = \Delta z_i^*(x) \in \{0,1\}$ for all $i \in J$. For $x > \sum_{i\in J} \underline{y}_i^*$, take $z_i^*(x) = \underline{y}_i^*$ $\forall i \in J$; as a result $\Delta z_i^*(x) = 0$ for all $i \in J$. In case (ii), an optimal solution $\{z_i^*(x)\}_{i\in J}$ of (2.6)-(2.7) can be determined for some given $x \in \mathbb{Z}$ by Lagrange relaxation (see Everett (1963)). Based on $\{z_i^*(x)\}_{i\in J}$, optimal solutions for $x+1$ and $x-1$ can be constructed utilizing parts (ii) and (iii) of Lemma 2.3, respectively. Continuing in this manner, $z_i^*(x)$ is determined for all $x \in \mathbb{Z}$ and $i \in J$ such that $\Delta z_i^*(x) \in \{0,1\}$. □

Remark 2.1 *Theorem 2.4 shows the existence of nondecreasing optimal allocation functions, but not all optimal allocation functions have to be nondecreasing. Consider the case with three identical retailers, i.e., three retailers with identical leadtimes, cost parameters and demand distributions. Define* $\bar{z}^*(x) = (z_1^*(x), z_2^*(x), z_3^*(x))$*. There are three optimal alternatives for rationing 4 units:* $\bar{z}^*(4) \in \{(2,1,1),(1,2,1),(1,1,2)\}$*. For* $x = 5$*,* $\bar{z}^*(5) \in \{(2,2,1),(1,2,2),(2,1,2)\}$*. Given* $z^*(4) = (2,1,1)$*, if one follows part (ii) of Lemma 2.3, then* $z^*(5)$ *is* $(2,2,1)$ *or* $(2,1,2)$*, which leads to nondecreasing optimal allocation functions at* $x = 4$*. Consider the following optimal allocations:* $z^*(4) = (2,1,1)$ *and* $z^*(5) = (1,2,2)$*; observe that then* $z_1^*(x)$ *is decreasing for* $x = 4$*.*

The following corollary follows directly from the proof of Theorem 2.4.

Corollary 2.5 *There exist optimal allocation functions* $\tilde{\mathbf{z}}^*$*.*

Proof : The additional property in the definition of $\tilde{\boldsymbol{z}}^*$ simply tells that for all the retailers with multiple optimums minimizing $G_i(\cdot)$ (i.e., for all $i \in J$ with $|Y_i^*| > 1$), $z_i^*(x) \neq \overline{y}_i^*$ for all $x > \sum_{i \in J} \underline{y}_i^*$. In fact, the optimal allocation functions constructed in the proof of Theorem 2.4 do have the additional property.

We can show the following properties of the function $H^*(\cdot)$.

Lemma 2.6 *Let* $x \in \mathbb{Z}$*.*

(i) $\Delta H^*(x) < 0$ *for* $x < \sum_{i \in J} \underline{y}_i^*$*,*

(ii) $\Delta H^*(x) = 0$ *for* $x \geq \sum_{i \in J} \underline{y}_i^*$*,*

(iii) $H^*(x)$ *is convex in* x*.*

Proof : See §2.4.

Lemma 2.6 gives the shape of the optimal objective function of the myopic allocation problem given in (2.6)-(2.7) as a function of the amount to allocate. $H^*(x)$ is convex, strictly decreasing in the region $(-\infty, \sum_{i \in J} \underline{y}_i^*)$, and constant over $[\sum_{i \in J} \underline{y}_i^*, +\infty)$.

Let

$G_0(y_0)$ = expected value of the cost attached to the echelon of the warehouse at the end of period $t + l_0$ given $IP_0(t) = y_0$ for $y_0 \in \mathbb{Z}$, and $t \in \mathbb{Z}_0^+$, i.e., $G_0(y_0) = \mathbf{E}\left[C_0(t+l_0)|IP_0(t) = y_0\right] = h_0(y_0 - (l_0+1)\mu_0)$

$\tau(y_0)$ = $\left[y_0 - \sum_{i \in J} \underline{y}_i^* + 1\right]^+$, $y_0 \in \mathbb{Z}$.

Under the balance assumption, let us denote the expected cycle cost of period t given $IP_0(t) = y_0$ and $\mathbf{z}$ for $y_0 \in \mathbb{Z}$, $t \in \mathbb{Z}_0^+$, by $G_{cyc}(y_0, \mathbf{z})$. Thus,

$$\begin{aligned} G_{cyc}(y_0, \mathbf{z}) &= \mathbf{E}\left[C_0(t+l_0) + \sum_{i \in J} C_i(t+l_0+l_i) \mid IP_0(t) = y_0, \mathbf{z}\right] \\ &= G_0(y_0) + \sum_{x=0}^{\infty} \sum_{i \in J} G_i(z_i(y_0 - x)) \Pr\{D_0^{(l_0)} = x\}. \end{aligned} \tag{2.8}$$

Lemma 2.7 .

(i) $G_{cyc}(y_0, \mathbf{z}^*) = G_0(y_0) + \sum\limits_{x=0}^{\infty} H^*(y_0 - x) \Pr\{D_0^{(l_0)} = x\}$, $y_0 \in \mathbb{Z}$,

(ii) $G_{cyc}(y_0, \mathbf{z}^*) \leq G_{cyc}(y_0, \mathbf{z})$, $y_0 \in \mathbb{Z}$,

(iii) $\Delta G_{cyc}(y_0, \mathbf{z}^*) = h_0 + \sum\limits_{x=\tau(y_0)}^{\infty} \Delta H^*(y_0 - x) \Pr\{D_0^{(l_0)} = x\}$, $y_0 \in \mathbb{Z}$,

(iv) $G_{cyc}(y_0, \mathbf{z}^*)$ *is convex in* y_0,

(v) $G_{cyc}(y_0, \mathbf{z}^*)$ *is minimized at all* $y_0 \in Y_0^* = \{\underline{y}_0^*, \underline{y}_0^* + 1, ..., \overline{y}_0^*\}$ *where* $\underline{y}_0^* = \min\{y_0 | \Delta G_{cyc}(y_0, \mathbf{z}^*) \geq 0\}$, *and* $\overline{y}_0^* = \min\{y_0 | \Delta G_{cyc}(y_0, \mathbf{z}^*) > 0\}$.

Proof : See §2.4.

Part (ii) of Lemma 2.7 implies that whatever ordering decision is made at the beginning of the cycle, utilizing $\mathbf{z}^*$ for allocation leads to expected cycle costs as good as any other set of allocation functions. The expressions for the optimal order-up-to levels minimizing $G_{cyc}(y_0, \mathbf{z}^*)$ are given in part (v); note that if $\Delta G_{cyc}(y_0, \mathbf{z}^*) = 0$ then $|Y_0^*| > 1$.

Corollary 2.8 *Under the balance assumption, the minimum expected cycle cost of an arbitrary period* $t \in \mathbb{Z}_0^+$ *is* $G_{cyc}(\underline{y}_0^*, \mathbf{z}^*)$.

2.3.4 Analysis of the Infinite Horizon Problem

In the previous section, we studied the cycle cost of an arbitrary period, which is shown to be convex under the balance assumption. Now, we return to the infinite horizon problem given in (2.2) and study it under the balance assumption.

Denote a *base stock policy* by a tuple $(y_0, \mathbf{z})$, where y_0 is the target echelon inventory position of the warehouse, and $\{z_i(x)\}_{i \in J}$ are the (state-dependent) target inventory positions of the retailers when the system-wide on-hand stock (state) is x. The decisions are made so that, at the beginning of each period t:

- the echelon inventory position of the warehouse is increased up to y_0, i.e., $IP_0(t) = y_0$,
- the inventory position of retailer i is raised to $z_i(I_0(t))$, i.e., $IP_i(t) = z_i(I_0(t)) \ \forall i \in J$.

Theorem 2.9 *Under the balance assumption, the minimization of the average expected cost of the system in an infinite horizon (see (2.2)) can be accomplished by following a base stock policy $(y_0, \mathbf{z}^*)$ with $y_0 \in Y_0^*$.*

Proof : Corollary 2.8 shows that base stock policy $(y_0 \in Y_0^*, \mathbf{z}^*)$ minimizes the expected cycle cost of period t. Due to the fact that warehouse order-up-to level $(y_0 \in Y_0^*)$, and optimal allocation functions $(\mathbf{z}^*)$ are independent of time, the proposed control policy can be applied to optimize each period's cycle cost within the horizon; as a result minimizing (2.2). □

Remark 2.2 *There are two causes for an imbalance situation. On one hand, the retailers might face disproportionate demands in the previous period and the amount of stock at the warehouse (at the beginning of the current period) is not enough to preclude the allocation of a negative quantity to at least one retailer. On the other hand, imbalance may emanate from decreasing allocation functions. Recall the example in Remark 2.1. Take some period $t \in \mathbb{Z}_0^+$. Assume that: (i) at the beginning of period t, the amount of stock to allocate is 4, (ii) the amount of stock the warehouse will receive in period $t+1$ is 1, (iii) $\bar{z}^*(4) = (2,1,1)$, $\bar{z}^*(5) = (1,2,2)$. If no demand occurs at any of the retailers in period t, then an imbalance occurs in period $t+1$ due to decreasing $z_1^*(x)$ at $x = 4$. This kind of imbalance can be prevented by using a base stock policy $(y_0 \in Y_0^*, \hat{\mathbf{z}}^*)$ with nondecreasing allocation functions $\hat{\mathbf{z}}^*$ (i.e., $\{\hat{z}_i^*\}_{i \in J}$ such that $\hat{z}_i^*(x)$ is optimal and nondecreasing for all $x \in \mathbb{Z}$ and for all $i \in J$).*

2.3.5 Newsboy Inequalities

The optimality of base stock policies has been proven in §2.3.4. In this subsection, we identify necessary conditions for an optimal warehouse base stock level, which constitute newsboy inequalities.

Define

$P_i(y_0, \mathbf{z})$ = probability of no-stockout at retailer i in period $t + l_0 + l_i$ given $\mathbf{z}$, and $IP_0(t) = y_0$ for $y_0 \in \mathbb{Z}$ and $t \in \mathbb{Z}_0^+$, i.e.,

$$P_i(y_0, \mathbf{z}) = \sum_{x=0}^{\infty} F_i^{(l_i+1)}(z_i(y_0 - x))\Pr\{D_0^{(l_0)} = x\}. \tag{2.9}$$

By Corollary 2.5, the existence of nondecreasing allocation functions with the additional property, $\tilde{\mathbf{z}}^*$ is assured.

Next, we derive upper and lower bounds on $\Delta G_{cyc}(y_0, \tilde{\mathbf{z}}^*)$.

Lemma 2.10 *For all $i \in J$, and $y_0 \in \mathbb{Z}$:*

$$\left[(h_0 + p_i) - (h_0 + h_i + p_i)F_i^{(l_i+1)}(\underline{y}_i^*)\right] F_0^{(l_0)}(\tau(y_0) - 1) - p_i$$

$$+(h_0 + h_i + p_i)\left[P_i(y_0, \tilde{\mathbf{z}}^*) - \sum_{x=\tau(y_0)}^{\infty} \Pr\{D_i^{(l_i+1)} = \tilde{z}_i^*(y_0 - x)\}\Pr\{D_0^{(l_0)} = x\}\right]$$

$$\leq \;\; \Delta G_{cyc}(y_0, \tilde{\mathbf{z}}^*) \;\; \leq$$

$$\left[(h_0 + p_i) - (h_0 + h_i + p_i)F_i^{(l_i+1)}(\underline{y}_i^*)\right] F_0^{(l_0)}(\tau(y_0) - 1) - p_i$$

$$+(h_0 + h_i + p_i)P_i(y_0, \tilde{\mathbf{z}}^*).$$

Proof : See §2.4.

If material availability is always guaranteed by the warehouse then the optimal order-up-to levels at the retailers are such that the no-stockout probability at each retailer $i \in J$ is at least $\frac{h_0+p_i}{h_0+h_i+p_i}$, see Lemma 2.1. Utilizing the result of Lemma 2.10, similar newsboy inequalities can also be derived for an optimal warehouse order-up-to level.

Theorem 2.11 *For each* $y_0 \in Y_0^*$ *and for all* $i \in J$*:*

$$\begin{aligned} P_i(y_0, \tilde{\mathbf{z}}^*) \quad &\geq \quad \frac{p_i}{h_0 + h_i + p_i} \\ &\quad + \left[F_i^{(l_i+1)}(\underline{y}_i^*) - \frac{h_0 + p_i}{h_0 + h_i + p_i} \right] F_0^{(l_0)}(\tau(y_0) - 1), \qquad (2.10) \\ P_i(\underline{y}_0^* - 1, \tilde{\mathbf{z}}^*) \quad &< \quad \frac{p_i}{h_0 + h_i + p_i} \\ &\quad + \left[F_i^{(l_i+1)}(\underline{y}_i^*) - \frac{h_0 + p_i}{h_0 + h_i + p_i} \right] F_0^{(l_0)}(\tau(\underline{y}_0^*) - 2) \\ &\quad + \sum_{x=\tau(\underline{y}_0^*)-1}^{\infty} Pr\{D_i^{(l_i+1)} = \tilde{z}_i^*(\underline{y}_0^* - 1 - x)\} Pr\{D_0^{(l_0)} = x\}. \end{aligned}$$

Proof : These inequalities follow directly from the result of Lemma 2.10, and the properties that $\Delta G_{cyc}(\underline{y}_0^*, \tilde{\mathbf{z}}^*) \geq 0$ and $\Delta G_{cyc}(\underline{y}_0^* - 1, \tilde{\mathbf{z}}^*) < 0$. □

The message of Theorem 2.11 is that an optimal order-up-to level of the warehouse leads to a no-stockout probability at each retailer $i \in J$, which is at least $\frac{p_i}{h_0+h_i+p_i}$. Note that $F_0^{(l_0)}(\tau(y_0) - 1)$ in (2.10) corresponds to the probability that retailers can reach inventory positions $\underline{y}_i^*$ via shipments from the warehouse (i.e., there is no shortage at the warehouse). $F_i^{(l_i+1)}(\underline{y}_i^*) - \frac{h_0+p_i}{h_0+h_i+p_i}$ is the overshoot from the target newsboy level for retailer i due to discreteness. In case of continuous demand, there is no overshoot; moreover, the newsboy inequalities for the retailers and (2.10) can be satisfied with equality. Thus, (2.10) can be streamlined as $P_i(y_0^*, \tilde{\mathbf{z}}^*) = \frac{p_i}{h_0+h_i+p_i}$ for all $i \in J$ (cf. Diks and de Kok (1998)).

Corollary 2.12 $P_i(y_0, \tilde{\mathbf{z}}^*) \geq \frac{p_i}{h_0+h_i+p_i} \quad \forall i \in J, \ \forall y_0 \in Y_0^*$.

Proof : Result follows directly from (2.10) and the definition of $\underline{y}_i^*$. □

The newsboy inequalities derived in Theorem 2.11 allow us to see the following direct relations between the holding cost parameters and the order-up-to levels under an optimal policy.

Corollary 2.13 *If there exists a retailer* $i \in J$ *with* $h_i = 0$ *and an infinite support for its demand distribution* $F_i^{(1)}$*, then the warehouse becomes a cross-docking point under an optimal policy.*

Proof : From Lemma 2.1, $\underline{y}_i^* = \infty$. Thus, in each period, all available stock at the warehouse is allocated to the retailers under an optimal policy. □

Lemma 2.14 .

(i) If $h_0 = 0$ *then the inventory position of retailer* i *can always be increased to at least* $\underline{y}_i^*$ *for all* $i \in J$ *under an optimal policy* $(y_0, \tilde{\mathbf{z}}^*)$ *with* $y_0 \in Y_0^*$.

(ii) If $h_0 = 0$ *and there is at least one retailer* $i \in J$ *with an infinite support for its demand distribution* $F_i^{(1)}$, *then* $\underline{y}_0^* = \infty$ *under an optimal policy* $(y_0, \tilde{\mathbf{z}}^*)$, $y_0 \in Y_0^* = \{\infty\}$. *Thus, infinite stock is kept at the warehouse.*

Proof : See §2.4.

For $N = 1$, the model reduces to a two-echelon serial system facing discrete demand. The newsboy inequalities discussed in this subsection hold for this system as well. For a detailed analysis of newsboy characterizations in serial systems, see Chapter 5.

2.3.6 Computational Issues

The results of the previous subsections are used to develop an efficient optimization scheme. The general line is reminiscent of the technique developed for serial systems by Clark and Scarf (1960). First, $\underline{y}_i^*$ $\forall i \in J$ are determined utilizing part (iv) of Lemma 2.1. Second, following the arguments in the proof of Theorem 2.4 and using parts (ii) and (iii) of Lemma 2.3, $\tilde{\mathbf{z}}^*$ is constructed. Finally, a simple search procedure can be run to find $\underline{y}_0^*$; details are as follows. Take a retailer $i \in J$, preferably one with $|Y_i^*| > 1$. Start the search at y_0 for which $P_i(y_0, \tilde{\mathbf{z}}^*) \geq \frac{p_i}{h_0+h_i+p_i}$ for the first time. Unless $\Delta G_{cyc}(y_0, \tilde{\mathbf{z}}^*) \geq 0$, increase y_0 by a suitable step size (depending on the distribution of demand at retailer i) until $\Delta G_{cyc}(y_0, \tilde{\mathbf{z}}^*) \geq 0$. Initiate a bisection procedure and terminate it when $\underline{y}_0^*$ is determined. Once $\underline{y}_0^*$, and $\tilde{\mathbf{z}}^*$ are obtained, the values are substituted into (2.8) and the optimal long-run average cost of the system under the balance assumption is obtained.

2.4 Appendix: Proofs

Proof of Lemma 2.2:

(i) Observe that (2.6) consists of N independent components that are convex functions. In the absence of (2.7), the problem is separable and the minimization of each component solves the problem; i.e., $z_i^*(x) \in Y_i^*$ for all $i \in J$. In case $x \geq \sum_{i \in J} \underline{y}_i^*$, $z_i^*(x) = \underline{y}_i^*$ $\forall i \in J$ constitutes an optimal solution, but any other $z_i^*(x) \in Y_i^*$ is also possible as long as $\sum_{i \in J} z_i^*(x) \leq x$.

(ii) For $x \leq \sum_{i \in J} \underline{y}_i^*$, consider a solution $\{z_i(x)\}_{i \in J}$ such that $\sum_{i \in J} z_i(x) < x$. Since $\sum_{i \in J} z_i(x) < \sum_{i \in J} \underline{y}_i^*$, there exists a retailer j with $z_j(x) < \underline{y}_j^*$. Allocate one unit extra to retailer j. Since $\Delta G_i(y_i) < 0$ for $y_i < \underline{y}_i^*$, $\forall i \in J$, the objective function improves. Thus, it is suboptimal to allocate less than x units when $x \leq \sum_{i \in J} \underline{y}_i^*$, independent of how the rationing is carried out. This makes (2.7) binding. □

Proof of Lemma 2.3: Note that (2.7) is binding since we assume $x < \sum_{i \in J} \underline{y}_i^*$ (see Lemma 2.2).

(i) Gross (1956) considered a slightly different resource allocation problem and derived the same necessary and sufficient conditions. His proof applies to our problem and goes as follows. First, we prove the *necessity* of the condition by contradiction. Suppose $\Delta G_k(z_k^*(x)) < \Delta G_m(z_m^*(x) - 1)$ for $k, m \in J$ and $k \neq m$. The solution $z_k(x) = z_k^*(x) + 1$, $z_m(x) = z_m^*(x) - 1$, $z_j(x) = z_j^*(x)$ $\forall j \in J \setminus \{k, m\}$ leads to a lower value for (2.6), which contradicts to the optimality of $\{z_i^*(x)\}_{i \in J}$. The *sufficiency* follows from the convexity of the function $G_i(\cdot)$ for $i \in J$. For the details, see Gross (1956) or Saaty (1970, pp. 184-186). See also Fox (1966) for a more general study.

(ii)-(iii) Substitute the proposed solutions in parts (ii) and (iii) in the necessary and sufficient optimality condition given in part (i), and verify them. Parts (ii) and (iii) constitute the greedy steps in an incremental (marginal) allocation algorithm, which was first proposed by Gross (1956). See Ibaraki and Katoh (1988) for an extensive discussion on incremental analysis. □

Proof of Lemma 2.6:

(i) Take $x < \sum_{i \in J} \underline{y}_i^*$, $x \in \mathbb{Z}$ and a corresponding optimal solution $\{z_i^*(x)\}_{i \in J}$ for (2.6)-(2.7). An optimal solution of (2.6)-(2.7) for $x+1$ can be constructed using part (ii) of Lemma 2.3. This is a solution $z_k^*(x+1) = z_k^*(x) + 1$, and $z_i^*(x+1) = z_i^*(x)$ $\forall i \in J \setminus \{k\}$ with $z_k^*(x) < \underline{y}_k^*$. Thus, $\Delta H^*(x) = H^*(x+1) - H^*(x) = \Delta G_k(z_k^*(x)) < 0$.

(ii) This is a direct result that follows from part (i) of Lemma 2.2, and part (iv) of Lemma 2.1.

(iii) Distinguish between three regions: $x < \sum_{i \in J} \underline{y}_i^* - 1$, $x = \sum_{i \in J} \underline{y}_i^* - 1$, and $x \geq \sum_{i \in J} \underline{y}_i^*$. Firstly, consider $\{z_i^*(x)\}_{i \in J}$ for $x < \sum_{i \in J} \underline{y}_i^* - 1$, and $x \in \mathbb{Z}$. By implementing part (ii) of Lemma 2.3, optimal solutions for $x+1$ and $x+2$ can be determined, which are, $z_k^*(x+1) = z_k^*(x) + 1$ for some $k \in J$ and no change $\forall i \in J \setminus \{k\}$, and $z_m^*(x+2) = z_m^*(x+1) + 1$ for some $m \in J$ and no change $\forall i \in J \setminus \{m\}$. Note that $k = m$ is possible. We find

$$\begin{aligned} \Delta^2 H^*(x) &= [H^*(x+2) - H^*(x+1)] - [H^*(x+1) - H^*(x)] \\ &= \Delta G_m(z_m^*(x+1)) - \Delta G_k(z_k^*(x)) \geq 0. \end{aligned}$$

The last step follows from part (i) of Lemma 2.3 in case $k \neq m$, and from the convexity of $G_k(\cdot)$ in case $k = m$. Secondly, for $x = \sum_{i \in J} \underline{y}_i^* - 1$, by parts (i) and (ii), $\Delta^2 H^* = -\Delta H^*(\sum_{i \in J} \underline{y}_i^* - 1) > 0$. Thirdly, for $x \geq \sum_{i \in J} \underline{y}_i^*$, $\Delta H^*(x) = 0$ resulting in $\Delta^2 H^*(x) = 0$. □

Proof of Lemma 2.7:

(i) Follows directly from (2.8) and the definition of $H^*(\cdot)$.

(ii) For all $y_0 \in \mathbb{Z}$:

$$G_{cyc}(y_0, \mathbf{z}^*) = G_0(y_0) + \sum_{x=0}^{\infty} H^*(y_0 - x) \Pr\{D_0^{(l_0)} = x\} \leq$$

$$G_0(y_0) + \sum_{x=0}^{\infty} \sum_{i \in J} G_i(z_i(y_0 - x)) \Pr\{D_0^{(l_0)} = x\} = G_{cyc}(y_0, \mathbf{z}).$$

(iii) For all $y_0 \in \mathbb{Z}$:

$$\begin{aligned} \Delta G_{cyc}(y_0, \mathbf{z}^*) &= \Delta G_0(y_0) + \sum_{x=0}^{\infty} \Delta H^*(y_0 - x) \Pr\{D_0^{(l_0)} = x\} \\ &= h_0 + \sum_{x=0}^{\infty} \Delta H^*(y_0 - x) \Pr\{D_0^{(l_0)} = x\}. \end{aligned} \tag{2.11}$$

This proves part (iii) for $y_0 < \sum_{i\in J} \underline{y}_i^*$. If $y_0 \geq \sum_{i\in J} \underline{y}_i^*$ then the lower limit of the summation in (2.11) can be reduced to $y_0 - \sum_{i\in J} \underline{y}_i^* + 1$ utilizing part (ii) of Lemma 2.6.

(iv) Since $G_0(x)$ and $H^*(x)$ are convex functions, so is $G_{cyc}(y_0, \mathbf{z}^*)$.

(v) Due to the fact that $G_{cyc}(y_0, \mathbf{z}^*)$ is convex with respect to y_0, the *first* minimizing value ($\underline{y}_0^*$) is obtained at the point where $\Delta G_{cyc}(y_0, \mathbf{z}^*) \geq 0$ for the first time. If $\Delta G_{cyc}(y_0, \mathbf{z}^*) = 0$, then there are multiple optimal values. The maximum of such points is the one where $\Delta G_{cyc}(y_0, \mathbf{z}^*)$ turns positive. □

Proof of Lemma 2.10: Let $y_0 \in \mathbb{Z}$. We continue from part (iii) of Lemma 2.7:

$$\begin{aligned}
\Delta G_{cyc}(y_0, \tilde{\mathbf{z}}^*) &= h_0 + \sum_{x=\tau(y_0)}^{\infty} \Delta H^*(y_0 - x)\Pr\{D_0^{(l_0)} = x\} \\
&= h_0 + \sum_{x=\tau(y_0)}^{\infty} \left[\sum_{j\in J} G_j(\tilde{z}_j^*(y_0 + 1 - x)) \right. \\
&\qquad \left. - \sum_{j\in J} G_j(\tilde{z}_j^*(y_0 - x))\right] \Pr\{D_0^{(l_0)} = x\} \\
&= h_0 + \sum_{x=\tau(y_0)}^{\infty} \min_{j\in J} \left\{\Delta G_j(\tilde{z}_j^*(y_0 - x))\right\} \Pr\{D_0^{(l_0)} = x\}. \qquad (2.12)
\end{aligned}$$

Note that for $x \geq \tau(y_0)$, $x \in \mathbb{Z}$:

$$\Delta G_i(\tilde{z}_i^*(y_0 - x) - 1) \leq \min_{j\in J} \{\Delta G_j(\tilde{z_j}^*(y_0 - x))\} \leq \Delta G_i(\tilde{z}_i^*(y_0 - x)), \quad (2.13)$$

for all $i \in J$. While the upper bound in (2.13) is obvious, lower bound follows from part (i) of Lemma 2.3. Substituting (2.13) into (2.12) leads to

$$\begin{aligned}
h_0 + \sum_{x=\tau(y_0)}^{\infty} \Delta G_i(\tilde{z}_i^*(y_0 - x) - 1)\Pr\{D_0^{(l_0)} = x\} &\leq \Delta G_{cyc}(y_0, \tilde{\mathbf{z}}^*) \\
&\leq h_0 + \sum_{x=\tau(y_0)}^{\infty} \Delta G_i(\tilde{z}_i^*(y_0 - x))\Pr\{D_0^{(l_0)} = x\},
\end{aligned}$$

for all $i \in J$. The lower bound may be rewritten in terms of $P_i(y_0, \tilde{\mathbf{z}}^*)$, which is defined in (2.9):

$$
\begin{aligned}
& h_0 + \sum_{x=\tau(y_0)}^{\infty} \Delta G_i(\tilde{z}_i^*(y_0 - x) - 1)\Pr\{D_0^{(l_0)} = x\} \\
= \;& h_0 + \\
& \sum_{x=\tau(y_0)}^{\infty} \Big[(h_0 + h_i + p_i)F_i^{(l_i+1)}(\tilde{z}_i^*(y_0 - x) - 1) - (h_0 + p_i)\Big] \Pr\{D_0^{(l_0)} = x\} \\
= \;& (h_0 + p_i)F_0^{(l_0)}(\tau(y_0) - 1) - p_i + \\
& \sum_{x=\tau(y_0)}^{\infty} \Big[(h_0 + h_i + p_i)F_i^{(l_i+1)}(\tilde{z}_i^*(y_0 - x) - 1)\Big] \Pr\{D_0^{(l_0)} = x\} \\
= \;& (h_0 + p_i)F_0^{(l_0)}(\tau(y_0) - 1) - p_i \\
& +(h_0 + h_i + p_i) \sum_{x=\tau(y_0)}^{\infty} \Big(F_i^{(l_i+1)}(\tilde{z}_i^*(y_0 - x)) \\
& \qquad - \Pr\{D_i^{(l_i+1)} = \tilde{z}_i^*(y_0 - x)\}\Big) \Pr\{D_0^{(l_0)} = x\} \\
= \;& (h_0 + p_i)F_0^{(l_0)}(\tau(y_0) - 1) - p_i \\
& +(h_0 + h_i + p_i) \Bigg[P_i(y_0, \tilde{\mathbf{z}}^*) - \sum_{x=0}^{\tau(y_0)-1} F_i^{(l_i+1)}(\tilde{z}_i^*(y_0 - x))\Pr\{D_0^{(l_0)} = x\} \\
& \qquad - \sum_{x=\tau(y_0)}^{\infty} \Pr\{D_i^{(l_i+1)} = \tilde{z}_i^*(y_0 - x)\}\Pr\{D_0^{(l_0)} = x\}\Bigg].
\end{aligned}
$$

Recall from Lemma 2.1 that $F_i^{(l_i+1)}(y_i) = F_i^{(l_i+1)}(\underline{y}_i^*)$ for $y_i \in Y_i^* \setminus \{\overline{y}_i^*\}$. Hence, the expression $\sum_{x=0}^{\tau(y_0)-1} F_i^{(l_i+1)}(\tilde{z}_i^*(y_0 - x))\Pr\{D_0^{(l_0)} = x\}$ reduces to $F_i^{(l_i+1)}(\underline{y}_i^*)F_0^{(l_0)}(\tau(y_0) - 1)$ and rearranging the terms leads to:

$$
\begin{aligned}
h_0 \;&+\; \sum_{x=\tau(y_0)}^{\infty} \Delta G_i(\tilde{z}_i^*(y_0 - x) - 1)\Pr\{D_0^{(l_0)} = x\} \\
&= \Big[(h_0 + p_i) - (h_0 + h_i + p_i)F_i^{(l_i+1)}(\underline{y}_i^*)\Big] F_0^{(l_0)}(\tau(y_0) - 1) - p_i \\
&+ (h_0 + h_i + p_i) \Bigg[P_i(y_0, \tilde{\mathbf{z}}^*) - \\
&\sum_{x=\tau(y_0)}^{\infty} \Pr\{D_i^{(l_i+1)} = \tilde{z}_i^*(y_0 - x)\}\Pr\{D_0^{(l_0)} = x\}\Bigg].
\end{aligned}
$$

Similarly, the upper bound can be expressed in terms of $P_i(y_0, \tilde{\mathbf{z}}^*)$:

$$\begin{aligned} h_0 \quad &+ \sum_{x=\tau(y_0)}^{\infty} \Delta G_i(\tilde{z}_i^*(y_0 - x)) \Pr\{D_0^{(l_0)} = x\} \\ &= \left[(h_0 + p_i) - (h_0 + h_i + p_i) F_i^{(l_i+1)}(\underline{y}_i^*)\right] F_0^{(l_0)}(\tau(y_0) - 1) - p_i \\ &\quad + (h_0 + h_i + p_i) P_i(y_0, \tilde{\mathbf{z}}^*). \quad \Box \end{aligned}$$

Proof of Lemma 2.14:

(i) Take retailer $i \in J$ such that $\tilde{z}_i^*(\sum_{i\in J} \underline{y}_i^* - 1) = \underline{y}_i^* - 1$. From (2.9) and (2.10), for each $y_0 \in Y_0^*$:

$$\begin{aligned} P_i(y_0, \tilde{\mathbf{z}}^*) &= \sum_{x=0}^{\tau(y_0)-1} F_i^{(l_i+1)}(\tilde{z}_i^*(y_0 - x)) \Pr\{D_0^{(l_0)} = x\} \\ &\quad + \sum_{x=\tau(y_0)}^{\infty} F_i^{(l_i+1)}(\tilde{z}_i^*(y_0 - x)) \Pr\{D_0^{(l_0)} = x\} \\ &\geq \frac{p_i}{h_i + p_i} + \left[F_i^{(l_i+1)}(\underline{y}_i^*) - \frac{p_i}{h_i + p_i}\right] F_0^{(l_0)}(\tau(y_0) - 1), \end{aligned}$$

which can be rewritten as

$$\sum_{x=\tau(y_0)}^{\infty} F_i^{(l_i+1)}(\tilde{z}_i^*(y_0 - x)) \Pr\{D_0^{(l_0)} = x\} \geq \frac{p_i}{h_i + p_i}\left(1 - F_0^{(l_0)}(\tau(y_0) - 1)\right) \qquad (2.14)$$

using the property that $F_i^{(l_i+1)}(\tilde{z}_i^*(x)) = F_i^{(l_i+1)}(\underline{y}_i^*)$ for $x \geq \sum_{i\in J} \underline{y}_i^*$, $x \in \mathbb{Z}$. Further, the inequality in (2.14) may be rewritten as

$$\sum_{x=\tau(y_0)}^{\infty} F_i^{(l_i+1)}(\tilde{z}_i^*(y_0 - x)) \Pr\{D_0^{(l_0)} = x\} \geq \sum_{x=\tau(y_0)}^{\infty} \frac{p_i}{h_i + p_i} \Pr\{D_0^{(l_0)} = x\}. \qquad (2.15)$$

From $\tilde{z}_i^*(\sum_{i\in J} \underline{y}_i^* - 1) = \underline{y}_i^* - 1$ and Lemma 2.1 , $F_i^{(l_i+1)}(\tilde{z}_i^*(y_0 - x)) < \frac{p_i}{h_i+p_i}$ for $x \geq \tau(y_0)$. Thus, the inequality in (2.15) can *only* be satisfied if

$\Pr\{D_0^{(l_0)} \geq \tau(y_0)\} = 0$, i.e., $F_0^{(l_0)}(\tau(y_0) - 1) = 1$. This implies that $y_0 \in Y_0^*$ is greater than or equal to any possible realization of $D_0^{(l_0)}$ plus $\sum_{i \in J} \underline{y}_i^*$.

(ii) An infinite support for $F_i^{(1)}$, $i \in J$ implies that there is also an infinite support for $F_0^{(1)}$. From part (i), $F_0^{(l_0)}(\tau(y_0) - 1) = 1$ for $y_0 \in Y_0^*$ can *only* be attained when $\underline{y}_0^* = \infty$. $\square$

Chapter 3

Relative Gap between the Upper and Lower Bound

Abstract: *The balance assumption is widely used in the analysis of one-warehouse multi-retailer inventory systems, and accepted to provide solutions of good quality in the literature. The balance assumption leads to a relaxed version of the original optimization problem, so the corresponding cost is a lower bound for the true optimal cost. An upper bound can be obtained by simulating the solution of the relaxed problem under a modified allocation rule. The relative gap between these bounds is used as a measure to assess the impact of the balance assumption on the average expected cost. The effect of lead-times, holding and penalty costs, mean and variance of the demand processes, and number of retailers on the relative gap is identified. The parameter settings resulting in small gaps are determined explicitly. Further, the relation between the imbalance probability and the relative gap is analyzed numerically. Our results point out that for several practically relevant cases, the balance assumption may lead to large gaps unlike the generally established belief in the literature that this assumption is not a serious limitation. In the light of the insights obtained, we conclude that researchers should put more effort in developing good heuristics for these relevant cases.*

3.1 Introduction

In this chapter, we consider the simplest form of a single-item divergent (arborescent) multi-echelon inventory system: one-warehouse multi-retailer system. A divergent structure is characterized by the property that each stock point is supplied by exactly one other stock point. In the system under study, there is a single stock point, called the warehouse, supplying downstream points, called retailers, which face stochastic demand of customers. The warehouse orders a single-item from an external supplier with ample stock and ships to the retailers. There are fixed leadtimes between the external supplier and the warehouse, and in between the warehouse and the retailers. Any unfulfilled demand of a customer is backlogged. Costs consist of linear inventory holding and penalty costs. We assume centralized control under periodic review and average cost criterion.

The literature on divergent multi-echelon inventory systems can be divided into two streams on the basis of the review policy used: continuous review and periodic review. In the former stream, in which *continuous review* is employed, opportunities to review inventory levels and to implement the derived policies exist continuously over time. While the early work in this stream is mainly on repairable items, the same line may be followed for consumable goods, too (see Axsäter (2003)). For the models in this stream, the structure of the optimal policies is not known. In general, base stock or (R, Q) policies are assumed, and optimization is carried out within the class. A common assumption in this line of research is that the backlogged retailer orders are fulfilled on *first-come, first-served* basis.

There are three main approaches for the evaluation of base stock policies in divergent systems under continuous review:

(i) approximating the mean resupply time of a retailer order, which consists of a deterministic leadtime and a random waiting time due to the stockouts at the warehouse. This approximation is the heart of the approach of Sherbrooke (1968) for the METRIC model.

(ii) considering all retailers as a single one and determining the outstanding orders of the aggregate retailer. By splitting the aggregate outstanding orders among the retailers, one can compute the inventory and backorder levels of the retailers exactly. Simon (1971) provided an exact evaluation for the METRIC model. The exact evaluation can also be used as a part of an optimization routine to determine the inventory levels at all stock points that optimizes some criterion, see Graves (1985). Further, Graves developed a two-moment fit for the number of outstanding orders at a retailer and the numerical results show that this approximation outperforms the METRIC model. Following this approach, Chen and Zheng (1997) developed an exact evaluation scheme for echelon stock (R, Q) policies.

(iii) following a unit from the moment it enters the system until it exits by fulfilling a demand. This allows one to determine the holding and penalty costs associated with this unit (see Axsäter (1990)). Further, Axsäter derives upper and lower bounds on the optimal base stock levels that lead to an efficient optimization procedure.

Lately, some research in this stream is focusing on how to relax the restrictive assumption of the first-come, first-served rule at the warehouse; see Axsäter and Marklund (2004), and Marklund (2004). We refer to Axsäter (2003) and the references therein for more information on continuous review divergent multi-echelon models and the extensions of the approaches discussed above.

The seminal work on *periodic review* multi-echelon models is by Clark and Scarf (1960). They developed a discounted dynamic program for the inventory control of an N-echelon *serial* system in a finite horizon. By introducing the concepts echelon stock and induced penalty cost, they were able to decompose the resulting multi-dimensional dynamic program (DP) into a series of single-dimensional programs (this is known as the *decomposition property*), and prove the optimality of base stock policies. Although they developed a similar DP for a two-echelon distribution system, decomposition is not possible due to the so-called *allocation (rationing) problem*: the decision of how to distribute the on-hand stock at the warehouse among the retailers at the beginning of each period.

Although the optimal policy for the inventory control of a distribution system is unknown, there are some structural results for slightly modified models. The key assumption in all these models is that the inventory positions of the retailers may be *balanced completely* by the allocation of warehouse stock at the beginning of each period. This is equivalent to allowing negative quantities to be apportioned to the retailers. We refer to this assumption as the *balance assumption*. Under the balance assumption, similar to the serial case, the optimality of base stock policies can be shown, and the decomposition property can be obtained, i.e., the optimal base stock levels are determined by solving single-dimensional optimization problems.

Though Clark and Scarf (1960) were the first to discuss the balance issue, it was Eppen and Schrage (1981) who made it explicit. Eppen and Schrage (1981) considered a one-warehouse multi-retailer system with a *stockless* warehouse (cross-docking point). The function of such a warehouse is to benefit from quantity discounts through consolidation and/or to exploit *risk pooling* by carrying a single inventory during the warehouse leadtime rather than individual retailer inventories. In their context, the balance assumption (which they referred to as the allocation assumption) implies that the quantities allocated by rationing are always sufficient to ensure equal stockout probabilities for the retailers. Making the balance assumption, and taking the retailers identical in terms of cost and leadtime parameters allowed them to derive closed-form expressions for optimal inventory control parameters of two policies they considered: base stock policy, and (T, S) policy (ordering every T periods up to the base stock level S). Under the balance assumption, Federgruen and Zipkin (1984b) characterized the optimal allocation and replenishment policies for a divergent system composed of a stockless warehouse serving N (possibly nonidentical) retailers in a finite horizon. The optimality results were extended to the infinite horizon case under an average cost criterion for a system with identical retailers and a stock keeping warehouse by Federgruen and Zipkin (1984c). Under the balance assumption, Diks and de Kok (1998) generalized the decomposition property and the optimality of base stock policies to general N-echelon divergent structures with nonidentical cost and leadtime parameters. In addition, they derived newsboy equalities for the optimal base stock levels. In Chapter 2 of this dissertation, these results have been extended to one-warehouse multi-retailer systems with *discrete* demand, in which case newsboy inequalities instead of equalities are obtained. For more information about periodic review distribution systems, we refer to the review papers by van Houtum *et al.* (1996) and Axsäter (2003).

The balance assumption also plays a central role in the development of heuristics for distribution systems under periodic review. Up to our knowledge, all heuristics in the literature utilize the balance assumption in some form. See Federgruen and Zipkin (1984a), Jackson and Muckstadt (1989), Kumar and Jacobson (1998), Axsäter *et al.* (2002), Cao and Silver (2005), and Lystad and Ferguson (2005).

Until recently, there was a generally established belief that the balance assumption provides solutions of very good quality (see Axsäter (2003, p. 544)). Although Axsäter *et al.* (2002) demonstrated numerically that the analysis under the balance assumption may lead to large errors in some cases, there is no clear-cut study interrogating the appropriateness of the balance assumption. Our numerical study tries to fill this hiatus by a numerical study (conducted over a wide range of parameters) that specifies the parameter settings where the balance assumption leads to a good approximation.

Our method can be explained as follows. The balance assumption leads to a *relaxed version* of the original problem, so the optimal expected cost of the relaxed problem is a lower bound (LB) for the optimal cost of the original one. When the optimal base stock levels of the relaxed problem are coupled with a myopic allocation procedure (the warehouse on-hand stock is distributed such that the sum of the expected holding and penalty costs of the retailers in the periods the allocated quantities reach their destinations is minimized), the resulting policy is a feasible heuristic policy, referred to as LB heuristic policy. The simulation of the LB heuristic policy serves as an upper bound (UB) for the true optimal cost. The relative gap ($\epsilon\% = 100\frac{UB-LB}{LB}$) is used as a measure to assess the impact of the balance assumption. The following input parameters are considered in the numerical study: holding and penalty costs, warehouse and retailer leadtimes, the number of retailers, and the mean and the coefficient of variation of the demand processes. We generated two test beds consisting of 2000 and 3888 problem instances for identical (in terms of costs, leadtimes, and demand distributions) and nonidentical retailers cases, respectively. The number of retailers is restricted to two in the nonidentical retailers case.

Due to the fact that the optimal cost of the original problem is between LB and UB, a *small relative gap* implies that LB value is close to the optimal cost of the original problem. Thus, the balance assumption leads to an accurate approximation of the true optimal cost. Further, one can infer that the LB heuristic policy is a good heuristic. In sum, a *small relative gap* justifies the use of the balance assumption for that input parameter setting. On the other hand, we cannot come to concrete conclusions when there is a *moderate* or a *large relative gap* because the distance between the true optimum and LB becomes an important issue. Due to the curse of dimensionality, true optimal cost can only be determined (numerically) for small problems where the number of retailers and the warehouse leadtime are limited, and the demand processes are discrete and distributed over a small set of points. For settings with moderate and large gaps, the computation of the true optimal cost by value iteration and the assessment of the precise effect of the balance assumption are carried out in Chapter 4.

The results of this chapter directed us to a clear and complete overview on when the relative gap is *small* and when *not*. In the identical retailers case, the relative gap is small when one of the following conditions hold: (i) the coefficient of variation is low or moderate, (ii) the added value at the warehouse is very low compared to the retailers, (iii) the warehouse leadtime is short while the retailer leadtimes are long. Large gaps up to 38.6% are observed when the coefficient of variation is high and the relative added value at the retailers is moderate or low. In the nonidentical retailers case, the relative gap turns out to be small when one of the following conditions is satisfied: (i) the added value at each stock point is positive and equal, (ii) the warehouse leadtime is short together with long retailer leadtimes, (iii) the warehouse leadtime is short while the big retailer's (in terms of mean demand) leadtime is long and the leadtime of the small one is short. The main determinants of large gaps (which are up to 186.9%) are positive added value at one retailer and zero at the other, and long warehouse leadtimes.

Further analysis routed us to interesting and new insights together with the justification of the previous findings in the literature. The retailer size turns out to be an important factor; as the asymmetry in the means of the retailers grows, relative gaps tend to increase. Coefficient of variation, echelon holding cost of the warehouse, and warehouse leadtime are positively correlated to relative gaps. Further, a phenomenon coined as *forwarding-to-the-small-retailer* is identified. When there is no added value at the small retailer, overstocking takes place at this retailer, which increases the relative gaps.

During the simulation runs, we also estimated the imbalance probabilities. It is found out that a considerable number of problem instances with high imbalance probability display relatively low $\epsilon\%$, but not the reverse.

Our contribution to the literature is twofold. To the best of our knowledge, this is the most comprehensive numerical study *specifically* targeting to assess the impact of the balance assumption on the expected long-run cost. We explicitly identify the parameter settings under which the relative gaps are small and under which not. Our results point out that the scenarios with large relative gaps are practically relevant, so there is a need for research on developing heuristics for these cases. Second, analysis of the relations between the relative gaps and the input parameters led us to new and interesting insights.

The rest of the chapter is organized as follows. The literature review on the balance assumption is given in §3.2. We present the notation, the model and a brief analysis in §3.3. The balance assumption and the relaxed version of the optimization problem resulting in the lower bound model are also discussed in this section. §3.4 is dedicated to the presentation and discussion of the numerical results. We close with a brief conclusion and directions for further research in §3.5.

3.2 Literature on the Balance Assumption

There are some analytical and numerical studies that interrogate the appropriateness of the balance assumption. The first of these is an analytical study by Zipkin (1984) who considered a one-warehouse multi-retailer system with zero leadtimes for orders and shipments, and a cross-docking warehouse. The author formulated a DP to determine inventory policies in which one dimension of the DP is a measure to assess the stock imbalance in the system. A numerical study shows the accuracy of the approximation for the limited number of scenarios considered. It is concluded that imbalance can be significant when demand variances are large.

Federgruen and Zipkin (1984a) conducted a numerical study to evaluate the performance of some heuristics developed under the balance assumption for a one-warehouse multi-retailer system with a stockless warehouse. They used the relative gap between an upper and a lower bound as a measure to evaluate the performance of their heuristics. The effects of fixed ordering cost, warehouse leadtime and penalty cost are investigated. Their results for identical retailers (with respect to cost parameters, leadtimes and demand distributions) show that the balance assumption does not impose a restriction. It has been identified that high coefficient of variation or high penalty cost results in larger relative gaps. The test bed consisting of 40 problem instances with nonidentical retailers with zero fixed ordering cost (these results are comparable to ours because the underlying models are equivalent) also exhibits low relative gaps (the maximum relative gap reported is 4.14%). Among these, the scenarios with long warehouse leadtimes display larger relative gaps.

A recent study in the same spirit of Federgruen and Zipkin (1984a) is by Axsäter *et al.* (2002), in which a system composed of a stock keeping warehouse and multiple (possibly nonidentical) retailers is considered. While the retailers follow base stock policies, the warehouse applies an (R, Q) policy; each period the echelon inventory position drops to or below R, one or more batches of size Q are ordered to bring the inventory position above R. Considering two heuristics for the warehouse replenishment policy (optimal policy under the balance assumption and virtual assignment) and two heuristics for the allocation problem (optimal allocation policy under the balance assumption, i.e., relaxed myopic allocation, and two-step allocation), the average expected holding and penalty costs of four different combinations of these heuristics are determined via simulation for various parameter settings. The relative gap between the simulation result and the optimal expected cost under the balance assumption is used to assess the performance of the heuristics. A more sophisticated heuristic, virtual assignment/two-step allocation, exhibits a better performance than the classical approach (optimal ordering and allocation policies under the balance assumption). Although the numerical study is set up for testing the heuristics considered, the results also give insight into the impact of the balance assumption. As the demand variability in the system, the warehouse leadtime and the warehouse batch size increase, the relative gap between the analytical cost under the balance assumption and the cost obtained via simulating the optimal policy (under the balance assumption) grows.

For the effect of imbalance on the P_1 service level (probability of stockout) at the retailers, see van Donselaar and Wijngaard (1987), and van Donselaar (1990). Jönsson and Silver (1987), McGavin *et al.* (1993, 1997), and van der

Heijden *et al.* (1997) are other studies on the issue of balanced retailer inventories. For risk-pooling effect, see Schwarz (1989), and Jackson and Muckstadt (1989).

In this study, we follow the same line of thought of Federgruen and Zipkin (1984a) and Axsäter *et al.* (2002), but our study is different in certain aspects. The main dissemblances are: (i) While their focal point is the performance of their heuristics, we focus particulary on the effect of the balance assumption. (ii) Federgruen and Zipkin consider a fixed cost for ordering from the external supplier and Axsäter *et al.* assume a fixed batch size for warehouse orders; we do not consider these issues. (iii) Federgruen and Zipkin consider a warehouse that cannot hold stock; Axsäter *et al.* and ourselves relax this restriction. (iv) Federgruen and Zipkin assume normal demands at the retailers; when the coefficient of variation is higher than 0.25, the normal distribution assumption creates complications because negative demands appear in the analysis. In real life, the coefficient of variation is higher than 0.5 in many cases, and can even be more than 3; see §3.6. We incorporate instances with coefficients of variation up to 3, and model all demand processes as mixtures of Erlang distributions. (v) For nonidentical retailers, Federgruen and Zipkin require the holding and penalty costs to be proportional[1]. Axsäter *et al.* and ourselves relax this restriction. Especially, the points (iv) and (v) allow us to consider some extreme scenarios (e.g., coefficient of variation of demand equal to 3, or added value of zero at a retailer) for the purpose of finding out the extent of the impact of the balance assumption.

3.3 Model and Analysis

In §3.3.1, we introduce the notation and the basic assumptions. Then we review some results from Chapter 2 where the system is analyzed in detail. A lower and an upper bound model for the original optimization problem are described in §3.3.3 and §3.3.4, respectively. Finally, we discuss the gap between the bounds and its interpretation in §3.3.5.

3.3.1 Preliminaries and Notation

Consider a one-warehouse multi-retailer inventory system controlled centrally under a periodic review setting. The retailers are replenished by shipments from the warehouse (indexed as stock point 0), which in turn orders from an

[1]It is assumed that there exist positive constants p, h, and c_i, $i = 1, 2, ..., N$, such that $p_i = pc_i$, $h_i = hc_i$ for $i = 1, 2, ..., N$ where N is the number of retailers.

exogenous supplier with ample stock. Leadtimes between the supplier and the warehouse, and in between the warehouse and the retailers are assumed to be constant. The retailers face the stochastic i.i.d. demand of the customers that are stationary and continuous on $(0, \infty)$ with no probability mass at zero. Costs consist of linear inventory holding and penalty costs. Time is divided into periods of equal length and we assume that the following sequence of events takes place during a period: (i) inventory levels are observed and the current period's replenishment decisions are made (at the beginning of the period), (ii) order/shipments arrive following their respective leadtimes (at the beginning of the period), (iii) demand occurs, (iv) holding and penalty costs are assessed on the period ending inventory and backorder levels (at the end of the period).

The objective is to minimize the expected holding and penalty costs of the system in the long-run; i.e., our performance criterion is the expected average system-wide cost. We refer to Chapter 1 for the definitions of echelon stock and echelon inventory position. We use the following notation:

$\mathbb{R}$	=	set of real numbers.
$\mathbb{Z}$	=	set of integer numbers; $\mathbb{Z}^+ = \{1, 2, ...\}$, and $\mathbb{Z}_0^+ = \mathbb{Z}^+ \cup \{0\}$.
t	=	index for time. Period t is defined as the time interval between epochs t and $t+1$ for $t \in \mathbb{Z}_0^+$.
N	=	number of retailers, $N \in \mathbb{Z}^+$.
i	=	index for stock points, $i = 0$ is the warehouse and $i = 1, 2, ..., N$ are the retailers.
J	=	set of retailers, $J = \{1, 2, ..., N\}$.
h_i	=	additional inventory holding cost parameter for stock point i. At the end of a period: (i) cost $h_0 \geq 0$ is charged for each unit on stock at the warehouse or in transit to any of the retailers. (ii) cost $h_0 + h_i$ is charged for each unit on stock at retailer i ($h_i \geq 0$, $i \in J$).
p_i	=	penalty cost parameter for retailer i. A cost p_i is charged for each unit of backlog at the end of a period at retailer i ($p_i > 0$, $i \in J$).
l_i	=	leadtime parameter for stock point i ($l_i \in \mathbb{Z}_0^+$ for $i = 1, ...N$, and $l_0 \in \mathbb{Z}^+$).
μ_i	=	mean of one-period demand faced by retailer i ($\mu_i > 0$, $i \in J$).
μ_0	=	mean of one-period demand faced by the system, $\mu_0 = \sum_{i \in J} \mu_i$.
cv_i	=	coefficient of variation of one-period demand faced by retailer i ($cv_i > 0$, $i \in J$).
$F_i^{(l)}$	=	cumulative distribution function of l-period demand of retailer i.

$I_i(t)$ = echelon stock of stock point i at the beginning of period t just after the receipt of the incoming order/shipment.

$IP_i(t)$ = echelon inventory position of stock point i at the beginning of period t just after the ordering/shipment decision.

We assume that the underlying demand distribution per period of a retailer is a mixture of Erlang distributions with the same scale parameter (cf. Tijms (2003, pp. 444-446)). An Erlang-k distributed random variable (which is a sum of $k \in \mathbb{Z}^+$ independent exponentially distributed random variables with the same mean λ^{-1}) is denoted by $E_{k,\lambda}$ where the cumulative probability distribution function is given by $E_{k,\lambda}(x) = 1 - \sum_{j=0}^{k-1} \frac{(\lambda x)^j}{j!} e^{-\lambda x}$ for $x \geq 0$.

When $cv_i \leq 1$, a mixture of $E_{k-1,\lambda}$ and $E_{k,\lambda}$ distributions is fitted for the one-period demand distribution of retailer i; i.e., $F_i^{(1)}(x) = pE_{k-1,\lambda}(x) + (1-p)E_{k,\lambda}(x)$ for $x \geq 0$. The parameters $k \geq 2$, $0 \leq p \leq 1$ and λ are chosen such that:

$$\frac{1}{k} \leq cv_i^2 \leq \frac{1}{k-1}, \quad p = \frac{1}{1+cv_i^2}\left[kcv_i^2 - \sqrt{k(1+cv_i^2) - k^2 cv_i^2}\,\right], \quad \lambda = \frac{k-p}{\mu_i}.$$

When $cv_i > 1$, a mixture of $E_{1,\lambda}$ and $E_{k,\lambda}$ distributions is fitted; i.e., $F_i^{(1)}(x) = pE_{1,\lambda}(x)+(1-p)E_{k,\lambda}(x)$ for $x \geq 0$. The smallest $k \geq 3$ that satisfies $cv_i^2 \leq \frac{k^2+4}{4k}$ is chosen. The values of p and λ are determined by:

$$p = \frac{2kcv_i^2+k-2-\sqrt{k^2+4-4kcv_i^2}}{2(k-1)(1+cv_i^2)}, \quad \lambda = \frac{p+k(1-p)}{\mu_i}.$$

3.3.2 Analysis of the System

The one-warehouse multi-retailer inventory system is analyzed in detail in Chapter 2. The concept of cost attached to an echelon (which allows one to rewrite the costs in terms of echelon stock), the interdependence between the ordering and allocation decisions, the cycle cost definition, and the optimization problem faced are discussed in §2.3.1. In §2.3.2, the myopic allocation problem and the mathematical interpretation of the balance assumption are introduced. Finally, the derivation of the optimal policy is given in §2.3.3 and §2.3.4. Note that the demand distributions in this chapter are continuous, which is the only departure from the analysis in Chapter 2.

Next, we review some of the results from Chapter 2. Consider the myopic allocation problem given in (2.3)-(2.5). The *balance assumption* implies that the quantities allocated to the retailers may be negative; i.e., (2.5) is omitted. This relaxation helps the analysis in two important ways. First, the myopic allocation problem turns into a convex, separable, nonlinear optimization problem subject to a linear constraint, which can be solved easily by the Lagrangian technique (see Bertsekas (1995, pp. 253-282)). Second, in the absence of (2.5), $C_{cyc}(t)$ depends *only* on ordering and allocation decisions that start with the ordering decision in period t; not on decisions of other periods. Then, it can be proved that myopic allocation is optimal (see Federgruen and Zipkin (1984b,c)). As a result, it is possible to decompose the system like in Clark and Scarf (1960), and characterize the optimal ordering and allocation decisions.

As long as there is sufficient stock at the warehouse, the retailers follow base stock policies where the optimal order-up-to levels, $y_i^*, i \in J$, satisfy the following newsboy equations (see Lemma 2.1):

$$F_i^{(l_i+1)}(y_i^*) = \frac{h_0 + p_i}{h_0 + h_i + p_i} \quad \text{for} \quad i = 1, 2, ..., N. \tag{3.1}$$

The interpretation of the expression above is that y_i^* corresponds to a level at which the probability of being out of stock is $\frac{h_i}{h_0+h_i+p_i}$ for retailer i.

Let $z_i : \mathbb{R} \to \mathbb{R}$, $i \in J$ be an allocation function such that $z_i(x)$ is the portion of x allocated to retailer i for $x \in \mathbb{R}$. When the system-wide stock ($x \in \mathbb{R}$) is not sufficient, i.e., $x < \sum_{i \in J} y_i^*$, the myopic allocation problem under the balance assumption is solved:

$$\min_{z_i(x),\ \forall i \in J} \left\{ \sum_{i \in J} G_i(z_i(x)) : \sum_{i \in J} z_i(x) \leq x, z_i(x) \in \mathbb{R} \text{ for } i \in J \right\} \tag{3.2}$$

where a solution is denoted by $\{z_i(x)\}_{i \in J}$. Let $\{z_i^*(x)\}_{i \in J}$ be an optimal solution of (3.2) for a given x. Define

$\mathbf{z}$ = set of allocation functions, i.e., $\{z_i\}_{i \in J}$.
$\mathbf{z}^*$ = set of optimal allocation functions, i.e., $\{z_i^*\}_{i \in J}$ such that $z_i^*(x)$ is optimal for all $x \in \mathbb{R}$ and for all $i \in J$.

Note that the objective function of (3.2) is separable and consists of N convex components. The solution $\{z_i^*(x) = y_i^*\}_{i \in J}$ is optimal for (3.2) when the constraint is not binding (i.e., for $x > \sum_{i \in J} y_i^*$) or when $x = \sum_{i \in J} y_i^*$. For $x < \sum_{i \in J} y_i^*$, Lagrangian relaxation can be used to find an optimal solution for (3.2); see Bertsekas (1995, pp. 253-282).

3.3.3 Lower Bound Model

The balance assumption is the relaxation of a nonnegativity constraint in the original model; thus, (3.2) is used for the allocation decision instead of (2.3)-

(2.5). Due to this relaxation, the expected average cost obtained by an optimal policy derived under the balance assumption serves as a lower bound for the original optimization problem given in (2.2).

Denote a *base stock policy* by a tuple $(y_0, \mathbf{z})$, where y_0 is the target echelon inventory position of the warehouse, and $z_i(x)$ for all $i \in J$ are the target levels of the retailers when the system-wide stock is x. The decisions are made such that, at the beginning of each period $t \in \mathbb{Z}_0^+$:

- the echelon inventory position of the warehouse is increased up to y_0, i.e., $IP_0(t) = y_0$,
- the inventory position of retailer i is raised to $z_i(I_0(t))$ where $\{z_i(I_0(t))\}_{i\in J}$ is a feasible solution for (3.2).

Consider the expected cycle cost attached to period t. As mentioned before, under the balance assumption, $\mathbf{E}[C_{cyc}(t)]$ depends only on the ordering and allocation decisions that start with an order given by the warehouse in period t. As a result, it can be shown that base stock policy $(y_0^*, \mathbf{z}^*)$ is the optimal replenishment policy with y_0^* satisfying the following newsboy equations:

$$\int_0^\infty F_i^{(l_i+1)}(z_i^*(y_0^* - u))dF_0^{(l_0)}(u) = \frac{p_i}{h_0 + h_i + p_i} \quad \forall i \in J, \tag{3.3}$$

(cf. Diks and de Kok (1998)). (3.3) indicates that y_0^* is the level at which the probability of being out of stock at retailer i is $\frac{h_0+h_i}{h_0+h_i+p_i}$.

Due to the fact that warehouse order-up-to level (y_0^*) and optimal allocation functions $(\mathbf{z}^*)$ are independent of time, policy $(y_0^*, \mathbf{z}^*)$ can be applied to optimize each period's cycle cost within the horizon. Thus, the expected average cost obtained by following $(y_0^*, \mathbf{z}^*)$ is a lower bound (LB) for (2.2).

Remark 3.1 *Consider the following two extreme scenarios: (i)* $h_0 = 0$ *and* $h_i > 0 \; \forall \; i \in J$*, (ii)* $h_0 > 0$*,* $h_i = 0$*, and* $h_j > 0$ *for* $j \in J \setminus \{i\}$*. In scenario (i), right-hand sides of the equalities in (3.1) and (3.3) become the same. Since the demand is distributed over* $(0, \infty)$*, left-hand sides of (3.1) and (3.3) can only be equal when* $y_0^* = \infty$*. In other words, the warehouse keeps infinite stock. This is intuitive because there is no cost of keeping stock at the warehouse (*$h_0 = 0$*). In such a case, imbalance of the retailer inventories is not an issue. In scenario (ii), the right-hand side of (3.1) is 1 for retailer i, which requires* $y_i^* = \infty$*. For the other retailers, there exists finite optimal order-up-to levels, i.e.,* $y_j^* < \infty$ *for* $j \in J \setminus \{i\}$*. Due to the fact that the right-hand side of (3.3) is less than 1,*

y_0^* is also finite. As a result, under the optimal policy, the warehouse becomes a cross-docking point, forwarding any stock that might have been kept at the warehouse to retailer i. Note that if the warehouse does not keep stock at the end of an arbitrary period, then there might be an imbalance among retailer inventories. It is not hard to expect that the imbalance of retailer inventories becomes an important issue in scenario (ii).

3.3.4 Constructing an Upper Bound

Consider the base stock policy $(y_0^*, \mathbf{z}^*)$ which is discussed in §3.3.3. This policy is indeed the optimal replenishment policy for (2.2) if the system never experiences imbalance. When policy $(y_0^*, \mathbf{z}^*)$ is implemented, there might be negative shipments, which make it infeasible to apply the solution of (3.2). In such a situation, one may ship nothing to retailers with negative shipments, deduct their inventory positions from system-wide stock, and allocate the remaining among the rest of the retailers by solving (3.2). This way of allocation is indeed the optimal solution of the myopic allocation problem in (2.3)-(2.5), cf. Axsäter *et al.* (2002, p. 79) or Zipkin (2000, pp. 340-342). The replenishment/allocation policy described, which we refer to as LB heuristic policy, is feasible and can be applied in each period within the horizon. This policy is not optimal, so the average expected cost of it is an upper bound (UB) for (2.2).

3.3.5 The Gap between LB and UB

For a given problem instance, the optimal inventory control parameters can be calculated for the lower bound model since analytical results are available. An estimate for an upper bound can be determined by simulating the LB heuristic policy described in §3.3.4. Since the value of (2.2) lies between LB and UB, the gap can be used to measure the impact of the balance assumption on the expected system-wide cost. Let $\epsilon\% = 100\frac{UB-LB}{LB}$ where $\epsilon\%$ is defined as the relative gap. If $\epsilon\%$ is small for a problem instance, then one can conclude that the balance assumption is not restrictive for that setting. While LB is an appropriate proxy for the true optimal cost, the LB heuristic policy is an accurate heuristic. On the other hand, a moderate or a large $\epsilon\%$ requires a further investigation which is the purpose of Chapter 4. We also consider $UB - LB$ and LB values in the numerical study since they might provide additional information apart from $\epsilon\%$.

3.4 Numerical Study and the Results

Our main objective is to identify the input parameter settings for which the resulting relative gaps are small and for which not. In addition, we try to illustrate the effect of leadtimes, holding and penalty costs, and the demand parameters of the retailers on the relative gaps. In order to achieve these goals, two test beds are generated; one for the case where retailers are identical, and the other for nonidentical retailers. The results and the discussions are presented in §3.4.1 and §3.4.2 for identical and nonidentical retailers cases, respectively. This section is concluded in §3.4.3 with a summary of the insights obtained from the results, and a comparison of these insights against the previous findings in the literature.

We use the words *scenario* and *problem instance* interchangeably for each combination of the input parameters N, l_0, h_0 and $h_i, p_i, \mu_i, l_i, cv_i$ for $i = 1, ..., N$.

In order to determine the optimal order-up-to level of each stock point, an integral with an upper limit of $+\infty$ has to be evaluated. For the calculations, we terminated when cdf value of any demand distribution had reached $1-10^{-8}$. The random number generator used is a prime modulus multiplicative linear congruential generator given in Law and Kelton (2000). The method of *batch means* is used for constructing a point estimate and a confidence interval for the steady-state mean of the cost. The batch size is fixed at 10,000 periods and the observations of the first batch were deleted (i.e., the warmup period consists of 10,000 periods). We ran each scenario for at least 200 batches and terminated as soon as the width of a 95% confidence interval about the average cost function was within 1% of the average cost. In scenarios with low coefficient of variation, LB and UB figures are very close. In some runs, UB turned out to be slightly lower than the corresponding LB value, an effect entirely caused by sampling errors[2].

3.4.1 Identical Retailers

For this subsection, the retailers are taken identical in terms of leadtime, holding and penalty costs, and the mean and coefficient of variation of the demand processes, i.e., $l_i = l_{i+1}$, $h_i = h_{i+1}$, $p_i = p_{i+1}$, $\mu_i = \mu_{i+1}$, and $cv_i = cv_{i+1}$ for $i = 1, ..., N-1$. Without loss of generality, the mean demand (μ_i) and the holding cost $(h_0 + h_i)$ at each retailer i is kept at 1 for all scenarios.

[2]Federgruen and Zipkin (1984a, p. 115) report similar observations where the simulated cost is lower than the corresponding LB value. See also Table 9 in Axsäter *et al.* (2002).

The following set of parameters is used (for $i = 1, 2, ..., N$):

$$h_i = 0, 0.1, 0.5, 0.9, 0.99 \qquad p_i = 4, 9, 19, 99 \qquad N = 2, 3, 4, 5$$
$$(l_0, l_i) = (1,1), (1,3), (3,1), (1,5), (5,1) \qquad cv_i = 0.25, 0.5, 1, 2, 3$$

The penalty costs are chosen to assure no-stockout probabilities of 80%, 90%, 95%, and 99% at each retailer under an optimal policy in the lower bound model. Additional inventory holding cost at a stock point reflects the added value at that point. Notice that as h_i increases from 0 to 0.99, the added value at the warehouse decreases since $h_0 + h_i = 1$. A full factorial design is used to generate a test bed that consists of 2000 problem instances.

During the simulation runs for computing UB, the probability of imbalance was also estimated. The probability of imbalance is defined as the fraction of periods in which a negative quantity is allocated to a retailer when (3.2) is solved.

The results are given in Table 3.1, which is organized in such a way that the results are summarized with respect to one parameter at a time. For example, the first part of the table is dedicated to display the effect of the coefficient of variation of demand. The first column gives the values of various measures for a set of 400 problem instances in which $cv_i = 0.25$ for $i = 1, 2, ..., N$. The *measures* used in the analysis are minimum, maximum and average $\epsilon\%$ (denoted by *min* $\epsilon\%$, *max* $\epsilon\%$ and *ave.* $\epsilon\%$, respectively), minimum, maximum and average probability of imbalance (denoted by *min* π, *max* π and *ave.* π, respectively), maximum and average $UB - LB$ (denoted by *max* δ and *ave.* δ, respectively), and average LB (denoted by *ave.* LB). All probability figures are given as *percentages*. Due to sampling errors, $UB - LB$ can be negative for a few scenarios. When calculating *ave.* $\epsilon\%$ and *ave.* δ for such scenarios, $UB - LB$ is taken as zero.

The main findings are summarized below.

1. In order to identify the impact of the balance assumption, we looked at *max* $\epsilon\%$ and *ave.* $\epsilon\%$ values in Table 3.1. An input parameter resulting in *max* $\epsilon\% \leq 10$ and *ave.* $\epsilon\% \leq 2$ is considered to be a setting with a small relative gap. The results for identical retailers show that the relative gap is small when

- the coefficient of variation of the retailers is moderate or low (i.e., cv_i= 0.25, 0.5 or 1), *or*
- the added value at the retailers is very high compared to the one at the warehouse (i.e., $h_i = 0.99$), *or*

- the warehouse leadtime is short, and retailer leadtimes are long (i.e., (l_0, l_i)= (1,3) or (1,5)).

Especially in scenarios with low coefficient of variation, the relative gap is negligible, see *max* $\epsilon\%$ and *ave.* $\epsilon\%$ figures for $cv_i = 0.25$ and 0.5 in Table 3.1.

Out of the 2000 problem instances studied, 1713, 1487 and 1275 of them have $\epsilon\%$ less than or equal to 5, 2, and 1, respectively. When the $\epsilon\%$ values (of all instances) are sorted, the scenarios with high relative gap figures have the following common parameter settings:

(i) 40 scenarios with $\epsilon\% > 25$; all having $cv_i = 3$, $l_0 = 3$ or 5, and h_i =0, 0.1 or 0.5.

(ii) 12 scenarios with $25 \geq \epsilon\% > 20$; all having $cv_i = 3$, and h_i =0, 0.1 or 0.5.

(iii) 36 scenarios with $20 \geq \epsilon\% > 15$; all having $cv_i = 2$ or 3, and h_i =0, 0.1 or 0.5.

(iv) 70 scenarios with $15 \geq \epsilon\% > 10$; all having $cv_i = 2$ or 3.

The figures above suggest that a high coefficient of variation is the main determinant of large gaps; see also Figure 3.1. Especially, when a high coefficient of variation is combined with moderate or low added value at the retailers, the gaps grow.

2. The imbalance probabilities are scattered against the corresponding $\epsilon\%$ values in Figure 3.1, where a different symbol is used for each coefficient of variation value. First, the results show that a high probability of imbalance does not necessarily lead to high $\epsilon\%$, but a high π is a necessity for a large relative gap. Among all the instances, 273 and 85 of them have $\epsilon\% < 2$ and $\pi > 15$, and $\epsilon\% < 1$ and $\pi > 20$, respectively. Second, when the points with very high $\epsilon\%$ and π values are analyzed, it turns out that they have long warehouse leadtimes and high coefficient of variations in common. There are 44 points in the region $\epsilon\% \geq 24.03$ and $\pi \geq 37.39$, all coming from scenarios with $cv_i = 2$ or 3, and (l_0, l_i) = (3,1) or (5,1).

3. The probability of having imbalanced retailer inventories increases when

- the warehouse leadtime extends (compare *ave.* π and *max* π values of columns (1,1), (3,1) and (5,1) in Table 3.1), *or*

Table 3.1: The summary of the results from scenarios with identical retailers.

	cv_i				
	0.25	0.5	1	2	3
$min\ \epsilon\%$	0.00	0.00	0.00	0.00	0.00
$max\ \epsilon\%$	0.15	0.36	4.93	16.63	38.55
$ave.\ \epsilon\%$	0.01	0.04	0.93	4.21	7.56
$min\ \pi$	0.00	0.00	0.13	0.08	0.09
$max\ \pi$	0.00	4.55	41.03	63.96	63.87
$ave.\ \pi$	0.00	2.22	21.06	30.91	27.69
$max\ \delta$	0.01	0.03	0.76	8.52	25.81
$ave.\ \delta$	0.00	0.00	0.17	1.53	4.50
$ave.\ LB$	7.29	11.13	19.98	39.71	63.50

	$h_i\ (= 1 - h_0)$				
	0	0.1	0.5	0.9	0.99
$min\ \epsilon\%$	0.00	0.00	0.00	0.00	0.00
$max\ \epsilon\%$	37.12	37.40	38.55	13.73	1.86
$ave.\ \epsilon\%$	4.25	4.29	3.32	0.82	0.08
$min\ \pi$	0.00	0.00	0.00	0.00	0.00
$max\ \pi$	63.96	63.96	63.87	51.31	13.35
$ave.\ \pi$	25.78	25.65	21.65	7.68	1.12
$max\ \delta$	25.81	25.81	25.15	10.08	1.13
$ave.\ \delta$	2.09	2.09	1.57	0.42	0.04
$ave.\ LB$	32.68	31.91	28.71	24.79	23.53

	p_i				N			
	4	9	19	99	2	3	4	5
$min\ \epsilon\%$	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
$max\ \epsilon\%$	31.44	38.55	20.13	27.74	30.08	37.19	37.86	38.55
$ave.\ \epsilon\%$	2.47	3.21	2.14	2.39	2.46	2.73	2.60	2.41
$min\ \pi$	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
$max\ \pi$	63.96	63.96	63.96	63.96	49.56	58.75	62.32	63.96
$ave.\ \pi$	15.81	17.10	17.01	15.58	12.06	15.93	18.03	19.48
$max\ \delta$	13.56	25.15	14.73	25.81	15.01	22.01	24.75	25.81
$ave.\ \delta$	0.74	1.39	1.13	1.70	0.72	1.16	1.44	1.64
$ave.\ LB$	17.68	24.41	30.35	40.85	16.64	24.43	32.22	40.01

Table 3.1: Continued.

	(l_0, l_i)				
	(1,1)	(1,3)	(1,5)	(3,1)	(5,1)
min $\epsilon\%$	0.00	0.00	0.00	0.00	0.00
max $\epsilon\%$	21.67	8.62	6.66	37.63	38.55
ave. $\epsilon\%$	2.31	1.12	0.82	4.13	4.38
min π	0.00	0.00	0.00	0.00	0.00
max π	48.73	48.73	48.73	61.96	63.96
ave. π	15.30	16.09	16.31	17.01	17.17
max δ	12.45	10.47	6.60	23.16	25.81
ave. δ	0.91	0.67	0.60	1.86	2.16
ave. LB	22.33	32.04	40.13	23.20	23.91

- the number of retailers increases (compare *ave.* π and *max* π values for $N = 2, 3, 4, 5$ in Table 3.1), *or*

- the added value at the warehouse increases (compare *ave.* π and *max* π values for $h_i = 0.01, 0.1, 0.5, 0.9, 1$ in Table 3.1).

In the rest of this subsection, we further discuss the effect of the coefficient of variation, leadtimes, holding and penalty costs, and the number of retailers on the performance measures.

<u>Coefficient of Variation</u>

The coefficient of variation and the imbalance of inventories are positively correlated. As the coefficient of variation increases, *max* $\epsilon\%$, *ave.* $\epsilon\%$, *max* δ and *ave.* δ all increase (see Table 3.1). This phenomenon is intuitive, suggesting that as the variance in the demand process gets larger, the retailer inventories become more imbalanced. The values of *max* π and *ave.* π seem to support this argument. For *max* π and *ave.* π, a steady rise is observed as the coefficient of variation increases from 0.25 to 2, but the figures drop for 3. In order to explain this decline, we compared the order-up-to level of each of the 400 scenarios with $cv_i = 2$ against the corresponding one with $cv_i = 3$. It turns out that more stock is kept at the warehouse in the scenarios with $cv_i = 3$, which results in lower imbalance probability. Though the imbalance probability figures decrease for $cv_i = 3$, it does not affect the trend for gap measures.

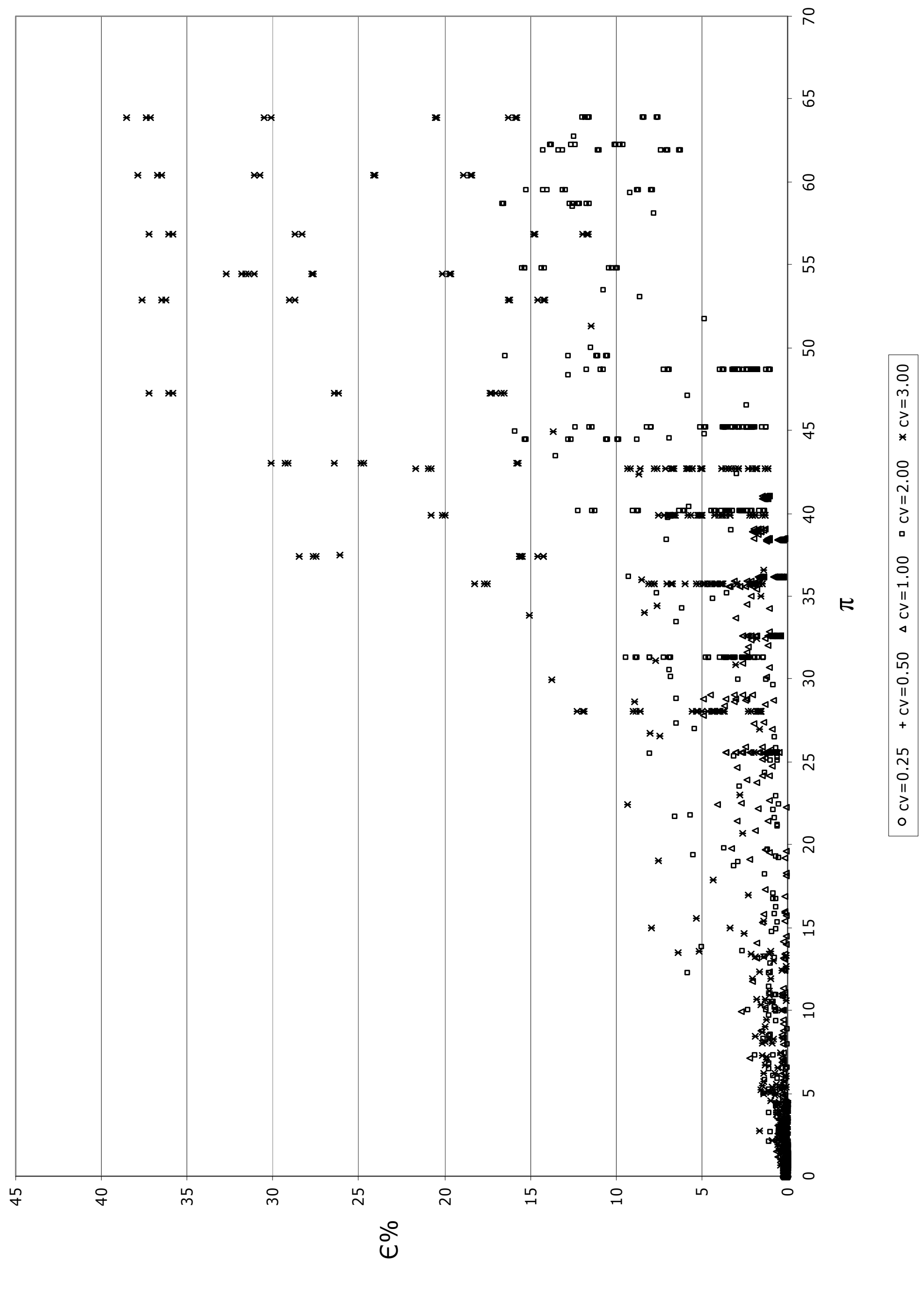

Figure 3.1: Probability of imbalance vs. relative gap for scenarios with identical retailers.

Holding Costs

In order to see the effect of the holding costs better, we extended the study for more h_i values between 0.1 and 0.8, see Table 3.2. The cost figures in Table 3.2 are chosen in a way that at one extreme there is no added value at the retailers ($h_i = 0$); at the other extreme, 99% of an item's value is attained at the retailers ($h_i = 0.99$). Our observations are as follows:

- Compare *max* π figures for different h_i values in Table 3.2. Between 0 and 0.3, *max* π is level at 63.96. For $h_i = 0.5$ and 0.6, there is a slightly lower level (63.87), and *max* π decreases as h_i grows starting from $h_i = 0.7$. For *ave.* π, there is a steady decline as h_i increases. The trend in imbalance probability measures is due to keeping more stock at the warehouse than

Table 3.2: The summary of the results for various values of h_i with identical retailers.

	h_i $(= 1 - h_0)$					
	0.00	0.10	0.20	0.30	0.40	0.50
min ϵ%	0.00	0.00	0.00	0.00	0.00	0.00
max ϵ%	37.12	37.40	37.68	37.96	38.25	38.55
ave. ϵ%	4.25	4.29	4.24	3.75	3.50	3.32
min π	0.00	0.00	0.00	0.00	0.00	0.00
max π	63.96	63.96	63.96	63.96	63.95	63.87
ave. π	25.78	25.65	25.13	23.89	22.79	21.65
max δ	25.81	25.81	25.15	25.15	25.15	25.15
ave. δ	2.09	2.09	2.04	1.82	1.68	1.57
ave. LB	32.68	31.91	31.13	30.35	29.54	28.71

	0.60	0.70	0.80	0.90	0.99
min ϵ%	0.00	0.00	0.00	0.00	0.00
max ϵ%	20.15	18.99	17.30	13.73	1.86
ave. ϵ%	2.38	1.88	1.41	0.82	0.08
min π	0.00	0.00	0.00	0.00	0.00
max π	63.87	63.75	58.52	51.31	13.35
ave. π	18.94	15.91	12.37	7.68	1.12
max δ	14.74	14.72	13.87	10.08	1.13
ave. δ	1.13	0.93	0.72	0.42	0.04
ave. LB	27.85	26.92	25.91	24.79	23.53

at the retailers as h_0 decreases (compared to h_i) since it is less costly. As a result, the chance of imbalance decreases.

- Compare *max* $\epsilon\%$ figures in Table 3.2. There is an increasing trend starting from $h_i = 0$ until $h_i = 0.5$. Then a sharp decline occurs at $h_i = 0.6$ (*max* $\epsilon\% = 38.55$ for $h_i = 0.5$ and *max* $\epsilon\% = 20.15$ for $h_i = 0.6$). From this point on, *max* $\epsilon\%$ values decrease with another sharp decline at $h_i = 0.99$. For *ave.* $\epsilon\%$, there is a steady decreasing trend as h_i increases starting from $h_i = 0.1$ with sharp declines at $h_i = 0.6$ and $h_i = 0.99$.

These observations point out a general tendency that the relative gap decreases as the added value at the upper echelon gets smaller with respect to the one at the lower echelon (i.e., h_0 decreases with respect to h_i). This pattern is more apparent when the added value at the warehouse is less than the one at the retailers (i.e., for h_i =0.6,0.7,0.8,0.9,0.99).

Leadtimes

Warehouse Leadtime: When l_0 extends, *max* $\epsilon\%$, *ave.* $\epsilon\%$, *max* δ, and *ave.* δ all increase (compare (1,1), (3,1) and (5,1) in Table 3.1). One can conclude that the longer the warehouse leadtime, the larger the relative gap. As the warehouse leadtime extends, it takes a longer time to recover from an imbalance situation. When the system stays in imbalance for more periods in a row, the effect on the expected cost becomes considerable.

Retailer Leadtime: Consider columns (1,1), (1,3) and (1,5) in Table 3.1. While *max* $\epsilon\%$, *ave.* $\epsilon\%$, *ave.* δ and *ave.* δ decrease, *ave.* LB shows a considerable increase. We conclude that gaps shrink as retailer leadtimes grow. Although the variance faced by the system grows and the warehouse is forced to ration far in advance as retailer leadtimes lengthen, the coefficient of variation over the retailer leadtime decreases. This is the reason for decreasing relative gaps.

Penalty Costs

The results do not show a clear relation between p_i and the measures considered.

Number of Retailers

As N gets larger, *max* $\epsilon\%$, *max* π, *ave.* π, *max* δ, and *ave.* δ figures grow. However, the trend in these measures is not reflected on *ave.* $\epsilon\%$.

3.4.2 Nonidentical Retailers

The number of retailers is restricted to two ($N = 2$) for the nonidentical retailers case since all the effects that we are after can be captured. The following parameter settings are used:

$$\begin{array}{lcl}(h_0, h_1, h_2) & = & (0.5, 0, 0.5), (0.5, 0.5, 0), (0.5, 0.5, 0.5), (0.9, 0, 0.1), (0.9, 0.1, 0), \\ & & (0.9, 0.1, 0.1) \\ (cv_1, cv_2) & = & (0.5, 0.5), (0.5, 1), (0.5, 2), (1, 0.5), (1, 1), (1, 2), (2, 0.5), (2, 1), (2, 2) \\ (l_0, l_1, l_2) & = & (1, 1, 1), (1, 1, 5), (1, 5, 1), (1, 5, 5), (5, 1, 1), (5, 5, 5) \\ (\mu_1, \mu_2) & = & (0.05, 0.95), (0.2, 0.8), (0.5, 0.5) \\ (p_1, p_2) & = & (9, 9), (9, 19), (19, 9), (19, 19)\end{array}$$

All combinations of the parameters generate a test bed of 3888 problem instances. All through the chapter, we treat the mean demand as a representant of the size of a retailer. The values for mean demands are chosen to see the effect of big/small retailer and without loss of generality, the sum is kept at 1. Note that in all the problem instances considered, the first retailer is equal or smaller in size compared to the second. Three values are selected for the coefficient of variation: 0.5 (low), 1 (moderate), 2 (high). This leads to nine distinct combinations. We picked leadtime values to reflect the effect of short warehouse and retailer leadtimes [(1,1,1)], long warehouse and retailer leadtimes [(5,5,5)], long warehouse leadtime-short retailer leadtime [(5,1,1)] and vice versa [(1,5,5)], and short warehouse leadtime and asymmetric retailer leadtimes [(1,1,5) and (1,5,1)]. We interpret the holding costs as a way of reflecting the added value. While (0.5,0.5,0.5) combination corresponds to an equal added value at all stock points, (0.9,0.1,0.1) represents high added value at the warehouse. The other combinations are chosen to see the effect of having zero added value at one of the retailers. From two values selected for penalty costs, high (19) and moderate (9), four combinations are generated to see the effect of equal and asymmetric penalty costs.

A summary of the results can be found in Table 3.3 (identical to Table 3.1 in terms of organization); the main findings are given below.

1. We still stick to the conditions $max\ \epsilon\% \leq 10$ and $ave.\ \epsilon\% \leq 2$ as a criterion for a relative gap to be small. The results in Table 3.3 show that for nonidentical retailers, the relative gap is small when

- there is positive and equal added value at each stock point (i.e., (h_0, h_1, h_2)= (0.5,0.5,0.5)), *or*
- the warehouse leadtime is short, and the leadtime of the second retailer (equal or larger in size) is long (i.e., (l_0, l_1, l_2)=(1,5,5) or (1,1,5)).

There are 1888, 2652 and 3390 problem instances having $\epsilon\%$ less than or equal to 1, 2, and 5, respectively (out of the 3888 scenarios studied). The instances with high $\epsilon\%$ figures have the following common parameter settings:

(i) 27 scenarios with $\epsilon\% \geq 69.76$; all having (l_0, l_1, l_2)=(5,1,1), (h_0, h_1, h_2)= (0.5,0,0.5), and (μ_1, μ_2)=(0.05,0.95).

(ii) 36 scenarios with $69.76 > \epsilon\% \geq 43.65$; all having $h_1 = 0$ and (l_0, l_1, l_2)= (5,1,1).

(iii) 55 scenarios with $43.65 > \epsilon\% > 25$; all having $h_1 = 0$ and $l_0 = 5$.

(iv) 23 scenarios with $25 \geq \epsilon\% > 17.85$; all having $h_1 = 0$ or $h_2 = 0$, and $l_0 = 5$.

(v) 35 scenarios with $17.85 \geq \epsilon\% > 15.3$; all having $h_1 = 0$ or $h_2 = 0$.

The figures above reveal the fact that long warehouse leadtime and having zero added value at one of the retailers cause significantly high gaps; especially no added value at the small retailer (a through discussion on this issue is presented under the heading *Holding Costs* in item 3).

It is interesting to notice that although the coefficient of variation turned out to be the main determinant of gaps in the identical retailers case, even $(cv_1, cv_2) = (0.5, 0.5)$ may lead to $\epsilon\%$ as high as 178.17, depending on the values of the other parameters, in the nonidentical retailers case.

Unlike in the identical retailers case, it is striking that the relative gap is small only in a few parameter settings. In many practically relevant parameter combinations, we observe large gaps. Especially, when there is asymmetry in the size of the retailers, a long warehouse leadtime, or a low added value at the small retailer, large gaps may occur.

2. The imbalance probabilities are scattered against the corresponding $\epsilon\%$ values as can be seen in Figure 3.2. The figure shows that a high $\epsilon\%$ requires a high π, but a high π does not necessarily correspond to a high $\epsilon\%$. For example, out of all scenarios considered, 1437 of them have $\epsilon\% < 2$ and $\pi > 15$, and 601 of them have $\epsilon\% < 1$ and $\pi > 20$. Now, we look at the points in the region with high $\epsilon\%$ and π. There are 118 problem instances with $\epsilon\% \geq 25.94$ and $\pi\% \geq 62.51$. It turns out to be that all these points have $l_0 = 5$ and zero added value at the small retailer (μ_1=0.05 or 0.2).

In the rest of this subsection, we delve into the relationships between the input parameters and the measures on the basis of trends.

Table 3.3: The summary of the results from scenarios with nonidentical retailers, $N = 2$.

	(cv_1, cv_2)					
	(0.5,0.5)	(0.5,1)	(0.5,2)	(1,0.5)	(1,1)	(1,2)
min ϵ%	0.00	0.00	0.00	0.03	0.02	0.00
max ϵ%	178.17	155.96	69.76	179.57	156.47	69.81
ave. ϵ%	3.15	3.24	1.95	3.92	4.17	2.91
min π	0.67	1.68	2.03	3.03	7.29	6.90
max π	85.96	89.79	85.57	86.00	89.82	85.67
ave. π	11.96	15.60	17.90	24.54	28.84	27.82
max δ	5.322	9.327	8.830	5.394	9.380	8.852
ave. δ	0.105	0.193	0.215	0.139	0.253	0.308
ave. LB	4.294	6.303	10.832	4.898	6.860	11.347

	(2,0.5)	(2,1)	(2,2)
min ϵ%	0.17	0.22	0.09
max ϵ%	186.86	159.42	70.23
ave. ϵ%	5.48	5.84	4.56
min π	6.19	12.80	13.53
max π	86.21	89.96	86.16
ave. π	40.00	42.73	35.51
max δ	5.705	9.614	8.943
ave. δ	0.238	0.409	0.538
ave. LB	6.329	8.216	12.605

h_0	0.5			0.9		
(h_1, h_2)	(0,0.5)	(0.5,0)	(0.5,0.5)	(0,0.1)	(0.1,0)	(0.1,0.1)
min ϵ%	0.00	0.00	0.00	0.00	0.00	0.00
max ϵ%	186.86	24.76	8.12	67.90	15.35	14.53
ave. ϵ%	13.71	1.73	1.28	3.72	1.48	1.56
min π	3.92	0.69	0.67	2.97	1.97	2.02
max π	89.96	49.74	50.68	89.43	57.94	60.24
ave. π	43.21	20.11	18.98	31.61	24.35	25.01
max δ	9.614	2.145	0.867	2.934	1.638	1.633
ave. δ	0.862	0.113	0.100	0.260	0.126	0.137
ave. LB	7.196	5.921	7.827	8.990	8.749	9.106

Table 3.3: Continued.

	(l_0, l_1, l_2)					
	(1,1,1)	(1,1,5)	(1,5,1)	(1,5,5)	(5,1,1)	(5,5,5)
min $\epsilon\%$	0.00	0.00	0.05	0.00	0.00	0.00
max $\epsilon\%$	17.84	4.90	17.32	2.93	186.86	43.30
ave. $\epsilon\%$	2.43	0.81	2.19	0.59	13.98	3.48
min π	1.32	0.69	2.29	1.88	0.67	1.24
max π	74.60	66.95	74.42	67.94	89.96	89.94
ave. π	24.71	22.04	27.49	24.70	32.25	32.09
max δ	0.591	0.511	0.555	0.434	9.614	4.279
ave. δ	0.125	0.071	0.136	0.063	0.839	0.365
ave. LB	5.193	8.793	6.395	10.007	6.450	10.952

	(μ_1, μ_2)			(p_1, p_2)			
	(0.05,0.95)	(0.2,0.8)	(0.5,0.5)	(9,9)	(9,19)	(19,9)	(19,19)
min $\epsilon\%$	0.00	0.00	0.01	0.00	0.00	0.00	0.00
max $\epsilon\%$	186.86	69.66	24.76	129.53	186.86	127.00	182.49
ave. $\epsilon\%$	6.11	3.25	2.37	3.33	4.75	3.34	4.23
min π	0.69	0.87	0.67	0.84	0.86	0.69	0.67
max π	89.96	70.67	49.82	89.96	89.96	89.96	89.96
ave. π	30.59	26.79	24.25	27.28	27.84	26.57	27.16
max δ	9.614	5.236	2.145	6.305	9.589	6.304	9.614
ave. δ	0.373	0.229	0.197	0.206	0.336	0.217	0.307
ave. LB	8.159	7.946	7.790	7.389	8.236	7.670	8.565

Remark 3.2 *In the test beds for both the identical and nonidentical retailers, we considered some extreme input parameter values like coefficient of variation equal to three or zero added value. Such parameter figures have a significant effect on the max* $\epsilon\%$ *and ave.* $\epsilon\%$ *values that one sees in Table 3.1 and 3.3, but not on the trends observed.*

Retailer Size

The results show that as the size of the retailers become disproportionate, the relative gap escalates. When μ_1 is decreased with respect to μ_2, we observe that *max* $\epsilon\%$, *ave.* $\epsilon\%$, *max* δ and *ave.* δ all increase. The measures *max* π and *ave.* π side with this trend. The impact of retailer size will be more evident as we discuss the other parameters like holding costs and coefficient of variation.

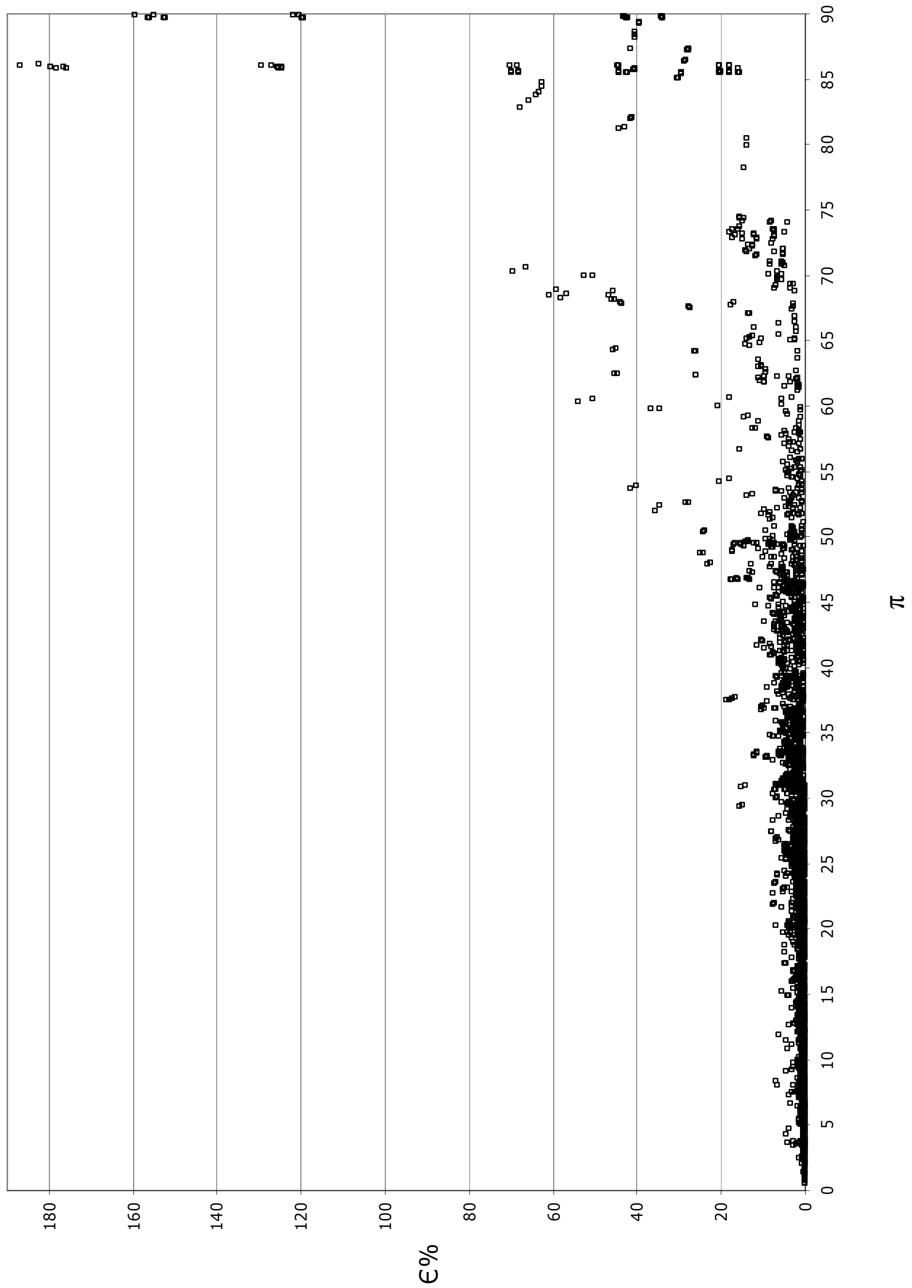

Figure 3.2: Probability of imbalance vs. relative gap for scenarios with non-identical retailers.

Holding Costs

- Similar to the identical retailers case, we observe that as the added value at the warehouse increases, the retailer inventories tend to become more imbalanced. Compare *max* $\epsilon\%$ and *ave.* $\epsilon\%$ values in columns (0.5,0.5,0.5) and (0.9,0.1,0.1) of Table 3.3. In both cases holding cost at the retailers is 1, but the warehouse holding cost differs. When h_0 gets larger, the system prefers to keep less stock at the warehouse, which in return increases the chance of imbalance. Larger *min* π, *max* π and *ave.* π in column (0.9,0.1,0.1) supports this reasoning.

- We can also discuss the combined effect of size and holding costs. Recall that the first retailer is equal to or smaller than the second one with respect to size. In Table 3.3, compare the following columns: (0.5,0,0.5) to (0.5,0.5,0.5), and (0.9,0,0.1) to (0.9,0.1,0.1). The results indicate that when the added value at the first retailer becomes zero, imbalance emerges as an important issue. Note that low added value at a retailer increases its order-up-to level (see (3.1)), so instead of retaining stock, the warehouse ships to this retailer. Since little or no inventory is kept at the warehouse, the chance of imbalance increases. There are two observations that verify this argument. First, all measures except *ave. LB* show a considerable increase when the holding cost at the first retailer becomes zero. Second, in terms of all measures the trend is stronger when $h_0 = 0.5$ because all the stock kept at the warehouse is routed to the small retailer. Since not much stock is kept at the warehouse when $h_0 = 0.9$ (due to high holding cost), the impact is weaker compared to $h_0 = 0.5$ case. We do not observe such a strong trend when the added value at the second retailer decreases; compare the following columns in Table 3.3: (0.5,0.5,0) to (0.5,0.5,0.5), and (0.9,0.1,0) to (0.9,0.1,0.1). Moreover, *ave.* $\epsilon\%$, *min* π, *max* π, *ave.* π values are lower in column (0.9,0.1,0) as compared to (0.9,0.1,0.1).

  Table 3.4 shows all scenarios with (h_0, h_1, h_2) equal to (0.5,0,0.5) and (0.5, 0.5,0), grouped with respect to the mean demands. When the added value at the first retailer is zero, *max* $\epsilon\%$, *ave.* $\epsilon\%$, *max* π, *ave.* π, *max* δ and *ave.* δ all show a sharp increase when the first retailer's mean demand decreases. In contrast, the aforementioned measures decrease as the second retailer's mean demand increases in case (0.5,0.5,0). In the light of these observations, we conclude that when the added value at one retailer is zero, both the probability and the extent of imbalance grows significantly as this retailer's mean demand decreases with respect to the other. Under the balance assumption, the amount of stock to keep at the warehouse is underestimated. Since the optimal order-up-to level for the retailer with zero added value is infinity, anything that might have been kept at the warehouse is sent to this retailer. As the mean demand of this retailer decreases it creates more and larger imbalance in the system. We call this phenomenon of the system getting severely imbalanced due to no added value at the small retailer as *forwarding-to-the-small-retailer*.

Table 3.4: Forwarding-to-the-small-retailer effect.

	(h_0, h_1, h_2)					
	(0.5,0,0.5)			(0.5,0.5,0)		
(μ_1, μ_2)	(0.05,0.95)	(0.2,0.8)	(0.5,0.5)	(0.5,0.5)	(0.2,0.8)	(0.05,0.95)
min $\epsilon\%$	0.00	0.00	0.08	0.11	0.00	0.00
max $\epsilon\%$	186.86	69.66	24.26	24.76	8.89	5.37
ave. $\epsilon\%$	27.17	10.63	3.33	3.39	1.25	0.55
min π	5.73	10.53	3.92	3.91	0.87	0.69
max π	89.96	70.67	49.82	49.74	39.85	43.22
ave. π	61.98	41.91	25.74	25.71	19.30	15.32
max δ	9.614	5.236	2.101	2.145	0.725	0.453
ave. δ	1.654	0.701	0.232	0.239	0.075	0.025
ave. LB	7.196	5.921	6.410	6.410	5.785	5.569

(Note that the table is organized in such a way that the mean demand of the retailer with zero additional inventory cost increases from left to right.)

Penalty Costs

Unlike the identical retailers case, interesting relationships are identified between the gaps and the penalty costs.

- The results confirm that as the penalty costs get larger, the relative gap grows significantly. When we compare columns (9,9) and (19,19) in Table 3.3, we observe that *max* $\epsilon\%$, *ave.* $\epsilon\%$, *max* δ and *ave.* δ all increase. Notice that *max* π and *ave.* π do not show a trend.
- For the effect of size, compare the following columns in Table 3.3: i. (9,9) to (9,19), ii. (9,9) to (19,9). In case (i), while probability measures do not show a trend, *max* $\epsilon\%$, *ave.* $\epsilon\%$, *max* δ and *ave.* δ all increase when the penalty cost of the second retailer is increased. However, in case (ii), there is hardly any change in any of the measures. We can conclude that as the

penalty cost of the second retailer increases, the relative gap widens. When the penalty cost of the big retailer is high, the negative effect of overstocking at the small retailer gets stronger, resulting in a larger gap.

Coefficient of Variation

In line with the identical retailers case, coefficient of variation and the imbalance of inventories are correlated. Compare the columns (0.5,0.5), (1,1) and (2,2) in Table 3.3. We observe an ascending trend in *ave.* $\epsilon\%$ and *ave.* δ. As the variation of demand rises, the inventories become more imbalanced; observe the significant increase in *min* π and *ave.* π figures. Interestingly, *max* $\epsilon\%$ shows a considerable decrease as the demand variance grows.

The joint effect of the retailer size and the coefficient of variation is somehow more complicated. Recall that the first retailer is smaller or equal to the second retailer in terms of size (mean demand). In Table 3.3, when the second retailer's coefficient of variation is kept constant and the first retailer's is varied (compare columns i. (0.5,0.5), (1,0.5), (2,0.5) ii. (0.5,1), (1,1), (2,1) iii. (0.5,2), (1,2), (2,2)), or the coefficient of variation of the first retailer is fixed and the second retailer's is varied (compare columns iv. (0.5,0.5), (0.5,1), (0.5,2) v. (1,0.5), (1,1), (1,2) vi. (2,0.5), (2,1), (2,2)), we observe different trends in the measures. In cases (i)-(iii), all measures show a positive trend as the coefficient of variation of the first retailer increases. For example, in case (i), *ave.* $\epsilon\%$ is 3.15, 3.92 and 5.48 in columns (0.5,0.5), (1,0.5) and (2,0.5), respectively. This behavior is intuitive since the demand variability faced by the system grows. However, in cases (iv)-(vi), we do not observe a similar trend when the coefficient of variation of the second retailer increases. While *max.* $\epsilon\%$ figures shrink considerably, *ave.* $\epsilon\%$ first inclines as cv_2 increases from 0.5 to 1, followed by a decline as cv_2 becomes 2. For example, in case (iv), *ave.* $\epsilon\%$ is 3.15, 3.24 and 1.95 in columns (0.5,0.5), (0.5,1) and (0.5,2), respectively. In order to distinguish the effects of size asymmetry and coefficient of variation on the balance of inventories, we prepared Tables 3.5 and 3.6.

Table 3.5 is constructed as follows. 1296 scenarios with $\mu_1 = \mu_2 = 0.5$ are clustered with respect to coefficient of variation, which leads to 9 cases ((cv_1, cv_2)=(0.5,0.5), (0.5,1), (0.5,2), (1,0.5), (1,1), (1,2), (2,0.5), (2,1), (2,2)) of 144 scenarios each. Observe that since retailer sizes are equal and there are two retailers, each scenario with (cv_1, cv_2)=(0.5,1) is equivalent to another scenario with (cv_1, cv_2)=(1,0.5). When cases with (cv_1, cv_2) equal to (0.5,1) or (1,0.5), (0.5,2) or (2,0.5), and (1,2) or (2,1) are aggregated, one is left with 6 cases: (cv_1, cv_2)=(0.5,0.5), (0.5,1) or (1,0.5), (0.5,2) or (2,0.5), (1,1), (1,2) or (2,1), (2,2). The measures for these 6 cases are calculated and presented in Table 3.5, which is organized as follows. In groups 1, 2 and 3, one retailer's coefficient of variation is kept constant at 0.5, 1 and 2, respectively. In each group, the other retailer's coefficient of variation obtains values 0.5, 1 and 2 as moved from left to right. In other words, in each group, while one retailer's coefficient of variation is constant, the other retailer's coefficient of variation is varied. Observe that within each group in Table 3.5, as one retailer's coefficient of variation increases, all measures show significant positive trends. We conclude that when there is no size asymmetry, as a retailer's coefficient of variation increases, the relative gap grows. This inference is intuitive because the variance of the total demand faced by the system gets larger; as a result, the relative gap increases.

In Table 3.6, the coefficient of variations of the retailer demands are kept constant and the retailer sizes are varied. Observe that *max* $\epsilon\%$, *ave.* $\epsilon\%$, *max* π and *ave.* π all increase significantly when the retailer sizes get disproportionate for (cv_1, cv_2) equal to (0.5,0.5) and (1,1). However, the trend is reverse for *ave.* $\epsilon\%$ and *ave.* π when $(cv_1, cv_2) = (2, 2)$. As observed, when there is no coefficient of variation asymmetry, the trend in relative gaps is not solely determined by the retailer size.

The trends of all gap and probability measures in cases (i)-(iii), i.e., inclining measures when the second retailer's coefficient of variation is kept constant and the first retailer's coefficient of variation is increased, can be explained by the tendency observed in Table 3.5. Unfortunately, the behavior in cases (iv)-(vi) is more complicated and Tables 3.5 and 3.6 do not provide enough insights to identify this behavior.

Leadtimes

Warehouse Leadtime: The impact of warehouse leadtime is in line with the observations from the identical retailers case. When the columns (1,1,1) and (5,1,1) of Table 3.3 are compared, one sees the drastic effect of warehouse leadtime. Both *max* $\epsilon\%$ and *ave.* $\epsilon\%$ obtain very high values when $l_0 = 5$. Notice that as *ave.* LB increases from 5.193 to 6.450 (by 128%), *ave.* δ jumps to 0.839 from 0.125 (571% rise). When the warehouse leadtime extends, the probability of having imbalanced inventories increases (compare *ave.* π and *max* π) and an imbalance situation takes a longer time to fix, which both expand the gaps.

Retailer Leadtimes: In Table 3.3, compare column (1,1,1) to (1,5,5). Although imbalance probability measures show a slight increase, *max* $\epsilon\%$, *ave.* $\epsilon\%$ and *ave.* δ figures point out a significant descent as both retailer leadtimes extend. Finally, when leadtime of the first retailer is increased, there is a tendency that inventories become more imbalanced; compare column (1,5,1) to

(1,1,5) and observe that *min* π, *max* π and *ave.* π all show a positive trend when the leadtime of the first retailer is longer. Further, all three relative gap measures follow the same trend.

Table 3.5: The effect of coefficient of variation on the balance of inventories when retailer sizes (mean demands) are equal.

Group	**1**			**2**		
(cv_1, cv_2)	(0.5,0.5)	(0.5,1) or (1,0.5)	(0.5,2) or (2,0.5)	(0.5,1) or (1,0.5)	(1,1)	(1,2) or (2,1)
min ϵ%	0.01	0.09	0.09	0.09	0.64	0.99
max ϵ%	1.43	7.34	15.72	7.34	12.10	18.56
ave. ϵ%	0.25	0.98	1.62	0.98	2.48	3.66
min π	0.67	3.03	6.17	3.03	8.71	12.80
max π	9.82	24.27	33.98	24.27	33.62	46.94
ave. π	4.27	14.09	26.10	14.09	26.02	35.66
max δ	0.031	0.266	1.104	0.266	0.508	1.372
ave. δ	0.008	0.043	0.120	0.043	0.138	0.310
ave. LB	4.209	5.491	8.474	5.491	6.670	9.548

Group	**3**		
(cv_1, cv_2)	(0.5,2) or (2,0.5)	(1,2) or (2,1)	(2,2)
min ϵ%	0.09	0.99	1.91
max ϵ%	15.72	18.56	24.76
ave. ϵ%	1.62	3.66	6.03
min π	6.17	12.80	21.72
max π	33.98	46.94	49.82
ave. π	26.10	35.66	36.23
max δ	1.104	1.372	2.145
ave. δ	0.120	0.310	0.677
ave. LB	8.474	9.548	12.210

Table 3.6: The effect of retailer size on the balance of inventories when coefficient of variations are fixed.

(cv_1, cv_2)	(0.5,0.5)			(1,1)		
(μ_1, μ_2)	(0.05,0.95)	(0.2,0.8)	(0.5,0.5)	(0.05,0.95)	(0.2,0.8)	(0.5,0.5)
min $\epsilon\%$	0.00	0.00	0.01	0.02	0.35	0.64
max $\epsilon\%$	178.17	35.56	1.43	156.47	60.90	12.10
ave. $\epsilon\%$	7.70	1.51	0.25	6.66	3.41	2.48
min π	0.69	0.87	1.32	11.16	16.88	21.12
max π	74.60	29.26	6.44	73.57	40.05	26.06
ave. π	14.92	6.77	3.91	25.89	24.41	25.28
max δ	0.378	0.096	0.019	0.368	0.249	0.175
ave. δ	0.034	0.011	0.008	0.042	0.071	0.126
ave. LB	4.383	4.290	4.209	7.059	6.851	6.670

(cv_1, cv_2)	(2,2)		
(μ_1, μ_2)	(0.05,0.95)	(0.2,0.8)	(0.5,0.5)
min $\epsilon\%$	0.09	0.98	1.91
max $\epsilon\%$	70.23	46.73	24.76
ave. $\epsilon\%$	3.26	4.38	6.03
min π	13.53	21.97	30.78
max π	53.02	39.47	31.74
ave. π	25.61	29.55	31.21
max δ	0.262	0.436	0.591
ave. δ	0.072	0.266	0.465
ave. LB	13.024	12.582	12.210

3.4.3 Summary and Insights

In this subsection, we summarize the results and insights obtained, and discuss their implications. We also compare our results against the previous findings in the literature.

The results for the identical retailers show that coefficient of variation, warehouse leadtime and relative added value at the warehouse are positively correlated to relative gaps. High coefficient of variation, and moderate or high added value at the warehouse (relative to the retailers) may lead to large relative gaps, as high as 38.6%. The relative gap is small (irrespective of the values of the other input parameters) when one of the following conditions is satisfied:

(i) the coefficient of variation is low or moderate,

(ii) the added value at the warehouse is insignificant with respect to the retailers,

(iii) short warehouse leadtime and long retailer leadtimes.

Under these settings, the use of the balance assumption is justified. If none of these conditions hold, high relative gaps may occur. In such settings, one has to be cautious in either utilizing LB as a proxy for the true optimal cost or using the LB heuristic policy as a control tool.

The results from the test bed of 3888 problem instances with nonidentical retailers indicate that the relative gap is small in a confined set of input parameters. The relative gap is small when one of the following conditions is satisfied:

(i) positive and equal added value at the warehouse and the retailers,

(ii) short warehouse leadtime, and long retailer leadtime at the big retailer or at both retailers.

Large gaps (as high as 186.9%) are observed in scenarios with a long warehouse leadtime and/or positive added value at one retailer and zero at the other. The latter situation is relevant to production environments where a finished product or a semi-finished assembly is stored centrally to supply two downstream stock points: at one, negligible value is added (e.g., just transportation or packaging), and at the other there is a positive added value (e.g., a module is added or further assembly operations are carried out). The relative gaps tend to expand as warehouse leadtime, holding cost at the warehouse and penalty costs at the retailers increase. Moreover, large differences in cost and demand parameters among the retailers augment the gaps, which conform to the results of Axsäter *et al.* (2002). In addition, we identified a phenomenon coined as forwarding-to-the-small-retailer, which is a result of overstocking at the small retailer due to negligible added value at this retailer. Forwarding-to-the-small-retailer leads to a severely imbalanced system, which results in big gaps. In such a case, one may consider putting an upper bound on the order-up-to level of the small retailer. Further, the combined effect of the retailer size and the coefficient of variation of demand turned out to be more interesting and complicated than expected.

The scenarios with high relative gaps also exhibit high imbalance probability, but there are some cases with a high imbalance probability and a quite small relative gap. These results may be important for developing heuristics that are based on or use imbalance probability (e.g., Verrijdt and de Kok (1995)). Eppen and Schrage (1981) derived an approximate term for the probability of having a balanced allocation, and the findings reveal the fact that probability of imbalance grows as the number of retailers increases; our results for the imbalance probabilities are in line with this fact.

Remark 3.3 *Schwarz (1989) has demonstrated that under the balance assumption, the benefit of risk pooling is high in situations with high demand variability, or long warehouse leadtime together with short retailer leadtimes. Our results reveal the fact that these are the settings where the gaps grow significantly. Although, it is not possible to conclude that the solution of the lower bound model is mediocre for these settings without knowing the optimal solution of the original problem, the prospective savings that are shown to exist under the balance assumption may not be attainable in a realistic setting.*

3.5 Conclusion

In this chapter, we studied the effect of a widely used assumption, the balance assumption, in the analysis of periodic review divergent inventory systems on the average expected cost. The balance assumption leads to a relaxation of the original problem, and the corresponding average expected cost is a lower bound. When the optimal policy of the relaxed model is modified (LB heuristic policy) and simulated, an upper bound is obtained. We used the relative gap between these bounds as an indicator for the effect of the balance assumption. We explicitly determined the scenarios that lead to small relative gaps (i.e., the settings under which the use of the balance assumption is justified); see 3.4.3 for a summary. In many practically relevant scenarios, which are identified expressly, the relative gaps are found to be moderate or large. These cases require the exact positioning of the optimal cost with respect to the upper and lower bounds.

In the next chapter, we solve the original optimization problem given in (2.2) by dynamic programming. Due to the curse of dimensionality, it is not possible to consider the problem instances of this chapter (even for the simple one-warehouse two-retailers system). Thus, we restrict ourselves to simple discrete demand structures and limited number of retailers. The results allow us to assess the precise impact of the balance assumption and shed more light on the optimal policy behavior.

3.6 Appendix: Demand Data from Practice

There are some studies in the literature providing real life demand data that exhibit large variability. One of the oldest reference is a paper by Burgin and Wild (1967) who mention actual demand data having a coefficient of variation slightly less than 2. Muckstadt (1997) argues that the manufacturing and distribution environments have gone through drastic changes since 90's, and the customers demand shorter delivery times (leadtimes). As the leadtimes shrink, the customer demand exhibits larger variability. Muckstadt provides data from six different companies; we mention two of them here:

- The weekly demand data (over a year) for 39 products manufactured by a cell at an aerospace component manufacturer exhibits high demand variability. While the coefficient of variations of demands of the products vary between 0.77 and 7.14, 28 products have coefficient of variations above 2.
- The second demand data comes from an assembly cell at a manufacturer supplying truck, bus, construction, mining and machine tool sectors throughout North America. About 76% of the unit demand for finished products have leadtimes of two weeks or less. The biweekly demand data over a year for 26 products assembled at this cell are given. While the coefficient of variation of demand of a product varies between 0.51 and 4.39, 17 products exhibit coefficient of variations larger than 1.

In Table 3.7, we provide weekly demand data obtained from a multi-national high volume electronic goods producer (having headquarters located in the Netherlands) for a product family consisting of 22 end products. Demand originates from retailers and wholesalers. For confidentiality reasons, means and standard deviations are manipulated, but the coefficient of variation values are exact. The figures show that demands for all end products have a coefficient of variation greater than 0.5; even two with more than 3.

Table 3.7: Weekly demand data for a product family from a consumer electronics company.

end product	μ	σ	cv	end product	μ	σ	cv
1	1493	1737	1.16	12	15175	11956	0.79
2	68	175	2.58	13	1286	1810	1.41
3	2158	2025	0.94	14	13184	12909	0.98
4	90	236	2.60	15	909	3460	3.81
5	93	237	2.53	16	17676	12632	0.71
6	1829	1910	1.04	17	3055	6106	2.00
7	3761	2467	0.66	18	1208	2927	2.42
8	7291	5301	0.73	19	405	645	1.59
9	1178	1025	0.87	20	12397	13115	1.06
10	9428	12542	1.33	21	787	1253	1.59
11	1444	2371	1.64	22	1034	3997	3.87

(μ= mean demand, σ= standard deviation of demand, cv= coefficient of variation of demand)

Chapter 4

Relative Gap between the Optimal Cost and the Bounds

Abstract: *This study is a continuation of the one in Chapter 3. For the settings with moderate or large relative gaps between the upper and lower bounds (as identified in the previous chapter), we position the optimal cost between the bounds, and determine the precise effect of the balance assumption. The one-warehouse multi-retailer inventory system is modelled as a multi-dimensional stochastic dynamic program and the optimal expected average cost is calculated by solving the resulting dynamic program by value-iteration algorithm. By comparing the optimal cost against the bounds, we (i) quantify the impact of the balance assumption on the long-run expected average cost, (ii) evaluate the quality of the lower bound as a proxy for the true optimal cost, (iii) determine the performance of the LB heuristic policy. We provide the first, concrete evidence in the literature that the balance assumption may lead to considerable errors. Further, the LB heuristic policy performs poorly compared to the optimal cost under various settings, which are identified explicitly. This indicates the need for efficient, accurate and robust heuristics. Finally, for the first time in the literature, we analyze the optimal policy behavior numerically, which facilitates the development of heuristics.*

4.1 Introduction

This chapter deals with a distribution system that consists of a warehouse and multiple (possibly nonidentical) retailers. The retailers are replenished by *shipments* from the warehouse, which in return *orders* from an exogenous supplier with ample stock. The retailers face stationary stochastic demand of the customers. All unfulfilled demand is backlogged and a penalty cost is charged. Inventory transfers between the retailers (lateral transhipment) are not permitted. We assume fixed replenishment leadtimes between the warehouse and the retailers, and between the supplier and the warehouse. Costs consist of linear holding and penalty costs. The system is controlled centrally and periodic review is employed. We consider the minimization of the expected total holding and penalty cost of the system in the long-run both under the discounted and average cost criteria.

The one-warehouse multi-retailer system is a relevant model in inventory, manufacturing and hierarchical production planning contexts (see Chapter 1). The problem of determining order and shipment sizes and their frequencies can be formulated as a stochastic dynamic program (DP). Unfortunately, the resulting DP is a multi-dimensional one where the dimension grows in the number of retailers and the warehouse leadtime. Therefore, it is too complex to compute an optimal policy in a practical setting due to the curse of dimensionality. However, introducing an assumption (called the balance assumption) simplifies the structure of the DP. The balance assumption is a relaxation of a constraint that prohibits the shipment of negative quantities to the retailers. This is equivalent to allowing instantaneous return of stock to the warehouse with no cost. Under the balance assumption, the decomposition result[1] of Clark and Scarf (1960) shown for serial systems also holds for one-warehouse multi-retailer systems. Hence, instead of a multi-dimensional DP, it is sufficient to solve single-dimensional DPs sequentially. Besides a tremendous reduction in the complexity of the problem, the balance assumption leads to important structural results. The optimal policy can be characterized, which turns out to be an echelon base stock policy for ordering (of the warehouse) and myopic allocation policy for shipment decisions. The parameters of this optimal policy can be computed efficiently and the corresponding average cost can be determined. Due to the fact that the balance assumption is a relaxation, the cost calculated under this assumption is a lower bound for the true optimal cost.

The balance assumption is first introduced by Eppen and Schrage (1981) who used the term allocation assumption. It was Federgruen and Zipkin (1984b) who made it explicit in the dynamic programming context and derived the

[1]Clark and Scarf (1960) developed a discounted dynamic program for the inventory control of a multi-echelon serial system in a finite horizon. By introducing the concepts echelon-stock and induced penalty cost, they were able to decompose the resulting multi-dimensional DP into a series of single-dimensional programs, which is known as the decomposition property. See Chapter 1 for a detailed discussion.

optimality results for the relaxed model[2].

The balance assumption is a widely used presupposition in the analysis of one-warehouse multi-retailer systems under periodic review. It is an essential assumption to obtain structural results. Moreover, up to our knowledge, all heuristics developed in the literature are based on the balance assumption. The following list of references utilize the balance assumption in various versions of the standard one-warehouse multi-retailer model discussed in this chapter: Eppen and Schrage (1981), Federgruen and Zipkin (1984a,b,c), Jönsson and Silver (1987), Jackson (1988), Schwarz (1989), Erkip *et al.* (1990), Chen and Zheng (1994b), Kumar *et al.* (1995), Bollapragada *et al.* (1998), Diks and de Kok (1998), Kumar and Jacobson (1998), Cachon and Fisher (2000), Axsäter *et al.* (2002), Özer (2003), Cao and Silver (2005). In the literature, there is an established belief that the balance assumption is a good approximation. In his extensive survey study, Axsäter (2003, p. 544) states:

> *"The balance assumption has been used extensively in the inventory literature and has been shown to produce solutions of very good quality in many different situations, see for example ..."*

There are several studies that investigate the appropriateness of the balance assumption; we refer to §3.2 and the references therein for a detailed literature review. Albeit the fact that the balance assumption is considered to be a good approximation, Axsäter *et al.* (2002) and Chapter 3 of this dissertation cast the first doubt on this widely accepted conviction. In Chapter 3, we explicitly identify the settings leading to moderate or large relative gaps between the upper and lower bounds. In this chapter, we further investigate these settings (with moderate or large relative gaps) by calculating the true optimal cost. Due to the curse of dimensionality, it is unrealistic to consider the scenarios[3] of Chapter 3 as they are, so the demand processes of the current chapter are discrete and distributed over a limited set of points. In this chapter, by calculating the true optimal cost, we are able to study the precise impact of the balance assumption for the first time in the literature.

Our methodology is as follows. First of all, we assume that retailer demand distributions are discrete and have finite supports. For a given problem instance, using the analytical results from Chapter 2, we compute the optimal policy parameters for the relaxed model and determine the average cost, which

[2] We use the terms *relaxed model* and *lower bound model* interchangeably to refer to the model under the balance assumption.

[3] We use the term scenario and problem instance interchangeably to refer to a combination of input parameters (number of retailers, leadtimes, holding and penalty cost parameters, mean and coefficient of variation of demand).

is a lower bound (LB) for the true optimal cost. The myopic allocation (the optimal shipment policy for the relaxed model) may lead to negative shipments. Hence, we modify the myopic allocation policy to make it feasible for the original model. When the optimal base stock policy for ordering in the relaxed model is coupled with the modified allocation policy, one obtains a policy that is feasible for the original model. This policy is a heuristic and gives an upper bound (UB) that may be estimated via simulation. Since this heuristic is based on the relaxed model, we call it as *LB heuristic policy*. The upper and lower bounds envelope the true optimal cost (g^*). If the relative gap between the bounds ($\epsilon\% = 100\frac{UB-LB}{LB}$) is small for a scenario, then we can conclude that the use of the balance assumption is justified for that setting. Federgruen and Zipkin (1984a), Kumar and Jacobson (1998), Axsäter *et al.* (2002) and ourselves in Chapter 3 use $\epsilon\%$ as a performance measure in the numerical experiments; while they use $\epsilon\%$ to test the heuristics they consider, we use it to assess the appropriateness of the balance assumption. This measure does not say much when the relative gap is considerable. For such problem instances, we calculate the optimal average cost by solving the multi-dimensional DP using the value-iteration algorithm. The relative gap between LB and the optimal cost ($\epsilon^*\% = 100\frac{g^*-LB}{LB}$) shows the effect of the balance assumption; if $\epsilon^*\%$ is significant, we conclude that the balance assumption has a considerable impact on the optimal cost. Also, the comparison of the relative gap measures provide additional insights (note that $\epsilon^*\% \leq \epsilon\%$ by definition):

(i) if both $\epsilon^*\%$ and $\epsilon\%$ are small, we conclude that LB is a proxy for the optimal cost, the LB heuristic policy is an appropriate heuristic, and UB approximates the optimal cost well for that particular setting.

(ii) if both $\epsilon^*\%$ and $\epsilon\%$ are significant and close, then we conclude that LB cannot approximate the optimal cost, but the LB heuristic policy is a good heuristic for that particular setting.

(iii) if both $\epsilon^*\%$ and $\epsilon\%$ are significant and considerably different, then we conclude that the balance assumption is not appropriate for that particular setting. In such a case, neither LB is a proxy for the optimal cost nor the LB heuristic policy is an appropriate heuristic.

(iv) if $\epsilon^*\%$ is small whereas $\epsilon\%$ is large, then we conclude that LB is a proxy for the optimal cost, but the LB heuristic policy is a mediocre heuristic for that particular setting.

We treated the identical and nonidentical retailers cases separately. Due to the curse of dimensionality, the number of retailers is restricted to two. A test bed of 81 scenarios is set up for the identical retailers case. The scenarios have gap measures that fall into categories (i), (ii) and (iii) of the four categories listed above. The results support the conclusion of Chapter 3 such that the balance assumption is not a restriction when the retailers are identical and the coefficient of variation is low or moderate (0.5 or 1). However, when the coefficient of variation is high (2), the effect of the balance assumption can be significant; $\epsilon^*\%$ values up to 12.25 have been found. We report problem instances that fall into category (iii) (e.g., one scenario has $\epsilon\% = 20.98$ and $\epsilon^*\% = 12.12$), where the balance assumption is a serious restriction. There are also scenarios for which the LB heuristic policy is a very accurate one. We conclude that when the coefficient of variation is high in the identical retailers case, the relative gap measures comply with categories (ii) and (iii).

For the nonidentical retailers case, we analyzed 37 scenarios. Unlike in the identical retailers case, there are scenarios that fall into category (iv) of the four categories listed above. For example, we report a problem instance with $\epsilon\% = 86.73$ and $\epsilon^*\% = 0.61$. The following settings conform to category (iv):

- when there is negligible added value at the small retailer or at both retailers (keeping other parameters equal),
- when there is size asymmetry between the retailers (keeping other parameters equal).

In case of asymmetric size and coefficient of variations, the relative gap measures comply to category (iii).

The results also provide valuable information about the optimal policy behavior. In a few scenarios, we analyzed the optimal policy numerically. On one hand, the behavior of the optimal policy coincides with the LB heuristic policy structure, but with different base stock levels. On the other hand, in some scenarios, the optimal policy uses detailed state information.

The main contributions of this study are as follows:

- Up to our knowledge, for the first time in the literature, we determine the exact error made in the calculation of the long-run average expected cost of a one-warehouse multi-retailer system by utilizing the balance assumption. Our numerical results show that the error can be significant, which implies that LB may be an inaccurate approximation for the optimal cost. Further, we identified several problem instances for which neither LB nor the LB heuristic policy is acceptable, so the balance assumption is an inappropriate presupposition under such conditions.

- The studies up to now have used $\epsilon\%$ as a measure to evaluate the performance of the heuristics proposed or the effect of the balance assumption. Our results show that $\epsilon\%$ is inadequate to anticipate the behavior of $\epsilon^*\%$. Further, we conclude that the *LB* heuristic policy is not robust because while the performance of this heuristic is comparable to the optimal policy in some scenarios, it is extremely poor in others. Note that this conclusion is in line with the results of Axsäter *et al.* (2002).

- As a direct consequence of the two items above, we have identified a need for good, robust and efficient heuristics for the control of one-warehouse multi-retailer systems.

- We analyze the solutions of the DPs for a few scenarios, which provides valuable insights about the optimal policy behavior. This information can be useful for developing good heuristics.

The rest of the chapter is organized as follows. We introduce the notation and formulate a stochastic dynamic program for the model in §4.2. The analysis of the dynamic program and the approach for quantifying the effect of the balance assumption are discussed in §4.3. §4.4 is dedicated to the explanation of the computational procedure followed and the results of the numerical study. In §4.4.4, the optimal policy behavior is investigated in several scenarios. Finally, the insights obtained from the study are summarized in §4.4.5, which can be read independent of the other parts of the chapter. We conclude and discuss the directions for future research in §4.5. The proofs of the lemmas can be found in §4.6.

4.2 Model

Consider a two-stage distribution system composed of a warehouse serving N retailers. The warehouse *orders* from an exogenous supplier with ample stock and the retailers are supplied by *shipments* from the warehouse. Time is divided into periods of equal length. The periods are numbered as 0,1,2, ... Define

$\mathbb{Z}$ = set of integer numbers. $\mathbb{Z}^- = \{..., -2, -1\}$, $\mathbb{Z}^+ = \{1, 2, ...\}$, and $\mathbb{Z}_0^+ = \mathbb{Z}^+ \cup \{0\}$.

t = index for time. Period t is defined as the time interval between epochs t and $t + 1$ for $t \in \mathbb{Z}_0^+$.

The following parameters describe the system:

N = number of retailers, $N \in \mathbb{Z}^+$, $N \geq 2$.
i = index for stock points, $i = 0$ is the warehouse and $i = 1, 2, ..., N$ are the retailers.
J = set of retailers, i.e., $J = \{1, 2, ..., N\}$.
l_i = leadtime parameter for stock point i, $l_i \in \mathbb{Z}_0^+$ $\forall i \in J$ and $l_0 \in \mathbb{Z}^+$.
α = discount factor, $0 < \alpha \leq 1$.

We refer to Chapter 1 for the definitions of echelon stock and echelon inventory position.

The following events occur in each period t. At the beginning of the period, ordering and shipment decisions are made:

$y(t)$ = size of the order placed by the warehouse at the beginning of period t.
$\mathbf{z}(t)$ = vector of quantities shipped to the retailers at the beginning of period t, $\mathbf{z}(t) = (z_1(t), ..., z_N(t))$ where $z_i(t)$ is the shipment size for retailer i.

Next, also at the beginning of the period, the incoming order and shipments arrive, i.e., $y(t - l_0)$, $z_1(t - l_1)$, ..., $z_N(t - l_N)$ are received at stock points $0, 1, ..., N$, respectively.

During the period, retailers face stochastic, stationary and independent demands of the customers. Demands in different periods are discrete i.i.d. random variables distributed over $\mathbb{Z}_0^+$. Let

$D_i(t, t+s)$ = discrete random variable denoting the demand faced by retailer i during the periods $t, t+1, ..., t+s$ for $t, s \in \mathbb{Z}_0^+$.
$D_0(t, t+s)$ = discrete random variable denoting the aggregate demand faced by the system during the periods $t, t+1, ..., t+s$ for $t, s \in \mathbb{Z}_0^+$, i.e., $D_0(t, t+s) = \sum_{i=1}^{N} D_i(t, t+s)$.
$D_i^{(l)}$ = discrete random variable denoting l-period demand faced by retailer i, $l \in \mathbb{Z}_0^+$.
$D_0^{(l)}$ = discrete random variable denoting l-period aggregate demand faced by the system, $l \in \mathbb{Z}^+$.
$F_i^{(l)}$ = cumulative distribution function of l-period demand of retailer i defined over $\mathbb{Z}_0^+$.
$F_0^{(l)}$ = cumulative distribution function of l-period demand faced by the system defined over $\mathbb{Z}_0^+$, i.e., $F_0^{(l)} = F_1^{(l)} * F_2^{(l)} * ... * F_N^{(l)}$.
μ_i = mean of one-period demand faced by retailer i, $\mu_i > 0$.

μ_0 = mean of one-period demand faced by the system, i.e., $\mu_0 = \sum_{i=1}^{N} \mu_i$.
cv_i = coefficient of variation of one-period demand faced by retailer i.

Notice we assume that $D_i(t,t)$ is distributed over integers for all $i \in J$, i.e., $D_i(t,t) \in \mathbb{Z}_0^+$. Further, we impose the following mild assumptions[4]: (i) there is a finite support for $D_i(t,t)$, $i \in J$, such that $D_i(t,t) \in [0, A_i]$, where A_i is a finite integer, (ii) $\Pr\{D_i(t,t) = 1\} > 0$. As a result, $D_0(t,t) \in \mathbb{Z}_0^+$ and has a finite support $[0, A_0]$ with $A_0 = \sum_{i \in J} A_i$.

At the end of the period, holding and penalty costs are incurred on the period-ending stock and backorder levels. The cost parameters are:

h_i = additional inventory holding cost parameter for stock point i. At the end of a period:
(i) cost h_0 is charged for each unit on stock at the warehouse or in transit to any retailer,
(ii) cost $h_0 + h_i$ is charged for each unit on stock at retailer i.
p_i = penalty cost parameter for retailer i. A cost p_i is charged for each unit of backlog at the end of a period at retailer i.

Define

$I_0(t)$ = echelon stock of the warehouse at the beginning of period t just before the receipt of the incoming order.
$IP_i(t)$ = inventory position of retailer i at the beginning of period t just before the shipment decision, $i \in J$.
$\hat{I}_i(t)$ = echelon stock of stock point i at the end of period t, $i \in J \cup \{0\}$.

The total cost of the system at the end of a period t is equal to

$$h_0 \left(\hat{I}_0(t) - \sum_{i \in J} \hat{I}_i(t) \right) + \sum_{i \in J} (h_0 + h_i) \hat{I}_i^+(t) + \sum_{i \in J} p_i \hat{I}_i^-(t),$$

where $a^+ = \max\{0, a\}$ and $a^- = -\min\{0, a\}$ for $a \in \mathbb{Z}$. (Note that we incur holding cost for the pipeline inventories in between the warehouse and the

[4]The regularity condition on the demand distribution, assumption (ii), is widely used; see Hadley and Whitin (1961), Chen and Zheng (1997), Chen (1998), Cachon and Fisher (2000), Cachon (2001).

retailers.) As carried out in §2.3.1, the expression above can be rewritten as

$$h_0\hat{I}_0(t) + \sum_{i\in J} h_i\hat{I}_i(t) + \sum_{i\in J}(h_0 + h_i + p_i)\hat{I}_i^-(t).$$

Define

$C_0(t)$ = cost attached to the echelon of the warehouse (echelon 0) at the end of period t, $C_0(t) = h_0\hat{I}_0(t)$.

$C_i(t)$ = cost attached to the echelon of retailer i (echelon i) at the end of period t, $C_i(t) = h_i\hat{I}_i(t) + (h_0 + h_i + p_i)\hat{I}_i^-(t)$.

Notice that the total cost of the system at the end of period t is reformulated into echelon holding costs using the concept of *cost attached to an echelon* (cf. Federgruen and Zipkin (1984c), Rosling (1989)).

The decisions of period $t \in \mathbb{Z}_0^+$ directly affect the costs $C_0(t+l_0)$ and $C_i(t+l_i)$ for $i \in J$. Thus, we account the discounted values of the costs $C_0(t + l_0)$ and $\sum_{i\in J} C_i(t + l_i)$ to period t. The expected one-period holding cost for the system-wide inventory at the end of period $t + l_0$ discounted to period t is

$$\begin{aligned}
\mathrm{E}\left[\alpha^{l_0}C_0(t + l_0)\right] &= \mathrm{E}\left[\alpha^{l_0}h_0\left(I_0(t) + \sum_{k=0}^{l_0} y(t-k) - D_0(t, t + l_0)\right)\right] \\
&= \alpha^{l_0}h_0\left(I_0(t) + \sum_{k=0}^{l_0} y(t-k) - (l_0 + 1)\mu_0\right) \\
&= G_0\left(I_0(t) + \sum_{k=0}^{l_0} y(t-k)\right),
\end{aligned}$$

where $G_0(\cdot)$ is defined by

$$G_0(x) = \alpha^{l_0}h_0(x - (l_0 + 1)\mu_0), \; x \in \mathbb{Z}. \tag{4.1}$$

The first order forward difference function of $G_0(x)$ is

$$\Delta G_0(x) \stackrel{\text{def}}{=} G_0(x + 1) - G_0(x) = \alpha^{l_0}h_0. \tag{4.2}$$

Similarly, the expected one-period holding and penalty cost for retailer i at the end of period $t + l_i$ discounted to period t is

$$\begin{aligned}
\mathrm{E}\left[\alpha^{l_i} C_i(t+l_i)\right] &= \mathrm{E}\left[\alpha^{l_i}\left\{h_i[IP_i(t) + z_i(t) - D_i(t, t+l_i)]\right.\right.\\
&\qquad \left.\left. + (h_0 + h_i + p_i)[IP_i(t) + z_i(t) - D_i(t, t+l_i)]^-\right\}\right]\\
&= \alpha^{l_i}\left\{h_i(IP_i(t) + z_i(t) - (l_i+1)\mu_i)\right.\\
&\qquad \left. + (h_0 + h_i + p_i)\mathrm{E}\left[\left(IP_i(t) + z_i(t) - D_i^{(l_i+1)}\right)^-\right]\right\}\\
&= G_i(IP_i(t) + z_i(t)),
\end{aligned}$$

where $G_i(\cdot)$ is defined by

$$\begin{aligned}
G_i(x) &= \alpha^{l_i}\left\{h_i(x - (l_i+1)\mu_i)\right.\\
&\qquad \left. + (h_0 + h_i + p_i)\mathrm{E}\left[\left(x - D_i^{(l_i+1)}\right)^-\right]\right\}, \ x \in \mathbb{Z}. \quad (4.3)
\end{aligned}$$

Observe that $G_i(x)$ is a newsboy type function, which is known to be convex. The first order forward difference function of $G_i(x)$ is $\Delta G_i(x) = G_i(x+1) - G_i(x)$, which can be expressed as

$$\Delta G_i(x) = \begin{cases} \alpha^{l_i}\left(-(h_0 + p_i) + (h_0 + h_i + p_i)F_i^{(l_i+1)}(x)\right) & \text{if} \quad x \geq 0 \\ -\alpha^{l_i}(h_0 + p_i) & \text{o/w.} \end{cases} \quad (4.4)$$

4.2.1 Stochastic Dynamic Programming Formulation

In this subsection, we develop a stochastic dynamic program for the system under consideration. The beginning of each period is a decision epoch, i.e., the decision epochs are $t \in \mathbb{Z}_0^+$. At the beginning of each period t, the following sequence of events takes place. First, the state of the system, $(I_0(t), \mathbf{y}(t), \mathbf{IP}(t))$, is observed, where

$\mathbf{y}(t)$ = vector of outstanding orders at the beginning of period t, $\mathbf{y}(t) = (y(t-l_0), ..., y(t-1))$

$\mathbf{IP}(t)$ = vector of inventory positions of the retailers at the beginning of period t, $\mathbf{IP}(t) = (IP_1(t), ..., IP_N(t))$.

Second, ordering and shipment decisions $(y(t), \mathbf{z}(t))$ are made. Third, incoming order $(y(t-l_0))$, and shipments $(z_i(t-l_i), \forall i \in J)$ arrive following their respective leadtimes. The ordering and shipment decisions (actions) are constrained by the following inequalities:

$$0 \leq y(t), \quad (4.5)$$

$$0 \leq z_i(t) \quad \forall i \in J, \quad (4.6)$$

$$\sum_{i\in J} z_i(t) \leq I_0(t) + y(t - l_0) - \sum_{i\in J} IP_i(t). \quad (4.7)$$

Each state-action pair leads to an expected immediate cost:

$$G_0\left(I_0(t) + \sum_{k=1}^{l_0} y(t-k) + y(t)\right) + \sum_{i\in J} G_i(IP_i(t) + z_i(t)).$$

The realization of demands, $D_i(t,t)$ for all $i \in J$, determine the transitions. The system begins the next period with the following state variables:

$$\begin{aligned}
I_0(t+1) &= I_0(t) + y(t - l_0) - D_0(t,t), \\
\mathbf{y}(t+1) &= (y(t - l_0 + 1), ..., y(t)), \\
\mathbf{IP}(t+1) &= (IP_1(t) + z_1(t) - D_1(t,t), ..., IP_N(t) + z_N(t) - D_N(t,t)).
\end{aligned}$$

4.3 Analysis

The stochastic dynamic program formulated in §4.2.1 is analyzed in this section. Under the discounted cost criterion, some properties of an optimal policy are derived, and these results are used to bound the state and action spaces in §4.3.1.2 where further implications of this result for the average cost criterion are also discussed. We explain the lower and upper bound models in §4.3.2 and §4.3.3, respectively. We close this section with a discussion on how we quantify the effect of the balance assumption in §4.3.4.

4.3.1 Discounted Cost Criterion

In this subsection, we analyze the infinite horizon problem with $0 < \alpha < 1$. The results obtained also have implications on the long-run average cost of the system.

Given a finite initial state $(I_0(0), \mathbf{y}(0), \mathbf{IP}(0))$, following the accounting scheme described in §4.2, the expected total discounted cost of the system in an infinite horizon under some policy f can be formulated as

$$V_f \stackrel{\text{def}}{=} \mathrm{E}\left[\sum_{t=0}^{\infty}\alpha^t\left\{G_0\left(I_0(t)+\sum_{k=0}^{l_0}y(t-k)\right)\right.\right.$$
$$\left.\left.+\sum_{i\in J}G_i\left(IP_i(t)+z_i(t)\right)\right\}\right]. \quad (4.8)$$

Note that the holding costs for the system-wide stock in periods $0, ..., l_0 - 1$, and holding and penalty costs for retailer $i \in J$ in periods $0, ..., l_i - 1$ are the same for any given policy f. Thus, they are not incorporated in (4.8).

We are interested in finding an optimal policy f^*, i.e., a policy f^* among the set of all feasible policies $\mathcal{F}$ such that $V_{f^*} \leq V_f$, for all $f \in \mathcal{F}$.

We restrict the cost parameters as follows:

Assumption 1 .

(i) $h_i > 0 \quad \forall i \in J$,

(ii) $h_0 > 0$,

(iii) $p_i > \frac{1-\alpha^{l_i}}{\alpha^{l_i}} h_0 \quad \forall i \in J$.

It is assumed that all additional holding costs are positive, which implies that there is a positive added value at each stock point in the system. Further, we do not want the system to backlog forever; (iii) serves for this purpose. Suppose policy f does not ship anything to retailer $i \in J$ and at the end of period $t-1$, retailer i has a negative inventory position, which implies that retailer i is in a backlog situation. Construct policy $\widetilde{f}$ that imitates all actions of f except that it orders one unit at the beginning of period t in order to send it to retailer i in period $t + l_0$. Compared to policy f, under policy $\widetilde{f}$, there is one unit more in the inventory position of echelon 0 starting from period t and onwards, and a unit more in the inventory position of retailer i starting from period $t + l_0$ and onwards. Thus, the cost difference between the two policies is

$$\begin{aligned} V_f - V_{\widetilde{f}} &= -\sum_{m=t}^{\infty}\alpha^m \Delta G_0\left(I_0(m)+\sum_{k=0}^{l_0}y(m-k)\right) \\ &\quad - \sum_{m=t+l_0}^{\infty}\alpha^m \Delta G_i(IP_i(m)) \\ &= -\sum_{m=t}^{\infty}\alpha^m(\alpha^{l_0}h_0) + \sum_{m=t+l_0}^{\infty}\alpha^m(\alpha^{l_i}(h_0+p_i)) \end{aligned}$$

$$= \frac{\alpha^{t+l_0}}{1-\alpha}\left(\alpha^{l_i}(h_0+p_i)-h_0\right),$$

where the second equality follows from (4.2) and (4.4). Since we want policy f to be suboptimal, $V_f - V_{\tilde{f}}$ should be positive, which leads to (iii). Note that (ii) and (iii) implies that all penalty costs are positive. Moreover, as $\alpha \uparrow 1$, (iii) simplifies as $p_i > 0 \ \forall i \in J$, which is a standard assumption in inventory theory.

As mentioned before, $G_i(x)$ for $i \in J$ is a convex function, so it is minimized at the first point for which $\Delta G_i(\cdot) \geq 0$. Let $\underline{x}_i$ be the first point such that $\Delta G_i(\underline{x}_i) \geq 0$, i.e., $\underline{x}_i = \min\left\{x | F_i^{(l_i+1)}(x) \geq \frac{h_0+p_i}{h_0+h_i+p_i}\right\}$. If $F_i^{(l_i+1)}(\underline{x}_i) = \frac{h_0+p_i}{h_0+h_i+p_i}$ then there are multiple optima. These are elements of the set $\{\underline{x}_i, \underline{x}_i + 1, ..., \overline{x}_i\}$ where $\overline{x}_i = \min\left\{x | F_i^{(l_i+1)}(x) > \frac{h_0+p_i}{h_0+h_i+p_i}\right\}$. Otherwise, there is a single optimum and $\underline{x}_i = \overline{x}_i$. Note for all $i \in J$, $\underline{x}_i \in Z_0^+$, and $\overline{x}_i$ is finite because $h_i > 0$.

For the sake of brevity, in the rest of this section, we make the following assumption with respect to the initial state.

Assumption 2 .
$(I_0(0), \mathbf{y}(0), \mathbf{IP}(0)) = (0, (0, ..., 0), (0, ..., 0))$.

This assumption on the initial state is used in our derivations in the next subsection, but it is not key for our results, and can easily be relaxed, see Remark 4.1.

4.3.1.1 Properties of an Optimal Policy

As discussed in §4.1, it is very hard to characterize the optimal policy for the system under consideration because it heavily depends on the full state description $(I_0(\cdot), \mathbf{y}(\cdot), \mathbf{IP}(\cdot))$. In this subsection, we give some properties of an optimal policy.

Lemma 4.1 *Under any optimal policy, for $t \in \mathbb{Z}_0^+$ and $i \in J$:*

(i) $z_i(t) = 0$ *if* $IP_i(t) > \overline{x}_i$,

(ii) $z_i(t) \leq \overline{x}_i - IP_i(t)$ *if* $IP_i(t) \leq \overline{x}_i$.

Proof : See §4.6.

Lemma 4.1 simply tells that an optimal policy does not ship anything to retailer i having inventory position greater than or equal to $\overline{x}_i$, the largest point minimizing $G_i(\cdot)$. Further, if there will be a shipment to retailer i, then the inventory position cannot be increased above $\overline{x}_i$. This is a strong result, which is valid for any finite initial state. The next corollary is a direct result of Lemma 4.1 and the initial state given by Assumption 2.

Corollary 4.2 *Under any optimal policy, $IP_i(t) + z_i(t) \leq \overline{x}_i$ for all $t \in \mathbb{Z}_0^+$ and $i \in J$.*

The following lemma bounds the ordering decisions from above.

Lemma 4.3 *Under any optimal policy, for $t \in \mathbb{Z}_0^+$:*

(i) $y(t) = 0$ *if* $I_0(t) + \sum_{k=1}^{l_0} y(t-k) \geq l_0 A_0 + \sum_{i \in J} \overline{x}_i$,

(ii) $y(t) \leq l_0 A_0 + \sum_{i \in J} \overline{x}_i - \left[I_0(t) + \sum_{k=1}^{l_0} y(t-k) \right]$ *if* $I_0(t) + \sum_{k=1}^{l_0} y(t-k) \leq l_0 A_0 + \sum_{i \in J} \overline{x}_i$.

Proof : See §4.6.

The result implies an upper bound $l_0 A_0 + \sum_{i \in J} \overline{x}_i$ for the system-wide inventory position. If a policy, which complies with Lemma 4.1, orders such that this level is exceeded, a portion of the order stays at the warehouse at least for one period for sure and results in an extra holding cost. Similarly, if a policy (conforming to Lemma 4.1) places a positive order when the system-wide inventory position is above this upper bound, then the entire order is certainly kept at the warehouse at least for one period.

We utilize Assumption 2 in the proof of Lemma 4.3, but it is straightforward to extend the result for any initial state as long as $IP_i(0) \leq \overline{x}_i$ for all $i \in J$. Notice that this assumption ($IP_i(0) \leq \overline{x}_i$ for all $i \in J$) is crucial here because when the retailer inventories are highly imbalanced, it might be beneficial to order even though the system-wide inventory position is above $l_0 A_0 + \sum_{i \in J} \overline{x}_i$. For example, let $N = 2$, $IP_1(0) < 0$, $IP_2(0) > l_0 A_0 + \overline{x}_1 + \overline{x}_2 - IP_1(0)$, $I_0(0) - (IP_1(0) + IP_2(0)) = 0$ and $\mathbf{y}(0) = (0, ..., 0)$. There is no on-hand stock at the warehouse and no order in the pipeline. While retailer 1 is in a backlog situation, retailer 2 is overstocking. Although $I_0(t) + \sum_{k=1}^{l_0} y(t-k) > l_0 A_0 + \sum_{i \in J} \overline{x}_i$, it can easily be shown that under an optimal policy $y(0) > 0$.

The next corollary follows from Lemma 4.3 and Assumption 2.

Corollary 4.4 *Under any optimal policy,* $I_0(t) + \sum_{k=0}^{l_0} y(t-k) \leq l_0 A_0 + \sum_{i \in J} \overline{x}_i$ *for all* $t \in \mathbb{Z}_0^+$.

Remark 4.1 *Note that as long as the initial state is finite and* $\mu_i > 0\ \forall i \in J$*, due to the result of Lemma 4.1, an optimal policy eventually leads the system into states where inventory position of each retailer* i *is less than or equal to* $\overline{x}_i$*, and keeps it there. Once this is realized, by Lemma 4.3, under an optimal policy, the system drifts into states where the echelon inventory position of the warehouse does not exceed* $l_0 A_0 + \sum_{i \in J} \overline{x}_i$*. Hence, as* $t \to \infty$*, Corollaries 4.2 and 4.4 hold. In that respect, Assumption 2 is not necessary, but improves the presentation. For a similar reasoning, we refer to the definition of long-run balance and the discussion on the realization of long-run balance in Rosling (1989).*

4.3.1.2 Bounding the State and Action Spaces

In §4.3.1.1, we identified some characteristics of an optimal policy. These properties bound the state variables from above:

$$I_0(t) \leq l_0 A_0 + \sum_{i \in J} \overline{x}_i, \quad IP_j(t) \leq \overline{x}_j \quad \forall j \in J,$$

for all $t \in \mathbb{Z}_0^+$. In a similar fashion, the assumptions on costs (Assumption 1) and the finite supports for demand distributions make us suspect that the state variables are also bounded from below. We make the following assumption:

Assumption 3 .
Under an optimal policy, there exists finite integers $\underline{I}_0$ *and* $\underline{IP}_j$ *for all* $j \in J$ *such that* $\underline{I}_0 \leq I_0(t)$ *and* $\underline{IP}_j \leq IP_j(t)$ *for all* $t \in \mathbb{Z}_0^+$.

Recall that a positive penalty cost forces a single-stage inventory system to order when the inventory position becomes negative (independent of the future demand realizations). An optimal policy for our model is also expected to order and ship to the retailers in order to satisfy customer demand. Although the system may fall into a backlogging situation, an optimal policy would not let the state of the system drift to $-\infty$. (Note that having demands with finite supports is crucial for this reasoning.) In parallel to this intuitive logic, the computational results in §4.4 support the validity of Assumption 3. Moreover, any computational procedure like value iteration require finite state spaces, e.g., see Xu *et al.* (1992, p. 1135).

The compactness of the state space follows immediately from Corollaries 4.2-4.4 and Assumption 3:

$$\begin{array}{rcccl} \underline{I}_0 & \leq & I_0(t) & \leq & l_0 A_0 + \sum_{i \in J} \overline{x}_i, \\ 0 & \leq & y(t) & \leq & l_0 A_0 + \sum_{i \in J} \overline{x}_i - \underline{I}_0, \\ \underline{IP}_j & \leq & IP_j(t) & \leq & \overline{x}_j \qquad \forall j \in J, \end{array} \tag{4.9}$$

for all $t \in \mathbb{Z}_0^+$. For any given state $(I_0(t), \mathbf{y}(t), \mathbf{IP}(t))$ satisfying (4.9), the action space is also compact with

$$\begin{array}{c} 0 \leq z_j(t) \leq \overline{x}_j - IP_j(t) \leq \overline{x}_j - \underline{IP}_j \qquad \forall j \in J, \\ 0 \leq y(t) \leq l_0 A_0 + \sum_{i \in J} \overline{x}_i - \left(I_0(t) + \sum_{k=1}^{l_0} y(t-k) \right) \leq l_0 A_0 + \sum_{i \in J} \overline{x}_i - \underline{I}_0. \end{array} \tag{4.10}$$

Note that all bounds are state independent. Under the assumptions made, any optimal policy has finite and compact discrete state and action spaces. Due to this, immediate costs ($G_0(\cdot)$, $G_i(\cdot)$) associated with each state-action pair are also bounded. Further, we assume stationary costs and transition probabilities. As a result, there exists an optimal deterministic stationary policy. (See Puterman (1994, p. 154).) This finding has another important consequence related to the average cost criterion. Define the expected long-run average cost of the system under policy f and an initial state $(I_0(0), \mathbf{y}(0), \mathbf{IP}(0))$ as

$$\begin{aligned} g_f \stackrel{\text{def}}{=} \lim_{T \to \infty} \frac{1}{T} \mathrm{E} \left[\sum_{t=0}^{T-1} \left\{ G_0 \left(I_0(t) + \sum_{k=0}^{l_0} y(t-k) \right) \right. \right. \\ \left. \left. + \sum_{i \in J} G_i \left(IP_i(t) + z_i(t) \right) \right\} \right], \end{aligned} \tag{4.11}$$

where $G_0(x)$ and $G_i(x)$ are given with $\alpha = 1$ in (4.1) and (4.3), respectively. Since an optimal policy in the discounted case has (i) discrete and finite state and action spaces, (ii) bounded costs, (iii) stationary transition probabilities, there exists $\alpha_0 \in (0, 1)$ and a stationary policy $f^*_{\alpha_0}$ such that $f^*_{\alpha_0}$ is α-discount optimal for $\alpha \in (\alpha_0, 1)$. Further, $f^*_{\alpha_0}$ is average cost optimal. (See Proposition 6.2.3 in Sennott (1999, p. 99).) Therefore, the bounds developed for the state and action spaces in (4.9) and (4.10) are also valid for the average cost criterion when $\alpha = 1$. Moreover, $f^*_{\alpha_0}$ is initial state independent under the average cost criterion. These are important results because we will use the average cost criterion in our computations in §4.4.

4.3.2 Lower Bound Model

Although the stochastic dynamic program under consideration is multi-dimensional with $N + l_0 + 1$ state variables, the relaxation of a constraint in the model leads to a great simplification. When the nonnegativity constraint in (4.6) is omitted, the infinite horizon dynamic program can be investigated using a single cycle analysis as carried out in Chapter 2. As shown there, the optimal policy can be characterized completely. The relaxation of (4.6), equivalent to allowing negative shipments to the retailers, is referred to as the *balance assumption*. Under the balance assumption, the ordering and shipment decisions can be made in isolation of the previous and subsequent ordering and shipment decisions.

Besides the characterization of the optimal policy, the balance assumption leads to an efficient computational procedure for determining the optimal policy parameters and the associated expected system-wide cost. Note that the expected cost calculated under the balance assumption is a lower bound (LB) for the true optimal cost because the resulting model is a relaxation of the original one. We refer to Chapter 2 and §3.3.3 for details of the optimal policy for the lower bound model.

4.3.3 Upper Bound

The long-run expected cost of any feasible policy is an upper bound for the true optimal cost. Other than the difficulty in characterizing the optimal policy, the complexity of the model usually does not permit the determination of the expected cost of a given policy analytically. Instead, this policy can be simulated long enough to give an estimate for the expected long-run cost.

Notice that the optimal policy for the lower bound model described in §4.3.2 is indeed optimal for the original model provided (4.6) is never violated, but this is not always the case in many settings. The optimal policy for the lower bound model can be modified so that it is feasible for the original model. This policy is *not* optimal, but it is a heuristic widely used in the literature. We refer to this policy as LB heuristic policy since it is based on the lower bound model. The LB heuristic policy follows an echelon base stock policy for ordering and myopic allocation policy for shipments; see §3.3.4 for the details of this policy. The simulation of the LB heuristic policy gives an upper bound (UB) for the true optimal cost.

4.3.4 Quantifying the Effect of the Balance Assumption

Our main objective in this study is to demonstrate the precise effect of the balance assumption on the long-run average expected cost by comparing the true optimal cost to the lower bound obtained under the balance assumption. For a given input parameter combination[5] (which we refer to as *scenario* or *problem instance*):

(i) LB can be calculated using the analytical results of Chapter 2,

(ii) the true optimal cost (g^*) can be computed by solving the DP using a numerical technique (e.g., policy iteration, value iteration),

(iii) UB can be determined by simulating the LB heuristic policy.

Define $\epsilon\% = 100\frac{UB-LB}{LB}$ as the relative gap between the lower and upper bound, and $\epsilon^*\% = 100\frac{g^*-LB}{LB}$ as the relative gap between the optimal cost and the lower bound. Recall that we used $\epsilon\%$ to analyze the effect of the balance assumption in Chapter 3. Since g^* lies in between LB and UB, a small $\epsilon\%$ implies that the impact of the balance assumption is insignificant and the LB heuristic policy (resulting in UB) is a good heuristic. In Chapter 3, we could not come to strong conclusions about scenarios with moderate or large $\epsilon\%$ because the exact positioning of g^* with respect to the bounds becomes important. In this study, we consider the settings leading to moderate or large $\epsilon\%$ and calculate $\epsilon^*\%$ for the purpose of (i) quantifying the error made by using LB as an approximation for g^*, and (ii) determining the appropriateness of the LB heuristic policy as a control mechanism. If a scenario with significant $\epsilon\%$ exhibits

- a small $\epsilon^*\%$, then LB is a good proxy for g^*, but the LB heuristic policy is not an appropriate heuristic.
- an $\epsilon^*\%$ close to $\epsilon\%$, then the LB heuristic policy is a good heuristic and UB approximates g^* well.
- an $\epsilon^*\%$ that is not close to zero nor $\epsilon\%$, then neither LB is an accurate proxy for g^* nor the LB heuristic policy is an appropriate heuristic.

[5] Any given combination of values of N, l_0, h_0 and $h_i, p_i, \mu_i, l_i, cv_i$ for $i = 1, ..., N$.

4.4 Computational Results

This section is dedicated to the results obtained from the numerical study conducted. Due to the curse of dimensionality, we restricted ourselves to $N = 2$ and one-period retailer demands distributed over integers in [0,3], see Table 4.1 for the distributions used for various mean and coefficient of variation values. The identical and nonidentical retailers cases are treated separately as done in §3.4, and the numerical results are given in §4.4.2 and §4.4.3, respectively. Next, we discuss the details of the computational procedure.

Table 4.1: The probability distribution functions used in the numerical study for one-period retailer demand.

cv_i	0.5				1		2	
μ_i	0.89		2.11		1.14		0.45	
	x	P_x	x	P_x	x	P_x	x	P_x
	0	0.15	0	0.14	0	0.42	0	0.78
	1	0.82	1	0.11	1	0.20	1	0.07
	2	0.02	2	0.25	2	0.20	2	0.07
	3	0.01	3	0.50	3	0.18	3	0.08

($P_x \overset{\text{def}}{=} \Pr\{D_i(t,t) = x\}$ for $i = 1, 2$.)

4.4.1 Computational Procedure

We used the average cost criterion for the computations. For each scenario, LB was calculated using the analytical results of Chapter 2. We simulated the LB heuristic policy for the purpose of computing UB, and used a prime modulus multiplicative linear congruential random number generator given in Law and Kelton (2000) in the simulations. As in Chapter 3, we used the method of batch means for constructing a point estimate and a confidence interval for the steady-state mean of the expected cost. The batch size is fixed at 10,000 periods and the observations of the first batch were deleted for the warmup procedure. Each problem instance is run for at least 200 batches and terminated as soon as the width of a 95% confidence interval about the average cost function is within 1% of the average cost.

We applied value iteration (successive approximation) algorithm for solving the stochastic dynamic program developed. The accuracy number used for termination is 10^{-4}. The absolute error made with this accuracy number depends on the cost estimated with the value iteration algorithm. In this numerical study, we present the optimal cost (g^*) values with three decimal places, and g^* figures can deviate at most $\pm 10^{-3}$ from the theoretical optimal.

Albeit the fact that Assumption 3 provides lower bounds for some of the state variables, we do not know the values of $\underline{I}_0$, $\underline{IP}_1$ and $\underline{IP}_2$ for an arbitrary scenario. Our approach is to truncate the state space and approximate the value function along the boundaries. First, we assign values for $\underline{I}_0$, $\underline{IP}_1$ and $\underline{IP}_2$. Any transition beyond the boundaries is redirected to a state along the boundary. For example, let $l_0 = 1$, $\underline{I}_0 = -6$, $\underline{IP}_1 = -9$, $\underline{IP}_2 = -9$, and the state of the system (after the actions are taken) be $(I_0(0), \mathbf{y}(0), \mathbf{IP}(0)) = (-6, (0), (-9, 3))$. Note that any positive demand at a retailer drifts the system beyond the boundary. If both retailers experience demand of 2, instead of evolving to $(-10, (0), (-11, 1))$, the transition is routed to $(-6, (0), (-9, 1))$. Manipulating transitions in this way prohibits visiting states with larger expected costs. Hence, the optimal cost value obtained from the value iteration algorithm underestimates the true optimal cost if $\underline{I}_0$ and $\underline{IP}_i$ do not cover the entire class of recurrent states. Next, we run the value iteration algorithm with lower $\underline{I}_0$ and $\underline{IP}_i$ values than before and repeat the computational procedure until the calculated cost does not differ from the previous iteration. Table 4.2 shows the average cost values obtained by applying the computational procedure described above for a scenario with identical retailers. Starting with $(\underline{I}_0, \underline{IP}_1, \underline{IP}_2) = (0, -6, -6)$, the cost figures increase until $(\underline{I}_0, \underline{IP}_1, \underline{IP}_2) = (-6, -12, -12)$ where g^* is achieved. We do not provide the data of each scenario (like in Table 4.2) regarding the convergence, but it was observed in all scenarios considered in the study. This is a numerical evidence supporting Assumption 3.

The convergence of the value iteration algorithm is reported to be problem dependent and the number of iterations typically increases in the number of states (see Tijms (2003, p. 259)). However, our experiences are quite satisfactory. The number of iterations required for convergence with accuracy number 10^{-4} ranges from 7 to 55; most of the scenarios converge in 20 to 40 iterations.

4.4.2 Identical Retailers

We developed a test bed of 81 scenarios with both retailers being identical in terms of cost and leadtime parameters, and demand distributions. In other

Table 4.2: The results of the computational procedure for various values of the lower bounds of the state variables from a scenario with $l_0 = 1$, $l_1 = l_2 = 3$, $h_0 = 0.90$, $h_1 = h_2 = 0.10$, $p_1 = p_2 = 4$ and $cv_1 = cv_2 = 2$.

I_0	IP_1	IP_2	Optimal Cost
0	-6	-6	8.79803391
-1	-7	-7	8.80378125
-2	-8	-8	8.80539633
-3	-9	-9	8.80594393
-4	-10	-10	8.80609908
-5	-11	-11	8.80611030
-6	-12	-12	8.80611031
-7	-13	-13	8.80611031

words, $l_1 = l_2$, $h_1 = h_2$, $p_1 = p_2$, $\mu_1 = \mu_2$ and $cv_1 = cv_2$. All possible combinations (resulting in 81 problem instances) of the following parameters are used:

$$h_i = 0.01, 0.1, 0.5 \qquad p_i = 4, 9, 19 \qquad (l_0, l_i) = (1,0), (1,1), (1,3) \qquad cv_i = 0.5, 1, 2,$$

for $i = 1, 2$. Without loss of generality the holding cost at each retailer is kept at 1 in all scenarios, i.e., $h_0 = 1 - h_i$. The one-period demand distribution with a mean of 0.89 is utilized in scenarios with $cv_i = 0.5$.

We first calculated LB and UB values for each scenario. When 81 scenarios are ranked with respect to $\epsilon\%$, 29 of them have $\epsilon\% > 1.85$. Among these 29 problem instances, 27 and 2 of them have coefficient of variation 2 and 1, respectively. The two scenarios with coefficient of variation 1 have $\epsilon\%$ values 1.87 and 2.13. Hence, we decided to study all problem instances with coefficient of variation equal to 2, which are listed in Table 4.3 where scenarios are numbered on the basis of decreasing $\epsilon\%$. Next, for each scenario in Table 4.3, we determined g^*. The results of the computations are also tabulated in Table 4.3.

In order to see the influence of the warehouse leadtime, we also considered 9 additional scenarios having $cv_i = 2$ for $i = 1, 2$, which are listed with the computational results in Table 4.4. Similar to Table 4.3, the scenarios in Table

4.4 are also numbered on the basis of decreasing $\epsilon\%$. The relative gap measures ($\epsilon^*\%$ and $\epsilon\%$) for scenarios 1- 36 are graphically depicted in Figure 4.1.

Table 4.3: The results for scenarios 1-27 with $cv_i = 2$. (i=1,2)

Sce.	l_0	l_i	h_0	h_i	p_i	LB	g^*	UB	$\epsilon^*\%$	$\epsilon\%$
1	1	0	0.50	0.50	4	3.828	4.127	4.632 ± 0.006	7.81	21.00
2	1	0	0.90	0.10	4	3.828	4.292	4.632 ± 0.006	12.12	21.00
3	1	0	0.99	0.01	4	3.828	4.297	4.632 ± 0.006	12.25	21.00
4	1	1	0.90	0.10	19	8.340	8.783	9.053 ± 0.014	5.31	8.55
5	1	1	0.99	0.01	19	8.421	8.915	9.134 ± 0.015	5.87	8.47
6	1	1	0.90	0.10	9	6.987	7.243	7.440 ± 0.010	3.66	6.48
7	1	0	0.90	0.10	19	5.990	6.181	6.378 ± 0.010	3.19	6.48
8	1	0	0.99	0.01	19	5.990	6.232	6.378 ± 0.010	4.04	6.48
9	1	1	0.99	0.01	9	7.068	7.372	7.521 ± 0.010	4.30	6.41
10	1	0	0.50	0.50	9	5.095	5.288	5.289 ± 0.005	3.79	3.81
11	1	0	0.90	0.10	9	5.095	5.289	5.289 ± 0.005	3.81	3.81
12	1	0	0.99	0.01	9	5.095	5.289	5.289 ± 0.005	3.81	3.81
13	1	1	0.50	0.50	19	7.918	8.188	8.201 ± 0.010	3.41	3.57
14	1	3	0.50	0.50	4	7.485	7.725	7.728 ± 0.010	3.21	3.25
15	1	3	0.50	0.50	9	9.374	9.598	9.677 ± 0.014	2.39	3.23
16	1	1	0.50	0.50	4	5.226	5.380	5.383 ± 0.005	2.95	3.00
17	1	3	0.90	0.10	9	10.454	10.756	10.757 ± 0.015	2.89	2.90
18	1	3	0.90	0.10	4	8.565	8.806	8.808 ± 0.011	2.81	2.84
19	1	3	0.99	0.01	9	10.697	10.999	11.000 ± 0.015	2.82	2.83
20	1	1	0.90	0.10	4	5.586	5.741	5.743 ± 0.005	2.77	2.81
21	1	1	0.99	0.01	4	5.667	5.822	5.824 ± 0.005	2.74	2.77
22	1	3	0.99	0.01	4	8.808	9.049	9.051 ± 0.011	2.74	2.76
23	1	3	0.50	0.50	19	11.006	11.227	11.303 ± 0.019	2.01	2.70
24	1	0	0.50	0.50	19	5.799	5.919	5.946 ± 0.006	2.07	2.53
25	1	3	0.90	0.10	19	12.086	12.381	12.384 ± 0.020	2.44	2.47
26	1	3	0.99	0.01	19	12.329	12.624	12.627 ± 0.020	2.39	2.42
27	1	1	0.50	0.50	9	6.528	6.635	6.649 ± 0.009	1.64	1.85

The findings can be summarized as follows:

1. In the test bed of 81 problem instances with $l_0 = 1$, there are 25 scenarios with $\epsilon\% > 2.5$; all have coefficient of variation equal to 2, see Table 4.3. Further, scenarios 28-36 have $\epsilon\%$ figures greater than 2. This is in line with the finding of Chapter 3 that the main determinant of considerable $\epsilon\%$ is high coefficient of variation of demand when the retailers are identical.

Table 4.4: The results for scenarios 28-36. ($i = 1, 2$)

Sce.	l_0	l_i	h_0	h_i	p_i	cv_i	LB	g^*	UB	$\epsilon^*\%$	$\epsilon\%$
28	2	0	0.90	0.10	4	2	4.232	4.644	4.914 ± 0.006	9.74	16.12
29	2	0	0.99	0.01	4	2	4.232	4.675	4.914 ± 0.006	10.47	16.12
30	2	0	0.90	0.10	9	2	5.535	5.959	6.316 ± 0.011	7.66	14.11
31	2	0	0.99	0.01	9	2	5.535	5.990	6.316 ± 0.011	8.22	14.11
32	2	0	0.50	0.50	9	2	5.415	5.703	5.831 ± 0.007	5.32	7.68
33	2	0	0.50	0.50	4	2	4.195	4.467	4.509 ± 0.005	6.48	7.49
34	2	0	0.90	0.10	19	2	6.616	6.895	6.947 ± 0.009	4.22	5.00
35	2	0	0.99	0.01	19	2	6.695	7.014	7.026 ± 0.009	4.76	4.94
36	2	0	0.50	0.50	19	2	6.190	6.308	6.317 ± 0.006	1.91	2.05

2. In scenarios 1-36, the relative gap between g^* and LB, i.e., $\epsilon^*\%$, is in the range [1.64,12.25] with 11 scenarios having $\epsilon^*\% > 5$, which implies that the error made by using LB as a proxy for the optimal average expected cost can be significant.

3. The performance of the LB heuristic policy is scenario dependent. We calculated $\frac{\epsilon\%}{\epsilon^*\%}$ ratios for all scenarios in Tables 4.3 and 4.4. For scenarios 10-14, 16-22, 25-26, and 35, $\frac{\epsilon\%}{\epsilon^*\%} \leq 1.05$. For these settings, the LB heuristic policy is a good heuristic. However, the performance deteriorates significantly in scenarios 1-9 and 28-33, see Figure 4.1. Unfortunately, the results do not reveal any specific parameter value that may cause good/bad performance.

4. Observe from Tables 4.3 and 4.4, scenarios 1-26 and 28-35 have $\epsilon^*\%$ figures greater than 2. In these scenarios, $\epsilon\% > 2.42$. Further, we do not have any problem instance with significant $\epsilon\%$ and negligible $\epsilon^*\%$. Hence, based on the results in identical retailers case, we conclude that when $\epsilon\%$ is high in a problem instance, the corresponding $\epsilon^*\%$ is also considerable.

5. In order to investigate whether a relationship exists between the input parameters and $\epsilon^*\%$, we set up Figures 4.2 and 4.3. In Figure 4.2 (a), $\epsilon^*\%$ values are scattered against penalty costs for 9 scenarios with $(l_0, l_i) = (1, 0)$. Each serie in the graph corresponds to a specific h_0. Similarly, Figure 4.2 (b), Figure 4.3 (a) and Figure 4.3 (b) depicts $\epsilon^*\%$ values of scenarios with (l_0, l_i) equal to (2,0), (1,1) and (1,3), respectively. A trend is only observed

in Figure 4.2. For a given penalty cost, $\epsilon^*\%$ grows as h_0 increases in Figure 4.2 (a) and (b). Further, in Figure 4.2 (b), for a given h_0 level, $\epsilon^*\%$ shrinks as the penalty cost increases.

4.4.3 Nonidentical Retailers

We developed a test bed consisting of 1280 scenarios for nonidentical retailers case. The following input parameter settings were used:

$$
\begin{aligned}
(h_0, h_1, h_2) &= (0.5, 0.5, 0.5), (0.5, 0.5, 0.01), (0.5, 0.01, 0.5), (0.5, 0.01, 0.01),\\
&\quad (0.9, 0.1, 0.1), (0.9, 0.1, 0.01), (0.9, 0.01, 0.1), (0.9, 0.01, 0.01)\\
(cv_1, cv_2, \mu_1, \mu_2) &= (0.5, 0.5, 0.89, 2.11), (0.5, 0.5, 0.89, 0.89), (0.5, 0.5, 2.11, 2.11),\\
&\quad (0.5, 1, 0.89, 1.14), (0.5, 1, 2.11, 1.14), (0.5, 2, 0.89, 0.45),\\
&\quad (0.5, 2, 2.11, 0.45), (1, 1, 1.14, 1.14), (1, 2, 1.14, 0.45),\\
&\quad (2, 2, 0.45, 0.45)\\
(l_0, l_1, l_2) &= (1, 0, 0), (1, 1, 0), (1, 0, 1), (1, 1, 1)\\
(p_1, p_2) &= (4, 4), (4, 19), (19, 4), (19, 19).
\end{aligned}
$$

Similar to how we set up the test beds in §3.4 and §4.4.2, a full factorial design using the input parameters given above results in a total of 1280 scenarios.

We interpret additional holding costs as parameters that represent the distribution of the added value in a supply chain. For example, the added value is higher at the warehouse compared to the retailers in a scenario with $(h_0, h_1, h_2) = (0.9, 0.1, 0.1)$. Moreover, we take the mean of the demand at a retailer as an indicator of its size. As an example, if $\mu_1 = 2.11$ and $\mu_2 = 0.45$ then the first retailer is almost five times larger than the second one in size.

The results of the numerical study show that many problem instances have high $\epsilon\%$; relative gap value $\epsilon\% = 530$ is found. We decided to focus on 510 of them, which have $\epsilon\% > 5$. Out of these 510 scenarios, while 148 of them have an added value of 0.01 at both retailers (i.e., $(h_0, h_1, h_2) =$ (0.5,0.01,0.01) or (0.9,0.01,0.01)), 261 of them have an added value of 0.01 at only one of the retailers, i.e., $(h_0, h_1, h_2) =$(0.5,0.5,0.01), (0.5,0.01,0.5), (0.9,0.1,0.01) or (0.9,0.01,0.1). The rest of the scenarios (101 out of 510) with $(h_0, h_1, h_2) =$(0.5,0.5,0.5) or (0.9,0.1,0.1) and $\epsilon\% > 5$ are enumerated with respect to coefficient of variation composition and the results are tabulated in Table 4.5. When we analyzed these 101 scenarios, the ones with $(cv_1, cv_2) =$ (0.5,1) have $(h_0, h_1, h_2) =$ (0.9,0.1,0.1), and the ones with $(cv_1, cv_2) =$ (0.5,0.5) have $(\mu_1, \mu_2)=$ (0.89,0.89) or (0.89,2.11). In the light of these observations, we decided to set up a new test bed that consists of 37 scenarios, which are given in Table 4.6.

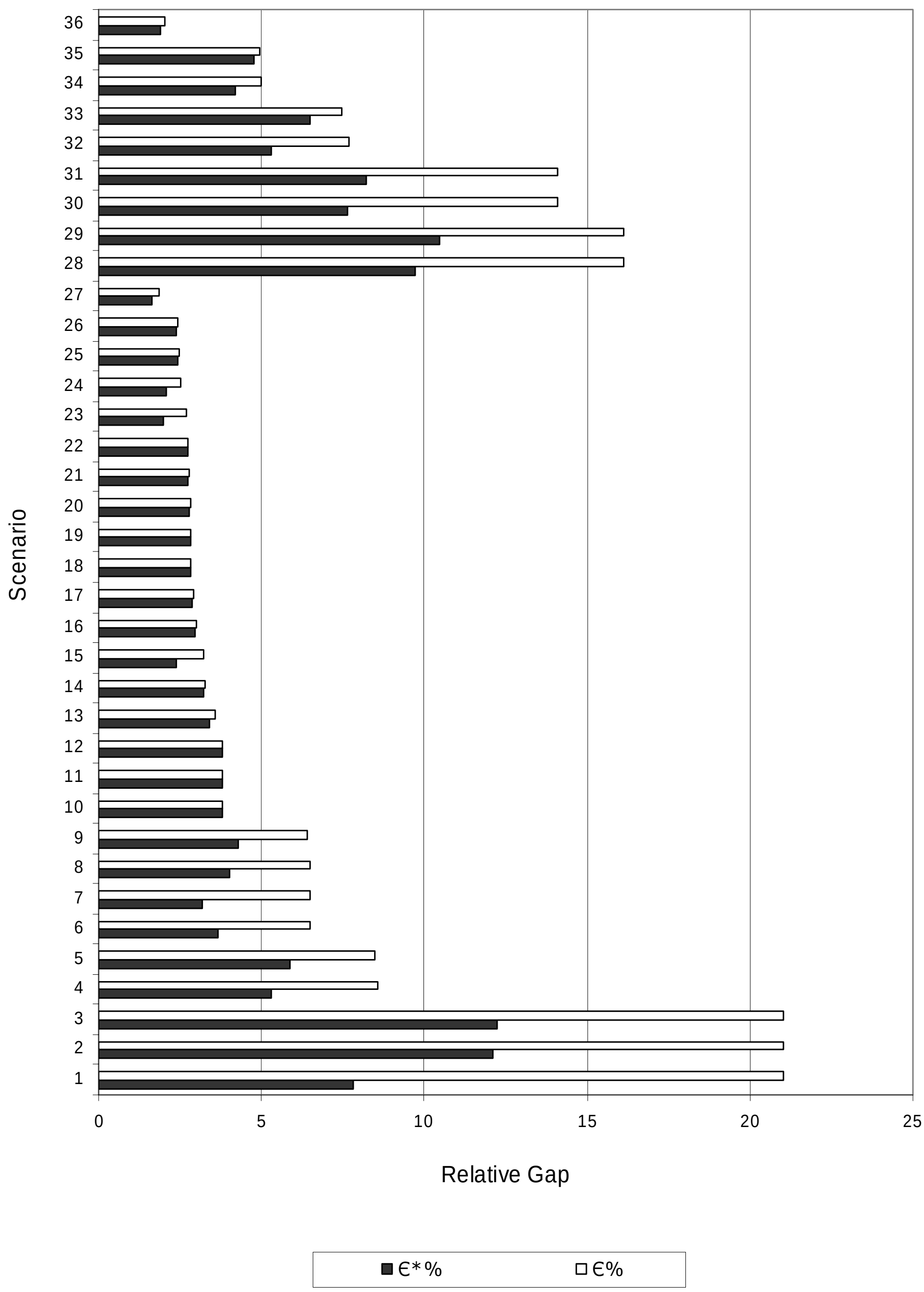

Figure 4.1: Relative gaps for scenarios 1-36.

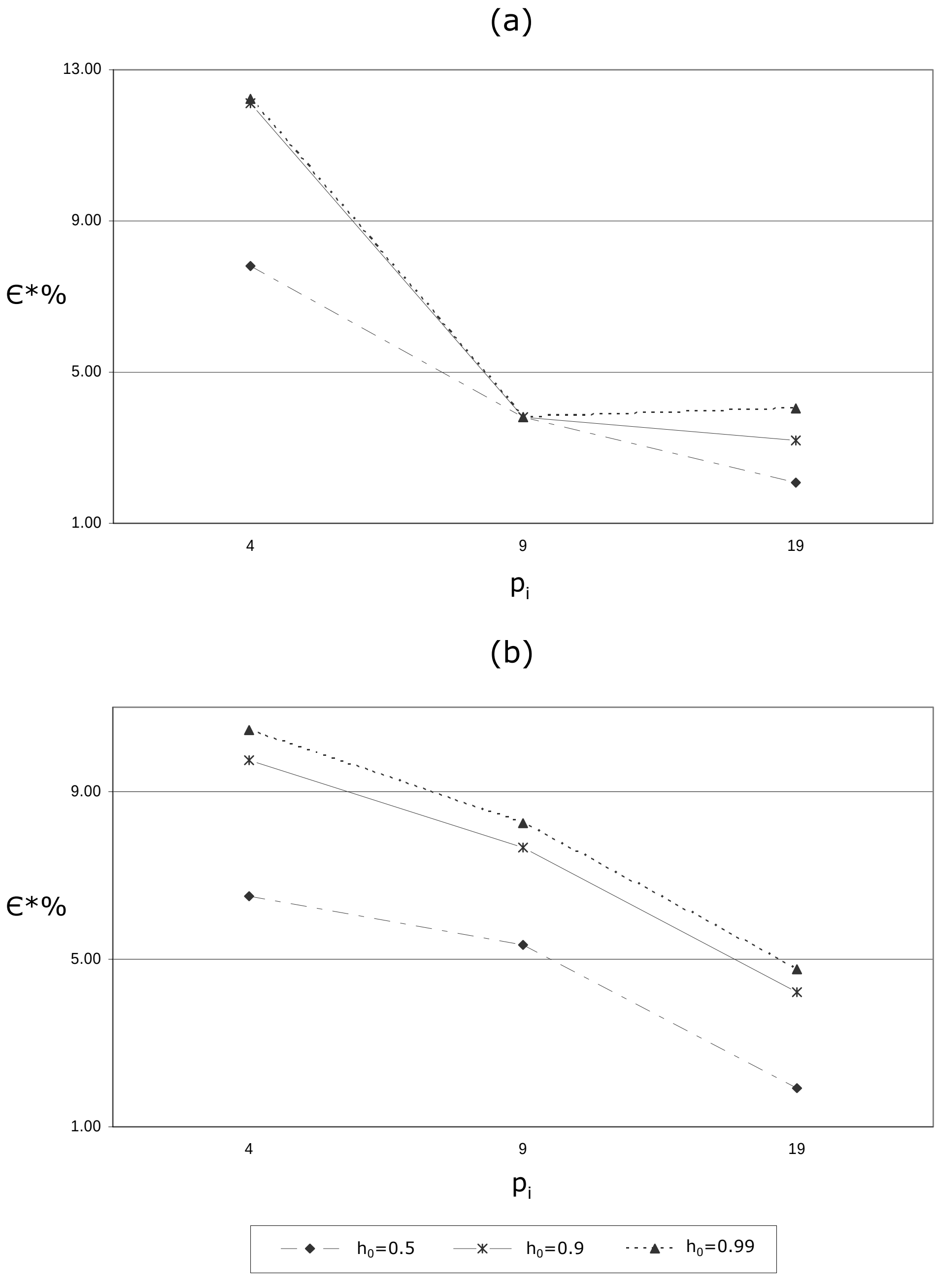

Figure 4.2: Relative gap vs. penalty cost for various h_0 with (a): $(l_0, l_i) = (1, 0)$, (b): $(l_0, l_i) = (2, 0)$ for scenarios 1-36. (i=1,2)

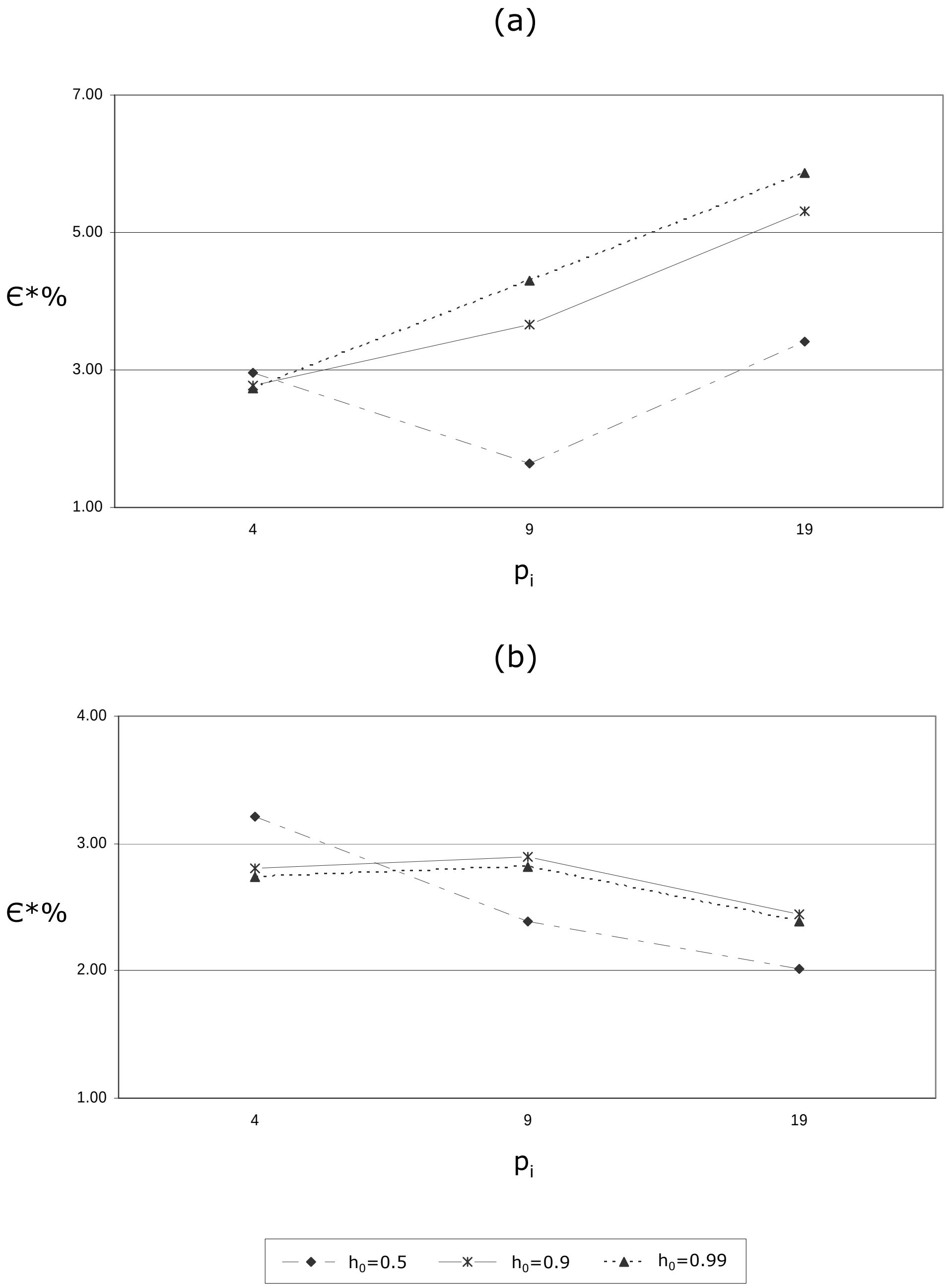

Figure 4.3: Relative gap vs. penalty cost for various h_0 with (a): $(l_0, l_i) = (1, 1)$, (b): $(l_0, l_i) = (1, 3)$ for scenarios 1-36. (i=1,2)

Table 4.5: The composition of 101 scenarios having (h_0, h_1, h_2) =(0.5,0.5,0.5) or (0.9,0.1,0.1) with respect to coefficient of variation.

cv_1, cv_2	(0.5,2)	(0.5,1)	(0.5,0.5)	(1,1)	(1,2)	(2,2)
Number of scenarios	32	7	14	2	20	26

Table 4.6 is organized as follows. There are 7 sets of scenarios, the first six of which target a specific interaction between the input parameters. In set 1, the relationship between the added value (h_1, h_2) and the size of the retailers (μ_1, μ_2) is investigated. While h_0=0.5, negligible added value (0.01) is assigned to one or both of the retailers. Leadtimes, penalty costs and coefficient of variations are set to equal and lowest possible values. Set 2 scenarios are identical to set 1 scenarios except the holding costs; all scenarios in set 2 have $h_0 = 0.9$. Sets 1 and 2 allow us to study the forwarding-to-the-small-retailer phenomenon (see §3.4.2) further. In set 3, we inquire into the effect of size. All parameters except mean demands are kept constant and equal. The difference between set 3 scenarios, and scenarios 40-41 and 45-46 is that the added values at the retailers are not negligible (0.1) and penalty costs are higher (19). The scenarios in set 4 are developed to see the impact of asymmetric penalty costs; in each scenario, the retailers are identical except the penalty costs. Set 5 show us the combined effect of size, coefficient of variation, and penalty cost asymmetry. Although the input parameters in scenarios 54-59 might seem too specific, the relative gaps provide interesting implications. The seven scenarios in set 6 are chosen to see the impact of the coefficient of variation. The common aspect among the problem instances in set 7 is high ϵ%.

The computational results for the scenarios of Table 4.6 are tabulated in Table 4.7. Note that in scenarios 41, 42, 46, 49, 51 and 53, UB is slightly lower than LB, and in scenario 50 UB is slightly lower than g^*, but in all these settings aforementioned LB and g^* figures are within the calculated confidence intervals.

Next, we analyze the results by comparing ϵ% and ϵ^*% figures from different scenarios and draw conclusions.

1. Compare scenario 37 against 38. While the larger retailer has an insignif-

Table 4.6: The parameters of scenarios 37-73.

Set	Sce.	l_0	l_1	l_2	h_0	h_1	h_2	p_1	p_2	cv_1	cv_2	μ_1	μ_2
1	37	1	0	0	0.50	0.50	0.01	4	4	0.5	0.5	0.89	2.11
	38	1	0	0	0.50	0.01	0.50	4	4	0.5	0.5	0.89	2.11
	39	1	0	0	0.50	0.01	0.01	4	4	0.5	0.5	0.89	2.11
	40	1	0	0	0.50	0.01	0.01	4	4	0.5	0.5	0.89	0.89
	41	1	0	0	0.50	0.01	0.01	4	4	0.5	0.5	2.11	2.11
2	42	1	0	0	0.90	0.10	0.01	4	4	0.5	0.5	0.89	2.11
	43	1	0	0	0.90	0.01	0.10	4	4	0.5	0.5	0.89	2.11
	44	1	0	0	0.90	0.01	0.01	4	4	0.5	0.5	0.89	2.11
	45	1	0	0	0.90	0.01	0.01	4	4	0.5	0.5	0.89	0.89
	46	1	0	0	0.90	0.01	0.01	4	4	0.5	0.5	2.11	2.11
3	47	1	0	0	0.90	0.10	0.10	19	19	0.5	0.5	0.89	2.11
	48	1	0	0	0.90	0.10	0.10	19	19	0.5	0.5	0.89	0.89
	49	1	0	0	0.90	0.10	0.10	19	19	0.5	0.5	2.11	2.11
4	50	1	0	0	0.50	0.50	0.50	4	19	0.5	0.5	0.89	0.89
	51	1	0	0	0.50	0.50	0.50	4	19	0.5	0.5	2.11	2.11
	52	1	0	0	0.90	0.10	0.10	4	19	0.5	0.5	0.89	0.89
	53	1	0	0	0.90	0.10	0.10	4	19	0.5	0.5	2.11	2.11
5	54	1	0	0	0.90	0.10	0.10	4	19	0.5	2	2.11	0.45
	55	1	0	0	0.90	0.10	0.10	19	4	0.5	2	2.11	0.45
	56	1	1	1	0.90	0.10	0.10	4	19	0.5	2	2.11	0.45
	57	1	1	1	0.90	0.10	0.10	19	4	0.5	2	2.11	0.45
	58	1	1	1	0.90	0.10	0.10	4	19	0.5	1	2.11	1.14
	59	1	1	1	0.90	0.10	0.10	19	4	0.5	1	2.11	1.14
6	60	1	0	0	0.50	0.50	0.50	4	4	0.5	2	2.11	0.45
	61	1	0	0	0.90	0.10	0.10	4	4	0.5	2	2.11	0.45
	62	1	0	0	0.50	0.50	0.50	4	4	0.5	2	0.89	0.45
	63	1	0	0	0.90	0.10	0.10	4	4	0.5	2	0.89	0.45
	64	1	0	0	0.50	0.50	0.50	4	4	1	2	1.14	0.45
	65	1	0	0	0.90	0.10	0.10	4	4	1	2	1.14	0.45
	66	1	0	0	0.50	0.50	0.50	4	4	2	2	0.45	0.45
	67	1	0	0	0.90	0.10	0.10	4	4	2	2	0.45	0.45
7	68	1	0	0	0.90	0.10	0.10	19	4	1	2	1.14	0.45
	69	1	0	0	0.90	0.01	0.10	19	4	1	2	1.14	0.45
	70	1	0	1	0.50	0.50	0.01	19	4	0.5	2	2.11	0.45
	71	1	0	1	0.50	0.50	0.50	19	4	0.5	2	2.11	0.45
	72	1	0	1	0.90	0.10	0.10	19	4	0.5	2	2.11	0.45
	73	1	0	0	0.50	0.01	0.50	4	19	0.5	0.5	0.89	2.11

Table 4.7: The results of scenarios 37-73.

Set	Sce.	LB	g^*	UB	$\epsilon^*\%$	$\epsilon\%$
1	37	1.320	1.320	1.320 ± 0.002	0.00	0.00
	38	1.597	1.607	1.876 ± 0.002	0.63	17.47
	39	1.166	1.176	1.482 ± 0.002	0.86	27.10
	40	0.818	0.825	0.825 ± 0.003	0.86	0.86
	41	1.798	1.798	1.797 ± 0.002	0.00	-0.06
2	42	2.053	2.053	2.051 ± 0.003	0.00	-0.10
	43	2.058	2.069	2.394 ± 0.003	0.53	16.33
	44	1.979	1.990	2.322 ± 0.003	0.56	17.33
	45	1.047	1.054	1.054 ± 0.003	0.67	0.67
	46	2.933	2.933	2.931 ± 0.003	0.00	-0.07
3	47	2.636	2.664	4.041 ± 0.008	1.06	53.30
	48	2.611	2.643	2.656 ± 0.008	1.23	1.72
	49	3.382	3.382	3.380 ± 0.003	0.00	-0.06
4	50	1.583	1.702	1.701 ± 0.007	7.52	7.45
	51	2.670	2.670	2.668 ± 0.002	0.00	-0.07
	52	1.590	1.711	1.785 ± 0.007	7.61	12.26
	53	3.082	3.082	3.080 ± 0.003	0.00	-0.06
5	54	4.414	4.414	4.416 ± 0.004	0.00	0.05
	55	3.471	3.603	11.194 ± 0.021	3.80	222.50
	56	7.978	8.187	9.428 ± 0.008	2.62	18.17
	57	6.964	7.128	11.217 ± 0.017	2.35	61.07
	58	8.839	8.852	8.890 ± 0.006	0.15	0.58
	59	7.938	7.951	8.703 ± 0.006	0.16	9.64
6	60	3.119	3.204	3.618 ± 0.003	2.73	16.00
	61	3.184	3.452	4.682 ± 0.004	8.42	47.05
	62	2.521	2.616	2.617 ± 0.003	3.77	3.81
	63	2.533	2.663	3.657 ± 0.006	5.13	44.37
	64	3.838	4.019	4.352 ± 0.004	4.72	13.39
	65	3.838	4.134	4.352 ± 0.004	7.71	13.39
	66	3.828	4.127	4.632 ± 0.006	7.81	21.00
	67	3.828	4.292	4.632 ± 0.006	12.12	21.00
7	68	4.366	5.002	8.184 ± 0.015	14.57	87.45
	69	4.205	4.847	8.068 ± 0.015	15.27	91.87
	70	2.898	2.974	15.748 ± 0.033	2.62	443.41
	71	3.884	3.911	6.854 ± 0.010	0.70	76.47
	72	4.175	4.339	12.088 ± 0.021	3.93	189.53
	73	1.635	1.645	3.053 ± 0.005	0.61	86.73

icant added value in the former, it is the reverse in the latter; all other parameters (leadtimes, penalty costs and coefficient of variations) are kept equal and at lowest values (within the parameter combinations of the test bed). Conceding the fact that the impact of this difference is big on $\epsilon\%$ (0.00 vs. 17.47), respective $\epsilon^*\%$ values 0.00 and 0.63 for scenarios 37 and 38 implies that the performance of the *LB* heuristic policy is poor when there is a negligible added value at the small retailer. Now, consider scenario 39, which has insignificant added value at both retailers (the retailer sizes are not equal). Observe that $\epsilon\% = 27.10$ whereas $\epsilon^*\% = 0.86$, which shows the mediocre performance of the *LB* heuristic policy. However, when the size effect is removed, look at scenarios 40 and 41, $\epsilon\%$ drops considerably. Analogues observations can be made for set 2 scenarios. In the light of these, we conclude that the *LB* heuristic policy does not perform well when the retailer sizes are disproportionate, and the added value only at the small retailer or at both retailers are insignificant. Moreover, *LB* approximates g^* accurately under these settings. Recall from Chapter 3 that $\epsilon\%$ grows considerably when there is a negligible added value at the small retailer. We call this phenomenon as forwarding-to-the-small-retailer due to the overstocking at the small retailer under the *LB* heuristic policy. The results for scenarios 38 and 39 show that the error made by forwarding-to-the-small-retailer is substantial, so the performance of the *LB* heuristic policy is poor under such settings.

2. Although we see the effect of retailer size in sets 1 and 2, it is a combined effect with small added value at the retailers. Thus, we decided to study the sole impact of retailer size and considered set 3 and the following scenarios:

l_0	l_1	l_2	h_0	h_1	h_2	p_1	p_2	cv_1	cv_2	μ_1	μ_2
1	0	0	0.50	0.50	0.50	4	4	0.5	0.5	0.89	2.11
1	0	0	0.50	0.50	0.50	4	4	0.5	0.5	0.89	0.89
1	0	0	0.50	0.50	0.50	4	4	0.5	0.5	2.11	2.11
1	0	0	0.50	0.50	0.50	19	19	0.5	0.5	0.89	2.11
1	0	0	0.50	0.50	0.50	19	19	0.5	0.5	0.89	0.89
1	0	0	0.50	0.50	0.50	19	19	0.5	0.5	2.11	2.11
1	0	0	0.90	0.10	0.10	4	4	0.5	0.5	0.89	2.11
1	0	0	0.90	0.10	0.10	4	4	0.5	0.5	0.89	0.89
1	0	0	0.90	0.10	0.10	4	4	0.5	0.5	2.11	2.11

Note that except retailer size, all other input parameters are kept constant. Further, $(h_0, h_1, h_2) = (0.5,0.5,0.5)$ and $(0.9,0.1,0.1)$, which is the main deviation from the scenarios of sets 1 and 2. For the nine scenarios listed above, $\epsilon\% < 0.2$, thus, we concentrate on the problem instances in set 3. Compare $\epsilon\%$ figures in scenarios 47-49. When there is a size asymmetry (scenario 47), $\epsilon\% = 53.3$, but $\epsilon^*\% = 1.06$, which shows g^* is close to LB. Further, $\epsilon\%$ and $\epsilon^*\%$ figures become close when size asymmetry is removed, see scenarios 48 and 49. This observation leads to the conclusion that the LB heuristic policy may perform poorly in the presence of size asymmetry.

3. Consider the following comparisons between the scenarios: 40 vs. 41, 45 vs. 46, and 48 vs. 49. In these problem instances, there is no size asymmetry. The difference between scenarios 40 and 41 is the mean demands of the retailer, similar observation holds for scenarios 45 and 46, and 48 and 49. While $\epsilon^*\% = 0.00$ in scenarios 41, 46 and 49 , $\epsilon^*\% > 0.5$ in scenarios 40, 45 and 48. This indicates the impact of the demand distribution on the quality of LB in approximating g^*.

4. Set 4 is developed to investigate the penalty cost asymmetry. All input variables except the penalty costs are identical in scenarios 50-53. The results for scenarios 51 and 53 suggest how good is LB as a proxy for g^*, but the performance deteriorates substantially in scenarios 50 and 52 where retailer mean demands are 0.89. Similarly, the LB heuristic policy functions well in scenarios 50, 51 and 53, but poorly in scenario 52. However, results point out a joint effect in scenario 52. Compare scenario 51 against 50 and 53 against 52. When mean demand at the retailers drops from 2.11 to 0.89, g^* escalates above 7.5 from 0. This shows the significant effect of demand distribution, which is also pointed out in item 3. These observations lead one to expect that $\epsilon\% = 12.26$ in scenario 52 is a joint effect of the demand distribution and the asymmetry in penalty costs. Nevertheless, the LB heuristic policy may not be an appropriate heuristic in the presence of penalty cost asymmetry.

5. Consider scenarios 54 and 55. The difference between the two settings is that the smaller retailer (retailer 2) has a higher penalty cost (19) in scenario 54 than in scenario 55. The impact on $\epsilon\%$ values is huge; observe the sudden jump from 0.05 to 222.5. Next, consider scenarios 56 and 57, which are identical to 54 and 55, respectively except (l_1, l_2) = (1,1). The performance of the LB heuristic policy is mediocre in both scenarios 56 and 57 ($\epsilon\% = 18.17$ in scenario 56 and $\epsilon\% = 61.07$ in scenario 57). Now compare scenario 56 against 58 and 57 against 59; except the coefficient of variation and the mean demand of the second retailer (cv_2=1, μ_2=1.14 in scenarios 58 and 59), the problem instances are identical. Notice that the $\epsilon\%$ and $\epsilon^*\%$ values are larger in scenarios 56 and 57, which implies that the effect of the balance assumption increases as the retailers become

more unlike. The results lead us to the conclusion that the LB heuristic policy performs inferiorly when the small retailer has a larger coefficient of variation and penalty cost.

6. Notice that in each set 6 scenario, retailers are identical except the coefficient of variations and mean demands. Further, scenarios 66 and 67 are identical to scenarios 1 and 2, respectively. $\epsilon^*\%$ figures in set 6 are scattered against four coefficient of variation and mean demand combinations in Figure 4.4. The scenarios with $h_0 = 0.5$ and $h_0 = 0.9$ are represented in two distinct series. The first retailer's coefficient of variation increases as moved from left to right along the x-axis in Figure 4.4. First of all, observe that $\epsilon^*\%$ values are higher when $h_0 = 0.9$. When the added value at the warehouse increases (with respect to the retailers) both $\epsilon\%$ and $\epsilon^*\%$ increase substantially. Second, when $h_0 = 0.5$, $\epsilon^*\%$ exhibits an escalating trend as the coefficient of variation of the first retailer increases. For $h_0 = 0.9$, a similar trend is observed starting from $(cv_1, cv_2, \mu_1, \mu_2) =$ (0.5,2,0.89,0.45). These observations lead one to expect that the more variable the demand faced by the system, the larger the relative gap between g^* and LB. The performance of the LB heuristic policy is poor in all problem instances of set 6 in general; moreover, the performance deteriorates significantly when h_0 is high, compare scenario 61 to 60 and 63 to 62.

7. Consider the problem instances in set 7. All scenarios have high $\epsilon\%$, but $\epsilon^*\%$ does not exhibit a similar behavior. While $\epsilon^*\%$ figures are high in scenarios 68 and 69, scenarios 71 and 73 have low $\epsilon^*\%$. Unlike in the identical retailers case, there are many problem instances in Table 4.7 with negligible $\epsilon^*\%$. Hence, although $\epsilon^*\%$-$\epsilon\%$ relation is clear in certain settings, we conclude that $\epsilon\%$ is *not sufficient* to understand how $\epsilon^*\%$ behaves in general.

4.4.4 Optimal Policy Behavior

In this subsection, we investigate the optimal policy derived by the value iteration algorithm in a couple of selected scenarios, which are found to give interesting insights. We do not intend to analyze the optimal policy behavior thoroughly; instead, we would like to compare the optimal policy to the LB heuristic policy and identify the similarities and the main differences. We consider scenarios 1, 2, 13, 39 and 55, which fall into the respective categories (iii), (iii), (ii), (iv) and (iv), as defined in §4.1.

When the value iteration terminates, the stationary policy derived is a ε-optimal policy where ε is the accuracy number used; in our study $\varepsilon = 10^{-4}$. The Markov chain induced by this ε-optimal policy is unichain, so composed

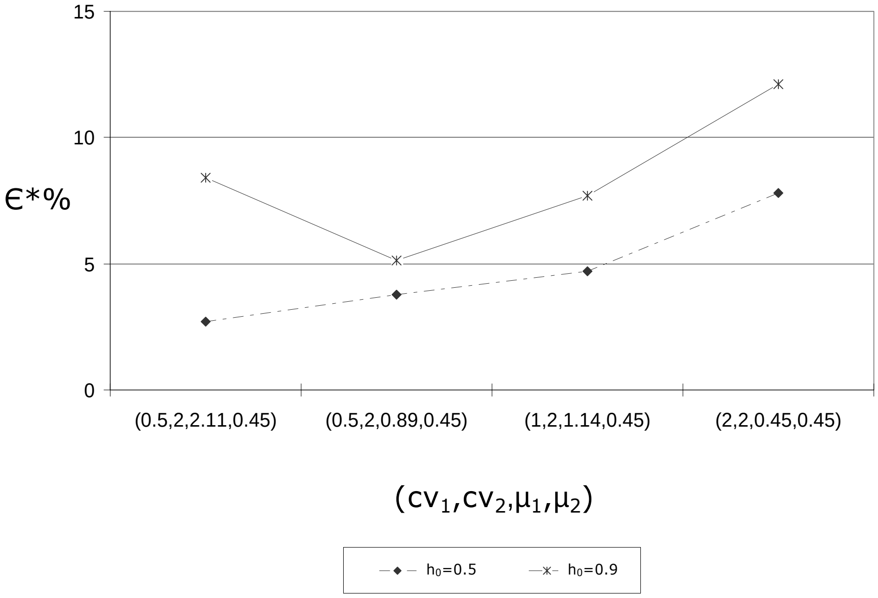

Figure 4.4: Relative gap vs. coefficient of variation-mean demand combination for the scenarios of set 6.

of recurrent and transient states. Let R be the set of recurrent states. In order to analyze the optimal policy behavior in the steady state, we need to determine R and the corresponding optimal actions for each state $s \in \mathrm{R}$. The algorithm developed by Fox and Landi (1968) can be utilized to determine the set of recurrent states, but in our numerical experiments, there are Markov chains with more than 500 thousand states. Since the algorithm by Fox and Landi is based on matrix structure, we decided to follow another route. Our approach is described in the next paragraph.

Let Ω denote the set of all states used in the value iteration. Recall that Ω is finite and compact. Define $a^*(s)$, $s \in \Omega$ as the optimal actions (ordering and shipment decisions) under the ε-optimal policy derived by value iteration. For a scenario, the system is simulated under ε-optimal policy given by the value iteration for 4 million periods. In this run, after the first 2 million periods, the states visited by the system in the next 2 million periods are recorded. The set of states recorded are denoted by R_{sim}. Then, a state $s \in \mathrm{R}_{sim}$ is picked. $a^*(s)$ and all possible demand realizations, $(d_1, ..., d_N)$ with

$\Pr\{D_1(t,t) = d_1, ..., D_N(t,t) = d_N\} > 0$ where $d_i \in \{0, 1, ..., A_i\}$ for $i = 1, 2$, determine the states visited in one transition starting from s. Let $S^1(s)$ be the set of states that are accessible from s in one transition under ε-optimal policy. Next, for each state $i \in S^1(s)$ we determine $S^1(i)$, and continue in this manner until no new state is identified. This recursion provides all states that are accessible from s, which is denoted by the set $\mathcal{A}(s)$. If $s \in \mathrm{R}$, i.e., s is a recurrent state, then all states in $\mathcal{A}(s)$ are also recurrent. Further, if for all $s \in \mathrm{R}_{sim}$, $\mathcal{A}(s) = \mathrm{R}_{sim}$ then we can conclude $\mathrm{R} = \mathrm{R}_{sim}$.

We analyzed scenarios 1, 2, 13, 39 and 55. For each problem instance, first R_{sim} was identified. Then we checked for all $s \in \mathrm{R}_{sim}$, whether $\mathcal{A}(s) = \mathrm{R}_{sim}$ holds. In all scenarios, this was demonstrated. Next, we analyzed the recurrent states, R, and the corresponding optimal actions. In all the scenarios considered $l_0 = 1$, so each state $s \in \mathrm{R}$ can be represented by a tuple (I_0, o_1, IP_1, IP_2) where I_0 is the echelon stock of the warehouse, o_1 is the incoming order, and IP_1 and IP_2 are the inventory positions of retailer 1 and 2, respectively. The optimal actions for s are denoted by another tuple (o, r_1, r_2) where o is the order size, and r_1 and r_2 are the respective shipment quantities for retailer 1 and 2. In every scenario, for every state $s \in \mathrm{R}$, we calculated $I_0 + o_1 - (IP_1 + IP_2 + r_1 + r_2)$ and plotted these quantities against $I_0 + o_1$. In other words, for each scenario, we depicted the amount of stock retained at the warehouse after a shipment decision against the echelon stock of the warehouse just before the shipments. We also plotted the order size, o, against the echelon stock of the warehouse just before the shipments, $I_0 + o_1$. In addition, we scattered the amount of stock retained at the warehouse after the shipment decision and the order size under the *LB* heuristic policy for $I_0 + o_1$ values on these graphs. Next, we discuss each selected scenario in detail.

Scenario 1

In R of scenario 1, echelon stock of the warehouse at the beginning of a period $(I_0 + o_1)$ varies over the integers in [-3,5]. By Lemma 2.1 (iv) and Lemma 2.7 (iv), $\underline{y}_1^* = \overline{y}_1^* = \underline{y}_2^* = \overline{y}_2^* = 2$ and $Y_0^* = \{3\}$. Hence, under the *LB* heuristic policy, no stock is kept at the warehouse and the ordering is carried out such that the echelon inventory position of the warehouse is increased up to 3.

Now, we investigate the optimal ordering behavior by analyzing Figure 4.5 (b). Observe from the figure that the warehouse increases its echelon inventory position up to three distinct levels by ordering. These levels depend on the echelon stock of the warehouse; for $(I_0 + o_1) \in \{-3, -2, -1\}$ level is 3, for $(I_0 + o_1) \in \{1, 2, 3\}$ level is 4, and for $(I_0 + o_1) \in \{4, 5\}$ level is 5. When $I_0 + o_1 = 0$, depending on the inventory positions of the retailers, the level becomes 3 or 4. On one hand, if inventory position of zero can be realized at both retailers after shipments (i.e., $IP_1 + r_1 = IP_2 + r_2 = 0$) then the warehouse orders 3 units. On the other hand, if $(IP_1 + r_1, IP_2 + r_2) =$ (1,-1) or (-1,1), then order size is 4. Note that in case of imbalance between the retailers (the latter case), warehouse orders more. This is an interesting point showing the sensitivity of the optimal policy on individual inventory positions of the retailers.

Unlike the LB heuristic policy, the optimal policy keeps stock at the warehouse depending on $I_0 + o_1$, IP_1 and IP_2, see Figure 4.5 (a). When $I_0 + o_1 = 3$, inventory positions of the retailers determine the decision of retaining a unit or not at the warehouse. At all states corresponding to $(I_0 + o_1, I_0 + o_1 - (IP_1 + IP_2 + r_1 + r_2) =$ (3,0) in Figure 4.5 (a), inventory position of one retailer is 2, i.e., $IP_1 = 2$ or $IP_2 = 2$. However, when it is possible to attain $(IP_1 + r_1, IP_2 + r_2) =$ (1,1) with the shipment decisions (these are the states corresponding to $(I_0 + o_1, I_0 + o_1 - (IP_1 + IP_2 + r_1 + r_2)) =$ (3,1) in Figure 4.5 (a)), then one unit is retained at the warehouse. Further, at states with $I_0 + o_1 = 5$, it is always possible to reach $(IP_1 + r_1, IP_2 + r_2) =$ (2,2), and the optimal policy keeps a unit at the warehouse.

The differences in ordering and shipment decisions detailed above results in a considerable improvement in terms of costs as can be seen from Table 4.3: $\epsilon^*\% = 7.81$ and $\epsilon\% = 21.00$.

<u>Scenario 2</u>

The optimal base stock levels for the relaxed model are $\underline{y}_1^* = \overline{y}_1^* = \underline{y}_2^* = \overline{y}_2^* = 3$ and $Y_0^* = \{3\}$. As a consequence, no stock is hold at the warehouse under the LB heuristic policy. As can be seen from Figure 4.6 (a), under the optimal policy, the warrehouse keeps stock when $I_0 + o_1 = 5$.

The optimal ordering policy structure is somehow similar to the one in scenario 1. When $(I_0 + o_1) \in \{-3, -2, -1, 0\}$, $(I_0 + o_1) \in \{2, 3\}$ and $(I_0 + o_1) \in \{4\}$, the optimal ordering scheme raises the inventory position of the warehouse up to 3, 4 and 5, respectively; see Figure 4.6 (b). If $I_0 + o_1 = 5$ then no order is placed, and if $I_0 + o_1 = 1$ then the order size depends on the individual retailer inventory positions:

- at the states corresponding to $(I_0 + o_1, o) =$ (1,2) in Figure 4.6 (b), $(IP_1 + r_1, IP_2 + r_2) =$ (1,0) or (0,1) after the shipment decisions,
- at the states corresponding to $(I_0 + o_1, o) =$ (1,3) in Figure 4.6 (b), one retailer has an inventory position of 2 just before the shipment, i.e., $IP_1 = 2$ or $IP_2 = 2$. Thus, the other retailer's inventory position can be increased up to -1, and 3 units are ordered.

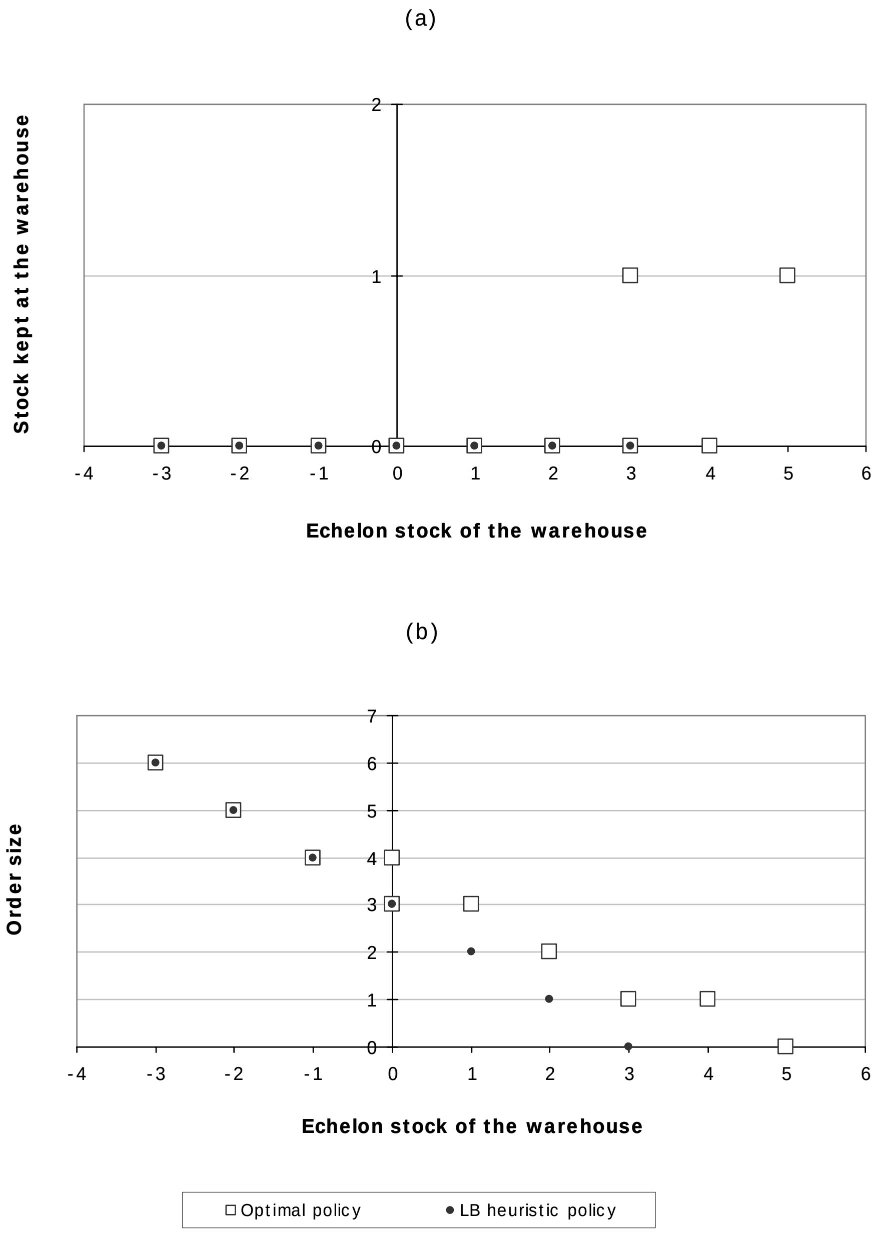

Figure 4.5: Behavior of optimal and LB heuristic policies in scenario 1: (a) Amount of stock retained at the warehouse after the shipment decision vs. echelon stock of the warehouse just before the shipment, (b) Order size vs. echelon stock of the warehouse just before the shipment.

Even though the base stock level of the warehouse for the relaxed model is different in scenarios 1 and 2, the behavior of the LB heuristic policy is identical in these problem instances because the sum of the base stock levels of the retailers is greater than the base stock level of the system. Thus, no stock is kept at the warehouse. Since the installation holding costs of the retailers are equal ($h_0 + h_i = 1$ for $i = 1, 2$) under both settings, LB values are equal and $\epsilon\% = 21.00$ for both scenarios. On the contrary, the optimal policy behavior in scenarios 1 and 2 is different and this has an impact on the long-run expected average cost, which can be observed from the respective $\epsilon\%$ figures 7.81 and 12.12.

<u>Scenario 13</u>

We have chosen this scenario on purpose. Note that $\epsilon\%$ and $\epsilon^*\%$ values are close, see Table 4.3. Since the cost of the LB heuristic policy is close to g^*, the differences between the structure of the LB heuristic policy and the optimal policy can give important insights. The optimal base stock levels for the relaxed model are $\underline{y}_1^* = \overline{y}_1^* = \underline{y}_2^* = \overline{y}_2^* = 4$ and $Y_0^* = \{9\}$. In the recurrent class R, I_0 takes integer values in [-3,9], and $(I_0 + o_1) \in \{3, 4, ..., 9\}$. We start with the analysis of the optimal ordering policy. As can be seen from Figure 4.7 (b), the echelon inventory position of the warehouse is raised to a single level 9 at all recurrent states. This coincides with the ordering behavior of the LB heuristic policy. Hence, we can conclude that under the optimal policy the warehouse follows an echelon base stock policy with a base stock level 9.

At all states with $I_0 + o_1 \geq 6$, both retailers' inventory positions are increased to 3, i.e., $IP_1 + r_1 = IP_2 + r_2 = 3$. Starting from $I_0 + o_1 = 7$, the warehouse retains any stock in excess of 6 units, see Figure 4.7 (a). Moreover, at states with $I_0 + o_1 = 3$, 4 and 5, the inventory positions of the retailers after the shipment decision, $(IP_1 + r_1, IP_2 + r_2)$, are (1,2) or (2,1), (2,2), and (2,3) or (3,2), respectively. These observations indicate that the optimal shipment policy also has a base stock policy structure with base stock levels 3. The main difference between the optimal policy and the LB heuristic policy is the base stock levels for the retailers.

<u>Scenario 39</u>

We have developed scenario 39 to see the joint effect of negligible added value at the retailers and asymmetric size. Respective relative gap figures for $\epsilon\%$ and $\epsilon^*\%$ are 27.10 and 0.86. g^* is close to LB and the performance of the LB heuristic policy is mediocre. For the relaxed model, the optimal base stock levels are $\underline{y}_1^* = \overline{y}_1^* = \underline{y}_2^* = \overline{y}_2^* = 3$ and $Y_0^* = \{8\}$. Next, we analyze the optimal policy and investigate the causes of the large difference between the relative

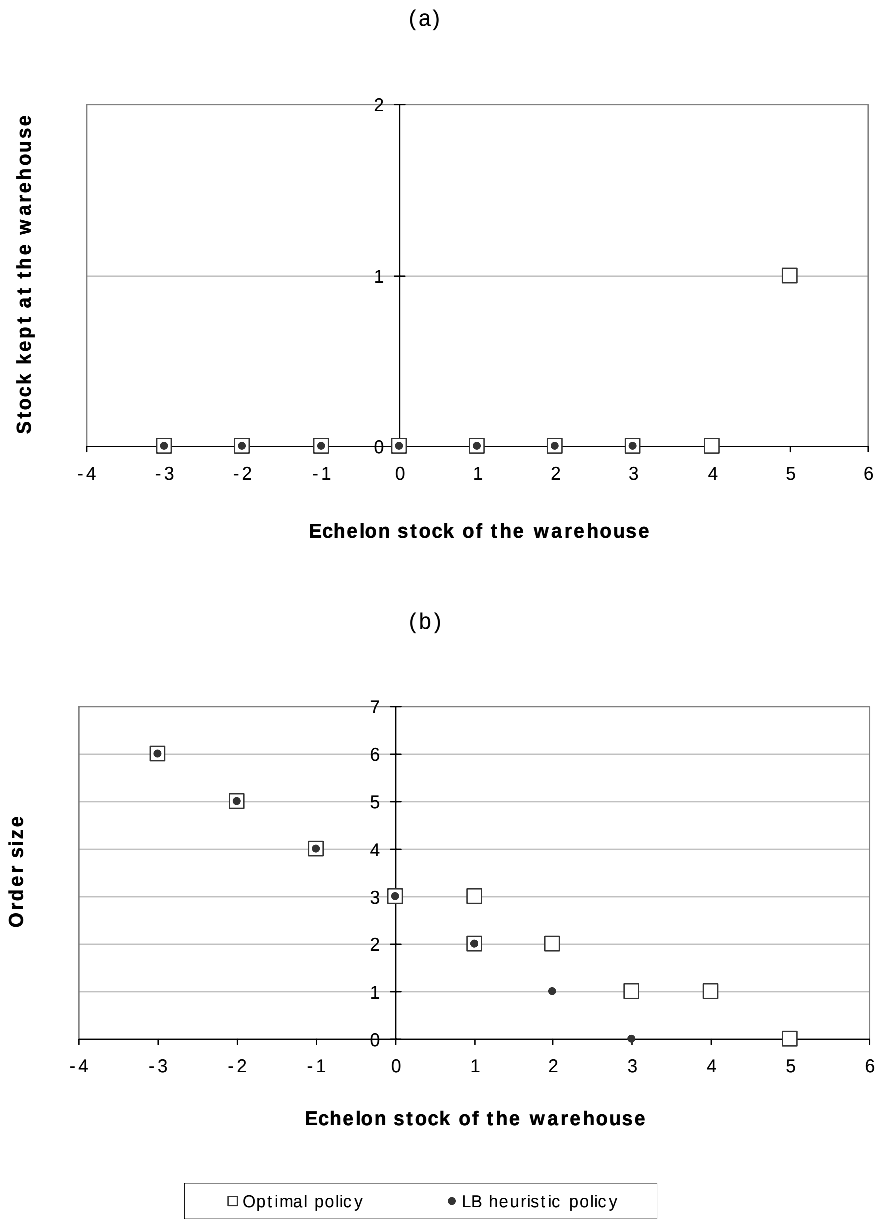

Figure 4.6: Behavior of optimal and LB heuristic policies in scenario 2: (a) Amount of stock retained at the warehouse after the shipment decision vs. echelon stock of the warehouse just before the shipment, (b) Order size vs. echelon stock of the warehouse just before the shipment.

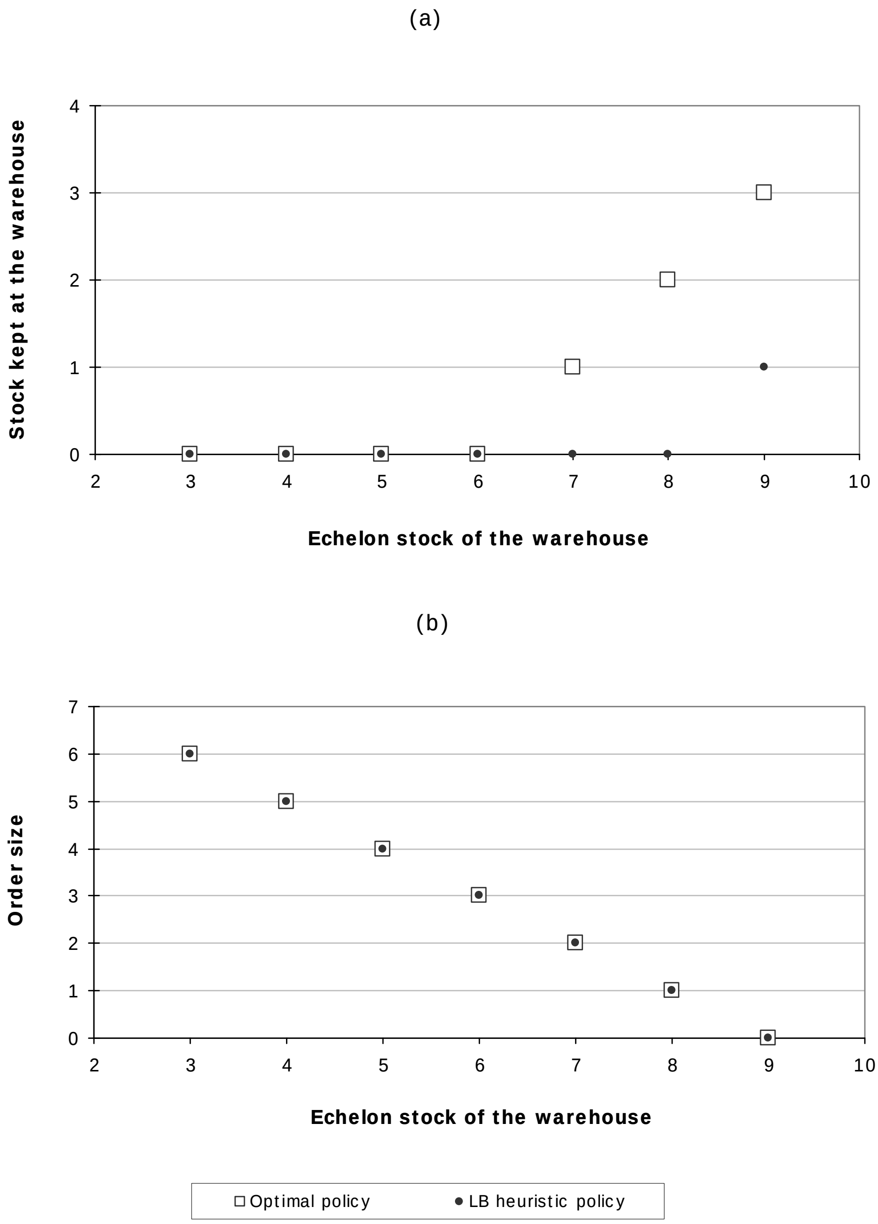

Figure 4.7: Behavior of optimal and LB heuristic policies in scenario 13: (a) Amount of stock retained at the warehouse after the shipment decision vs. echelon stock of the warehouse just before the shipment, (b) Order size vs. echelon stock of the warehouse just before the shipment.

gap measures.

First, the recurrent states have $I_0 \in \{-4, -3, ..., 8\}$ and $(I_0 + o_1) \in \{2, 3, ..., 8\}$. Figure 4.8 (b) suggests that the optimal ordering strategy has a base stock policy structure. When $I_0 + o_1$ is below 8 at a recurrent state, the inventory position of the warehouse is increased up to 8. This ordering scheme coincides with the ordering behavior of the LB heuristic policy in that range.

The optimal shipment policy has an interesting structure. We tabulated $(IP_1 + r_1, IP_2 + r_2)$ figures for each $I_0 + o_1$:

$I_0 + o_1$	$(IP_1 + r_1, IP_2 + r_2)$
2	(1,1)
3	(1,2)
4	(1,3)
5	(2,3)
6	(2,3)
7	(2,3)
8	(2,3)

Note that until the second retailer's inventory position reaches 3, the optimal shipment policy prioritizes the second retailer, which is the larger retailer. This behavior might be related to the probability distributions other than the second retailer being larger in size. As can be seen from Table 4.1, while the probability of having demand of 2 or 3 is 0.03 for the first retailer, it is 0.75 for the second one. Further, unlike the LB heuristic policy, the optimal policy does not increase the first retailer's inventory position above 2. As a result, the warehouse keeps stock when $I_0 + o_1 \geq 6$, see Figure 4.8 (a).

As mentioned before, the ordering structures in the optimal and LB heuristic policies are identical. It is interesting to observe how the shipment policy leads to a significant difference in terms of long-run expected average cost.

Scenario 55

This scenario's relative gap figures are $\epsilon^*\% = 3.8$ and $\epsilon\% = 222.5$, which leads one to expect a considerable difference between the optimal and the LB heuristic policy. The optimal base stock levels are $\underline{y}_1^* = \overline{y}_1^* = \underline{y}_2^* = \overline{y}_2^* = 3$ and $Y_0^* = \{7\}$ for the relaxed model. In R, $I_0 \in \{-5, -4, ..., 8\}$ and $(I_0 + o_1) \in \{1, 2, ..., 8\}$. The optimal ordering policy raises the echelon inventory position of the warehouse to 7 when $I_0 + o_1 = 1$, 2, 3 or 4, and to 8 when $I_0 + o_1 = 5$, 6 or 7. No order is placed when when $(I_0 + o_1) = 8$, see Figure 4.9 (b).

The optimal shipment policy is similar to the one in scenario 39. $(IP_1 + r_1, IP_2 + r_2)$ and $I_0 + o_1$ figures are tabulated for the recurrent states:

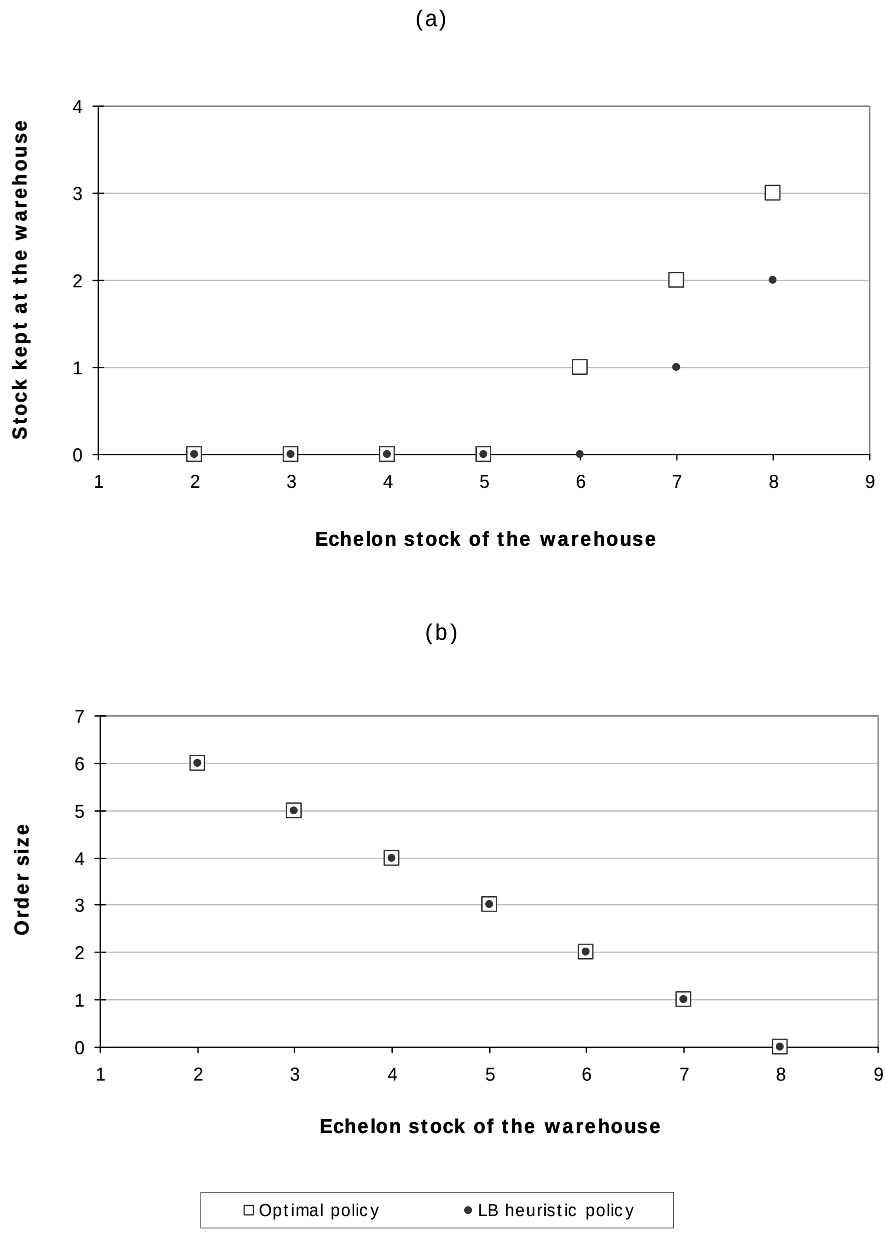

Figure 4.8: Behavior of optimal and LB heuristic policies in scenario 39: (a) Amount of stock retained at the warehouse after the shipment decision vs. echelon stock of the warehouse just before the shipment, (b) Order size vs. echelon stock of the warehouse just before the shipment.

$I_0 + o_1$	$(IP_1 + r_1, IP_2 + r_2)$
1	(-2,3)
2	(-1,3)
3	(0,3)
4	(1,3)
5	(2,3)
6	(2,3)
7	(2,3)
8	(2,3)

Under the optimal policy, the inventory position of the second retailer (larger in size, and having higher penalty cost and less variable demand) just after the shipment is always maintained at 3. For $I_0 + o_1 \geq 6$, stock in excess of 5 units is retained at the warehouse, see Figure 4.9 (a).

4.4.5 Summary

The insights obtained from the results of this study can be summarized as follows:

- As discussed in detail in §4.1, the balance assumption is accepted as a well-established assumption in the analysis of one-warehouse multi-retailers systems. There are many past and recent studies that utilize this presupposition. Our results show that the error in calculating the long-run expected average cost made by introducing the balance assumption can be significant. We report $\epsilon^*\%$ values more than 10, see scenarios 2, 3, 29, 67-69.
- The *LB* heuristic policy, which is built on the optimal policy for the relaxed model, is not a robust heuristic. While it is accurate in some scenarios, it performs poorly in others.
- The results show that $\epsilon\%$ is solely not enough to explain the behavior of $\epsilon^*\%$ in *general*. There are scenarios with
  - (i) low $\epsilon^*\%$ and $\epsilon\%$, which implies that *LB* is an accurate proxy for the optimal long-run expected average cost, and the *LB* heuristic policy is a proper heuristic, e.g., scenarios 37, 41, 42, 51.
  - (ii) considerable and close $\epsilon^*\%$ and $\epsilon\%$, which implies that the *LB* heuristic policy performs well, e.g., scenarios 10, 35, 50, 62.
  - (iii) considerable $\epsilon^*\%$ and much larger $\epsilon\%$, which implies neither *LB* nor the *LB* heuristic policy is appropriate, e.g., scenarios 2, 30, 63-69. The use of the balance assumption leads to substantial errors under such settings.

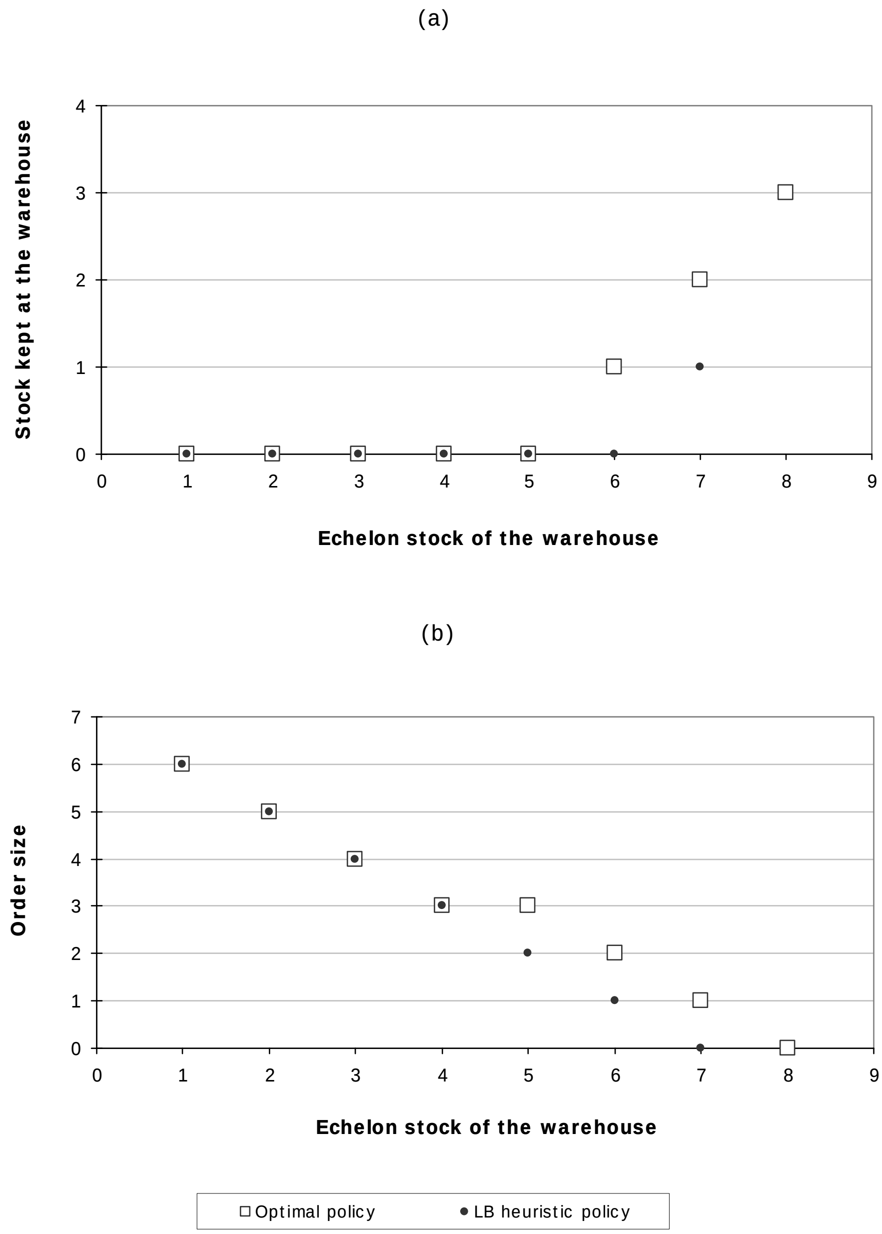

Figure 4.9: Behavior of optimal and *LB* heuristic policies in scenario 55: (a) Amount of stock retained at the warehouse after the shipment decision vs. echelon stock of the warehouse just before the shipment, (b) Order size vs. echelon stock of the warehouse just before the shipment.

(iv) low $\epsilon^*\%$ and significant $\epsilon\%$, which implies that although LB is a proper proxy, the LB heuristic policy is a mediocre heuristic, e.g., scenarios 39, 44, 59, 73.

Albeit the fact $\epsilon\%$ is not sufficient to understand the behavior of $\epsilon^*\%$, we determined several settings that fit into one or more of the items listed above. When retailers are identical, the main determinant of $\epsilon\%$ is the coefficient of variation. The scenarios with identical retailers, and low (0.5) or moderate (1) coefficient of variation exhibit $\epsilon\% < 2.1$. This observation conforms to the findings of Chapter 3. In the case of identical retailers, both $\epsilon\%$ and $\epsilon^*\%$ are significant when there is high coefficient of variation (2). The comparison of the relative gap measures ($\epsilon\%$ and $\epsilon^*\%$) falls into items (ii) and (iii) given above. We report $\epsilon^*\%$ figures as high as 12.25.

Recall forwarding-to-the-small-retailer phenomenon introduced in §3.4.2. Having negligible added value at the small retailer expands $\epsilon\%$ substantially. The results for optimal cost show that the settings with negligible value added at the small retailer or at both retailers (while keeping other parameters equal) fall into item (iv). In these type of settings, putting an upper bound on the order-up-to level of the small retailer can be considered (the optimal policy of scenario 39 supports this idea). Similarly, when there is only a size difference between the retailers, the relative gaps comply with item (iv). The joint effect of size asymmetry and coefficient of variation asymmetry conforms to item (iii).

- There is a need for good heuristics under the settings that fall into items (iii) and (iv). Although Axsäter *et al.* (2002) report that their heuristics perform better than the classical approach (which refers to the LB heuristic policy in our system setting) and show $\epsilon\%$ can be reduced to half in some scenarios, the difference between the cost of their best heuristic and the optimal may still be substantial. For example take scenario 55 with $\epsilon\% = 222.50$ and $\epsilon^*\% = 3.80$. Even if another heuristic leads to a 50% reduction in $\epsilon\%$, having $\epsilon^*\% = 3.80$ suggests that there is still a substantial potential improvement that can be realized.

- The numerical analysis of the optimal policy in some problem instances show that the optimal policy behavior resembles the LB heuristic policy behavior, but with different base stock levels. On the other hand, we also identified more complicated optimal ordering and shipment behavior where the full state description of the system is utilized.

4.5 Conclusion and Further Research

In this study, we investigate the effect of the balance assumption on the long-run average expected cost in one-warehouse multi-retailer inventory/production systems. The balance assumption is a widely used presupposition in the analysis of the aforementioned systems under periodic review. Although there are a few studies suggesting that this assumption *might* not be appropriate for some system parameter settings, the general belief in the multi-echelon inventory literature is that the assumption is well-founded. The balance assumption leads to a relaxation of the original model, which is a multi-dimensional stochastic dynamic program, and analytical results are available for the resulting (relaxed) model. As a consequence, the cost of the relaxed model is a lower bound for the optimal cost of the system. The optimal policy for the relaxed model can be modified to give a feasible policy for the original model, and the simulation of this modified policy leads to an upper bound for the optimal cost of the system. We conducted a numerical study where the upper and lower bounds, and the optimal cost are calculated for various scenarios. For the first time in the literature, our results show that the error made by utilizing the balance assumption can be substantial. There are numerous problem instances having considerable $\epsilon^*\%$ and substantially higher $\epsilon\%$, which suggests that neither the lower bound is a proxy for the optimal cost nor the LB heuristic policy is an accurate heuristic. Further, $\epsilon\%$, which is used in the literature to assess the performance of heuristics and quantify the effect of the balance assumption, can be a misleading measure because we found problem instances with high $\epsilon\%$ and very low $\epsilon^*\%$, or very close $\epsilon\%$ and $\epsilon^*\%$ figures. These results suggest a need for good, robust and efficient heuristics for the control of one-warehouse multi-retailer inventory systems. The numerical analysis of the optimal policy for some scenarios show that the optimal policy behavior is similar to the behavior of the LB heuristic policy. Keeping the ordering and shipment behavior of the LB heuristic policy, but determining the base stock levels differently might be a potential candidate for a good heuristic. As a future research, we plan to develop new heuristics. The performance of these heuristics can be tested by comparing them against the optimal cost and the costs of the available heuristics in the literature.

4.6 Appendix: Proofs

Proof of Lemma 4.1:

(i) Take an arbitrary policy f. For some $t \in \mathbb{Z}_0^+$ and $i \in J$, let $IP_i(t) = \overline{x}_i + b$ and $z_i(t) = a$, $a \in \mathbb{Z}^+$ and $b \in \mathbb{Z}_0^+$. Consider policy $\widetilde{f}$ (all variables of this policy are represented with $^\sim$) that imitates all actions of f except that $\widetilde{z}_i(t) = 0$ and $\widetilde{z}_i(t+1) = z_i(t+1) + a$. In other words, policy $\widetilde{f}$ postpones the shipment of a units to retailer i in period t by one period. Verify that $IP_i(s) + z_i(s) = \widetilde{IP}_i(s) + \widetilde{z}_i(s)$ for $s \leq t-1$, $IP_i(t) + z_i(t) = \widetilde{IP}_i(t) + \widetilde{z}_i(t) + a$, and $IP_i(s) + z_i(s) = \widetilde{IP}_i(s) + \widetilde{z}_i(s)$ for $s \geq t+1$. Thus, the expected total discounted costs of both policies differ *only* by the expected costs attached to retailer i for period t; as a result,

$$V_f - V_{\widetilde{f}} = \alpha^t \left\{ G_i(\overline{x}_i + a + b) - G_i(\overline{x}_i + b) \right\} > 0,$$

which follows from the convexity of $G_i(\cdot)$ and the definition of $\overline{x}_i$. Hence, policy f is suboptimal.

(ii) Take an arbitrary policy f. For some $t \in \mathbb{Z}_0^+$ and $i \in J$, let $IP_i(t) = \overline{x}_i - b$ and $z_i(t) = a$ such that $a, b \in \mathbb{Z}^+$ and $a > b$. Consider policy $\widetilde{f}$ (all variables of this policy are represented with $^\sim$) that imitates all actions of f except that $\widetilde{z}_i(t) = b$ and $\widetilde{z}_i(t+1) = z_i(t+1) + a - b$. In other words, policy $\widetilde{f}$ postpones the shipment of $a - b$ units to retailer i in period t by one period. Verify that $IP_i(s) + z_i(s) = \widetilde{IP}_i(s) + \widetilde{z}_i(s)$ for $s \leq t-1$, $IP_i(t) + z_i(t) = \widetilde{IP}_i(t) + \widetilde{z}_i(t) + a - b$, and $IP_i(s) + z_i(s) = \widetilde{IP}_i(s) + \widetilde{z}_i(s)$ for $s \geq t+1$. Therefore,

$$V_f - V_{\widetilde{f}} = \alpha^t \left\{ G_i(\overline{x}_i + a - b) - G_i(\overline{x}_i) \right\} > 0,$$

which shows that policy f is suboptimal. $\square$

Proof of Lemma 4.3:

(i) Proof is by contradiction. Take an optimal policy f. For some $t \in \mathbb{Z}_0^+$, let $y(t) = a$, $a \in \mathbb{Z}^+$ while $I_0(t) + \sum_{k=1}^{l_0} y(t-k) \geq l_0 A_0 + \sum_{i \in J} \overline{x}_i$. Due to Corollary 4.2, $IP_i(s) \leq \overline{x}_i$ for all $s \in \mathbb{Z}_0^+$. Observe that $I_0(t + l_0) = I_0(t) + \sum_{k=0}^{l_0} y(t-k) - D_0(t, t+l_0-1) \geq \sum_{i \in J} \overline{x}_i + a$ because $I_0(t) + \sum_{k=0}^{l_0} y(t-k) \geq l_0 A_0 + \sum_{i \in J} \overline{x}_i + a$ and the maximum realization of $D_0(t, t+l_0-1)$ is $l_0 A_0$. Thus, by Lemma 4.3, $y(t)$ as a whole stays at the warehouse in period $t + l_0$. Consider policy $\widetilde{f}$ (all variables of this policy are represented with $^\sim$) that imitates all actions of f except that $\widetilde{y}(t) = 0$ and $\widetilde{y}(t+1) = y(t+1) + a$. In other words, policy $\widetilde{f}$ postpones the ordering of a units in period t by one period. Since no portion of $y(t)$ is shipped in period $t + l_0$, any shipment under f can be replicated by $\widetilde{f}$. The expected total discounted costs of both policies differ *only* by the expected system-wide holding cost in period t. Therefore,

$$\begin{aligned} V_f - V_{\widetilde{f}} &= \alpha^t \left\{ G_0 \left(I_0(t) + \sum_{k=1}^{l_0} y(t-k) + a \right) \right. \\ &\qquad \left. - G_0 \left(\widetilde{I}_0(t) + \sum_{k=1}^{l_0} \widetilde{y}(t-k) \right) \right\} \\ &= \alpha^{t+l_0} h_0 a \\ &> 0, \end{aligned}$$

which contradicts the optimality of f.

(ii) Proof is by contradiction. Take an optimal policy f. For some $t \in \mathbb{Z}_0^+$, let $I_0(t) + \sum_{k=1}^{l_0} y(t-k) = l_0 A_0 + \sum_{i \in J} \overline{x}_i - b$ and $y(t) = a$ such that $a, b \in \mathbb{Z}^+$ and $a > b$. By Corollary 4.2, $IP_i(s) \leq \overline{x}_i$ for all $s \in \mathbb{Z}_0^+$. Note that $I_0(t+l_0) + y(t) = I_0(t) + \sum_{k=0}^{l_0} y(t-k) - D_0(t, t+l_0-1) \geq \sum_{i \in J} \overline{x}_i + (a-b)$ because $I_0(t) + \sum_{k=0}^{l_0} y(t-k) = l_0 A_0 + \sum_{i \in J} \overline{x}_i + (a-b)$ and the maximum realization of $D_0(t, t+l_0-1)$ is $l_0 A_0$. Thus, by Lemma 4.3, *at least* $a - b$ units of $y(t)$ are kept at the warehouse in period $t + l_0$. Consider policy $\widetilde{f}$ (all variables of this policy are represented with $\sim$) that imitates all actions of f except that $\widetilde{y}(t) = y(t) - (a-b) = b$ and $\widetilde{y}(t+1) = y(t+1) + (a-b)$. In other words, policy $\widetilde{f}$ postpones the ordering of $a - b$ units in period t by one period. Since $a - b$ units of $y(t)$ is not shipped in period $t + l_0$, any shipment under f can be replicated by $\widetilde{f}$.

The expected total discounted costs of both policies differ *only* by the expected system-wide holding cost in period t. Therefore,

$$\begin{aligned} V_f - V_{\widetilde{f}} &= \alpha^t \left\{ G_0 \left(l_0 A_0 + \sum_{i \in J} \overline{x}_i + (a-b) \right) - G_0 \left(l_0 A_0 + \sum_{i \in J} \overline{x}_i \right) \right\} \\ &= \alpha^{t+l_0} h_0 (a-b) \\ &> 0, \end{aligned}$$

which contradicts the optimality of f. □

Chapter 5

Newsboy Characterizations for Serial Inventory Systems

Abstract: *This chapter considers an N-stage serial production/inventory system where materials flow from one stage to another in fixed batches. Linear holding and penalty costs (for backorders) are assumed. By Chen (2000), echelon stock (R, Q) policies are optimal for such systems. Based on the results of Chen (1998), we show that the optimal reorder levels satisfy newsboy inequalities (equalities) when the demand has a discrete (continuous) distribution. The newsboy inequalities/equalities show a direct relation between the probability of no-stockout at the most downstream point and the cost parameters. Thus, they contribute to the understanding of optimal control. Also, they are easy to explain to managers and non-mathematical oriented students.*

5.1 Introduction

This chapter considers an N-stage serial inventory/production system facing stochastic demand of the customers at the most downstream stage (stage 1). The stages are numbered such that stage 1 orders from stage 2, 2 from 3, ..., and stage N from an external supplier with ample stock. The order size at each stage is required to be a nonnegative integer multiple of a base quantity specific for that stage. Further, there is an integer-ratio constraint implying that the base quantity at some stage should be a positive integer multiple of the base quantity of the immediate successor stage. The leadtimes between the stages are constant. Any unfulfilled demand is backlogged and a penalty cost is incurred. We assume centralized control and periodic review of the inventories. The objective is to minimize the average expected holding and penalty costs of the system in an infinite horizon.

For the system under study, Chen (2000) has shown that an optimal ordering policy for each stage is to follow an echelon stock (R, Q) policy: whenever the echelon inventory position at stage i is at or below the reorder level R_i, a minimum integer multiple of its base quantity (Q_i) that brings the echelon inventory position above R_i is ordered from stage $i + 1$. By Chen (1998), the optimal reorder points are calculated by solving N single-stage (R, Q) models sequentially. In other words, N single dimensional average cost functions are minimized successively. This study focuses on the optimal reorder points. We heavily draw on the results of Chen (1998). Introducing a new representation based on the concept of *shortfall*, we are able to derive alternative expressions for cost functions, which lead to newsboy characterizations (newsboy inequalities /equalities) for the optimal reorder levels. Newsboy inequalities/equalities are expressions that show a direct relation between the probability of no-stockout at stage 1 (as a result of a given reorder level at some stage i) and the cost parameters.

Our contribution in this study is as follows. First, we develop a new cost formulation for the long-run average expected cost of the system based on the shortfall concept. Second, we show that the optimal reorder levels in an N-stage serial inventory/production system with fixed batch sizes satisfy newsboy inequalities (equalities) when the demand distribution is discrete (continuous). Newsboy characterizations are appealing because: (i) they provide new insights and contribute to the understanding of optimal control, (ii) they are relatively easy to explain to non-mathematical oriented students (e.g., MBA students) and managers when compared to recursive expressions that one finds in the literature.

Under continuous demand, newsboy equalities have been derived for multi-echelon serial systems without batching (Clark and Scarf (1960) model), serial systems with fixed replenishment intervals, and divergent systems under the balance assumption by van Houtum and Zijm (1991), van Houtum *et al.* (2003), and Diks and de Kok (1998), respectively. In Chapter 2, we extended the results of Diks and de Kok (1998) to the discrete demand case where newsboy inequalities instead of newsboy equalities are obtained. This study generalizes the newsboy characterizations of van Houtum and Zijm (1991) in two directions; our model incorporates fixed batch sizes and can handle discrete demand distributions.

In a recent paper, Shang and Song (2005) consider the same model that we study in this chapter. They construct upper and lower bound functions for the cost functions that have to be minimized sequentially in order to calculate optimal reorder points. The upper and lower bound functions are single-stage newsboy functions with modified holding and penalty costs. The minimization of these functions leads to lower and upper bounds for the optimal reorder levels, which follow from newsboy characterizations. In contrast, we derive newsboy characterizations for the optimal reorder levels themselves. By our newsboy characterizations, one obtains an alternative proof for the property that the variables r_i^l as defined by Shang and Song (2005) are lower bounds for the optimal reorder levels; see Remark 5.3.

The rest of the chapter is organized as follows. We introduce the model in §5.2. The results from Chen and Zheng (1994a), and Chen (1998) that we use in our analysis are presented in §5.3.1. Newsboy characterizations are derived in §5.3.2. We dedicate §5.3.3 for the discussion of our results when the demand process is continuous, and conclude in §5.4.

5.2 Model

Consider a serial N-stage inventory/production system under periodic review. The most downstream stage, stage 1, orders from stage 2, 2 from 3, ..., $N-1$ from N, and stage N from an exogenous supplier (called stage N+1) with ample stock. Any order of stage i is an integer multiple of a base order quantity Q_i. Further, we assume that $Q_{i+1} = n_i Q_i$ for $i = 1, ..., N-1$ where n_i is a positive integer (*integer-ratio assumption*). Because of the integer-ratio assumption, we presuppose that the initial on-hand stock at stage i is an integer multiple of Q_{i-1} for $i = 2, ..., N$. Stage 1 faces the stochastic demand of the customers. Demands in different periods are i.i.d., discrete, nonnegative random variables. Any unfulfilled customer demand at stage 1 is backlogged and a penalty cost is incurred. There are deterministic leadtimes between the stages, and between the supplier and stage N. Holding cost at every stage, and penalty cost at stage 1 are linear. We assume centralized control and the objective is to minimize the average expected holding and penalty costs of the system in the long-run.

The following sequence of events takes place during a period: (i) inventory levels at all stages are observed and the current period's ordering decisions are made (at the beginning of the period), (ii) orders arrive following their respective leadtimes (at the beginning of the period), (iii) demand occurs during the period, (iv) holding and penalty costs are assessed on the period ending inventory and backorder levels (at the end of the period).

We follow the same notation and assumptions as Chen (1998), except that we consider a periodic review setting with i.i.d. demand. For details, we refer to §2 and §3 of Chen (1998). As indicated by Chen, his analysis and the results also hold for periodic review models with i.i.d. demand. However, we need to modify the definitions of some variables and introduce new notation. In addition to the notation introduced above, we define:

$\mathbb{Z}$	=	set of integers; $\mathbb{Z}^+$ is the set of positive integers, and $\mathbb{Z}_0^+ = \mathbb{Z}^+ \cup \{0\}$.
$\mathbb{R}$	=	set of real numbers.
t	=	index for periods, $t \in \mathbb{Z}^+$.
L_i	=	leadtime from stage $i+1$ to stage i, $L_1 \in \mathbb{Z}_0^+$ and $L_i \in \mathbb{Z}^+$ for $i = 2, ..., N$.
l_i	=	total leadtime from the outside supplier to stage i, $l_i = \sum_{j=i}^N L_j$ for $i = 1, ..., N$.
h_i	=	echelon holding cost per unit per period at stage i, $h_i > 0$ for $i = 1, ..., N$.
H_i	=	installation holding cost per unit per period at stage i, $H_i = \sum_{j=i}^N h_j$ for $i = 1, ..., N$.
p	=	penalty cost per backlogged unit per period, $p > 0$.
$D(t)$	=	discrete demand in period t, which is distributed over $\mathbb{Z}_0^+$ with $\Pr\{D(t) = 1\} > 0$.
μ	=	expected one-period demand, $\mathrm{E}[D(t)] = \mu \ \ \forall t$, $\mu > 0$.
$D_i(t)$	=	demand during the periods $t + l_{i+1}$, ...,$t + l_i$.
$D_i^-(t)$	=	demand during the periods $t + l_{i+1}$, ...,$t + l_i - 1$.
F	=	cumulative distribution function of one-period demand defined over $\mathbb{Z}_0^+$.
$B(t)$	=	backorder level at stage 1 at the end of period t.
$IL_i(t)$	=	echelon inventory of stage i at the end of period t, i.e., on-hand inventory at stage i plus inventories in transit to or on-hand at stages 1, ...,$i-1$ minus backorders at stage 1, $i = 1, ..., N$.
$IL_i^-(t)$	=	echelon inventory of stage i at the beginning of period t just after the receipt of the incoming order, but before the demand, $IL_i(t) = IL_i^-(t) - D(t)$, $i = 1, ..., N$.
$IP_i(t)$	=	echelon inventory position at stage i at the beginning of period t just after ordering, but before the demand, i.e., $IL_i^-(t)$ + inventories in transit to stage i, $i = 1, ..., N$.

When the period index t in variables $D_i(t)$, $D_i^-(t)$, $B(t)$, $IL_i(t)$, $IL_i^-(t)$ and $IP_i(t)$ is suppressed, the notation represents the corresponding steady state variables. While D_i^- denotes L_i period demand in steady state, D_i stands for $L_i + 1$ period demand (in steady state).

Remark 5.1 *We assume* $Pr\{D(t) = 1\} > 0$ *in order to obtain an irreducible Markov chain that describes the behavior of the system. All results derived in the rest of this chapter are valid under this condition. However, the results can be shown to hold under weaker conditions on the demand, see the appendix of Chen (2000).*

Example 5.1: In order to illustrate our results, we use a system with $N = 2$, $L_1 = L_2 = 1$, $Q_1 = 2$ and $Q_2 = 4$ as an example throughout the chapter. The data implies that $l_1 = 2$, $l_2 = 1$ and $n_1 = 2$. See Figure 5.1 for a visual representation of the system. □

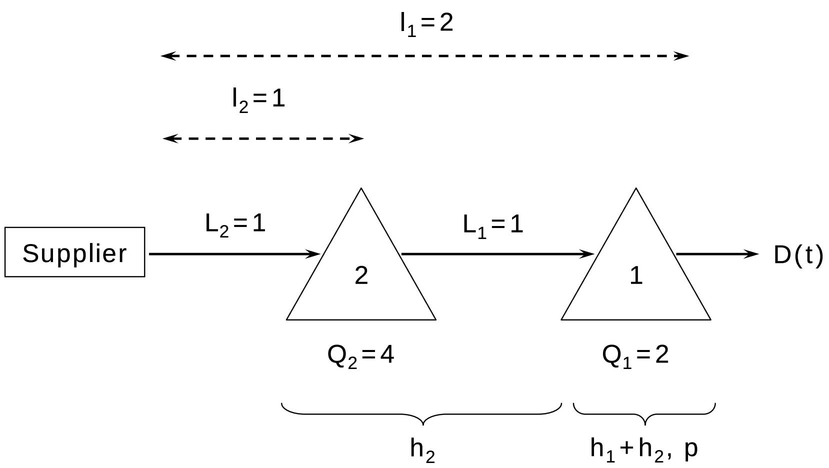

Figure 5.1: The representation of the system considered in Example 5.1.

5.3 Analysis

This section is composed of three parts. We review some of the main results from Chen (1998) in §5.3.1; as a matter of fact, some results stem from Chen and Zheng (1994a). These are used in §5.3.2 to derive the newsboy characterizations. The case of continuous demand and its consequences on the newsboy characterizations are discussed in §5.3.3.

5.3.1 Preliminaries

We express the expected holding costs via echelon holding cost parameters. At the end of period t, the expected holding and penalty cost of the system is

$$\sum_{i=1}^{N} h_i IL_i(t) + (p + H_1)B(t). \tag{5.1}$$

Consider the following chain of actions that starts with the ordering decision of stage N in period t. After ordering, the echelon inventory position of stage N is $IP_N(t)$. This order is received by stage N at the beginning of period $t+l_N$ and the echelon inventory of stage N at that epoch is $IL_N^-(t+l_N)$. Note that $IL_N^-(t+l_N)$

(i) bounds the ordering decision of stage $N-1$ in period $t+l_N$ from above, i.e., $IP_{N-1}(t+l_N) \leq IL_N^-(t+l_N)$, and

(ii) determines the echelon holding cost of stage N at the end of period $t+l_N$, i.e., $h_N IL_N(t+l_N)$.

Similarly, the order placed by stage $N-1$ in period $t+l_N$ arrives in period $t+l_{N-1}$ and it bounds the ordering decision of stage $N-2$ from above, i.e., $IP_{N-2}(t+l_{N-1}) \leq IL_{N-1}^-(t+l_{N-1})$, and affects the echelon $N-1$ cost $h_{N-1}IL_{N-1}(t+l_{N-1})$. Apply this reasoning for the rest of the stages. The sum of the costs as a consequence of the chain of actions that starts with the ordering decision of stage N in period t is

$$\sum_{i=1}^{N} h_i IL_i(t+l_i) + (p + H_1)B(t+l_1). \tag{5.2}$$

We call the cost $h_i IL_i(t+l_i)$ for $i = 2, ..., N$ as the *cost attached to echelon i*, and the cost $h_1 IL_1(t+l_1) + (p+H_1)B(t+l_1)$ as the *cost attached to echelon 1*. Note that, under the average cost criterion in an infinite horizon, (5.1) and (5.2) are equivalent. Further, when the system reaches the steady state, the time index can be suppressed for the calculation of the expected value of (5.2).

Example 5.1 (continued): The chain of events (and their consequences) that starts with the ordering decision of stage 2 in period t is depicted in Figure 5.2. □

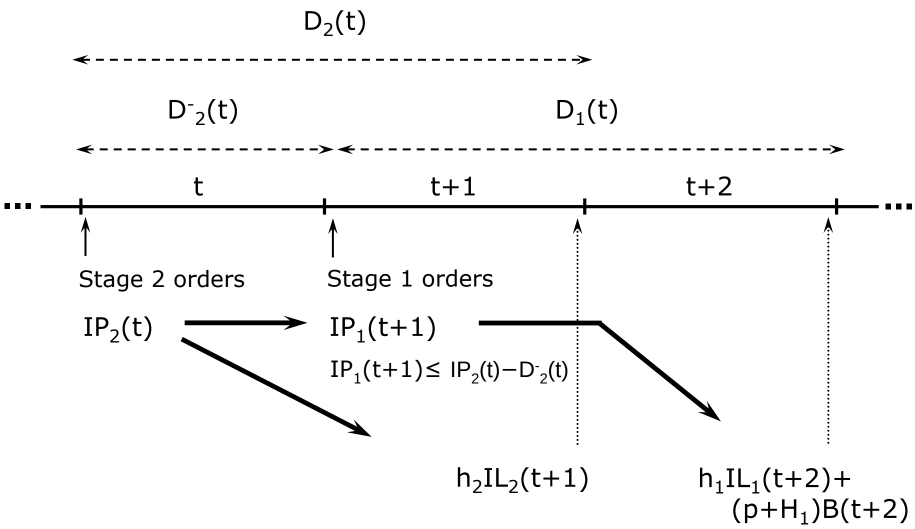

Figure 5.2: The relationship between the decisions and their cost-wise consequences in the system of Example 5.1.

Chen (2000) showed that echelon stock (R, Q) policies are optimal for each stage of the system under study. Hence we are interested in finding optimal echelon reorder levels, $\mathbf{R}^* = (R_1^*, ..., R_N^*)$, that minimize

$$C(\mathbf{R}) \stackrel{\text{def}}{=} \mathbf{E}\left[\sum_{i=1}^{N} h_i IL_i + (p + H_1)B\right], \qquad (5.3)$$

where $C(\mathbf{R})$ is the expected average cost of the system in a steady state when echelon reorder points $\mathbf{R} = (R_1, ..., R_N)$ are used (cf. Chen (1998, p. S225)).

In order to set the stage for new results, we now review some important findings that originate from Chen and Zheng (1994a), and Chen (1998). First, note that

$$IL_i = IP_i - D_i, \quad i = 1, ..., N \qquad (5.4)$$

From Lemma 1 of Chen and Zheng (1994a),

$$IP_i = O_i[IL_{i+1}^-], \quad i = 1, ..., N$$

where

$$O_i[x] \stackrel{\text{def}}{=} \begin{cases} x & \text{if} \quad x \leq R_i + Q_i \\ x - mQ_i & \text{if} \quad x > R_i + Q_i, \end{cases}$$

and $m = \max\{b | x - bQ_i > R_i,\ b \in \mathbb{Z}^+\}$.

We define the following random variables:

$$\begin{aligned} Pr\{U_i = u\} &= \frac{1}{Q_i}, \quad u = 1, ..., Q_i,\ i = 1, ..., N \\ Pr\{Z_i = z\} &= \frac{1}{n_i}, \quad z = 0, ..., n_i - 1,\ i = 1, ..., N - 1. \end{aligned}$$

These random variables are independent of each other and independent of the demand process. If two random variables X and Y have the same distribution, then we denote it by $X \stackrel{\text{d}}{=} Y$. From Lemma 1 in Chen (1998), it holds that

$$U_{i+1} \stackrel{\text{d}}{=} Z_i Q_i + U_i \quad \text{for } i = 1, ..., N - 1. \tag{5.5}$$

Hadley and Whitin (1961) has shown that the inventory position of a single-stage (R, Q) system is distributed uniformly over $[R + 1, ..., R + Q]$. Similar to this result, due to the assumption of ample stock at the supplier, it is not hard to see that IP_N has a uniform distribution over $[R_N + 1, ..., R_N + Q_N]$, i.e., $IP_N \stackrel{\text{d}}{=} R_N + U_N$. Unfortunately, such a property does not hold for the lower echelons. However, Chen (1998) has shown a similar behavior such that $IP_i \stackrel{\text{d}}{=} V_i + U_i$ for $i = 1, ..., N$, where

$$V_N = R_N, \quad V_i = \min\{R_i, V_{i+1} + Z_i Q_i - D^-_{i+1}\} \ \text{ for } \ i = N - 1, ..., 1.$$

(Notice that when a recursion starts from some stage i and continues until stage j for $i > j$, we use a descending index.) Chen (1998) calls V_i as the *effective reorder point* at stage i. This result is very important because a characteristic known for a single-stage system also holds in a multi-echelon setting, but in a slightly different way: At stages $1, ..., N-1$, IP_i is uniformly distributed over $[V_i + 1, ..., V_i + Q_i]$ where V_i is a random variable instead of the fixed reorder point R_i.

5.3.2 Newsboy Characterizations

We are now prepared for new results. First, we develop an alternative average cost formula for $C(\mathbf{R})$. Then, subsystem costs are introduced and the first order difference functions of these costs are derived. These are explicit difference functions that lead to the newsboy characterizations.

Alternative average cost formula
Define

B_i = shortfall of the effective reorder point from the reorder level at stage i, i.e., $B_i = R_i - V_i$ for $i = 1, ..., N$.
B_0 = backorder level at stage 1, i.e., $B_0 = B$.

Due to the infinite stock at the supplier,

$$B_N = 0. \tag{5.6}$$

For $i = N - 1, ..., 1$:

$$\begin{aligned} B_i &= R_i - V_i \\ &= R_i - \min\{R_i, V_{i+1} + Z_i Q_i - D_{i+1}^-\} \\ &= R_i + \max\{-R_i, -V_{i+1} - Z_i Q_i + D_{i+1}^-\} \\ &= \max\{0, R_i - V_{i+1} - Z_i Q_i + D_{i+1}^-\} \\ &= \max\{0, (R_{i+1} - V_{i+1}) - (R_{i+1} - R_i) - Z_i Q_i + D_{i+1}^-\} \\ &= \left[B_{i+1} + D_{i+1}^- - (R_{i+1} - R_i) - Z_i Q_i\right]^+, \end{aligned} \tag{5.7}$$

where $[x]^+ = \max\{0, x\}$ for $x \in \mathbb{Z}$. Since $B = [D_1 - IP_1]^+$ and $IP_1 \stackrel{\text{d}}{=} V_1 + U_1$, $B = [D_1 - V_1 - U_1]^+$. Substituting $V_1 = R_1 - B_1$ in the expression for B leads to

$$B_0 = B = [B_1 + D_1 - R_1 - U_1]^+ . \tag{5.8}$$

Lemma 5.1 *$IP_i \stackrel{\text{d}}{=} R_i - B_i + U_i$ for $i = 1, ..., N$ where the B_i are defined by (5.6)-(5.7).*

Proof : By Theorem 1 of Chen (1998), $IP_i \stackrel{\text{d}}{=} V_i + U_i$ for $i = 1, ..., N$. By definition, $B_i = R_i - V_i$ for $i = 1, ..., N$. Substituting $V_i = R_i - B_i$ into the result of Theorem 1 in Chen (1998) leads to the result given in the lemma. □

The following lemma gives an alternative formula for the expected long-run average cost of the system.

Lemma 5.2 *Let* $\mathbf{R} \in \mathbb{Z}^N$. *The long-run average expected cost under the echelon stock (R, Q) policy with reorder levels $\mathbf{R}$ is*

$$C(\mathbf{R}) = \sum_{i=1}^{N} h_i \left(R_i - \mathrm{E}[B_i] + \frac{Q_i + 1}{2} - (L_i + 1)\mu \right) + (p + H_1)\mathrm{E}[B_0],$$

where B_i for $i = N, ..., 0$ are defined by (5.6)-(5.8).

Proof : The result is obtained by the substitution of (5.4) and the result of Lemma 5.1 into (5.3):

$$\begin{aligned} C(\mathbf{R}) &= \mathrm{E}\left[\sum_{i=1}^{N} h_i IL_i + (p+H_1)B\right] \\ &= \mathrm{E}\left[\sum_{i=1}^{N} h_i\left(IP_i - D_i\right) + (p+H_1)B_0\right] \\ &= \mathrm{E}\left[\sum_{i=1}^{N} h_i\left(R_i - B_i + U_i - D_i\right) + (p+H_1)B_0\right]. \\ &= \sum_{i=1}^{N} h_i\left(R_i - \mathrm{E}[B_i] + \frac{Q_i+1}{2} - (L_i+1)\mu\right) + (p+H_1)\mathrm{E}[B_0]. \ \square \end{aligned}$$

Example 5.1 (continued): We continue with the alternative average cost formula for the system in Example 1:

$$\begin{aligned} C(\mathbf{R}) &= h_2\left(R_2 + \frac{5}{2} - 2\mu\right) + h_1\left(R_1 - \mathrm{E}[B_1] + \frac{3}{2} - 2\mu\right) \\ &\quad + (p+h_1+h_2)\mathrm{E}[B_0], \end{aligned} \tag{5.9}$$

where $B_1 = \left[D_2^- - (R_2 - R_1) - 2Z_1\right]^+$ with $Pr\{Z_1 = 0\} = Pr\{Z_1 = 1\} = 0.5$, and $B_0 = [B_1 + D_1 - R_1 - U_1]^+$ with $Pr\{U_1 = 1\} = Pr\{U_1 = 2\} = 0.5$. As can be seen in Figure 5.2, while D_2^- is a random variable representing one-period demand, D_1 stands for two-period demand. □

Subsystem costs and forward difference functions
Consider the following i-stage serial subsystem with $i \in \{1, ..., N\}$: stage k follows echelon stock (R_k, nQ_k) policy for $k = 1, ..., i$, and stage $i+1$ has ample stock. In the i-stage subsystem, only the costs attached to the echelons $1, 2, ..., i$ are taken into account. A similar expected long-run average cost expression is obtained for this subsystem as for the full system in Lemma 5.2:

$$\begin{aligned} C_i(R_1, ..., R_i) &\stackrel{\text{def}}{=} \sum_{k=1}^{i} h_k\left(R_k - \mathrm{E}[B_k^{(i)}] + \frac{Q_k+1}{2} - (L_k+1)\mu\right) \\ &\quad + (p+H_1)\mathrm{E}[B_0^{(i)}], \end{aligned} \tag{5.10}$$

where the $B_k^{(i)}$ for $k = i, ..., 0$ are defined by

$$B_i^{(i)} = 0, \tag{5.11}$$

$$B_k^{(i)} = \left[B_{k+1}^{(i)} + D_{k+1}^{-} - (R_{k+1} - R_k) - Z_k Q_k\right]^{+}, \; k = i-1, ..., 1, \tag{5.12}$$

$$B_0^{(i)} = \left[B_1^{(i)} + D_1 - R_1 - U_1\right]^{+}. \tag{5.13}$$

Remark 5.2 *Note that* $B_k^{(N)} = B_k$ *for* $k = 0, ..., N$*. For* $i = 1, ..., N-1$*:* $B_k^{(i)} = (B_k | B_i = 0)$ *for* $k = 0, ..., i$*. Further,* $C_N(\mathbf{R}) = C(\mathbf{R})$*.*

Example 5.1 (continued): In our example, we distinguish two subsystems: 2-stage subsystem and 1-stage subsystem. The 2-stage subsystem is identical to the full system. The 1-stage subsystem is equivalent to a *single-stage system* (the parameters are represented with $\sim$) having a leadtime $\widetilde{L} = L_1 = 1$, a fixed batch size $\widetilde{Q} = Q_1 = 2$, identical demands, i.e., $\widetilde{D}(t) = D(t)$, a holding cost $\widetilde{h} = h_1$, and a penalty cost $\widetilde{p} = p + H_2 = p + h_2$. We depict the 1-stage subsystem in Figure 5.3. □

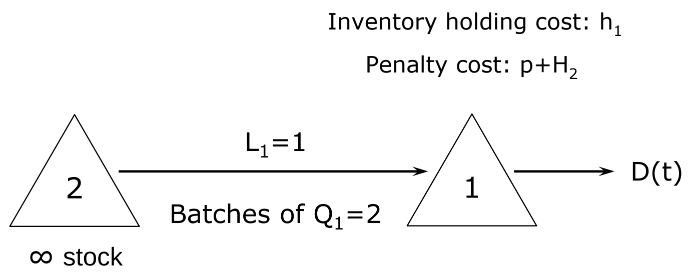

Figure 5.3: The representation of the 1-stage subsystem in Example 5.1.

Define $c_i(R_1, ..., R_i) = C_i(R_1, ..., R_i + 1) - C_i(R_1, ..., R_i)$, which is the first order forward difference function of $C_i(\cdot)$ in R_i.

Lemma 5.3 *Let $i \in \{1,...,N\}$, and $(R_1,...,R_i) \in \mathbb{Z}^i$. Then,*

$$\begin{aligned} c_i(R_1,...,R_i) &= h_i + \sum_{k=1}^{i-1} h_k \Pr\{B_j^{(i)} > 0 \text{ for } j = i-1,...,k\} \\ &\quad -(p+H_1)\Pr\{B_j^{(i)} > 0 \text{ for } j = i-1,...,0\}, \quad (5.14) \end{aligned}$$

where the $B_k^{(i)}$ for $k = i-1,...,0$ are defined by (5.11)-(5.13), and the sum on the righthand side is taken to be a null sum when $i = 1$.

Proof : Let $\hat{B}_k^{(i)}$ be defined by (5.11)-(5.13), but with R_i replaced by $R_i + 1$. Then, $\hat{B}_i^{(i)} = B_i^{(i)} = 0$. Note that $\hat{B}_{i-1}^{(i)} = [\hat{B}_i^{(i)} + D_i^- - (R_i + 1 - R_{i-1}) - Z_{i-1}Q_{i-1}]^+ = [B_{i-1}^{(i)} - 1]^+$. If $B_{i-1}^{(i)} = 0$ then $\hat{B}_{i-1}^{(i)} - B_{i-1}^{(i)} = 0$; else if $B_{i-1}^{(i)} > 0$ then $\hat{B}_{i-1}^{(i)} - B_{i-1}^{(i)} = -1$. Next, $\hat{B}_{i-2}^{(i)} = [\hat{B}_{i-1}^{(i)} + D_{i-1}^- - (R_{i-1} - R_{i-2}) - Z_{i-2}Q_{i-2}]^+$. If $B_{i-1}^{(i)} > 0$ and $B_{i-2}^{(i)} > 0$, then $\hat{B}_{i-2}^{(i)} - B_{i-2}^{(i)} = -1$; if $B_{i-1}^{(i)} = 0$ or $B_{i-2}^{(i)} = 0$, then $\hat{B}_{i-2}^{(i)} = B_{i-2}^{(i)}$. Continuing in this fashion shows that $\mathrm{E}[\hat{B}_k^{(i)}] - \mathrm{E}[B_k^{(i)}] = \Pr\{B_j^{(i)} > 0 \text{ for } j = i-1,...,k\}$, $k = i-1,...,0$. This result in combination with (5.10) leads to (5.14). □

We are able to rewrite (5.14) in a recursive way, which is given in the next lemma.

Lemma 5.4 *For $i \in \{1,...,N\}$, and $(R_1,...,R_i) \in \mathbb{Z}^i$:*

$$\begin{aligned} c_i(R_1,...,R_i) &= \sum_{k=1}^{i} h_k - (p+H_1)\Pr\{B_0^{(i)} > 0\} \\ &\quad - \sum_{k=1}^{i-1} \Pr\{B_k^{(i)} = 0\} c_k(R_1,...,R_k), \quad (5.15) \end{aligned}$$

where the $B_k^{(i)}$ for $k = i-1,...,0$ are defined by (5.11)-(5.13), and the second sum on the righthand side is taken to be a null sum when $i = 1$.

Proof : For $i = 1$, (5.15) is read as $c_1(R_1) = h_1 - (p+H_1)\Pr\{B_0^{(1)} > 0\}$. This is the expression given in Lemma 5.3. In the rest of the proof, let $i \geq 2$, and any summation with the lower limit greater than the upper limit (e.g., $\sum_{m=i}^{i-1}$) be a null sum. For $k \in \{1,...,i-1\}$, we may rewrite $\Pr\{B_j^{(i)} > 0 \text{ for } j = i-1,...,k\}$ as

$\Pr\{B_j^{(i)} > 0 \text{ for } j = i-1, ..., k\}$

$$\begin{aligned} &= 1 - \Big(\Pr\{B_k^{(i)} = 0\} + \Pr\{B_{k+1}^{(i)} = 0, B_k^{(i)} > 0\} + ... \\ &\qquad + \Pr\{B_{i-1}^{(i)} = 0, B_j^{(i)} > 0 \text{ for } j = i-2, ..., k\}\Big) \\ &= 1 - \Big(\Pr\{B_k^{(i)} = 0\} \\ &\quad + \sum_{m=k+1}^{i-1} \Pr\{B_m^{(i)} = 0, B_j^{(i)} > 0 \text{ for } j = m-1, ..., k\}\Big) \\ &= 1 - \Big(\Pr\{B_k^{(i)} = 0\} + \sum_{m=k+1}^{i-1} \Pr\{B_j^{(i)} > 0 \\ &\qquad \text{for } j = m-1, ..., k | B_m^{(i)} = 0\}\Pr\{B_m^{(i)} = 0\}\Big). \end{aligned} \tag{5.16}$$

Similarly,

$\Pr\{B_j^{(i)} > 0 \text{ for } j = i-1, ..., 0\}$

$$\begin{aligned} &= 1 - \Big(\Pr\{B_0^{(i)} = 0\} + \Pr\{B_1^{(i)} = 0, B_0^{(i)} > 0\} + ... \\ &\qquad + \Pr\{B_{i-1}^{(i)} = 0, B_j^{(i)} > 0 \text{ for } j = i-2, ..., 0\}\Big) \\ &= \Pr\{B_0^{(i)} > 0\} \\ &\quad - \sum_{m=1}^{i-1} \Pr\{B_j^{(i)} > 0 \text{ for } j = m-1, ..., 0 | B_m^{(i)} = 0\}\Pr\{B_m^{(i)} = 0\}. \end{aligned} \tag{5.17}$$

Substituting (5.16) and (5.17) into (5.14), and rearranging the terms results in

$$\begin{aligned} c_i(R_1, ..., R_i) = &\sum_{k=1}^{i} h_k - (p + H_1)\Pr\{B_0^{(i)} > 0\} - \sum_{k=1}^{i-1} h_k \Pr\{B_k^{(i)} = 0\} \\ &- \left[\sum_{k=1}^{i-1} h_k \sum_{m=k+1}^{i-1} \Pr\{B_j^{(i)} > 0 \text{ for } j = m-1, ..., k | B_m^{(i)} = 0\}\Pr\{B_m^{(i)} = 0\} \right. \\ &\left. -(p + H_1) \sum_{m=1}^{i-1} \Pr\{B_j^{(i)} > 0 \text{ for } j = m-1, ..., 0 | B_m^{(i)} = 0\}\Pr\{B_m^{(i)} = 0\}\right]. \end{aligned}$$

Changing the order of summation gives

$$c_i(R_1,...,R_i) = \sum_{k=1}^{i} h_k - (p+H_1)\Pr\{B_0^{(i)} > 0\} - \sum_{k=1}^{i-1} h_k \Pr\{B_k^{(i)} = 0\}$$
$$- \left[\sum_{m=2}^{i-1} \Pr\{B_m^{(i)} = 0\} \sum_{k=1}^{m-1} h_k \Pr\{B_j^{(i)} > 0 \text{ for } j = m-1,...,k | B_m^{(i)} = 0\}\right.$$
$$\left. -(p+H_1) \sum_{m=1}^{i-1} \Pr\{B_j^{(i)} > 0 \text{ for } j = m-1,...,0 | B_m^{(i)} = 0\} \Pr\{B_m^{(i)} = 0\}\right].$$

Next,

$$\begin{aligned}
&c_i(R_1,...,R_i)\\
&= \sum_{k=1}^{i} h_k - (p+H_1)\Pr\{B_0^{(i)} > 0\} - \sum_{k=1}^{i-1} h_k \Pr\{B_k^{(i)} = 0\}\\
&\quad - \left[\sum_{m=2}^{i-1} \Pr\{B_m^{(i)} = 0\} \sum_{k=1}^{m-1} h_k \Pr\{B_j^{(m)} > 0 \text{ for } j = m-1,...,k\}\right.\\
&\quad \left. -(p+H_1) \sum_{m=1}^{i-1} \Pr\{B_j^{(m)} > 0 \text{ for } j = m-1,...,0\} \Pr\{B_m^{(i)} = 0\}\right]\\
&= \sum_{k=1}^{i} h_k - (p+H_1)\Pr\{B_0^{(i)} > 0\}\\
&\quad -\Pr\{B_1^{(i)} = 0\}\left[h_1 - (p+H_1)\Pr\{B_0^{(1)} > 0\}\right]\\
&\quad - \sum_{m=2}^{i-1} \Pr\{B_m^{(i)} = 0\} \left[h_m + \sum_{k=1}^{m-1} h_k \Pr\{B_j^{(m)} > 0 \text{ for } j = m-1,...,k\}\right.\\
&\quad \left. -(p+H_1)\Pr\{B_j^{(m)} > 0 \text{ for } j = m-1,...,0\}\right]\\
&= \sum_{k=1}^{i} h_k - (p+H_1)\Pr\{B_0^{(i)} > 0\} - \sum_{k=1}^{i-1} \Pr\{B_k^{(i)} = 0\} c_k(R_1,...,R_k),
\end{aligned}$$

where the first equality follows from the fact that $\{B_j^{(i)} > 0 \text{ for } j = m-1,...,k | B_m^{(i)} = 0\} \equiv \{B_j^{(m)} > 0 \text{ for } j = m-1,...,k\}$ for $k \in \{m-1,...,0\}$, second equality from rewriting some of the expressions under a single summation, and the last equality from the equivalence of (5.14) for $i = 1$ and $i = m$ to the expressions within the first and the second brackets, respectively. □

Newsboy inequalities
Let $\overline{Y}_1$ be a minimizing point of $C_1(R_1)$. For $i = 2, ..., N$, define recursively $\overline{Y}_i \in \mathbb{Z}$ as a point that minimizes $C_i(\overline{Y}_1, ...\overline{Y}_{i-1}, R_i)$. The function $C_i(\overline{Y}_1, ...\overline{Y}_{i-1}, R_i)$ is equal to $\overline{G}_i(R_i)$ of Chen (1998) for $i = 1, ..., N$ and $R_i \in \mathbb{Z}$. As shown there in Lemma 3, $\overline{G}_i(R_i)$ is convex in R_i (and thus also $C_i(\overline{Y}_1, ..., \overline{Y}_{i-1}, R_i)$). Due to Theorem 1 of Chen (1998), the optimal reorder levels are found by minimizing $C_i(\overline{Y}_1, ...\overline{Y}_{i-1}, R_i)$ for $i = 1, ..., N$ recursively.

Theorem 5.5 *Let $i \in \{1, ..., N\}$. An optimal reorder level $\overline{Y}_i \in \mathbb{Z}$ is an element of the set $\mathbf{y}_i \stackrel{\text{def}}{=} \{y_i^l, y_i^l + 1, ..., y_i^u\}$ where*

$$y_i^l = \min\left\{R_i | \Pr\{B_0^{(i)} = 0\} \geq \frac{p + H_{i+1}}{p + H_1} + \frac{1}{p + H_1} \sum_{k=1}^{i-1} \Pr\{B_k^{(i)} = 0\} c_k(\overline{Y}_1, ..., \overline{Y}_k)\right\}, \quad (5.18)$$

$$y_i^u = \min\left\{R_i | \Pr\{B_0^{(i)} = 0\} > \frac{p + H_{i+1}}{p + H_1} + \frac{1}{p + H_1} \sum_{k=1}^{i-1} \Pr\{B_k^{(i)} = 0\} c_k(\overline{Y}_1, ..., \overline{Y}_k)\right\}, \quad (5.19)$$

and

$$B_i^{(i)} = 0, \quad (5.20)$$

$$B_{i-1}^{(i)} = \left[B_i^{(i)} + D_i^- - (R_i - \overline{Y}_{i-1}) - Z_{i-1}Q_{i-1}\right]^+, \quad (5.21)$$

$$B_k^{(i)} = \left[B_{k+1}^{(i)} + D_{k+1}^- - (\overline{Y}_{k+1} - \overline{Y}_k) - Z_kQ_k\right]^+, \quad k = i-2, ..., 1, \quad (5.22)$$

$$B_0^{(i)} = \left[B_1^{(i)} + D_1 - (\overline{Y}_1 + U_1)\right]^+. \quad (5.23)$$

(For $i = 1$: $B_1^{(1)} = 0$, $B_0^{(1)} = [B_1^{(1)} + D_1 - (R_1 + U_1)]^+$, and the summations $\sum_{k=1}^{0}$ are taken to be null sums.)
The set

$$\left\{R_i | \Pr\{B_0^{(i)} = 0\} \geq \frac{p + H_{i+1}}{p + H_1} + \frac{1}{p + H_1} \sum_{k=1}^{i-1} \Pr\{B_k^{(i)} = 0\} c_k(\overline{Y}_1, ..., \overline{Y}_k)\right\} \neq \emptyset;$$

thus, y_i^l is finite. The set

$$\left\{R_i | \Pr\{B_0^{(i)} = 0\} > \frac{p + H_{i+1}}{p + H_1} + \frac{1}{p + H_1} \sum_{k=1}^{i-1} \Pr\{B_k^{(i)} = 0\} c_k(\overline{Y}_1, ..., \overline{Y}_k)\right\}$$

may be empty; then, $y_i^u = +\infty$.

Proof : From the convexity of $C_i(\overline{Y}_1, ..., \overline{Y}_{i-1}, R_i)$ in R_i, $\overline{Y}_i \in \mathbf{y}_i = \{y_i^l, y_i^l + 1, ..., y_i^u\}$ where y_i^l is the minimum R_i satisfying $c_i(\overline{Y}_1, ..., \overline{Y}_{i-1}, R_i) \geq 0$, and y_i^u is the minimum R_i satisfying $c_i(\overline{Y}_1, ..., \overline{Y}_{i-1}, R_i) > 0$. By Lemma (5.4),

$$\begin{aligned} c_i(\overline{Y}_1, ..., \overline{Y}_{i-1}, R_i) &= \sum_{k=1}^{i} h_k - (p + H_1)\Pr\{B_0^{(i)} > 0\} \\ &\quad - \sum_{k=1}^{i-1} \Pr\{B_k^{(i)} = 0\} c_k(\overline{Y}_1, ..., \overline{Y}_k) \\ &\geq 0 \end{aligned}$$

with the $B_k^{(i)}$ defined by (5.20)-(5.23). By substituting $\Pr\{B_0^{(i)} > 0\} = 1 - \Pr\{B_0^{(i)} = 0\}$ and rearranging the terms, this inequality may be rewritten as

$$\Pr\{B_0^{(i)} = 0\} \geq \frac{p + H_{i+1}}{p + H_1} + \frac{1}{p + H_1} \sum_{k=1}^{i-1} \Pr\{B_k^{(i)} = 0\} c_k(\overline{Y}_1, ..., \overline{Y}_k).$$

The expression for y_i^u can be derived similarly.

By Lemma 5.3, $\lim_{R_i \to +\infty} c_i(\overline{Y}_1, ..., \overline{Y}_{i-1}, R_i) = h_i$ for $i = 1, ..., N$. Further, due to the convexity of $C_i(\overline{Y}_1, ..., \overline{Y}_{i-1}, R_i)$ in R_i, $c_i(\overline{Y}_1, ..., \overline{Y}_{i-1}, R_i)$ is a non-decreasing function of R_i. Since $h_i > 0$, there exists finite R_i values that satisfy $c_i(\overline{Y}_1, ..., \overline{Y}_{i-1}, R_i) \geq 0$. Hence, y_i^l is a finite point. □

The intuitive message of Theorem 5.5 is informative. Assume that $N = 1$, i.e., there is a single-stage system. Note that (5.18) becomes $\Pr\{B_0^{(1)} = 0\} \geq \frac{p+H_2}{p+H_1} = \frac{p}{p+h_1}$. The optimal reorder level, $\overline{Y}_1$, is chosen such that the probability of having no-stockout (as a consequence of this reorder level) is greater than or equal to $\frac{p}{p+h_1}$, which is the newsboy fractile in a single-stage inventory system. For a general N-stage system, an optimal reorder level at each stage $i \geq 1$ is chosen such that the probability of no-stockout at stage 1 in the i-stage subsystem is at least equal to $\frac{p+H_{i+1}}{p+H_1}$ plus a term that depends on the extent the newsboy fractiles are met at the stages $1, ..., i-1$, see (5.18) and (5.19). The second term is nonnegative, by definition, since $c_k(\overline{Y}_1, ..., \overline{Y}_k) \geq 0$ for $k = 1, ..., N$. This leads to the following corollary.

Corollary 5.6 *For* $i = 1, ..., N$*:* $\overline{Y}_i$ *satisfies*

$$\Pr\{B_0^{(i)} = 0\} \geq \frac{p + H_{i+1}}{p + H_1},$$

where $B_k^{(i)}$ *for* $k = i, ..., 0$ *are defined in (5.20)-(5.23).*

We can verify the following intuitive relationship between the holding cost parameters and the reorder levels under an optimal policy.

Corollary 5.7 *There exists an optimal* (R, Q) *policy under which no safety stock is held at stage* $i+1$ *when* $h_i \downarrow 0$, $i = 1, ..., N-1$.

Proof : Let $i \in \{1, ..., N-1\}$. Note that $\lim_{R_i \to +\infty} c_i(\overline{Y}_1, ..., \overline{Y}_{i-1}, R_i) = h_i$ and $c_i(\overline{Y}_1, ..., \overline{Y}_{i-1}, R_i)$ is a nondecreasing function of R_i due to Lemma 5.3 and the convexity of $C_i(\overline{Y}_1, ..., \overline{Y}_{i-1}, R_i)$ in R_i, respectively. If $h_i \downarrow 0$, then $y_i^u \to +\infty$; thus, we may choose $\overline{Y}_i = +\infty$. This then implies that all goods arriving at stage $i+1$ are immediately forwarded to stage i at the beginning of each period; resulting in no-stock-keeping at stage $i+1$. □

When $Q_i = 1$ for $i = 1, ..., N$, the system reduces to the Clark-Scarf model for which the optimality of base stock policies has been known for more than four decades (see Clark and Scarf (1960), Federgruen and Zipkin (1984c), and Chen and Zheng (1994b)). The results of Theorem 5.5 still apply for the calculation of optimal base stock levels, $\mathbf{S}^* \stackrel{\text{def}}{=} (\overline{S}_1, ..., \overline{S}_N)$, with $\overline{S}_i = \overline{Y}_i + 1$ for $i = 1, ..., N$.

For $N = 2$ with $Q_1 = Q_2 = 1$, our newsboy inequalities are equivalent to the findings in Chapter 2 when the one-warehouse multi-retailer system has a single retailer (cf. (2.9)).

Example 5.1 (continued): Assume that $h_1 = 1$, $h_2 = 2$ and $p = 7$; hence, $H_1 = 3$ and $H_2 = 2$. Further, one-period demand has the following distribution:

a	0	1	2
$\Pr\{D(t) = a\}$	0.2	0.5	0.3

An optimal reorder level for stage 1 is determined as follows. The event $\{B_0^{(1)} = [D_1 - R_1 - U_1]^+ = 0\}$ is equivalent to the event $\{D_1 - U_1 \leq R_1\}$. The random variable D_1 has a distribution of two-period demand. The probability distributions of D_1 and $D_1 - U_1$, and the cumulative probability distribution of $D_1 - U_1$ are given below:

a	-2	-1	0	1	2	3	4
$\Pr\{D_1 = a\}$	0	0	0.04	0.2	0.37	0.3	0.09
$\Pr\{D_1 - U_1 = a\}$	0.02	0.12	0.285	0.335	0.195	0.045	0
$\Pr\{D_1 - U_1 \leq a\}$	0.02	0.14	0.425	0.76	0.955	1	1

Note that $\Pr\{B_0^{(1)} = 0 | R_1 = a\} = \Pr\{D_1 - U_1 \leq a\}$. From (5.18) and (5.19), an optimal reorder level for stage 1 ($\overline{Y}_1$) must satisfy $\Pr\{B_0^{(1)} = 0\} \geq \frac{p+H_2}{p+H_1} = \frac{9}{10}$. Since $\Pr\{B_0^{(1)} = 0 | R_1 = 2\} = 0.955 > 0.9$, $y_1^l = y_1^u = \overline{Y}_1 = 2$.

Next, we find an optimal reorder point for stage 2. By (5.15), $c_1(\overline{Y}_1 = 2) = 1 - (10)(0.045) = 0.55$. In order to find $\overline{Y}_2$, we need to determine the first point satisfying

$$\begin{aligned} \Pr\{B_0^{(2)} = 0\} &\geq \frac{p}{p+H_1} + \frac{1}{p+H_1}\Pr\{B_1^{(2)} = 0\}c_1(\overline{Y}_1) \qquad (5.24) \\ &= 0.7 + 0.055\Pr\{B_1^{(2)} = 0\}, \end{aligned}$$

where $B_1^{(2)} = [D_2^- - (R_2 - 2) - 2Z_1]^+$ and $B_0^{(2)} = [B_1^{(2)} + D_1 - 2 - U_1]^+$. For $R_2 = 1$, the distributions of $B_1^{(2)}$ and $B_0^{(2)}$ are tabulated below:

a	$\Pr\{D_2^- - 2Z_1 + 1 = a\}$	$\Pr\{B_1^{(2)} = a\}$	$Pr(a)$	$\Pr\{B_0^{(2)} = a\}$
-4	0	0	0.007	0
-3	0	0	0.047	0
-2	0	0	0.13475	0
-1	0.1	0	0.2215	0
0	0.25	0.35	0.24125	0.6515
1	0.25	0.25	0.191	0.191
2	0.25	0.25	0.11025	0.11025
3	0.15	0.15	0.0405	0.0405
4	0	0	0.00675	0.00675

where $Pr(a) \stackrel{\text{def}}{=} \Pr\{B_1^{(2)} + D_1 - 2 - U_1 = a\}$. Note that $\Pr\{B_1^{(2)} = 0\} = 0.35$ and $\Pr\{B_0^{(2)} = 0\} = 0.6515$, so (5.24) does not hold for $R_2 = 1$. For $R_2 = 2$, the distributions of $B_1^{(2)}$ and $B_0^{(2)}$ are as follows:

a	$\Pr\{D_2^- - 2Z_1 = a\}$	$\Pr\{B_1^{(2)} = a\}$	$Pr(a)$	$\Pr\{B_0^{(2)} = a\}$
-4	0	0	0.012	0
-3	0	0	0.077	0
-2	0.1	0	0.204	0
-1	0.25	0	0.29025	0
0	0.25	0.6	0.2435	0.82675
1	0.25	0.25	0.126	0.126
2	0.15	0.15	0.0405	0.0405
3	0	0	0.00675	0.00675

Substituting $\Pr\{B_1^{(2)} = 0\} = 0.6$ and $\Pr\{B_0^{(2)} = 0\} = 0.82675$ in (5.24) leads to $\Pr\{B_0^{(2)} = 0\} = 0.82675 > 0.733$. Hence, $y_2^l = y_2^u = \overline{Y}_2 = 2$, which follows from (5.18) and (5.19).

The optimal reordering points are $(\overline{Y}_1, \overline{Y}_2) = (2, 2)$. In order to determine the optimal long-run average cost of the system (see (5.9)), we calculate $\mathrm{E}[B_1] = \mathrm{E}\left[[D_2^- - 2Z_1]^+\right]$ and $\mathrm{E}[B_0] = \mathrm{E}\left[[B_1 + D_1 - 2 - U_1]^+\right]$ using the distributions given above; the corresponding values are 0.55 and 0.22725, respectively. Substituting these values in (5.9) leads to

$$\begin{aligned} C(\overline{Y}_1 = 2, \overline{Y}_2 = 2) &= 2\left(2 + \frac{5}{2} - 2(1.1)\right) + 1\left(2 - 0.55 + \frac{3}{2} - 2(1.1)\right) \\ &\quad +10(0.22725) \\ &= 7.6225. \end{aligned}$$

Under the optimal policy, $(\overline{Y}_1, \overline{Y}_2) = (2, 2)$, the distributions of $IP_2 = \overline{Y}_2 + U_2$, $IP_1 = \overline{Y}_1 - B_1 + U_1$ and $V_1 = R_1 - B_1$ are given below:

a	$\Pr\{IP_2 = a\}$	$\Pr\{IP_1 = a\}$	$\Pr\{V_1 = a\}$
0	0	0	0.15
1	0	0.075	0.25
2	0	0.2	0.6
3	0.25	0.425	0
4	0.25	0.3	0
5	0.25	0	0
6	0.25	0	0

While IP_2 is distributed uniformly over the integers in $[\overline{Y}_2 + 1, \overline{Y}_2 + Q_2]$, IP_1 does not exhibit such a behavior. However, IP_1 is distributed uniformly over the integers in $[V_1 + 1, V_1 + Q_1]$. Unlike at stage 2 where $\overline{Y}_2 = 2$ is fixed, V_1 is a random variable. (cf. the discussion at the end of §5.3.1) □

5.3.3 Continuous Demand

The newsboy characterizations in Theorem 5.5 can be further sharpened when the demand process is continuous. For the rest of this section, assume that the demand distribution F is continuous on $(0,\infty)$ with $F(0) = 0$. Then, $C_i(R_1, ..., R_i)$ becomes a continuous function. Further, $c_i(R_1, ..., R_i)$ is now defined as $\frac{\partial C_i(R_1,...,R_i)}{\partial R_i}$. The results of Lemmas 5.3 and 5.4 still hold.

Contrary to the discrete demand case, we now know for $i = 1, ..., N$ that there is a $\overline{Y}_i \in \mathbb{R}$ such that $c_i(\overline{Y}_1, ..., \overline{Y}_i) = 0$, because of the continuity and convexity of $C_i(\overline{Y}_1, ..., \overline{Y}_{i-1}, R_i)$. Also, for $(R_1, ..., R_{i-1}) = (\overline{Y}_1, ..., \overline{Y}_{i-1})$, the second summation in (5.15) vanishes. Thus, we find $\overline{Y}_i$ such that

$$c_i(\overline{Y}_1, ..., \overline{Y}_i) = \sum_{k=1}^{i} h_k - (p + H_1)\Pr\{B_0^{(i)} > 0\} = 0.$$

This leads to the following simplified newsboy characterization.

Theorem 5.8 *If F is a continuous cdf on $(0,\infty)$ with $F(0) = 0$, then the optimal reorder level $\overline{Y}_i \in \mathbb{R}$ satisfies*

$$Pr\{B_0^{(i)} = 0\} = \frac{p + H_{i+1}}{p + H_1}, \quad i = 1, ..., N, \tag{5.25}$$

where the $B_k^{(i)}$ for $k = i, ..., 0$ are defined by (5.20)-(5.23).

The interpretation of this theorem is that the optimal reorder level at some stage i (assuming ample stock at stage $i + 1$) leads to a probability of no-stockout at stage 1 that is equal to $\frac{p+H_{i+1}}{p+H_1}$. This result is a generalization of the newsboy equalities shown by van Houtum and Zijm (1991) for the Clark and Scarf model.

Remark 5.3 ***On the connection to the lower bounds for the optimal reorder levels as derived by Shang and Song (2005).***

For $i \in \{2, ..., N\}$, define:

$$\begin{aligned}
\widehat{B}_i^{(i)} &= 0, \\
\widehat{B}_{i-1}^{(i)} &= \widehat{B}_i^{(i)} + D_i^- - (R_i - \overline{Y}_{i-1}) - Z_{i-1}Q_{i-1}, \\
\widehat{B}_k^{(i)} &= \widehat{B}_{k+1}^{(i)} + D_{k+1}^- - (\overline{Y}_{k+1} - \overline{Y}_k) - Z_kQ_k \quad \textit{for} \;\; k = i-2, ..., 1, \\
\widehat{B}_0^{(i)} &= \left[\widehat{B}_1^{(i)} + D_1 - (\overline{Y}_1 + U_1)\right]^+.
\end{aligned}$$

For $j = i, ..., 0$, one can show recursively that $\Pr\{\widehat{B}_j^{(i)} \leq a\} \geq \Pr\{B_j^{(i)} \leq a\}$ *$\forall a \in \mathbb{Z}$; i.e., $B_j^{(i)}$ is stochastically larger than $\widehat{B}_j^{(i)}$, which is denoted by $B_j^{(i)} \geq_{st} \widehat{B}_j^{(i)}$. In particular, $B_0^{(i)} \geq_{st} \widehat{B}_0^{(i)}$. The expression for $\widehat{B}_0^{(i)}$ may be rewritten as,*

$$\begin{aligned}
\widehat{B}_0^{(i)} &= \left[D_1 + \sum_{j=2}^{i} D_j^- - R_i - \left(\sum_{j=1}^{i-1} Z_jQ_j + U_1\right)\right]^+ \\
&= \left[D_1 + \sum_{j=2}^{i} D_j^- - (R_i + U_i)\right]^+,
\end{aligned}$$

where the second equality follows from (5.5). Define

$$r_i^l = \min\left\{ R_i | \Pr\{\widehat{B}_0^{(i)} = 0\} \geq \frac{p + H_{i+1}}{p + H_1} \right\}.$$

This r_i^l is identical to the lower bound for the optimal reorder level $\overline{Y}_i$ as defined by Shang and Song (2005, §3.5). By Corollary 5.6, $\overline{Y}_i$ is such that $\Pr\{B_0^{(i)} = 0\} \geq \frac{p+H_{i+1}}{p+H_1}$*. Since $B_0^{(i)} \geq_{st} \widehat{B}_0^{(i)}$, it follows that $\overline{Y}_i \geq r_i^l$, $i \in \{2, ..., N\}$. Note that r_1^l defined by Shang and Song is equal to the optimal reorder point $\overline{Y}_1$; see their Theorem 4.*

5.4 Concluding Remarks

In this chapter, we considered a multi-stage serial inventory/production system where each stage follows an (R, Q) policy. We developed a new cost formula for the long-run average expected cost based on the concept of shortfall, which allowed us to show that the optimal reorder levels satisfy newsboy inequalities (equalities) when demand distribution is discrete (continuous). These results add insights to our knowledge in the behavior of such systems under an optimal control, and generalize the newsboy characterizations found for the Clark and Scarf model.

We studied a periodic review model with i.i.d. demand, but it is straightforward to extend the results to a continuous review model with compound Poisson demand. Further, Chen (2000) has shown that a pure assembly system with fixed batch sizes can be transformed into an equivalent serial system under a specific integer ratio assumption, and, as a result of this, the optimality of echelon stock (R, Q) policies (with a slight modification) still holds. For the assembly systems that the serial transformation is possible, our results are also valid.

Chapter 6

Conclusions and Further Research

The one-warehouse multi-retailer inventory system is a well-known model with clear-cut applications in inventory, manufacturing and hierarchal production planning contexts. The model can be analyzed under continuous or periodic review, but it is rather complex in both settings. Up to now, no one has been able to characterize the optimal policy structure in any review setting. Under continuous review, the main approach is to assume first-come, first-served for the backlogged retailer orders at the warehouse and optimize the control parameters of a given policy. On the other hand, under periodic review, one faces the problem of solving a multi-dimensional stochastic dynamic program (DP). Unfortunately, the size of the DP grows in the number of retailers and the warehouse leadtime, which makes it impossible to solve the DP for real life applications. Moreover, there is no special structure of the DP that enables the reduction of the number of dimensions or simplification of the overall problem. However, relaxing a constraint, called the *balance assumption*, decomposes the problem of solving a multi-dimensional DP into a problem where single-dimensional DPs are solved sequentially. In addition, the structure of the optimal policy is characterized.

The balance assumption is used extensively in the analysis of the standard one-warehouse multi-retailer inventory system and its extensions. Further, in the light of limited numerical experiments reported in the literature, the balance assumption is accepted to be an accurate approximation. The main aim of this dissertation was to challenge the established conviction that the balance assumption leads to good performance under general conditions. We quantified the effect of the balance assumption on the long-run average cost of the system, and identified the settings where it is justified/not justified.

This dissertation is a collection of four papers, which constitute chapters 2-5. We mainly focused on studying the balance assumption in one-warehouse multi-retailer systems. In addition, we derived results regarding newsboy characterizations. Thus, the conclusions and suggestions for further research are discussed under two separate headings, Balance Assumption and Newsboy Characterizations in §6.1 and §6.2, respectively.

6.1 Balance Assumption

Our objective was to measure the error that emanates from making the balance assumption. The optimal cost of the system under the balance assumption (the relaxed model) is a lower bound for the true optimal cost and it can be calculated analytically. The optimal policy of the relaxed model may not be feasible for the original one, but it can be modified to satisfy the constraints of the original model. The simulation of this modified policy, which is referred to as LB heuristic policy, gives an upper bound for the true optimal cost. For the purpose of measuring the error (resulting from the balance assumption), it was necessary to solve a multi-dimensional DP by a numerical technique. Due to the curse of dimensionality, we had to confine ourselves to discrete demand distributions and limited input parameters. Thus, we first extended the optimality results in the relaxed model that exist for continuous demand distributions to discrete demand distributions. This gave us the theoretical basis for the comparison. Next, we conducted a rather extensive numerical study where instead of comparing the optimal cost against the lower bound, we used the relative gap between the upper and lower bounds ($\epsilon\%$) as a measure. Since these bounds envelope the optimal cost, a small $\epsilon\%$ value implies a small error due to the balance assumption. We wanted to distinguish the settings with small $\epsilon\%$ from the ones with significant $\epsilon\%$, so that we could focus on the scenarios with substantial relative gaps. We developed two test beds; one for identical retailers consisting of 2000 problem instances, another having 3888 problem instances for nonidentical retailers. The results from the identical retailers case show that the main determinant of a large relative gap ($\epsilon\%$) is the coefficient of variation. Independent of the other input parameters, $\epsilon\%$ is small as long as

- the coefficient of variation of the retailers is low or moderate (0.25,0.5,1),
- the added value at the warehouse is negligible with respect to the one at the retailers ($(h_0, h_i) = (0.01, 0.99)$ for $i = 1, 2, ..., N$),

- the warehouse leadtime is short and the retailer leadtimes are long ($(l_0, l_i) = (1,3)$ or $(1,5)$ for $i = 1, 2, ..., N$).

In the nonidentical retailers case, relative gaps are small when one of the following conditions satisfied:

- the added value at the warehouse is equal to the added value at the retailers ($(h_0, h_1, h_2) = (0.5, 0.5, 0.5)$),
- there is a short leadtime at the warehouse and a long retailer leadtime at the big retailer ($(l_0, l_1, l_2) = (1, 5, 5)$ or $(1, 1, 5)$).

Under most of the settings in nonidentical retailers case, $\epsilon\%$ is moderate or large. Hence, the relative gap between the lower bound and the optimal cost ($\epsilon^*\%$) had to be computed.

Next, we formulated a DP and solved it by value iteration algorithm for systems with two retailers. The results show that $\epsilon\%$ is not sufficient to interpret the behavior of $\epsilon^*\%$ *in general*; a scenario with a significant $\epsilon\%$ may have a $\epsilon^*\%$ that is (i) small, (ii) close to $\epsilon\%$, or (iii) significant and substantially less than $\epsilon\%$. The relative gap measures fall into

- item (ii) and (iii) when the retailers are identical and the coefficient of variation is high (2),
- item (i) when the small retailer has a negligible added value or both retailers have negligible added values,
- item (i) when there is an asymmetry between the retailers in terms of size,
- item (iii) when there is an asymmetry between the retailers in terms of coefficient of variation and size.

We reported $\epsilon^*\%$ values up to 15.27. We provided the first concrete evidence in the literature that the balance assumption may have a significant impact. Further, the results show that the LB heuristic policy is not robust; its performance is scenario dependent. In the scenarios that fall into item (iii), neither the lower bound estimates the optimal cost well nor the LB heuristic policy performs satisfactorily. The use of the balance assumption in the analysis under these settings is inappropriate.

Our results indicate the need for efficient and accurate heuristics for the control of one-warehouse multi-retailer systems. We also analyzed the behavior of the optimal policy in some scenarios, which gives valuable and interesting insights. As a follow-up research, we plan to develop new heuristics.

6.2 Newsboy Characterizations

The newsboy model is developed by Arrow *et al.* (1951). The main idea behind the model is the trade-off between holding stock and backlogging demand under demand uncertainty. When costs are assigned to the associated actions, these cost figures together with the demand distribution determines the optimal ordering policy in a single-echelon inventory system. The optimal policy is base stock policy and the optimal base stock level results in a probability of no-stockout that is equal to a ratio of the holding and penalty costs. We refer to such conditions for optimal policy parameters as newsboy characterizations.

Since 1951, the modelling concept introduced by Arrow *et al.* (1951) has been the dominating paradigm in the inventory theory. Although the systems studied have evolved into more complex structures and cost forms have been enriched, the main trade-off has been between the demand overage (e.g., holding cost) and underage costs (e.g., penalty costs).

In this dissertation, we first showed that newsboy characterizations hold for one-warehouse multi-retailer systems facing discrete demand under the balance assumption. Although it may not be surprising that the optimal base stock levels for the retailers satisfy newsboy characterizations when the demand is discrete, we did not directly expect such a behavior for the optimal order-up-to level of the warehouse. Second, we proved that the optimal reorder levels in a multi-echelon serial system with fixed batch quantities conform to newsboy characterizations.

The results for newsboy characterizations led us to an ambitious direction for further research. Since the newsboy characterizations hold in a variety of multi-echelon inventory systems, we plan to develop the underlying idea (an optimal inventory level at a stock point leads to a no-stockout probability at a down stream stock point facing demand, which is at least a certain ratio of the related cost parameters) to a framework for analyzing general multi-echelon inventory models. Even if the framework does not produce optimal solutions for general structures, it might still be a candidate for a good heuristic.

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Summary

In this dissertation, we consider the inventory control problem for a two-echelon distribution system that is composed of a single warehouse serving N retailers. The warehouse is replenished by an external supplier, which has ample stock. The retailers face the stochastic demand of the customers. Any unfulfilled demand is backlogged and a penalty cost is incurred. There are fixed leadtimes of ordering from the supplier and shipping to the retailers. We assume periodic review of the inventories and centralized control. The objective is to minimize the average expected holding and penalty costs of the system in the long-run. The one-warehouse multi-retailer model has important applications in inventory, manufacturing and hierarchial production planning contexts.

The system can be modeled by dynamic programming, but the resulting dynamic program (DP) is a multi-dimensional one where the dimension grows in the number of retailers and the warehouse leadtime. Unfortunately, no approach to decrease the dimension of the DP or to decompose the DP into simpler ones has been reported in the literature. Hence, the analysis of a one-warehouse multi-retailer system is intricate and the optimal policy is not known. However, relaxation of a constraint in the original optimization problem, which is referred to as the *balance assumption*, simplifies the analysis considerably and leads to structural results. The balance assumption is widely used in the analysis of one-warehouse multi-retailer systems and there is an established belief in the literature that it produces solutions of good quality. In addition, all heuristics for the control of one-warehouse multi-retailer systems under periodic review are based on the balance assumption. There are a few studies that investigate the impact of the balance assumption, but both the numerical experiments considered and the insights obtained are limited. This dissertation aims to quantify the effect of the balance assumption on the long-run average expected cost of the system, and to determine the settings under which this presupposition is justified/unjustified.

The balance assumption results in a relaxed model of the original optimization

problem, and the optimal policy for this relaxed model can be characterized. The optimal cost for the relaxed model is calculated through analytical functions and this cost is a lower bound (LB) for the true optimal cost (g^*). The optimal policy of the relaxed model is based on the balance assumption, and is not feasible for the original problem. However, it can be slightly modified to give a suboptimal, but feasible policy, which is referred to as the LB heuristic policy. The long-run average expected cost of the LB heuristic policy may be estimated by simulation, and this is an upper bound (UB) for g^*. For a given problem instance, g^* can be determined by solving the DP by a numerical technique. However, the curse of dimensionality compels one to consider discrete demand distributions with finite supports and limited input parameters (e.g., small number of retailers and short warehouse leadtime). Thus, it was necessary to extend some analytical results for one-warehouse multi-retailer systems, and we decided to conduct two numerical studies to investigate the impact of the balance assumption.

First, we analyzed a one-warehouse multi-retailer system with discrete demand distributions under the balance assumption. The optimality of base stock policies for the continuous demand has been extended to the discrete demand case. Further, we developed an efficient algorithm for the computation of the optimal control parameters and the optimal long-run average expected cost for the relaxed model. These results gave us the necessary analytical grounds for the calculation of LB for a given problem instance.

Next, we conducted an extensive numerical study where the relative gap between LB and UB, $\epsilon\% = 100\frac{UB-LB}{LB}$, is used as a measure to assess the impact of the balance assumption. The results give a clear picture of the input parameter (number of retailers, leadtimes, holding and penalty cost parameters, mean and coefficient of variation of demand) combinations resulting in small/moderate/large relative gaps. For the settings with small $\epsilon\%$, one may conclude that LB is an accurate proxy for g^* and the LB heuristic policy is a good heuristic. In the numerical study, we developed separate test beds for identical and nonidentical retailers. The test bed of 2000 scenarios with identical retailers shows that the main determinant of large $\epsilon\%$ is high coefficient of variation of demand. Relative gap ($\epsilon\%$) values up to 38.55 were found. The relative gap is small when one of the following conditions is satisfied:

- the coefficient of variation of demand is low (0.25,0.5) or moderate (1),
- the added value at the warehouse is very small compared to the added values at the retailers,
- the warehouse leadtime is short and the retailer leadtimes are long.

The test bed of 3888 scenarios for the nonidentical retailers case shows that the group of settings with guaranteed small $\epsilon\%$ is quite limited. The relative gaps are small when one of the following conditions is satisfied:

- the added value at the warehouse is equal to the added value at the retailers,
- there is a short warehouse leadtime and long retailer leadtimes,
- the leadtimes of the warehouse and the small retailer are short, and the leadtime of the big retailer is long.

Many of the settings in the nonidentical retailers case exhibit moderate or large $\epsilon\%$. 118 scenarios have $\epsilon\% > 25$ and the maximum $\epsilon\%$ reported is 186.86.

In both test beds, we also analyzed the relationship between the input parameters and $\epsilon\%$, which provided interesting insights. Further, we estimated the probability of imbalance (π) during the simulation runs for computing the UB, and we investigated the relation between π and $\epsilon\%$. The probability of imbalance is defined as the fraction of periods in which a negative quantity is allocated to a retailer under the optimal policy of the relaxed model. The results show that a high $\epsilon\%$ requires a high π, but a high π does not necessarily correspond to a high $\epsilon\%$.

When the relative gap is moderate or high, the assessment of the impact of the balance assumption based on the measure $\epsilon\%$ is limited because the relative position of g^* between the bounds LB and UB becomes important. Hence, we decided to study the settings with moderate/high $\epsilon\%$ further. We developed a DP for the control of the system. In order to be able to solve the resulting DP numerically, discrete demand distributions with finite supports are assumed. We developed two test beds (one for identical retailers, and another for nonidentical retailers case) to analyze the settings with moderate or high $\epsilon\%$, which were identified in the previous numerical study. Due to the curse of dimensionality, we were not able to consider the scenarios of the previous study as they are, but with restricted values for input variables. We selected one-period retailer demands that are distributed over four points, two retailers ($N = 2$), and short warehouse leadtimes (1 or 2 periods). For each scenario, LB, g^*, UB, $\epsilon\%$ and $\epsilon^*\%$ (the relative gap between LB and g^*), which is defined as $\epsilon^*\% = 100\frac{g^*-LB}{LB}$ are calculated. A scenario with a significant $\epsilon\%$ may have a $\epsilon^*\%$ that is (i) small, (ii) close to $\epsilon\%$, or (iii) significant and substantially less than $\epsilon\%$. In category (ii), the performance of the LB heuristic policy is good, but this heuristic is poor in categories (i) and (iii). The LB model leads to a proper proxy for the true optimal cost in category (i), but to an inaccurate proxy in categories (ii) and (iii). Observe that in category (iii) the balance assumption leads to neither a good heuristic policy nor a proper proxy for the optimal cost. We considered several input parameter settings and found that scenarios fall into

- category (ii) or (iii) when the retailers are identical and the coefficient of variation is high (2),
- category (i) when there is a small retailer with negligible added value or when both retailers have negligible added values,
- category (i) when there is asymmetry between the retailers in terms of size,
- category (iii) when there is asymmetry between the retailers in terms of coefficient of variation and size.

We found $\epsilon^{*}\%$ values up to 15.27 and 20 scenarios out of 73 have $\epsilon^{*}\% > 5$. These results are the first concrete evidence in the literature that the balance assumption may have a significant impact. In addition, the performance of the LB heuristic policy is scenario dependent. Hence, it is not a robust heuristic.

Our results indicate the need for more research on efficient and accurate heuristics for the control of one-warehouse multi-retailer systems. We also analyzed the behavior of the optimal policy in some scenarios with high $\epsilon^{*}\%$, which has led to valuable and interesting insights for what goes wrong under the LB heuristic policy.

Besides the results discussed up to now, we also derived *newsboy characterizations* for two basic models of multi-echelon production/inventory systems. Newsboy characterizations are equations/inequalities for optimal policy parameters that directly connect the probability of no-stockout at a stock point facing customer demand to the inventory holding and penalty cost parameters. These characterizations are appealing because they facilitate intuition development, and they are easy to convey to students and managers. First, we derived newsboy inequalities for the optimal base stock levels in a one-warehouse multi-retailer system with discrete demands and under the balance assumption. This result extends the newsboy equations that hold when the demand distributions are continuous. Next, we considered an N-echelon serial inventory system where stock flows from one echelon to the other in given fixed batch quantities. This system is a generalization of the Clark and Scarf model. We showed that the optimal reorder levels in this system satisfy newsboy equations (inequalities) when the demand distribution is continuous (discrete).

Samenvatting

In dit proefschrift beschouwen we de voorraadbeheersing voor een twee-echelon distributiesysteem bestaande uit een centraal voorraadpunt en N retailers. Het centraal voorraadpunt wordt bevoorraad door een externe leverancier, die altijd voldoende voorraad heeft. De retailers hebben te maken met stochastische vraag van klanten. Vraag die niet direct kan worden beantwoord, komt in de 'backlog' en wordt zo snel als mogelijk nageleverd. Voor na te leveren vraag worden boetekosten betaald. De doorlooptijden voor leveringen van de externe leverancier naar het centraal voorraadpunt en voor zendingen van het centraal voorraadpunt naar de retailers zijn vast. We veronderstellen discrete tijd. Dat wil zeggen dat de tijd is verdeeld in perioden van gelijke lengte. Voorraadhoogten worden bepaald aan het begin van iedere periode en op die momenten worden bestellingen geplaatst door zowel het centraal voorraadpunt als de retailers. Het doel is het minimaliseren van de gemiddelde voorraad- en boetekosten. Het twee-echelon distributiesysteem vindt haar toepassing binnen voorraadbeheersing, productiebesturing en hiërarchische planningsomgevingen.

Het voorraadbeheersingsprobleem kan worden gemodelleerd als Dynamisch Programmeringsprobleem, maar het resulterende DP probleem is multi-dimensionaal, waarbij de dimensie stijgt als functie van het aantal retailers en de levertijd van de externe leverancier. Er is jammer genoeg geen methode bekend om de dimensie van het DP probleem te verlagen of om het DP probleem te laten uiteen vallen in eenvoudigere, kleine problemen. Het voorraadbeheersingsprobleem voor het twee-echelon distributiesysteem is derhalve lastig. Echter, via de relaxatie van een bepaalde constraint, aangeduid als de *balansaanname*, vereenvoudigt de analyse van het oorspronkelijke probleem aanzienlijk en in dat geval kunnen mooie, analytische resultaten worden afgeleid. De balansaanname wordt vaak toegepast in analyses van distributiesystemen en onderzoekers in het algemeen denken dat deze relaxatie tot goede oplossingen leidt voor het oorspronkelijke probleem. Alle heuristieken die tot nog toe ontwikkeld zijn voor distributiesystemen met discrete tijd, zijn gebaseerd op de balansaanname. Er zijn een paar studies bekend waarin de impact van de balansaanname is onderzocht. Echter, de inzichten uit die studies en de uitgevoerde numerieke experimenten zijn beperkt. Het doel van dit proefschrift is om het effect van de balansaanname grondig te onderzoeken en om vast te stellen onder welke omstandigheden de balansaanname gerechtvaardigd is en wanneer niet.

De balansaanname resulteert in een gerelaxeerd model voor het oorspronkelijke probleem, en voor dit gerelaxeerde model kan een optimale strategie worden afgeleid. De optimale kosten voor het gerelaxeerde model kunnen aan de hand van analytische kostenfuncties worden berekend, en ze vormen een ondergrens (LB) voor de gemiddelde kosten van het oorspronkelijke model (g^*). De optimale strategie voor het gerelaxeerde model is gebaseerd op de balansaanname, en is geen toegelaten strategie voor het oorspronkelijke model. Echter, uit deze strategie is een toegelaten strategie te verkrijgen via een kleine aanpassing. Deze licht aangepaste strategie heet de LB heuristiek. De gemiddelde kosten van de LB heuristiek kunnen worden bepaald via simulatie en ze vormen een bovengrens UB voor g^*. Voor een gegeven probleeminstantie kan g^* worden bepaald via Dynamische Programmering. Echter, vanwege de rekencomplexiteit, kan men dit alleen doen voor problemen met discreet verdeelde vraag op een beperkt aantal punten en voor kleine waarden voor andere inputparameters (zoals het aantal retailers en de levertijd van de externe leverancier). Om die reden hebben we besloten om twee numerieke experimenten uit te voeren.

Als eerste analyseren we een twee-echelon distributiesysteem met discreet verdeelde vraag en we maken de balansaanname. We bewijzen voor dit systeem de optimaliteit van basestock strategieën. Dit is een uitbreiding van hetzelfde resultaat voor distributiesystemen met continu verdeelde vraag. Daarna ontwikkelen we een efficiënt algoritme voor de berekening van de optimale basestock niveau's en de optimale kosten voor het gerelaxeerde model. Daarmee zijn we in staat om de ondergrens LB te berekenen voor een willekeurige instantie.

Vervolgens voeren we een uitgebreid numeriek experiment uit waarin de relatieve afstand tussen LB en UB, $\epsilon\% = 100\frac{UB-LB}{LB}$, wordt gebruikt als maat voor de impact van de balansaanname. De resultaten geven een helder beeld van de combinaties van input parameters (aantal retailers, doorlooptijden, voorraad- en boetekostenparameters, gemiddelde en variatiecoefficient van de vraag) die leiden tot een kleine/gematigde/ grote relatieve afstand $\epsilon\%$. Voor instanties met kleine $\epsilon\%$ kan men concluderen dat de ondergrens LB een nauwkeurige benadering is voor g^* en dat de LB heuristiek een goede heuristiek is. Het numerieke experiment bestaat uit een testbed voor systemen met identieke retailers en een testbed voor niet-indentieke retailers. Het testbed voor identieke retailers bevat 2000 instanties en laat zien dat een hoge variatiecoefficient van de vraag de belangrijkste verklarende factor vormt voor een grote $\epsilon\%$ waarde. We hebben relatieve afstanden ($\epsilon\%$) tot en met 38.55 gevonden. De relatieve afstand is klein als aan één van de volgende voorwaarden wordt voldaan:

- de variatiecoefficient van de vraag is laag (0.25,0.5) of gematigd (1),
- de toegevoegde waarde bij het centraal voorraadpunt is erg klein ten opzichte van de toegevoegde waarde bij de retailers,
- de levertijd van de externe leverancier is kort en de doorlooptijden voor verzendingen naar de retailers zijn lang.

Het testbed met 3888 instanties voor niet-identieke retailers laat zien dat binnen dit testbed de groep instanties met een gegarandeerd lage $\epsilon\%$ waarde tamelijk beperkt is. De relatieve afstand is klein als aan één van de volgende voorwaarden wordt voldaan:

- de toegevoegde waarde bij het centraal voorraadpunt is gelijk aan de toegevoegde waarde bij de retailers,
- de levertijd van de externe leverancier is kort en de doorlooptijden voor verzendingen naar de retailers zijn lang,
- de levertijd van de externe leverancier en de doorlooptijd richting de kleine retailer zijn kort en de doorlooptijd richting de grote retailer is lang.

Voor de meeste instanties met niet-identieke retailers zijn gematigde tot grote $\epsilon\%$ waarden gevonden. In 118 scenarios vonden we $\epsilon\% > 25$ en de grootste gevonden $\epsilon\%$ waarde was 186.86.

In beide gevallen hebben we ook het gedrag van $\epsilon\%$ als functie van input parameters onderzocht, wat tot interessante resultaten heeft geleid. Verder hebben we de onbalanskans (π) bepaald tijdens de simulatieruns ter bepaling van UB, en we hebben de relatie tussen π en $\epsilon\%$ onderzocht. De onbalanskans is gedefinieerd als de fractie van de perioden waarin een negatieve hoeveelheid aan een retailer zouden worden gealloceerd onder de LB heuristiek, d.w.z. onder de strategie die je krijgt via de balansaanname. Dat heeft laten zien dat alleen een grote $\epsilon\%$ waarde wordt gevonden in instanties met een hoge π waarde, maar een hoge π waarde impliceert niet dat dan de $\epsilon\%$ waarde hoog is.

Als de relatieve afstand tussen LB en UB gematigd of hoog is, dan is er nog geen duidelijke conclusie over de impact van de balansaanname. De positie van

g^* tussen de grenzen LB en UB is dan belangrijk. Daarom hebben we besloten om instanties met gematigde/grote $\epsilon\%$ verder te onderzoeken. Voor de bepaling van de optimale kosten g^* via DP is een efficiënt programma ontwikkeld. Voor de vraagverdelingen zijn discrete verdelingen op een eindig aantal punten genomen. Op basis van de voorgaande studie is een serie instanties met identieke en niet-identieke retailers geselecteerd met een gematigde/grote relatieve afstand $\epsilon\%$. Ten behoeve van de rekencomplexiteit is daarbij gewerkt met vraagverdelingen op 4 punten, 2 retailers ($N = 2$) en korte levertijden van de externe leverancier (1 of 2 perioden). Voor elke instantie zijn LB, g^*, UB, en $\epsilon\%$ bepaald, en daarnaast ook $\epsilon^*\% = 100\frac{g^*-LB}{LB}$, de relatieve afstand tussen LB en g^*. Voor een instantie met significante $\epsilon\%$ onderscheiden we drie gevallen voor $\epsilon^*\%$: (i) $\epsilon^*\%$ is klein; (ii) $\epsilon^*\%$ is bijna even groot als $\epsilon\%$; (iii) $\epsilon^*\%$ is significant maar substantieel kleiner dan $\epsilon\%$. In geval (ii) geldt dat de prestatie van de LB heuristiek goed is, maar deze heuristiek is matig in de gevallen (i) en (iii). Het LB model leidt tot een goede benadering voor de optimale kosten g^* in geval (i), en tot een onnauwkeurige benadering in de gevallen (ii) en (iii). In geval (iii) leidt de balansaanname dus noch tot een goede heuristiek noch tot een goede benadering voor de optimale kosten g^*. We hebben vele instanties bekeken en onderzocht wanneer welk geval verkregen wordt. Een instantie resulteert in:

- geval (ii) of (iii) als de retailers identiek zijn en de variatiecoefficient van de vraag hoog (2) is,
- geval (i) als er een kleine retailer is met verwaarloosbaar kleine toegevoegde waarde of als beide retailers verwaarloosbaar kleine toegevoegde waarde hebben,
- geval (i) als de retailers asymmetrisch zijn met betrekking tot de grootte,
- geval (iii) als de retailers asymmetrisch zijn met betrekking tot de variatiecoefficient en de grootte.

We hebben $\epsilon^*\%$ waarden tot en met 15.27 gevonden en bij 20 van de 73 instanties was $\epsilon^*\% > 5$. Deze resultaten vormen het eerste concrete bewijs in de literatuur van de significante impact die de balansaanname kan hebben. Bovendien tonen deze resultaten aan dat de kwaliteit van de LB heuristiek instantie-afhankelijk is. Het is dus geen robuuste heuristiek.

Onze resultaten tonen de noodzaak aan van meer onderzoek naar efficiënte, nauwkeurige heuristieken voor twee-echelon distributiesystemen. We hebben ook de structuur onderzocht van optimale strategieën voor instanties met een hoge $\epsilon^*\%$. Hieruit hebben we geleerd wat er onder de LB heuristiek mis gaat, en dat heeft aanwijzingen opgeleverd voor de ontwikkeling van nieuwe heuristieken.

Naast de bovenstaande resultaten met betrekking tot de balansaanname, hebben we *Newsboy karakteriseringen* afgeleid voor twee multi-echelon modellen. Newsboy karakteriseringen laten het verband zien tussen optimale basestock niveau's en stockout kansen bij de meest stroomafwaartse voorraadpunten waar de vraag van klanten plaats vindt. Deze karakteriseringen dragen bij aan het begrip van wat er onder optimale voorraadbeheersing gebeurt, en ze zijn relatief eenvoudig uit te leggen aan studenten en managers. We hebben allereerst Newsboy ongelijkheden afgeleid voor de optimale basestock niveau's in een twee-echelon distributiesysteem met discrete vraag en onder de balansaanname. Deze resultaten vormen een uitbreiding van reeds bestaande resultaten voor distributiesystemen met continue vraag. Vervolgens hebben we een N-echelon, serieel voorraadsysteem bestudeerd waarbij voor ieder voorraadpunt een vaste bestelgrootte is gegeven. Dit systeem is een directe generalisatie van het Clark-en-Scarf model. We hebben aangetoond dat de optimale basestock niveau's voldoen aan Newsboy vergelijkingen (ongelijkheden) in het geval van continu (discreet) verdeelde vraag.

Curriculum Vitae

Mustafa Kemal Doğru was born in Besni, Turkey on September 14th, 1977. He graduated from İzmir American Collegiate Institute and İzmir Science High School in 1992 and 1995, respectively. In 1995, he was admitted to the Department of Industrial Engineering at Middle East Technical University. After obtaining a BS degree in 1999, he continued his graduate education at the same department where he also worked as a teaching assistant.

In 2001, he started as a research assistant at the Department of Technology Management in Operations Planning, Accounting, and Control group under the supervision of A.G. de Kok and G.J.J.A.N. van Houtum. At the same time, he was enrolled to the PhD program of Beta Research School. His research topic is in the field of multi-echelon inventory theory. The research activities conducted between October 2001–November 2005 led to this PhD dissertation.

As extracurricular activities, he was the secretary of the board of Beta Research School PhD Students Council for one year. He served in the board of PromoVE (an association that protects the interests of PhD students who are connected to TU/e), and acted as the treasurer and foreign PhD students contact person.