

Positive operator-valued measures and phase-space representations

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Positive Operator-Valued Measures and Phase-Space Representations

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Contents

Chapter 1. Introduction	7
1. Context	7
2. Probability distributions on the outcome space	8
3. Motivation	8
4. Results	10
5. Outline	11
Chapter 2. Preliminaries	13
1. Conventions, terminology and notation	13
2. Tensor product of two Hilbert spaces	13
3. Spaces of operators on a Hilbert space	14
4. Topologies on $\mathcal{B}_\infty(\mathbf{H})$ and $L_\infty(\mu)$	14
Chapter 3. Positive operator-valued measures	17
1. Introduction	17
2. Conventions, terminology and notation	17
3. Preliminaries	18
4. Naimark's theorem	19
5. Separability	21
6. Operator densities	23
7. Integration with respect to a POVM	26
8. Dominance of POVMs	29
9. Four remarks about the definition of \leftarrow	29
10. Dominance between PVMs	32
11. Isomorphic POVMs	33
12. Unitary equivalence	34
13. Image measures	35
14. Cones generated by POVMs	35
15. Maximal PVMs	37
16. Examples of maximal PVMs	39
17. Maximal POVMs	39
18. Bargmann POVM	46
19. Susskind-Glogower phase POVM	46
20. Density of the span of the range of a POVM	47
21. Prototypical POVMs	50
22. Multiplicity theory	51
23. Bounded subnormal operators	55
24. Restriction algebra	56
25. Constructing POVMs from PVMs	57
26. POVMs on Federer metric spaces	61
27. Integration of unbounded functions with respect to a POVM	64
Chapter 4. Wigner and Husimi representations, and in between	69

1. Introduction	69
2. Conventions and notation	69
3. Wigner representation	70
4. A family of representations interpolating between the Wigner and Husimi representations	72
5. The range of \mathbf{W}_s	74
6. The spaces \mathbf{B}_+ and \mathbf{B}_-	75
7. Characterization of $\mathbf{W}_s(\mathbf{B}_+)$ and $\mathbf{W}_s(\mathbf{B}_-)$	76
Chapter 5. Phase-space analogues for quantum mechanical expectation values	79
1. Introduction	79
2. Wigner representation	80
3. Existence of phase-space analogues for quantum mechanical expectation values	80
4. Non-uniformity of approximation of expectation values	81
5. A method to approximate expectation values	82
Chapter 6. Extended Bargmann space	87
1. Introduction	87
2. Hermite polynomials	87
3. Laguerre polynomials	89
4. Relation between Hermite and Laguerre polynomials	92
5. Analytic and Harmonic functions	94
6. Extended Bargmann space	95
Appendix A. Miscellaneous results	99
1. Measurable covers and partitions	101
2. Hilbert-Schmidt operators	101
3. Trace-class operators	101
4. Convergence and σ -additivity of measures	101
5. Monotone convergence theorem and a partial converse	102
6. Radon-Nikodym theorem	102
7. Strong topological dual of $L_1(\Omega, \Sigma, \mu)$	103
Appendix B. A criteria for the density of subspaces	105
Appendix C. Weak-star topology on dual Banach spaces	107
Appendix D. Bounded sesquilinear forms on Hilbert spaces	111
Appendix E. Hilbertian semi-norms	113
Appendix F. Bargmann space	117
Appendix G. C^* -algebras	119
1. Von Neumann algebras	119
2. Commutative von Neumann algebras	119
3. Maximal commutative von Neumann algebras	119
Appendix H. Operator ranges	121
Appendix I. Measurable Banach space-valued functions	123
Appendix J. Carleman operators	127
1. Generating vectors	127

Appendix K. Application of Gauss's theorem	129
1. On the real line	129
2. On the complex plane	130
3. Gaussian convolution on $L_p(\mathbb{R}^m)$	132
Appendix L. Integral operators with a Gaussian kernel	135
1. Calculations	135
2. The integral operators \mathcal{S}_{pqr}	136
Appendix M. Gelfand-Shilov space $S_{1/2}^{1/2}$.	139
1. Introduction	139
2. The spaces H_+ and H_- .	139
3. The spaces $H_+^{(s)}$ and $H_-^{(s)}$	140
4. Duality between $H_-^{(s)}$ and $H_+^{(s)}$	141
5. Approximation in H_-	141
Appendix N. Estimates	143
1. Estimates for Bessel functions on $(0, \infty)$	143
2. Estimates for the Gamma function	146
3. Estimates for Laguerre polynomials on $(0, \infty)$	146
Appendix O. Cambell-Baker-Hausdorff formulas	149
1. Introduction	149
2. $e^{[\mathcal{A}, \cdot]}$ if \mathcal{A} is nilpotent	149
3. A generalization	150
4. If operators \mathcal{A} and \mathcal{B} are skew-adjoint	151
5. If operators \mathcal{A} and \mathcal{B} commute with $[\mathcal{A}, \mathcal{B}]$.	151
Bibliography	153
Index	157
Acknowledgement	159
Samenvatting	161
Curriculum Vitae	163

CHAPTER 1

Introduction

1. Context

In the standard formalism of quantum mechanics as formulated by Dirac [36] and von Neumann [106], a measurement with outcome space Ω corresponds to a spectral measure M concentrated on Ω and acting on a complex separable Hilbert space \mathbf{H} . States of a quantum mechanical system are described mathematically by density operators ρ , which are non-negative trace-class operators on \mathbf{H} with trace 1. The set of all density operators, the ‘state space,’ is denoted by $\mathbf{S}(\mathbf{H})$. The following is postulated:

(P) The number $\text{Tr}(\rho M(\Delta))$, i.e. the trace of the composition of operators ρ and $M(\Delta)$, is to be interpreted as the probability that in state ρ the outcome of the measurement lies in Δ .

Two spectral measures E and F acting on the same Hilbert space, and defined on the Borel subsets of outcome spaces Ω_1 and Ω_2 respectively, can be combined to get a spectral measure $E \otimes F$, defined on the Borel subsets of $\Omega_1 \times \Omega_2$ by $E \otimes F(\Delta_1 \times \Delta_2) = E(\Delta_1)F(\Delta_2)$ (and unique extension), if and only if E and F commute. Von Neumann shows (Section III.3 in [106]) that this condition is necessary and sufficient in order for the quantities corresponding to E and F to be simultaneously measurable with arbitrary high accuracy. Such measurements are not always desirable because they are very incomplete in the sense that to each density operator ρ corresponds a large class $[\rho]_M$ of states ρ' satisfying $\text{Tr}((\rho - \rho')M(\Delta)) = 0$ for all Δ . In other words: The range of M , considered as a subspace of the vector space of bounded self-adjoint operators, has a large ‘orthogonal complement’ in the space of self-adjoint trace-class operators.

More recent investigations [74, 73], [31] (among others) have led to the so-called ‘operational approach to quantum measurement.’ This theory is about the compound system formed by object and measurement instrument, and the interaction between these two parts plays an essential role. (See e.g. [58], [55], [30], [15] and [17].) A measurement is described mathematically by an operator-valued measure M , defined on a σ -field Σ of subsets of a set Ω and taking values in the set of non-negative bounded operators on a separable complex Hilbert space, and normalized by the condition $M(\Omega) = \mathcal{I}$, the identity operator. A measure M with these properties, is called a normalized positive operator-valued measure, which we abbreviate to ‘POVM.’ A POVM M whose range consists of projection operators only, is called a projection-valued measure, which we abbreviate to ‘PVM.’ Measurements corresponding to POVMs are called generalized measurements, and measurements corresponding to PVMs (spectral measures in particular) are called simple measurements. Many experiments performed in practice are generalized measurements: For example [83]. Again (P) is assumed, but it is no longer demanded that M has only projection operators as values. Now there is the possibility that the numbers $\text{Tr}(\rho M(\Delta))$, $\Delta \in \Sigma$ determine ρ completely: I.e. $[\rho]_M$ consists of ρ only. Such measurements are called complete measurements, and the corresponding POVMs are called complete. Eight-port optical homodyning (see e.g. [86, 87], [47], [34]) is an example; The POVM corresponding to this measurement is called the

Bargmann measure in this thesis, and it will be discussed a.o. in Sections 3.2 and 4 of this introduction.

2. Probability distributions on the outcome space

To a POVM M , defined on a σ -field Σ of subsets of an outcome space Ω , and taking values in the set of non-negative bounded operators on \mathbf{H} , corresponds a function

$$V_M: \mathbf{S}(\mathbf{H}) \rightarrow \text{Prob}(\Omega, \Sigma), \quad \rho \mapsto V_M[\rho],$$

from $\mathbf{S}(\mathbf{H})$ into $\text{Prob}(\Omega, \mathfrak{B})$, the set of probability measures on (Ω, Σ) , defined by

$$(1) \quad V_M[\rho](\Delta) = \text{Tr}(\rho M(\Delta)).$$

Transformation V_M commutes with taking finite (and countable) convex combinations. Conversely, if $V: \mathbf{S}(\mathbf{H}) \rightarrow \text{Prob}(\Omega, \Sigma)$ commutes with taking finite convex combinations, then a unique POVM M exists such that $V = V_M$. This follows from the following two facts:

- If $V: \mathbf{S}(\mathbf{H}) \rightarrow \text{Prob}(\Omega, \Sigma)$ commutes with taking finite convex combinations and $\Delta \in \Sigma$, then $L_\Delta: \mathbf{S}(\mathbf{H}) \rightarrow [0, 1]$, defined by $L_\Delta[\rho] = V[\rho](\Delta)$, commutes with taking finite convex combinations;
- To every function $L: \mathbf{S}(\mathbf{H}) \rightarrow [0, 1]$ that commutes with taking finite convex combinations, corresponds a unique $\mathcal{B} \in \mathbf{B}_+(\mathbf{H})$ such that $L[\rho] = \text{Tr}(\rho \mathcal{B})$ for all $\rho \in \mathbf{S}(\mathbf{H})$.

The properties of the mapping V_M are investigated in [16]. From the separability of \mathbf{H} follows the existence of a probability measure μ on Ω with the same sets of measure zero as M . From the Radon-Nikodym theorem it follows that probability measure $V_M[\rho]$ has a probability density $p_\rho: \Omega \rightarrow [0, \infty)$ with respect to μ on Ω :

$$V_M[\rho](\Delta) = \int_{\Delta} p_\rho(x) \mu(dx).$$

For some POVMs a function $\varphi(x): \Omega \rightarrow [0, \infty)$ exists such that

$$(\forall \rho \in \mathbf{S}(\mathbf{H})) (\forall_\mu x \in \Omega) \quad p_\rho(x) \leq \varphi(x),$$

where $\forall_\mu x \in \Omega$ abbreviates ‘for μ -almost all $x \in \Omega$.’ In that case there is (according to Lemma 24 below) a family (\mathcal{M}_x) of bounded non-negative operators \mathcal{M}_x such that

$$(\forall \rho \in \mathbf{S}(\mathbf{H})) (\forall_\mu x \in \Omega) \quad p_\rho(x) = \text{Tr}(\mathcal{M}_x \rho).$$

Family (\mathcal{M}_x) is called an operator density of POVM M with respect to measure μ . If, for example, $\Omega = \mathbb{N}$ and Σ is the collection of all subsets of Ω , then $p_\rho(x) \leq \varphi(x)$ with $\varphi(x) = 1/\mu(\{x\})$, and $\mathcal{M}_x = M(\{x\})/\mu(\{x\})$. An example of a POVM for which no operator density exists is the spectral measure of the ‘position operator’ \mathcal{Q} on $L_2(\mathbb{R})$, which is densely defined on its domain by $\mathcal{Q}f(x) = xf(x)$. More generally, the spectral measure of a self-adjoint operator on $L_2(\mathbb{R})$ has an operator density if, and only if, it has pure point spectrum; In that case the operator density (with respect to the counting measure) consists of the orthogonal projections on the eigenspaces.

3. Motivation

The investigations reported in this thesis were motivated by the following questions:

- (M i) How can we compare the amount of information about the density operator that can be obtained by two distinct POVMs?

- (M ii) Is there a generalization to the case of generalized measurements, of the pre-order relation on the set of simple measurements corresponding to the partial order (i.e. inclusion) on the set of von Neumann algebras generated by the spectral measures?
- (M iii) Is there a generalization of Dirac's concept of a complete set of observables to the case of generalized measurements?
- (M iv) Which POVMs correspond to complete measurements?

The last question will not be answered (completely) in this thesis. The second and the third questions are answered in a (sufficiently) general context: Namely for POVMs on countably generated measurable spaces, which includes all POVMs on the Borel subsets of \mathbb{R}^m with $m \in \mathbb{N}$. This includes also POVMs which have no operator density. (This provides at least a partial answer to the first question.)

3.1. Reconstruction of the density operator. If M is a POVM defined on the subsets of $\Omega = \mathbb{N}$ and having operators on \mathbf{H} as values, and (\mathcal{M}_k) , related to M as in Section 2 by $\mathcal{M}_k = M(\{k\})$, is a weak-star Schauder basis (this concept is defined in [97], Definitions 13.2 and 14.2) of the Banach space of bounded operators on \mathbf{H} , then

$$(2) \quad \rho = \sum_{k \in \Omega} p_\rho(k) \mathcal{M}^k,$$

where $p_\rho(k) = \text{Tr}(\mathcal{M}_k \rho)$, and (\mathcal{M}^k) is the basis (whose existence is guaranteed by [97], Theorem 14.1) of the Banach space of trace-class operators on \mathbf{H} satisfying $\text{Tr}(\mathcal{M}^k \mathcal{M}^\ell) = \delta_\ell^k$ for all $k, \ell \in \Omega$. (The expression δ_ℓ^k is called the Kronecker delta: It is equal to one if $k = \ell$ and zero otherwise.)

If M does not correspond to a complete measurement, then a complete reconstruction of ρ is impossible. If (\mathcal{M}_k) is a weak-star Schauder basis of a weak-star closed linear subspace of the Banach space of bounded operators on \mathbf{H} , then a partial reconstruction of ρ similar to (2) is possible.

In this thesis the possibility of a reconstruction of ρ similar to (2) is considered for POVMs for which the above requirements are not satisfied. The investigations were motivated by following questions:

- (R i) Given a particular POVM M corresponding to a complete measurement, is there a simple formula expressing the numbers $\text{Tr}(\rho B)$, for bounded non-negative operators B which are not in the range of M , in terms of the numbers $\text{Tr}(\rho M(\Delta))$, where Δ is contained in the σ -field on which M is defined. A similar question can be formulated for a particular phase-space representation instead of a particular measurement.
- (R ii) Is a reconstruction of ρ similar to (2) possible if (\mathcal{M}_k) does have a weak-star dense linear span, but is not a weak-star basis?
- (R iii) Is a reconstruction of ρ similar to (2) possible if, for example, $\Omega = \mathbb{C}$ and M is non-atomic?

The first three questions are considered, in this thesis, only for one particular POVM, namely the Bargmann measure which will be discussed below. For this POVM the outcome space is \mathbb{C} . Because this is a complete POVM, the possibility of a complete reconstruction of ρ (in stead of a partial reconstruction) is considered. We give a positive answer to question (R i) for this POVM and for the case of a particular family of phase-space representations.

3.2. Bargmann measure. The investigations reported in this thesis were motivated by POVM $M^{(\text{Bargmann})}$ on the Borel subsets $\mathfrak{B}_{\mathbb{C}}$ of \mathbb{C} , and taking values in the operators on $L_2(\mathbb{R})$. This POVM is determined by the probability densities $p_\rho: \mathbb{C} \rightarrow [0, \infty)$ with respect to measure μ given by

$$p_\rho^{(\text{Bargmann})}(z) = (g_z, \rho g_z), \quad \mu(dz) = \pi^{-1} d\text{Re}(z) d\text{Im}(z),$$

where (g_z) is the family of normalized (complex conjugate) coherent state vectors (also called Gabor functions [48]):

$$g_z(x) = \lambda(z) e^{\sqrt{2}zx - x^2/2},$$

where $\lambda(z)$ realizes the normalization. We have

$$(\forall z \in \mathbb{C}) \quad \mathcal{M}_z^{(\text{Bargmann})} = g_z \otimes g_z,$$

where $g_z \otimes g_z$ is the operator of orthogonal projection on the one dimensional space generated by the vector g_z (of unit length). In [9] it has been shown that

$$M^{(\text{Bargmann})}(\mathbb{C}) = \int_{\mathbb{C}} \mathcal{M}_z^{(\text{Bargmann})} \mu(dz) = \mathcal{I},$$

or equivalently,

$$(\forall \rho \in \mathfrak{S}(\mathbb{H})) \quad \int_{\mathbb{C}} p_\rho(z) \mu(dz) = 1.$$

4. Results

In [37] (and in Theorem 68 below) it has been shown that the concept of maximality of POVMs, which was introduced in [80], generalizes the concept of a complete set of commuting self-adjoint operators (which is a mathematical interpretation of the concept of a complete set of observables introduced in [36]). We have been able to give a useful characterization (Theorem 77) of the set of maximal POVMs, from which it follows e.g. that $M^{(\text{Bargmann})}$ is a maximal POVM: In particular, for each relation of $M^{(\text{Bargmann})}$ with another POVM of the form

$$(\forall_\mu z \in \mathbb{C}) \quad \mathcal{M}_z^{(\text{Bargmann})} = \int_{\mathbb{R}^m} K(z, x) \mathcal{N}_x \nu(dx),$$

where

- $m \in \mathbb{N}$ and ν is a finite positive Borel measure on \mathbb{R}^m ;
- (\mathcal{N}_x) is a family of non-negative bounded operators;
- $K: \mathbb{C} \times \mathbb{R}^m \rightarrow [0, \infty)$ satisfies $\int_{\mathbb{C}} K(z, x) \mu(dz) = 1$ for ν -almost all $x \in \mathbb{R}^m$,

there is also a relation of the form

$$(\forall_\nu x \in \mathbb{R}^m) \quad \mathcal{N}_x = \int_{\mathbb{C}} \tilde{K}(x, z) \mathcal{M}_z^{(\text{Bargmann})} \mu(dz),$$

where $\tilde{K}: \mathbb{R}^m \times \mathbb{C} \rightarrow [0, \infty)$ satisfies $\int_{\mathbb{R}^m} \tilde{K}(x, z) \nu(dx) = 1$ for μ -almost all $z \in \mathbb{C}$.

We have exemplified the characterization of maximality by a POVM that has an operator density. Our result is, however, applicable to the (more general) class of POMVs on countably generated measurable spaces. An answer to question (M ii) is given in the same context.

Questions (R ii) and (R iii) are answered only for the case of the Bargmann measure: We prove (Theorem 161) that for the case of the Bargmann measure, a reconstruction

of the density operator as in (2) is not possible: A family (\mathcal{M}^z) of trace-class operators such that

$$\rho = \int_{\mathbb{C}} p_{\rho}^{(\text{Bargmann})}(z) \mathcal{M}^z \mu(dz)$$

does not exist. However similar approximative reconstructions of ρ are possible: There exist families (\mathcal{M}_n^z) of trace-class operators such that

$$(3) \quad \rho = \lim_{n \rightarrow \infty} \int_{\mathbb{C}} p_{\rho}^{(\text{Bargmann})}(z) \mathcal{M}_n^z \mu(dz)$$

weakly in the Banach space of trace-class operators. This provides a positive answer to question (R i) for the case of the Bargmann measure: We show (Theorem 164) that for every bounded operator \mathcal{B} and density operator ρ , we can approximate $\text{Tr}(\mathcal{B}\rho)$ (with arbitrary high accuracy) by integrals over \mathbb{C} in terms of probability density $p_{\rho}^{(\text{Bargmann})}$: Let \mathfrak{c} be an infinitely differentiable function on \mathbb{C} with compact support such that $\mathfrak{c}(0) = 1$. For every bounded operator \mathcal{B} on $L_2(\mathbb{R})$,

$$\text{Tr}(\mathcal{B}\rho) = \lim_{n \rightarrow \infty} \int_{\mathbb{C}} p_{\rho}^{(\text{Bargmann})}(z) \text{Tr}(\mathcal{B}\mathcal{M}_z^{(n)}) \mu(dz)$$

where

$$\mathcal{M}_z^{(n)} = \int_{\mathbb{C}} \mathfrak{c}\left(\frac{w}{n}\right) \frac{e^{-|w|^2}}{(g_{z+w}, g_{z-w})} g_{z-w} \otimes g_{z+w} \mu(dw), \quad n \in \mathbb{N}.$$

(The operators $\mathcal{M}_z^{(n)}$ depend on \mathfrak{c} . For each choice of \mathfrak{c} we get a solution of (3).) We have obtained similar results for a certain family of phase space representations: The Bargmann measure is closely related to the so-called Husimi representation, representing density operators ρ by functions $z \mapsto \pi^{-1}(g_z, \rho g_z)$ on \mathbb{C} . For a family of phase-space representations interpolating between the Husimi and the Wigner representations, we present a unifying approach to approximate $\text{Tr}(\mathcal{B}\rho)$ by integrals over \mathbb{C} in terms of functions on phase-space. This provides, for these particular cases, a positive answer to question (R i).

We have been able to give characterizations (Chapter 4, Section 7) of certain subspaces of the space of Hilbert-Schmidt operators whose integral kernels are contained in the Gelfand-Shilov space $\mathbf{S}_{1/2}^{1/2}(\mathbb{R}^2)$ (these subspaces are considered e.g. by De Bruijn in [32]) by means of growth conditions on the analytic continuation of the functions on phase-space corresponding to one of the phase-space representations considered above. For the special case of the Husimi representation, this characterization of operators \mathcal{B} can be put into the form

$$(\exists M, A > 0)(\forall z, w \in \mathbb{C}) \quad |(g_{z+w}, \mathcal{B}g_{z-w})| \leq M e^{-A(|z|^2 + |w|^2)}.$$

5. Outline

This thesis is organized as follows. In Chapter 3 we investigate the mathematical properties of a pre-order relation, denoted by \leftarrow , on the collection of POVMs which was introduced in [80] and further investigated in [37]. We show that \leftarrow provides the generalization considered in question (M ii) above, and that the related concept of maximality provides the generalization considered in question (M iii). In [80] and [37] a characterization of the set of discrete maximal POMVs is given. This result is generalized to the set of countably generated POVMs. (This includes the POVMs defined on the Borel subsets of \mathbb{R}^m .) To exemplify the theoretical results, we consider the Bargmann measure and the Susskind-Glogower phase POVM, which is a POVM defined on the Borel subsets of interval $[0, 2\pi)$.

In Chapter 4, we consider the definition and elementary properties of a family of phase-space representations, interpolating between the Wigner and the Husimi representations. We characterize particular subspaces of the range of these transformation in terms of the growth conditions on the analytic continuations of functions on phase-space. In Chapter 5, we give explicit formulas for the approximation of quantum mechanical expectation values by integrals over phase-space. This provides a positive answer to question (R i) for the case of the phase-space representations considered in Chapter 4.

In Chapter 6 we consider an extension of the usual orthonormal basis of the Bargmann space, which consists of properly normalized analytic monomials, to an orthonormal basis of a functional Hilbert space, densely contained in a space of square integrable function classes on the complex plane. These basis functions are, up to a constant factor, the Wigner functions of the operators $h \mapsto (\varphi_\ell, h)\varphi_k$ on $L_2(\mathbb{R})$, where (φ_k) is the Hermite basis, and have been investigated earlier in the context of Weyl-(de)quantization: [27, 28] and [56].

CHAPTER 2

Preliminaries

1. Conventions, terminology and notation

A sesquilinear form on a vector space over \mathbb{C} is linear in the second argument.

An operator \mathcal{A} on a Hilbert space \mathbf{H} is a linear transformation $\mathcal{A}: \mathfrak{D}(\mathcal{A}) \rightarrow \mathbf{H}$, where domain $\mathfrak{D}(\mathcal{A})$ is a linear subspace of \mathbf{H} . The linear hull of a subset S of a vector space is denoted by $\text{span}(S)$.

An isometry from a normed space \mathbf{X} to a normed space \mathbf{Y} is a mapping $\mathbf{V}: \mathbf{X} \rightarrow \mathbf{Y}$ such that $\|\mathbf{V}[x]\|_{\mathbf{Y}} = \|x\|_{\mathbf{X}}$ for all $x \in \mathbf{X}$. Note that a linear isometry $\mathbf{V}: \mathbf{H}_1 \rightarrow \mathbf{H}_2$ from a Hilbert space \mathbf{H}_1 to another Hilbert space \mathbf{H}_2 is a bounded operator satisfying $\mathbf{V}^*\mathbf{V} = \mathcal{I}_1$, the identity on \mathbf{H}_1 , and that consequently $\mathbf{V}\mathbf{V}^*$ is the operator of orthogonal projection on $\text{range}(\mathbf{V})$. A unitary mapping from a Hilbert space \mathbf{X} onto a Hilbert space \mathbf{Y} is a linear isometry from \mathbf{X} onto \mathbf{Y} . A contraction from a normed space \mathbf{X} to a normed space \mathbf{Y} is a mapping $\mathbf{V}: \mathbf{X} \rightarrow \mathbf{Y}$ such that $\|\mathbf{V}[x]\|_{\mathbf{Y}} \leq \|x\|_{\mathbf{X}}$ for all $x \in \mathbf{X}$.

By an algebra we mean a linear algebra and by a subalgebra we mean a linear subalgebra. Related concepts that are used in this paper are defined in Appendix G.

The σ -field generated by the open sets of a topological space is called the Borel σ -field. Elements of a Borel σ -field are called Borel subsets of the topological space. Let Σ be a σ -field of subsets of a set Ω . A measurable partition of Ω is a family (ω_i) of pairwise disjoint elements of Σ such that $\Omega = \cup_i \omega_i$. The contraction of a measure μ defined on a σ -field Σ by an element $\omega \in \Sigma$ is the measure $\mu|_{\omega}$ on Σ , defined by $\mu|_{\omega}(\Delta) = \mu(\omega \cap \Delta)$. For (operator-valued) measures μ_1 and μ_2 defined on the same σ -field Σ we write $\mu_1 \ll \mu_2$ to denote that μ_1 is absolutely continuous with respect to μ_2 : If $\Delta \in \Sigma$ and $\mu_2(\Delta) = 0$ then $\mu_1(\Delta) = 0$. We use the following abbreviation: ‘ $\forall_{\mu} x \in \Omega$ ’ means ‘for μ -almost all $x \in \Omega$.’

If φ_n , $n \in \mathbb{I}$ is an orthonormal basis of $L_2(\mathbb{R})$ and $m \in \mathbb{N}$, then φ_n , with $n \in \mathbb{I}^m$, is defined almost everywhere on \mathbb{R}^m by

$$\varphi_n(x) = \prod_{j=1}^m \varphi_{n_j}(x_j).$$

2. Tensor product of two Hilbert spaces

For Hilbert spaces \mathbf{H}_1 and \mathbf{H}_2 the algebraic sesquilinear tensor product is denoted by $\mathbf{H}_1 \otimes \mathbf{H}_2$. This is a pre-Hilbert space with respect to its usual inner-product (See e.g. Section 3.4 of [108]), and its completion is denoted by $\mathbf{H}_1 \hat{\otimes} \mathbf{H}_2$. For a Hilbert space \mathbf{H} , we identify $\mathbf{H} \hat{\otimes} \mathbf{H}$ with the space of Hilbert-Schmidt operators on \mathbf{H} . For example: $f \otimes g$, where $f, g \in \mathbf{H}$, is identified with the operator $f \otimes g$ on \mathbf{H} defined by $f \otimes g[h] = (g, h)_{\mathbf{H}} f$.

3. Spaces of operators on a Hilbert space

Let \mathbf{H} be a Hilbert space. We use the following notation:

- $\mathbf{B}_\infty(\mathbf{H})$: The bounded operators on \mathbf{H} .
- $\mathbf{B}_2(\mathbf{H})$: The Hilbert-Schmidt operators on \mathbf{H} .
- $\mathbf{B}_1(\mathbf{H})$: The trace-class operators on \mathbf{H} .
- $\mathbf{B}_0(\mathbf{H})$: The compact operators on \mathbf{H} .
- $\mathbf{B}_{00}(\mathbf{H})$: The operators with finite dimensional range.
- $\mathbf{B}_+(\mathbf{H})$: The non-negative operators in $\mathbf{B}_\infty(\mathbf{H})$.
- \mathbf{O}_p : The orthogonal projection operators contained in subset \mathbf{O} of $\mathbf{B}_\infty(\mathbf{H})$.
- \mathbf{O}' : The commutant of a subset \mathbf{O} of $\mathbf{B}_\infty(\mathbf{H})$.

We have:

$$\mathbf{B}_{00}(\mathbf{H}) \subset \mathbf{B}_0(\mathbf{H}) \subset \mathbf{B}_1(\mathbf{H}) \subset \mathbf{B}_2(\mathbf{H}) \subset \mathbf{B}_\infty(\mathbf{H}).$$

The space \mathbf{B}_2 is a Hilbert space with inner product

$$(\mathcal{A}, \mathcal{B})_{\mathbf{B}_2} = \text{Tr}(\mathcal{A}^* \mathcal{B}),$$

the trace of the composition $\mathcal{A}^* \mathcal{B}$ of the Hilbert-Schmidt operators \mathcal{A}^* and \mathcal{B} . The space $\mathbf{B}_1(\mathbf{H})$ consists of all compositions of pairs of Hilbert-Schmidt operators, and is a Banach space with respect to the trace norm

$$\|\mathcal{T}\|_1 = \text{Tr} |\mathcal{T}|, \quad |\mathcal{T}| = \sqrt{\mathcal{T}^* \mathcal{T}}.$$

A bounded operator \mathcal{B} acts as a linear form on $\mathbf{B}_1(\mathbf{H})$ through $\mathcal{T} \mapsto \text{Tr}(\mathcal{T} \mathcal{B})$. This linear form is continuous; its norm is $\|\mathcal{B}\|_\infty$. Conversely, every continuous linear form on $\mathbf{B}_1(\mathbf{H})$ is of this form.

For $n \in \mathbb{N}$, the inner product on $\mathbf{L}_2(\mathbb{R}^n)$ is defined by $(f, g) = \int_{\mathbb{R}^n} \overline{f(x)} g(x) dx$. Define $\mathbf{K}: \mathbf{L}_2(\mathbb{R}) \hat{\otimes} \mathbf{L}_2(\mathbb{R}) \rightarrow \mathbf{L}_2(\mathbb{R}^2)$ by $\mathbf{K}[f \otimes g](x, y) = f(x) \overline{g(y)}$ and linear and isometric extension. Hilbert-Schmidt operators on $\mathbf{L}_2(\mathbb{R}^n)$ are integral operators with an integral kernel from $\mathbf{L}_2(\mathbb{R}^2)$. The unitary operator \mathbf{K} maps Hilbert-Schmidt operators onto their integral kernels. We will identify $\mathbf{L}_2(\mathbb{R}) \hat{\otimes} \mathbf{L}_2(\mathbb{R})$ with $\mathbf{L}_2(\mathbb{R}^2)$ through \mathbf{K} .

Reference: The above spaces of operators are introduced, for example, in [23]. Information about integral kernels of Hilbert-Schmidt operators can be found, for example, in [108].

4. Topologies on $\mathbf{B}_\infty(\mathbf{H})$ and $\mathbf{L}_\infty(\mu)$

WOT is short for weak operator topology on $\mathbf{B}_\infty(\mathbf{H})$, which is the locally convex topology generated by seminorms $\mathcal{A} \mapsto |(g, \mathcal{A}h)|$, $g, h \in \mathbf{H}$. Because we assume that \mathbf{H} is a complex Hilbert space, the WOT is generated already by the seminorms $\mathcal{A} \mapsto |(h, \mathcal{A}h)|$, $h \in \mathbf{H}$. SOT is short for strong operator topology on $\mathbf{B}_\infty(\mathbf{H})$, which is the locally convex topology generated by seminorms $\mathcal{A} \mapsto \|\mathcal{A}h\|$, $h \in \mathbf{H}$. The weak-star topology on the topological dual \mathbf{X}' of a normed space \mathbf{X} is the locally convex topology generated by the seminorms $x' \mapsto |x'(x)|: \mathbf{X}' \rightarrow \mathbb{C}$ with $x \in \mathbf{X}$. Examples are the weak-star topology on $\mathbf{B}_\infty(\mathbf{H})$ which is generated by the seminorms $\mathcal{B} \mapsto |\text{Tr}(\mathcal{B} \mathcal{T})|$ with $\mathcal{T} \in \mathbf{B}_1(\mathbf{H})$, and the weak-star topology on $\mathbf{L}_\infty(\mu)$, for a finite measure μ on a σ -field of subsets of a set Ω , which is generated by the seminorms $\varphi \mapsto |\int_\Omega f \varphi d\mu|$, with $f \in \mathbf{L}_1(\mu)$. In stead of weak-star we write sometimes weak*. The following facts are well-known (for example [23, 25]):

- (i) The space of operators with finite dimensional range is a representation of the topological dual of $(\mathbf{B}_\infty(\mathbf{H}), \mathcal{T})$ where $\mathcal{T} \in \{\text{WOT}, \text{SOT}\}$.
- (ii) The SOT closure of a convex subset of $\mathbf{B}_\infty(\mathbf{H})$ is equal to its WOT closure.
- (iii) The weak* and the WOT agree on bounded subsets of $\mathbf{B}_\infty(\mathbf{H})$.

- (iv) If H is a separable Hilbert space, then the closed unit ball of $B_\infty(H)$ with the weak-star topology is compact and metrizable.
- (v) A convex subset of the topological dual of a separable Banach space is weak-star closed if, and only if, it is weak-star sequentially closed.
- (vi) A Banach space X is separable if, and only if, the closed unit ball of X' with the weak-star topology is metrizable.

CHAPTER 3

Positive operator-valued measures

1. Introduction

Positive operator-valued measures (POVMs, for shortness) are used to model quantum mechanical measurements. In this paper their mathematical properties are investigated. A pre-order relation on the set of POVMs is considered. An answer of the following question is given: Which POVMs belong to an equivalence class that is maximal with respect to the partial order induced by the pre-order. Attention is paid also to POVMs associated to subnormal operators. The Bargmann measure, the Susskind-Glogower phase POVM and the Pegg-Barnett phase PVMs are considered as examples of maximal and subnormal POVMs.

2. Conventions, terminology and notation

In this chapter, \mathbf{H} denotes a complex and separable Hilbert space. Unless stated otherwise, Σ denotes a σ -field of subsets of a non-empty set Ω . We will explain what this means:

A non-empty collection of subsets of a set Ω is called a field of subsets, or a (Boolean) algebra of subsets, if it contains Ω and is closed under complementation and under the formation of finite unions and, consequently, under the formation of finite intersections. A function M defined on a field Σ of subsets of a set Ω and taking values in a vector space, is called additive (or finitely additive) if it satisfies

$$M(\Delta_1 \cup \dots \cup \Delta_n) = M(\Delta_1) + \dots + M(\Delta_n)$$

for every $n \in \mathbb{N}$ and n -tuple (Δ_k) of pairwise disjoint sets in Σ .

A σ -field of subsets of a set Ω is a field of subsets of Ω which is closed under the formation of countable unions (and consequently under the formation of countable intersections). A function M defined on a σ -field Σ of subsets of a set Ω and taking values in a vector space, is called σ -additive if it satisfies

$$M\left(\bigcup_{k=1}^{\infty} \Delta_k\right) = \sum_{k=1}^{\infty} M(\Delta_k)$$

for every sequence (Δ_k) of pairwise disjoint sets in Σ . The pair (Ω, Σ) is called a measurable space. A measure (or positive measure) on Σ (or (Ω, Σ)) is a function $\mu: \Sigma \rightarrow [0, \infty]$ that satisfies $\mu(\emptyset) = 0$ and is countably additive. The triple (Ω, Σ, μ) is called a measure space. If $\mu(\Omega) < \infty$ then μ is called a finite measure and (Ω, Σ, μ) is called a finite measure space. A probability measure on (Ω, Σ) is a measure μ on (Ω, Σ) satisfying $\mu(\Omega) = 1$. The triple (Ω, Σ, μ) is then called a probability space. A complex (valued) measure is a linear combination of finite positive measures. (Every complex measure can be written as a linear combination of four finite positive measures. This statement is usually presented as a theorem, because usually, for example in [22], complex measures are defined directly in stead of indirectly in terms of positive measures.)

For a collection \mathcal{C} of subsets of a set Ω , there exists a smallest σ -field of subsets of Ω that includes \mathcal{C} . This σ -field is unique and is called the σ -field generated by \mathcal{C} . If

for $k \in \{1, 2\}$, Σ_k is a σ -field of subsets of a set Ω_k , then $\Sigma_1 \times \Sigma_2$ denotes the σ -field of subsets of $\Omega_1 \times \Omega_2$ generated by the collection of sets $\Delta_1 \times \Delta_2$, with $\Delta_1 \in \Sigma_1$ and $\Delta_2 \in \Sigma_2$.

The difference between two sets A and B is denoted and defined by

$$A \setminus B = A \cap B^c = \{x \in A : x \notin B\}.$$

The symmetric difference of two sets A and B is denoted and defined by

$$A \Delta B = (A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B) = (A \cup B) \cap (A^c \cup B^c).$$

Note that $1_{A \Delta B} = (1_A - 1_B)^2$. For a field of subsets Σ , the triple $(\Sigma, \Delta, \emptyset)$ is a group.

3. Preliminaries

A finite positive operator-valued measure (we will call it an FPOVM for shortness) is a function $M: \Sigma \rightarrow \mathbf{B}_+(\mathbf{H})$ with the property that for each $h \in \mathbf{H}$, the function $M_h: \Sigma \rightarrow \mathbb{R}_+$, defined by

$$(4) \quad M_h(\Delta) = (h, M(\Delta)h)_{\mathbf{H}}, \quad \Delta \in \Sigma,$$

is a finite measure. A normalized positive operator-valued measure (we will call it a POVM for shortness) is an FPOVM M satisfying $M(\Omega) = \mathcal{I}$, the identity operator on \mathbf{H} . We have introduced FPOVMs only for technical reasons; this paper is primarily concerned with POVMs. It is shown e.g. in Remarks 1 and 2 of [37] that the separability of \mathbf{H} implies the existence of a (scalar-valued) probability measure μ on Σ with the same sets of measure zero as M . A PVM is a projection-valued POVM. A POVM M on Σ is projection-valued if and only if

$$(\forall A, B \in \Sigma) \quad M(A \cap B) = M(A)M(B).$$

(See for example Theorem 2 of Section 36 in [54].) An FPOVM M is uniquely defined by the associated measures M_h , $h \in \mathbf{H}$: For $f, g \in \mathbf{H}$ and $\Delta \in \Sigma$,

$$(5) \quad \begin{aligned} \operatorname{Re}(g, M(\Delta)f) &= \frac{1}{4}\{M_{f+g}(\Delta) - M_{f-g}(\Delta)\}, \\ \operatorname{Im}(g, M(\Delta)f) &= \frac{1}{4}\{M_{f+ig}(\Delta) - M_{f-ig}(\Delta)\}. \end{aligned}$$

(We use the convention that an inner-product is linear in the second argument. If the inner-product is linear in the first argument, then an extra factor -1 is needed for the last equality.) The following theorem is similar to Theorem 2 in [10]; The essential difference is that triangle inequality $\sqrt{\mu_{f+g}(\Delta)} \leq \sqrt{\mu_f(\Delta)} + \sqrt{\mu_g(\Delta)}$ is not part of conditions (a),(b) below; it is implied by (a) and (b):

THEOREM 1. *Suppose that for each $h \in \mathbf{H}$ there is given a positive measure μ_h on Σ satisfying $\mu_h(\Omega) = \|h\|^2$. In order that there exist a POVM M on Σ such that $\mu_h = M_h$ for all $h \in \mathbf{H}$, it is necessary and sufficient that for all $f, g, h \in \mathbf{H}$,*

- (a) $\mu_{ch} = |c|^2 \mu_h$ for every $c \in \mathbb{C}$;
- (b) $\mu_{f+g} + \mu_{f-g} = 2\mu_f + 2\mu_g$

PROOF. Theorem 225 below. □

We use the following abbreviation: ' $\forall_M x \in \Omega$ ' means 'for M -almost all $x \in \Omega$.'

4. Naimark's theorem

Proofs of Naimark's theorem on the existence of a projection-valued dilation (for which we shall use the term 'Naimark extension') of a POVM can be found in e.g. [45], [3], [99] and [25]. In [25] only POVMs with compact support are considered. In [3] and [99] POVMs on the Borel subsets of \mathbb{R} (called 'generalized spectral families') are considered. The proof in [99] is adapted in [61] for a version of Naimark's theorem concerning POVMs on arbitrary measurable spaces. From the statement of Naimark's theorem in [45] it is clear that σ -additivity of a positive operator valued set function is not needed for the existence of a Naimark extension.

THEOREM 2 ([45], Theorem II of Section 8). *Let \mathbf{H} be a Hilbert space. Let Σ be a field of subsets of a set Ω . Let $M: \Sigma \rightarrow \mathbf{B}_+(\mathbf{H})$ be a finitely additive set function with $M(\Omega) = \mathcal{I}$. There is a Hilbert space \mathbf{K} , a projection-valued additive set function $N: \Sigma \rightarrow \mathbf{B}_+(\mathbf{K})$ with $N(\Omega) = \mathcal{I}$ and an isometry $\mathbf{V}: \mathbf{H} \rightarrow \mathbf{K}$ such that $M(\Delta) = \mathbf{V}^*N(\Delta)\mathbf{V}$ for all $\Delta \in \Sigma$.*

DEFINITION 3. A Naimark extension of a positive operator-valued additive set function M is a triple $(N, \mathbf{K}, \mathbf{V})$ related to M as in Theorem 2.

REMARK 4. Theorem 2, together with Proposition 7 below, implies Naimark's theorem which says that every POVM M has a Naimark extension $(N, \mathbf{K}, \mathbf{V})$ where N is a PVM.

DEFINITION 5. A Naimark extension $(N, \mathbf{K}, \mathbf{V})$ of an additive set function $M: \Sigma \rightarrow \mathbf{B}_+(\mathbf{H})$ is called minimal if the only subspace of \mathbf{K} containing $\text{range}(\mathbf{V})$ and reducing $N(\Sigma)$ is \mathbf{K} itself. This is the case if, and only if, the only closed subspace of \mathbf{K} , containing $N(\Delta)\mathbf{V}[h]$ for all $\Delta \in \Sigma$ and $h \in \mathbf{H}$, is \mathbf{K} itself: This condition can be formulated as

$$\mathbf{K} = \text{clspan}\{N(\Delta)\mathbf{V}[h] : \Delta \in \Sigma, h \in \mathbf{H}\}.$$

PROPOSITION 6. *For $k \in \{1, 2\}$, let $(N_k, \mathbf{K}_k, \mathbf{V}_k)$ be a minimal Naimark extension of POVM $M: \Sigma \rightarrow \mathbf{B}_+(\mathbf{H})$. There exists a unitary operator $\mathbf{U}: \mathbf{K}_1 \rightarrow \mathbf{K}_2$ such that $\mathbf{U}\mathbf{V}_1 = \mathbf{V}_2$ and $\mathbf{U}N_1(\Delta) = N_2(\Delta)\mathbf{U}$ for all $\Delta \in \Sigma$.*

PROOF. For finite sequences $(\Delta_n), (h_n)$ we have

$$\begin{aligned} \left\| \sum_n N_1(\Delta_n)\mathbf{V}_1[h_n] \right\|^2 &= \sum_{n,m} (N_1(\Delta_n \cap \Delta_m)\mathbf{V}_1[h_n], \mathbf{V}_1[h_m]) \\ &= \sum_{n,m} (M(\Delta_n \cap \Delta_m)h_n, h_m) \\ (6) \quad &= \sum_{n,m} (N_2(\Delta_n \cap \Delta_m)\mathbf{V}_2[h_n], \mathbf{V}_2[h_m]) = \\ &= \left\| \sum_n N_2(\Delta_n)\mathbf{V}_2[h_n] \right\|^2. \end{aligned}$$

Hence

$$\sum_n N_1(\Delta_n)\mathbf{V}_1[h_n] = 0 \quad \Leftrightarrow \quad \sum_n N_2(\Delta_n)\mathbf{V}_2[h_n] = 0.$$

Hence there exists a linear mapping \mathbf{U} from $\text{span}\{N_1(\Delta)\mathbf{V}_1[h] : h \in \mathbf{H}, \Delta \in \Sigma\}$ to \mathbf{K}_2 satisfying

$$(7) \quad \mathbf{U}\left[\sum_n N_1(\Delta_n)\mathbf{V}_1[h_n]\right] = \sum_n N_2(\Delta_n)\mathbf{V}_2[h_n]$$

for finite sequences (Δ_n) and (h_n) . By (6), \mathbf{U} is isometric. The minimality of Naimark extension $(N_1, \mathbf{K}_1, \mathbf{V}_1)$ implies that there exists a unique extension of \mathbf{U} to an isometric

operator from \mathbf{K}_1 to \mathbf{K}_2 . This extension is denoted again by \mathbf{U} . From (7) we see that $\mathbf{U}\mathbf{V}_1 = \mathbf{V}_2$. From (7) and the multiplicativity of N_1 and N_2 , it follows that $\mathbf{U}N_1(\Delta) = N_2(\Delta)\mathbf{U}$ for $\Delta \in \Sigma$. The minimality of Naimark extension $(N_2, \mathbf{K}_2, \mathbf{V}_2)$ implies that \mathbf{U} is surjective. \square

PROPOSITION 7. *Let $(N, \mathbf{K}, \mathbf{V})$ be a minimal Naimark extension of a (finitely) additive set function M on a σ -field Σ . If M is σ -additive, then N is σ -additive.*

PROOF. Let $\Delta_n, n \in \mathbb{N}$ be a pairwise disjoint sequence of sets from Σ . For $A, B \in \Sigma$ and $h \in \mathbf{H}$,

$$\begin{aligned} (N(A)\mathbf{V}[h], N(\cup_{n \in \mathbb{N}} \Delta_n)N(B)\mathbf{V}[h]) &= (\mathbf{V}[h], N(A \cap \cup_{n \in \mathbb{N}} \Delta_n \cap B)\mathbf{V}[h]) \\ &= (h, M(A \cap \cup_{n \in \mathbb{N}} \Delta_n \cap B)h) = \sum_{n \in \mathbb{N}} (h, M(A \cap \Delta_n \cap B)h) \\ &= \sum_{n \in \mathbb{N}} (\mathbf{V}[h], N(A \cap \Delta_n \cap B)\mathbf{V}[h]) \\ &= \sum_{n \in \mathbb{N}} (N(A)\mathbf{V}[h], N(\Delta_n)N(B)\mathbf{V}[h]). \end{aligned}$$

Because of the minimality of the Naimark extension, $D = \text{span}\{N(\Delta)\mathbf{V}[h] : \Delta \in \Sigma, h \in \mathbf{H}\}$ is dense in \mathbf{K} . By Lemma 204, this implies that N is σ -additive. \square

PROPOSITION 8. *Let $M: \Sigma \rightarrow \mathbf{B}_+(\mathbf{H})$ be a POVM and let $(N, \mathbf{K}, \mathbf{V})$ be a minimal Naimark extension of M . Then*

$$N(\Delta) = 0 \quad \text{if and only if} \quad M(\Delta) = 0.$$

PROOF. We have $N_{\mathbf{V}[h]}(\Delta) = M_h(\Delta)$ for every $\Delta \in \Sigma$. Hence $N(\Delta) = 0$ implies $M(\Delta) = 0$. The minimality of N is used for the converse: $M(\Delta) = 0$ implies

$$(\forall A \in \Sigma)(\forall h \in \mathbf{H}) \quad N_{N(A)\mathbf{V}[h]}(\Delta) = N_{\mathbf{V}[h]}(A \cap \Delta) = M_h(A \cap \Delta) = 0.$$

Because N is minimal, this implies $N_k(\Delta) = 0$ for every $k \in \mathbf{K}$. Hence $N(\Delta) = 0$. \square

4.1. Example. Let Σ be a σ -field of subsets of a set Ω . Let $\mu: \Sigma \rightarrow [0, 1]$ be a probability measure. Define POVM $M: \Sigma \rightarrow \mathbf{B}_+(\mathbf{H})$ by

$$M(\Delta) = \mu(\Delta)\mathcal{I}.$$

We will give a Naimark extension for M .

DEFINITION 9. For a finite positive measure space (Ω, Σ, μ) let $\mathbf{1}_\Delta$, for $\Delta \in \Sigma$, be the operator of multiplication with the indicator function 1_Δ on $\mathbf{L}_2(\Omega, \Sigma, \mu)$, and let PVM

$$\mathbf{1}: \Sigma \rightarrow \mathbf{B}_+(\mathbf{L}_2(\Omega, \Sigma, \mu))$$

be defined by $\mathbf{1}(\Delta) = \mathbf{1}_\Delta$ for $\Delta \in \Sigma$.

Let $\mathbf{K} = \mathbf{H} \hat{\otimes} \mathbf{L}_2(\Omega, \Sigma, \mu)$. For $\Delta \in \Sigma$ define orthogonal projection operator $N(\Delta)$ on \mathbf{K} by

$$N(\Delta) = \mathcal{I} \otimes \mathbf{1}_\Delta.$$

Define linear isometry $\mathbf{V}: \mathbf{H} \rightarrow \mathbf{K}$ by

$$\mathbf{V}[h] = h \otimes \mathbf{1}_\Omega,$$

It is easily seen that $(N, \mathbf{K}, \mathbf{V})$ is a minimal Naimark extension of M .

Hilbert space \mathbf{K} can be identified with $\mathbf{L}_2(\Omega, \Sigma, \mu; \mathbf{H})$, the Hilbert space of \mathbf{H} -valued μ -square-integrable function classes on Ω . We will give a Naimark extension of M related to $\mathbf{L}_2(\Omega, \Sigma, \mu; \mathbf{H})$.

DEFINITION 10. For a finite positive measure space (Ω, Σ, μ) let PVM

$$\mathbb{1}^{(\mathbb{H})}: \Sigma \rightarrow \mathbf{B}_+(\mathbf{L}_2(\Omega, \Sigma, \mu; \mathbb{H}))$$

be defined by $\mathbb{1}^{(\mathbb{H})}(\Delta)f(x) = 1_\Delta(x)f(x)$.

Define linear isometry $\mathbf{V}: \mathbb{H} \rightarrow \mathbf{L}_2(\Omega, \Sigma, \mu; \mathbb{H})$ by

$$\mathbf{V}[h](x) = h(x) \quad \text{for } \mu\text{-almost all } x \in \Omega.$$

Then $(\mathbb{1}^{(\mathbb{H})}, \mathbf{L}_2(\Omega, \Sigma, \mu; \mathbb{H}), \mathbf{V})$ is a minimal Naimark extension of M .

5. Separability

The σ -field generated by a collection \mathfrak{C} of subsets of a set Ω is the smallest σ -field of subsets of Ω including \mathfrak{C} .

DEFINITION 11. A σ -field Σ of subsets of a set Ω is countably generated if there exists a countable subcollection \mathfrak{C} of Σ that generates Σ .

LEMMA 12 (Lemma III.8.4 of [42]). *The field of subsets generated by a countable subcollection of Σ is again countable.*

PROPOSITION 13. *Let $m \in \mathbb{N}$. The Borel σ -field on \mathbb{R}^m is countably generated. The collection of all subsets of \mathbb{N} is countably generated.*

DEFINITION 14. Let Σ be a σ -field of subsets of a set Ω . A positive measure μ on Σ is countably generated if and only if $\Sigma(\mu)$ is countably generated: I.e., if there exists a countable subcollection \mathfrak{C} of Σ such that the σ -field Σ' generated by \mathfrak{C} has the following property: For every $A \in \Sigma$, there exists a $B \in \Sigma'$ such that $B \in [A]_\mu$. (This is the case if and only if $\mu(A) = \mu(A \cap B) = \mu(B)$.)

DEFINITION 15. A positive measure μ on Σ is separable if there exists a countable subcollection \mathfrak{C} of Σ such that for every $A \in \Sigma$ and $\epsilon > 0$, there exists a $B \in \mathfrak{C}$ such that $\mu(A \Delta B) \leq \epsilon$.

THEOREM 16. *Let (Ω, Σ, μ) be a finite positive measure space and let $1 \leq p < \infty$. Let $q \in (1, \infty]$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. The following conditions are equivalent.*

- (a) $\mathbf{L}_p(\Omega, \Sigma, \mu)$ is a separable Banach space.
- (b) $\text{ball}(\mathbf{L}_q(\Omega, \Sigma, \mu))$ is weak-star metrizable.
- (c) μ is separable.
- (d) μ is countably generated.

PROOF. (a) \Leftrightarrow (b): Theorem 5.1 of Chapter V in [23], or Theorem V.5.1 in [42].

(c) implies (a): This follows from $|1_A(x) - 1_B(x)| = 1_{A \Delta B}(x)$ because the simple functions are dense in $\mathbf{L}_p(\Omega, \Sigma, \mu)$.

(d) implies (c): Lemma 3.4.6. in [22].

(a) implies (d): We use the same method as in the proof of Lemma III.8.5 in [42]: Let $\{f_n : n \in \mathbb{N}\}$ be dense in $\mathbf{L}_p(\Omega, \Sigma, \mu)$ and let $f_n^{(m)}$, with $m, n \in \mathbb{N}$, be simple functions such that $\lim_{m \rightarrow \infty} \|f_n^{(m)} - f_n\|_p = 0$. Let X_0 be the countable set of the non-zero values of the functions $f_n^{(m)}$. Let \mathfrak{C} be the countable collection of sets $E \in \Sigma$ of the form $E = \{x : f_n^{(m)}(x) = x_0\}$, where m, n are arbitrary positive integers and $x_0 \in X_0$. Let $\Omega_1 = \cup\{E : E \in \mathfrak{C}\}$. Let Σ_1 be the σ -field of subsets of Ω_1 generated by \mathfrak{C} . All the functions $f_n^{(m)}$ vanish on the complement of Ω_1 and are Σ_1 -measurable. Hence $\mu(\Omega_1^c) = 0$. Let μ_1 be the restriction of μ to Σ_1 . Let $A \in \Sigma$. There is a sequence (g_n) of elements of $\{f_n^{(m)} : n, m \in \mathbb{N}\}$ such that $g_n \rightarrow 1_A \in \mathbf{L}_p(\Omega, \Sigma, \mu)$. Then (g_n) is an $\mathbf{L}_p(\Omega, \Sigma, \mu)$ -Cauchy sequence, and hence an $\mathbf{L}_p(\Omega_1, \Sigma_1, \mu_1)$ -Cauchy sequence. Because $\mathbf{L}_p(\Omega_1, \Sigma_1, \mu_1)$ is complete, there is a $g \in \mathbf{L}_p(\Omega_1, \Sigma_1, \mu_1)$ such that $g_n \rightarrow g \in \mathbf{L}_p(\Omega_1, \Sigma_1, \mu_1)$. This implies

that $g_n \rightarrow g \in L_p(\Omega, \Sigma, \mu)$, and hence that $g(x) = 1_A(x)$ for μ -almost all x . This implies that $g(x) = |g(x)|^2$ for μ -almost all x , and hence for μ_1 -almost all x . Hence there is a $\tilde{A} \in \Sigma_1$ such that $1_{\tilde{A}}(x) = g(x)$ for μ_1 -almost all x . Hence $1_{\tilde{A}}(x) = g(x) = 1_A(x)$ for μ -almost all x . Hence $\mu(A \Delta \tilde{A}) = \int_{\Omega} |1_A(x) - 1_{\tilde{A}}(x)|^p \mu(dx) = 0$. \square

REMARK 17. In e.g. [14] a metric space is associated to each positive measure space (Ω, Σ, μ) . It is shown (Theorem 17.7 of [14]) that this metric space is complete if and only if $L_p(\Omega, \Sigma, \mu)$ is complete for some $p \in [1, \infty)$ if and only if $L_p(\Omega, \Sigma, \mu)$ is complete for every $p \in [1, \infty)$.

DEFINITION 18. Let Σ be a σ -field of subsets of a set Ω . A POVM $M: \Sigma \rightarrow \mathbf{B}_+(\mathbf{H})$ is countably generated if and only if $\Sigma(M)$ is countably generated: I.e., if there exists a countable subcollection \mathfrak{C} of Σ such that the σ -field Σ' generated by \mathfrak{C} has the following property: For every $A \in \Sigma$, there exists a $B \in \Sigma'$ such that $B \in [A]_M$. (This is the case if and only if $M(A) = M(A \cap B) = M(B)$.)

DEFINITION 19. A POVM $M: \Sigma \rightarrow \mathbf{B}_+(\mathbf{H})$ is separable if for every $h \in \mathbf{H}$ the measure M_h is separable.

LEMMA 20. A POVM M is separable if and only if there exists a countable subcollection \mathfrak{C} of Σ such that for every $A \in \Sigma$, $h \in \mathbf{H}$ and $\epsilon > 0$, there exists a $B \in \mathfrak{C}$ such that $\|M(A \Delta B)h\| < \epsilon$.

PROOF. Assume that such a \mathfrak{C} exists. By the Cauchy-Bunyakovskii-Schwarz inequality, $M_h(\Delta) \leq \|h\| \|M(\Delta)h\|$. Hence M_h is separable.

Assume that M is separable: For every h there exists a countable subcollection \mathfrak{C}_h of Σ such that for every $A \in \Sigma$ and $\epsilon > 0$, there exists a $B \in \mathfrak{C}_h$ such that $M_h(A \Delta B) \leq \epsilon$. From $0 \leq M(\Delta) \leq \mathcal{I}$ follows $M(\Delta)^2 \leq M(\Delta)$ and consequently $\|M(\Delta)h\|^2 = (h, M(\Delta)^2 h) \leq M_h(\Delta)$. Let \mathbf{H}_0 be a countable dense subset of \mathbf{H} and let $\mathfrak{C} = \cup\{\mathfrak{C}_h : h \in \mathbf{H}_0\}$. Then \mathfrak{C} is countable. Let $\epsilon > 0$, and $h \in \mathbf{H}$ with $\|h\| = 1$, and $A \in \Sigma$. There exists a $h_0 \in \mathbf{H}_0$ such that $\|h - h_0\| \leq \epsilon$, and there exists a $B \in \mathfrak{C}$ such that $\|M(A \Delta B)h_0\| \leq \epsilon$. Then

$$\begin{aligned} \|M(A \Delta B)h\|^2 &= \|M(A \Delta B)(h - h_0)\|^2 - \|M(A \Delta B)h_0\|^2 + 2(h, M(A \Delta B)h_0) \\ &\leq \epsilon^2 + \epsilon^2 + 2\epsilon. \end{aligned}$$

\square

PROPOSITION 21. Let Σ be a σ -field of subsets of a set Ω . Let $(N, \mathbf{K}, \mathbf{V})$ be a minimal Naimark extension of a POVM $M: \Sigma \rightarrow \mathbf{B}_+(\mathbf{H})$, and let μ be a finite positive measure with the same sets of measure zero as M . The following conditions are equivalent:

- (a) \mathbf{K} is separable.
- (b) N is separable.
- (c) N is countably generated.
- (d) μ is countably generated.
- (e) M is countably generated.
- (f) M is separable.

PROOF. (e) implies (f): The proof of Lemma 3.4.6. in [22].

(a) implies (b): By Proposition 230 there is a bounded self-adjoint operator \mathcal{E} on \mathbf{K} that generates commutative von Neumann algebra $N(\Sigma)''$. The spectral measure $E: \mathfrak{B}_{\mathbf{C}} \rightarrow \mathbf{B}_+(\mathbf{K})$ of \mathcal{E} is countably generated and hence separable. Let $h \in \mathbf{H}$. There exists a countable subcollection $\mathfrak{C}_{\mathbf{C}}$ of $\mathfrak{B}_{\mathbf{C}}$ such that for every $A \in \mathfrak{B}_{\mathbf{C}}$ and $\epsilon > 0$, there exists a $B \in \mathfrak{C}_{\mathbf{C}}$ such that

$$\|E(A)h - E(B)h\| = \|E(A \Delta B)h\| < \epsilon.$$

By Lemma 196, for every $A \in \mathfrak{C}_{\mathbb{C}}$ there exists a sequence (A_n) of sets in Σ such that

$$(\forall n \in \mathbb{N}) \quad \|N(A_n)h - E(A)h\| \leq \frac{1}{n}.$$

Let $\mathfrak{C}' = \{A_n : A \in \mathfrak{C}_{\mathbb{C}}, n \in \mathbb{N}\}$. For every $\epsilon > 0$ and $\Delta \in \Sigma$ there is an $A \in \mathfrak{B}_{\mathbb{C}}$, an $A' \in \mathfrak{C}_{\mathbb{C}}$, and a $\Delta' \in \mathfrak{C}'$ such that

$$\|N(\Delta)h - E(A)h\| \leq \epsilon, \quad \|E(A)h - E(A')h\| \leq \epsilon, \quad \|E(A')h - N(\Delta')h\| \leq \epsilon.$$

By the triangle inequality,

$$\|N(\Delta\Delta')h\| = \|N(\Delta)h - N(\Delta')h\| \leq 3\epsilon.$$

Because \mathfrak{C}' is countable, this implies that N_h is separable.

(b) implies (a): Let \mathfrak{C} be as in Lemma 20. Let \mathbf{H}_0 be a countable dense subset of \mathbf{H} . Then \mathbf{K} is the closure of the countable set formed by linear combinations with rational coefficients of $N(\Delta)\mathbf{V}[h]$ with $\Delta \in \mathfrak{C}$ and $h \in \mathbf{H}_0$.

(c) implies (b): The proof of Lemma 3.4.6. in [22].

(b) implies (c): Let (k_n) be an orthonormal basis of \mathbf{K} . (The sequence (k_n) is countable by (a).) For $\Delta \in \Sigma$ let $\tilde{\mu}(\Delta) = \sum_{n=1}^{\infty} 2^{-n} N_{k_n}(\Delta)$. By (b), $\tilde{\mu}$ is separable. By Theorem 16, $\tilde{\mu}$ is countably generated. Hence N is countably generated.

(c) \Leftrightarrow (d) \Leftrightarrow (e): μ and M and N have the same sets of measure zero.

(f) implies (d): Let (h_n) be an orthonormal basis of \mathbf{H} . (The sequence (h_n) is countable because \mathbf{H} is separable.) For $\Delta \in \Sigma$ let $\tilde{\mu}(\Delta) = \sum_{n=1}^{\infty} 2^{-n} M_{h_n}(\Delta)$. Because M is separable, $\tilde{\mu}$ is separable. By Theorem 16, $\tilde{\mu}$ is countably generated. Hence μ is countably generated. Hence M is countably generated. \square

5.1. Example. It is clear that \mathbf{K} in Example 4.1 is separable if and only if $L_2(\Omega, \Sigma, \mu)$ is separable.

6. Operator densities

In this section we introduce a technical tool which is needed in Section 17: We prove that with the help of a Hilbert-Schmidt operator \mathcal{R} , we can transform a POVM M into an FPOVM $\Delta \mapsto \mathcal{R}^*M(\Delta)\mathcal{R}$ which has the special property that there exist a family (\mathcal{M}_x) of operators and a probability measure μ such that

$$\mathcal{R}^*M(\Delta)\mathcal{R} = \int_{\Delta} \mathcal{M}_x \mu(dx)$$

for all $\Delta \in \Sigma$. The operators (\mathcal{M}_x) and the measure μ both depend on the choice of \mathcal{R} .

DEFINITION 22. Let Σ be a σ -field of subsets of a set Ω . An FPOVM $M: \Sigma \rightarrow \mathbf{B}_+(\mathbf{H})$ is said to have operator density \mathcal{M}_x , $x \in \Omega$ with respect to a positive measure μ on Σ if

$$(\forall \Delta \in \Sigma)(\forall h \in \mathbf{H}) \quad M_h(\Delta) = \int_{\Delta} (h, \mathcal{M}_x h) \mu(dx).$$

The operator density is called bounded if $\sup\{\|\mathcal{M}_x\|_{\infty} : x \in \Omega\} < \infty$.

LEMMA 23. *If $M: \Sigma \rightarrow \mathbf{B}_+(\mathbf{H})$ is an FPOVM with operator density (\mathcal{M}_x) with respect to a positive measure μ on Σ , then $\mathcal{M}_x \geq 0$ for μ -almost all x .*

PROOF. Let $h \in \mathbf{H}$. For all $\Delta \in \Sigma$, $\int_{\Delta} (h, \mathcal{M}_x h) \mu(dx) = M_h(\Delta) \geq 0$. This implies that $(h, \mathcal{M}_x h) \geq 0$ for μ -almost all x . This is true for all h : For every $h \in \mathbf{H}$ there exists a μ -null set \mathfrak{N}_h such that $(h, \mathcal{M}_x h) \geq 0$ for all $x \in \Omega \setminus \mathfrak{N}_h$. Let \mathbf{H}_0 be a countable dense subset of \mathbf{H} . From the σ -additivity of μ it follows that countable unions of μ -null sets are again μ -null sets. Hence there exists a μ -null set \mathfrak{N} such that $(h, \mathcal{M}_x h) \geq 0$ for

$x \in \Omega \setminus \mathfrak{N}$ and all $h \in \mathbf{H}_0$. Because \mathcal{M}_x is a bounded operator, this implies that $\mathcal{M}_x \geq 0$ for $x \in \Omega \setminus \mathfrak{N}$. \square

LEMMA 24. *Let Σ be a σ -field of subsets of a set Ω . Let $M: \Sigma \rightarrow \mathbf{B}_+(\mathbf{H})$ be an FPOVM and let μ be a finite positive measure on Σ with the same sets of measure zero as M . For $h \in \mathbf{H}$ let $Q[h]$ be the Radon-Nikodym derivative of M_h with respect to μ :*

$$(\forall \Delta \in \Sigma)(\forall h \in \mathbf{H}) \quad M_h(\Delta) = \int_{\Delta} Q[h](x) \mu(dx).$$

The following conditions are equivalent:

(a) *There is a function $\varphi: \Omega \rightarrow [0, \infty)$ such that*

$$(\forall h \in \mathbf{H})(\forall_{\mu} x \in \Omega) \quad |Q[h](x)| \leq \varphi(x) \|h\|^2.$$

(b) *There is a family (\mathcal{M}_x) of non-negative bounded operators on \mathbf{H} such that*

$$(\forall h \in \mathbf{H})(\forall_{\mu} x \in \Omega) \quad Q[h](x) = (h, \mathcal{M}_x h).$$

PROOF. (b) implies (a): Let $\varphi(x) = \|\mathcal{M}_x\|$.

(a) implies (b): For every $f, g, h \in \mathbf{H}$ there exists a μ -null set $\mathfrak{N}_{f,g,h}$ such that for all $x \in \Omega \setminus \mathfrak{N}_{f,g,h}$ the following conditions are satisfied:

- $0 \leq Q[h](x) \leq \varphi(x) \|h\|^2$,
- $Q[ch](x) = |c|^2 Q[h](x)$ for $c \in \mathbb{C}$,
- $Q[f+g](x) + Q[f-g](x) = 2Q[f](x) + 2Q[g](x)$.

Let \mathbf{H}_0 be a countable dense subset of \mathbf{H} . From the σ -additivity of μ it follows that countable unions of μ -null sets are again μ -null sets. Hence there exists a μ -null set \mathfrak{N} such that for $x \in \Omega \setminus \mathfrak{N}$ and all $f, g, h \in \mathbf{H}_0$ the three conditions above are satisfied. By Theorem 225, there is, for every $x \in \Omega \setminus \mathfrak{N}$, an $\mathcal{M}_x \in \mathbf{B}_+(\mathbf{H})$ with $\|\mathcal{M}_x\|_{\infty} \leq \varphi(x)$ such that $Q[h](x) = (h, \mathcal{M}_x h)$ for all $h \in \mathbf{H}_0$. Hence

$$(\forall h \in \mathbf{H}_0)(\forall \Delta \in \Sigma) \quad M_h(\Delta) = \int_{\Delta} (h, \mathcal{M}_x h) \mu(dx).$$

Because \mathbf{H}_0 is dense in \mathbf{H} , this implies (b). \square

EXAMPLE 25. Let $\mathfrak{B}_{\mathbb{R}}$ be the Borel subsets of \mathbb{R} , let μ be ordinary Lebesgue measure on \mathbb{R} , and let $\mathbf{H} = \mathbf{L}_2(\mathbb{R}, \mu)$. Define $M: \mathfrak{B}_{\mathbb{R}} \rightarrow \mathbf{B}_+(\mathbf{H})$ by $M = \mathbf{1}$. From

$$M(\Delta) = \int_{\Delta} |h(x)|^2 \mu(dx)$$

it follows that the Radon-Nikodym derivative $Q[h]$ of M_h with respect to μ satisfies $Q[h](x) = |h(x)|^2$ for μ -almost all $x \in \mathbb{R}$. It is known that a function $\varphi: \mathbb{R} \rightarrow [0, \infty)$ such that

$$(\forall h \in \mathbf{H})(\forall_{\mu} x \in \Omega) \quad |h(x)|^2 \leq \varphi(x) \|h\|^2$$

does not exist: At page 23 of Section 3 of Chapter I in [106] it is shown that the identity operator \mathcal{I} on $\mathbf{L}_2(\mathbb{R})$ is not an integral operator. This implies, in particular, that \mathcal{I} is not a Carleman operator. (This concept is defined e.g. in [52] and [108] and in Appendix J below.) By Korotkov's theorem (Theorem 6.14 in [108] or Theorem 17.2 in [52]) there is no function $\varphi: \mathbb{R} \rightarrow [0, \infty)$ such that $|\mathcal{I}[h](x)| \leq \|h\| \varphi(x)$ almost everywhere for every $h \in \mathbf{L}_2(\mathbb{R})$. This implies, by Lemma 24, that M does not have an operator density.

LEMMA 26. *Let $M: \Sigma \rightarrow \mathbf{B}_+(\mathbf{H})$ be an FPOVM such that $k_{\Omega} = \text{Tr}(M(\Omega)) < \infty$. There exists a probability measure μ on Σ with the same sets of measure zero as M*

and a family (\mathcal{M}_x) of non-negative bounded operators on \mathbf{H} such that $\|\mathcal{M}_x\|_\infty \leq k_\Omega$ for μ -almost all $x \in \Omega$, and

$$(\forall \Delta \in \Sigma)(\forall h \in \mathbf{H}) \quad M_h(\Delta) = \int_{\Delta} (h, \mathcal{M}_x h) \mu(dx).$$

PROOF. Let probability measure μ on Σ be defined by

$$\mu(\Delta) = \text{Tr}(M(\Delta))/k_\Omega.$$

If $\mu(\Delta) = 0$ then $\text{Tr}(M(\Delta)) = 0$. Because $M(\Delta) \geq 0$, this implies that $M(\Delta) = 0$. Hence μ has the same sets of measure zero as M . Let $Q[h]$ be the Radon-Nikodym derivative of M_h with respect to μ :

$$(\forall \Delta \in \Sigma)(\forall h \in \mathbf{H}) \quad M_h(\Delta) = \int_{\Delta} Q[h](x) \mu(dx).$$

We will proof that

$$(8) \quad (\forall h \in \mathbf{H})(\forall \mu x \in \Omega) \quad |Q[h](x)| \leq k_\Omega \|h\|^2.$$

Assume that it is not true: There exists a $h \in \mathbf{H}$ and a $\Delta \in \Sigma$ with $\mu(\Delta) > 0$ such that $Q[h](x) > k_\Omega \|h\|^2$. This implies that

$$M_h(\Delta) = \int_{\Delta} Q[h](x) \mu(dx) > \mu(\Delta) k_\Omega \|h\|^2 = \text{Tr}(M(\Delta)) \|h\|^2.$$

Hence $\text{Tr}(M(\Delta)) < (\frac{h}{\|h\|}, M(\Delta) \frac{h}{\|h\|})$. This is impossible because $M(\Delta) \geq 0$. Consequently, (8) is satisfied.

By Lemma 24, there is a family (\mathcal{M}_x) of non-negative bounded operators on \mathbf{H} such that

$$(\forall h \in \mathbf{H})(\forall \mu x \in \Omega) \quad Q[h](x) = (h, \mathcal{M}_x h).$$

From (8) follows $\|\mathcal{M}_x\| \leq k_\Omega$ for μ -almost all $x \in \Omega$. \square

PROPOSITION 27. Let $M: \Sigma \rightarrow \mathbf{B}_+(\mathbf{H})$ be an FPOVM. Let \mathcal{R} be a Hilbert-Schmidt operator on \mathbf{H} . There exists a probability measure μ on Σ with the same sets of measure zero as M and a family (\mathcal{M}_x) of non-negative bounded operators on \mathbf{H} such that $\|\mathcal{M}_x\|_\infty \leq \|M(\Omega)\|_\infty \text{Tr}(\mathcal{R}^* \mathcal{R})$ for μ -almost all $x \in \Omega$, and

$$(9) \quad (\forall \Delta \in \Sigma)(\forall h \in \mathbf{H}) \quad M_{\mathcal{R}h}(\Delta) = \int_{\Delta} (h, \mathcal{M}_x h) \mu(dx).$$

PROOF. Define FPOVM $M^{\mathcal{R}}$ on Σ by $M^{\mathcal{R}}(\Delta) = \mathcal{R}^* M(\Delta) \mathcal{R}$. Then $M_{\mathcal{R}h} = M_h^{\mathcal{R}}$ for all $h \in \mathbf{H}$ and $M^{\mathcal{R}}(\Omega)$ is a trace-class operator. Existence of μ and (\mathcal{M}_x) satisfying (9) and $\|\mathcal{M}_x\|_\infty \leq \text{Tr}(M^{\mathcal{R}}(\Omega))$ follows from Lemma 26. By Proposition 201, $\text{Tr}(M^{\mathcal{R}}(\Omega)) \leq \|M(\Omega)\|_\infty \text{Tr}(\mathcal{R}^* \mathcal{R})$. \square

REMARK 28. Assume that $\mathbf{H}, \Sigma, \Omega, \mathcal{R}, \mu, \mathcal{M}_x$ are as in Proposition 27. Let $\mathcal{T} = \mathcal{R} \mathcal{R}^*$ and $\mathcal{T}_x = \mathcal{R} \mathcal{M}_x \mathcal{R}^*$. Then

$$(\forall \Delta \in \Sigma)(\forall h \in \mathbf{H}) \quad M_{\mathcal{T}h}(\Delta) = \int_{\Delta} (h, \mathcal{T}_x h) \mu(dx),$$

and $\text{Tr}(\mathcal{T}_x) \leq \|M(\Omega)\|_\infty (\text{Tr} \mathcal{R}^* \mathcal{R})^2 < \infty$ for μ -almost all $x \in \Omega$.

EXAMPLE 29. Let $\mathbf{H} = \mathbf{L}_2(\mathbb{R})$ and let $\mathfrak{B}_{\mathbb{R}}$ be the Borel subsets of \mathbb{R} . PVM $\mathbb{1}: \mathfrak{B}_{\mathbb{R}} \rightarrow \mathbf{B}_+(\mathbf{H})$ does not have an operator density. An operator \mathcal{R} on \mathbf{H} for which there is a family (e_x) of vectors in \mathbf{H} such that

$$(10) \quad \mathbb{1}_{\mathcal{R}h}(\Delta) = \int_{\Delta} |(e_x, h)|^2 dx$$

can easily be found. Take for example $\mathcal{R} = \mathcal{F}\mathcal{M}_f$, where \mathcal{F} is Fourier transformation on \mathbf{H} and \mathcal{M}_f is the operator of multiplication with a bounded and square-integrable function f . Then (10) is satisfied with $e_x(y) = \overline{f(y)}e^{ixy}/\sqrt{2\pi}$. The family $(e_x \otimes e_x)$ is an operator density (with respect to the Lebesgue measure on \mathbb{R}) for PVM $\mathbf{1}$.

7. Integration with respect to a POVM

DEFINITION 30. Let Σ be a σ -field of subsets of a set Ω . Let $M: \Sigma \rightarrow \mathbf{B}_+(\mathbf{H})$ be a POVM. The M -essential supremum of a Σ -measurable function $\varphi: \Omega \rightarrow \mathbb{C}$ is denoted and defined by

$$\begin{aligned} \|\varphi\|_\infty &= \inf\{\sup\{|\varphi(x)| : x \notin \Delta\} : \Delta \in \Sigma, M(\Delta) = 0\} \\ &= \inf\{c > 0 : M(\{x \in \Omega : |\varphi(x)| > c\}) = 0\} \\ &= \inf\{c > 0 : (\forall_M x \in \Omega) \quad |\varphi(x)| \leq c\}. \end{aligned}$$

If this is finite then φ is M -essentially bounded. Let

$$\mathbf{L}_\infty(\Omega, \Sigma, M) = \{\varphi: \Omega \rightarrow \mathbb{C} : \varphi \text{ is an } M\text{-essentially bounded measurable function}\}.$$

The M -essential supremum is a norm on $\mathbf{L}_\infty(\Omega, \Sigma, M)$ when functions that agree M -almost everywhere are identified. The normed space thus obtained is denoted again by $\mathbf{L}_\infty(\Omega, \Sigma, M)$. It is a Banach space and it is (partially) ordered pointwise modulo sets of measure zero.

DEFINITION 31. Let $M: \Sigma \rightarrow \mathbf{B}_+(\mathbf{H})$ be a POVM. Let $\rho_M: \mathbf{L}_\infty(\Omega, \Sigma, M) \rightarrow \mathbf{B}_\infty(\mathbf{H})$ be defined by

$$(11) \quad (h, \rho_M(\varphi)h) = \int_\Omega \varphi(x) M_h(dx), \quad h \in \mathbf{H}$$

and polarization.

PROPOSITION 32. Let $M: \Sigma \rightarrow \mathbf{B}_+(\mathbf{H})$ be a POVM. Then $\rho_M: \mathbf{L}_\infty(\Omega, \Sigma, M) \rightarrow \mathbf{B}_\infty(\mathbf{H})$ is a contractive operator.

For a POVM $M: \Sigma \rightarrow \mathbf{B}_+(\mathbf{H})$ and probability measure μ on Σ with the same sets of measure zero as M , there is a contractive operator $\rho'_M: \mathbf{B}_1(\mathbf{H}) \rightarrow \mathbf{L}_1(\Omega, \Sigma, \mu)$ satisfying

$$(12) \quad (\forall \varphi \in \mathbf{L}_\infty(\Omega, \Sigma, M))(\forall \mathcal{T} \in \mathbf{B}_1(\mathbf{H})) \quad \text{Tr}(\rho_M(\varphi)\mathcal{T}) = \int_\Omega \varphi(x) \rho'_M[\mathcal{T]}(x) \mu(dx).$$

A special case is

$$(\forall \Delta \in \Sigma)(\forall h \in \mathbf{H}) \quad M_h(\Delta) = \int_\Delta \rho'_M[h \otimes h](x) \mu(dx).$$

If M has an operator density (\mathcal{M}_x) with respect to μ , then $\rho'_M[\mathcal{T]}(x) = \text{Tr}(\mathcal{T}\mathcal{M}_x)$ for μ -almost all x .

PROOF. Let $\mathcal{T} \in \mathbf{B}_1(\mathbf{H})$. From (11) follows

$$\text{Tr}(\rho_M(\varphi)\mathcal{T}) = \int_\Omega \varphi(x) \text{Tr}(M(dx)\mathcal{T}) \quad \forall \varphi \in \mathbf{L}_\infty(\Omega, \Sigma, M).$$

If $\Delta \in \Sigma$ and $\mu(\Delta) = 0$ then $M(\Delta) = 0$ and hence $\text{Tr}(M(\Delta)\mathcal{T}) = 0$. Let $\rho'_M[\mathcal{T}]$ be the Radon-Nikodym derivative of $\Delta \mapsto \text{Tr}(M(\Delta)\mathcal{T})$ with respect to μ : $\rho'_M[\mathcal{T}] \in \mathbf{L}_1(\Omega, \Sigma, \mu)$ and $\text{Tr}(M(dx)\mathcal{T}) = \rho'_M[\mathcal{T]}(x) \mu(dx)$. Now (12) is satisfied. From the linearity of $\mathcal{T} \mapsto \text{Tr}(M(\Delta)\mathcal{T})$ it follows that $\rho'_M: \mathbf{B}_1(\mathbf{H}) \rightarrow \mathbf{L}_1(\Omega, \Sigma, \mu)$ is linear. By Theorem 207,

$$\|\rho'_M[\mathcal{T}]\|_1 \leq \|M(\Delta)\|_\infty \|\mathcal{T}\|_1 \leq \|\mathcal{T}\|_1.$$

This means that ρ'_M is contractive. By (12), this implies that ρ_M is also contractive.

If M has an operator density (\mathcal{M}_x) with respect to μ , then

$$\mathrm{Tr}(M(\Delta)\mathcal{T}) = \int_{\Delta} \mathrm{Tr}(\mathcal{M}_x\mathcal{T}) \mu(dx)$$

for all $\Delta \in \Sigma$. Hence $\rho'_M[\mathcal{T}](x) = \mathrm{Tr}(\mathcal{T}\mathcal{M}_x)$ for μ -almost all x . \square

PROPOSITION 33. *If $M: \Sigma \rightarrow \mathbf{B}_+(\mathbf{H})$ is a POVM, then*

$$\rho_M: (\mathbf{L}_\infty(\Omega, \Sigma, M), \text{weak}^*) \rightarrow (\mathbf{B}_\infty(\mathbf{H}), \text{weak}^*)$$

is continuous. Conversely, if μ is a finite positive measure on Σ and

$$\rho: (\mathbf{L}_\infty(\Omega, \Sigma, \mu), \text{weak}^*) \rightarrow (\mathbf{B}_\infty(\mathbf{H}), \text{WOT})$$

is positive, linear, sequentially continuous and satisfies $\rho(1) = \mathcal{I}$, then $\rho = \rho_M$ for a POVM M on Σ .

PROOF. Let (φ_α) be a net in $\mathbf{L}_\infty(\Omega, \Sigma, \mu)$ converging to 0 with respect to the weak-star topology. From (12) it follows that $\lim_\alpha \mathrm{Tr}(\mathcal{T}\rho_M(\varphi_\alpha)) = 0$ for every $\mathcal{T} \in \mathbf{B}_1(\mathbf{H})$. This means that $\rho_M(\varphi_\alpha)$ converges to 0 with respect to the weak-star topology of $\mathbf{B}_\infty(\mathbf{H})$.

By Theorem 1, there exists a POVM $M: \Sigma \rightarrow \mathbf{B}_+(\mathbf{H})$ such that $\rho(1_\Delta) = \rho_M(1_\Delta)$ for $\Delta \in \Sigma$. This implies, together with the first part of the proposition and the fact that the indicator functions form a dense subset of $\mathbf{L}_\infty(\Omega, \Sigma, M)$, that $\rho = \rho_M$. \square

REMARK 34. Using Theorem 208 and Proposition 209, it is easily seen that M is an injective POVM if, and only if, $\{\rho'_M[\mathcal{T}] : \mathcal{T} \in \mathbf{B}_1(\mathbf{H})\}$ is a dense subset of $\mathbf{L}_1(\Omega, \Sigma, \mu)$, where μ is a finite positive measure with the same sets of measure zero as M .

Let, for example, $\mathbf{H} = \mathbf{L}_2(\mathbb{R})$, and $M = \mathbf{1}$, and $\mathcal{T} = g \otimes h$. Then $\mathrm{Tr}(\rho_M(\varphi)\mathcal{T}) = \int_{\mathbb{R}} \varphi(x)g(x)\bar{h}(x) dx$. Hence $f_{\mathcal{T}}(x) = g(x)\bar{h}(x)$. The space formed by functions $x \mapsto g(x)\bar{h}(x)$, with $g, h \in \mathbf{L}_2(\mathbb{R})$, is equal to (and hence dense in) $\mathbf{L}_1(\mathbb{R})$. Hence M is injective.

THEOREM 35. *Let $M: \Sigma \rightarrow \mathbf{B}_+(\mathbf{H})$ be a PVM. Then*

- (a) $\rho_M(\varphi \cdot \psi) = \rho_M(\varphi)\rho_M(\psi)$ for all $\varphi, \psi \in \mathbf{L}_\infty(\Omega, \Sigma, M)$;
- (b) $\|\rho_M(\varphi)h\|^2 = \int_{\Omega} |\varphi(x)|^2 M_h(dx)$ for all $\varphi \in \mathbf{L}_\infty(\Omega, \Sigma, M)$;
- (c) ρ_M is injective
- (d) $\rho_M: \mathbf{L}_\infty(\Omega, \Sigma, M) \rightarrow \mathbf{B}_\infty(\mathbf{H})$ is a linear isometry;
- (e) $\mathrm{range}(\rho_M)$ is a C^* -subalgebra of $\mathbf{B}_\infty(\mathbf{H})$,
and $\varphi \leq \psi$ if, and only if, $\rho_M(\varphi) \leq \rho_M(\psi)$;
- (f) $\mathrm{range}(\rho_M)$ is weak-star closed;
- (g) $\rho_M: \mathbf{L}_\infty(\Omega, \Sigma, \mu) \rightarrow \mathrm{range}(\rho_M)$ has a weak-star continuous linear inverse.

PROOF. (a): Theorem 15 in [10]. (The general case can be reduced to the case where φ and ψ are simple functions, for which (a) is easily verified.)

(a) implies (b): Take $\varphi = \bar{\psi}$ and use $\rho_M(\bar{\psi}) = \rho_M(\psi)^*$.

(b) implies (c): If $\int_{\Omega} |\varphi(x)|^2 M_h(dx) = 0$ for all $h \in \mathbf{H}$, then $|\varphi(x)|^2 = 0$ for M -almost all x . Hence $\varphi(x) = 0$ for M -almost all x .

(b) implies (d): Let $\varphi \in \mathbf{L}_\infty(\Omega, \Sigma, M)$ and $\|\varphi\|_\infty = 1$. By (b),

$$\|\rho_M(\varphi)h\|^2 = \int_{\Omega} |\varphi(x)|^2 M_h(dx) \leq M_h(\Omega) = \|h\|^2.$$

Hence $\|\rho_M(\varphi)\| \leq 1$. Let $\epsilon > 0$. There is a $\Delta \in \Sigma$ with $M(\Delta) \neq 0$ such that $|\varphi(x)| \geq 1 - \epsilon$ for $x \in \Delta$. By (b),

$$\|\rho_M(\varphi)h\|^2 \geq \int_{\Delta} |\varphi(x)|^2 M_h(dx) \geq (1 - \epsilon)^2 M_h(\Delta).$$

for $h \in \mathbf{H}$. Hence

$$(\forall h \in \mathbf{H}) \quad \|\rho_M(\varphi)h\| \geq (1 - \epsilon)\|M(\Delta)h\|.$$

Operator $M(\Delta)$ is a non-zero orthogonal projection operator and hence has norm 1. Hence $\|\rho_M(\varphi)\| \geq 1 - \epsilon$. This is true for every $\epsilon > 0$; hence $\|\rho_M(\varphi)\| \geq 1$. Combining both results gives: $\|\rho_M(\varphi)\| = \|\varphi\|_\infty$ if $\|\varphi\|_\infty = 1$. This implies (d).

(a) implies (e): From (a) it follows that $\text{range}(\rho_M)$ is a $*$ -subalgebra of $\mathbf{B}_\infty(\mathbf{H})$. From (b) it follows that $\text{range}(\rho_M)$ is a C^* -subalgebra. The second part of (e) can be reduced to the case $\varphi = 0$. If $\psi \geq 0$ then clearly $\rho_M(\psi) \geq 0$. Assume that ψ is Σ -measurable and that $\rho_M(\psi) \geq 0$. Then $\rho_M(\psi) = \rho_M(\psi)^* = \rho_M(\bar{\psi})$ hence $\rho_M(\psi - \bar{\psi}) = 0$. Because ρ_M is injective, this implies $\psi = \bar{\psi}$ i.e. ψ is real-valued. Let $\Delta = \{x \in \Omega : \psi(x) \leq 0\}$. Then $\Delta \in \Sigma$ and $\rho_M(1_\Delta\psi) \leq 0$. But for $h \in \mathbf{H}$ we have

$$(h, \rho_M(1_\Delta\psi)h) = (M(\Delta)h, \rho_M(\psi)M(\Delta)h) \geq 0 \quad \forall h \in \mathbf{H}.$$

Hence $\rho_M(1_\Delta\psi) = 0$. By the injectivity of ρ_M , this implies $1_\Delta\psi = 0$. Hence $\psi \geq 0$.

(e) implies (f): Theorem 229.

(c) and (f) imply (g): Proposition 216 □

In the following proposition we give relations between ρ_M and ρ_N , where N is the minimal Naimark extension of POVM M .

PROPOSITION 36. *Let $M: \Sigma \rightarrow \mathbf{B}_+(\mathbf{H})$ be a POVM and let $(N, \mathbf{K}, \mathbf{V})$ be a minimal Naimark extension of M . Then $\mathbf{L}_\infty(\Omega, \Sigma, M) = \mathbf{L}_\infty(\Omega, \Sigma, N)$. We have*

$$(13) \quad (\forall \varphi \in \mathbf{L}_\infty(\Omega, \Sigma, M)) \quad \rho_M(\varphi) = \mathbf{V}^* \rho_N(\varphi) \mathbf{V}.$$

Hence

$$(\forall \varphi \in \mathbf{L}_\infty(\Omega, \Sigma, M)) (\forall h \in \mathbf{H}) \quad \|\rho_M(\varphi)h\|^2 \leq \int_\Omega |\varphi(x)|^2 M_h(dx).$$

PROOF. By Proposition 8, M and N have the same sets of measure zero. Hence $\mathbf{L}_\infty(\Omega, \Sigma, M) = \mathbf{L}_\infty(\Omega, \Sigma, N)$. For $\varphi \in \mathbf{L}_\infty(\Omega, \Sigma, M)$ and $h \in \mathbf{H}$,

$$\|\rho_M(\varphi)h\|^2 \leq \|\rho_N(\varphi) \mathbf{V}[h]\|^2 = \int_\Omega |\varphi(x)|^2 N_{\mathbf{V}[h]}(dx) = \int_\Omega |\varphi(x)|^2 M_h(dx).$$

□

DEFINITION 37. A POVM $M: \Sigma \rightarrow \mathbf{B}_+(\mathbf{H})$ is called injective if

$$\rho_M: \mathbf{L}_\infty(\Omega, \Sigma, M) \rightarrow \mathbf{B}_\infty(\mathbf{H})$$

is injective (i.e. if $\rho_M(\varphi) = 0$ implies $\varphi = 0$).

REMARK 38. Every PVM is injective. Not every injective POVM is projection-valued: In Remark 8 of [37] an example is given of a POVM $M: \Sigma \rightarrow \mathbf{B}_+(\mathbb{R}^2)$ which is injective and not projection-valued.

PROPOSITION 39. *Let $M: \Sigma \rightarrow \mathbf{B}_+(\mathbf{H})$ be a POVM and let $M(\Sigma)''$ be the von Neumann algebra generated by its range $M(\Sigma)$. Then $\text{range}(\rho_M) \subset M(\Sigma)''$.*

PROOF. The simple functions form a dense subset S of $(\mathbf{L}_\infty(\Omega, \Sigma, M), \text{weak}^*)$. The continuity of ρ_M implies that $\rho_M(S)$ is dense in $(\text{range}(\rho_M), \text{weak}^*)$. Because $\rho_M(S) \subset M(\Sigma)''$ and $M(\Sigma)''$ is weak-star closed, this implies that $\text{range}(\rho_M) \subset M(\Sigma)''$. □

8. Dominance of POVMs

In [80] and [78] a pre-order on the collection of POVMs is explored. In [37] some of the mathematical properties of this pre-order are investigated even further. In this section we repeat some definitions and elementary results from [37]. In Section 17 we characterize the set of POVMs that belong to an equivalence class which is maximal with respect to the partial order induced by the pre-order. A similar characterization was already given in [78] for the case of POVMs on finite outcome sets and in [37] for discrete POVMs.

DEFINITION 40 ([37], Section 2). For $k \in \{1, 2\}$, let Σ_k be a σ -field of subsets of a set Ω_k . Let $M_2: \Sigma_1 \rightarrow \mathbf{B}_+(\mathbf{H})$ be a POVM. Let $\mathbf{M}(\Sigma_1; \Omega_2, \Sigma_2, M_2)$ be the set of families (p_Δ) , indexed by Σ_1 and consisting of equivalence classes of M_2 -measurable functions on Ω_2 , such that

- $p_\Delta \in \mathbf{L}_\infty(\Omega_2, \Sigma_2, M_2)$ for all $\Delta \in \Sigma_1$,
- $p_{\Omega_1}(x) = 1$ and $0 \leq p_\Delta(x) \leq 1$ for M_2 -almost all $x \in \Omega_2$ and $\Delta \in \Sigma_1$,
- $p_\Delta(x) = \sum_{n=1}^{\infty} p_{\Delta_n}(x)$ for M_2 -almost all $x \in \Omega_2$ for every disjoint union $\Delta = \cup_{n=1}^{\infty} \Delta_n$ with $\Delta_n \in \Sigma_1$.

DEFINITION 41 ([37], Definitions 2 and 3). Let Σ_1, Σ_2 be σ -fields of subsets of sets Ω_1, Ω_2 respectively. For POVMs $M_1: \Sigma_1 \rightarrow \mathbf{B}_+(\mathbf{H})$ and $M_2: \Sigma_2 \rightarrow \mathbf{B}_+(\mathbf{H})$ we say that M_1 is dominated by M_2 , denoting $M_1 \leftarrow M_2$, if there exists $(p_\Delta) \in \mathbf{M}(\Sigma_1; \Omega_2, \Sigma_2, M_2)$ such that

$$(\forall \Delta \in \Sigma_1) \quad M(\Delta) = \rho_{M_2}(p_\Delta) = \int_{\Omega_2} p_\Delta(y) M_2(dy).$$

If both $M_1 \leftarrow M_2$ and $M_2 \leftarrow M_1$ then we say that M_1 and M_2 are equivalent which is denoted by $M_1 \leftrightarrow M_2$.

DEFINITION 42 ([80]). A POVM M is maximal if $M \leftarrow E$ for another POVM E , implies $M \leftrightarrow E$.

LEMMA 43 ([37], Lemma 2.2). \leftarrow is a pre-order and \leftrightarrow is the associated equivalence relation.

PROPOSITION 44. A POVM has commutative range if and only if it is dominated by a PVM.

PROOF. Let $M: \Sigma \rightarrow \mathbf{B}_+(\mathbf{H})$ be a POVM and let $E: \Sigma_2 \rightarrow \mathbf{B}_+(\mathbf{H})$ be a PVM that dominates M . Then $M(\Sigma)$ is commutative because it is contained in $\text{range}(\rho_E)$. Corollary 3.8 of [37] provides the remaining part of the proof. \square

9. Four remarks about the definition of \leftarrow

LEMMA 45. The three conditions in Definition 40 are equivalent to

- (a) $p_\Delta \in \mathbf{L}_\infty(\Omega_2, \Sigma_2, M_2)$ for all $\Delta \in \Sigma$,
- (b) $p_\Omega = 1$ and $0 \leq p_\Delta \leq 1$ in $\mathbf{L}_\infty(\Omega_2, \Sigma_2, M_2)$ for all $\Delta \in \Sigma_1$,
- (c) If Δ is the disjoint union of family $\Delta_n, n \in \mathbb{N}$ of sets in Σ , then $p_\Delta = \sum_{n=1}^{\infty} p_{\Delta_n}$ in $(\mathbf{L}_\infty(\Omega_2, \Sigma_2, M_2), \text{weak}^*)$.

PROOF. Conditions (a), (b) are clearly equivalent to the first two condition in Definition 40. Assume that these conditions are satisfied and that $p_\Delta = p_{\Delta_1} + p_{\Delta_2}$ for every disjoint union Δ of $\Delta_1, \Delta_2 \in \Sigma_1$.

It suffices to prove that (under the above assumptions) the third condition of Definition 40 is equivalent to condition (c):

Assume that we are given a disjoint union $\Delta = \cup_{n=1}^{\infty} \Delta_n$ with $\Delta_n \in \Sigma_1$. Let μ be a probability measure on Σ_2 with the same sets of measure zero as M_2 . For $N \in \mathbb{N}$ let

$\varphi_N = \sum_{n=1}^N p_{\Delta_n}$. From the assumptions it follows that $N \mapsto \varphi_N$ is a monotone increasing sequence of elements of $L_\infty(\Omega_2, \Sigma_2, \mu)$ and that $0 \leq \varphi_N \leq p_\Delta$. We will prove that the following conditions are equivalent:

- (i) $\lim_{N \rightarrow \infty} \varphi_N(x) = p_\Delta(x)$ for μ -almost all x ;
- (ii) $\lim_{N \rightarrow \infty} \varphi_N = p_\Delta$ in $(L_\infty(\Omega_2, \Sigma_2, \mu), \text{weak}^*)$.

(i) implies (ii): This follows from the monotone convergence theorem

(ii) implies (i): Because $1_{\Omega_2} \in L_1(\Omega_1, \Sigma_1, \mu)$, it follows from (ii) that

$$\lim_{N \rightarrow \infty} \int_{\Omega_2} \varphi_N(x) \mu(dx) = \int_{\Omega_2} p_\Delta(x) \mu(dx).$$

By Theorem 205, this implies (i). □

LEMMA 46. For $k \in \{1, 2\}$, let (Ω_k, Σ_k) be a measurable space, let $M_k: \Sigma_k \rightarrow \mathbf{B}_+(\mathbf{H})$ be a POVM, and let μ_k be a probability measure with the same sets of measure zero as M_k . The following conditions are equivalent.

- (a) $M_2 \leftarrow M_1$;
- (b) There exists a positive and weak-star continuous operator

$$\mathbf{K}: L_\infty(\Omega_2, \Sigma_2, M_2) \rightarrow L_\infty(\Omega_1, \Sigma_1, M_1)$$

such that $\mathbf{K}[1_{\Omega_2}] = 1_{\Omega_1}$ and $\rho_{M_2} = \rho_{M_1} \circ \mathbf{K}$, or equivalently,

$$(\forall \Delta_2 \in \Sigma_2) \quad M_2(\Delta_2) = \int_{\Omega_1} \mathbf{K}[1_{\Delta_2}](x) M_1(dx).$$

- (c) There exists a positive operator $\mathbf{R}: L_1(\Omega_1, \Sigma_1, M_1) \rightarrow L_1(\Omega_2, \Sigma_2, M_2)$ such that

$$(14) \quad \int_{\Omega_2} \mathbf{R}[f](y) \mu_2(dy) = \int_{\Omega_1} f(x) \mu_1(dx)$$

for all $f \in L_1(\Omega_1, \Sigma_1, \mu_1)$ and $\rho'_{M_2} = \mathbf{R} \circ \rho'_{M_1}$.

PROOF. (b) implies (a): For $\Delta_2 \in \Sigma_2$ let $p_{\Delta_2}(x) = \mathbf{K}[1_{\Delta_2}](x)$. From the weak-star continuity of \mathbf{K} it follows that $(p_{\Delta_2}) \in \mathbf{M}(\Sigma_2, \Omega_1, \Sigma_1, M_1)$. By construction, $M_2(\Delta_2) = \rho_{M_1}(p_{\Delta_2})$. Hence $M_2 \leftarrow M_1$.

(a) implies (b): Let \mathbf{S} be the subspace of $L_\infty(\Omega_2, \Sigma_2, M_2)$ consisting of simple functions (i.e. linear combinations of measurable indicator functions). Let $s \in \mathbf{S}$. There are $N \in \mathbb{N}$, an N -tuple (Δ_n) of pairwise disjoint sets from Σ_2 , and $c_n \in \mathbb{C}$ such that $s = \sum_{n=1}^N c_n 1_{\Delta_n}$. Define $\mathbf{K}[s] \in L_\infty(\Omega_1, \Sigma_1, M_1)$ by

$$\mathbf{K}[s] = \sum_{n=1}^N c_n p_{\Delta_n}.$$

It is easily seen that the properties of $B \mapsto p_B$ imply that this does not depend on the particular representation of s . Hence \mathbf{K} is the unique linear function from \mathbf{S} to $L_\infty(\Omega_1, \Sigma_1, M_1)$ such that $\mathbf{K}[1_B] = p_B$ for all $B \in \Sigma_2$. It is easily seen that \mathbf{K} is positive (i.e. that $\mathbf{K}[s] \geq 0$ if $s \geq 0$), and that $\mathbf{K}[1_{\Omega_2}] = 1_{\Omega_1}$. Hence $\|\mathbf{K}[f]\|_\infty \leq \|f\|_\infty$ for $f \geq 0$. Lemma 197 says that this implies that \mathbf{K} is a bounded operator. Because \mathbf{S} is a dense linear subspace of $L_\infty(\Omega_2, \Sigma_2, M_2)$, the bounded operator \mathbf{K} has a unique extension to a bounded operator from $L_\infty(\Omega_2, \Sigma_2, M_2)$ to $L_\infty(\Omega_1, \Sigma_1, M_1)$. By construction,

$$(\forall B \in \Sigma_2) \quad M_2(B) = \int_{\Omega_1} \mathbf{K}[1_B](x) M_1(dx).$$

If $B = \cup_{k=1}^{\infty} B_k$ is a disjoint union with $B_k \in \Sigma_2$ then, by Lemma 45,

$$\sum_{k=1}^{\infty} \mathbf{K}[1_{B_k}] = \mathbf{K}[1_B]$$

with respect to the weak-star topology. Theorem 46.4 of [25], says that this implies that \mathbf{K} is weak-star continuous.

(c) implies (b): If ρ_{M_k} is considered as

$$\rho_{M_k}: (\mathbf{L}_{\infty}(\Omega_1, \Sigma_1, M_1), \text{weak}^*) \rightarrow (\mathbf{B}_{\infty}(\mathbf{H}), \text{weak}^*)$$

then $(\rho'_{M_k})' = \rho_{M_k}$. Hence $\rho'_{M_2} = \mathbf{R} \circ \rho'_{M_1}$ implies $\rho_{M_2} = \rho_{M_1} \circ \mathbf{K}$, where $\mathbf{K} = \mathbf{R}'$. By Lemma 215, \mathbf{K} is weak-star continuous. From (14) follows $\mathbf{K}[1_{\Omega_2}] = 1_{\Omega_1}$.

(b) implies (c): By Lemma 215, there is a bounded operator $\mathbf{R}: \mathbf{L}_1(\Omega_1, \Sigma_1, M_1) \rightarrow \mathbf{L}_1(\Omega_2, \Sigma_2, M_2)$ such that $\mathbf{K} = \mathbf{R}'$. We have $\mathbf{K}' = \mathbf{R}$ if we consider \mathbf{K} as operator from $(\mathbf{L}_{\infty}(\Omega_2, \Sigma_2, M_2), \text{weak}^*)$ to $(\mathbf{L}_{\infty}(\Omega_1, \Sigma_1, M_1), \text{weak}^*)$. Hence $\rho_{M_2} = \rho_{M_1} \circ \mathbf{K}$ implies $\rho'_{M_2} = \mathbf{R} \circ \rho'_{M_1}$. From $\mathbf{K}[1_{\Omega_2}] = 1_{\Omega_1}$ follows (14). \square

DEFINITION 47. For $k \in \{1, 2\}$, let (Ω_k, Σ_k) be a measurable space.

$\mathbf{M}((\Omega_2, \Sigma_2), (\Omega_1, \Sigma_1))$ denotes the set of functions $K: \Sigma_2 \times \Omega_1 \rightarrow [0, 1]$ satisfying

- For each $x \in \Omega_1$, $K(\cdot, x)$ is a probability measure on (Ω_2, Σ_2) ;
- For each $\Delta \in \Sigma_2$, $K(\Delta, \cdot)$ is a Σ_1 -measurable function on Ω_1 .

DEFINITION 48. For $k \in \{1, 2\}$, let $(\Omega_k, \Sigma_k, \mu_k)$ be two measure spaces.

$\mathbf{M}((\Omega_2, \Sigma_2, \mu_2), (\Omega_1, \Sigma_1, \mu_1))$ denotes the set of functions $K: \Sigma_2 \times \Omega_1 \rightarrow [0, \infty)$ satisfying:

- For each $x \in \Omega_1$, $K(\cdot, x)$ is a μ_2 -measurable function on Ω_2 ;
- For each $y \in \Omega_2$, $K(y, \cdot)$ is a μ_1 -measurable function on Ω_1 ;
- $\int_{\Omega_2} K(y, x) \mu_2(dy) = 1$ for μ_1 -almost all $x \in \Omega_1$.

In this definition, μ_1 may be replaced by a POVM on Σ_1 .

LEMMA 49. For $k \in \{1, 2\}$, let (Ω_k, Σ_k) be a measurable space, and let $M_k: \Sigma_k \rightarrow \mathbf{B}_+(\mathbf{H})$ be a POVM. Consider the following conditions:

- (i) $M_2 \leftarrow M_1$;
- (ii) There exists a $K \in \mathbf{M}((\Omega_2, \Sigma_2), (\Omega_1, \Sigma_1))$ such that

$$(\forall \Delta_2 \in \Sigma_2) \quad M_2(\Delta_2) = \int_{\Omega_1} K(\Delta_2, x) M_1(dx),$$

and $K(\cdot, x) \ll M_2$ for M_1 -almost all $x \in \Omega_1$.

Condition (ii) implies (i). If Σ_2 is countably generated, then (i) implies (ii).

If, moreover, M_2 has an operator density $(\mathcal{M}_y^{(2)})$ with respect to probability measure μ_2 on Σ_2 , then condition (ii) is equivalent to the following: There exists a function $K \in \mathbf{M}((\Omega_2, \Sigma_2, \mu_2), (\Omega_1, \Sigma_1, M_1))$ such that

$$(\forall_{\mu_2} y \in \Omega_2) \quad \mathcal{M}_y^{(2)} = \int_{\Omega_1} K(y, x) M_1(dx).$$

PROOF. (ii) implies (i): For $\Delta_2 \in \Sigma_2$ let $p_{\Delta_2}(x) = K(\Delta_2, x)$.

Then $(p_{\Delta_2}) \in \mathbf{M}(\Sigma_2; \Omega_1, \Sigma_1, M_1)$ and $M_2(\Delta_2) = \rho_{M_1}(p_{\Delta_2})$. Hence $M_2 \leftarrow M_1$.

(i) implies (ii): Assume that Σ_2 is countably generated: There exists a countable subfamily \mathfrak{C} of Σ_2 such that Σ_2 is the smallest σ -field containing \mathfrak{C} . The algebra of subsets generated by \mathfrak{C} and Ω_2 is again countable. We denote this algebra by \mathfrak{A} . Define $K: \mathfrak{A} \times \Omega_1 \rightarrow [0, 1]$ by

$$K(\Delta, x) = p_{\Delta}(x),$$

where p_Δ is a representative satisfying $0 \leq p_\Delta(x) \leq 1$. The union of a countable family of countable sets is again countable. The union of a countable family of M_1 -null sets is again an M_1 -null set. Hence there exists an M_1 -null set \mathfrak{N} such that for every $x \in \mathfrak{N}^c$,

- $K(\cdot, x): \mathfrak{A} \rightarrow [0, 1]$ is a countably additive set function;
- $K(\Omega_2, x) = 1$;
- $K(\Delta, x) \geq 0$ for every $\Delta \in \mathfrak{A}$;
- $K(\Delta, x) = 0$ for every $\Delta \in \mathfrak{A}$ with $M_2(\Delta) = 0$.

By Theorem III.5.8 in [42], $K(\cdot, x)$ has a unique extension to a probability measure on Σ_2 , for every $x \in \mathfrak{N}^c$. This extension is again denoted by $K(\cdot, x)$. Let \mathfrak{F} be the collection of sets $\Delta \in \Sigma_2$ for which the function $K(\Delta, \cdot): \mathfrak{N}^c \rightarrow [0, 1]$ is measurable. Then \mathfrak{F} is a σ -field which includes \mathfrak{A} . Hence $K(\Delta, \cdot): \mathfrak{N}^c \rightarrow [0, 1]$ is measurable for all $\Delta \in \Sigma_2$. For $x \in \mathfrak{N}$ we redefine $K(\cdot, x)$ in such a way that $\Delta \mapsto K(\Delta, x)$ becomes a probability measure. (This can be done in many ways, but the choice is irrelevant because \mathfrak{N} is a M_1 -null set.) Then $K \in \mathbf{M}((\Omega_2, \Sigma_2), (\Omega_1, \Sigma_1))$ and $K(\cdot, x) \ll M_2$ for M_1 -almost all $x \in \Omega_1$. By construction,

$$(\forall \Delta_2 \in \mathfrak{A}) \quad M_2(\Delta_2) = \int_{\Omega_1} K(\Delta_2, x) M_1(dx).$$

The uniqueness part of Corollary III.5.9 in [42] implies that this is true also for $\Delta_2 \in \Sigma_2$.

The final part of the lemma follows from the Radon-Nikodym theorem and Fubini's theorem. Define $y \mapsto K(y, x)$ as the Radon-Nikodym derivative of $\Delta \mapsto K(\Delta, x)$ with respect to μ_2 . \square

REMARK 50. An element $K \in \mathbf{M}((\Omega_2, \Sigma_2), (\Omega_1, \Sigma_1))$ is called (in Exercise 6 of Section 4 of Chapter 2 in [22]) a kernel from (Ω_1, Σ_1) to (Ω_2, Σ_2) . For some properties of kernels, see also Exercise 6 of Section 1 of Chapter 5 in [22]. In [15], K is called a conditional confidence measure.

For $k \in \{1, 2\}$, let Σ_k be a σ -field of subsets of a set Ω_k . Define $\Sigma_1 \times \Sigma_2$ as the σ -field of subsets of $\Omega_1 \times \Omega_2$ generated by the sets $\Delta_1 \times \Delta_2$ with $\Delta_k \in \Sigma_k$.

LEMMA 51. For $k \in \{1, 2\}$, let Σ_k be a σ -field of subsets of a set Ω_k . Let $M_1: \Sigma_1 \rightarrow \mathbf{B}_+(\mathbf{H})$ be a POVM. Let $p \in \mathbf{M}(\Sigma_2; \Omega_1, \Sigma_1, M_1)$. For $\Delta \in \Sigma_1 \times \Sigma_2$ let

$$q_\Delta(x) = p_{\Delta_x}(x), \quad \text{where } \Delta_x = \{y \in \Omega_2 : (x, y) \in \Delta\}.$$

Then $(q_\Delta) \in \mathbf{M}(\Sigma_1 \times \Sigma_2; \Omega_1, \Sigma_1, M_1)$. Define POVM $M: \Sigma_1 \times \Sigma_2 \rightarrow \mathbf{B}_+(\mathbf{H})$ by

$$M(\Delta) = \int_{\Omega_1} q_\Delta(x) M_1(dx).$$

Then $M(A \times B) = \int_A p_B(x) M_1(dx)$. In particular, $M(A \times \Omega_2) = M_1(A)$.

PROOF. By Lemma 5.1.1 in [22], $\Delta_x \in \Sigma_2$ for all x . Let \mathfrak{F} be the collection of sets $\Delta \in \Sigma_1 \times \Sigma_2$ for which q_Δ is Σ_1 -measurable. Because $q_{A \times B} = 1_A(x)p_B(x)$, $A \times B \in \mathfrak{F}$ for all $A \in \Sigma_1$ and $B \in \Sigma_2$. It is easily seen that \mathfrak{F} is a σ -field. Hence \mathfrak{F} contains $\Sigma_1 \times \Sigma_2$. Hence $q_\Delta \in \mathbf{L}_\infty(\Omega_1, \Sigma_1, M)$ for all $\Delta \in \Sigma_1 \times \Sigma_2$. That $(q_\Delta) \in \mathbf{M}(\Sigma_1 \times \Sigma_2; \Omega_1, \Sigma_1, M_1)$ follows easily from the definition of Δ_x . \square

10. Dominance between PVMs

THEOREM 52. Let \mathbf{H} be a Hilbert space and let Σ be a σ -field of subsets of a set Ω . Let $E: \Sigma \rightarrow \mathbf{B}_+(\mathbf{H})$ be a PVM. Then

$$(15) \quad \text{range}(\rho_E) = E(\Sigma)''.$$

This is a maximal commutative von Neumann algebra if and only if

$$(16) \quad (E(\Sigma)')_p = E(\Sigma).$$

PROOF. Let μ be a non-negative finite measure on Ω with the same sets of measure zero as E . By Theorem 35, $\text{range}(\rho_E)$ is a weak-star closed C^* -subalgebra of $\mathbf{B}_\infty(\mathbf{H})$. By von Neumann's double commutant theorem, $\text{range}(\rho_E)$ is a von Neumann algebra. We will prove that

$$(17) \quad \text{range}(\rho_E)_p = E(\Sigma).$$

Let $\mathcal{A} = \rho_E(\varphi)$ be an orthogonal projection operator: $\mathcal{A}^2 = \mathcal{A} = \mathcal{A}^*$. Because every PVM is injective, this implies that $(\varphi(x))^2 = \varphi(x) = \overline{\varphi(x)}$ for E -almost all $x \in \Omega$. Hence $\varphi(x) \in \{0, 1\}$ for E -almost all $x \in \Omega$. Hence $\mathcal{A} = E(\text{supp}(\varphi))$. Hence (17). Every von Neumann algebra is the norm closed linear span of its projections. Hence $\text{range}(\rho_E)$ is the norm closed linear span of $E(\Sigma)$. Hence $\text{range}(\rho_E)$ is the von Neumann algebra generated by $E(\Sigma)$. By von Neumann's double commutant theorem, this is equal to $E(\Sigma)''$. This is a maximal commutative von Neumann algebra if and only if $E(\Sigma)'' = E(\Sigma)'$. We will prove that conditions

- (a) $(E(\Sigma)')_p = E(\Sigma)$,
- (b) $E(\Sigma)'' = E(\Sigma)'$

are equivalent: (a) implies (b): $E(\Sigma)'$ and $E(\Sigma)''$ are von Neumann algebras and hence are equal to the norm closed linear span of their projections $E(\Sigma)$. (b) implies (a): $(E(\Sigma)')_p = (E(\Sigma)'')_p = \text{range}(\rho_E)_p = E(\Sigma)$. \square

PROPOSITION 53. Let Σ_1 and Σ_2 be σ -fields of subsets of sets Ω_1 and Ω_2 respectively. Let $E_1: \Sigma_1 \rightarrow \mathbf{B}_\infty(\mathbf{H})$ and $E_2: \Sigma_2 \rightarrow \mathbf{B}_\infty(\mathbf{H})$ be two PVMs. Then $E_1 \leftarrow E_2$ if and only if $E_1(\Sigma_1) \subset E_2(\Sigma_2)''$.

PROOF. By Theorem 52, $E_2(\Sigma_2)'' = \text{range}(\rho_{E_2})$. Assume that $E_1(\Sigma_1) \subset E_2(\Sigma_2)''$. Then for every $\Delta \in \Sigma_1$ there exists a $\varphi_\Delta \in \mathbf{L}_\infty(\Omega_2, \Sigma_2, E_2)$ such that $E_1(\Delta) = \rho_{E_2}(\varphi_\Delta)$. We have $\rho_{E_2}(\varphi_{\Omega_1}) = \mathcal{I} = \rho_{E_2}(1)$. Hence $\varphi_{\Omega_1} = 1$ by the injectivity of E_2 . By Theorem 35, $\varphi_\Delta \geq 0$. The injectivity of E_2 and the σ -additivity of E_1 implies that $(\varphi_\Delta) \in \mathbf{M}(\Sigma_1; \Omega_2, \Sigma_2, E_2)$. Hence $E_1 \leftarrow E_2$.

Assume that $E_1 \leftarrow E_2$. Then $E_1(\Sigma_1) \subset \text{range}(\rho_{E_2}) = E_2(\Sigma_2)''$. \square

11. Isomorphic POVMS

Let M be a POVM defined on a σ -field Σ . Two sets $A, B \in \Sigma$ are said to be M -equivalent if $M(A \setminus B) = M(B \setminus A) = 0$. The class of sets M -equivalent to $A \in \Sigma$ is denoted by $[A]_M$. We put $\Sigma(M) = \{[A]_M : A \in \Sigma\}$. Then $\Sigma(M)$ is a Boolean σ -algebra with operations defined by $[A]_M \cup [B]_M = [A \cup B]_M$ for $A, B \in \Sigma$, etc. We define M on $\Sigma(M)$ by $M([A]_M) = M(A)$.

DEFINITION 54. A boolean isomorphism $\Phi: \Sigma_1(M_1) \rightarrow \Sigma_2(M_2)$ is a bijective mapping such that

$$\Psi([A]_M \setminus [B]_M) = \Psi([A]_M) \setminus \Psi([B]_M)$$

for $A, B \in \Sigma_1$, and

$$\Psi\left(\bigcup_{i=1}^{\infty} [A_i]_M\right) = \bigcup_{i=1}^{\infty} \Psi([A_i]_M)$$

for $(A_i) \subset \Sigma_1$.

DEFINITION 55 ([37]). Two POVMs $M_1: \Sigma_1 \rightarrow \mathbf{B}_+(\mathbf{H})$ and $M_2: \Sigma_2 \rightarrow \mathbf{B}_+(\mathbf{H})$ are isomorphic, denoting $M_1 \cong M_2$, if there exists a Boolean isomorphism

$$\Psi: \Sigma_1(M_1) \rightarrow \Sigma_2(M_2)$$

with the property

$$M_1(\Delta) = M_2(\Psi(\Delta)) \quad \forall \Delta \in \Sigma_1.$$

LEMMA 56. *If POVMs M_1 and M_2 are isomorphic ($M_1 \cong M_2$) and M_1 is countably generated, then M_2 is also countably generated.*

PROOF. Let Φ be as in Definition 55. Let $\mathfrak{A} \subset \Sigma_1$ is a field of subsets of Ω_1 that generates $\Sigma_1(M_1)$. Then $\Phi[[\mathfrak{A}]_{M_1}]$ generates $\Sigma_2(M_2)$. \square

Isomorphic POVMs are equivalent: For injective POVMs the converse is true:

THEOREM 57 ([37], Theorem 2.6). *Two equivalent POVMs which are both injective are isomorphic.*

REMARK 58. Two PVMs which generate the same von Neumann algebra are equivalent (by Proposition 53) and hence are isomorphic (by Theorem 57 and Remark 38).

PROPOSITION 59. *Let Σ be a σ -field of subsets of a set Ω . The following conditions are equivalent:*

- (a) *M is countably generated.*
- (b) *M is isomorphic to a POVM on the Borel subsets of $[0, 1]$.*

PROOF. Let $(N, \mathbf{K}, \mathbf{V})$ be a minimal Naimark extension of a POVM $M: \Sigma \rightarrow \mathbf{B}_+(\mathbf{H})$. Consider the following conditions:

- (1) *N is countably generated.*
- (2) *N is isomorphic to a POVM on the Borel subsets of $[0, 1]$.*

We will prove that the four conditions are equivalent: (1) \Leftrightarrow (a): Proposition 21.

(2) \Leftrightarrow (b): Naimark's theorem.

(1) \Leftrightarrow (2): By Proposition 21, \mathbf{K} is separable. By Proposition 230 there is a bounded self-adjoint operator \mathcal{E} on \mathbf{K} that generates commutative von Neumann algebra $N(\Sigma)''$. We can assume without loss of generality that $\|\mathcal{E}\|_\infty = 1$. Let $E: \mathfrak{B} \rightarrow \mathbf{B}_+(\mathbf{K})$ be the spectral measure of \mathcal{E} . By Remark 58, $N \cong E$.

(2) \Leftrightarrow (1): This follows from Lemma 56 because the σ -field of Borel subsets of $[0, 1]$ is countably generated. \square

12. Unitary equivalence

PROPOSITION 60. *Let $\mathbf{U}: \mathbf{H} \rightarrow \mathbf{K}$ be a linear isometry from Hilbert space \mathbf{H} into Hilbert space \mathbf{K} . Let $M_1: \Sigma_1 \rightarrow \mathbf{B}_+(\mathbf{K})$ and $M_2: \Sigma_2 \rightarrow \mathbf{B}_+(\mathbf{K})$ be two POVMs. For $\ell \in \{1, 2\}$ and $\Delta \in \Sigma_\ell$ let $N_\ell(\Delta) = \mathbf{U}^* M_\ell(\Delta) \mathbf{U}$. Then $N_\ell: \Sigma_\ell \rightarrow \mathbf{B}_+(\mathbf{H})$, $\ell \in \{1, 2\}$ are POVMs and $M_1 \leftarrow M_2$ implies $N_1 \leftarrow N_2$. If, moreover, \mathbf{U} is surjective then*

$$M_1 \leftarrow M_2 \quad \Leftrightarrow \quad N_1 \leftarrow N_2$$

and M_1 is maximal if and only if N_1 is maximal.

REMARK 61. Let $M_1: \Sigma_1 \rightarrow \mathbf{B}_\infty(\mathbf{H}_1)$ and $M_2: \Sigma_2 \rightarrow \mathbf{B}_\infty(\mathbf{H}_2)$ be two PVMs. The generated von Neumann algebras $M_1(\Sigma_1)''$ and $M_2(\Sigma_2)''$ are spatially isomorphic (as defined in Definition 235) if and only if the ranges $M_1(\Sigma_1)$ and $M_2(\Sigma_2)$ are spatially isomorphic. Assume that this is the case, i.e. that there exists a unitary operator $\mathbf{U}: \mathbf{H}_1 \rightarrow \mathbf{H}_2$ such that $M_1(\Sigma_1) = \mathbf{U}^* M_2(\Sigma_2) \mathbf{U}$. Define a PVM $M_3: \Sigma_2 \rightarrow \mathbf{B}_\infty(\mathbf{H}_1)$ by $M_3(\Delta) = \mathbf{U}^* M_2(\Delta) \mathbf{U}$. Then M_1 and M_3 are PVMs with the same range. By Remark 58, M_1

and M_3 are isomorphic; i.e. there exists a Boolean isomorphism $\Psi: \Sigma_1 \rightarrow \Sigma_2$ such that $M_1(\Delta) = M_3(\Psi(\Delta))$ for $\Delta \in \Sigma_1$. Hence the original PVMs are related by

$$M_1(\Delta) = \mathbf{U}^* M_2(\Psi(\Delta)) \mathbf{U} \quad \forall \Delta \in \Sigma_1.$$

13. Image measures

Let (Ω_1, Σ_1) and (Ω_2, Σ_2) be two measurable space. Let $M: \Sigma_1 \rightarrow \mathbf{B}_+(\mathbf{H})$ be a POVM. If $\Psi: \Omega_1 \rightarrow \Omega_2$ is a measurable function, then $\Psi(M): \Sigma_2 \rightarrow \mathbf{B}_+(\mathbf{H})$, defined by

$$\Psi(M)(\Delta) = M(\Psi^{-1}(\Delta)) = M(\{x \in \Omega_1 : \Psi(x) \in \Delta\})$$

is a POVM. $\Psi(M)$ is called the image measure corresponding to M and transformation Ψ . Marginals are examples of image measures.

REMARK 62. In probability theory (for example [20]), Ψ is called an Ω_2 -valued random variable. For each $h \in \mathbf{H}$, we have $\Psi(M)_h = \Psi(M_h)$. If $\|h\| = 1$, then $\Psi(M_h)$ is sometimes called the distribution of Ψ (corresponding to the probability measure M_h).

PROPOSITION 63. *Let (Ω_1, Σ_1) and (Ω_2, Σ_2) be two measurable spaces and let \mathbf{H} be a Hilbert space. Let $M: \Sigma_1 \rightarrow \mathbf{B}_+(\mathbf{H})$ be a POVM and $\Psi: \Omega_1 \rightarrow \Omega_2$ a measurable function. Then $\Psi(M) \leftarrow M$. If M is injective then $\Psi(M)$ is injective.*

PROOF. We have

$$\Psi(M)(\Delta) = \int_{\Omega_1} 1_{\Delta}(\Psi(x)) M(dx).$$

Hence $\Psi(M) \leftarrow M$. Let $\varphi \in \mathbf{L}_{\infty}(\Omega_2, \Sigma_2, \Psi(M))$. We have

$$\int_{\Omega_2} \varphi(y) \Psi(M)(dy) = \int_{\Omega_1} \varphi(\Psi(x)) M(dx)$$

If this is zero and M is injective, then $\varphi(\Psi(x)) = 0$ for M -almost all $x \in \Omega_1$. Equivalently, $\varphi(y) = 0$ for $\Psi(M)$ -almost all $y \in \Omega_2$. \square

14. Cones generated by POVMS

In this section we give a geometric interpretation of dominance of a POVM by an injective POVM.

The proof of the following proposition is an adaptation of the proof of Lemma 3 of Chapter IX in [35]. See also [64].

PROPOSITION 64. *Let $M: \Sigma \rightarrow \mathbf{B}_+(\mathbf{H})$ be a POVM. We have*

$$\overline{\text{co}}(M(\Sigma)) = \{\rho_M(\varphi) : 0 \leq \varphi \leq 1, \varphi \in \mathbf{L}_{\infty}(\Omega, \Sigma, M)\},$$

where $\overline{\text{co}}(M(\Sigma))$ is the weak-star closure of the convex hull of $M(\Sigma)$.

PROOF. By Proposition 33, $\rho_M: (\mathbf{L}_{\infty}(\Omega, \Sigma, M), \text{weak}^*) \rightarrow (\mathbf{B}_{\infty}, \text{weak}^*)$ is continuous. Let $U = \{\varphi \in \mathbf{L}_{\infty}(\Omega, \Sigma, M) : 0 \leq \varphi \leq 1\}$. Then U is a compact convex subset of $(\mathbf{L}_{\infty}(\Omega, \Sigma, M), \text{weak}^*)$. Hence $\rho_M(U)$ is a compact convex subset of $(\mathbf{B}_{\infty}, \text{weak}^*)$. Together with $M(\Sigma) \subset \rho_M(U)$, this implies that $\overline{\text{co}}(M(\Sigma)) \subset \rho_M(U)$. The proof of the reverse inclusion: The simple functions in U form a dense subset S of (U, weak^*) . The continuity of ρ_M implies that $\rho_M(S)$ is dense in $(\rho_M(U), \text{weak}^*)$. In the proof of Lemma 3 of Chapter IX in [35] it is shown that $\rho_M(S) \subset \text{co}(M(\Sigma))$. Hence $\rho_M(U) \subset \overline{\text{co}}(M(\Sigma))$. \square

DEFINITION 65. Let $M: \Sigma \rightarrow \mathbf{B}_+(\mathbf{H})$ be a POVM. The cone generated by M is denoted and defined by

$$\begin{aligned} \text{cone}(M) &= \{\rho_M(\varphi) : \varphi \in \mathbf{L}_{\infty}(\Omega, \Sigma, M), \varphi \geq 0\} \\ &= \bigcup_{r>0} r \overline{\text{co}}(M(\Sigma)). \end{aligned}$$

PROPOSITION 66. *Let Σ, Σ_2 be two σ -fields of subsets of sets Ω, Ω_2 respectively. Let $N: \Sigma \rightarrow \mathbf{B}_+(\mathbf{H})$ and $M: \Sigma_2 \rightarrow \mathbf{B}_+(\mathbf{H})$ be two POVMs. If $N \leftarrow M$ then $\overline{\text{co}}(N(\Sigma)) \subset \overline{\text{co}}(M(\Sigma))$; in particular $\text{cone}(N) \subset \text{cone}(M)$. If M is equivalent to an injective POVM then*

$$N \leftarrow M \quad \Leftrightarrow \quad \text{cone}(N) \subset \text{cone}(M).$$

PROOF. Assume that $N \leftarrow M$. By Proposition 64, $N(\Sigma) \subset \overline{\text{co}}(M(\Sigma_2))$. Hence $\overline{\text{co}}(N(\Sigma)) \subset \overline{\text{co}}(M(\Sigma_2))$. By Proposition 64, this implies $\text{cone}(N) \subset \text{cone}(M)$.

Assume that M is injective and $\text{cone}(N) \subset \text{cone}(M)$. This implies that there is a function $\Delta \mapsto \varphi_\Delta$ from Σ to $(\mathbf{L}_\infty(\Omega_2, \Sigma_2, M))_+$ such that $N(\Delta) = \rho_M(\varphi_\Delta)$ for all Δ . Because M is injective, $\varphi_\Omega = 1$ and $\varphi_\emptyset = 0$.

Let $\Delta = \Delta_1 \cup \Delta_2$, with $\Delta_n \in \Sigma$, be a disjoint union. Because N is additive,

$$\rho_M(\varphi_\Delta) = \rho_M(\varphi_{\Delta_1} + \varphi_{\Delta_2}).$$

Because M is injective, this implies that $\varphi_\Delta = \varphi_{\Delta_1} + \varphi_{\Delta_2}$. We consider Σ to be ordered by inclusion, and we considered $\mathbf{L}_\infty(\Omega_2, \Sigma_2, M)$ to be ordered as usual; i.e. pointwise and disregarding sets of measure zero. Then $\Delta \mapsto \varphi_\Delta$ is monotone increasing. Hence $0 \leq \varphi_\Delta \leq 1$ for all $\Delta \in \Sigma$.

Let $\Delta = \cup_{n=1}^\infty \Delta_n$, with $\Delta_n \in \Sigma$, be a disjoint union. Then

$$\sum_{n=1}^N \varphi_{\Delta_n} = \varphi_{\cup_{n=1}^N \Delta_n} \leq \varphi_\Delta \quad \forall N \in \mathbb{N}$$

Let μ be a finite positive measure with the same sets of measure zero as M . For every positive $f \in \mathbf{L}_1(\Omega_2, \Sigma_2, \mu)$, the sequence of positive numbers

$$N \mapsto \sum_{n=1}^N \int_{\Omega} f(x) \varphi_{\Delta_n}(x) \mu(dx)$$

is monotone increasing and bounded, and hence a Cauchy sequence. Every integrable function can be written as a linear combination of positive integrable functions. Hence $N \mapsto \sum_{n=1}^N \varphi_{\Delta_n}$ is a weak-star Cauchy sequence. By Proposition 212, there exists a $\varphi \in \mathbf{L}_\infty(\Omega_2, \Sigma_2, M)$ such that $\varphi = \lim_{N \rightarrow \infty} \sum_{n=1}^N \varphi_{\Delta_n}$ w.r.t. the weak-star topology. By Proposition 33 and the linearity of ρ_M ,

$$\rho_M(\varphi) = \sum_{n=1}^{\infty} \rho_M(\varphi_{\Delta_n}).$$

Because N is σ -additive,

$$\rho_M(\varphi_\Delta) = \sum_{n=1}^{\infty} \rho_M(\varphi_{\Delta_n}) = \rho_M(\varphi).$$

The injectivity of M implies that $\varphi_\Delta = \varphi$. By Lemma 45, this implies that $(\varphi_\Delta) \in \mathbf{M}(\Sigma; \Omega_2, \Sigma_2, M)$. Hence $N \leftarrow M$.

Assume now that M is equivalent to an injective POVM M_2 . From $M \leftarrow M_2$ and the first part of the proposition it follows that $\text{cone}(M) \subset \text{cone}(M_2)$. Because M_2 is injective, $\text{cone}(N) \subset \text{cone}(M)$ implies $N \leftarrow M_2$. Because $M_2 \leftarrow M$, this implies $N \leftarrow M$. \square

PROPOSITION 67. *Let $M: \Sigma \rightarrow \mathbf{B}_+(\mathbf{H})$ be a POVM. Every extreme point of $\overline{\text{co}}(M(\Sigma))$ belongs to $M(\Sigma)$. If M is injective then $M(\Sigma) = \text{ext}(\overline{\text{co}}(M(\Sigma)))$, the set of extreme points of $\overline{\text{co}}(M(\Sigma))$.*

PROOF. The first part of the proposition is a special case of Theorem 1 of Chapter IX of [35]. Assume that M is injective and $\Delta \in \Sigma$. If $\varphi_1, \varphi_2 \in L_\infty(\Omega, \Sigma, M)$, $0 \leq \varphi_1, \varphi_2 \leq 1$ and $M(\Delta) = \frac{1}{2}(\rho_M(\varphi_1) + \rho_M(\varphi_2))$ then $1_\Delta = \frac{1}{2}(\varphi_1 + \varphi_2)$. Hence $\varphi_1 = \varphi_2 = 1_\Delta$. Hence $\rho_M(\varphi_1) = \rho_M(\varphi_2) = M(\Delta)$. By Proposition 64, this shows that $M(\Delta) \subset \text{ext}(\overline{\text{co}}(M(\Sigma)))$. \square

15. Maximal PVMs

In this section we show that a finite set of strongly commuting self-adjoint operators on a separable complex Hilbert space \mathbf{H} is a complete set of operators if, and only if, the joint spectral measure is a maximal PVM. A finite set of strongly commuting self-adjoint operators is called complete if the joint spectral measure generates a maximal commutative von Neumann algebra, or equivalently, if the set is of uniform multiplicity one ([104]). The related concept of a complete set of observables was introduced by Dirac ([36]). In a Hilbert space formulation of quantum mechanics ([106], [90], [15]), Dirac's heuristic formulation of this concept becomes rigorous only in the case of a set $\{\mathcal{A}_1, \dots, \mathcal{A}_n\}$ of self-adjoint operators on a separable Hilbert space \mathbf{H} having pure point spectra. In that case $\{\mathcal{A}_1, \dots, \mathcal{A}_n\}$ is called a complete set of operators if:

- (a) To each n -tuple $(\lambda_1, \dots, \lambda_n)$ in the joint spectrum belongs a vector $\Psi_{\lambda_1, \dots, \lambda_n}$ from the common domain of $\mathcal{A}_1, \dots, \mathcal{A}_n$ which satisfies

$$\mathcal{A}_k \Psi_{\lambda_1, \dots, \lambda_n} = \lambda_k \Psi_{\lambda_1, \dots, \lambda_n}, \quad k = 1, \dots, n,$$

and

- (b) $(\Psi_{\lambda_1, \dots, \lambda_n})$ is an orthonormal basis of \mathbf{H} .

Even without the limitation to pure point spectra, a mathematically rigorous interpretation of Dirac's formulation (of the concept of a complete set of observables) is possible: [104] and [102]. To do this, Hilbert space is replaced by a system of two topological vector spaces: One consisting of bras, the other consisting of kets. In this section an extension of the Hilbert space is not needed because conditions (a) and (b) are equivalent to the existence of a unitary operator \mathbf{U} from \mathbf{H} to the space $\ell_2(\sigma)$ of square summable \mathbb{C} -valued functions on the joint spectrum σ such that for all k , $\mathbf{U}\mathcal{A}_k\mathbf{U}^* = \mathcal{Q}_k$, the operator on $\ell_2(\sigma)$ of multiplication with $(\lambda_1, \dots, \lambda_n) \mapsto \lambda_k$. This condition is easily generalized (Proposition 70) to the case where the joint spectral measure on σ is replaced by a PVM on a measurable space (Ω, Σ) .

THEOREM 68. *A PVM is maximal if, and only if, it generates a maximal commutative von Neumann algebra.*

PROOF. Let $M: \Sigma \rightarrow \mathbf{B}_+(\mathbf{H})$ be a PVM.

It follows from Proposition 53 that maximality of M is a necessary condition for M to generate a maximal commutative von Neumann algebra.

Assume now that M generates a maximal commutative von Neumann algebra. Then Proposition 53 implies that M is maximal if we can prove that every POVM M_2 that dominates M is projection-valued. We combine the proofs of Theorem 3.7 and Remark 3 in [37] to do this: Let $\mathcal{P} = M(\Delta)$. We will show that $\mathcal{P} \in M_2(\Sigma)$: There exists a M_2 -measurable function p taking values in $[0, 1]$ such that

$$\mathcal{P} = \int_{\Omega_2} p(y) M_2(dy).$$

Let $n \in \mathbb{N}$ and

$$Y_n = \left\{ y : \frac{1}{n} \leq p(y) \leq 1 - \frac{1}{n} \right\}.$$

Then

$$\frac{1}{n}M_2(Y_n) = \int_{Y_n} \frac{1}{n} M_2(dy) \leq \int_{Y_n} p(y) M_2(dy) \leq \mathcal{P}.$$

and

$$\begin{aligned} \frac{1}{n}M_2(Y_n) &= \mathcal{I} - \left(1 - \frac{1}{n}\right)M_2(Y_n) - M_2(Y_n^c) \\ &\leq \mathcal{I} - \int_{Y_n} p(y) M_2(dy) - \int_{Y_n^c} p(y) M_2(dy) \\ &= \mathcal{I} - \mathcal{P}. \end{aligned}$$

Hence $M_2(Y_n) = 0$. This is true for all $n \in \mathbb{N}$. Hence $p(y) \in \{0, 1\}$ for M_2 -almost all y . Hence $\mathcal{P} \in M_2(\Sigma_2)$.

We will now show that $\mathcal{P} = M_2(A) \in M_2(\Sigma_2)'$: We have

$$M_2(A)M_2(B) = M_2(A)M_2(B \cap A) + M_2(A)M_2(B \cap A^c).$$

Because $M_2(A)$ is a projection, $M_2(A^c)$ is also a projection and

$$M_2(A^c)M_2(A) = M_2(A)M_2(A^c) = 0.$$

From this, together with Corollary 239, we see that: $0 \leq M_2(B \cap A) \leq M_2(A)$ implies $M_2(A)M_2(B \cap A) = M_2(B \cap A)$, and $0 \leq M_2(B \cap A^c) \leq M_2(A^c)$ implies $M_2(A)M_2(B \cap A^c) = 0$. Hence

$$M_2(A)M_2(B) = M_2(B \cap A).$$

It follows similarly that $M_2(B)M_2(A) = M_2(B \cap A)$. Hence $\mathcal{P} \in M_2(\Sigma_2)'$.

This is true for all $\mathcal{P} \in M_1(\Sigma)$. Hence $M_1(\Sigma) \subset M_2(\Sigma_2)'$. Hence $M_2(\Sigma_2)'' \subset M_1(\Sigma)' = M_1(\Sigma)$. Hence M_2 is projection-valued. \square

PROPOSITION 69. *Let (Ω, Σ, μ) be a finite measure space. $\mathbf{1}(\Sigma)''$ is a maximal commutative von Neumann algebra; More precisely: $\mathbf{1}(\Sigma)'' = \mathbf{A}_\mu$, where \mathbf{A}_μ is defined by Definition 233.*

PROOF. By Theorem 52, $(\mathbf{1}(\Sigma)'')_p = \mathbf{1}(\Sigma) = (\mathbf{A}_\mu)_p$. Because every von Neumann algebra is the norm closed linear span of its projections, this implies that $\mathbf{1}(\Sigma)'' = \mathbf{A}_\mu$. \square

PROPOSITION 70. *Let $E: \Sigma \rightarrow \mathbf{B}_+(\mathbf{H})$ be a PVM. The following conditions are equivalent:*

- (a) E is maximal.
- (b) There exists a probability measure μ on Σ and a unitary operator $\mathbf{U}: \mathbf{H} \rightarrow \mathbf{L}_2(\Omega, \Sigma, \mu)$ such that $E(\Delta) = \mathbf{U}^* \mathbf{1}(\Delta) \mathbf{U}$ for $\Delta \in \Sigma$; i.e.

$$(18) \quad E_h(\Delta) = \int_{\Delta} |\mathbf{U}[h](x)|^2 \mu(dx) \quad \forall \Delta \in \Sigma, h \in \mathbf{H}.$$

PROOF. (b) implies (a): This follows from Proposition 69, Theorem 68 and Proposition 60.

(a) implies (b): Combining Theorem 68, Proposition 69 and Theorem 236 and Remark 61, we see that there is a compact metric space Ω_2 , a regular Borel measure μ_2 on the Borel subsets \mathfrak{B} of Ω_2 and a unitary operator $\mathbf{W}: \mathbf{H} \rightarrow \mathbf{L}_2(\Omega_2, \mathfrak{B}, \mu_2)$ such that $E(\Sigma) = \mathbf{W}^* \mathbf{1}(\mathfrak{B}) \mathbf{W}$. By Remark 61 there is a Boolean isomorphism $\Psi: \Sigma(E) \rightarrow \mathfrak{B}(\mathbf{1})$ such that

$$(\forall \Delta \in \Sigma) \quad E(\Delta) = \mathbf{W}^* \mathbf{1}(\Psi(\Delta)) \mathbf{W}.$$

Hence

$$(\forall \Delta \in \Sigma)(\forall h \in \mathbf{H}) \quad E_h(\Delta) = \int_{\Psi(\Delta)} |\mathbf{W}[h](x)|^2 \mu_2(dx)$$

Define probability measure μ on Σ by $\mu(\Delta) = \mu_2(\Psi(\Delta))/\mu_2(\Omega_2)$. Define unitary operator $\mathbf{V}_\Psi: \mathbf{L}_2(\Omega, \Sigma, \mu) \rightarrow \mathbf{L}_2(\Omega_2, \mathfrak{B}, \mu_2)$ by $\mathbf{V}_\Psi[\mathbf{1}_\Delta] = \mu_2(\Omega_2)^{-1} \cdot \mathbf{1}_{\Psi(\Delta)}$ for $\Delta \in \Sigma$ and linear and isometric extension. It is easily seen that $\mathbf{U} = \mathbf{V}_\Psi^* \mathbf{W}$ satisfies (18). \square

16. Examples of maximal PVMs

Let $\Phi: \mathbb{N}_0 \rightarrow \mathbf{L}_2(\mathbb{R})$ be the Hermite basis. Let $\mathbf{1}: \mathfrak{B}_{\mathbb{R}} \rightarrow \mathbf{B}_+(\mathbf{L}_2(\mathbb{R}))$ be the PVM of definition 9. Then

$$\mathbf{1}_h(\Delta) = \int_{\Delta} |h(q)|^2 dq.$$

By Proposition 70, $\mathbf{1}$ is a maximal POVM. Let $\hat{\mathbf{1}}: \mathfrak{B}_{\mathbb{R}} \rightarrow \mathbf{B}_+(\mathbf{L}_2(\mathbb{R}))$ be the PVM defined by $\hat{\mathbf{1}}(\Delta) = \mathcal{F}^* \mathbf{1}(\Delta) \mathcal{F}$, where \mathcal{F} is Fourier transform on $\mathbf{L}_2(\mathbb{R})$. Then $\hat{\mathbf{1}}_h(\Delta) = \mathbf{1}_{\mathcal{F}[h]}(\Delta)$:

$$\hat{\mathbf{1}}_h(\Delta) = \int_{\Delta} |\mathcal{F}[h](p)|^2 dp,$$

By Proposition 70, $\hat{\mathbf{1}}$ is a maximal POVM. The σ -field of all subsets of \mathbb{N}_0 is denoted by $2^{\mathbb{N}_0}$. Let $N: 2^{\mathbb{N}_0} \rightarrow \mathbf{B}_+(\mathbf{L}_2(\mathbb{R}))$ be the PVM defined by on the singletons by $N(\{n\}) = \varphi_n \otimes \varphi_n$. Then

$$N_h(\Delta) = \sum_{n \in \Delta} |(\varphi_n, h)|^2$$

where (φ_n) is the Hermite basis of $\mathbf{L}_2(\mathbb{R})$. By Proposition 70, N is a maximal POVM.

The PVMs $\mathbf{1}$, $\hat{\mathbf{1}}$ and N are (the) mathematical representations of quantum mechanical measurements of position, momentum and number observables respectively. The (self-adjoint) position, momentum and number operators are defined as

$$\mathcal{Q} = \int_{-\infty}^{\infty} q \mathbf{1}(dq), \quad \mathcal{P} = \int_{-\infty}^{\infty} p \hat{\mathbf{1}}(dp), \quad \mathcal{N} = \sum_{n=0}^{\infty} n N(\{n\})$$

respectively.

17. Maximal POVMs

In this section we present the main result of this chapter. The following six lemmas are used to prove it.

Let (Ω, Σ, μ) be a measure space. Let \mathbf{V} be a vector space. A family $L_x: \mathbf{V} \rightarrow \mathbb{C}$, $x \in \Omega$ is called weakly μ -measurable if for each $v \in \mathbf{V}$, the function $x \mapsto L_x(v)$ is μ -measurable. The same terminology is used for families of elements of a Hilbert space (the elements of a Hilbert space are considered as linear forms on the Hilbert space), and for families of bounded operators on a Hilbert space (the bounded operators are considered as linear forms on the space of trace-class operators).

LEMMA 71. *Let \mathbf{H} be a separable complex Hilbert space. Let (Ω, Σ, μ) be a measure space. Let \mathcal{A}_x , $x \in \Omega$ be a weakly μ -measurable family of bounded operators on \mathbf{H} . Let e_x , $x \in \Omega$ be a weakly μ -measurable family of points of \mathbf{H} . Assume that for all $h \in \mathbf{H}$,*

$$(h, \mathcal{A}_x h) \leq |(e_x, h)|^2 \quad \text{for } \mu\text{-almost all } x.$$

Then there exists a Σ -measurable function $f: \Omega \rightarrow [0, 1]$ such that

$$\mathcal{A}_x = f(x) e_x \otimes e_x \quad \text{for } \mu\text{-almost all } x.$$

PROOF. Let $\mathbf{H}_0 = \{g_n : n \in \mathbb{N}\}$ be a countable dense subset of \mathbf{H} . The union of a countable family of μ -null sets is again a μ -null set. Hence there exists a μ -null set \mathfrak{N} such that

$$(h, \mathcal{A}_x h) \leq |(e_x, h)|^2 \quad \forall x \in \mathfrak{N}^c, h \in \mathbf{H}_0.$$

This implies that

$$\mathcal{A}_x \leq e_x \otimes e_x \quad \forall x \in \mathfrak{N}^c.$$

By Corollary 239, there is a function $\tilde{f}: \Omega \rightarrow [0, 1]$ such that

$$\mathcal{A}_x = \tilde{f}(x) e_x \otimes e_x \quad \forall x \in \mathfrak{N}^c.$$

We will prove that there is a Σ -measurable function $f: \Omega \rightarrow [0, 1]$ such that $\tilde{f}(x) = f(x)$ for all $x \in \mathfrak{N}^c$ with $e_x \neq 0$. For all $h \in \mathbf{H}$, there exists a Σ -measurable function $x \mapsto f_x(h)$ from Ω to $[0, 1]$ such that

$$(h, \mathcal{A}_x h) = f_x(h) |(e_x, h)|^2 \quad \text{for } \mu\text{-almost all } x.$$

Let

$$\Omega_n = \{x : (e_x, g_n) \neq 0\} \setminus \bigcup_{k=1}^{n-1} \Omega_k.$$

Then (Ω_n) is a family of pairwise disjoint sets in Σ such that

$$\Omega = \{x : e_x = 0\} \cup \bigcup_{n \in \mathbb{N}} \Omega_n \quad \text{and} \quad (e_x, g_n) \neq 0 \quad \forall x \in \Omega_n$$

for all $n \in \mathbb{N}$. Define the Σ -measurable function $f: \Omega \rightarrow [0, 1]$ by

$$f(x) = \begin{cases} f_x(g_n) & \text{if } x \in \Omega_n; \\ 0 & \text{if } e_x = 0. \end{cases}$$

Then $f(x) |(e_x, g_n)|^2 = \tilde{f}(x) |(e_x, g_n)|^2$ for $x \in \Omega_n \setminus \mathfrak{N}$. Hence $f(x) = \tilde{f}(x)$ for $x \in \Omega_n \setminus \mathfrak{N}$. This is true for all n . Hence $f(x) = \tilde{f}(x)$ for all $x \in \mathfrak{N}^c$ with $e_x \neq 0$. \square

LEMMA 72. *Let (Ω, Σ, ν) be a finite measure space, let $M: \Sigma \rightarrow \mathbf{B}_+(\mathbf{H})$ be an FPOVM, and let $\mathbf{V}: \mathbf{H} \rightarrow \mathbf{L}_2(\Omega, \Sigma, \nu)$ be a linear isometry. Define POVM $N: \Sigma \rightarrow \mathbf{B}_+(\mathbf{H})$ by*

$$N_h(\Delta) = \int_{\Delta} |\mathbf{V}[h](x)|^2 \nu(dx), \quad \Delta \in \Sigma, h \in \mathbf{H}.$$

Assume that $M(\Delta) \leq N(\Delta)$ for all $\Delta \in \Sigma$. Then there is a ν -measurable function φ such that $0 \leq \varphi \leq 1$ and

$$(19) \quad (\forall \Delta \in \Sigma) \quad M(\Delta) = \int_{\Delta} \varphi(x) N(dx).$$

PROOF. By Lemma 199, we can (and will) assume without loss of generality that ν and N have the same sets of measure zero.

Let \mathcal{R} be a Hilbert-Schmidt operator on \mathbf{H} with dense range. By Proposition 27, there is a probability measure μ on Σ with the same sets of measure zero as M and there are $\mathcal{M}_x \in \mathbf{B}_+(\mathbf{H})$ such that

$$(\Delta \in \Sigma)(\forall h \in \mathbf{H}) \quad M_{\mathcal{R}h}(\Delta) = \int_{\Delta} (h, \mathcal{M}_x h) \mu(dx).$$

From $M \ll N$ follows $\mu \ll \nu$. Let f_ν be the Radon-Nikodym derivative of μ with respect to ν : $\mu = f_\nu \cdot \nu$. Let $\tilde{\mathcal{M}}_x = f_\nu(x) \mathcal{M}_x$. Then

$$(20) \quad (\forall \Delta \in \Sigma)(\forall h \in \mathbf{H}) \quad M_{\mathcal{R}h}(\Delta) = \int_{\Delta} (h, \tilde{\mathcal{M}}_x h) d\nu(x).$$

\mathcal{R} is a Hilbert-Schmidt operator. Hence $h \mapsto \mathbf{V}[\mathcal{R}h]$ is a Hilbert-Schmidt operator and hence a Carleman operator: There are $e_x \in \mathbf{H}$ such that $x \mapsto (e_x, h)$ is a ν -measurable function for all $h \in \mathbf{H}$ and

$$\mathbf{V}[\mathcal{R}h](x) = (e_x, h) \quad \nu\text{-almost all } x.$$

This, $M(\Delta) \leq N(\Delta)$ for $\Delta \in \Sigma$, and (20), implies that for each $h \in \mathbf{H}$,

$$(21) \quad (h, \tilde{\mathcal{M}}_x h) \leq |(e_x, h)|^2 \quad \text{for } \nu\text{-almost all } x.$$

By Lemma 71, there exists a Σ -measurable function $\varphi: \Omega \rightarrow [0, 1]$ such that $(h, \tilde{\mathcal{M}}_x h) = \varphi(x)|(e_x, h)|^2$ for ν -almost all x . Let $\Delta \in \Sigma$. For $h \in \mathbf{H}$,

$$M_{\mathcal{R}h}(\Delta) = \int_{\Delta} \varphi(x) N_{\mathcal{R}h}(dx).$$

Because $\text{range}(\mathcal{R})$ is a dense subset of \mathbf{H} , this implies that

$$(\forall h \in \mathbf{H}) \quad M_h(\Delta) = \int_{\Delta} \varphi(x) N_h(dx).$$

□

LEMMA 73. *Let Σ and Σ_2 be σ -fields of subsets of sets Ω and Ω_2 respectively. Let $M: \Sigma \rightarrow \mathbf{B}_+(\mathbf{H})$ and $N: \Sigma_2 \rightarrow \mathbf{B}_+(\mathbf{H})$ be two POVMs. If $\text{cone}(N) \subset \text{cone}(M)$ and there is a finite measure ν on Σ_2 and a linear isometry $\mathbf{W}: \mathbf{H} \rightarrow \mathbf{L}_2(\Omega_2, \Sigma_2, \nu)$ such that*

$$(\forall \Delta_2 \in \Sigma_2)(\forall h \in \mathbf{H}) \quad N_h(\Delta_2) = \int_{\Delta_2} |\mathbf{W}[h](y)|^2 d\nu(y)$$

then there are $(p_{\Delta_2}) \in \mathbf{M}(\Sigma_2; \Omega, \Sigma, M)$ and $(q_{\Delta}) \in \mathbf{M}(\Sigma; \Omega_2, \Sigma_2, N)$ such that

$$(22) \quad (\forall \Delta \in \Sigma, \Delta_2 \in \Sigma_2) \quad \int_{\Delta} p_{\Delta_2}(x) M(dx) = \int_{\Delta_2} q_{\Delta}(y) N(dy).$$

In particular $\text{cone}(N) = \text{cone}(M)$ and $M \leftrightarrow N$.

PROOF. For every $\Delta_2 \in \Sigma_2$ there is a function $p_{\Delta_2}: \Omega \rightarrow [0, \infty)$ such that

$$\int_{\Omega} p_{\Delta_2}(x) M(dx) = N(\Delta_2).$$

Let $\Delta \in \Sigma$. Then

$$(\forall \Delta_2 \in \Sigma_2) \quad \int_{\Delta} p_{\Delta_2}(x) M(dx) \leq N(\Delta_2).$$

By Lemma 72 there exists a function $q_{\Delta}: \Omega_2 \rightarrow [0, 1]$ such that

$$(\forall \Delta_2 \in \Sigma_2) \quad \int_{\Delta} p_{\Delta_2}(x) M(dx) = \int_{\Delta_2} q_{\Delta}(y) N(dy).$$

This is true for all $\Delta \in \Sigma$. Hence (22). It is easily seen that $(p_{\Delta_2}) \in \mathbf{M}(\Sigma_2; \Omega, \Sigma, M)$ and $(q_{\Delta}) \in \mathbf{M}(\Sigma; \Omega_2, \Sigma_2, N)$. □

LEMMA 74. *Let $s: \mathbf{H} \times \mathbf{H} \rightarrow \mathbb{C}$ be a positive bounded non-zero sesquilinear form and let q be the associated quadratic form: $q(h) = s(h, h)$. The following conditions are equivalent:*

- (a) $|s(f, g)|^2 = q(f)q(g)$ for all $f, g \in \mathbf{H}$.
- (b) $\dim(\ker(q)^\perp) = 1$.

PROOF. If $q(g) = 0$ then $q(f + cg) = q(f) + 2 \text{Re } s(f, cg) \geq 0$ for all $c \in \mathbb{C}$ and $f \in \mathbf{H}$. This implies that $s(f, g) = 0$ and $q(f + g) = q(f)$ for all $f \in \mathbf{H}$. This implies that $\ker(q)$ is a linear subspace of \mathbf{H} . Because s is bounded, $\ker(q)$ is a closed linear subspace of \mathbf{H} . Because s is non-zero, q is also non-zero: $\ker(q) \neq \mathbf{H}$.

(b) implies (a): Assume that (b) is satisfied: There exists an $h \in \mathbf{H}$ such that $\ker(q)^\perp = \text{span}\{h\}$. Every $f \in \mathbf{H}$ has a unique decomposition of the form $f = \alpha_f h + f_0$, where $f_0 \in \ker(q)$ and $\alpha_f \in \mathbb{C}$. This implies the following:

$$\begin{aligned} \text{Re } s(f, g) &= \frac{1}{4}\{q(f+g) - q(f-g)\} = \frac{1}{4}\{q((\alpha_f + \alpha_g)h) - q((\alpha_f - \alpha_g)h)\} \\ &= \frac{1}{4}\{|\alpha_f + \alpha_g|^2 - |\alpha_f - \alpha_g|^2\} q(h) = \text{Re}(\alpha_f \bar{\alpha}_g) q(h) \end{aligned}$$

and similarly $\text{Im } s(f, g) = \text{Im}(\alpha_f \bar{\alpha}_g) q(h)$. Hence $s(f, g) = \alpha_f \bar{\alpha}_g q(h)$. Hence $|s(f, g)|^2 = q(f)q(g)$.

(a) implies (b): If $c \in \mathbb{C}$ then

$$q(f + cg) = q(f) + |c|^2 q(g) + 2 \text{Re } s(f, cg), \quad f, g \in \mathbf{H}.$$

Hence if (a) is satisfied, $c \in \mathbb{R}$ and $s(f, g) \neq 0$ then

$$q(f + c \frac{|s(f, g)|}{s(f, g)} g) = (\sqrt{q(f)} + c\sqrt{q(g)})^2.$$

This is zero for some $c \in \mathbb{R}$. Hence $q(g) \neq 0$ implies that for every $f \in \mathbf{H}$ there exists an $\alpha \in \mathbb{C}$ such that $f + \alpha g \in \ker(q)$. (If $s(f, g) = 0$ we take $\alpha = 0$.) Hence $\mathbf{H} = \ker(q) \oplus \text{span}\{g\}$. Hence $\ker(q)^\perp = \text{span}\{g\}$. This implies (b), because the existence of a $g \perp \ker(q)$ with $q(g) \neq 0$ is implied by $\ker(q) \neq \mathbf{H}$. \square

LEMMA 75. *Let (Ω, Σ, μ) be a finite measure space. Let $s_x: \mathbf{H} \times \mathbf{H} \rightarrow \mathbb{C}$, $x \in \Omega$ be a family of positive bounded non-zero sesquilinear forms such that $x \mapsto s_x(f, g)$ is a Σ -measurable function for all $f, g \in \mathbf{H}$. Let q_x , $x \in \Omega$ be the associated quadratic forms: $q_x(h) = s_x(h, h)$ for all $h \in \mathbf{H}$. The following conditions are equivalent:*

(a) $\dim(\ker(q_x)^\perp) = 1$ for μ -almost all $x \in \Omega$.

(b) *There exists a finite measure space $(\Omega_2, \Sigma_2, \mu_2)$ and a surjective measurable function $\Psi: \Omega_2 \rightarrow \Omega_1$ and a family \mathcal{P}_y , $y \in \Omega_2$ of operators with one-dimensional range such that $\mu_1 = \Psi(\mu_2)$ and for all $f, g \in \mathbf{H}$,*

$$s_{\Psi(y)}(f, g) = s_{\Psi(y)}(\mathcal{P}_y f, \mathcal{P}_y g) \quad \mu_2\text{-almost all } y \in \Omega_2.$$

PROOF. (a) implies (b): Let $\Omega_2 = \Omega$, and $\Psi(y) = y$ for all y , and $\Sigma_2 = \Sigma$, and $\mu_2 = \mu$, and let \mathcal{P}_y be the operator of orthogonal projection on $\ker(q_y)^\perp$.

(b) implies (a): Follows from Lemma 74 and

$$\begin{aligned} \int_{\Psi(\Delta)} |s_x(f, g)|^2 \mu(dx) &= \int_{\Delta} |s_{\Psi(y)}(f, g)|^2 \mu_2(dy) \\ &= \int_{\Delta} |s_{\Psi(y)}(\mathcal{P}_y f, \mathcal{P}_y g)|^2 \mu_2(dy) \\ &= \int_{\Delta} q_{\Psi(y)}(\mathcal{P}_y f) q_{\Psi(y)}(\mathcal{P}_y(g)) \mu_2(dy) \\ &= \int_{\Delta} q_{\Psi(y)}(f) q_{\Psi(y)}(g) \mu_2(dy) \\ &= \int_{\Psi(\Delta)} q_x(f) q_x(g) \mu(dx). \end{aligned}$$

\square

LEMMA 76. *Let (Ω, Σ, μ) and $(\tilde{\Omega}, \tilde{\Sigma}, \nu)$ be two finite positive measure spaces. Let $M: \Sigma \rightarrow \mathbf{B}_+(\mathbf{H})$ and $N: \tilde{\Sigma} \rightarrow \mathbf{B}_+(\mathbf{H})$ be two POVMs. Assume that M has a bounded operator density (\mathcal{M}_x) with respect to μ , and that N has a bounded operator density (\mathcal{N}_y) with respect to ν . The following two conditions are equivalent:*

(a) There are $(p_{\tilde{\Delta}}) \in \mathbf{M}(\tilde{\Sigma}; \Omega, \Sigma, \mu)$ and $(q_{\tilde{\Delta}}) \in \mathbf{M}(\Sigma; \tilde{\Omega}, \tilde{\Sigma}, \nu)$ such that

$$(\forall \Delta \in \Sigma, \tilde{\Delta} \in \tilde{\Sigma}) \quad \int_{\Delta} p_{\tilde{\Delta}}(x) \mathcal{M}_x \mu(dx) = \int_{\tilde{\Delta}} q_{\tilde{\Delta}}(y) \mathcal{N}_y \nu(dy)$$

(b) There exist measures m and n on $\Sigma \times \tilde{\Sigma}$ such that

$$m(\cdot \times \tilde{\Delta}) \ll \mu \quad \text{and} \quad n(\Delta \times \cdot) \ll \nu \quad \forall \Delta \in \Sigma, \tilde{\Delta} \in \tilde{\Sigma},$$

and

$$\mathcal{M}_x m(dx, dy) = \mathcal{N}_y n(dx, dy).$$

Condition (b) implies

(c) $\mathcal{M}_x m(dx, dy) = \tilde{\mathcal{P}}_y \mathcal{M}_x \tilde{\mathcal{P}}_y m(dx, dy)$ and $\mathcal{N}_x n(dx, dy) = \mathcal{P}_x \mathcal{N}_y \mathcal{P}_x n(dx, dy)$, where $\tilde{\mathcal{P}}_y$ is the operator of orthogonal projection on $\text{range}(\mathcal{N}_y)$, and \mathcal{P}_x is the operator of orthogonal projection on $\text{range}(\mathcal{M}_x)$.

Condition (c) implies

(d) $\text{range}(\mathcal{M}_x)$ is one-dimensional for μ -almost all $x \in \Omega$ if, and only if, $\text{range}(\mathcal{N}_y)$ is one-dimensional for ν -almost all y .

PROOF. (b) implies (a): Let $p_{\tilde{\Delta}}$ be the Radon-Nikodym derivative of $\Delta \mapsto m(\Delta \times \tilde{\Delta})$ with respect to μ : $m(\Delta \times \tilde{\Delta}) = \int_{\tilde{\Delta}} p_{\tilde{\Delta}}(x) \mu(dx)$. Let $q_{\tilde{\Delta}}$ be the Radon-Nikodym derivative of $\tilde{\Delta} \mapsto n(\Delta \times \tilde{\Delta})$ with respect to ν : $n(\Delta \times \tilde{\Delta}) = \int_{\tilde{\Delta}} q_{\tilde{\Delta}}(y) \nu(dy)$.

(a) implies (b): By Lemma 51, there are measures m and n on $\Sigma \times \tilde{\Sigma}$ such that

$$m(\Delta \times \tilde{\Delta}) = \int_{\Delta} p_{\tilde{\Delta}}(x) \mu(dx) \quad \text{and} \quad n(\Delta \times \tilde{\Delta}) = \int_{\tilde{\Delta}} q_{\tilde{\Delta}}(y) \nu(dy).$$

(b) implies (c): By Lemma 243, $x \mapsto \mathcal{M}_x$ is a SOT μ -measurable function from Ω to $\mathbf{B}_+(\mathbf{H})$ and $y \mapsto \mathcal{N}_y$ is a SOT ν -measurable function from $\tilde{\Omega}$ to $\mathbf{B}_+(\mathbf{H})$. By Proposition 246, $y \mapsto \tilde{\mathcal{P}}_y$ is SOT ν -measurable. Let $h \in \mathbf{H}$. By Proposition 241, there exists a sequence of simple ν -measurable \mathbf{H} -valued functions $y \mapsto h_y^{(n)}$, $n \in \mathbb{N}$ such that $\lim_{n \rightarrow \infty} \|h_y^{(n)} - \tilde{\mathcal{P}}_y h\| = 0$ and $\|h_y^{(n)}\| \leq 2\|\tilde{\mathcal{P}}_y h\| \leq 2\|h\|$ for all n for ν -almost all y .

Let $f, g \in \mathbf{H}$. We have

$$\begin{aligned} (f, \mathcal{M}_x g) m(dx, dy) &= (f, \mathcal{N}_y g) n(dx, dy) = (\tilde{\mathcal{P}}_y f, \mathcal{N}_y \tilde{\mathcal{P}}_y g) n(dx, dy) \\ &= \lim_{k \rightarrow \infty} \lim_{\ell \rightarrow \infty} (f_y^{(k)}, \mathcal{N}_y g_y^{(\ell)}) n(dx, dy) \\ (23) \quad &= \lim_{k \rightarrow \infty} \lim_{\ell \rightarrow \infty} (f_y^{(k)}, \mathcal{M}_x g_y^{(\ell)}) m(dx, dy) \\ &= (\mathcal{P}_y f, \mathcal{M}_x \mathcal{P}_y g) m(dx, dy), \end{aligned}$$

where the dominated convergence theorem is used for the third and fifth equality and (b) for the first and fourth equality.

(c) implies (d): If $\text{range}(\mathcal{N}_y)$ is one-dimensional for ν -almost all y , then this, together with Lemma 75, implies that $\text{range}(\mathcal{M}_x) = \ker(\mathcal{M}_x)^\perp$ is one-dimensional for μ -almost all x . \square

THEOREM 77. *Let Σ be a σ -field of subsets of a set Ω . Let $M: \Sigma \rightarrow \mathbf{B}_+(\mathbf{H})$ be a separable POVM. The following conditions are equivalent:*

(a) M is maximal.

(b) There exists a probability measure μ on Σ and a linear isometry $\mathbf{V}: \mathbf{H} \rightarrow \mathbf{L}_2(\Omega, \Sigma, \mu)$ such that $M(\Delta) = \mathbf{V}^* \mathbf{1}(\Delta) \mathbf{V}$ for $\Delta \in \Sigma$; i.e.

$$(24) \quad (\forall \Delta \in \Sigma)(\forall h \in \mathbf{H}) \quad M_h(\Delta) = \int_{\Delta} |\mathbf{V}[h](x)|^2 \mu(dx).$$

If M has an operator density (\mathcal{A}_x) with respect to a measure m on Σ with the same sets of measure zero as M , then M is maximal if and only if $\dim \text{range}(\mathcal{A}_x) = 1$ for m -almost all $x \in \Omega$.

PROOF. (a) implies (b): By Naimark's theorem there is a Hilbert space \mathbf{K} , an isometric operator $\mathbf{V}: \mathbf{H} \rightarrow \mathbf{K}$ and a PVM $F: \Sigma \rightarrow \mathbf{B}_+(\mathbf{K})$ such that $M(\Delta) = \mathbf{V}^*F(\Delta)\mathbf{V}$ for every $\Delta \in \Sigma$. Every commutative von Neumann algebra is contained in a maximal commutative von Neumann algebra and (Proposition 230) every commutative von Neumann algebra is generated by a single self-adjoint operator. Hence Proposition 53 implies that there is a σ -field of subsets of a set Ω_2 and a maximal PVM $E: \Sigma_2 \rightarrow \mathbf{B}_+(\mathbf{K})$ such that $F \leftarrow E$. Define $N: \Sigma_2 \rightarrow \mathbf{B}_+(\mathbf{H})$ by $N(\Delta_2) = \mathbf{V}^*E(\Delta_2)\mathbf{V}$ for every $\Delta_2 \in \Sigma_2$. Then $M \leftarrow N$. (See Proposition 60.) Because M is maximal, $N \leftarrow M$. By Proposition 70, there exists a probability measure ν on Σ_2 and a unitary operator $\mathbf{U}: \mathbf{K} \rightarrow \mathbf{L}_2(\Omega_2, \Sigma_2, \nu)$ such that

$$(\forall \Delta_2 \in \Sigma_2)(\forall k \in \mathbf{K}) \quad E_k(\Delta_2) = \int_{\Delta_2} |\mathbf{U}[k](y)|^2 d\nu(y)$$

Define linear isometry $\mathbf{W}: \mathbf{H} \rightarrow \mathbf{L}_2(\Omega_2, \Sigma_2, \nu)$ by $\mathbf{W} = \mathbf{U}\mathbf{V}$. Then

$$(\forall \Delta_2 \in \Sigma_2)(\forall h \in \mathbf{H}) \quad N_h(\Delta_2) = \int_{\Delta_2} |\mathbf{W}[h](y)|^2 d\nu(y).$$

Let \mathcal{R} be a Hilbert-Schmidt operator on \mathbf{H} with dense range. By Proposition 27 there is a probability measure λ_m on Σ with the same sets of measure zero as M , there is a probability measure λ_n on Σ_2 with the same sets of measure zero as N , and there are operators $\mathcal{M}_x, \mathcal{N}_y \in \mathbf{B}_+(\mathbf{H})$ such that

$$\sup\{\|\mathcal{M}_x\|_\infty : x \in \Omega\} < \infty, \quad \sup\{\|\mathcal{N}_y\|_\infty : y \in \Omega_2\} < \infty$$

and

$$(\mathcal{M}_x h, h) \lambda_m(dx) = M_{\mathcal{R}h}(dx), \quad (\mathcal{N}_y h, h) \lambda_n(dy) = N_{\mathcal{R}h}(dy) \quad \forall h.$$

By Lemma 73 there are $(p_{\Delta_2}) \in \mathbf{M}(\Sigma_2; \Omega, \Sigma, M)$ and $(q_\Delta) \in \mathbf{M}(\Sigma; \Omega_2, \Sigma_2, N)$ such that

$$(\forall \Delta \in \Sigma, \Delta_2 \in \Sigma_2) \quad \int_{\Delta} p_{\Delta_2}(x) \mathcal{M}_x \lambda_m(dx) = \int_{\Delta_2} q_\Delta(y) \mathcal{N}_y \lambda_n(dy)$$

It is easily seen that $\text{range}(\mathcal{N}_y)$ is one-dimensional for λ_n -almost all $y \in \Omega_2$. By Lemma 76, this implies that $\text{range}(\mathcal{M}_x)$ is one-dimensional for λ_m -almost all $x \in \Omega$: There are $e_x \in \mathbf{H}$ such that $\mathcal{M}_x = e_x \otimes e_x$ for λ_m -almost all $x \in \Omega$. Let $\mu = \lambda_m$. Define $\mathbf{W}: \mathbf{H} \rightarrow \mathbf{L}_2(\Omega, \Sigma, \mu)$ by $\mathbf{W}[h](x) = (e_x, h)$. Then

$$(\forall \Delta \in \Sigma)(\forall h \in \mathbf{H}) \quad \int_{\Delta} |\mathbf{W}[h](x)|^2 \mu(dx) = M_{\mathcal{R}h}(\Delta).$$

Define the isometric operator $\mathbf{V}: \mathbf{H} \rightarrow \mathbf{L}_2(\Omega, \Sigma, \mu)$ on $\text{range}(\mathcal{R})$ by $\mathbf{V}[h] = \mathbf{W}[\mathcal{R}^{-1}h]$. Because $\text{range}(\mathcal{R})$ is a dense, the linear isometry \mathbf{V} is determined by its restriction to $\text{range}(\mathcal{R})$ and satisfies (24).

If M has an operator density (\mathcal{A}_x) with respect to a measure m on Σ with the same sets of measure zero as M , then $\mathcal{M}_x \mu(dx) = \mathcal{R}\mathcal{A}_x\mathcal{R}m(dx)$. Let f_μ be the Radon-Nikodym derivative of μ with respect to m . Then $\mathcal{M}_x f_\mu(x)m(dx) = \mathcal{R}\mathcal{A}_x\mathcal{R}m(dx)$. Hence $\mathcal{R}\mathcal{A}_x\mathcal{R} = f_\mu(x)\mathcal{M}_x$ for m -almost all x . Hence

$$\dim \text{range}(\mathcal{M}_x) = \dim \text{range}(\mathcal{R}\mathcal{A}_x\mathcal{R}) = \dim \text{range}(\mathcal{A}_x)$$

for m -almost all x .

(b) implies (a): This follows from Lemma 73.

Assume finally that M has an operator density (\mathcal{A}_x) with respect to a measure m on Σ satisfying $\dim \text{range}(\mathcal{A}_x) = 1$ for m -almost all x . There are $a_x \in \mathbf{H}$ such that $\mathcal{A}_x = a_x \otimes a_x$. Let $\mathbf{V}[h](x) = (a_x, h)$ and $\mu = m$. Then (24) is satisfied. \square

PROPOSITION 78. *Let $M: \Sigma \rightarrow \mathbf{B}_+(\mathbf{H})$ be a separable POVM and let \mathcal{R} be a Hilbert-Schmidt operator on \mathbf{H} with dense range. Let (\mathcal{M}_x) be an operator density of the FPOVM $\Delta \mapsto \mathcal{R}^* M(\Delta) \mathcal{R}$ with respect to finite positive measure μ on Σ with the same sets of measure zero as M . POVM M is maximal if, and only if, $\dim(\text{range}(\mathcal{M}_x)) = 1$ for μ -almost all $x \in \Omega$.*

PROOF. Assume that M is maximal. By Theorem 77 and Lemma 199, there is a probability measure $\tilde{\mu}$ on Σ with the same sets of measure zero as M , and there is an isometry $\mathbf{V}: \mathbf{H} \rightarrow \mathbf{L}_2(\Omega, \Sigma, \mu)$ such that $M_h(\Delta) = \mathbf{1}_{\mathbf{V}[h]}(\Delta)$ for $\Delta \in \Sigma$ and $h \in \mathbf{H}$. Let $f \in \mathbf{L}_1(\Omega, \Sigma, \mu)$ be the Radon-Nikonym derivative of $\tilde{\mu}$ with respect to μ : $\tilde{\mu}(dx) = f(x)\mu(dx)$. There are $e_x \in \mathbf{H}$ such that $\mathbf{V}[\mathcal{R}h](x) = (e_x, h)$ for $\tilde{\mu}$ -almost all x . Then

$$(h, \mathcal{M}_x h) = f(x) |(e_x, h)|^2 \quad \text{for } \mu\text{-almost all } x$$

This implies that $\mathcal{M}_x = f(x) e_x \otimes e_x$, and hence that $\dim(\text{range}(\mathcal{M}_x)) = 1$ for μ -almost all x .

Assume now that $\dim(\text{range}(\mathcal{M}_x)) = 1$ for μ -almost all x . There are $e_x \in \mathbf{H}$ such that $\mathcal{M}_x = e_x \otimes e_x$ for μ -almost all x . Define $\mathbf{V}: \text{range}(\mathbf{R}) \rightarrow \mathbf{L}_2(\Omega, \Sigma, \mu)$ by $\mathbf{V}[h](x) = (e_x, \mathbf{R}^{-1}[h])$. Then

$$M_{\mathcal{R}h}(\Delta) = \int_{\Delta} |\mathbf{V}[\mathbf{R}h](x)|^2 \mu(dx)$$

for all $h \in \mathbf{H}$ and $\Delta \in \Sigma$. From $M_{\mathcal{R}h}(\Omega) = \|\mathcal{R}h\|^2$ it follows that $\mathbf{V}: \text{range}(\mathbf{R}) \rightarrow \mathbf{L}_2(\Omega, \Sigma, \mu)$ is isometric. Because $\text{range}(\mathbf{R})$ is dense in $\mathbf{L}_2(\Omega, \Sigma, \mu)$, this implies that \mathbf{V} has a unique extension to a linear isometry $\mathbf{V}: \mathbf{H} \rightarrow \mathbf{L}_2(\Omega, \Sigma, \mu)$. By continuity,

$$M_h(\Delta) = \int_{\Delta} |\mathbf{V}[h](x)|^2 \mu(dx)$$

for all $h \in \mathbf{H}$ and $\Delta \in \Sigma$. \square

THEOREM 79. *Every POVM $M: \Sigma \rightarrow \mathbf{B}_+(\mathbf{H})$ is dominated by a maximal POVM.*

PROOF. Let $M: \Sigma \rightarrow \mathbf{B}_+(\mathbf{H})$ be a POVM. Let \mathcal{T} be an injective non-negative trace-class operator on \mathbf{H} . By Proposition 27 and Remark 28 there is a probability measure μ on Σ with the same sets of measure zero as M , and there is a family (\mathcal{M}_x) of non-negative bounded operators, such that $\text{Tr}(\mathcal{M}_x) < \infty$ and

$$(h, \mathcal{M}_x h) \mu(dx) = M_{\mathcal{T}h}(dx) \quad \forall h \in \mathbf{H}.$$

For every $x \in \Omega$, there are $\varphi_n^{(x)} \in \mathbf{H}$, $n \in \mathbb{N}$ such that

$$\mathcal{M}_x = \sum_{n=1}^{\infty} \varphi_n^{(x)} \otimes \varphi_n^{(x)}.$$

Hence

$$M_{\mathcal{T}h}(\Delta) = \int_{\Delta} \sum_{n=1}^{\infty} |\mathbf{W}[h](x, n)|^2 \mu(dx)$$

where $\mathbf{W}[h](x, n) = (\varphi_n^{(x)}, h)$. Let τ be counting measure on \mathbb{N} . Then \mathbf{W} maps \mathbf{H} into

$$\mathbf{K} = \mathbf{L}_2(\Omega \times \mathbb{N}, \Sigma \times 2^{\mathbb{N}}, \mu \otimes \tau),$$

where $2^{\mathbb{N}}$ denotes the σ -field of all subsets of \mathbb{N} . From $M_h(\Omega) = \|h\|^2$ it follows that $h \mapsto \mathbf{W}[\mathcal{T}^{-1}h]$ maps the dense linear subspace $\text{range}(\mathcal{T})$ of \mathbf{H} isometrically into \mathbf{K} . Denote the extension to \mathbf{H} of this linear isometry by $\mathbf{U}: \mathbf{H} \rightarrow \mathbf{K}$. We have

$$M_h(\Delta) = \int_{\Delta} \sum_{n=1}^{\infty} |\mathbf{U}[h](x, n)|^2 \mu(dx) \quad \forall h \in \mathbf{H}.$$

Define POVM $N: \Sigma \times 2^{\mathbb{N}} \rightarrow \mathbf{B}_+(\mathbf{H})$ by $N(\tilde{\Delta}) = \mathbf{U}^* \mathbf{1}(\tilde{\Delta}) \mathbf{U}$ for $\tilde{\Delta} \in \Sigma \times 2^{\mathbb{N}}$. Then

$$M(\Delta) = N(\Delta \times \mathbb{N}) \quad \forall \Delta \in \Sigma.$$

In particular, $M \leftarrow N$. By Theorem 77, N is a maximal POVM. \square

18. Bargmann POVM

In this section we give an example of a maximal POVM. This POVM is discussed in the context of quantum optics in [70], [96], [47], and [71]. A measurement scheme is proposed in [87, 86].

Let $\mathfrak{B}_{\mathbb{C}}$ be the σ -field of Borel subsets of \mathbb{C} . Let $g: \mathbb{C} \rightarrow \mathbf{H}$ be the family of normalized coherent state vectors in \mathbf{H} (with squeezing parameter 1) and let $g_z = g(z)$:

$$g_z = e^{-|z|^2/2} \sum_{n=0}^{\infty} \frac{\bar{z}^n}{\sqrt{n!}} \varphi_n,$$

where (φ_n) is the Hermite basis. Define linear isometry $\mathbf{V}_g: \mathbf{H} \rightarrow \mathbf{L}_2(\mathbb{C}, \mu)$, where $\mu(dz) = d \text{Re}(z) d \text{Im}(z)$, by

$$\mathbf{V}_g[h](z) = \frac{1}{\sqrt{\pi}} (g_z, h).$$

We have

$$(25) \quad \mathbf{V}_g[\varphi_n](z) = e^{-|z|^2/2} \frac{z^n}{\sqrt{n!}}.$$

The range of \mathbf{V}_g consists of the function classes that contain a function φ with the property that $z \mapsto e^{|z|^2/2} \varphi(z)$ is entire analytic.

Let $M^{(\text{Bargmann})}$ be the POVM on $\mathfrak{B}_{\mathbb{C}}$ defined by

$$M_h^{(\text{Bargmann})}(\Delta) = \int_{\Delta} |\mathbf{V}_g[h](z)|^2 \mu(dz).$$

This POVM is called the Bargmann POVM because \mathbf{V}_g is up to multiplication by $e^{-|z|^2/2}$, the integral transform (denoted by \mathbf{U}_B in appendix F) introduced in [9] by Bargmann.

By Theorem 77, $M^{(\text{Bargmann})}$ is a maximal POVM. A minimal Naimark extension of $M^{(\text{Bargmann})}$ is $(\mathbf{1}, \mathbf{L}_2(\mathbb{C}, \mu), \mathbf{V}_h)$.

19. Susskind-Glogower phase POVM

The Susskind-Glogower phase POVM, introduced in [98], is another example of a maximal POVM. In this section we give its definition. Let $\mathfrak{B}_{(0,2\pi)}$ be the Borel σ -field of $(0, 2\pi)$.

Let $\mathbf{e}: \mathbb{Z} \rightarrow \mathbf{L}_2([0, 2\pi])$ be the Fourier basis of $\mathbf{L}_2([0, 2\pi])$:

$$\mathbf{e}_n(\theta) = \frac{1}{\sqrt{2\pi}} e^{in\theta}$$

The Susskind-Glogower phase POVM $S: \mathfrak{B}_{(0,2\pi)} \rightarrow \mathbf{B}_+(\mathbf{L}_2(\mathbb{R}))$ is defined by

$$S_h(\Delta) = \int_{\Delta} |\mathbf{V}_S[h](\theta)|^2 d\theta,$$

where linear isometry $\mathbf{V}_S: \mathbf{L}_2(\mathbb{R}) \rightarrow \mathbf{L}_2([0, 2\pi])$ is defined by its action

$$\mathbf{V}_S[\varphi_n] = \mathbf{e}_n \quad \forall n \in \mathbb{N}_0$$

on the Hermite functions. The range of linear isometry \mathbf{V}_S is the Hardy space. By Theorem 77, S is a maximal POVM.

A minimal Naimark extension of S is $(N, \mathbf{L}_2([0, 2\pi]), \mathbf{V}_S)$, where $N(\Delta) = \mathcal{F}^* \mathbf{1}(\Delta) \mathcal{F}$, where $\mathcal{F}: \mathbf{L}_2(\mathbb{R}) \rightarrow \ell_2(\mathbb{Z})$ is Fourier transformation: $\mathcal{F}[h](n) = (\mathbf{e}_n, h)$.

REMARK 80. The Susskind-Glogower POVM is an example of a so-called covariant phase-observable: [68]. More recent is the investigation of the Weyl quantization of the angle function in phase-space: [39, 41].

20. Density of the span of the range of a POVM

Let $M: \Sigma \rightarrow \mathbf{B}_+(\mathbf{H})$ be a POVM. By Proposition 39, $\text{range}(\rho_M) \subset M(\Sigma)''$, the von Neumann algebra generated by $M(\Sigma)$. This section is motivated by the following questions: Is $\text{range}(\rho_M)$ weak-star dense in $M(\Sigma)''$? Is $M(\Sigma)''$ equal to $\mathbf{B}_\infty(\mathbf{H})$? Everything that we will say about the last question is the following: If $M(\Sigma)'' = \mathbf{B}_\infty(\mathbf{H})$ then $M(\Sigma)' = \mathbb{C}\mathcal{I}$. For a commutative POVM this is only the case if its range is contained in $\mathbb{C}\mathcal{I}$.

PROPOSITION 81. *Let $M: \Sigma \rightarrow \mathbf{B}_+(\mathbf{H})$ be a POVM and let μ be a probability measure on Σ with the same sets of measure zero as M . The following conditions are equivalent:*

- $\text{span}(M(\Sigma))$ is weak-star dense in $M(\Sigma)''$,
- $\text{range}(\rho_M)$ is weak-star dense in $M(\Sigma)''$,
- $\rho'_M[\mathcal{T}] = 0$ implies $\text{Tr}(\mathcal{T}\mathcal{A}) = 0$ for all $\mathcal{A} \in M(\Sigma)''$.

PROOF. From the fact that the indicator functions form a weak-star dense linear subspace of $\mathbf{L}_\infty(\Omega, \Sigma, M)$, it follows that $\text{span}(M(\Sigma))$ is a weak-star dense subset $\text{range}(\rho_M)$, and that consequently the first two conditions are equivalent. Using Proposition 209 together with the fact that

$$\mathbf{B}_1(\mathbf{H}) / \{\mathcal{T} \in \mathbf{B}_1(\mathbf{H}) : \text{Tr}(\mathcal{T}\mathcal{A}) = 0 \text{ for all } \mathcal{A} \in M(\Sigma)''\}$$

is (a representation of) the topological dual of $(M(\Sigma)'', \text{weak}^*)$, we see that the last two conditions of the proposition are equivalent. \square

PROPOSITION 82. *Let Σ be a σ -field of subsets of a set Ω . Let $M: \Sigma \rightarrow \mathbf{B}_+(\mathbf{H})$ be a maximal POVM, and let probability measure μ on Σ and linear isometry $\mathbf{V}: \mathbf{H} \rightarrow \mathbf{L}_2(\Omega, \Sigma, \mu)$ be such that $\mathbf{1}(\Delta) = \mathbf{V}^* M(\Delta) \mathbf{V}$ for $\Delta \in \Sigma$. The following conditions are equivalent*

- $\text{range}(\rho_M)$ is dense in $(\mathbf{B}_\infty(\mathbf{H}), \text{SOT})$.
- $\text{range}(\rho_M)$ is dense in $(\mathbf{B}_\infty(\mathbf{H}), \text{WOT})$.
- For finite subsets Λ and Γ of \mathbf{H} , $\sum_{h \in \Lambda} \sum_{g \in \Gamma} \overline{\mathbf{V}[h](x)} \mathbf{V}[g](x) = 0$ for μ -almost all $x \in \Omega$, implies that $\sum_{h \in \Lambda} \sum_{g \in \Gamma} g \otimes h = 0$.

PROOF. The following conditions are equivalent:

- $\sum_{h \in \Lambda} \sum_{g \in \Gamma} \overline{\mathbf{V}[h](x)} \mathbf{V}[g](x) = 0$ for μ -almost all $x \in \Omega$,
- $\sum_{h \in \Lambda} \sum_{g \in \Gamma} (h, M(\Delta)g) = 0$ for all $\Delta \in \Sigma$.
- $\text{Tr}(\mathcal{T}M(\Delta)) = 0$ for all $\Delta \in \Sigma$ where $\mathcal{T} = \sum_{h \in \Lambda} \sum_{g \in \Gamma} g \otimes h$.

By Proposition 209 and the fact that the space of operators with finite dimensional range is (a representation of) the topological dual of $(\mathbf{B}_\infty(\mathbf{H}), \mathcal{T})$ where $\mathcal{T} \in \{\text{WOT}, \text{SOT}\}$, this is satisfied by all such operators \mathcal{T} if, and only if, $\text{span}(M(\Sigma))$ is dense in $(\mathbf{B}_\infty(\mathbf{H}), \mathcal{T})$, where $\mathcal{T} \in \{\text{WOT}, \text{SOT}\}$. \square

In the following proposition we formulate a sufficient condition for $\text{range}(\rho_M)$ to be a weak-star sequentially dense subset of $M(\Sigma)''$. This condition entails the existence of a large $*$ -subalgebra in $\text{range}(\rho_M)$. In the following proposition we give a sufficient condition for the sequential density of $\text{range}(\rho_M)$ (for a POVM M defined on Σ) in $M(\Sigma)''$.

PROPOSITION 83. *Let Σ be a σ -field of subsets of a set Ω . Let $M: \Sigma \rightarrow \mathbf{B}_+(\mathbf{H})$ be a POVM. Let \mathbf{A} be a weak-star sequentially dense subset of $\mathbf{L}_\infty(\Omega, \Sigma, M)$ such that $\rho_M(\mathbf{A})$ is a non-degenerate $*$ -subalgebra of $\mathbf{B}_\infty(\mathbf{H})$. Then $\rho_M(\mathbf{A})$ is weak-star sequentially dense in $M(\Sigma)''$.*

PROOF. Because $\rho_M: (\mathbf{L}_\infty(\Omega, \Sigma, M), \text{weak}^*) \rightarrow (\mathbf{B}_\infty(\mathbf{H}), \text{weak}^*)$ is continuous, the assumptions imply that $\rho_M(\mathbf{A})$ is weak-star sequentially dense in $\text{range}(\rho_M)$. By von Neumann's double commutant theorem, this implies that $\rho_M(\mathbf{A})$ is weak-star dense in $M(\Sigma)''$. because $\mathbf{B}_1(\mathbf{H})$ is separable, a convex subset of $\mathbf{B}_\infty(\mathbf{H})$ is weak-star closed if, and only if, it is weak-star sequentially closed. This implies that the weak-star sequential closure of $\rho_M(\mathbf{A})$ is equal to $M(\Sigma)''$. \square

In the rest of this section we formulate (and prove) a sufficient condition for a $*$ -subalgebra of $\mathbf{L}_\infty(\Omega, \Sigma, M)$ to be weak-star dense in $\mathbf{L}_\infty(\Omega, \Sigma, M)$. First we need some preparatory results.

Let Ω be a topological Hausdorff space. Let $\mathbf{C}_b(\Omega)$ be the algebra of complex bounded continuous functions on Ω . A finite positive Radon measure on Ω is a finite positive Borel measure which is inner regular with respect to compact sets.

THEOREM 84 ([101], Section 14). *Let μ be a finite positive Radon measure on a topological Hausdorff space Ω . Let \mathbf{A} be a unital $*$ -subalgebra of $\mathbf{C}_b(\Omega)$. If \mathbf{A} separates the points of Ω then \mathbf{A} is dense in $\mathbf{L}_1(\Omega, \mu)$.*

LEMMA 85. *Let (Ω, Σ, μ) be a σ -finite positive measure. For a subset \mathbf{A} of $\mathbf{L}_\infty(\Omega, \Sigma, \mu)$, the following conditions are equivalent:*

- (a) \mathbf{A} is dense in $\mathbf{L}_1(\Omega, \Sigma, \nu)$ for all finite positive measures ν on Σ with $\nu \ll \mu$.
- (b) \mathbf{A} is weak-star dense in $\mathbf{L}_\infty(\Omega, \Sigma, \mu)$.

PROOF. The topological dual of $\mathbf{L}_1(\Omega, \Sigma, \nu)$ is $\mathbf{L}_\infty(\Omega, \Sigma, \nu)$. By Proposition 209, this implies that the following conditions are equivalent:

- \mathbf{A} is dense in $\mathbf{L}_1(\Omega, \Sigma, \nu)$.
- $f \in \mathbf{L}_\infty(\Omega, \Sigma, \nu)$ and $\int_\Omega f\varphi d\nu = 0$ for all $\varphi \in \mathbf{A}$ implies $f = 0$.

The topological dual of $(\mathbf{L}_\infty(\Omega, \Sigma, \mu), \text{weak}^*)$ is $\mathbf{L}_1(\Omega, \Sigma, \mu)$. By Proposition 209, this implies that the following conditions are equivalent:

- \mathbf{A} is weak-star dense in $\mathbf{L}_\infty(\Omega, \Sigma, \mu)$.
- $f \in \mathbf{L}_1(\Omega, \Sigma, \mu)$ and $\int_\Omega f\varphi d\mu = 0$ for all $\varphi \in \mathbf{A}$ implies $f = 0$.

(b) implies (a): Assume that $f \in \mathbf{L}_\infty(\Omega, \Sigma, \nu)$ and $\int_\Omega f\varphi d\nu = 0$ for all $\varphi \in \mathbf{A}$. If $\nu = g\mu$ with $g \in \mathbf{L}_1(\Omega, \Sigma, \mu)$ then $fg \in \mathbf{L}_1(\Omega, \Sigma, \mu)$ and $\int_\Omega fg\varphi d\mu = 0$ for all $\varphi \in \mathbf{A}$. Hence $f(x)g(x) = 0$ for μ -almost all x . Hence $f(x) = 0$ for ν -almost all x .

(a) implies (b): Assume that $f \in \mathbf{L}_1(\Omega, \Sigma, \mu)$ and $\int_\Omega f\varphi d\mu = 0$ for all $\varphi \in \mathbf{A}$. Let $\mu_f = |f|\mu$. There exists a $g \in \mathbf{L}_1(\Omega, \Sigma, \mu_f) \cap \mathbf{L}_\infty(\mu_f)$ such that $|g| = 1$ and $g\mu_f = f\mu$. The map $\varphi \mapsto \int_\Omega \varphi g d\mu_f$ is continuous on $\mathbf{L}_1(\Omega, \Sigma, \mu_f)$, and being zero on the dense subspace \mathbf{A} of $\mathbf{L}_1(\Omega, \Sigma, \mu_f)$, it is zero for all $\varphi \in \mathbf{L}_1(\Omega, \Sigma, \mu_f)$, in particular for $\varphi = \bar{g}$: Thus $\int_\Omega \bar{g}g d\mu_f = \int_\Omega d\mu_f = 0$. Hence $\mu_f = 0$. Hence $|f| = 0$. Hence $f = 0$. \square

THEOREM 86. *Let μ be a finite positive Radon measure on a topological Hausdorff space Ω . Let \mathbf{A} be a unital $*$ -subalgebra of $\mathbf{C}_b(\Omega)$. If \mathbf{A} separates the points of Ω then*

then \mathbf{A} is weak-star dense in $L_\infty(\Omega, \Sigma, \mu)$. If, in addition, $L_1(\Omega, \mu)$ is separable, then \mathbf{A} is weak-star sequentially dense in $L_\infty(\Omega, \mu)$.

PROOF. Assume that \mathbf{A} separates the points of Ω . From Lemma 85 and Theorem 84 it follows that \mathbf{A} is weak-star dense in $L_\infty(\Omega, \mu)$. If $L_1(\Omega, \mu)$ is separable, then a convex subset of $L_\infty(\Omega, \mu)$ is weak-star closed if, and only if, it is weak-star sequentially closed. This implies that the weak-star sequential closure of \mathbf{A} is equal to $L_\infty(\Omega, \mu)$. \square

20.1. Example. Let $M^{(\text{Bargmann})}: \mathfrak{B}_\mathbb{C} \rightarrow \mathbf{B}_\infty(L_2(\mathbb{R}))$ be the Bargmann measure defined in Section 18. Define $\rho_{\text{Bargmann}}: L_\infty(\mathbb{C}) \rightarrow \mathbf{B}_\infty(L_2(\mathbb{R}))$ by $\rho_{\text{Bargmann}} = \rho_{M^{(\text{Bargmann})}}$. We will show that $\text{range}(\rho_{\text{Bargmann}})$ is weak-star sequentially dense in $\mathbf{B}_\infty(L_2(\mathbb{R}))$.

For $\tau > 0$ let

$$\mathcal{N}_\tau = \int_{\mathbb{C}} e^{(1-e^\tau)|z|^2} M^{(\text{Bargmann})}(dz).$$

We have

$$\begin{aligned} \mathcal{N}_\tau &= 2 \sum_{n=0}^{\infty} \int_0^{\infty} e^{-e^\tau r^2} \frac{r^{2n+1}}{n!} dr \varphi_n \otimes \varphi_n = \sum_{n=0}^{\infty} \int_0^{\infty} e^{-e^\tau x} \frac{x^n}{n!} dx \varphi_n \otimes \varphi_n \\ &= \sum_{n=0}^{\infty} e^{-(1+n)\tau} \varphi_n \otimes \varphi_n. \end{aligned}$$

Let

$$\mathcal{E} = \int_{\mathbb{C}} \frac{z}{|z|} M^{(\text{Bargmann})}(dz).$$

We have

$$\begin{aligned} \mathcal{E} &= \sum_{n=0}^{\infty} 2 \int_0^{\infty} \frac{r^{2n+2}}{\sqrt{n!(n+1)!}} e^{-r^2} dr \varphi_{n+1} \otimes \varphi_n \\ &= \sum_{n=0}^{\infty} \int_0^{\infty} \frac{x^n \sqrt{x}}{\sqrt{n!(n+1)!}} e^{-x} dx \varphi_{n+1} \otimes \varphi_n \\ &= \sum_{n=0}^{\infty} \omega_n \varphi_{n+1} \otimes \varphi_n, \quad \text{where } \omega_n = \frac{\Gamma(\frac{3}{2} + n)}{\sqrt{n!(n+1)!}}. \end{aligned}$$

If $\mathcal{A} \in M^{(\text{Bargmann})}(\mathfrak{B}_\mathbb{C})'$ then $\mathcal{A} \in \{\mathcal{E}, \mathcal{N}_\tau\}'$. Hence

$$\mathcal{N}_\tau \mathcal{A} \varphi_n = \mathcal{A} \mathcal{N}_\tau \varphi_n = e^{-(1+n)\tau} \mathcal{A} \varphi_n.$$

Hence $\mathcal{A} \varphi_n = \lambda_n \varphi_n$ with $\lambda_n \in \mathbb{C}$. But also

$$\lambda_n \omega_n \varphi_{n+1} = \mathcal{E} \mathcal{A} \varphi_n = \mathcal{A} \mathcal{E} \varphi_n = \omega_n \lambda_{n+1} \varphi_{n+1}.$$

Hence $\lambda_n = \lambda_{n+1}$ for all n . Hence $\mathcal{A} \in \text{span}\{\mathcal{I}\}$. This is true for all $\mathcal{A} \in M^{(\text{Bargmann})}(\mathfrak{B}_\mathbb{C})'$: Thus $M^{(\text{Bargmann})}(\mathfrak{B}_\mathbb{C})' = \text{span}\{\mathcal{I}\}$. Hence $H(\mathfrak{B}_\mathbb{C})'' = \mathbf{B}_\infty(L_2(\mathbb{R}))$.

Define *-subalgebra \mathbf{A} of $\mathbf{C}_b(\mathbb{C})$ by

$$\mathbf{A} = \text{span}\{z \mapsto e^{z\bar{w} - \bar{z}w} : w \in \mathbb{C}\}.$$

By Theorem 86, \mathbf{A} is weak-star sequentially dense in $L_\infty(\mathbb{C})$. Let

$$\mathcal{D}_w = e^{|w|^2/2} \int_{\mathbb{C}} e^{z\bar{w} - \bar{z}w} M^{(\text{Bargmann})}(dz)$$

Using

$$(g_a, g_b) = \exp\{-\frac{1}{2}(|a|^2 + |b|^2)\} e^{\bar{b}a} \quad \forall a, b \in \mathbb{C},$$

it is easily seen that

$$\begin{aligned} (g_a, \mathcal{D}_w g_b) &= \exp\left\{\frac{1}{2}(|w|^2 - |a|^2 - |b|^2)\right\} \frac{1}{\pi} \int_{\mathbb{C}} \exp\{\bar{z}(a-w) + z(\bar{b} + \bar{w}) - |z|^2\} \mu(dz) \\ &= \exp\left\{\frac{1}{2}(|w|^2 - |a|^2 - |b|^2) + (\bar{b} + \bar{w})(a-w)\right\} \\ &= e^{(b\bar{w} - \bar{b}w)/2} (g_a, g_{b+w}). \end{aligned}$$

Hence $\mathcal{D}_w g_b = e^{(b\bar{w} - \bar{b}w)/2} g_{b+w}$ and $\mathcal{D}_z \mathcal{D}_w = e^{(w\bar{z} - \bar{w}z)/2} \mathcal{D}_{z+w}$. Hence

$$\rho_{\text{Bargmann}}(\mathbf{A}) = \text{span}\{\mathcal{D}_z : z \in \mathbb{C}\}$$

is a $*$ -subalgebra of $\mathbf{B}_{\infty}(\mathbf{L}_2(\mathbb{R}))$. Because $\mathcal{D}_0 = \mathcal{I}$, $\rho_{\text{Bargmann}}(\mathbf{A})$ is a unital $*$ -subalgebra. This implies that $\rho_{\text{Bargmann}}(\mathbf{A})$ is non-degenerate. By Proposition 83, $\rho_{\text{Bargmann}}(\mathbf{A})$ is weak-star sequentially dense in $M^{(\text{Bargmann})}(\mathfrak{B}_{\mathbb{C}})'' = \mathbf{B}_{\infty}(\mathbf{L}_2(\mathbb{R}))$. This implies in particular that $\rho_{\text{Bargmann}}(\mathbf{C}_b^{\infty}(\mathbb{C}))$ is weak-star sequentially dense in $\mathbf{B}_{\infty}(\mathbf{L}_2(\mathbb{R}))$, where $\mathbf{C}_b^{\infty}(\mathbb{C})$ are the bounded infinitely differentiable functions on \mathbb{C} .

20.2. Example. The Susskind-Glogower phase POVM $S: \mathfrak{B}_{(0,2\pi)} \rightarrow \mathbf{B}_{\infty}(\mathbf{L}_2(\mathbb{R}))$ was introduced in Section 19. We will show that $S(\mathfrak{B}_{(0,2\pi)})'' = \mathbf{B}_{\infty}(\mathbf{L}_2(\mathbb{R}))$, but that $\text{range}(\rho_S)$ is not weak-star dense in $\mathbf{B}_{\infty}(\mathbf{L}_2(\mathbb{R}))$. Let

$$\mathcal{S}_k = \int_0^{2\pi} e^{-ik\theta} S(d\theta).$$

It is easily seen that

$$\mathcal{S}_k = \sum_{n=0}^{\infty} \varphi_{n+k} \otimes \varphi_n,$$

Let $\mathcal{A} \in \{\mathcal{S}_k, \mathcal{S}_k^* : k \in \mathbb{N}\}'$. Then $\mathcal{A} \in \{\mathcal{S}_k^* \mathcal{S}_k : k \in \mathbb{N}\}'$. But $\mathcal{I} - \mathcal{S}_k^* \mathcal{S}_k$ is the operator of orthogonal projection on $\text{span}\{\varphi_n : 0 \leq n \leq k\}$. Hence $\mathcal{A} \in \{\varphi_n \otimes \varphi_n : n \in \mathbb{N}_0\}'$. Hence $\mathcal{A} \in \{\mathcal{S}_1, \varphi_n \otimes \varphi_n : n \in \mathbb{N}_0\}' = \text{span}\{\mathcal{I}\}$. Hence $\{\mathcal{S}_k, \mathcal{S}_k^* : k \in \mathbb{N}\}' = \text{span}\{\mathcal{I}\}$. Hence $S(\mathfrak{B}_{(0,2\pi)})'' = \mathbf{B}_{\infty}(\mathbf{L}_2(\mathbb{R}))$.

If $k \neq 0$ then $(\varphi_n, \rho_S(\mathbf{e}_k) \varphi_n) = 0$ for all n . Hence if $\varphi \in \text{span}\{\mathbf{e}_k : k \in \mathbb{Z}\}$ then $(\varphi_n, \rho_S(\varphi) \varphi_n) = (\varphi_0, \rho_S(\varphi) \varphi_0)$ for all n . By Theorem 86, $\text{span}\{\mathbf{e}_k : k \in \mathbb{Z}\}$ is weak-star sequentially dense in $\mathbf{L}_{\infty}([0, 2\pi], \mathfrak{B}_{(0,2\pi)}, S)$. Hence $(\varphi_n, \mathcal{A} \varphi_n) = (\varphi_0, \mathcal{A} \varphi_0)$ for all n and $\mathcal{A} \in \text{range}(\rho_S)$. Hence $(\varphi_n, \mathcal{A} \varphi_n) = (\varphi_0, \mathcal{A} \varphi_0)$ for all n and \mathcal{A} in the weak-star closure of $\text{range}(\rho_S)$. Hence $\text{range}(\rho_S)$ is not weak-star dense in $\mathbf{B}_{\infty}(\mathbf{L}_2(\mathbb{R}))$. This follows also from Proposition 82: $|\mathbf{V}_S[\varphi_n](\theta)| = |\mathbf{V}_S[\varphi_0](\theta)|$ for every $n \in \mathbb{N}$ and almost all θ . Hence $|\mathbf{V}_S[\varphi_n](\theta)|^2 - |\mathbf{V}_S[\varphi_0](\theta)|^2 = 0$ for every $n \in \mathbb{N}$ and almost all θ . Hence $\text{range}(\rho_S)$ is not dense in $(\mathbf{B}_{\infty}(\mathbf{L}_2(\mathbb{R})), \text{WOT})$. This implies that $\text{range}(\rho_S)$ is not dense in $(\mathbf{B}_{\infty}(\mathbf{L}_2(\mathbb{R})), \text{weak}^*)$.

21. Prototypical POVMs

Theorem 87 below provides Naimark extensions of POVMs in the form of multiplication operators.

THEOREM 87. *Let $M: \Sigma \rightarrow \mathbf{B}_+(\mathbf{H})$ be a POVM. There exists a separable Hilbert space \mathbf{X} , a probability measure μ on Σ with the same sets of measure zero as M and an isometric operator $\mathbf{V}: \mathbf{H} \rightarrow \mathbf{L}_2(\Omega, \Sigma, \mu; \mathbf{X})$ such that $M(\Delta) = \mathbf{V}^* \mathbf{1}^{(\mathbf{X})}(\Delta) \mathbf{V}$ for $\Delta \in \Sigma$; i.e.*

$$(26) \quad M_h(\Delta) = \int_{\Delta} \|\mathbf{V}[h](x)\|_{\mathbf{X}}^2 \mu(dx)$$

for all $h \in \mathbf{H}$ and $\Delta \in \Sigma$. POVM M is projection-valued if, and only if, $\text{range}(\mathbf{V})$ reduces the operators $\mathbf{1}^{(\mathbf{X})}(\Delta)$, $\Delta \in \Sigma$.

PROOF. By the proof of Theorem 79, there is a probability measure μ on Σ with the same sets of measure zero as M , and a linear isometry $\mathbf{U}: \mathbf{H} \rightarrow \mathbf{L}_2(\Omega \times \mathbb{N}, \Sigma \times 2^{\mathbb{N}}, \mu \otimes \tau)$ such that

$$M_h(\Delta) = \int_{\Delta} \sum_{n=1}^{\infty} |\mathbf{U}[h](x, n)|^2 \mu(dx) \quad \forall h \in \mathbf{H}.$$

Corresponding to Fubini's theorem, we can identify $\mathbf{L}_2(\Omega \times \mathbb{N}, \Sigma \times 2^{\mathbb{N}}, \mu \otimes \tau)$ with $\mathbf{L}_2(\Omega, \Sigma, \mu; \ell_2(\mathbb{N}))$: $\mathbf{U}[h]$ is identified with $\mathbf{V}[h] \in \mathbf{L}_2(\Omega, \Sigma, \mu; \ell_2(\mathbb{N}))$ defined by $\mathbf{V}[h](x) = (\mathbf{U}[h](x, n))_{n \in \mathbb{N}} \in \ell_2(\mathbb{N})$. We have

$$M_h(\Delta) = \int_{\Delta} \|\mathbf{V}[h](x)\|_{\ell_2(\mathbb{N})}^2 \mu(dx) \quad \forall h \in \mathbf{H}.$$

Let $\mathbf{X} = \ell_2(\mathbb{N})$. This proves the first part of the theorem.

Now we will prove that M is projection-valued if, and only if, $\text{range}(\mathbf{V})$ reduces $\mathbf{1}^{(\mathbf{X})}(\Sigma)$. Assume first that $\text{range}(\mathbf{V})$ reduces $\mathbf{1}^{(\mathbf{X})}(\Sigma)$. Then

$$\begin{aligned} M(\Delta)^2 &= \mathbf{V}^* \mathbf{1}^{(\mathbf{X})}(\Delta) \mathbf{V} \mathbf{V}^* \mathbf{1}^{(\mathbf{X})}(\Delta) \mathbf{V} \\ &= \mathbf{V}^* \mathbf{1}^{(\mathbf{X})}(\Delta) \mathbf{1}^{(\mathbf{X})}(\Delta) \mathbf{V} = M(\Delta). \end{aligned}$$

Hence M is projection-valued.

Assume now that M is projection-valued. Let

$$\tilde{\mathbf{H}} = \text{clspan}\{\mathbf{1}^{(\mathbf{X})}(\Delta) \mathbf{V}[h] : \Delta \in \Sigma, h \in \mathbf{H}\}.$$

Then $(M, \mathbf{H}, \mathcal{I})$ and $(\mathbf{1}^{(\mathbf{X})}, \tilde{\mathbf{H}}, \mathbf{V})$ are two minimal Naimark extensions of POVM M . By Proposition 6, $\tilde{\mathbf{H}} = \text{range}(\mathbf{V})$. \square

22. Multiplicity theory

In this section we provide a suitable context for the characterization of maximality given in Theorem 77: Maximal POVMs are the POVMs of uniform multiplicity one as defined below. This section refines the results of Section 21: Theorem 94 provides minimal Naimark extensions of POVMs in terms of multiplication operators.

DEFINITION 88. Let $m \in \mathbb{N} \cup \{\infty\}$. A POVM $M: \Sigma \rightarrow \mathbf{B}_{\infty}(\mathbf{H})$ is of uniform multiplicity m , if there exist a separable Hilbert space \mathbf{X} with $\dim(\mathbf{X}) = m$, a probability measure μ on Σ , and a linear isometry $\mathbf{V}: \mathbf{H} \rightarrow \mathbf{L}_2(\Omega, \Sigma, \mu; \mathbf{X})$ such that $(\mathbf{1}^{(\mathbf{X})}, \mathbf{V}, \mathbf{L}_2(\Omega, \Sigma, \mu; \mathbf{X}))$ is a minimal Naimark extension of M .

This generalizes the usual concept of uniform multiplicity for a finite set of commuting self-adjoint operators:

PROPOSITION 89. A PVM $E: \Sigma \rightarrow \mathbf{B}_{\infty}(\mathbf{H})$ is of uniform multiplicity m if, and only if, there exist a separable Hilbert space \mathbf{X} with $\dim(\mathbf{X}) = m$, a probability measure μ on Σ , and a unitary operator $\mathbf{U}: \mathbf{H} \rightarrow \mathbf{L}_2(\Omega, \Sigma, \mu; \mathbf{X})$ such that $E(\Delta) = \mathbf{U}^* \mathbf{1}^{(\mathbf{X})}(\Delta) \mathbf{U}$ for all $\Delta \in \Sigma$.

PROOF. If such \mathbf{X}, μ and \mathbf{U} exist, then $(\mathbf{1}^{(\mathbf{X})}, \mathbf{U}, \mathbf{L}_2(\Omega, \Sigma, \mu; \mathbf{X}))$ is a minimal Naimark extension of E , and hence E is of uniform multiplicity m .

Assume now that E is of uniform multiplicity m : There exists a separable Hilbert space \mathbf{X} with $\dim(\mathbf{X}) = m$, a probability measure μ on Σ and a linear isometry $\mathbf{V}: \mathbf{H} \rightarrow \mathbf{L}_2(\Omega, \Sigma, \mu; \mathbf{X})$ such that $(\mathbf{1}^{(\mathbf{X})}, \mathbf{V}, \mathbf{L}_2(\Omega, \Sigma, \mu; \mathbf{X}))$ is a minimal Naimark extension of E . The fact that E is projection-valued implies (by the second part of Theorem 87) that $\text{range}(\mathbf{V})$ reduces the operators $\mathbf{1}^{(\mathbf{X})}(\Delta)$, $\Delta \in \Sigma$. This, together with the minimality of the Naimark extension, implies that $\text{range}(\mathbf{V}) = \mathbf{L}_2(\Omega, \Sigma, \mu; \mathbf{X})$. Hence \mathbf{V} is unitary. \square

THEOREM 90. *A POVM $M: \Sigma \rightarrow \mathbf{B}_+(\mathbf{H})$ is maximal if, and only if, it is of uniform multiplicity 1.*

PROOF. M is of uniform multiplicity 1 if, and only if, there exist a probability measure μ on Σ and a linear isometry $\mathbf{V}: \mathbf{H} \rightarrow \mathbf{L}_2(\Omega, \Sigma, \mu)$ such that $(\mathbf{1}, \mathbf{V}, \mathbf{L}_2(\Omega, \Sigma, \mu))$ is a minimal Naimark extension of M .

By Theorem 77, M is maximal if, and only if, there exist a probability measure μ on Σ and a linear isometry $\mathbf{V}: \mathbf{H} \rightarrow \mathbf{L}_2(\Omega, \Sigma, \mu)$ such that $(\mathbf{1}, \mathbf{V}, \mathbf{L}_2(\Omega, \Sigma, \mu))$ is a Naimark extension of M . By Lemma 199, μ can be replaced by a measure with the same sets of measure 0 as M .

To prove the equivalence of the two concepts, we show that every Naimark extension of the form $(\mathbf{1}, \mathbf{V}, \mathbf{L}_2(\Omega, \Sigma, \mu))$ such that μ and M have the same sets of measure 0, is minimal. Assume that it is not minimal: There exists a μ -square integrable Σ -measurable function f such that $\int_{\Delta} \overline{\mathbf{V}[h](x)} f(x) \mu(dx) = 0$ for all $\Delta \in \Sigma$ and $h \in \mathbf{H}$, but $\mu(\Delta_f) > 0$, where $\Delta_f = \{x \in \Omega : f(x) \neq 0\}$. This implies, in particular, that the restriction of $\mathbf{1}(\Delta_f)$ to $\text{range}(\mathbf{V})$ is zero, and hence that $M(\Delta_f) = 0$. This is impossible because μ and M have the same sets of measure zero. \square

LEMMA 91. *Let Σ be a σ -field of subsets of a set Ω , let $M: \Sigma \rightarrow \mathbf{B}_+(\mathbf{H})$ be an FPOVM, and let $m \in \mathbb{N} \cup \{\infty\}$. If*

- μ is a probability measure on Σ with the same sets of measure zero as M ;
- \mathbf{X} is an m dimensional complex separable Hilbert space;
- $\mathbf{V}: \mathbf{H} \rightarrow \mathbf{L}_2(\Omega, \Sigma, \mu; \mathbf{X})$ is a contractive operator such that $M(\Delta) = \mathbf{V}^* \mathbf{1}^{(\mathbf{X})}(\Delta) \mathbf{V}$ for each $\Delta \in \Sigma$;
- \mathcal{T}^{ext} is an injective non-negative trace-class operator on $\mathbf{L}_2(\Omega, \Sigma, \mu; \mathbf{X})$ satisfying $\mathcal{T}^{\text{ext}}(\text{range}(\mathbf{V})) \subset \text{range}(\mathbf{V})$;
- $\mathcal{T} = \mathbf{V}^* \mathcal{T}^{\text{ext}} \mathbf{V}$;
- (\mathcal{M}_x) is a family of non-negative trace-class operators on \mathbf{H} such that $\mathcal{T} M(\Delta) \mathcal{T} = \int_{\Delta} \mathcal{M}_x \mu(dx)$ for all $\Delta \in \Sigma$,

then the following conditions are equivalent:

- (a) $\dim(\text{range}(\mathcal{M}_x)) = m$ for μ -almost all $x \in \Omega$,
- (b) $\mathbf{L}_2(\Omega, \Sigma, \mu; \mathbf{X}) = \text{cl span}\{\mathbf{1}^{(\mathbf{X})}(\Delta) \mathbf{V}[h] : h \in \mathbf{H}, \Delta \in \Sigma\}$.

PROOF. \mathcal{T} is an injective non-negative trace-class operator on \mathbf{H} and $\mathbf{V} \mathcal{T} = \mathcal{T}^{\text{ext}} \mathbf{V}$. (b) implies (a): There is a family (\mathcal{N}_x) of non-negative trace-class operators such that $\mathcal{T}^{\text{ext}} \mathbf{1}^{(\mathbf{X})}(\Delta) \mathcal{T}^{\text{ext}} = \int_{\Delta} \mathcal{N}_x \mu(dx)$ for all $\Delta \in \Sigma$. From

$$(\forall \Delta \in \Sigma) \quad \int_{\Delta} \|\mathcal{T}^{\text{ext}}[f](x)\|_{\mathbf{X}}^2 \mu(dx) = \int_{\Delta} (f, \mathcal{N}_x f) \mu(dx)$$

for all $f \in \mathbf{L}_2(\Omega, \Sigma, \mu; \mathbf{X})$ follows $\dim(\text{range}(\mathcal{N}_x)) = m$ for μ -almost all x . For μ -almost all $x \in \Omega$, there is an m -tuple $\varphi_n^{(x)}$, $n \in \mathbb{I}$ of non-zero pairwise orthogonal vectors of $\mathbf{L}_2(\Omega, \Sigma, \mu; \mathbf{X})$ such that

$$\mathcal{N}_x = \sum_{n=1}^m \varphi_n^{(x)} \otimes \varphi_n^{(x)}.$$

We have

$$(\forall \mu x \in \Omega) \quad \mathcal{M}_x = \sum_{n=1}^m \mathbf{V}^*[\varphi_n^{(x)}] \otimes \mathbf{V}^*[\varphi_n^{(x)}].$$

From (b) follows

$$\mathbf{L}_2(\Omega, \Sigma, \mu; \mathbf{X}) = \text{cl span}\{\text{range}\left(\int_{\Delta} \mathcal{N}_x \mathbf{V} \mu(dx)\right) : \Delta \in \Sigma\}.$$

Hence $\dim(\text{range}(\mathcal{N}_x \mathbf{V})) = m$ for μ -almost all x . Hence $\dim(\text{range}(\mathcal{M}_x)) = m$ for μ -almost all $x \in \Omega$.

(a) implies (b): Let $\mathbb{I} = \{n \in \mathbb{N} : 1 \leq n < m + 1\}$. Let $\mathbf{X} = \ell_2(\mathbb{I})$. For every $x \in \Omega$, there is an m -tuple $\varphi_n^{(x)}$, $n \in \mathbb{I}$ of non-zero pairwise orthogonal vectors of \mathbf{H} such that

$$\mathcal{M}_x = \sum_{n=1}^m \varphi_n^{(x)} \otimes \varphi_n^{(x)}.$$

We have $\mathcal{T} \mathbf{V}^* \mathbf{1}^{(\mathbf{X})}(\Delta) \mathbf{V} \mathcal{T} = \int_{\Delta} \mathcal{M}_x \mu(dx)$. This can be written as

$$\int_{\Delta} |\mathbf{V}[\mathcal{T}h](x)|^2 \mu(dx) = \sum_{n=1}^m \int_{\Delta} |(\varphi_n^{(x)}, h)|^2 \mu(dx) \quad \forall h \in \mathbf{H}.$$

Condition (b) is equivalent with the following condition:

$$\mathbf{L}_2(\Omega, \Sigma, \mu; \mathbf{X}) = \text{cl span}\{\mathbf{1}^{(\mathbf{X})}(\Delta) \mathbf{V}[\mathcal{T}h] : h \in \mathbf{H}, \Delta \in \Sigma\}.$$

Assume it is not satisfied: There are $f_n \in \mathbf{L}_2(\Omega, \Sigma, \mu)$ such that $0 < \sum_{n=1}^m \|f_n\|^2 < \infty$ but

$$\sum_{n=1}^m \int_{\Delta} f_n(x) \varphi_n^{(x)} \mu(dx) = 0$$

for all $\Delta \in \Sigma$. The integral has meaning in the weak sense, and the order of summation and integration can be interchanged. Hence for all $h \in \mathbf{H}$,

$$\sum_{n=1}^m f_n(x) (\varphi_n^{(x)}, h) = 0 \quad \text{for } \mu\text{-almost all } x \in \Omega.$$

Because \mathbf{H} is separable and the countable union of μ -null sets is again a μ -null set, this implies that

$$\sum_{n=1}^m f_n(x) \varphi_n^{(x)} = 0 \quad \text{for } \mu\text{-almost all } x \in \Omega.$$

Because $(\varphi_k^{(x)}, \varphi_\ell^{(x)}) = 0$ unless $k = \ell$,

$$f_n(x) \|\varphi_n^{(x)}\|^2 = 0 \quad \text{for } \mu \text{ almost all } x \in \Omega$$

for all $n \in \mathbb{I}$. Because $\|\varphi_n^{(x)}\|^2 > 0$ for all n , $f_n = 0$ for all n . Because this is impossible, condition (b) must be satisfied. \square

THEOREM 92 (Commutative multiplicity theorem). *Let $E: \Sigma \rightarrow \mathbf{B}_+(\mathbf{H})$ be a PVM. There exist pairwise disjoint sets $\Omega_m \in \Sigma$, $m \in \mathbb{I} = \mathbb{N} \cup \{\infty\}$ such that each PVM $E_m: \Sigma \rightarrow \mathbf{B}_+(\mathbf{H}_m)$ in the countable direct sum*

$$E = E_\infty \oplus E_1 \oplus E_2 \oplus \cdots$$

corresponding to orthogonal decomposition

$$\mathbf{H} = \mathbf{H}_\infty \oplus \mathbf{H}_1 \oplus \mathbf{H}_2 \oplus \cdots \quad \text{where } \mathbf{H}_m = \text{range}(E(\Omega_m)),$$

is of uniform multiplicity m .

PROOF. Let \mathcal{M}_x and μ be defined by Proposition 78. Let

$$\Omega_m = \{x \in \Omega : \dim \text{range}(\mathcal{M}_x) = m\}$$

for $m \in \mathbb{I}$. By Proposition 247, $\Omega_m \in \Sigma$ for all m . From Lemma 91 it follows that E_m has uniform multiplicity m . \square

LEMMA 93. *Let $M: \Sigma \rightarrow \mathbf{B}_+(\mathbf{H})$ be a POVM. Let $\mathbb{I} = \mathbb{N} \cup \{\infty\}$. There exists*

- A measurable partition Ω_m , $m \in \mathbb{I} \cup \{0\}$ of Ω with $M(\Omega_0) = 0$;
- For each $m \in \mathbb{I}$ a finite measure μ_m on Σ with the same sets of measure zero as $\Delta \mapsto M(\Omega_m \cap \Delta)$;
- For each $m \in \mathbb{I}$ an m -dimensional separable complex Hilbert space \mathbf{X}_m ;
- For each $m \in \mathbb{I}$ a contractive operator $\mathbf{V}_m: \mathbf{H} \rightarrow \mathbf{L}_2(\Omega, \Sigma, \mu_m; \mathbf{X}_m)$

such that for all $m \in \mathbb{I}$,

$$(27) \quad M(\Delta \cap \Omega_m) = \mathbf{V}_m^* \mathbf{1}^{(\mathbf{X}_m)}(\Delta) \mathbf{V}_m \quad \forall \Delta \in \Sigma$$

and

$$(28) \quad \mathbf{L}_2(\Omega, \Sigma, \mu_m; \mathbf{X}_m) = \text{cl span}\{\mathbf{1}^{(\mathbf{X}_m)}(\Delta) \mathbf{V}_m[h] : h \in \mathbf{H}, \Delta \in \Sigma\}.$$

PROOF. Let $\mathbf{X}, \mu, \mathbf{V}$ be defined by Theorem 87. Let $\mathcal{T}, \mathcal{M}_x$ be defined as in the proof of Theorem 87. Let $\Omega_m = \{x \in \Omega : \dim \text{range}(\mathcal{M}_x) = m\}$ for $m \in \mathbb{I}$. By Proposition 247, $\Omega_m \in \Sigma$ for all m . Let $\Omega_0 = \{x \in \Omega : \mathcal{M}_x = 0\}$. Let $\mathbf{X}_m = \mathbb{C}^m$ and $\mathbf{X}_\infty = \ell_2(\mathbb{N})$. Define μ_m by $\mu_m(\Delta) = \mu(\Omega_m \cap \Delta)$. Define $\mathbf{V}_m: \mathbf{H} \rightarrow \mathbf{L}_2(\Omega, \Sigma, \mu_m; \mathbf{X}_m)$ by $\mathbf{V}_m = \mathbf{1}^{(\mathbf{X})}(\Omega_m) \mathbf{V}$. Condition (27) follows from

$$\begin{aligned} M(\Delta \cap \Omega_m) &= \mathbf{V}^* \mathbf{1}^{(\mathbf{X})}(\Delta \cap \Omega_m) \mathbf{V} \\ &= \mathbf{V}_m^* \mathbf{1}^{(\mathbf{X})}(\Delta) \mathbf{V}_m. \end{aligned}$$

Condition (28) follows from Lemma 91. □

THEOREM 94. Let $M: \Sigma \rightarrow \mathbf{B}_+(\mathbf{H})$ be a POVM. Let $\mathbb{I} = \mathbb{N} \cup \{\infty\}$. There exist

- A measurable partition Ω_m , $m \in \mathbb{I} \cup \{0\}$ of Ω with $M(\Omega_0) = 0$;
- For each $m \in \mathbb{I}$ a finite measure μ_m on Σ with the same sets of measure zero as $\Delta \mapsto M(\Omega_m \cap \Delta)$;
- For each $m \in \mathbb{I}$ an m -dimensional separable complex Hilbert space \mathbf{X}_m ;
- A linear isometry $\mathbf{V}: \mathbf{H} \rightarrow \mathbf{K}$, where

$$\mathbf{K} = \bigoplus_{m \in \mathbb{I}} \mathbf{L}_2(\Omega, \Sigma, \mu_m; \mathbf{X}_m),$$

such that $(F, \mathbf{K}, \mathbf{V})$, where $F: \Sigma \rightarrow \mathbf{B}_+(\mathbf{K})$ is defined by

$$F(\Delta) \oplus_{m \in \mathbb{I}} f_m = \bigoplus_{m \in \mathbb{I}} \mathbf{1}^{(\mathbf{X}_m)}(\Delta) f_m,$$

is a minimal Naimark extension M .

PROOF. Let \mathbf{V}_m and \mathbf{X}_m be as in Lemma 93. Denote the inner-product of \mathbf{X}_m by $(\cdot, \cdot)_m$. Define linear isometry $\mathbf{V}: \mathbf{H} \rightarrow \mathbf{K}$ by $\mathbf{V}[h] = \bigoplus_{m \in \mathbb{I}} \mathbf{V}_m[h]$ for $h \in \mathbf{H}$. To prove minimality of the Naimark extension, it suffices to prove that $f_m \in \mathbf{L}_2(\Omega, \Sigma, \mu_m, \mathbf{X}_m)$ and $\sum_{n=0}^m \|f_n\|^2 < \infty$ and

$$(29) \quad \sum_{m \in \mathbb{I}} \int_{\Delta} (f_m(x), \mathbf{V}_m[h](x))_m \mu(dx) = 0 \quad \forall \Delta \in \Sigma, h \in \mathbf{H}$$

implies $f_m = 0$ for all m . Because μ_m is concentrated on Ω_m , (29) implies

$$\int_{\Delta} (f_m(x), \mathbf{V}_m[h](x))_m \mu(dx) = 0 \quad \forall \Delta \in \Sigma, h \in \mathbf{H}$$

for all $m \in \mathbb{I}$. By (28), this implies that $f_m = 0$ for all $m \in \mathbb{I}$. □

REMARK 95. POVMs of finite uniform multiplicity which have an operator density are sometimes called frames. These are investigated (together with their relation to the theory of reproducing kernel Hilbert spaces) in [6]. See also [7].

23. Bounded subnormal operators

DEFINITION 96. A bounded operator \mathcal{S} on a Hilbert space \mathbf{H} is subnormal if there is a Hilbert space \mathbf{K} , a bounded normal operator \mathcal{N} on \mathbf{K} , and an isometry $\mathbf{V}: \mathbf{H} \rightarrow \mathbf{K}$ such that $\mathbf{V}\mathcal{S}\mathbf{V}^* = \mathcal{N}$ on $\text{range}(\mathbf{V})$.

Triple $(\mathcal{N}, \mathbf{V}, \mathbf{K})$ is called a normal extension of subnormal operator \mathcal{S} on \mathbf{H} . If $\mathbf{H} \subset \mathbf{K}$ and \mathbf{V} is the identity operator, then \mathcal{N} is called a normal extension of \mathcal{S} .

EXAMPLE 97. Every linear isometry is a subnormal operator.

In [72] an example is given of two commuting subnormal operators \mathcal{R}, \mathcal{S} such that neither $\mathcal{R} + \mathcal{S}$ nor $\mathcal{R}\mathcal{S}$ is subnormal.

DEFINITION 98. A normal extension $(\mathcal{N}, \mathbf{V}, \mathbf{K})$ of a subnormal operator \mathcal{S} on \mathbf{H} is called minimal if the only subspace of \mathbf{K} , containing $\text{range}(\mathbf{V})$ and reducing \mathcal{N} , is \mathbf{K} itself. This is the case if, and only if, the only closed linear subspace of \mathbf{K} , containing $\mathcal{N}^{*k}\mathbf{V}[h]$ for all $k \in \mathbb{N}_0$ and $h \in \mathbf{H}$, is \mathbf{K} itself: This condition can be formulated as

$$\mathbf{K} = \text{cl span}\{\mathcal{N}^{*k}\mathbf{V}[h] : n \in \mathbb{N}_0, h \in \mathbf{H}\}.$$

PROPOSITION 99 (Proposition 2.5 of Chapter II in [24]). *Given two minimal normal extensions $(N_1, \mathbf{V}_1, \mathbf{K}_1)$ and $(N_2, \mathbf{V}_2, \mathbf{K}_2)$ of a subnormal operator \mathcal{S} on \mathbf{H} , there exists a unitary operator $\mathbf{U}: \mathbf{K}_1 \rightarrow \mathbf{K}_2$ such that $\mathbf{U}\mathbf{V}_1 = \mathbf{V}_2$ and $\mathbf{U}\mathcal{N}_1 = \mathcal{N}_2\mathbf{U}$.*

THEOREM 100 (Spectral inclusion theorem, Problem 200 in [53]). *If \mathcal{S} is subnormal and \mathcal{N} is a minimal normal extension, then $\sigma(\mathcal{N}) \subset \sigma(\mathcal{S})$.*

A bounded operator \mathcal{S} on a Hilbert space \mathbf{H} is called hyponormal if $\|\mathcal{S}^*h\| \leq \|\mathcal{S}h\|$ for $h \in \mathbf{H}$. An operator \mathcal{S} is normal if, and only if, both \mathcal{S} and \mathcal{S}^* are hyponormal. Every compact hyponormal operator is normal ([24]).

PROPOSITION 101 ([53]). *Every subnormal operator is hyponormal.*

LEMMA 102 ([45]). *If \mathcal{A} is a hyponormal operator then $\|\mathcal{A}^n\| = \|\mathcal{A}\|^n$ for all $n \in \mathbb{N}$. Consequently, $\|\mathcal{A}\| = r(\mathcal{A})$, the spectral radius of \mathcal{A} .*

PROPOSITION 103. *If \mathcal{S} is subnormal and \mathcal{N} is its minimal normal extension, then $\|\mathcal{S}\| = \|\mathcal{N}\|$.*

PROOF. By the spectral inclusion theorem we have $r(\mathcal{N}) \leq r(\mathcal{A})$. By Lemma 102 implies $\|\mathcal{N}\| \leq \|\mathcal{A}\|$. The inequality $\|\mathcal{A}\| \leq \|\mathcal{N}\|$ follows directly from the appropriate definitions. \square

REMARK 104. Let \mathcal{N} be a (possibly unbounded) normal operator on a Hilbert space \mathbf{K} . If the domain $\mathfrak{D}(\mathcal{N})$ of \mathcal{N} contains a closed linear subspace \mathbf{H} of \mathbf{K} and if $\mathcal{N}(\mathbf{H}) \subset \mathbf{H}$, then the restriction $\mathcal{N}|_{\mathbf{H}}$ is a subnormal operator with minimal normal extension $\mathcal{N}|_{\mathbf{J}}$, where

$$\mathbf{J} = \text{cl span}\{\mathcal{N}^{*k}h : h \in \mathbf{H}, k \in \mathbb{N}_0\}.$$

23.1. Susskind-Glogower phase POVM. The positive moments of the Susskind-Glogower phase POVM S are

$$(30) \quad \mathcal{S}_n = \int_0^{2\pi} e^{-in\theta} S(d\theta) = \sum_{\ell=0}^{\infty} \varphi_{\ell+n} \otimes \varphi_{\ell},$$

for $n \in \mathbb{N}_0$. These are linear isometries and hence subnormal operators. A minimal Naimark extension of S is $(\mathbf{1}, \mathbf{L}_2([0, 2\pi]), \mathbf{V}_S)$. The moments of this Naimark extension are the bilateral shifts w.r.t. the Fourier basis:

$$\mathcal{Z}_n = \int_0^{2\pi} e^{-in\theta} \mathbf{1}(d\theta) = \sum_{\ell \in \mathbb{Z}} \mathbf{e}_{\ell+n} \otimes \mathbf{e}_{\ell}, \quad n \in \mathbb{Z}.$$

A minimal normal extension of S_1 is $(\mathcal{Z}_1, L_2([0, 2\pi]), \mathbf{V}_S)$.

24. Restriction algebra

In §11 of Chapter II of [24], the restriction algebra of a subnormal operator is investigated. In this section we introduce a similar concept for POVMs.

DEFINITION 105. For a POVM M on σ -field Σ of subsets of set Ω , let

$$\mathbf{R}(M) = \{\varphi \in L_\infty(\Omega, \Sigma, M) : \|\rho_M(\varphi)h\|^2 = \int_\Omega |\varphi(x)|^2 M_h(dx) \quad \forall h \in \mathbf{H}\}.$$

This is called the restriction algebra for M .

REMARK 106. Apart from the constant functions, the restriction algebra of a POVM might be empty. However, if $M: \Sigma \rightarrow \mathbf{B}_+(\mathbf{H})$ is a PVM then $\mathbf{R}(M) = L_\infty(\Omega, \Sigma, M)$.

PROPOSITION 107. Let $M: \Sigma \rightarrow \mathbf{B}_+(\mathbf{H})$ be a POVM on σ -field Σ of subsets of set Ω . Let $(N, \mathbf{K}, \mathbf{V})$ be a minimal Naimark extension of M . Then

$$(31) \quad \mathbf{R}(M) = \{\varphi \in L_\infty(\Omega, \Sigma, M) : \rho_N(\varphi)\text{range}(\mathbf{V}) \subset \text{range}(\mathbf{V})\}$$

and $\mathbf{R}(M)$ is a weak-star closed subalgebra of $L_\infty(\Omega, \Sigma, M)$.

PROOF. Let $\varphi \in L_\infty(\Omega, \Sigma, N)$. For all $h \in \mathbf{H}$,

$$\|\mathbf{V}^* \rho_N(\varphi) \mathbf{V}[h]\| = \|\rho_M(\varphi)h\|$$

and

$$\|\rho_N(\varphi) \mathbf{V}[h]\| = \int_\Omega |\varphi(x)|^2 N_{\mathbf{V}[h]}(dx) = \int_\Omega |\varphi(x)|^2 M_h(dx).$$

The left-hand sides are equal for all h if, and only if, $\rho_N(\varphi)\text{range}(\mathbf{V}) \subset \text{range}(\mathbf{V})$. The right-hand sides are equal for all h if, and only if, $\varphi \in \mathbf{R}(M)$. Hence (31). From (31) it follows that $\mathbf{R}(M)$ is an algebra. Let φ_i be a net in $\mathbf{R}(M)$ and suppose that $\varphi_i \rightarrow \varphi$ in $(L_\infty(\Omega, \Sigma, N), \text{weak}^*)$. By Proposition 33, $\rho_N(\varphi_i) \rightarrow \rho_N(\varphi)$ in $(\mathbf{B}_\infty(\mathbf{K}), \text{weak}^*)$. Hence for each $h \in \mathbf{H}$, $\rho_N(\varphi_i) \mathbf{V}[h] \rightarrow \rho_N(\varphi) \mathbf{V}[h]$ weakly. Because $\rho_N(\varphi_i) \mathbf{V}[h] \in \text{range}(\mathbf{V})$, this implies that $\rho_N(\varphi)h \in \text{range}(\mathbf{V})$. Hence $\varphi \in \mathbf{R}(M)$. \square

REMARK 108. Let \mathcal{S} be a bounded subnormal operator on \mathbf{H} with normal extension $(\mathcal{N}, \mathbf{V}, \mathbf{K})$. Let $N: \mathfrak{B} \rightarrow \mathbf{B}_+(\mathbf{K})$ be the spectral measure of \mathcal{N} , and define POVM $M: \mathfrak{B} \rightarrow \mathbf{B}_+(\mathbf{H})$ by $M(\Delta) = \mathbf{V}^* N(\Delta) \mathbf{V}$. Then $\mathcal{S} = \rho_M(z)$ (this is an abbreviation for $\rho_M(\varphi)$, where φ is the function $\varphi(z) = z$) and from (31) it follows that $\mathbf{R}(M)$ contains the analytic polynomials.

PROPOSITION 109. For a POVM M on σ -field Σ of subsets of set Ω , $\mathbf{R}(M) \subset \{\varphi \in L_\infty(\Omega, \Sigma, M) : \rho_M(\varphi) \text{ is a subnormal operator}\}$.

PROOF. Let $(N, \mathbf{K}, \mathbf{V})$ be a minimal Naimark extension of POVM M . Let $\varphi \in \mathbf{R}(M)$. Then $\rho_M(\varphi) = \mathbf{V}^* \rho_N(\varphi) \mathbf{V}$. From (31) it follows that $\mathbf{V} \rho_M(\varphi) \mathbf{V}^* = \rho_N(\varphi)$ on $\text{range}(\mathbf{V})$. Hence $\rho_M(\varphi)$ is a subnormal operator with normal extension $\rho_N(\varphi)$ on \mathbf{K} . \square

THEOREM 110. The map

$$\rho_M: (\mathbf{R}(M), \text{weak}^*) \rightarrow (\mathbf{B}_\infty(\mathbf{H}), \text{weak}^*)$$

is multiplicative, isometric and has closed range.

PROOF. This follows from Proposition 109 above, together with Proposition 11.2 of Chapter II in [24]. \square

REMARK 111. Theorem 110 implies that

$$\mathbf{R}(M) \subset \{\varphi \in L_\infty(\Omega, \Sigma, M) : \|\rho_M(\varphi)\| = \|\varphi\|_\infty\}.$$

THEOREM 112. *Let $M: \Sigma \rightarrow \mathbf{B}_+(\mathbf{H})$ be a POVM on σ -field Σ of subsets of set Ω . If $\mathbf{R}(M)$ contains an injective function, then M is an injective POVM.*

PROOF. By Proposition 109, $\int_{\mathbb{C}} z \varphi(M)(dz) = \rho_M(\varphi)$ is a subnormal operator. By Proposition 30.20 in [25], this implies that $\varphi(M)$ is an injective POVM. If f is a measurable function on \mathbb{C} such that $f \circ \varphi \in \mathbf{L}_\infty(\Omega, \Sigma, M)$, then

$$\rho_M(f \circ \varphi) = \int_{\mathbb{C}} f(z) \varphi(M)(dz) = 0$$

implies $f(z) = 0$ for $\varphi(M)$ -almost all z . Hence $f(\varphi(x)) = 0$ for M -almost all x . Conclusion: If $\varphi \in \mathbf{R}(M)$ and f is a measurable function on \mathbb{C} such that $f \circ \varphi \in \mathbf{L}_\infty(\Omega, \Sigma, M)$, then $\rho_M(f \circ \varphi) = 0$ implies $f \circ \varphi = 0$. If φ is injective then every $g \in \mathbf{L}_\infty(\Omega, \Sigma, M)$ can be written in the form $f \circ \varphi$ for some f . \square

24.1. Susskind-Glogower phase POVM. Let S be the Susskind-Glogower phase POVM. Then $\text{span}\{\mathbf{e}_k : k \in \mathbb{N}_0\} \subset \mathbf{R}(S)$.

25. Constructing POVMs from PVMs

In this section we consider two operations on the set of POVMs: Projection and smearing. We also consider taking the limit of a sequence of POVMs and taking convex combinations of POVMs. The result of these operations is a POVM, but not always a projection-valued one.

25.1. Projection of a PVM. Given a POVM, Naimark's theorem guarantees the existence of a PVM from which the POVM can be recovered with the help of a linear isometry (or a projection operator, depending on the particular formulation of Naimark's theorem). The direction of this process can be reversed: Let $N: \Sigma \rightarrow \mathbf{B}_+(\mathbf{K})$ be a PVM with operators on a Hilbert space \mathbf{K} as values, and let \mathbf{V} be a linear isometry from a Hilbert space \mathbf{H} to \mathbf{K} . Then $M: \Sigma \rightarrow \mathbf{B}_+(\mathbf{H})$, defined by $M(\Delta) = \mathbf{V}^* \mathcal{N}(\Delta) \mathbf{V}$, is a POVM. (Similarly, if \mathbf{H} is a closed linear subspace of \mathbf{K} and \mathcal{P} is the projection operator with range \mathbf{H} then $M(\Delta) = \mathcal{P} N(\Delta)$ defines a POVM taking its values in the operators on \mathbf{H} .)

25.2. Smearing of a PVM. Let Σ be a σ -field of subsets of a set Ω and let Σ_1 be a σ -field of subsets of a set Ω_1 . From a POVM $E: \Sigma_1 \rightarrow \mathbf{B}_+(\mathbf{H})$ and $(p_\Delta) \in \mathbf{M}(\Sigma; \Omega_1, \Sigma_1, E)$ we can make a new POVM $M: \Sigma \rightarrow \mathbf{B}_+(\mathbf{H})$ by $M(\Delta) = \rho_E(p_\Delta)$. The word smearing is used in Section II.2.3 of [15], to indicate the process of making a POVM out of a PVM in this way.

The POVM M obtained in this way is not necessarily projection-valued, even if E is. The POVM M has however commutative range if E is a PVM. By Proposition 44, every POVM with commutative range is a smeared version of a PVM.

25.3. Convex combinations of PVMs. Given two POVMs $M_1: \Sigma \rightarrow \mathbf{B}_+(\mathbf{H})$ and $M_2: \Sigma \rightarrow \mathbf{B}_+(\mathbf{H})$ we can make new POVMs $M_\kappa: \Sigma \rightarrow \mathbf{B}_+(\mathbf{H})$ by taking convex combinations:

$$M_\kappa(\Delta) = \kappa M_1(\Delta) + (1 - \kappa) M_2(\Delta), \quad \kappa \in (0, 1).$$

The resulting POVMs are not necessarily projection-valued even if M_1 and M_2 are. Convex combinations (or probability averages) of uncountable sets of POVMs are considered in [4] and [5]: In the context of POVMs on the Borel subsets \mathfrak{B} of a separable metrizable locally compact Hausdorff space Ω , it is shown that every POVM with commutative range can be written as the probability average of a set of PVMs. More precisely: Given a POVM M , there is a compact metrizable topology on the convex set \mathbf{X} of all regular

FPOVMs on \mathfrak{B} taking values in $\text{ball}(M(\mathfrak{B})''_+)$, and there is a probability measure μ on $\text{ext}(\mathbf{X})$ (the extremal points of \mathbf{X}) concentrated on the subset of POVMs, such that

$$(\forall \Delta \in \mathfrak{B})(\forall h \in \mathbf{H}) \quad M_h(\Delta) = \int_{\text{ext}(\mathbf{X})} N_h(\Delta) \mu(dN).$$

If M has commutative range, then $\text{ext}(\mathbf{X})$ consists of PVMs.

25.3.1. *Example:* Let $(\Sigma_2, \Sigma_2, \mu_2)$ be a probability space. Let \mathfrak{B} be the Borel subsets of a separable metrizable locally compact Hausdorff space Ω . Let $(p_\Delta) \in \mathbf{M}(\mathfrak{B}; \Omega_2, \Sigma_2, \mu_2)$. Let $\mathbf{H} = \mathbf{L}_2(\Omega_2, \Sigma_2, \mu_2)$. Consider the PVM $\mathbb{1}: \Sigma_2 \rightarrow \mathbf{B}_+(\mathbf{H})$. By Theorem 35,

$$\rho_{\mathbb{1}}: \mathbf{L}_\infty(\Omega_2, \Sigma_2, \mu_2) \rightarrow \mathbf{B}_\infty(\mathbf{H})$$

is a linear isometry. Define POVM $M: \mathfrak{B} \rightarrow \mathbf{B}_+(\mathbf{H})$ by

$$M(\Delta) = \rho_{\mathbb{1}}(p_\Delta), \quad \Delta \in \mathfrak{B}.$$

The range $M(\mathfrak{B})$ of M is contained in the commutative von Neumann algebra $\mathbb{1}(\Sigma_2)''$ generated by the range $\mathbb{1}(\Sigma_2)$ of $\mathbb{1}$. Let \mathbf{X} be as above. Then $\text{ext}(\mathbf{X})$ consists of PVMs $P: \mathfrak{B} \rightarrow \mathbf{B}_+(\mathbf{H})$ whose ranges are contained in $\mathbb{1}(\Sigma_2)$. For every such P there is a function $\Phi_P: \mathfrak{B} \rightarrow \Sigma_2$ such that $P(\Delta) = \mathbb{1}(\Phi_P(\Delta))$ for $\Delta \in \mathfrak{B}$. Hence there is a probability measure μ on $\text{ext}(\mathbf{X})$ such that

$$(\forall \Delta \in \mathfrak{B}) \quad p_\Delta = \int_{\text{ext}(\mathbf{X})} 1_{\Phi_P(\Delta)} \mu(dP)$$

with respect to the weak-star topology of $\mathbf{L}_\infty(\Omega_2, \Sigma_2, \mu_2)$. In [4], a minimal Naimark extension of the POVM M is given in terms of $(\text{ext}(\mathbf{X}), \mu)$.

25.4. Limits of PVMs.

PROPOSITION 113. *Let $M^{(n)}: \Sigma \rightarrow \mathbf{B}_+(\mathbf{H})$, $n \in \mathbb{N}$ be a sequence of POVMs such that WOT-limits*

$$M(\Delta) = \lim_{n \rightarrow \infty} M^{(n)}(\Delta), \quad \Delta \in \Sigma$$

exist. Then M is a POVM on Σ .

PROOF. This follows from Theorem 1 and Proposition 203. \square

PROPOSITION 114. *Let $F^{(n)}: \Sigma \rightarrow \mathbf{B}_+(\mathbf{H})$, $n \in \mathbb{N}$ be a sequence of PVMs such that the SOT-limits*

$$F(\Delta) = \lim_{n \rightarrow \infty} F^{(n)}(\Delta), \quad \Delta \in \Sigma$$

exist. Then F is a PVM on Σ .

PROOF. The SOT-limit of a sequence of orthogonal projection operators is again an orthogonal projection operator. \square

Every POVM is the WOT-limit of a sequence of PVMs and the restriction algebra can be characterized in terms of the convergence of these limits.

PROPOSITION 115. *Let \mathbf{H} be an infinite dimensional complex separable Hilbert space. Let $M: \Sigma \rightarrow \mathbf{B}_+(\mathbf{H})$ be a POVM. There exists a sequence $N^{(m)}: \Sigma \rightarrow \mathbf{B}_+(\mathbf{H})$, $m \in \mathbb{N}$ of PVMs such that*

(a) $\lim_{m \rightarrow \infty} N^{(m)}(\Delta) = M(\Delta)$ in the WOT for every $\Delta \in \Sigma$.

(b) $\mathbf{R}(M) = \{\varphi \in \mathbf{L}_\infty(\Omega, \Sigma, M) : \lim_{m \rightarrow \infty} \rho_{N^{(m)}}(\varphi) = \rho_M(\varphi) \text{ in the SOT}\}$.

PROOF. Let $(N, \mathbf{K}, \mathbf{V})$ be a Naimark extension of M . Let $\varphi_n, n \in \mathbb{N}$ be an orthonormal basis of \mathbf{H} . For $m \in \mathbb{N}$ let $\mathbf{H}_m = \text{span}\{\varphi_n : 1 \leq n \leq m\}$. Because \mathbf{H} is infinite dimensional, there exists for each $m \in \mathbb{N}$ a unitary operator $\mathbf{U}_m: \mathbf{H} \rightarrow \mathbf{K}$ satisfying $\mathbf{U}_m = \mathbf{V}$ on \mathbf{H}_m . Define PVM $N^{(m)}: \Sigma \rightarrow \mathbf{B}_p(\mathbf{H})$ by $N^{(m)}(\Delta) = \mathbf{U}_m^* N(\Delta) \mathbf{U}_m$. Let $\varphi \in \mathbf{L}_\infty(\Omega, \Sigma, N)$. If $h \in \mathbf{H}_m$ then

$$\begin{aligned} (h, \rho_{N^{(m)}}(\varphi)h) &= (h, \mathbf{U}_m^* \rho_N(\varphi) \mathbf{U}_m h) \\ &= (\mathbf{U}_m h, \rho_N(\varphi) \mathbf{U}_m h) \\ &= (\mathbf{V}[h], \rho_N(\varphi) \mathbf{V}[h]). \end{aligned}$$

Let \mathcal{P}_m be orthogonal projection on \mathbf{H}_m . For $h \in \mathbf{H}$,

$$\begin{aligned} \lim_{m \rightarrow \infty} (h, \rho_{N^{(m)}}(\varphi)h) &= \lim_{m \rightarrow \infty} (\mathcal{P}_m h, \rho_{N^{(m)}}(\varphi) \mathcal{P}_m h) \\ &= \lim_{m \rightarrow \infty} (\mathbf{V}[\mathcal{P}_m h], \rho_N(\varphi) \mathbf{V}[\mathcal{P}_m h]) \\ &= (\mathbf{V}[h], \rho_N(\varphi) \mathbf{V}[h]). \end{aligned}$$

Hence

$$(h, \rho_M(\varphi)h) = (\mathbf{V}[h], \rho_N(\varphi) \mathbf{V}[h]) = \lim_{m \rightarrow \infty} (h, \rho_{N^{(m)}}(\varphi)h) \quad \forall h \in \mathbf{H}.$$

For $h \in \mathbf{H}$,

$$\begin{aligned} \lim_{m \rightarrow \infty} \|\rho_{N^{(m)}}(\varphi)h\|^2 &= \lim_{m \rightarrow \infty} (h, \rho_{N^{(m)}}(|\varphi|^2)h) \\ &= (\mathbf{V}[h], \rho_N(|\varphi|^2) \mathbf{V}[h]) \\ &= \|\rho_N(\varphi) \mathbf{V}[h]\|^2. \end{aligned}$$

Hence $\varphi \in \mathbf{R}(M)$ if, and only if,

$$\lim_{m \rightarrow \infty} \|\rho_{N^{(m)}}(\varphi)h\| = \|\rho_M(\varphi)h\|.$$

□

25.5. Example. Let $S: \mathfrak{B} \rightarrow \mathbf{B}_\infty(\mathbf{L}_2(\mathbb{R}))$ be the Susskind-Glogower phase POVM introduced in Section 19. We introduce a sequence of PVMS, called the Pegg-Barnett phase PVMS, that converge to S in the sense of Proposition 115.

25.5.1. *Pegg-Barnett phase PVMS.* The Pegg-Barnett phase PVMS ([88]) are maximal PVMS on finite dimensional spaces. For $M \in \mathbb{N}$ let $\mathbb{N}_M = \mathbb{N}_0 \cap [0, M]$, let $\Phi_M: \mathbb{N}_M \rightarrow \mathbf{L}_2(\mathbb{R})$ be the restriction of the Hermite basis Φ to \mathbb{N}_M , let the finite dimensional Hilbert space $\mathbf{H}_M \subset \mathbf{L}_2(\mathbb{R})$ be defined by $\mathbf{H}_M = \text{span}(\Phi_M)$.

LEMMA 116. For $m \in \mathbb{N}_M$ define $\theta_m \in [0, 2\pi)$ by $\theta_m = 2\pi m / (M + 1)$. For $\theta \in [0, 2\pi)$ let δ_θ be Dirac measure at θ . Let

$$\mu_M = \frac{2\pi}{M + 1} \sum_{m=0}^M \delta_{\theta_m}$$

Restriction $\mathbf{e}^{(M)}: \mathbb{N}_M \rightarrow \mathbf{L}_2([0, 2\pi])$ of the Fourier basis \mathbf{e} is an orthonormal basis of $\mathbf{L}_2([0, 2\pi], \mu_M)$.

PROOF. The lemma follows from

$$\frac{1}{M + 1} \sum_{m=0}^M e^{in\theta_m} = 1_{(M+1)\mathbb{Z}}(n) \quad \forall n \in \mathbb{Z},$$

which we will now prove: We have

$$(32) \quad \sum_{m=0}^M e^{ix\theta_m} = \frac{e^{2\pi ix} - 1}{e^{2\pi ix/(M+1)} - 1}.$$

This is clearly a continuous $(M+1)$ -periodic function on $\mathbb{R} \setminus (M+1)\mathbb{Z}$ which is 0 for $x \in \mathbb{Z} \setminus (M+1)\mathbb{Z}$. The function on \mathbb{R} in the left-hand side of (32) is the unique continuous extension of the right-hand side and equals $M+1$ at $x=0$ and hence (by periodicity) at the points $x \in (M+1)\mathbb{Z}$. \square

The Pegg-Barnett phase PVM $P^{(M)}: \mathfrak{B}_{\mathbb{C}} \rightarrow \mathbf{B}_+(\mathbf{H}_M)$ is defined by

$$P_h^{(M)}(\Delta) = \int_{\Delta} |\mathbf{V}_P^{(M)}[h](\theta)|^2 \mu_M(d\theta),$$

where the unitary operator $\mathbf{V}_P^{(M)}: \mathbf{H}_M \rightarrow \mathbf{L}_2([0, 2\pi], \mu_M)$ is defined in terms of its action on the Hermite functions by

$$\mathbf{V}_P^{(M)}[\varphi_n^{(M)}] = \mathbf{e}_n^{(M)}, \quad n \in \mathbb{N}_M.$$

By Theorem 77, $P^{(M)}$ is a maximal PVM.

25.5.2. *Moments.* In order to prove that $P^{(M)}$ converges to S as $M \rightarrow \infty$ in the sense of Proposition 115, we calculate the moments of $\mathcal{P}^{(M)}$. The moments of \mathcal{S} were given in (30). The moments of the Pegg-Barnett phase PVM are

$$\mathcal{P}_n^{(M)} = \int_0^{2\pi} e^{-in\theta} P^{(M)}(d\theta) = \sum_{\ell=0}^M \Phi(\gamma_M(\ell+n)) \otimes \Phi(\ell), \quad n \in \mathbb{Z}$$

where $\gamma_M: \mathbb{Z} \rightarrow \mathbb{N}_M$ is defined by

$$\gamma_M(x) = x - \left\lfloor \frac{x}{M+1} \right\rfloor (M+1).$$

25.5.3. *Approximation of the Susskind-Glogower phase POVM.* The definition of the Pegg-Barnett phase PVM can be extended to get a PVM $P^{(M)}: \mathfrak{B}_{\mathbb{C}} \rightarrow \mathbf{B}_+(\mathbf{L}_2(\mathbb{R}))$: Let $P_h^{(M)}(\Delta) = 0$ if $\mu_M(\Delta) = 0$, and

$$P_h^{(M)}(\Delta) = \int_{\Delta} |\mathbf{V}_P^{(M)}[\mathcal{P}_M h](\theta)|^2 \mu_M(d\theta) + \|(\mathcal{I} - \mathcal{P}_M)h\|^2,$$

where \mathcal{P}_M is the operator of orthogonal projection on the closed linear subspace \mathbf{H}_M of $\mathbf{L}_2(\mathbb{R})$, if $\mu_M(\Delta) > 0$. From

$$\begin{aligned} \mathcal{P}_n^{(2M)} &= \mathcal{S}_n \quad \text{on } \mathbf{H}_M \quad \forall n \in \mathbb{N}_M, \\ \mathcal{P}_{-n}^{(2M)} &= \mathcal{S}_n^* \quad \text{on } \mathbf{H}_M \quad \forall n \in \mathbb{N}_M \end{aligned}$$

it follows that

$$\rho_{P^{(M)}}(\varphi)h = \rho_S(\varphi)h \quad \forall h \in \mathbf{H}_M, \varphi \in \text{span}\{\mathbf{e}_k : -M \leq k \leq M\}.$$

By Theorem 86, $\text{span}\{\mathbf{e}_k : k \in \mathbb{Z}\}$ is weak-star sequentially dense in $\mathbf{L}_{\infty}(S)$. Hence

$$(\forall \Delta \in \mathfrak{B}_{\mathbb{C}})(\forall h \in \mathbf{L}_2(\mathbb{R})) \quad \lim_{M \rightarrow \infty} P_h^{(M)}(\Delta) = S_h(\Delta).$$

26. POVMs on Federer metric spaces

DEFINITION 117. A Federer metric space is a metric space (Ω, d) with the following property: For every regular Borel measure μ which is bounded on bounded Borel sets and every function $f: \Omega \rightarrow \mathbb{C}$ which is integrable on bounded Borel sets, there exists a μ -null set \mathfrak{N} such that for all $r > 0$ and all $x \in \Omega \setminus \mathfrak{N}$ the closed ball $\text{ball}(x, r)$ with radius r and center x has positive μ -measure and the limits

$$\lim_{r \downarrow 0} \frac{1}{\mu(\text{ball}(x, r))} \int_{\text{ball}(x, r)} f(y) \mu(dy), \quad x \in \Omega \setminus \mathfrak{N}$$

exist and define (pointwise) a μ -measurable function which is equal to f μ -almost everywhere.

Similar conditions on metric spaces are introduced in [43]. A proof of the following theorem can be found in [104].

THEOREM 118. *Let $n \in \mathbb{N}$ and let $|\cdot|$ be a norm on \mathbb{R}^n . Then \mathbb{R}^n with metric d , defined by $d(x, y) = |x - y|$, is a Federer metric space.*

DEFINITION 119. Let \mathcal{R} be a bounded injective operator on Hilbert space \mathbf{H} with dense range. Hilbert spaces \mathbf{H}_- and \mathbf{H}_+ and sesquilinear form $\langle \cdot, \cdot \rangle: \mathbf{H}_- \times \mathbf{H}_+ \rightarrow \mathbb{C}$ are defined as follows:

- $\mathbf{H}_+ = \text{range}(\mathcal{R})$, equipped with inner-product $(h, g)_+ = (\mathcal{R}^{-1}h, \mathcal{R}^{-1}g)$.
- \mathbf{H}_- is the completion of \mathbf{H} with respect to inner-product $(h, g)_- = (\mathcal{R}^*h, \mathcal{R}^*g)$.
- Operator $(\mathcal{R}^*)^{\text{ext}}: \mathbf{H}_- \rightarrow \mathbf{H}$ is the isometric extension of \mathcal{R}^* defined on \mathbf{H} , considered as a subspace of Hilbert space \mathbf{H}_- .
- $\langle H, g \rangle = ((\mathcal{R}^*)^{\text{ext}}[H], \mathcal{R}^{-1}g)_{\mathbf{H}}$, for $H \in \mathbf{H}_-$ and $g \in \mathbf{H}_+$.

The triple of Hilbert spaces $\mathbf{H}_+ \subset \mathbf{H} \subset \mathbf{H}_-$ is called ‘Gelfand triple associated with operator \mathcal{R} ’ and sesquilinear form $\langle \cdot, \cdot \rangle: \mathbf{H}_- \times \mathbf{H}_+ \rightarrow \mathbb{C}$ is called ‘the pairing between \mathbf{H}_- and \mathbf{H}_+ .’

The following theorem and proof are in essence not new; they are similar to results in [103] and [102] and to results about canonical Dirac bases in [104].

THEOREM 120. *Let (Ω, d) be a complete and separable and Federer metric space.*

Let \mathfrak{B} be the Borel subsets of Ω , and let $M: \mathfrak{B} \rightarrow \mathbf{B}_+(\mathbf{H})$ be a POVM.

Let $\mathbb{I}, (\Omega_m), (\mu_m)$ be defined by Theorem 94.

Let \mathcal{R} be an injective non-negative Hilbert-Schmidt operator on \mathbf{H} , and let $\mathbf{H}_+ \subset \mathbf{H} \subset \mathbf{H}_-$ be the Gelfand triple associated with \mathcal{R} , and let $\langle \cdot, \cdot \rangle: \mathbf{H}_- \times \mathbf{H}_+ \rightarrow \mathbb{C}$ be the pairing between \mathbf{H}_- and \mathbf{H}_+ . (See Definition 119.)

There is a M -null set \mathfrak{N} and there are $F_{x,j}^{(m)} \in \mathbf{H}_-$, $(m, x, j) \in \tilde{\Omega}$, where

$$\tilde{\Omega} = \{(m, x, j) : m \in \mathbb{I}, x \in \Omega_m \setminus \mathfrak{N}, 1 \leq j \leq m\}$$

such that for all $h \in \mathbf{H}_+$,

$$(\forall \Delta \in \mathfrak{B}) \quad M_h(\Delta) = \sum_{m \in \mathbb{I}} \sum_{j=1}^m \int_{\Delta} |\langle F_{x,j}^{(m)}, h \rangle|^2 \mu_m(dx)$$

and

$$(\forall (m, x, j) \in \tilde{\Omega}) \quad \lim_{r \downarrow 0} \left\| F_{x,j}^{(m)} - F_{x,j}^{(m)}(r) \right\|_- = 0$$

where

$$F_{x,j}^{(m)}(r) = \frac{1}{\mu_m(\text{ball}(x, r))} \int_{\text{ball}(x, r)} F_{y,j}^{(m)} \mu_m(dy).$$

PROOF. From the results of Chapter 8 in [22] it follows that \mathfrak{B} is countably generated, and that every finite Borel measure is regular.

There are $\rho_k > 0$ and an orthonormal basis (v_k) of \mathbf{H} such that $\sum_{k=1}^{\infty} \rho_k^2 < \infty$ and

$$\mathcal{R} = \sum_{k=1}^{\infty} \rho_k v_k \otimes v_k.$$

Let $(\mathbf{K}_m), (\mathbf{V}_m)$ be defined by Lemma 93. Let $\psi_j^{(m)}, 1 \leq j \leq m$ be an orthonormal basis of \mathbf{K}_m . Let $m \in \mathbb{I}$ and j be an integer in $[1, m]$. Define $\mathbf{V}_{m,j}: \mathbf{H} \rightarrow \mathbf{L}_2(\Omega, \mathfrak{B}, \mu_m)$ by $\mathbf{V}_{m,j}[h] = (\psi_j^{(m)}, \mathbf{V}_m[h](x))$. There exists an M -null set $\mathfrak{N}_{m,j} \subset \Omega_m$ such that for $x \in \Omega_m \setminus \mathfrak{N}_{m,j}$ and $k \in \mathbb{N}$ limit

$$(33) \quad \varphi_{j,k}^{(m)}(x) = \lim_{r \downarrow 0} \frac{1}{\mu_m(\text{ball}(x, r))} \int_{\text{ball}(x, r)} \mathbf{V}_{m,j}[v_k](y) \mu_m(dy)$$

exists and

$$(34) \quad F_{x,j}^{(m)} = \sum_{k=1}^{\infty} \overline{\varphi_{j,k}^{(m)}(x)} v_k$$

converges in \mathbf{H}_- .

Let $r > 0$ and $(m, x, j) \in \tilde{\Omega}$. Element $F_{x,j}^{(m)}(r)$ of \mathbf{H}_- is defined by its action

$$\langle F_{x,j}^{(m)}(r), h \rangle = \frac{1}{\mu_m(\text{ball}(x, r))} \int_{\text{ball}(x, r)} \langle F_{y,j}^{(m)}, h \rangle \mu_m(dy), \quad h \in \mathbf{H}_+$$

on \mathbf{H}_+ . If (34) is used, we get an integral of a sum; we show that integration can be done term-by-term: By Fubini's theorem the following estimation suffices

$$\begin{aligned} \int_{\text{ball}(x, r)} \sum_{k=1}^{\infty} |\varphi_{j,k}^{(m)}(y)(v_k, h)| \mu_m(dy) &\leq \left(\int_{\text{ball}(x, r)} \sum_{k=1}^{\infty} \rho_k^{-2} |(v_k, h)|^2 \mu_m(dy) \right)^{1/2} \\ &\quad \cdot \left(\int_{\text{ball}(x, r)} \sum_{k=1}^{\infty} \rho_k^2 |\varphi_{j,k}^{(m)}(y)|^2 \mu_m(dy) \right)^{1/2} \\ &\leq \sqrt{\mu_m(\text{ball}(x, r))} \|h\|_+ \left(\sum_{k=1}^{\infty} \rho_k^2 \|\varphi_{j,k}^{(m)}\|^2 \right)^{1/2} \\ &\leq \sqrt{\mu_m(\text{ball}(x, r))} \|h\|_+ \left(\sum_{k=1}^{\infty} \rho_k^2 \right)^{1/2}. \end{aligned}$$

This is finite. Hence

$$(35) \quad F_{x,j}^{(m)}(r) = \frac{1}{\mu_m(\text{ball}(x, r))} \sum_{k=1}^{\infty} \int_{\text{ball}(x, r)} \overline{\varphi_{j,k}^{(m)}(y)} \mu_m(dy) v_k.$$

By Lemma 248 (Lemma 4 in [103]) there exists a μ_m -null set \mathfrak{N}_m such that for $x \in \Omega_m \setminus \mathfrak{N}_m$

$$\lim_{r \downarrow 0} \left\| F_{x,j}^{(m)} - F_{x,j}^{(m)}(r) \right\|_- = 0, \quad 1 \leq j \leq m.$$

Let $\mathfrak{N} = \cup_{m \in \mathbb{I}} \mathfrak{N}_m$. □

26.1. Susskind-Glogower phase POVM. The Susskind-Glogower phase POVM $S: \mathfrak{B} \rightarrow \mathcal{B}_\infty(\mathbb{L}_2(\mathbb{R}))$ was introduced in Section 19. Let $r \in (0, 1)$. Let (φ_n) be the Hermite basis of $\mathbb{L}_2(\mathbb{R})$, and define operator \mathcal{R}_r in terms of its action on the Hermite basis functions by

$$(\forall n \in \mathbb{N}_0) \quad \mathcal{R}_r \varphi_n = r^n \varphi_n = \exp\left\{-\log\left(\frac{1}{r}\right)n\right\} \varphi_n.$$

This is a positive and injective Hilbert-Schmidt operator on $\mathbb{L}_2(\mathbb{R})$. Let

$$\mathbb{H}_+^{(r)} \subset \mathbb{L}_2(\mathbb{R}) \subset \mathbb{H}_-^{(r)}$$

denote the Gelfand triple associated with operator \mathcal{R}_r . Note that (φ_n) is an orthogonal basis of Hilbert spaces \mathbb{H}_+ , \mathbb{H} and \mathbb{H}_- . Let

$$\langle \cdot, \cdot \rangle: \mathbb{H}_-^{(r)} \times \mathbb{H}_+^{(r)} \rightarrow \mathbb{C}$$

be the pairing between $\mathbb{H}_-^{(r)}$ and $\mathbb{H}_+^{(r)}$. Define $F_\theta \in \mathbb{H}_-^{(r)}$, $\theta \in [0, 2\pi)$ by

$$(36) \quad F_\theta = \sum_{n=0}^{\infty} \overline{\epsilon_n(\theta)} \varphi_n.$$

We have

$$(\forall h \in \mathbb{H}_+^{(r)}) (\forall \theta \in [0, 2\pi)) \quad \mathbf{V}_S[h](\theta) = \langle F_\theta, h \rangle$$

27. Integration of unbounded functions with respect to a POVM

We will use the theory of [67] concerning the integration of unbounded measurable functions with respect to an operator valued measure. We only consider operator-valued measures with positive values.

DEFINITION 121. Let M be a POVM on σ -field Σ of subsets of a non-empty set Ω . Let $\varphi: \Omega \rightarrow \mathbb{C}$ be a Σ -measurable function. Let

$$\mathfrak{D}_M^\varphi = \{h \in \mathbf{H} : \int_{\Omega} |\varphi(x)|^2 M_h(dx) < \infty\}.$$

DEFINITION 122 ([67], Appendix). Let M be a POVM on σ -field Σ of subsets of a non-empty set Ω . Let $\varphi: \Omega \rightarrow \mathbb{C}$ be a Σ -measurable function. We let $\mathfrak{D}(L_M(\varphi))$ denote the set of those $h \in \mathbf{H}$ for which φ is integrable with respect to all complex measures $(g, M(\cdot)h)$, $g \in \mathbf{H}$.

LEMMA 123 ([67], Lemma A.1). *For every $h \in \mathfrak{D}(L_M(\varphi))$ there exist exactly one element $L_M(\varphi)h$ of \mathbf{H} satisfying*

$$(g, L_M(\varphi)h) = \int_{\Omega} \varphi(x) (g, M(dx)h)$$

for all $g \in \mathbf{H}$.

DEFINITION 124. Let $L_M(\varphi)$ be the operator, with domain $\mathfrak{D}(L_M(\varphi))$, such that

$$(g, L_M(\varphi)h) = \int_{\Omega} \varphi(x) (g, M(dx)h) \quad \forall g \in \mathbf{H}, h \in \mathfrak{D}(L_M(\varphi)).$$

We sometimes write $\int_{\Omega} \varphi(x) M(dx)$ in stead of $L_M(\varphi)$.

LEMMA 125 ([67], Lemma A.2). *Let M be a POVM on σ -field Σ of subsets of a non-empty set Ω . Let $\varphi: \Omega \rightarrow \mathbb{C}$ be a Σ -measurable function. Then*

- (1) $\mathfrak{D}_M^\varphi \subset \mathfrak{D}(L_M(\varphi))$.
- (2) *If M is projection-valued then $\mathfrak{D}_M^\varphi = \mathfrak{D}(L_M(\varphi))$.*

LEMMA 126 ([69], Section III, Part A). *Let $(N, \mathbf{K}, \mathbf{V})$ be a Naimark extension of POVM M on σ -field Σ of subsets of a non-empty set Ω . Let $\varphi: \Omega \rightarrow \mathbb{C}$ be a Σ -measurable function. Then*

$$L_M(\varphi) = \mathbf{V}^* L_N(\varphi) \mathbf{V} \quad \text{on } \mathfrak{D}_M^\varphi.$$

LEMMA 127. \mathfrak{D}_M^φ is dense in \mathbf{H} .

PROOF. Let $(N, \mathbf{K}, \mathbf{V})$ be a Naimark extension of M . For $n \in \mathbb{N}$ let $\Omega_n = \{x : n-1 \leq |\varphi(x)| \leq n\}$. Then \mathfrak{D}_N^φ contains $N(\Omega_n)\mathbf{K}$ for all n . This, together with the σ -additivity of N , implies that \mathfrak{D}_N^φ is dense in \mathbf{K} . From

$$\int_{\Omega} |\varphi(x)|^2 M_h(dx) = \int_{\Omega} |\varphi(x)|^2 N_{\mathbf{V}[h]}(dx)$$

it follows that \mathfrak{D}_M^φ contains $\mathbf{V}^* \mathfrak{D}_N^\varphi$, which is dense in \mathbf{H} because \mathfrak{D}_N^φ is dense in \mathbf{K} and \mathbf{V}^* is a bounded operator from \mathbf{K} onto \mathbf{H} . \square

LEMMA 128. *Let M be a POVM on σ -field Σ of subsets of a non-empty set Ω . Let $\varphi: \Omega \rightarrow \mathbb{C}$ be a Σ -measurable function. Then $L_M(\varphi)^*$ extends $L_M(\bar{\varphi})$.*

PROOF. This follows from Lemma A.4 of [67] because $\mathfrak{D}(L_M(\varphi))$ is a dense subset of \mathbf{H} . \square

PROPOSITION 129. Let M be a POVM on σ -field Σ of subsets of a non-empty set Ω . Let $\varphi: \Omega \rightarrow \mathbb{C}$ be a Σ -measurable function and let $h \in \mathfrak{D}_M^\varphi$. Then

$$(37) \quad \|L_M(\varphi)h\|^2 \leq \int_{\Omega} |\varphi(x)|^2 M_h(dx).$$

If M is projection-valued then both sides of this inequality are equal.

PROOF. Let (φ_n) be a sequence of simple functions converging pointwise to φ , with $|\varphi_n| \leq |\varphi|$ for all n .

Let $h \in \mathfrak{D}_M^\varphi$. From the proof of Lemma A.2 in [67], it follows that

$$\int_{\Omega} |\varphi_n(x)| |(g, M(dx)h)| \leq \|g\| \sqrt{\int_{\Omega} |\varphi_n(x)|^2 M_h(dx)}$$

for all $g \in \mathbf{H}$. By the dominated convergence theorem,

$$|(g, L_M(\varphi)h)| \leq \int_{\Omega} |\varphi(x)| |(g, M(dx)h)| \leq \|g\| \sqrt{\int_{\Omega} |\varphi(x)|^2 M_h(dx)}$$

for all $g \in \mathbf{H}$. Hence (37).

Now assume that M is projection-valued. By (37),

$$\|L_M(\varphi)k - L_M(\varphi_n)k\|^2 \leq \int_{\Omega} |\varphi(x) - \varphi_n(x)|^2 M_k(dx)$$

for all n . By the dominated convergence theorem, the right-hand-side converges to 0. Hence $(L_M(\varphi_n)k)$ converges to $L_M(\varphi)k$. By the dominated convergence theorem, $(\int_{\Omega} |\varphi_n(x)|^2 M_k(dx))$ converges to $\int_{\Omega} |\varphi(x)|^2 M_k(dx)$. Hence the result follows from

$$\|L_M(\varphi_n)k\|^2 = \int_{\Omega} |\varphi_n(x)|^2 M_k(dx) \quad \forall n.$$

□

PROPOSITION 130. Let $(N, \mathbf{K}, \mathbf{V})$ be a minimal Naimark extension of POVM M on σ -field Σ of subsets of a set Ω . Let $\varphi: \Omega \rightarrow \mathbb{C}$ be a Σ -measurable function. The following conditions are equivalent:

(a) $\|L_M(\varphi)h\|^2 = \int_{\Omega} |\varphi(x)|^2 M_h(dx)$ for all $h \in \mathfrak{D}_M^\varphi$.

(b) $L_N(\varphi)\mathbf{V}(\mathfrak{D}_M^\varphi) \subset \text{range } \mathbf{V}$.

(c) $\mathbf{V}L_M(\varphi) = L_N(\varphi)\mathbf{V}$ on \mathfrak{D}_M^φ .

If these conditions are satisfied for φ then

$$(38) \quad L_M(\varphi)(\mathfrak{D}_M^{\varphi\psi} \cap \mathfrak{D}_M^\varphi) \subset \mathfrak{D}_M^\psi \quad \text{and} \quad L_M(\psi)L_M(\varphi) = L_M(\psi\varphi) \quad \text{on} \quad \mathfrak{D}_M^{\varphi\psi} \cap \mathfrak{D}_M^\varphi.$$

for Σ -measurable functions $\psi: \Omega \rightarrow \mathbb{C}$.

REMARK 131. In Section 27.1, we give an example of a POVM on the Borel subsets of \mathbb{C} (which is not projection valued) such that (a) is satisfied not only by $\varphi(z) = 1$, but also by $\varphi(z) = z^n$ for all $n \in \mathbb{N}$.

PROOF. Let $\mathcal{M} = L_M(\varphi)$ and $\mathcal{N} = L_N(\varphi)$. Then $\mathcal{M} = \mathbf{V}^*\mathcal{N}\mathbf{V}$ on \mathfrak{D}_M^φ .

(b) implies (a): Let $h \in \mathfrak{D}_M^\varphi$. Then $\mathcal{N}\mathbf{V}[h] \subset \text{range } \mathbf{V}$ and

$$\begin{aligned} \|\mathcal{M}h\|^2 &= (\mathbf{V}^*\mathcal{N}\mathbf{V}h, \mathbf{V}^*\mathcal{N}\mathbf{V}h) = (\mathcal{N}\mathbf{V}h, \mathbf{V}\mathbf{V}^*\mathcal{N}\mathbf{V}h) \\ &= (\mathcal{N}\mathbf{V}h, \mathcal{N}\mathbf{V}h) = \|\mathcal{N}\mathbf{V}h\|^2 = \int_{\Omega} |\varphi(x)|^2 N_{\mathbf{V}[h]}(dx) \\ &= \int_{\Omega} |\varphi(x)|^2 M_h(dx). \end{aligned}$$

Hence (a).

(a) implies (b): Let $h \in \mathfrak{D}_M^\varphi$. Then

$$\begin{aligned} \|\mathcal{N}\mathbf{V}[h]\|^2 &= \int_{\Omega} |\varphi(x)|^2 N_{\mathbf{V}[h]}(dx) = \int_{\Omega} |\varphi(x)|^2 M_h(dx) \\ &= \|\mathcal{M}h\|^2 = \|\mathbf{V}^*\mathcal{N}\mathbf{V}[h]\|^2 \\ &= \|\mathbf{V}\mathbf{V}^*\mathcal{N}\mathbf{V}[h]\|^2. \end{aligned}$$

Hence $\mathcal{N}\mathbf{V}[h] \in \text{range}(\mathbf{V})$. Hence (b).

We have

$$\mathbf{V}(\mathfrak{D}_M^\psi) = \{f \in \text{range}(\mathbf{V}) : \int_{\Omega} |\psi(x)|^2 N_f(dx) < \infty\} = \text{range}(\mathbf{V}) \cap \mathfrak{D}_N^\psi.$$

and $L_N(\varphi)\mathfrak{D}_N^{\varphi\psi} \cap \mathfrak{D}_N^\varphi \subset \mathfrak{D}_N^\psi$. Together with (b) this implies the first part of (38).

We have $L_M(\psi) = \mathbf{V}^*L_N(\psi)\mathbf{V}$ on \mathfrak{D}_M^ψ . If (b) is satisfied then $\mathbf{V}\mathcal{M} = \mathcal{N}\mathbf{V}$ on \mathfrak{D}_M^φ . Hence

$$L_M(\psi)\mathcal{M} = \mathbf{V}^*L_N(\psi)\mathbf{V}\mathcal{M} = \mathbf{V}^*L_N(\psi)\mathcal{N}\mathbf{V} = \mathbf{V}^*L_N(\psi\varphi)\mathbf{V} = L_M(\psi\varphi)$$

on $\mathfrak{D}_M^{\varphi\psi} \cap \mathfrak{D}_M^\varphi$. \square

PROPOSITION 132. *Let \mathbf{H} be a Hilbert space. Let Σ be a σ -field of subsets of a set Ω . Let $M: \Sigma \rightarrow \mathbf{B}_+(\mathbf{H})$ be a POVM. Let $\varphi: \Omega \rightarrow \mathbb{C}$ be a Σ -measurable function. The operator $L_M(\varphi)$ on \mathbf{H} with domain \mathfrak{D}_M^φ is closable; the closure, denoted by $\overline{L_M(\varphi)}$, satisfies*

$$\mathfrak{D}(\overline{L_M(\varphi)}) = \{h \in \mathbf{H} : \mathfrak{D}_M^\varphi \ni g \mapsto (h, L_M(\bar{\varphi})g) \text{ is bounded}\}$$

and

$$(\overline{L_M(\varphi)}h, g) = (h, L_M(\bar{\varphi})g) \quad \forall h \in \mathfrak{D}(\overline{L_M(\varphi)}), g \in \mathfrak{D}_M^\varphi.$$

PROOF. The closure $\overline{L_M(\varphi)}$ of $L_M(\varphi)$ is the adjoint $(L_M(\varphi)^*)^*$ of $L_M(\varphi)^*$. By Lemma 128, $\overline{L_M(\varphi)} = L_M(\bar{\varphi})^*$. \square

LEMMA 133. *Let $(N, \mathbf{K}, \mathbf{V})$ be a Naimark extension of POVM M on σ -field Σ of subsets of a non-empty set Ω . Let $\varphi: \Omega \rightarrow \mathbb{C}$ be a Σ -measurable function such that*

$$\|L_M(\varphi)h\|^2 = \int_{\Omega} |\varphi(x)|^2 M_h(dx) \quad \forall h \in \mathfrak{D}_M^\varphi.$$

Then $\mathbf{V}^*(\mathfrak{D}_N^\varphi) \subset \mathfrak{D}(\overline{L_M(\bar{\varphi})})$ and

$$(39) \quad \overline{L_M(\bar{\varphi})}\mathbf{V}^* = \mathbf{V}^*L_N(\bar{\varphi}) \quad \text{on } \mathfrak{D}(L_N^\varphi).$$

PROOF. Let $f \in \mathfrak{D}_N^\varphi$ and $g \in \mathfrak{D}_M^\varphi$. By Proposition 130 and Lemma 128,

$$\begin{aligned} (\mathbf{V}^*[f], L_M(\varphi)g) &= (f, \mathbf{V}L_M(\varphi)g) = (f, L_N(\varphi)\mathbf{V}[g]) \\ &= (\mathbf{V}^*L_N(\bar{\varphi})f, g). \end{aligned}$$

By Proposition 132, $\mathbf{V}^*[f] \in \mathfrak{D}(\overline{L_M(\bar{\varphi})})$ and

$$(\overline{L_M(\bar{\varphi})}\mathbf{V}^*[f], g) = (\mathbf{V}^*[f], L_M(\varphi)g) = (\mathbf{V}^*L_N(\bar{\varphi})f, g), \quad g \in \mathfrak{D}_M^\varphi.$$

Hence (39). \square

27.1. Bargmann measure. Let $M^{(\text{Bargmann})} : \mathfrak{B} \rightarrow B_\infty(L_2(\mathbb{R}))$ be as in Section 18. This POVM is a mathematical representation of a simultaneous non-ideal measurement ([79]) of the position, momentum and number observables. An actual measurement (instrument) represented by this POVM is eight-port optical homodyning (detector). (See e.g. Sections 3.6 and 3.7 of Chapter VII in [15].) We will calculate the moments of this POVM. Because the support of $M^{(\text{Bargmann})}$ is not bounded, we need some of the integration theory just described.

27.1.1. *Naimark extension.* A minimal Naimark extension of the Bargmann POVM is $(\mathbb{1}, L_2(\mathbb{C}, \mu), \mathbf{V}_g)$.

27.1.2. *Moments.* Define the unbounded operator \mathcal{S} by

$$\mathcal{S} = \int_{\mathbb{C}} z M^{(\text{Bargmann})}(dz).$$

We have

$$\begin{aligned} \mathcal{S}\varphi_n &= \sum_{k=0}^{\infty} \left(\int_{\mathbb{C}} z \frac{\bar{z}^k z^n}{\sqrt{k!n!}} e^{-|z|^2} dz \right) \varphi_k \\ &= \sqrt{n+1} \varphi_{n+1}. \end{aligned}$$

Let $\mathcal{D} = \text{span}\{\varphi_n : n \in \mathbb{N}_0\}$. Let \mathcal{Z} be the operator of multiplication with the identity function, $\mathbb{C} \ni z \mapsto z$, on $L_2(\mathbb{C}, \mu)$: $\mathcal{Z} = \int_{\mathbb{C}} z \mathbb{1}(dz)$. We have $\mathcal{S} = \mathbf{V}_g^* \mathcal{Z} \mathbf{V}_g$. It follows from (25) that

$$\mathbf{V}_g[\mathcal{D}] = \text{span}\{\varphi : \varphi(z) = z^n, n \in \mathbb{N}_0\},$$

and hence that \mathcal{Z} maps $\mathbf{V}_g[\mathcal{D}]$ into itself. This, together with the fact that \mathcal{D} is a dense subset of $L_2(\mathbb{R})$, implies that the three conditions of Proposition 130 are satisfied for the functions in \mathcal{D} , and that

$$(40) \quad (\mathcal{S}^*)^k \mathcal{S}^\ell = \int_{\mathbb{C}} \bar{z}^k z^\ell M^{(\text{Bargmann})}(dz) \quad \forall k, \ell \in \mathbb{N}_0$$

on \mathcal{D} .

Wigner and Husimi representations, and in between

1. Introduction

E. Wigner initiated the investigations of a correspondence between the Hilbert space description of quantum mechanics, and a particular description by functions on phase-space: [110] [109], [82], [89], [32], [46], [27, 28], [56]. One part of this correspondence is to represent density operators ρ on $L_2(\mathbb{R})$ (which are non-negative trace-class operators with trace 1) by functions \mathbf{W}_ρ on phase-space (which is \mathbb{R}^2 or \mathbb{C}). This function \mathbf{W}_ρ is known as the Wigner function of ρ . The transformation $\rho \mapsto \mathbf{W}_\rho$ is linear. Phase-space pictures corresponding to other linear transformation from the space of density operators to functions on phase-space, are described e.g. in [1, 2], [21] and [40].

The Wigner function of a trace-class operator on $L_2(\mathbb{R})$ is an element of $M(\mathbb{R}^2)$, the Borel measurable functions on phase-space. There is no simple characterization of the subspace of $M(\mathbb{R}^2)$ that is formed by the Wigner functions of trace-class operators. It is also not known when the Weyl quantization of a function on phase-space is a bounded operator. Partial solutions of these problems are given in [46] and [29, 26]. In [89] it is proven that the ‘Wigner functions’ of Hilbert-Schmidt operators are the square integrable function classes on phase-space, and that this linear correspondence between B_2 (in this Chapter we write B_2 , B_1 and B_∞ in stead of $B_2(L_2(\mathbb{R}))$, etc.) and $L_2(\mathbb{R}^2)$ is, up to normalization, isometric. This linear transformation between B_2 and $L_2(\mathbb{R}^2)$ is the subject of Section 3. In Section 4 we introduce a family (\mathbf{W}_s) of phase space representations that interpolate between the Wigner and the Husimi [59] representations. In Section 5 we investigate the phase space pictures of B_1 and B_∞ operators. In Section 6 we introduce a particular family of subspaces of B_2 whose phase-space pictures we investigate in Section 7.

2. Conventions and notation

Define self-adjoint operators \mathcal{P} and \mathcal{Q} on their usual domains in $L_2(\mathbb{R})$ by $\mathcal{P}[f](x) = -if'(x)$ and $\mathcal{Q}[f](x) = xf(x)$. We have

$$[\mathcal{Q}, \mathcal{P}] = \mathcal{Q}\mathcal{P} - \mathcal{P}\mathcal{Q} = i\mathcal{I},$$

where \mathcal{I} is the identity operator. Let $\varphi_n \in L_2(\mathbb{R})$ be the n 'th Hermite basis function. We define φ_n as follows: $\varphi_0(q) = \pi^{-1/4}e^{-q^2/2}$ and $\varphi_n = (n!)^{-1/2}\mathcal{S}^n\varphi_0$, where $\mathcal{S} = (\mathcal{Q} - i\mathcal{P})/\sqrt{2}$. We have $\mathcal{S}\varphi_n = \sqrt{n+1}\varphi_{n+1}$, $\mathcal{S}^*\varphi_n = \sqrt{n}\varphi_{n-1}$ and $\mathcal{N}\varphi_n = n\varphi_n$ where $\mathcal{N} = \mathcal{S}\mathcal{S}^* = \frac{1}{2}(\mathcal{Q}^2 + \mathcal{P}^2 - \mathcal{I})$. The fractional Fourier transform \mathcal{F}_θ is defined by $\mathcal{F}_\theta = \exp(-i\theta\mathcal{N})$, and satisfies $\mathcal{F}_\theta\varphi_n = e^{-i\theta n}\varphi_n$ (see e.g. [9]). The ordinary Fourier transform $\mathcal{F} = \mathcal{F}_{\pi/2}$ satisfies

$$\mathcal{F}[f](p) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(q)e^{-ipq}dq.$$

We have $\mathcal{F}_\theta^*\mathcal{Q}\mathcal{F}_\theta = \cos(\theta)\mathcal{Q} + \sin(\theta)\mathcal{P}$. (This is a consequence of $[i\mathcal{N}, \mathcal{Q}] = \mathcal{P}$ and $[i\mathcal{N}, \mathcal{P}] = -\mathcal{Q}$, as is explained in the proof of lemma 134 below.) In particular, $\mathcal{P} =$

$\mathcal{F}^* \mathcal{Q} \mathcal{F} = -\mathcal{F} \mathcal{Q} \mathcal{F}^*$. We have $\mathcal{F} = e^{-i(\frac{\pi}{2})\mathcal{N}}$. The parity operator Π , defined on $L_2(\mathbb{R})$ by $\Pi[f](x) = f(-x)$, satisfies $\Pi = \mathcal{F}^2$, hence $\Pi\varphi_n = (-1)^n\varphi_n$.

The squeezing operator $\mathcal{Z}_\lambda: L_2(\mathbb{R}) \rightarrow L_2(\mathbb{R})$ is defined by

$$\mathcal{Z}_\lambda = \exp\left\{-i\frac{1}{2}\ln(\lambda)(\mathcal{Q}\mathcal{P} + \mathcal{P}\mathcal{Q})\right\}$$

and satisfies $(\mathcal{Z}_\lambda f)(x) = \lambda^{-1/2}f(\lambda^{-1}x)$.

For a bounded operator \mathcal{A} on $L_2(\mathbb{R})$ we will denote by $\mathcal{A}^{(k)}$ the operator on $L_2(\mathbb{R}^n)$ acting as \mathcal{A} on the k 'th variable only. For example $\mathcal{F}^{(1)}$ and $\mathcal{F}^{(2)}$ denote the Fourier transforms on $L_2(\mathbb{R}^2)$ in the first and second variable.

3. Wigner representation

Let

$$\mathbf{G}_\mathcal{Q} = \frac{1}{2}(\mathcal{Q}\mathcal{Q}\mathcal{I} - \mathcal{I}\mathcal{Q}\mathcal{Q}) \quad \text{and} \quad \mathbf{G}_\mathcal{P} = \frac{1}{2}(\mathcal{P}\mathcal{P}\mathcal{I} - \mathcal{I}\mathcal{P}\mathcal{P}).$$

This is a pair of commuting self-adjoint operators on \mathbf{B}_2 . Let

$$\tilde{\mathbf{G}}_\mathcal{Q} = \frac{1}{2}(\mathcal{Q}\mathcal{Q}\mathcal{I} + \mathcal{I}\mathcal{Q}\mathcal{Q}) \quad \text{and} \quad \tilde{\mathbf{G}}_\mathcal{P} = \frac{1}{2}(\mathcal{P}\mathcal{P}\mathcal{I} + \mathcal{I}\mathcal{P}\mathcal{P}).$$

This is also a pair of commuting self-adjoint operators on \mathbf{B}_2 .

LEMMA 134. Let $\mathbf{R}_\varphi = \exp\{i\varphi(\mathcal{Q}\mathcal{Q}\mathcal{P} + \mathcal{P}\mathcal{Q}\mathcal{Q})\}$. Then

$$(41) \quad \mathbf{K}[\mathbf{R}_\varphi[\mathcal{A}]](x, y) = \mathbf{K}[\mathcal{A}](x \cos \varphi + y \sin \varphi, y \cos \varphi - x \sin \varphi)$$

for $\mathcal{A} \in \mathbf{B}_2$ and almost all $x, y \in \mathbb{R}$. We have

$$(42) \quad \begin{aligned} \tilde{\mathbf{G}}_\mathcal{Q} &= \frac{1}{\sqrt{2}} \mathbf{R}_{\pi/4}(\mathcal{Q}\mathcal{Q}\mathcal{I})\mathbf{R}_{\pi/4}^{-1}, & \tilde{\mathbf{G}}_\mathcal{P} &= \frac{1}{\sqrt{2}} \mathbf{R}_{\pi/4}(\mathcal{I}\mathcal{P}\mathcal{P})\mathbf{R}_{\pi/4}^{-1}, \\ \mathbf{G}_\mathcal{Q} &= \frac{-1}{\sqrt{2}} \mathbf{R}_{\pi/4}(\mathcal{I}\mathcal{Q}\mathcal{Q})\mathbf{R}_{\pi/4}^{-1}, & \mathbf{G}_\mathcal{P} &= \frac{1}{\sqrt{2}} \mathbf{R}_{\pi/4}(\mathcal{P}\mathcal{P}\mathcal{I})\mathbf{R}_{\pi/4}^{-1} \end{aligned}$$

on $\text{span}\{\varphi_k \otimes \varphi_\ell : k, \ell \in \mathbb{N}_0\}$.

PROOF. (42) follows from (41) and

$$(43) \quad \begin{aligned} \mathbf{K}(\mathcal{Q}\mathcal{Q}\mathcal{I}) &= \mathcal{Q}^{(1)}\mathbf{K}, & \mathbf{K}(\mathcal{I}\mathcal{P}\mathcal{P}) &= -\mathcal{P}^{(2)}\mathbf{K} \\ \mathbf{K}(\mathcal{I}\mathcal{Q}\mathcal{Q}) &= \mathcal{Q}^{(2)}\mathbf{K}, & \mathbf{K}(\mathcal{P}\mathcal{P}\mathcal{I}) &= \mathcal{P}^{(1)}\mathbf{K}. \end{aligned}$$

Using the addition formulas for sine and cosine, it is easily seen that (41) is equivalent to

$$\mathbf{K}[\mathbf{R}_\varphi[\mathcal{A}]](r \cos \theta, r \sin \theta) = \mathbf{K}[\mathcal{A}](r \cos(\theta - \varphi), r \sin(\theta - \varphi)),$$

which follows, for $\mathcal{A} \in \text{span}\{\varphi_k \otimes \varphi_\ell : k, \ell \in \mathbb{N}_0\}$, from

$$\mathbf{K}[i(\mathcal{Q}\mathcal{Q}\mathcal{P} + \mathcal{P}\mathcal{Q}\mathcal{Q})\mathcal{A}](r \cos \theta, r \sin \theta) = -\frac{\partial}{\partial \theta} \mathbf{K}[\mathcal{A}](r \cos \theta, r \sin \theta).$$

□

PROPOSITION 135. There exists precisely one unitary operator $\mathbf{W}: \mathbf{B}_2 \rightarrow L_2(\mathbb{R}^2)$ such that

$$(44) \quad \mathbf{W}\tilde{\mathbf{G}}_\mathcal{Q} = \frac{1}{\sqrt{2}}\mathcal{Q}^{(1)}\mathbf{W} \quad \text{and} \quad \mathbf{W}\tilde{\mathbf{G}}_\mathcal{P} = \frac{1}{\sqrt{2}}\mathcal{Q}^{(2)}\mathbf{W}$$

and

$$(45) \quad \mathbf{W}\mathbf{G}_\mathcal{Q} = \frac{-1}{\sqrt{2}}\mathcal{P}^{(2)}\mathbf{W} \quad \text{and} \quad \mathbf{W}\mathbf{G}_\mathcal{P} = \frac{1}{\sqrt{2}}\mathcal{P}^{(1)}\mathbf{W}.$$

For an $f, g \in L_2(\mathbb{R})$ we have $\mathbf{W}[f \otimes g](q, p) = \sqrt{\pi}W_{f,g}(q/\sqrt{2}, p/\sqrt{2})$, where $W_{f,g}$ is the mixed Wigner function of f and g defined by

$$W_{f,g}(q, p) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f\left(q + \frac{y}{2}\right) \overline{g\left(q - \frac{y}{2}\right)} \frac{e^{-ipy}}{\sqrt{2\pi}} dy.$$

PROOF. From Lemma 134 and (43) it follows that $\mathbf{W} = (\mathcal{F}^{(2)})^* \mathbf{KR}_{\pi/4}^{-1}$ satisfies (44) and (45). Assume that unitary operator $\tilde{\mathbf{W}}$ from \mathbf{B}_2 to $L_2(\mathbb{R}^2)$ also satisfies (44) and (45). Then $\mathcal{Q}^{(1)}, \mathcal{Q}^{(2)}, \mathcal{P}^{(1)}$ and $\mathcal{P}^{(2)}$ commute with $\mathbf{W}\tilde{\mathbf{W}}^*$. Hence $\mathbf{W}\tilde{\mathbf{W}}^* = \mathcal{I}$. Hence $\mathbf{W} = \tilde{\mathbf{W}}$. \square

REMARK 136. (45) implies the following: If $\mathcal{A}\mathcal{Q} = \mathcal{Q}\mathcal{A}$ then $\frac{\partial}{\partial p}\mathbf{W}[\mathcal{A}](q, p) = 0$. If $\mathcal{A}\mathcal{P} = \mathcal{P}\mathcal{A}$ then $\frac{\partial}{\partial q}\mathbf{W}[\mathcal{A}](q, p) = 0$.

PROPOSITION 137. Let $\mathcal{A} \in \mathbf{B}_2$.

$$(a) \mathbf{W}[\exp\{2i(v\mathbf{G}_{\mathcal{Q}} - u\mathbf{G}_{\mathcal{P}})\}\mathcal{A}](q, p) = \mathbf{W}[\mathcal{A}](q - \sqrt{2}u, p - \sqrt{2}v).$$

$$(b) \mathbf{W}[\exp\{2i(v\tilde{\mathbf{G}}_{\mathcal{Q}} - u\tilde{\mathbf{G}}_{\mathcal{P}})\}\mathcal{A}](q, p) = e^{\sqrt{2}i(vq - up)}\mathbf{W}[\mathcal{A}](q, p).$$

$$(c) \mathbf{W}[\mathcal{F}_{\theta}\mathcal{A}\mathcal{F}_{\theta}^*](q, p) = \mathbf{W}[\mathcal{A}](q \cos \theta - p \sin \theta, q \sin \theta + p \cos \theta).$$

$$(d) \mathbf{W}[\mathcal{Z}_{\lambda}\mathcal{A}\mathcal{Z}_{\lambda}^*](q, p) = \mathbf{W}[\mathcal{A}](\lambda^{-1}q, \lambda p).$$

$$(e) \mathbf{W}[e^{-\tau\mathcal{N}}\mathcal{A}e^{-\tau\mathcal{N}}] = e^{-\tau\mathcal{N}^{(1)}}e^{-\tau\mathcal{N}^{(2)}}\mathbf{W}[\mathcal{A}].$$

$$(f) \mathbf{W}[\mathcal{F}_{\theta}\mathcal{A}\mathcal{F}_{\theta}] = \mathcal{F}_{\theta}^{(1)}\mathcal{F}_{\theta}^{(2)}\mathbf{W}[\mathcal{A}].$$

$$(g) \mathbf{W}[e^{-s(\mathbf{G}_{\mathcal{Q}}^2 + \mathbf{G}_{\mathcal{P}}^2)}\mathcal{A}] = e^{-\frac{s}{2}|\Delta|}\mathbf{W}[\mathcal{A}],$$

where $\Delta = \partial^2/\partial q^2 + \partial^2/\partial p^2$ and $|\Delta| = -\partial^2/\partial q^2 - \partial^2/\partial p^2$.

PROOF. The first and the last identities follow from (45). The second identity follows from (44). From

$$- \mathbf{G}_{\mathcal{Q}}^2 + \tilde{\mathbf{G}}_{\mathcal{Q}}^2 = \frac{1}{2}(\mathcal{Q}^2 \otimes \mathcal{I} + \mathcal{I} \otimes \mathcal{Q}^2) \text{ and } \mathbf{G}_{\mathcal{P}}^2 + \tilde{\mathbf{G}}_{\mathcal{P}}^2 = \frac{1}{2}(\mathcal{P}^2 \otimes \mathcal{I} + \mathcal{I} \otimes \mathcal{P}^2),$$

$$- \mathcal{N} \otimes \mathcal{I} - \mathcal{I} \otimes \mathcal{N} = 2(\tilde{\mathbf{G}}_{\mathcal{Q}}\mathbf{G}_{\mathcal{Q}} + \tilde{\mathbf{G}}_{\mathcal{P}}\mathbf{G}_{\mathcal{P}}),$$

$$- \mathcal{N} \otimes \mathcal{I} + \mathcal{I} \otimes \mathcal{N} = \mathbf{G}_{\mathcal{Q}}^2 + \tilde{\mathbf{G}}_{\mathcal{Q}}^2 + \mathbf{G}_{\mathcal{P}}^2 + \tilde{\mathbf{G}}_{\mathcal{P}}^2 - \mathbf{I},$$

$$- 2(\mathbf{G}_{\mathcal{Q}}\tilde{\mathbf{G}}_{\mathcal{P}} + \mathbf{G}_{\mathcal{P}}\tilde{\mathbf{G}}_{\mathcal{Q}}) = \frac{1}{2}(\mathcal{Q}\mathcal{P} + \mathcal{P}\mathcal{Q}) \otimes \mathcal{I} - \mathcal{I} \otimes \frac{1}{2}(\mathcal{Q}\mathcal{P} + \mathcal{P}\mathcal{Q})$$

follows

$$- \mathcal{F}_{\theta} \otimes \mathcal{F}_{\theta} = \exp\{-2i\theta(\tilde{\mathbf{G}}_{\mathcal{Q}}\mathbf{G}_{\mathcal{Q}} + \tilde{\mathbf{G}}_{\mathcal{P}}\mathbf{G}_{\mathcal{P}})\},$$

$$- \mathcal{Z}_{\lambda} \otimes \mathcal{Z}_{\lambda} = \exp\{-2i \ln(\lambda)(\mathbf{G}_{\mathcal{Q}}\tilde{\mathbf{G}}_{\mathcal{P}} + \mathbf{G}_{\mathcal{P}}\tilde{\mathbf{G}}_{\mathcal{Q}})\},$$

$$- \mathcal{N}_{\tau} \otimes \mathcal{N}_{\tau} = e^{\tau} \exp\{-\tau(\mathbf{G}_{\mathcal{Q}}^2 + \tilde{\mathbf{G}}_{\mathcal{Q}}^2 + \mathbf{G}_{\mathcal{P}}^2 + \tilde{\mathbf{G}}_{\mathcal{P}}^2)\},$$

$$- \mathcal{F}_{\theta} \otimes \mathcal{F}_{\theta}^* = e^{i\theta} \exp\{-i\theta(\mathbf{G}_{\mathcal{Q}}^2 + \tilde{\mathbf{G}}_{\mathcal{Q}}^2 + \mathbf{G}_{\mathcal{P}}^2 + \tilde{\mathbf{G}}_{\mathcal{P}}^2)\}.$$

Together with (44) and (45), this implies (d) and (e) and (f), and, together with some calculation, also (c). \square

LEMMA 138. Let $u, v \in \mathbb{R}$ and $\mathbf{G}_{u,v} = \exp\{2i(v\mathbf{G}_{\mathcal{Q}} - u\mathbf{G}_{\mathcal{P}})\}$. We have

$$(46) \quad \mathbf{G}_{u,v} = \exp\{i(v\mathcal{Q} - u\mathcal{P})\} \otimes \exp\{i(v\mathcal{Q} - u\mathcal{P})\} = e^{iv\mathcal{Q}}e^{-iu\mathcal{P}} \otimes e^{iv\mathcal{Q}}e^{-iu\mathcal{P}}$$

on \mathbf{B}_2 . Let $\tilde{\mathbf{G}}_{u,v} = \exp\{2i(v\tilde{\mathbf{G}}_{\mathcal{Q}} - u\tilde{\mathbf{G}}_{\mathcal{P}})\}$. We have

$$(47) \quad \tilde{\mathbf{G}}_{u,v} = \exp\{i(v\mathcal{Q} - u\mathcal{P})\} \otimes \exp\{-i(v\mathcal{Q} - u\mathcal{P})\} = e^{iv\mathcal{Q}}e^{-iu\mathcal{P}} \otimes e^{-iv\mathcal{Q}}e^{iu\mathcal{P}}$$

on \mathbf{B}_2 .

PROOF. The first equalities in (46) and (47) follow from Theorem 8.35 in [108]. The second equalities in (46) and (47) follow from Proposition 294. \square

REMARK 139. In the right-hand sides of (46) and (47), the order of composition of the two operators on both sides of the \otimes symbol may be interchanged on both sides simultaneously; on both sides this must be compensated with an extra factor, but the two factors neutralize each other.

4. A family of representations interpolating between the Wigner and Husimi representations

For $s > 0$ and $f \in L_2(\mathbb{R}^m)$ define the function $\mathcal{G}_s[f]$ on \mathbb{R}^m by

$$(48) \quad \mathcal{G}_s[f](x) = (2s\pi)^{-m/2} \int_{\mathbb{R}^m} e^{-(x-x')^2/(2s)} f(x') dx'.$$

This function is square-integrable. The operator \mathcal{G}_s on $L_2(\mathbb{R}^m)$ is called the m -dimensional Gaussian convolution operator with parameter s . It can be written as

$$\mathcal{G}_s = \exp\left\{-\frac{s}{2}|\Delta|\right\}, \quad \text{where} \quad |\Delta| = -\sum_{j=1}^m \partial^2/\partial x_j^2.$$

If $m = 1$ then $\mathcal{G}_s = \exp\{-\frac{s}{2}\mathcal{P}^2\}$.

DEFINITION 140. For $s \geq 0$ and $\mathcal{A} \in B_2$ let $\mathbf{W}_s[\mathcal{A}] = \mathcal{G}_s\mathbf{W}[\mathcal{A}]$, where \mathcal{G}_s is the 2-dimensional Gaussian convolution operator.

REMARK 141. The transforms \mathbf{W}_s have been investigated e.g. in [19], [11] and [40].

LEMMA 142. For $n \in \mathbb{N}_0$ and $z \in \mathbb{Z}$,

$$\mathcal{G}_1[\varphi_n](\sqrt{2}z) = 2^{-1/2}\pi^{-1/4} \frac{z^n}{\sqrt{n!}} e^{-z^2/2}.$$

Consequently,

$$(g_z, h) = \pi^{1/4}\sqrt{2}e^{-\text{Im}(z)^2+i\text{Re}(z)\text{Im}(z)}\mathcal{G}_1[h](\sqrt{2}z),$$

where

$$g_z = e^{-|z|^2/2} \sum_{n=0}^{\infty} \frac{\bar{z}^n}{\sqrt{n!}} \varphi_n,$$

PROOF. Let $\mathcal{S} = (\mathcal{Q} - i\mathcal{P})/\sqrt{2}$. Then $\mathcal{S}^n\varphi_0 = (n!)^{1/2}\varphi_n$. By Proposition 288 together with $[\mathcal{P}^2, \mathcal{Q}] = \mathcal{P}[\mathcal{P}, \mathcal{Q}] + [\mathcal{P}, \mathcal{Q}]\mathcal{P} = -2i\mathcal{P}$,

$$\begin{aligned} \mathcal{G}_1\mathcal{S} &= \mathcal{G}_1\mathcal{S}\mathcal{G}_1^{-1}\mathcal{G}_1 = 2^{-1/2} \exp\left\{-\frac{1}{2}\mathcal{P}^2\right\}(\mathcal{Q} - i\mathcal{P}) \exp\left\{\frac{1}{2}\mathcal{P}^2\right\}\mathcal{G}_1 \\ &= 2^{-1/2}(\mathcal{Q} + i\mathcal{P} - i\mathcal{P})\mathcal{G}_1 = 2^{-1/2}\mathcal{Q}\mathcal{G}_1 \end{aligned}$$

on $\text{span}\{\varphi_n : n \in \mathbb{N}_0\}$. Hence $\mathcal{G}_1\mathcal{S}^n = 2^{-n/2}\mathcal{Q}^n\mathcal{G}_1$. Hence

$$\mathcal{G}_1[\varphi_n] = (n!)^{-1/2}\mathcal{G}_1\mathcal{S}^n\varphi_0 = (2^n n!)^{-1/2}\mathcal{Q}^n\mathcal{G}_1[\varphi_0].$$

We have $\mathcal{G}_1[\varphi_0](x) = 2^{-1/2}\pi^{-1/4}e^{-x^2/4}$. □

LEMMA 143. Let $f, g \in L_2(\mathbb{R})$. Then

$$\begin{aligned} \mathbf{W}_1[f \otimes g](\sqrt{2}q, \sqrt{2}p) &= \mathcal{G}_1[f](q - ip)\overline{\mathcal{G}_1[g](q - ip)} e^{-p^2} \\ &= \frac{1}{2\sqrt{\pi}}(g, g_{(q-ip)/\sqrt{2}})(g_{(q-ip)/\sqrt{2}}, f). \end{aligned}$$

PROOF. For $s > 0$,

$$\begin{aligned} & \mathbf{W}_s[f \otimes g](q, p) \\ &= \frac{1}{2\pi\sqrt{s}} \int_{\mathbb{R}^2} \exp\left\{-\frac{1}{2s}\left(q - \frac{x+y}{\sqrt{2}}\right)^2 - \frac{s}{2}\left(\frac{x-y}{\sqrt{2}}\right)^2 - ip\left(\frac{x-y}{\sqrt{2}}\right)\right\} f(x)\overline{g(y)} \, dx dy. \end{aligned}$$

Hence

$$\begin{aligned} & \mathbf{W}_1[f \otimes g](q, p) \\ &= \frac{1}{2\pi} \int_{\mathbb{R}^2} \exp\left\{-\left(\frac{q-ip}{\sqrt{2}} - x\right)^2/2 - \left(\frac{q+ip}{\sqrt{2}} - y\right)^2/2 - p^2/2\right\} f(x)\overline{g(y)} \, dx dy. \end{aligned}$$

□

PROPOSITION 144. For $s \geq 0$,

$$(49) \quad \mathbf{W}_s \tilde{\mathbf{G}}_{\mathcal{Q}} = \frac{1}{\sqrt{2}}(\mathcal{Q}^{(1)} + is\mathcal{P}^{(1)})\mathbf{W}_s \quad \text{and} \quad \mathbf{W}_s \tilde{\mathbf{G}}_{\mathcal{P}} = \frac{1}{\sqrt{2}}(\mathcal{Q}^{(2)} + is\mathcal{P}^{(2)})\mathbf{W}_s$$

and

$$(50) \quad \mathbf{W}_s \mathbf{G}_{\mathcal{Q}} = \frac{-1}{\sqrt{2}}\mathcal{P}^{(2)}\mathbf{W}_s \quad \text{and} \quad \mathbf{W}_s \mathbf{G}_{\mathcal{P}} = \frac{1}{\sqrt{2}}\mathcal{P}^{(1)}\mathbf{W}_s.$$

PROOF. (50) follows from (45) and the fact that \mathcal{G}_s is a convolution operator, and hence commutes with \mathcal{P} . (49) follows from

$$\mathcal{G}_s \mathcal{Q} = (\mathcal{Q} + is\mathcal{P})\mathcal{G}_s.$$

(This follows easily from $\mathcal{G}_s = \exp\{-\frac{s}{2}\mathcal{P}^2\}$ and $[\mathcal{P}^2, \mathcal{Q}] = \mathcal{P}[\mathcal{P}, \mathcal{Q}] + [\mathcal{P}, \mathcal{Q}]\mathcal{P} = -2i\mathcal{P}$.)

□

LEMMA 145 ([19]). Let $\mathcal{A} \in \mathbf{B}_2$. Then

$$(51) \quad \mathbf{W}_s[\mathcal{A}](\sqrt{2}q, \sqrt{2}p) = \frac{1}{\sqrt{\pi}} \frac{1}{1+s} \sum_{n=0}^{\infty} \left(\frac{s-1}{s+1}\right)^n (\varphi_n^{(q,p)}, \mathcal{A}\varphi_n^{(q,p)}).$$

where $\varphi_n^{(q,p)} = \exp\{i(p\mathcal{Q} - q\mathcal{P})\}\varphi_n$.

PROOF. By Proposition (137) and (46), this can be reduced to the case $q = p = 0$. We have $\mathbf{W}_1[f \otimes g](0, 0) = (2\sqrt{\pi})^{-1}(g, \varphi_0 \otimes \varphi_0[f])$. For $s \geq 1$,

$$\mathbf{W}_s[f \otimes g](0, 0) = (2\sqrt{\pi})^{-1}(g, e^{(1-s)(\mathbf{G}_{\mathcal{Q}}^2 + \mathbf{G}_{\mathcal{P}}^2)}[\varphi_0 \otimes \varphi_0]f).$$

We have

$$\begin{aligned} \mathbf{G}_{\mathcal{Q}} + i\mathbf{G}_{\mathcal{P}} &= 2^{-1/2}(\mathcal{S}^* \otimes \mathcal{I} - \mathcal{I} \otimes \mathcal{S}), \\ \mathbf{G}_{\mathcal{Q}} - i\mathbf{G}_{\mathcal{P}} &= 2^{-1/2}(\mathcal{S} \otimes \mathcal{I} - \mathcal{I} \otimes \mathcal{S}^*), \end{aligned}$$

where \mathcal{S} is defined in terms of its action on the Hermite basis elements by $\mathcal{S}\varphi_n = \sqrt{n+1}\varphi_{n+1}$. From $\mathcal{S}^*\varphi_0 = 0$ and $\mathcal{S}^n\varphi_0 = (n!)^{1/2}\varphi_n$ and $\mathcal{S}^*\varphi_n = \sqrt{n}\varphi_{n-1}$ follows

$$\begin{aligned} \frac{1}{n!}(\mathbf{G}_{\mathcal{Q}}^2 + \mathbf{G}_{\mathcal{P}}^2)^n \varphi_0 \otimes \varphi_0 &= \frac{1}{n!}(\mathbf{G}_{\mathcal{Q}} - i\mathbf{G}_{\mathcal{P}})^n (\mathbf{G}_{\mathcal{Q}} + i\mathbf{G}_{\mathcal{P}})^n \varphi_0 \otimes \varphi_0 \\ &= (-1)^n 2^{-n/2} \frac{1}{\sqrt{n!}} (\mathbf{G}_{\mathcal{Q}} - i\mathbf{G}_{\mathcal{P}})^n \varphi_0 \otimes \varphi_n \\ &= 2^{-n} \frac{1}{\sqrt{n!}} \sum_{k=0}^n \binom{n}{k} (-1)^k \mathcal{S}^k [\varphi_0] \otimes \mathcal{S}^{(n-k)*} [\varphi_n] \\ &= 2^{-n} \sum_{k=0}^n \binom{n}{k} (-1)^k \varphi_k \otimes \varphi_k \\ &= \left(\frac{1 - \mathcal{S}_u \otimes \mathcal{S}_u}{2} \right)^n [\varphi_0 \otimes \varphi_0], \end{aligned}$$

where \mathcal{S}_u is defined on the Hermite basis functions by $\mathcal{S}_u\varphi_n = \varphi_{n+1}$ for all n . Hence

$$\begin{aligned} e^{(1-s)(\mathbf{G}_{\mathcal{Q}}^2 + \mathbf{G}_{\mathcal{P}}^2)} [\varphi_0 \otimes \varphi_0] &= \sum_{n=0}^{\infty} (1-s)^n \left(\frac{1 - \mathcal{S}_u \otimes \mathcal{S}_u}{2} \right)^n [\varphi_0 \otimes \varphi_0] \\ &= \frac{2}{1+s} \left(1 + \frac{1-s}{1+s} \mathcal{S}_u \otimes \mathcal{S}_u \right)^{-1} [\varphi_0 \otimes \varphi_0]. \end{aligned}$$

This implies

$$\mathbf{W}_s[\mathcal{A}](0,0) = \frac{1}{\sqrt{\pi}} \frac{1}{1+s} \sum_{n=0}^{\infty} \left(\frac{s-1}{s+1} \right)^n (\varphi_n, \mathcal{A}\varphi_n).$$

□

DEFINITION 146. For $s > 0$ and $\mathcal{B} \in \mathbf{B}_{\infty}$ define the function $\mathbf{W}_s[\mathcal{B}]$ on \mathbb{R}^2 by

$$\mathbf{W}_s[\mathcal{B}](\sqrt{2}q, \sqrt{2}p) = \frac{1}{\sqrt{\pi}} \frac{1}{1+s} \sum_{n=0}^{\infty} \left(\frac{s-1}{s+1} \right)^n (\varphi_n^{(q,p)}, \mathcal{B}\varphi_n^{(q,p)}).$$

5. The range of \mathbf{W}_s

THEOREM 147. Let $s > 0$. If $\mathcal{A} \in \mathbf{B}_1$ then $\mathbf{W}_s[\mathcal{A}] \in \mathbf{L}_1(\mathbb{R}^2)$ and

$$(52) \quad \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}^2} \mathbf{W}_s[\mathcal{A}](\sqrt{2}q, \sqrt{2}p) dqdp = \text{Tr}(\mathcal{A})$$

and

$$(53) \quad \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}^2} |\mathbf{W}_s[\mathcal{A}](\sqrt{2}q, \sqrt{2}p)| dqdp \leq \text{Tr}(\sqrt{\mathcal{A}^* \mathcal{A}}).$$

If $\mathcal{B} \in \mathbf{B}_{\infty}$ then $\mathbf{W}_s[\mathcal{B}]$ has an extension to an entire analytic function of two complex variables and

$$(54) \quad \frac{1}{\sqrt{\pi}} |\mathbf{W}_s[\mathcal{B}](q + iu, p + iv)| \leq \frac{e^{(u^2+v^2)/s}}{s\pi} \|\mathcal{B}\|_{\infty} \quad \forall q, p, u, v \in \mathbb{R}.$$

PROOF. Let $\mathcal{A} \in \mathbf{B}_1$ and $f, g \in \mathbf{L}_2(\mathbb{R})$. By Lemma 145,

$$\begin{aligned} |\mathbf{W}_s[f \otimes g](\sqrt{2}q, \sqrt{2}p)| &\leq \frac{1}{\sqrt{\pi}} \frac{1}{1+s} \sum_{n=0}^{\infty} \lambda^n |(\varphi_n^{(q,p)}, f)| |(g, \varphi_n^{(q,p)})| \\ &\leq \frac{1}{2\sqrt{\pi}} \frac{1}{1+s} \left(\sum_{n=0}^{\infty} \lambda^n |(\varphi_n^{(q,p)}, f)|^2 + \sum_{n=0}^{\infty} \lambda^n |(g, \varphi_n^{(q,p)})|^2 \right) \\ &= \frac{1}{2} \left(\mathbf{W}_{s'}[f \otimes f](\sqrt{2}q, \sqrt{2}p) + \mathbf{W}_{s'}[g \otimes g](\sqrt{2}q, \sqrt{2}p) \right) \end{aligned}$$

with $\lambda \in (0, 1)$ and $s' \in (1, \infty)$ such that

$$\lambda = \left| \frac{s-1}{s+1} \right| = \frac{s'-1}{s'+1}.$$

It is easily seen that precisely one such $s' \in (1, \infty)$ exists. Let $h \in \mathbf{L}_2(\mathbb{R})$. We have $\mathbf{W}_1[h \otimes h](q, p) \geq 0$ for all $(q, p) \in \mathbb{R}^2$. Hence $\mathbf{W}_{s'}[h \otimes h](q, p) \geq 0$ for all $(q, p) \in \mathbb{R}^2$. We have

$$\frac{1}{\sqrt{\pi}} \int_{\mathbb{R}^2} \mathbf{W}_{s'}[h \otimes h](q, p) dqdp = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}^2} \mathbf{W}_1[h \otimes h](q, p) dqdp = 2\|h\|^2.$$

Because \mathcal{A} is a trace-class operator, it can be written in the form:

$$\mathcal{A} = \sum_{n=1}^{\infty} a_n f_n \otimes g_n$$

with $(a_n) \in \ell_1(\mathbb{C})$ with $\sum_n |a_n| = \text{Tr}(\sqrt{\mathcal{A}^* \mathcal{A}})$, and with $(f_n), (g_n)$ orthonormal families in $\mathbf{L}_2(\mathbb{R})$. Hence

$$\begin{aligned} |\mathbf{W}_s[\mathcal{A}](q, p)| &\leq \sum_{n=1}^{\infty} |a_n| |\mathbf{W}_s[f_n \otimes g_n](q, p)| \\ &\leq \frac{1}{2} \sum_{n=1}^{\infty} |a_n| \left(\mathbf{W}_{s'}[f_n \otimes f_n](q, p) + \mathbf{W}_{s'}[g_n \otimes g_n](q, p) \right). \end{aligned}$$

By Fubini's theorem, this implies that $\mathbf{W}_s[\mathcal{A}] \in \mathbf{L}_1(\mathbb{R}^2)$ and that (53) is satisfied. The dominated convergence theorem can be used to prove (52).

Now we will prove (54). Let $\mathcal{B} \in \mathbf{B}_\infty$. We have $\mathbf{W}_s[\mathcal{B}] = \mathcal{G}_{s-\epsilon} \mathbf{W}_\epsilon[\mathcal{B}]$. By Lemma 253,

$$|\mathbf{W}_s[\mathcal{B}](q + iu, p + iv)| \leq e^{(u^2+v^2)/(2(s-\epsilon))} \|\mathbf{W}_\epsilon[\mathcal{B}]\|_\infty.$$

From Definition 146 follows $|\mathbf{W}_\epsilon[\mathcal{B}](q, p)| \leq \|\mathcal{B}\|_\infty (\sqrt{\pi} 2\epsilon)^{-1}$. Hence

$$|\mathbf{W}_s[\mathcal{B}](q + iu, p + iv)| \leq (\sqrt{\pi} 2\epsilon)^{-1} e^{(u^2+v^2)/(2(s-\epsilon))} \|\mathcal{B}\|_\infty$$

for every $\epsilon \in (0, s)$. □

6. The spaces \mathbf{B}_+ and \mathbf{B}_-

For $\tau > 0$ let \mathcal{N}_τ be the operator on $\mathbf{L}_2(\mathbb{R})$ which is characterized in terms of its action on the Hermite basis by

$$(55) \quad \mathcal{N}_\tau \varphi_n = e^{-(n+1/2)\tau} \varphi_n, \quad n \in \mathbb{N}_0.$$

Let $\mathbf{B}_+ = \cup \{ \mathcal{N}_\tau \mathcal{A} \mathcal{N}_\tau : \mathcal{A} \in \mathbf{B}_2, \tau > 0 \}$ and let \mathbf{B}_- be the space of linear forms on \mathbf{B}_+ that are bounded on each set $\cup \{ \mathcal{N}_\tau \mathcal{A} \mathcal{N}_\tau : \mathcal{A} \in \mathbf{B}_2, \|\mathcal{A}\|_2 \leq M \}$ with $\tau, M > 0$. We can identify \mathbf{B}_∞ as a space of linear forms on \mathbf{B}_1 : Identify $\mathcal{B} \in \mathbf{B}_\infty$ with linear form $L_{\mathcal{B}}$ on \mathbf{B}_1 , defined by $L_{\mathcal{B}}(\mathcal{A}) = \text{Tr}(\mathcal{B}^* \mathcal{A})$.

PROPOSITION 148. *We have*

$$\mathbf{B}_+ \subset \mathbf{B}_1 \subset \mathbf{B}_2 \subset \mathbf{B}_\infty \subset \mathbf{B}_-.$$

The space \mathbf{B}_+ is dense in \mathbf{B}_1 which is dense in \mathbf{B}_2 .

PROOF. Inclusion $\mathbf{B}_+ \subset \mathbf{B}_1$ follows from the fact that $\mathcal{N}_\tau \in \mathbf{B}_1$ for all $\tau > 0$. The space \mathbf{B}_+ is dense in \mathbf{B}_1 and in \mathbf{B}_2 because $\text{range}(\mathcal{N}_\tau)$ is dense in $\mathbf{L}_2(\mathbb{R})$ for every $\tau > 0$. Inclusion $\mathbf{B}_\infty \subset \mathbf{B}_-$: Let $\mathcal{B} \in \mathbf{B}_\infty$. Then $L_{\mathcal{B}}$ is a continuous linear form on \mathbf{B}_1 . Hence $L_{\mathcal{B}}$ is bounded on the subsets

$$\mathbf{B}_1(M) = \{\mathcal{A}\mathcal{B} : \mathcal{A}, \mathcal{B} \in \mathbf{B}_2, \|\mathcal{A}\|_2 \leq M, \|\mathcal{B}\|_2 \leq M\}$$

of \mathbf{B}_1 , where $M > 0$. Let $\tau > 0$. For every $M > 0$, the set

$$\mathbf{B}_+(M, \tau) = \{\mathcal{N}_\tau \mathcal{A} \mathcal{N}_\tau : \mathcal{A} \in \mathbf{B}_2, \|\mathcal{A}\| \leq M\}$$

is contained in a set $\mathbf{B}_1(M')$ for some M' . Hence $L_{\mathcal{B}} \in \mathbf{B}_-$. \square

We use the following notation: The application of a linear form $\mathcal{B} \in \mathbf{B}_-$ on $\mathcal{A} \in \mathbf{B}_+$ is denoted by $\langle \mathcal{B}, \mathcal{A} \rangle$. If $\mathcal{B} \in \mathbf{B}_\infty$ then $\langle \mathcal{B}, \mathcal{A} \rangle = \langle L_{\mathcal{B}}, \mathcal{A} \rangle = \text{Tr}(\mathcal{B}^* \mathcal{A})$.

LEMMA 149. *The space \mathbf{B}_+ can be characterized as follows:*

$$\mathbf{B}_+ = \left\{ \sum_{k,\ell=0}^{\infty} \alpha_{k,\ell} \varphi_k \otimes \varphi_\ell : \exists t > 0 \text{ such that } |\alpha_{k\ell}| = O(e^{-t(k+\ell)}) \right\}.$$

The space \mathbf{B}_- can be characterized as follows: A linear form L on \mathbf{B}_+ is an element of \mathbf{B}_- if, and only if,

$$|L(\varphi_k \otimes \varphi_\ell)| = O(e^{t(k+\ell)}) \quad \forall t > 0.$$

PROOF. This follows from (55). \square

If $s > 0$ then $\frac{s-1}{s+1} < 1$. Hence the following definition makes sense:

DEFINITION 150. For $s > 0$ and $\mathcal{B} \in \mathbf{B}_-$ define the function $\mathbf{W}_s[\mathcal{B}]$ on \mathbb{R}^2 by

$$\mathbf{W}_s[\mathcal{B}](\sqrt{2}q, \sqrt{2}p) = \frac{1}{\sqrt{\pi}} \frac{1}{1+s} \sum_{n=0}^{\infty} \left(\frac{s-1}{s+1} \right)^n \overline{\langle \mathcal{B}, \varphi_n^{(q,p)} \otimes \varphi_n^{(q,p)} \rangle}.$$

7. Characterization of $\mathbf{W}_s(\mathbf{B}_+)$ and $\mathbf{W}_s(\mathbf{B}_-)$

For $A, B \in \mathbb{R}$ and $M > 0$ let $\mathbf{A}(M, A, B)$ be the set of functions on \mathbb{R}^2 that have a continuation to an entire analytic function φ on \mathbb{C}^2 satisfying

$$|\varphi(q + ix, p + iy)|^2 \leq M \exp\{-A(q^2 + p^2) + B(x^2 + y^2)\} \quad \forall q, p, x, y \in \mathbb{R}$$

From Theorem 267 and Proposition 137 it follows that: For $s \geq 0$,

$$(56) \quad \mathbf{W}_s(\mathbf{B}_+) = \bigcup_{M>0} \left\{ \bigcup_{M>0} \mathbf{A}(M, A, B) : A > 0, B > 0, sB < 1 \right\},$$

and for $s > 0$,

$$\mathbf{W}_s(\mathbf{B}_-) = \bigcap_{M>0} \left\{ \bigcup_{M>0} \mathbf{A}(M, A, B) : A < 0, sB > 1 \right\}.$$

PROPOSITION 151. *The following operators on \mathbf{B}_2 leave \mathbf{B}_+ invariant:*

- (a) $\exp\{i(v\mathcal{Q} - u\mathcal{P})\} \otimes \mathcal{I}$ and $\mathcal{I} \otimes \exp\{i(v\mathcal{Q} - u\mathcal{P})\}$ with $u, v \in \mathbb{R}$;
- (b) $\mathcal{F}_\theta \otimes \mathcal{I}$ and $\mathcal{I} \otimes \mathcal{F}_\theta$ with $\theta \in [0, 2\pi)$.

The following operators on \mathbf{B}_2 map \mathbf{B}_+ into itself:

- (i) $\mathcal{Z}_\lambda \otimes \mathcal{I}$ and $\mathcal{I} \otimes \mathcal{Z}_\lambda$ with $\lambda > 0$;

(ii) $e^{-\tau\mathcal{N}} \otimes \mathcal{I}$ and $\mathcal{I} \otimes e^{-\tau\mathcal{N}}$ with $\operatorname{Re}(\tau) \geq 0$;

(iii) $e^{-s(\mathbf{G}_{\mathcal{Q}}^2 + \mathbf{G}_{\mathcal{P}}^2)}$ with $s \geq 0$.

The operators $\exp\{i(v\mathcal{Q} - u\mathcal{P})\} \otimes \mathcal{I}$ and $\mathcal{I} \otimes \exp\{i(v\mathcal{Q} - u\mathcal{P})\}$, with $u, v \in \mathbb{C} \setminus \mathbb{R}$, are unbounded on \mathbf{B}_2 , but map \mathbf{B}_+ into itself.

PROOF. (a): Let $u, v \in \mathbb{R}$. By Proposition 137 and (56), the operators $\mathbf{G}_{u,v}$ and $\tilde{\mathbf{G}}_{u,v}$ on \mathbf{B}_2 leave \mathbf{B}_+ invariant. By Lemma 138, this implies that the operators $\exp\{i(v\mathcal{Q} - u\mathcal{P})\} \otimes \mathcal{I}$ and $\mathcal{I} \otimes \exp\{i(v\mathcal{Q} - u\mathcal{P})\}$ leave \mathbf{B}_+ invariant.

(b) and (ii): This follows from Lemma 149.

(i): Let $\lambda > 0$. By Proposition 137 and (56), the operator $\mathcal{Z}_\lambda \otimes \mathcal{Z}_\lambda$ on \mathbf{B}_2 maps \mathbf{B}_+ into itself. Because $\mathcal{F}_{\pi/4} \mathcal{Z}_\lambda = \mathcal{Z}_{1/\lambda} \mathcal{F}_{\pi/4}$ it follows from (b) that the operator $\mathcal{Z}_\lambda \otimes \mathcal{Z}_{1/\lambda}$ on \mathbf{B}_2 maps \mathbf{B}_+ into itself. By taking a composition we see that $\mathcal{Z}_\lambda \otimes \mathcal{I}$ maps \mathbf{B}_+ into itself. The proof that $\mathcal{I} \otimes \mathcal{Z}_\lambda$ maps \mathbf{B}_+ into itself is similar.

(iii): By Proposition 137 and (56), $\mathbf{W}e^{-s(\mathbf{G}_{\mathcal{Q}}^2 + \mathbf{G}_{\mathcal{P}}^2)}[\mathbf{B}_+] = \mathcal{G}_s \mathbf{W}[\mathbf{B}_+] = \mathbf{W}_s[\mathbf{B}_+] \subset \mathbf{W}[\mathbf{B}_+]$. The proof of the final statement is similar to the proof of (a). \square

REMARK 152. All the operators of Proposition 151 can be defined on \mathbf{B}_- by contraposition: If \mathcal{A} is one of the above operators, and L is a linear form on \mathbf{B}_+ , then $\mathcal{A}[L]$ is defined as the composition $L \circ \mathcal{A}$. If L is in \mathbf{B}_- then $\mathcal{A}[L]$ is in \mathbf{B}_- .

Phase-space analogues for quantum mechanical expectation values

1. Introduction

It is shown e.g. in [18], [57], and [2] that quantum mechanical expectation values can be expressed as phase-space averages. At this point, insufficient care has been taken to provide a mathematically rigorous formulation.

For a density operator ρ , the quantum mechanical expectation value of a bounded operator \mathcal{B} is given by $\text{Tr}(\rho\mathcal{B})$. This chapter is concerned with the problem of expressing $\text{Tr}(\rho\mathcal{B})$ in terms of phase-space integrals involving a phase-space representation of ρ . The Hilbert space on which operators ρ and \mathcal{B} act is $L_2(\mathbb{R})$ in this section, and we write, as before, \mathbf{B}_2 , \mathbf{B}_1 and \mathbf{B}_∞ in stead of $\mathbf{B}_2(L_2(\mathbb{R}))$, etc. Only the phase-space representations of Section 4 of Chapter 4 are considered. We prove that:

- For every $\mathcal{B} \in \mathbf{B}_\infty$ and $s \in [0, 1]$, there exists a sequence of bounded, integrable and infinitely differentiable functions (b_n) on \mathbb{R}^2 such that

$$(57) \quad \text{Tr}(\rho\mathcal{B}) = \lim_{n \rightarrow \infty} \int_{\mathbb{R}^2} b_n(q, p) \mathbf{W}_s[\rho](q, p) dqdp$$

for every $\rho \in \mathbf{B}_1$. The convergence is uniform for ρ in compact subsets of \mathbf{B}_1 .

- For $s = 1$, there is a projection operator \mathcal{B} with the property that no sequence of functions (b_n) exists for which the above limit converges uniformly for $\rho \in \{\varphi_n \otimes \varphi_n : n \in \mathbb{N}\}$, where (φ_n) is the Hermite basis of $L_2(\mathbb{R})$.

After that we consider one particular (linear) way of obtaining a sequence of functions (b_n) satisfying (57).

REMARK 153. In [8], $\text{Tr}(\mathcal{B}\mathcal{A})$ is expressed as a sum over integrals containing the functions $\mathbf{W}_1[\mathcal{A}]$ and $\Delta^n \mathbf{W}_1[\mathcal{B}]$, with $n \in \mathbb{N}_0$.

REMARK 154. In [63], a construction is given for the (uniform) approximation of trace-class operators by integrals involving projections on the coherent state vectors.

REMARK 155. In Section 6 of [16], it is remarked that a POVM $M: \Sigma \rightarrow \mathbf{B}_+(\mathbf{H})$ on a σ -field Σ of subsets of a set Ω with the property that ρ_M has weak-star dense range, serves to define a kind of classical representation of quantum mechanics in the following sense: For any $\mathcal{B} \in \mathbf{B}_\infty(\mathbf{H})$, and $\epsilon > 0$, and a finite set ρ_1, \dots, ρ_n of density operators, there exists a complex bounded function f on Ω such that

$$\left| \int_{\Omega} f(x) \text{Tr}(M(dx)\rho_i) - \text{Tr}(\mathcal{B}\rho_i) \right| < \epsilon \quad \text{for all } i = 1, \dots, n.$$

This can be viewed as an approximate representation of the quantum mechanical expectation values $\text{Tr}(\rho\mathcal{A})$, for density operator ρ and bounded operators \mathcal{B} .

For the special case of the Bargmann measure, this is very similar to our objectives for $s = 1$. We consider, apart from the existence of such functions f , also a particular choice.

2. Wigner representation

THEOREM 156. *Let $\mathcal{A} \in \mathbf{B}_1$ and $\mathcal{B} \in \mathbf{B}_\infty$. Then*

$$\mathrm{Tr}(\mathcal{B}^* \mathcal{A}) = \lim_{n \rightarrow \infty} \int_{\mathbb{R}^2} \overline{b_n(q, p)} \mathbf{W}[\mathcal{A}](q, p) dq dp$$

where $b_n = \mathbf{W}[\mathcal{B}_{1/n}]$ with $\mathcal{B}_{1/n} = \mathcal{N}_{1/n} \mathcal{B} \mathcal{N}_{1/n}$.

PROOF. $\mathcal{N}_{1/n}$ is a Hilbert-Schmidt operator. Hence $\mathcal{B}_{1/n}$ is also a Hilbert-Schmidt operator. Because \mathbf{W} is unitary,

$$\int_{\mathbb{R}^2} \overline{b_n(q, p)} \mathbf{W}[\mathcal{A}](q, p) dq dp = \mathrm{Tr}(\mathcal{B}_{1/n}^* \mathcal{A}).$$

Let $h \in \mathbf{L}_2(\mathbb{R})$ then $\lim_{n \rightarrow \infty} \|\mathcal{N}_{1/n} h - h\| = 0$. This can be used together with

$$\|\mathcal{B}_{1/n} h - \mathcal{B} h\| = \|\mathcal{N}_{1/n} \mathcal{B} \mathcal{N}_{1/n} h - \mathcal{B} \mathcal{N}_{1/n} h\| + \|\mathcal{B} \mathcal{N}_{1/n} h - \mathcal{B} h\|$$

to prove that $\lim_{n \rightarrow \infty} \|\mathcal{B}_{1/n} h - \mathcal{B} h\| = 0$ for all $h \in \mathbf{L}_2(\mathbb{R})$. By the dominated convergence theorem, this implies that $\lim_{n \rightarrow \infty} \mathrm{Tr}(\mathcal{B}_{1/n}^* \mathcal{A}) = \mathrm{Tr}(\mathcal{B}^* \mathcal{A})$. \square

3. Existence of phase-space analogues for quantum mechanical expectation values

LEMMA 157. *Subspace*

$$\left\{ \int_{\mathbb{C}} \varphi(z) g_z \otimes g_z dz : \varphi \in \mathbf{L}_1(\mathbb{C}) \cap \mathbf{C}_b^\infty(\mathbb{C}) \right\}$$

of \mathbf{B}_∞ is weak-star sequentially dense in \mathbf{B}_∞ .

PROOF. This lemma improves the result of Example 20.1. We use the same method to prove it: We show that a non-degenerate $*$ -subalgebra is contained in the above subspace. Let $\mathcal{S} = \int_{\mathbb{C}} z M^{(\text{Bargmann})}(dz)$. In Section 27.1.2 of Chapter 3 we saw that $\mathcal{S} \varphi_n = \sqrt{n+1} \varphi_{n+1}$.

By Theorem 86, $\mathrm{span}\{\bar{z}^k z^\ell e^{-\tau|z|^2} : \tau > 0, k, \ell \in \mathbb{N}_0\}$ is weak-star sequentially dense in $\mathbf{L}_\infty(\mathbb{C})$. Let

$$\mathbf{A} = \mathrm{span}\left\{ \int_{\mathbb{C}} \bar{z}^k z^\ell g_z \otimes g_z e^{-\tau|z|^2} dz : \tau > 0, k, \ell \in \mathbb{N}_0 \right\}.$$

From $\mathcal{N}_\tau \varphi_n = e^{-\tau(n+1)} \varphi_n$ follows $\mathcal{N}_\tau \tilde{g}_z = e^{-\tau} \tilde{g}_{e^{-\tau} z}$, where $\tilde{g}_z = e^{|z|^2/2} g_z$. Together with (40), this implies that

$$\mathbf{A} = \mathrm{span}\{e^{-\tau \mathcal{N}} \mathcal{S}^{*k} \mathcal{S}^\ell e^{-\tau \mathcal{N}} : \tau > 0, k, \ell \in \mathbb{N}_0\}.$$

Using mathematical induction, $\mathcal{S} \mathcal{S}^* = \mathcal{S}^* \mathcal{S} - \mathcal{I}$ and $e^{-\tau \mathcal{N}} \mathcal{S} = e^{-\tau} \mathcal{S} e^{-\tau \mathcal{N}}$, it is easily seen that \mathbf{A} is a $*$ -subalgebra of \mathbf{B}_∞ . \mathbf{A} is non-degenerate, because it contains the injective operators $e^{-\tau \mathcal{N}}$, $\tau > 0$. By Proposition 83 and the results of Example 20.1, \mathbf{A} is weak-star sequentially dense in \mathbf{B}_∞ . \square

THEOREM 158. *For every $\mathcal{B} \in \mathbf{B}_\infty$ and $s \in [0, 1]$, there exists a sequence of bounded infinitely differentiable functions (b_n) on \mathbb{C} such that*

$$\mathrm{Tr}(\mathcal{B} \mathcal{T}) = \lim_{n \rightarrow \infty} \int_{\mathbb{C}} b_n(z) \mathbf{W}_s[\rho](z) dz$$

for every $\mathcal{T} \in \mathbf{B}_1$.

PROOF. Let $\mathcal{B} \in \mathbf{B}_\infty$. By Lemma 157, there exists a sequence of integrable functions (b_n) such that

$$\mathcal{B} = \lim_{n \rightarrow \infty} \frac{1}{\pi} \int_{\mathbb{C}} b_n(z) g_z \otimes g_z dz$$

in the weak-star sense. This means that

$$\mathrm{Tr}(\mathcal{B}\mathcal{T}) = \lim_{n \rightarrow \infty} \frac{1}{\pi} \int_{\mathbb{C}} b_n(z) (g_z, \mathcal{T}g_z) dz$$

for every $\mathcal{T} \in \mathbf{B}_1$. We have

$$\left| \int_{\mathbb{C}} b_n(z) (g_z, \mathcal{T}g_z) dz \right| \leq \|b_n\|_1 \|\mathcal{T}\|_\infty \leq \|b_n\|_1 \|\mathcal{T}\|_1.$$

Consequently, there are operators $\mathcal{B}_n \in \mathbf{B}_\infty$ such that

$$(\forall \mathcal{T} \in \mathbf{B}_1) \quad \mathrm{Tr}(\mathcal{B}_n \mathcal{T}) = \frac{1}{\pi} \int_{\mathbb{C}} b_n(z) (g_z, \mathcal{T}g_z) dz.$$

Let $s \in [0, 1]$. Because $\mathbf{W}[\mathcal{T}] \in \mathbf{L}_\infty(\mathbb{C})$ and $b_n \in \mathbf{L}_1(\mathbb{C})$, Fubini's theorem can be used to prove that

$$\int_{\mathbb{C}} b_n(z) \mathbf{W}_1[\mathcal{T}](z) dz = \int_{\mathbb{C}} \mathcal{G}_{1-s}[b_n](z) \mathbf{W}_s[\mathcal{T}](z) dz.$$

Hence everything can be reduced to the case $s = 1$. \square

4. Non-uniformity of approximation of expectation values

In this section we prove the existence of a projection operator \mathcal{P} whose expectation values $\mathrm{Tr}(\rho\mathcal{P})$ cannot be approximated uniformly in ρ by phase-space integrals.

LEMMA 159. *Let $\frac{1}{2}\pi < |\alpha| < \frac{3}{4}\pi$. There is no sequence of functions $(f_n^{(\alpha)})$ on \mathbb{C} such that*

$$(\varphi_k, \mathcal{F}_\alpha \varphi_k) = \lim_{n \rightarrow \infty} \int_{\mathbb{C}} f_n^{(\alpha)}(z) \mathbf{W}_1[\varphi_k \otimes \varphi_k](z) dz.$$

converges uniformly for $k \in \mathbb{N}$.

PROOF. We have $(\varphi_k, \mathcal{F}_\alpha[\varphi_k]) = e^{-ik\alpha}$ and $|(g_z, \varphi_k)|^2 = \frac{|z|^{2k}}{k!} e^{-|z|^2}$. It is assumed that

$$\int_{\mathbb{C}} |f_n^{(\alpha)}(z)| e^{-|z|^2} dz < \infty.$$

It suffices to proof that there is no sequence of functions $(g_n^{(\alpha)})$ on $(0, \infty)$ such that

$$e^{-ik\alpha} = \lim_{n \rightarrow \infty} \int_0^\infty g_n^{(\alpha)}(x) \frac{x^k}{k!} e^{-x} dx.$$

converges uniformly for $k \in \mathbb{N}$. Assume that there is such a sequence $(g_n^{(\alpha)})$. Then there exists an $N \in \mathbb{N}$ such that

$$\mathrm{Re} \int_0^\infty g_n^{(\alpha)}(x) \frac{(e^{i\alpha}x)^k}{k!} e^{-x} dx \geq \frac{1}{2}$$

for $n \geq N$ and all $k \in \mathbb{N}_0$. Then

$$\mathrm{Re} \int_0^\infty g_n^{(\alpha)}(x) \frac{(\lambda e^{i\alpha}x)^k}{k!} e^{-x} dx \geq \frac{\lambda^k}{2}$$

for $n \geq N$ and $\lambda \in (0, 1)$ and $k \in \mathbb{N}_0$. It is assumed that

$$(58) \quad \int_0^\infty |g_n^{(\alpha)}(x)| e^{-x} dx < \infty$$

Hence

$$\operatorname{Re} \int_0^\infty g_n^{(\alpha)}(x) e^{-x} \exp\{\lambda e^{i\alpha} x\} dx \geq \frac{1}{2} \sum_{k=0}^\infty \lambda^k.$$

for $n \geq N$ and $\lambda \in (0, 1)$. Hence

$$\lim_{\lambda \uparrow 1} \operatorname{Re} \int_0^\infty g_n^{(\alpha)}(x) e^{-x} \exp\{\lambda e^{i\alpha} x\} dx = +\infty$$

for $n \geq N$. Because (58) and $\operatorname{Re}(\lambda e^{i\alpha} x) = \lambda \cos(\alpha)x < 0$, this is impossible (by the dominated convergence theorem). \square

For a bounded subset S of \mathbf{B}_1 let \mathbf{B}_S be the set of all bounded operators \mathcal{B} with the property that there exists a sequence of functions (b_n) on \mathbb{C} such that

$$\operatorname{Tr}(\mathcal{T}\mathcal{B}) = \lim_{n \rightarrow \infty} \int_{\mathbb{C}} b_n(z) \mathbf{W}_1[\mathcal{T}](z) dz.$$

converges uniformly for $\mathcal{T} \in S$.

LEMMA 160. *For every bounded subset S of \mathbf{B}_1 , \mathbf{B}_S is a closed linear subspace of \mathbf{B}_∞ .*

PROOF. It is easily seen that \mathbf{B}_S is a linear subspace. Let $\mathcal{N} \in \operatorname{cl}(\mathbf{B}_S)$. There is a sequence (\mathcal{N}_j) of operators in \mathbf{B}_S such that $\|\mathcal{N}_j - \mathcal{N}\|_\infty$ converges to zero as $j \rightarrow \infty$. For every j there is a sequence $(f_n^{(j)})$ such that

$$\operatorname{Tr}(\mathcal{T}\mathcal{N}_j) = \lim_{n \rightarrow \infty} \int_{\mathbb{C}} f_n^{(j)}(z) \mathbf{W}_1[\mathcal{T}](z) dz.$$

converges uniformly for $\mathcal{T} \in S$. There is a subsequence (k_n) of \mathbb{N} such that

$$\operatorname{Tr}(\mathcal{T}\mathcal{N}) = \lim_{n \rightarrow \infty} \int_{\mathbb{C}} f_n(z) \mathbf{W}_1[\mathcal{T}](z) dz$$

where $f_n = f_{k_n}^{(n)}$. Hence $\mathcal{N} \in \mathbf{B}_S$. \square

THEOREM 161. *There exists an orthogonal projection operator \mathcal{P} for which there is no sequence of functions (p_n) such that*

$$(\varphi_k, \mathcal{P}\varphi_k) = \lim_{n \rightarrow \infty} \int_{\mathbb{C}} p_n(z) \mathbf{W}_1[\varphi_k \otimes \varphi_k](z) dz.$$

converges uniformly for $k \in \mathbb{N}$.

PROOF. Let $S = \{\varphi_k \otimes \varphi_k : k \in \mathbb{N}\}$. Assume that the statement in the theorem is not true. Then every projection operator is an element of \mathbf{B}_S . But the norm closed linear span of the projection operators is all of \mathbf{B}_∞ . Hence $\mathbf{B}_S = \mathbf{B}_\infty$, by Lemma 160. This is impossible by Lemma 159. \square

5. A method to approximate expectation values

LEMMA 162. *Let $\mathcal{A} \in \mathbf{B}_1$. There are $\mathcal{A}_{x,y} \in \mathbf{B}_1$ such that*

$$\mathbf{W}[\mathcal{A}_{x,y}](q, p) = \mathbf{W}[\mathcal{A}](q + x, y + p) \quad \forall q, p \in \mathbb{R}.$$

Let $\mathcal{B} \in \mathbf{B}_\infty$. The function $(x, y) \mapsto \operatorname{Tr}(\mathcal{B}\mathcal{A}_{x,y})$ on \mathbb{R}^2 is continuous and bounded.

PROOF. By Proposition 137,

$$\mathcal{A}_{x,y} = \exp\left\{\frac{-i}{\sqrt{2}}(y\mathcal{Q} - x\mathcal{P})\right\}\mathcal{A}\exp\left\{\frac{i}{\sqrt{2}}(y\mathcal{Q} - x\mathcal{P})\right\}.$$

Hence $\|\mathcal{A}_{x,y}\|_1 = \|\mathcal{A}\|_1$ for all $\mathcal{A} \in \mathcal{B}_1$. The operators with finite dimensional range form a dense subspace of \mathcal{B}_1 . Hence there is a sequence $(\mathcal{A}^{(n)})$, of operators with finite dimensional range, that converges to \mathcal{A} in \mathcal{B}_1 . For every n , the function $(x, y) \mapsto \text{Tr}(\mathcal{B}\mathcal{A}_{x,y}^{(n)})$ on \mathbb{R}^2 is bounded and continuous. From

$$|\text{Tr}(\mathcal{B}\mathcal{A}_{x,y}) - \text{Tr}(\mathcal{B}\mathcal{A}_{x,y}^{(n)})| \leq \|\mathcal{B}\|_\infty \|\mathcal{A} - \mathcal{A}^{(n)}\|_1$$

it follows that also $(x, y) \mapsto \text{Tr}(\mathcal{B}\mathcal{A}_{x,y})$ is bounded and continuous. \square

LEMMA 163. Let $\mathcal{A} \in \mathcal{B}_1$ and $\mathcal{B} \in \mathcal{B}_\infty$. Let \mathfrak{c} be a function on \mathbb{R}^2 with compact support such that $\mathfrak{c}(0) = 1$ and let

$$\mathfrak{d}(x, y) = \int_{\mathbb{R}^2} \mathfrak{c}(q, p) \frac{e^{i(xq+yp)}}{(2\pi)^2} dqdp.$$

Let $s > 0$. Then

$$\int_{\mathbb{R}^2} \overline{\mathbf{W}_{-s}^{(c)}[\mathcal{B}]}(q, p) \mathbf{W}_s[\mathcal{A}](q, p) dqdp = \int_{\mathbb{R}^2} \frac{1}{s^2} \mathfrak{d}\left(\frac{x}{s}, \frac{y}{s}\right) \text{Tr}(\mathcal{B}^* \mathcal{A}_{x,y}) e^{-(x^2+y^2)/(4s)} dx dy,$$

where

$$\mathbf{W}_{-s}^{(c)}[\mathcal{B}](q, p) = \int_{\mathbb{R}^2} \mathfrak{c}(u, v) \mathbf{W}_s[\mathcal{B}](q + 2iu, p + 2iv) \frac{\exp\{-(u^2 + v^2)/s\}}{s\pi} dudv.$$

PROOF. Let $\tau > 0$. We will prove first that

$$(59) \quad \int_{\mathbb{R}^2} \overline{\mathbf{W}_{-s}^{(c)}[\mathcal{B}_\tau]}(q, p) \mathbf{W}_s[\mathcal{A}](q, p) dqdp = \int_{\mathbb{R}^2} \frac{1}{s^2} \mathfrak{d}\left(\frac{x}{s}, \frac{y}{s}\right) \text{Tr}(\mathcal{B}_\tau^* \mathcal{A}_{x,y}) e^{-(x^2+y^2)/(4s)} dx dy,$$

where $\mathcal{B}_\tau = \mathcal{N}_\tau \mathcal{B} \mathcal{N}_\tau$. From Proposition 137 it follows that $\mathbf{W}[\mathcal{A}]$ is a bounded function on \mathbb{R}^2 . Because $\mathbf{W}[\mathcal{B}_\tau] \in \mathcal{L}_1(\mathbb{R}^2)$ it follows from Lemma 255 that

$$\begin{aligned} & \int_{\mathbb{R}^2} \overline{\mathbf{W}_{-s}^{(c)}[\mathcal{B}_\tau]}(q, p) \mathbf{W}_s[\mathcal{A}](q, p) dqdp \\ &= \int_{\mathbb{R}^2} \frac{1}{s^2} \mathfrak{d}\left(\frac{x}{s}, \frac{y}{s}\right) \mathbf{W}[\mathcal{B}_\tau]^* * \mathbf{W}[\mathcal{A}](x, y) e^{-(x^2+y^2)/(4s)} dx dy, \end{aligned}$$

where $\mathbf{W}[\mathcal{B}_\tau]^*(q, p) = \overline{\mathbf{W}[\mathcal{B}_\tau](-q, -p)}$ and $\cdot * \cdot$ denotes convolution. Because \mathcal{B}_τ and $\mathcal{A}_{x,y}$ are Hilbert-Schmidt operators, it follows from the unitarity of $\mathbf{W}: \mathcal{B}_2 \rightarrow \mathcal{L}_2(\mathbb{R}^2)$ that

$$\begin{aligned} \mathbf{W}[\mathcal{B}_\tau]^* * \mathbf{W}[\mathcal{A}](x, y) &= \int_{\mathbb{R}^2} \overline{\mathbf{W}[\mathcal{B}_\tau](q, p)} \mathbf{W}[\mathcal{A}_{x,y}](q, p) dqdp \\ &= \text{Tr}(\mathcal{B}_\tau^* \mathcal{A}_{x,y}). \end{aligned}$$

This implies (59).

Let $0 < \epsilon < \min(s, 1)$. We have $\mathbf{W}_s[\mathcal{B}_\tau] = \mathcal{G}_{s-\epsilon} \mathbf{W}_\epsilon[\mathcal{B}_\tau]$. From Definition 146 follows

$$|\mathbf{W}_\epsilon[\mathcal{B}_\tau](q, p)| \leq \|\mathcal{B}_\tau\|_\infty (2\epsilon)^{-1} \leq \|\mathcal{B}\|_\infty \epsilon^{-1}$$

and $\lim_{\tau \downarrow 0} \mathbf{W}_\epsilon[\mathcal{B}_\tau](q, p) = \mathbf{W}_\epsilon[\mathcal{B}](q, p)$. By the dominated convergence theorem,

$$\lim_{\tau \downarrow 0} \mathbf{W}_s[\mathcal{B}_\tau](q + 2iu, p + 2iv) = \mathbf{W}_s[\mathcal{B}](q + 2iu, p + 2iv).$$

By Lemma 253,

$$|\mathbf{W}_s[\mathcal{B}_\tau](q + 2iu, p + 2iv)| \leq \|\mathcal{B}\|_\infty \epsilon^{-1} e^{2(u^2+v^2)/s},$$

and

$$|\mathbf{W}_{-s}^{(c)}[\mathcal{B}_\tau](q, p)| \leq \|\mathcal{B}\|_\infty \frac{1}{\epsilon} \int_{\mathbb{R}^2} \mathbf{c}(u, v) \frac{\exp\{(u^2 + v^2)/s\}}{s\pi} dudv$$

for all $\tau > 0$. Hence

$$\lim_{\tau \downarrow 0} \mathbf{W}_{-s}^{(c)}[\mathcal{B}_\tau](q, p) = \mathbf{W}_{-s}^{(c)}[\mathcal{B}](q, p).$$

By the dominated convergence theorem and Theorem 147,

$$\lim_{\tau \downarrow 0} \int_{\mathbb{R}^2} \overline{\mathbf{W}_{-s}^{(c)}[\mathcal{B}_\tau](q, p)} \mathbf{W}_s[\mathcal{A}](q, p) dqdp = \int_{\mathbb{R}^2} \overline{\mathbf{W}_{-s}^{(c)}[\mathcal{B}](q, p)} \mathbf{W}_s[\mathcal{A}](q, p) dqdp.$$

It is easily seen that

$$|\mathrm{Tr}(\mathcal{B}_\tau^* \mathcal{A}_{x,y})| \leq \|\mathcal{B}_\tau\|_\infty \|\mathcal{A}\|_1 \leq \|\mathcal{B}\|_\infty \|\mathcal{A}\|_1.$$

and

$$\lim_{\tau \downarrow 0} \mathrm{Tr}(\mathcal{B}_\tau^* \mathcal{A}_{x,y}) = \mathrm{Tr}(\mathcal{B}^* \mathcal{A}_{x,y}).$$

The result follows by another application of the dominated convergence theorem. \square

THEOREM 164. *Let $\mathcal{A} \in \mathbf{B}_1$ and $\mathcal{B} \in \mathbf{B}_\infty$. Let $s > 0$. Let \mathbf{c} be an infinitely differentiable function on \mathbb{R}^2 with compact support such that $\mathbf{c}(0) = 1$. Then*

$$\mathrm{Tr}(\mathcal{B}^* \mathcal{A}) = \lim_{n \rightarrow \infty} \int_{\mathbb{R}^2} \overline{b_n(q, p)} \mathbf{W}_s[\mathcal{A}](q, p) dqdp$$

where

$$(60) \quad b_n(q, p) = \int_{\mathbb{R}^2} \mathbf{c}\left(\frac{u}{n}, \frac{v}{n}\right) \mathbf{W}_s[\mathcal{B}](q + 2iu, p + 2iv) \frac{\exp\{-(u^2 + v^2)/s\}}{s\pi} dudv.$$

PROOF. By Lemma 163,

$$\int_{\mathbb{R}^2} \overline{b_n(q, p)} \mathbf{W}_s[\mathcal{A}](q, p) dqdp = \int_{\mathbb{R}^2} \frac{1}{s^2} \mathfrak{d}\left(\frac{x}{s}, \frac{y}{s}\right) \mathrm{Tr}(\mathcal{B}^* \mathcal{A}_{\frac{x}{n}, \frac{y}{n}}) e^{-(x^2 + y^2)/(4sn^2)} dx dy,$$

Let $\mathbf{C}_{00}^\infty(\mathbb{R}^2)$ be the space of infinitely differentiable functions on \mathbb{R}^2 with compact support and let $\mathcal{S}(\mathbb{R}^m)$ be the Schwartz space of infinitely differentiable functions on \mathbb{R}^2 with derivatives that decay rapidly at infinity. It is well-known that

$$\mathbf{C}_{00}^\infty(\mathbb{R}^2) \subset \mathcal{S}(\mathbb{R}^2) \subset \mathbf{L}_1(\mathbb{R}^2)$$

and that $\mathcal{S}(\mathbb{R}^2)$ is invariant under Fourier transformation. Hence $\mathbf{c} \in \mathbf{C}_{00}^\infty(\mathbb{R}^2)$ implies $\mathfrak{d} \in \mathcal{S}(\mathbb{R}^2)$. Hence \mathfrak{d} is integrable. By the dominated convergence theorem and the fact that $(x, y) \mapsto \mathrm{Tr}(\mathcal{B}^* \mathcal{A}_{x,y}) \in \mathbf{C}_b(\mathbb{R}^2)$, we have

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^2} \overline{b_n(q, p)} \mathbf{W}_s[\mathcal{A}](q, p) dqdp = \int_{\mathbb{R}^2} \frac{1}{s^2} \mathfrak{d}\left(\frac{x}{s}, \frac{y}{s}\right) \mathrm{Tr}(\mathcal{B}^* \mathcal{A}) dx dy.$$

\square

There is an interesting expression for the analytic continuation of \mathbf{W}_1 :

REMARK 165. For $\mathcal{B} \in \mathbf{B}_\infty$ and $q, p, u, v \in \mathbb{R}$,

$$\mathbf{W}_1[\mathcal{B}](q + 2iu, p + 2iv) = \frac{1}{2\sqrt{\pi}} \frac{(g_{\bar{z}/2+i\bar{w}}, \mathcal{B}g_{\bar{z}/2-i\bar{w}})}{(g_{\bar{z}/2+i\bar{w}}, g_{\bar{z}/2-i\bar{w}})},$$

where $z = q + ip$ and $w = u + iv$.

THEOREM 166. *Let $\mathcal{A} \in \mathbf{B}_+$ and $\mathcal{B} \in \mathbf{B}_-$. Let $s > 0$. Let \mathbf{c} be a bounded function on \mathbb{R}^m which has a compact support, is continuous in 0, and satisfies $\mathbf{c}(0) = 1$. Then*

$$\mathrm{Tr}(\mathcal{B}^* \mathcal{A}) = \lim_{n \rightarrow \infty} \int_{\mathbb{R}^2} \overline{b_n(q, p)} \mathbf{W}_s[\mathcal{A}](q, p) dq dp$$

where (b_n) is given by (60). The convergence is uniform for \mathcal{A} in subsets

$$\{\mathcal{N}_\tau \mathcal{O} \mathcal{N}_\tau : \mathcal{O} \in \mathbf{B}_2, \|\mathcal{O}\| \leq M\},$$

with $M, \tau > 0$.

PROOF. This follows from the unitarity of $\mathbf{W}: \mathbf{B}_2 \rightarrow \mathbf{L}_2(\mathbb{R}^2)$, Proposition 137, and Theorem 273. \square

CHAPTER 6

Extended Bargmann space

1. Introduction

The Bargmann space is the closed linear subspace of entire analytic functions in $L_2(\mathbb{C}, \mu)$, where

$$\mu(dz) = \pi^{-1} \exp\{-|z|^2\} d \operatorname{Re}(z) d \operatorname{Im}(z).$$

We extend the Bargmann space to a functional Hilbert space W_2 which is a Sobolev space related to the Laplacian operator on $L_2(\mathbb{C}, \mu)$. The inner product of W_2 coincides with that of the Bargmann space. We extend the usual orthonormal basis of the Bargmann space to an orthonormal basis $(\Psi_{\ell,k})$ of $L_2(\mathbb{C}, \mu)$ which is orthogonal in the triple of Hilbert spaces $W_2 \subset L_2(\mathbb{C}, \mu) \subset W_{-2}$, where W_{-2} is the strong topological dual of W_2 .

REMARK 167. The functions $\Psi_{\ell,k}$ are used to investigate Weyl-(de)quantization in [27, 28] and [56].

2. Hermite polynomials

Let $\gamma > 0$ and let

$$\nu(dx) = (\pi\gamma)^{-1/2} e^{-x^2/\gamma} dx.$$

LEMMA 168. *The polynomials form a dense subspace in $L_2(\mathbb{R}, \nu)$.*

PROOF. This can be reduced to the case $\gamma = 1$. This case is treated e.g. in the proof of Theorem V.1.3.6 in [76]. \square

The adjoint of the operator $\frac{d}{dx}$, densely defined on the polynomials, can be calculated by integration by parts:

$$\left(\frac{d}{dx}\right)^* = \frac{2x}{\gamma} - \frac{d}{dx} \quad \text{on the polynomials.}$$

The operators $\frac{d}{dx}$ and $\left(\frac{d}{dx}\right)^*$ map the space of polynomials into itself. Consequently, finite compositions of these operators are again well defined on the polynomials. The (non-normal) operator $\frac{d}{dx}$ satisfies $[\frac{d}{dx}, \left(\frac{d}{dx}\right)^*] = \frac{2}{\gamma} [\frac{d}{dx}, x] = \frac{2}{\gamma}$ on the polynomials. Using $[\mathcal{A}^2, \mathcal{B}] = \mathcal{A}[\mathcal{A}, \mathcal{B}] + [\mathcal{A}, \mathcal{B}]\mathcal{A}$, we see that $[\frac{d^2}{dx^2}, \left(\frac{d}{dx}\right)^*] = \frac{4}{\gamma} \frac{d}{dx}$ on the polynomials. For $z \in \mathbb{C}$, $\exp\{z \frac{d^2}{dx^2}\}$ is defined on the space of polynomials p by

$$\exp\{z \frac{d^2}{dx^2}\} p = \sum_{n=0}^{\infty} \frac{z^n}{n!} \left(\frac{d}{dx}\right)^{2n} p,$$

in which only a finite number of terms are non-zero. By Proposition 288,

$$(61) \quad \exp\left\{\frac{\gamma}{4} \frac{d^2}{dx^2}\right\} \left(\frac{d}{dx}\right)^* \exp\left\{-\frac{\gamma}{4} \frac{d^2}{dx^2}\right\} = \frac{2x}{\gamma}.$$

DEFINITION 169. For $n \in \mathbb{N}_0$ let $H_n^{(\gamma)} = \left(\frac{d}{dx}\right)^{*n} 1$. Let $H_n = H_n^{(1)}$.

PROPOSITION 170. *We have the following expression for $H_n^{(\gamma)}$:*

$$H_n^{(\gamma)}(x) = \exp\left\{-\frac{\gamma}{4} \frac{d^2}{dx^2}\right\} \left(\frac{2x}{\gamma}\right)^n.$$

Hence

$$\frac{1}{\sqrt{\pi\gamma}} \int_{\mathbb{R}} e^{-(x-y)^2/\gamma} H_n^{(\gamma)}(y) dy = \left(\frac{2x}{\gamma}\right)^n.$$

Hence

$$H_n^{(\gamma)}(x) = \gamma^{-n/2} H_n\left(\frac{x}{\sqrt{\gamma}}\right)$$

PROOF. This follows from (61) and $\exp\left\{-\frac{\gamma}{4} \frac{d^2}{dx^2}\right\} 1 = 1$. \square

Operator $\frac{d}{dx}$ and its adjoint are densely defined and hence closable. The closure of $\frac{d}{dx}$ is denoted by $\bar{\frac{d}{dx}}$.

PROPOSITION 171. *Let $\mathcal{N} = \frac{\gamma}{2} \left(\frac{\bar{d}}{dx}\right)^* \bar{\frac{d}{dx}}$. Then \mathcal{N} is a self-adjoint operator on $L_2(\mathbb{R}, \nu)$ and $\mathcal{N}H_n^{(\gamma)} = nH_n^{(\gamma)}$. The polynomials $H_n^{(\gamma)}$, $n \in \mathbb{N}_0$ form an orthogonal family in $L_2(\mathbb{R}, \nu)$ for each (fixed) $\gamma > 0$.*

PROOF. From (61) follows

$$\exp\left\{\frac{\gamma}{4} \frac{d^2}{dx^2}\right\} \mathcal{N} \exp\left\{-\frac{\gamma}{4} \frac{d^2}{dx^2}\right\} = x \frac{d}{dx} \quad \text{on the polynomials}$$

From 170 and $x \frac{d}{dx} x^n = n x^n$ follows $\mathcal{N}H_n^{(\gamma)} = nH_n^{(\gamma)}$. Lemma 237 says that \mathcal{N} is self-adjoint. The orthogonality of $H_n^{(\gamma)}$, $n \in \mathbb{N}_0$, is a consequence. \square

PROPOSITION 172. *We have*

$$\sum_{n=0}^{\infty} \frac{\alpha^n}{n!} H_n^{(\gamma)}(x) = \exp\left\{\frac{2x\alpha}{\gamma} - \frac{\alpha^2}{\gamma}\right\}.$$

Hence

$$H_n^{(\gamma)}(x) = \left(\frac{\partial}{\partial \alpha}\right)^n \exp\left\{\frac{2x\alpha}{\gamma} - \frac{\alpha^2}{\gamma}\right\} \Big|_{\alpha=0}.$$

PROOF. This follows from Proposition 170 and

$$\frac{1}{\sqrt{\pi\gamma}} \int_{\mathbb{R}} e^{-(x-y)^2/\gamma} \exp\left\{\frac{2y\alpha}{\gamma} - \frac{\alpha^2}{\gamma}\right\} dy = \exp\left\{\frac{2x\alpha}{\gamma}\right\}.$$

and injectivity of Gaussian convolution. \square

PROPOSITION 173. $H_n(x) = 2^n \int_{\mathbb{R}} (x + iy)^n \nu(dy)$.

PROOF. This follows from Proposition 172 and

$$\exp\{2x\alpha - \alpha^2\} = \int_{\mathbb{R}} \exp\{2\alpha(x + iy)\} \nu(dy).$$

\square

PROPOSITION 174. *We have*

$$H_n(x) = n! \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k (2x)^{n-2k}}{k!(n-2k)!}.$$

PROOF. This follows by using in the integral expression for $H_n(x)$ in Proposition 173, the binomial formula on $(x + iy)^n$ and identity

$$\int_{\mathbb{R}} y^k \nu(dy) = \begin{cases} \pi^{-1/2} \Gamma(\frac{k}{2} + \frac{1}{2}) = 4^{-k} (2k)! / k! & \text{if } k = 0, 2, \dots \\ 0 & \text{if } k \text{ is odd.} \end{cases}$$

□

PROPOSITION 175. H_n is the n 'th Hermite polynomial. $(H_n^{(\gamma)} \sqrt{(\gamma/2)^n / n!})$ is an orthonormal basis of $L_2(\mathbb{R}, \nu)$.

PROOF. The first part follows from the explicit expression of H_n given in Proposition 174. By Proposition 170, the second part can be reduced to the case $\gamma = 1$. □

REMARK 176. In [77] and [76] one can find some other (but closely related) properties of the Hermite polynomials.

3. Laguerre polynomials

The differential operators defined by

$$(62) \quad \frac{\partial}{\partial z} = \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right)$$

are called Wirtinger derivatives (or operators).

Harmonic functions are the infinitely differentiable solutions of Laplace equation $\Delta f = 0$, where $\Delta = 4 \frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}}$. Entire analytic functions are the infinitely differentiable solutions of the Cauchy-Riemann equations on \mathbb{C} . The Cauchy-Riemann equations are equivalent to the single equation $\frac{\partial}{\partial \bar{z}} f = 0$. For $\gamma > 0$ let

$$\mu(dz) = (\pi\gamma)^{-1} e^{-|z|^2/\gamma} d \operatorname{Re}(z) d \operatorname{Im}(z).$$

From the density of the polynomials in $\operatorname{Re}(z)$ and $\operatorname{Im}(z)$ in $L_2(\mathbb{C}, \mu)$ follows the density of the polynomials in z and \bar{z} . The Wirtinger operators map the space of polynomials into itself. Consequently, finite compositions of these operators are again well defined on the polynomials. They commute on the space of polynomials. Their adjoints are given by

$$(63) \quad \left(\frac{\partial}{\partial z} \right)^* = \frac{z}{\gamma} - \frac{\partial}{\partial \bar{z}}, \quad \left(\frac{\partial}{\partial \bar{z}} \right)^* = \frac{\bar{z}}{\gamma} - \frac{\partial}{\partial z} \quad \text{on the polynomials in } z, \bar{z}.$$

The (non-normal) operators $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar{z}}$ satisfy $[\frac{\partial}{\partial z}, \frac{\partial}{\partial z}^*] = [\frac{\partial}{\partial \bar{z}}, \frac{\partial}{\partial \bar{z}}^*] = \mathcal{I}/\gamma$ on the polynomials. For $w \in \mathbb{C}$, $\exp\{w\Delta\}$ is defined, on the space of polynomials p in z and \bar{z} , by

$$\exp\{w\Delta\}p = \sum_{n=0}^{\infty} \frac{w^n}{n!} \Delta^n p,$$

in which only a finite number of terms are non-zero. From $[\frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}}, (\frac{\partial}{\partial z})^*] = \frac{1}{\gamma} \frac{\partial}{\partial \bar{z}}$ and $[\frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}}, (\frac{\partial}{\partial \bar{z}})^*] = \frac{1}{\gamma} \frac{\partial}{\partial z}$ it follows easily that

$$(64) \quad \exp\left\{\frac{\gamma}{4}\Delta\right\} \left(\frac{\partial}{\partial z}\right)^* \exp\left\{-\frac{\gamma}{4}\Delta\right\} = \frac{z}{\gamma}, \quad \exp\left\{\frac{\gamma}{4}\Delta\right\} \left(\frac{\partial}{\partial \bar{z}}\right)^* \exp\left\{-\frac{\gamma}{4}\Delta\right\} = \frac{\bar{z}}{\gamma}$$

on the polynomials in z and \bar{z} .

DEFINITION 177. For $k, \ell \in \mathbb{N}_0$ let

$$(65) \quad \Psi_{\ell,k}^{(\gamma)} = \gamma^{(k+\ell)/2} \frac{(\frac{\partial}{\partial z})^\ell (\frac{\partial}{\partial \bar{z}})^k 1}{\sqrt{\ell!k!}}.$$

Let $\Psi_{\ell,k} = \Psi_{\ell,k}^{(1)}$.

PROPOSITION 178. We have the following expression for $\Psi_{\ell,k}^{(\gamma)}$:

$$\Psi_{\ell,k}^{(\gamma)}(z) = \exp\left\{-\frac{\gamma}{4}\Delta\right\} \frac{z^\ell \bar{z}^k}{\sqrt{k!\ell!\gamma^{k+\ell}}}.$$

Hence

$$\frac{1}{\pi\gamma} \int_{\mathbb{C}} e^{-|w-z|^2/\gamma} \Psi_{\ell,k}^{(\gamma)}(z) dz = \frac{w^\ell \bar{w}^k}{\sqrt{k!\ell!\gamma^{k+\ell}}}.$$

Hence

$$\Psi_{\ell,k}^{(\gamma)}(z) = \Psi_{\ell,k}\left(\frac{z}{\sqrt{\gamma}}\right)$$

PROOF. This follows from (64) and $\exp\{-\frac{\gamma}{4}\Delta\}1 = 1$. \square

PROPOSITION 179. For every $n \in \mathbb{N}$,

$$\text{span}\{\Psi_{\ell,k}^{(\gamma)} : \ell, k \in \{1, \dots, n\}\} = \text{span}\{z \mapsto z^\ell \bar{z}^k : \ell, k \in \{1, \dots, n\}\}.$$

PROOF. Inclusion \subset follows from (63) and (65). Inclusion \supset follows from Proposition 178 and the fact that $\exp\{\frac{\gamma}{4}\Delta\}$ is injective and maps the set $\text{span}\{z \mapsto z^\ell \bar{z}^k : \ell, k \in \{1, \dots, n\}\}$ into itself. \square

Operators $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar{z}}$ and their adjoints are densely defined and hence closable. The closure of $\frac{\partial}{\partial z}$ is denoted by $\bar{\frac{\partial}{\partial z}}$, and the closure of $\frac{\partial}{\partial \bar{z}}$ is denoted by $\bar{\frac{\partial}{\partial \bar{z}}}$.

PROPOSITION 180. For $k, \ell \in \mathbb{N}_0$,

$$(66) \quad \gamma \left(\frac{\partial}{\partial z}\right)^* \frac{\partial}{\partial z} \Psi_{\ell,k}^{(\gamma)} = \ell \Psi_{\ell,k}^{(\gamma)}, \quad \gamma \left(\frac{\partial}{\partial \bar{z}}\right)^* \frac{\partial}{\partial \bar{z}} \Psi_{\ell,k}^{(\gamma)} = k \Psi_{\ell,k}^{(\gamma)}.$$

Polynomials $(\Psi_{\ell,k}^{(\gamma)})$ are orthogonal in $L_2(\mathbb{C}, \mu)$ for each (fixed) $\gamma > 0$.

PROOF. From $[\Delta, \frac{\partial}{\partial z}] = [\Delta, \frac{\partial}{\partial \bar{z}}] = 0$ and (64) it follows that

$$\exp\left\{\frac{\gamma}{4}\Delta\right\} \left(\frac{\partial}{\partial z}\right)^* \frac{\partial}{\partial z} \exp\left\{-\frac{\gamma}{4}\Delta\right\} = \frac{1}{\gamma} z \frac{\partial}{\partial z},$$

and

$$\exp\left\{\frac{\gamma}{4}\Delta\right\} \left(\frac{\partial}{\partial \bar{z}}\right)^* \frac{\partial}{\partial \bar{z}} \exp\left\{-\frac{\gamma}{4}\Delta\right\} = \frac{1}{\gamma} \bar{z} \frac{\partial}{\partial \bar{z}}$$

on the polynomials. By Proposition 178, the identities $z \frac{\partial}{\partial z} z^\ell \bar{z}^k = \ell z^\ell \bar{z}^k$ and $\bar{z} \frac{\partial}{\partial \bar{z}} \bar{z}^\ell z^k = k \bar{z}^\ell z^k$ imply (66). Lemma 237 says that $(\bar{\frac{\partial}{\partial z}})^* \bar{\frac{\partial}{\partial z}}$ and $(\bar{\frac{\partial}{\partial \bar{z}}})^* \bar{\frac{\partial}{\partial \bar{z}}}$ are self-adjoint. The orthogonality of $(\Psi_{\ell,k}^{(\gamma)})$ is a consequence. \square

PROPOSITION 181. We have

$$\sum_{k,\ell=0}^{\infty} \frac{\alpha^\ell \beta^k}{\sqrt{k!\ell!}} \Psi_{\ell,k}(z) = \exp\{\alpha z + \beta \bar{z} - \alpha\beta\}.$$

Hence

$$\Psi_{\ell,k}(z) = \frac{1}{\sqrt{k!\ell!}} \left(\frac{\partial}{\partial \alpha}\right)^\ell \left(\frac{\partial}{\partial \beta}\right)^k \exp\{\alpha z + \beta \bar{z} - \alpha\beta\} \Big|_{\alpha=\beta=0}.$$

PROOF. This follows from Proposition 178 and

$$\frac{1}{\pi} \int_{\mathbb{C}} e^{-|w-z|^2} \exp\{\alpha z + \beta \bar{z} - \alpha\beta\} dz = \exp\{\alpha w + \beta \bar{w}\}.$$

and injectivity of Gaussian convolution. \square

PROPOSITION 182. $\Psi_{\ell,k}(z) = \int_{\mathbb{C}} \frac{(z-w)^\ell (\bar{z}+\bar{w})^k}{\sqrt{\ell!k!}} \mu(dw)$.

PROOF. This follows from Proposition 181 and

$$\exp\{\alpha z + \beta \bar{z} - \alpha\beta\} = \int_{\mathbb{C}} \exp\{\alpha(z-w) + \beta(\bar{z}+\bar{w})\} \mu(dw)$$

\square

DEFINITION 183. For $n, m \in \mathbb{N}_0$, let

$$\ell_{m,n}(r) = (-1)^n \sqrt{\frac{n!}{(n+m)!}} r^{m/2} L_n^m(r),$$

where L_n^m is a generalized Laguerre polynomial:

$$(67) \quad L_n^m(x) = \sum_{k=0}^n \binom{n+m}{n-k} \frac{(-x)^k}{k!}.$$

PROPOSITION 184. For $n \in \mathbb{N}_0$ and $m \in \mathbb{N}_0$,

$$\Psi_{n+m,n}(re^{i\theta}) = e^{im\theta} \ell_{m,n}(r^2), \quad \Psi_{n,n+m}(re^{i\theta}) = e^{-im\theta} \ell_{m,n}(r^2).$$

PROOF. By Proposition 182, $\Psi_{\ell,k}(z) = \overline{\Psi_{k,\ell}(z)}$. Hence the second identity follows from the first. We will now prove the first identity: By expanding $(r-w)^{n+m}$ and $(r+\bar{w})^n$ using the binomial formula we get (using definition (67) of the Laguerre polynomials and $\pi^{-1} \int_{\mathbb{C}} w^k \bar{w}^\ell e^{-|w|^2} dw = \sqrt{k!\ell!} \delta_{k\ell}$)

$$(-1)^n n! r^m L_n^m(r^2) = \frac{1}{\pi} \int_{\mathbb{C}} (r-w)^{n+m} (r+\bar{w})^n e^{-|w|^2} dw.$$

This, together with Proposition 182, implies the desired result. \square

PROPOSITION 185. The family $\Psi_{\ell,k}^{(\gamma)}$, $k, \ell \in \mathbb{N}_0$ is an orthonormal basis of $L_2(\mathbb{C}, \mu)$.

PROOF. By Proposition 178, this can be reduced to the case $\gamma = 1$. Then the result follows from Proposition 184: For every $m \in \mathbb{N}_0$, the family $(\ell_{m,n})_{n \in \mathbb{N}_0}$ is an orthonormal basis of $L_2([0, \infty), e^{-r} dr)$. \square

PROPOSITION 186. For $z \in \mathbb{C}$ and $k, \ell \in \mathbb{N}_0$,

$$\sum_{n=0}^{\infty} \Psi_{\ell,n}(z) \Psi_{n,k}(z) = e^{|z|^2} \delta_{\ell k}.$$

PROOF. From Proposition 182 follows that the sum on the left-hand-side is

$$\int_{\mathbb{C}} \int_{\mathbb{C}} \frac{(z-w)^\ell}{\sqrt{\ell!}} \exp\{(\bar{z}+\bar{w})(z-w')\} \frac{(\bar{z}+\bar{w}')^k}{\sqrt{k!}} \mu(dw) \mu(dw').$$

This is

$$e^{|z|^2} \int_{\mathbb{C}} e^{\bar{w}z} \frac{(z-w)^\ell}{\sqrt{\ell!}} \frac{(\bar{z} - (\bar{z}+\bar{w}))^k}{\sqrt{k!}} \mu(dw).$$

By Proposition 252, the value of the integral does not change if we replace (w, \bar{w}) by $(w + z, \bar{w})$ in the integrand. Doing so, we get

$$e^{|z|^2} \frac{1}{\pi} \int_{\mathbb{C}} e^{\bar{w}z} \frac{(-w)^\ell}{\sqrt{\ell!}} \frac{(-\bar{w})^k}{\sqrt{k!}} e^{-(w+z)\bar{w}} dw.$$

This is

$$e^{|z|^2} \int_{\mathbb{C}} \frac{(-w)^\ell}{\sqrt{\ell!}} \frac{(-\bar{w})^k}{\sqrt{k!}} \mu(dw) = e^{|z|^2} \delta_{k\ell}.$$

□

3.1. Estimate for $\Psi_{\ell,k}$.

PROPOSITION 187. *There is a function $C: \mathbb{C} \setminus \{0\} \rightarrow (0, \infty)$ which is bounded on compact subsets of $\mathbb{C} \setminus \{0\}$ such that*

$$|\Psi_{\ell,k}(z)| \leq C(z) \frac{\Gamma(\frac{k+\ell}{2} + 1)}{\sqrt{k!\ell!}} (k + \ell + 1)^{-1/6}$$

for all $k, \ell \in \mathbb{N}_0$ and $z \in \mathbb{C} \setminus \{0\}$. We have

$$(68) \quad \frac{\Gamma^2(\frac{k+\ell}{2} + 1)}{k!\ell!} \leq \frac{1 + \min(k, \ell)}{1 + \min(k, \ell) + (k - \ell)^2/4}.$$

PROOF. The first part follows from Proposition 184 and the proof of Proposition 286. (68) follows from Proposition 283. □

4. Relation between Hermite and Laguerre polynomials

We have

$$\begin{aligned} \left(\frac{\partial}{\partial z}\right)^* \frac{\partial}{\partial z} &= \frac{1}{2\gamma} \left(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}\right) - \frac{i}{2\gamma} \left(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x}\right) - \frac{1}{4} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right), \\ \left(\frac{\partial}{\partial \bar{z}}\right)^* \frac{\partial}{\partial \bar{z}} &= \frac{1}{2\gamma} \left(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}\right) + \frac{i}{2\gamma} \left(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x}\right) - \frac{1}{4} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right). \end{aligned}$$

Hence

$$(69) \quad \gamma \left(\frac{\partial}{\partial z}\right)^* \frac{\partial}{\partial z} + \gamma \left(\frac{\partial}{\partial \bar{z}}\right)^* \frac{\partial}{\partial \bar{z}} = \mathcal{N} \otimes \mathcal{I} + \mathcal{I} \otimes \mathcal{N}.$$

Hence $\Psi_{\ell,k}^{(\gamma)}$ and $H_\ell^{(\gamma)} \otimes H_k^{(\gamma)}$ are simultaneous eigenvectors of $\left(\frac{\partial}{\partial z}\right)^* \frac{\partial}{\partial z} + \left(\frac{\partial}{\partial \bar{z}}\right)^* \frac{\partial}{\partial \bar{z}}$ with eigenvalue $k + \ell$.

PROPOSITION 188. *Define unitary operator $\mathbf{U}_\gamma: \mathbf{L}_2(\mathbb{R}, \nu) \hat{\otimes} \mathbf{L}_2(\mathbb{R}, \nu) \rightarrow \mathbf{L}_2(\mathbb{C}, \mu)$ by $\Psi_{\ell,k}^{(\gamma)} = \mathbf{U}_\gamma[\sqrt{\frac{\gamma^{k+\ell}}{2^\ell \ell! 2^k k!}} H_\ell^{(\gamma)} \otimes H_k^{(\gamma)}]$. Then*

$$(70) \quad \mathbf{U}_\gamma[f \otimes g](z) = \int_{\mathbb{R}} g\left(\frac{z}{\sqrt{2}} - x\right) f\left(\frac{z}{\sqrt{2}} + x\right) \nu(dx),$$

for $f, g \in \text{span}\{H_n^{(\gamma)} : n \in \mathbb{N}_0\}$. Measure ν on \mathbb{R} is defined, as before, by $\nu(dx) = (\pi\gamma)^{-1/2} e^{-x^2/\gamma} dx$.

PROOF. It suffices to prove that

$$(71) \quad \Psi_{\ell,k}^{(\gamma)}(z) = \int_{\mathbb{R}} \sqrt{\frac{\gamma^{k+\ell}}{2^\ell \ell! 2^k k!}} H_\ell^{(\gamma)}\left(\frac{\bar{z}}{\sqrt{2}} - x\right) H_k^{(\gamma)}\left(\frac{z}{\sqrt{2}} + x\right) \nu(dx)$$

for all $k, \ell \in \mathbb{N}_0$. This can be reduced to the case $\gamma = 1$. By Proposition 172,

$$\sum_{\ell, k=0}^{\infty} \frac{\alpha^\ell \beta^k}{\sqrt{\ell!k!}} \frac{H_\ell(x)H_k(y)}{\sqrt{2^\ell \ell! 2^k k!}} = \exp\{\sqrt{2}(\alpha x + \beta y)\} e^{-(\alpha^2 + \beta^2)/2}.$$

(71) (with $\gamma = 1$) follows from

$$\int_{\mathbb{R}} \exp\{\sqrt{2}(\alpha(\frac{z}{\sqrt{2}} + x) + \beta(\frac{\bar{z}}{\sqrt{2}} - x)) - \frac{\alpha^2 + \beta^2}{2}\} \nu(dx) = \exp\{\alpha z + \beta \bar{z} - \alpha\beta\}.$$

and Proposition 181. \square

PROPOSITION 189. For $f, g \in L_2(\mathbb{R}, \nu)$ and $q, p \in \mathbb{R}$,

$$\mathbf{U}_\gamma[f \otimes g](q + ip) = \gamma^{-1/2} e^{(q^2 + p^2)/(2\gamma)} \mathbf{W}[\mathcal{M}[f] \otimes \mathcal{M}[g]](q, -p/\gamma).$$

where \mathcal{M} is defined by $\mathcal{M}[f](x) = e^{-x^2/(2\gamma)} f(x)$.

PROOF. From (70) and Proposition 250 follows

$$\mathbf{U}_\gamma[f \otimes g](q + ip) = \frac{e^{(q^2 + p^2)/(2\gamma)}}{\sqrt{2\pi\gamma}} \int_{\mathbb{R}} \overline{\mathcal{M}[g](\frac{q-x}{\sqrt{2}})} \mathcal{M}[f](\frac{q+x}{\sqrt{2}}) e^{ixp/\gamma} dx$$

for $f, g \in \text{span}\{H_n^{(\gamma)} : n \in \mathbb{N}_0\}$. The rest follows from Proposition 135. \square

PROPOSITION 190. Let

$$\mathcal{U}_z^{(\gamma)} = \sum_{\ell, k=0}^{\infty} \overline{\Psi_{\ell, k}^{(\gamma)}(z)} \sqrt{\frac{\gamma^{k+\ell}}{2^\ell \ell! 2^k k!}} H_\ell^{(\gamma)} \otimes H_k^{(\gamma)}.$$

Then $\mathbf{U}_\gamma[f \otimes g](z) = (\mathcal{U}_z^{(\gamma)} g, f)$.

Operator $\mathcal{U}_z^{(\gamma)}$ is bounded, self-adjoint, and proportional to a unitary operator, and can be expressed alternatively as

$$(72) \quad \mathcal{U}_z^{(\gamma)} = (\exp\{z \frac{1}{\sqrt{2}} \frac{d}{dx}\})^* \Pi \exp\{z \frac{1}{\sqrt{2}} \frac{d}{dx}\}$$

where $\Pi = \exp\{i\pi\mathcal{N}\}$ (the parity operator). We have:

$$(73) \quad \frac{\partial}{\partial z} \mathbf{U}_\gamma = \mathbf{U}_\gamma \frac{1}{\sqrt{2}} \frac{d}{dx} \otimes \mathcal{I}, \quad \frac{\partial}{\partial \bar{z}} \mathbf{U}_\gamma = \mathbf{U}_\gamma \mathcal{I} \otimes \frac{1}{\sqrt{2}} \frac{d}{dx}$$

and

$$(74) \quad (\frac{\partial}{\partial z})^* \mathbf{U}_\gamma = \mathbf{U}_\gamma (\frac{1}{\sqrt{2}} \frac{d}{dx})^* \otimes \mathcal{I}, \quad (\frac{\partial}{\partial \bar{z}})^* \mathbf{U}_\gamma = \mathbf{U}_\gamma \mathcal{I} \otimes (\frac{1}{\sqrt{2}} \frac{d}{dx})^*$$

on $\text{span}\{H_\ell^{(\gamma)} \otimes H_k^{(\gamma)} : \ell, k \in \mathbb{N}_0\}$.

PROOF. From $\overline{\Psi_{\ell, k}^{(\gamma)}(z)} = \Psi_{k, \ell}^{(\gamma)}(z)$ follows $(\mathcal{U}_z^{(\gamma)})^* = \mathcal{U}_z^{(\gamma)}$. From Proposition 186 follows $(\mathcal{U}_z^{(\gamma)})^2 = e^{|z|^2} \mathcal{I}$. Hence $\mathcal{U}_z^{(\gamma)}$ is proportional to a unitary operator and is in particular a bounded operator.

If the function $f: \mathbb{R} \rightarrow \mathbb{C}$ has a continuation to an entire analytic function then $\exp\{z \frac{1}{\sqrt{2}} \frac{d}{dx}\}[f](q) = f(q + \frac{z}{\sqrt{2}})$. From (70) follows

$$(75) \quad (\mathcal{U}_z^{(\gamma)} g, f) = \mathbf{U}_\gamma[f \otimes g](z) = (\Pi \exp\{z \frac{1}{\sqrt{2}} \frac{d}{dx}\} g, \exp\{z \frac{1}{\sqrt{2}} \frac{d}{dx}\} f)$$

for real analytic functions $f, g \in L_2(\mathbb{R}, \nu)$ and (because real analytic functions form a dense subset of $L_2(\mathbb{R}, \nu)$) hence for all $f, g \in L_2(\mathbb{R}, \nu)$. Hence (72).

(73) follows from (75) and $\frac{\partial}{\partial z} \exp\{z \frac{1}{\sqrt{2}} \frac{d}{dx}\} = \frac{1}{\sqrt{2}} \frac{d}{dx} \exp\{z \frac{1}{\sqrt{2}} \frac{d}{dx}\}$. (74) follows from (73) and the unitarity of \mathbf{U}_γ . \square

REMARK 191. By Proposition 190,

$$\mathbf{U}_\gamma \mathcal{N} \otimes \mathcal{I} = \gamma \left(\frac{\partial}{\partial z}\right)^* \frac{\partial}{\partial z} \mathbf{U}_\gamma, \quad \mathbf{U}_\gamma \mathcal{I} \otimes \mathcal{N} = \gamma \left(\frac{\partial}{\partial \bar{z}}\right)^* \frac{\partial}{\partial \bar{z}} \mathbf{U}_\gamma$$

on $\text{span}\{H_\ell^{(\gamma)} \otimes H_k^{(\gamma)} : \ell, k \in \mathbb{N}_0\}$. Cf. (69).

5. Analytic and Harmonic functions

In this section we determine the kernels of the operators $\frac{\bar{\partial}}{\partial z}$, $\frac{\bar{\partial}}{\partial \bar{z}}$ and $\bar{\Delta}$, where $\bar{\Delta}$ is the closure of $4 \frac{\bar{\partial}}{\partial z} \frac{\bar{\partial}}{\partial \bar{z}}$, defined on the polynomials.

LEMMA 192.

$$(76) \quad \frac{\bar{\partial}}{\partial z} = \frac{1}{\sqrt{\gamma}} \sum_{k,\ell=0}^{\infty} \sqrt{\ell+1} \Psi_{\ell,k}^{(\gamma)} \otimes \Psi_{\ell+1,k}^{(\gamma)}$$

and $\mathcal{D}(\frac{\bar{\partial}}{\partial z}) = \{h \in \mathbf{L}_2(\mathbb{C}, \mu) : \sum_{\ell,k=0}^{\infty} \ell |(\Psi_{\ell,k}^{(\gamma)}, h)|^2 < \infty\}$.

$$\frac{\bar{\partial}}{\partial \bar{z}} = \frac{1}{\sqrt{\gamma}} \sum_{k,\ell=0}^{\infty} \sqrt{k+1} \Psi_{\ell,k}^{(\gamma)} \otimes \Psi_{\ell,k+1}^{(\gamma)}$$

and $\mathcal{D}(\frac{\bar{\partial}}{\partial \bar{z}}) = \{h \in \mathbf{L}_2(\mathbb{C}, \mu) : \sum_{\ell,k=0}^{\infty} k |(\Psi_{\ell,k}^{(\gamma)}, h)|^2 < \infty\}$.

$$\bar{\Delta} = \frac{4}{\gamma} \sum_{k,\ell=0}^{\infty} \sqrt{(\ell+1)(k+1)} \Psi_{\ell,k}^{(\gamma)} \otimes \Psi_{\ell+1,k+1}^{(\gamma)}$$

and $\mathcal{D}(\bar{\Delta}) = \{h \in \mathbf{L}_2(\mathbb{C}, \mu) : \sum_{\ell,k=0}^{\infty} \ell k |(\Psi_{\ell,k}^{(\gamma)}, h)|^2 < \infty\}$.

PROOF. We only treat the case of $\frac{\bar{\partial}}{\partial z}$. From (65) it follows that $\frac{\bar{\partial}}{\partial z}$ is equal to the right-hand side of (76) on $\text{span}\{\Psi_{\ell,k}^{(\gamma)} : \ell, k \in \mathbb{N}_0\}$.

Let $(\lambda_{\ell,k})$ be a double sequence of complex numbers such that $\sum_{\ell,k=0}^{\infty} \ell |\lambda_{\ell,k}|^2 < \infty$. Let $h = \sum_{\ell,k=0}^{\infty} \lambda_{\ell,k} \Psi_{\ell,k}^{(\gamma)}$ and let $h_n = \sum_{\ell,k=0}^n \lambda_{\ell,k} \Psi_{\ell,k}^{(\gamma)}$. Then $h_n \rightarrow h \in \mathbf{L}_2(\mathbb{C}, \mu)$ and $\frac{\bar{\partial}}{\partial z} h_n \rightarrow g \in \mathbf{L}_2(\mathbb{C}, \mu)$ where

$$g = \frac{1}{\sqrt{\gamma}} \sum_{\ell,k=0}^{\infty} \lambda_{\ell+1,k} \sqrt{\ell+1} \Psi_{\ell,k}^{(\gamma)}$$

Because $\frac{\bar{\partial}}{\partial z}$ is closed, $h \in \mathcal{D}(\frac{\bar{\partial}}{\partial z})$ and $\frac{\bar{\partial}}{\partial z} h = g$. This shows that $\frac{\bar{\partial}}{\partial z}$ extends the operator on the right-hand-side of (76). For equality one has to prove that the operator on the right-hand-side of (76) is closed. This follows from the fact that the operator

$$\frac{1}{\sqrt{\gamma}} \sum_{k,\ell=0}^{\infty} \sqrt{\ell} \Psi_{\ell,k}^{(\gamma)} \otimes \Psi_{\ell,k}^{(\gamma)}$$

(with the same domain) is self-adjoint. \square

It follows easily from the previous lemma that

$$\left| \frac{\bar{\partial}}{\partial z} \right| = \frac{1}{\sqrt{\gamma}} \sum_{k,\ell=0}^{\infty} \sqrt{\ell} \Psi_{\ell,k}^{(\gamma)} \otimes \Psi_{\ell,k}^{(\gamma)}, \quad \left| \frac{\bar{\partial}}{\partial \bar{z}} \right| = \frac{1}{\sqrt{\gamma}} \sum_{k,\ell=0}^{\infty} \sqrt{k} \Psi_{\ell,k}^{(\gamma)} \otimes \Psi_{\ell,k}^{(\gamma)},$$

$$|\bar{\Delta}| = \frac{4}{\gamma} \sum_{k,\ell=0}^{\infty} \sqrt{k\ell} \Psi_{\ell,k}^{(\gamma)} \otimes \Psi_{\ell,k}^{(\gamma)}.$$

Hence the kernels are given by

$$\ker\left(\frac{\bar{\partial}}{\partial z}\right) = \text{cl}(\text{span}\{\Psi_{0k}^{(\gamma)} : k \in \mathbb{N}_0\}), \quad \ker\left(\frac{\bar{\partial}}{\partial \bar{z}}\right) = \text{cl}(\text{span}\{\Psi_{\ell 0}^{(\gamma)} : \ell \in \mathbb{N}_0\}),$$

$$\ker(\bar{\Delta}) = \text{cl}(\text{span}\{\Psi_{\ell,k}^{(\gamma)} : k, \ell \in \mathbb{N}_0, k\ell = 0\}).$$

We write $L_2^a(\mathbb{C}, \mu) = \ker(\frac{\bar{\partial}}{\partial \bar{z}})$. This is the closed linear subspace of entire analytic functions in $L_2(\mathbb{C}, \mu)$. We write $L_2^h(\mathbb{C}, \mu) = \ker(\bar{\Delta})$. This is the closed linear subspace of harmonic functions in $L_2(\mathbb{C}, \mu)$.

6. Extended Bargmann space

In the following we take $\gamma = 1$ and simplify our notation: Let

$$\mathbf{U} = \mathbf{U}_1, \quad \mathcal{U}_z = \mathcal{U}_z^{(1)}, \quad z \in \mathbb{C}.$$

We will extend the Bargmann space to a functional Hilbert space densely contained in $L_2(\mathbb{C}, \mu)$.

6.1. Sobolov space \mathbf{W}_2 . We denote the inner product of $L_2(\mathbb{C}, \mu)$ by $(\cdot, \cdot)_0$.

DEFINITION 193. $\mathbf{W}_2 = \mathcal{D}(\bar{\Delta})$, equipped with inner-product

$$(f, g)_2 = (f, g)_0 + (\bar{\Delta}f, \bar{\Delta}g)_0.$$

In the previous section we saw that $\ker(\bar{\Delta}) = L_2^h(\mathbb{C}, \mu)$. It is clear that $(f, g)_2 = (f, g)_0$ if either f or g is contained in $L_2^h(\mathbb{C}, \mu)$. From Lemma 237 it follows that \mathbf{W}_2 is a Hilbert space. Let \mathbf{R} be the square root of the bounded operator $(\mathcal{I} + |\bar{\Delta}|^2)^{-1}$ on $L_2(\mathbb{C}, \mu)$. Note that $\mathbf{R}\Psi_{\ell,k} = (1 + 16k\ell)^{-1/2}\Psi_{\ell,k}$, that \mathbf{R} is not a compact operator, and that $\mathbf{R}: L_2(\mathbb{C}, \mu) \rightarrow \mathbf{W}_2$ is unitary.

PROPOSITION 194. *Point evaluation is continuous on \mathbf{W}_2 . Moreover, there exist $E_w \in \mathbf{W}_2$, $w \in \mathbb{C}$, such that $w \mapsto (E_w, f)_2$ is a continuous representative of $f \in \mathbf{W}_2$.*

PROOF. By Proposition 187,

$$|\Psi_{\ell,k}(z)|^2 \leq C(z)^2 \frac{(1 + k\ell)(k + \ell + 1)^{-1/3}}{1 + k\ell + (k - \ell)^2/4}.$$

Hence

$$\frac{|\Psi_{\ell,k}(z)|^2}{1 + 16k\ell} \leq C(z)^2 \frac{(k + \ell + 1)^{-1/3}}{1 + k\ell + (k - \ell)^2/4}.$$

Hence

$$\frac{|\Psi_{\ell,k}(z)|^2}{1 + 16k\ell} \leq 4C(z)^2 \frac{(k + \ell + 1)^{-1/3}}{4 + k^2 + \ell^2}.$$

Using polar coordinates and the estimate $\sin(\theta) + \cos(\theta) \geq 1$ as $0 \leq \theta \leq \pi/2$, it is easily seen that

$$\int_0^\infty \int_0^\infty \frac{(x + y + 1)^{-1/3}}{4 + x^2 + y^2} dx dy < \infty.$$

By calculating the first partial derivatives, it is easily seen that the integrand is a monotone decreasing function in x for fixed y and also in y for fixed x . Hence

$$\sum_{k,\ell=0}^{\infty} \frac{(k + \ell + 1)^{-1/3}}{4 + k^2 + \ell^2} < \infty.$$

Hence

$$\sum_{k,\ell=0}^{\infty} \frac{|\Psi_{\ell,k}(z)|^2}{1+16k\ell} < \infty.$$

This implies the convergence in \mathbf{W}_2 of the sum

$$E_w = \sum_{k,\ell=0}^{\infty} (1+16k\ell)^{-1} \overline{\Psi_{\ell,k}(w)} \Psi_{\ell,k}.$$

We have $f(w) = (E_w, f)_2$, which proves continuity of point evaluation. For $f \in \mathbf{W}_2$, the sum

$$(77) \quad \tilde{f}(w) = \sum_{k,\ell=0}^{\infty} (\Psi_{\ell,k}, f)_0 \Psi_{\ell,k}(w)$$

converges uniform on compact subsets of $\mathbb{C} \setminus \{0\}$. It is easily seen that \tilde{f} is a representative of f and that \tilde{f} is continuous on $\mathbb{C} \setminus \{0\}$. If $f \in \mathbf{W}_2$ then g , defined by $g(z) = f(\frac{1}{2}z - 1)$, is also contained in \mathbf{W}_2 . The continuity of f in 0 follows from the continuity of g in $z = 2$. \square

6.2. Sobolev space \mathbf{W}_{-2} . Define a sesquilinear form $(\cdot, \cdot)_{-2}$ on $\mathbf{L}_2(\mathbb{C}, \mu)$ by

$$(f, g)_{-2} = (\mathbf{R}f, \mathbf{R}g)_0.$$

Let \mathbf{W}_{-2} be the completion of $\mathbf{L}_2(\mathbb{C}, \mu)$ equipped with this inner-product.

The family $(\Psi_{\ell,k})$ is an orthogonal basis in each of the three Hilbert spaces in

$$\mathbf{W}_2 \subset \mathbf{L}_2(\mathbb{C}, \mu) \subset \mathbf{W}_{-2}.$$

Define \mathbf{R}^{ext} on \mathbf{W}_{-2} by

$$\mathbf{R}^{\text{ext}} = \sum_{\ell,k=0}^{\infty} (1+16k\ell)^{-1/2} \Psi_{\ell,k} \otimes \Psi_{\ell,k}.$$

\mathbf{W}_{-2} is a representation of the topological dual of \mathbf{W}_2 : For $F \in \mathbf{W}_{-2}$ we define a continuous linear form on \mathbf{W}_2 by $\langle F, w \rangle = (\mathbf{R}F, \mathbf{R}^{-1}w)_0$ for $w \in \mathbf{W}_2$. All continuous linear forms on \mathbf{W}_2 are represented this way. Denote the continuous representant of $w \in \mathbf{W}_2$ by \tilde{w} . For each $z \in \mathbb{C}$, there is an element $\delta_z \in \mathbf{W}_{-2}$ such that $\langle \delta_z, w \rangle = \tilde{w}(z)$ for all $w \in \mathbf{W}_2$. For all $z \in \mathbb{C}$,

$$\delta_z = \sum_{k,\ell=0}^{\infty} \overline{\Psi_{\ell,k}(z)} \Psi_{\ell,k}.$$

The sum converges in \mathbf{W}_{-2} . We have $E_z = \mathbf{R}^2 \delta_z$.

6.3. Bargmann projection. Define \mathbf{P}_a on $\mathbf{L}_2(\mathbb{C}, \mu)$ by

$$\mathbf{P}_a = \sum_{\ell=0}^{\infty} \Psi_{\ell 0} \otimes \Psi_{\ell 0}.$$

This is the operator of orthogonal projection on the closed linear subspace $\mathbf{L}_2^a(\mathbb{C}, \mu)$ of entire analytic functions. We have $\mathbf{P}_a \mathbf{R} = \mathbf{R} \mathbf{P}_a = \mathbf{P}_a$. The orthogonal projection operator \mathbf{P}_a maps E_w on

$$e_w = \sum_{n=0}^{\infty} \frac{\bar{w}^n}{\sqrt{n!}} \Psi_{n,0}.$$

We have $e_w(z) = e^{\bar{w}z}$. Define $\mathbf{P}_a^{\text{ext}}$ on W_{-2} by

$$\mathbf{P}_a^{\text{ext}} = \sum_{\ell=0}^{\infty} \Psi_{\ell 0} \otimes \Psi_{\ell 0}.$$

It satisfies

$$\mathbf{P}_a^{\text{ext}} = \mathbf{P}_a \mathbf{R}^{\text{ext}},$$

and maps W_{-2} onto closed linear subspace $L_2^a(\mathbb{C}, \mu)$ of W_2 , and maps $\delta_w \in W_{-2}$ on e_w .

6.4. Sobolev space V_2 . We identify the sesquilinear tensor product

$$L_2(\mathbb{R}, \nu)^{\hat{\otimes} 2} = L_2(\mathbb{R}, \nu) \hat{\otimes} L_2(\mathbb{R}, \nu)$$

with the space of Hilbert-Schmidt operators on $L_2(\mathbb{R}, \nu)$.

DEFINITION 195. $V_2 = \mathcal{D}(\mathcal{N}^{1/2} \hat{\otimes} \mathcal{N}^{1/2})$, equipped with inner-product

$$(\mathcal{A}, \mathcal{B})_2 = (\mathcal{A}, \mathcal{B})_0 + 16(\mathcal{N}^{1/2} \mathcal{A} \mathcal{N}^{1/2}, \mathcal{N}^{1/2} \mathcal{B} \mathcal{N}^{1/2})_0.$$

By Remark 191, $\mathcal{N}^{1/2} \hat{\otimes} \mathcal{N}^{1/2} = \mathbf{U}^* \bar{\Delta} \mathbf{U}$. Hence $\mathbf{U}: V_2 \rightarrow W_2$ is unitary and

$$\tilde{\mathbf{R}} = (\mathbf{I} + 16\mathcal{N} \hat{\otimes} \mathcal{N})^{-1/2}: L_2(\mathbb{R}, \nu)^{\hat{\otimes} 2} \rightarrow V_2$$

is unitary. Note that $\tilde{\mathbf{R}} H_k \otimes H_\ell = (1 + 16k\ell)^{-1/2} H_k \otimes H_\ell$.

6.5. Sobolev space V_{-2} . Define a sesquilinear form $(\cdot, \cdot)_{-2}$ on $L_2(\mathbb{R}, \nu)^{\hat{\otimes} 2}$ by

$$(f, g)_{-2} = (\tilde{\mathbf{R}} f, \tilde{\mathbf{R}} g)_0.$$

Let V_{-2} be the completion of $L_2(\mathbb{R}, \nu)^{\hat{\otimes} 2}$ equipped with this inner-product. The family $(H_k \otimes H_\ell)$ is an orthogonal basis in each of the three Hilbert spaces in

$$V_2 \subset L_2(\mathbb{R}, \nu)^{\hat{\otimes} 2} \subset V_{-2}.$$

Define unitary operator $\mathbf{U}^{\text{ext}}: V_{-2} \rightarrow W_{-2}$ by

$$\Psi_{\ell, k} = \mathbf{U}^{\text{ext}}[(2^\ell \ell! 2^k k!)^{-1/2} H_\ell^{(\gamma)} \otimes H_k^{(\gamma)}]$$

and linear and continuous extension. We have $\mathbf{R} \mathbf{U}^{\text{ext}} = \mathbf{U} \tilde{\mathbf{R}}^{\text{ext}}$, and $\mathbf{U}^{\text{ext}}[\mathcal{U}_z] = \delta_z$. V_{-2} is a representation of the topological dual of V_2 : For $H \in V_{-2}$ we define a continuous linear form on V_2 by $\langle H, v \rangle = (\tilde{\mathbf{R}} H, \tilde{\mathbf{R}}^{-1} v)_0$ for $v \in V_2$. For $v \in V_2$,

$$\mathbf{U}[v](z) = \langle \mathcal{U}_z, v \rangle.$$

6.6. Overview. The following diagram gives the relations between the spaces and operators involved in this section:

$$\begin{array}{ccccc} V_2 & \xleftarrow{\tilde{\mathbf{R}}} & L_2(\mathbb{R}, \nu)^{\hat{\otimes} 2} & \xleftarrow{\tilde{\mathbf{R}}} & V_{-2} \\ \mathbf{U} \downarrow & & \downarrow \mathbf{U} & & \downarrow \mathbf{U} \\ W_2 & \xleftarrow{\mathbf{R}} & L_2(\mathbb{C}, \mu) & \xleftarrow{\mathbf{R}} & W_{-2} \\ \mathbf{P}_a \downarrow & & \downarrow \mathbf{P}_a & & \downarrow \mathbf{P}_a^{\text{ext}} \\ L_2^a(\mathbb{C}, \mu) & \xlongequal{\quad} & L_2^a(\mathbb{C}, \mu) & \xlongequal{\quad} & L_2^a(\mathbb{C}, \mu) \end{array}$$

APPENDIX A

Miscellaneous results

LEMMA 196. *Let \mathcal{P} be a projection operator on a Hilbert space \mathbf{H} . Let (\mathcal{P}_k) be a finite sequence of projection operators on \mathbf{H} that commute with \mathcal{P} and have pairwise orthogonal ranges. Let $\epsilon > 0$ and $h \in \mathbf{H}$. If (λ_k) is a finite sequence of complex numbers and*

$$\|\mathcal{P}h - \sum_k \lambda_k \mathcal{P}_k h\| \leq \epsilon,$$

then

$$\|\mathcal{P}h - \sum_k \{\mathcal{P}_k h : |\lambda_k| \geq \epsilon^{1/4}\}\| \leq \epsilon + \epsilon^{1/4}\|h\| + \sqrt{(2\|h\| + \epsilon)\epsilon} + 2\sqrt{\epsilon}\epsilon^{1/4}.$$

PROOF. Because \mathcal{P} is a contraction, $\|\mathcal{P}h - \sum_k \lambda_k \mathcal{P}_k \mathcal{P}h\| \leq \epsilon$. By the triangle inequality, $|\|\sum_k \lambda_k \mathcal{P}_k \mathcal{P}h\| - \|\mathcal{P}h\|| \leq \epsilon$. This implies that $\|\sum_k \lambda_k \mathcal{P}_k \mathcal{P}h\| \leq \epsilon + \|h\|$. By the Cauchy-Bunyakovskii-Schwarz inequality, the triangle inequality, and the pairwise orthogonality of the sequence $(\mathcal{P}_k \mathcal{P}h)$, this implies that

$$\begin{aligned} \sum_k |1 - \lambda_k|^2 \|\mathcal{P}_k \mathcal{P}h\|^2 &= \sum_k (1 - \bar{\lambda}_k)(1 - \lambda_k)(\mathcal{P}_k \mathcal{P}h, \mathcal{P}_k \mathcal{P}h) \\ &= \sum_k (1 - \bar{\lambda}_k) (\mathcal{P}_k \mathcal{P}h, \mathcal{P}h - \sum_\ell \lambda_\ell \mathcal{P}_\ell h) \\ &\leq \left\| \sum_k (1 - \lambda_k) \mathcal{P}_k \mathcal{P}h \right\| \left\| \mathcal{P}h - \sum_\ell \lambda_\ell \mathcal{P}_\ell h \right\| \\ &\leq \left(\left\| \sum_k \mathcal{P}_k \mathcal{P}h \right\| + \left\| \sum_k \lambda_k \mathcal{P}_k \mathcal{P}h \right\| \right) \epsilon \\ &\leq (2\|h\| + \epsilon)\epsilon. \end{aligned}$$

By the pairwise orthogonality of the sequence $(\mathcal{P}_k \mathcal{P}h)$, we have, for every finite set Δ of indices,

$$\left\| \sum_{\Delta} \mathcal{P}_k \mathcal{P}h - \sum_{\Delta} \lambda_k \mathcal{P}_k \mathcal{P}h \right\|^2 \leq \sum_k |1 - \lambda_k|^2 \|\mathcal{P}_k \mathcal{P}h\|^2 \leq (2\|h\| + \epsilon)\epsilon.$$

Let $\Delta_\epsilon = \{k : |\lambda_k| \geq \epsilon^{1/4}\}$. By the pairwise orthogonality of the sequence $(\mathcal{P}_k(\mathcal{I} - \mathcal{P})h)$,

$$\begin{aligned} \left\| \sum_{\Delta_\epsilon} \mathcal{P}_k(\mathcal{I} - \mathcal{P})h \right\|^2 &= \sum_{\Delta_\epsilon} \|\mathcal{P}_k(\mathcal{I} - \mathcal{P})h\|^2 \leq \epsilon^{-1/2} \sum_k |\lambda_k|^2 \|\mathcal{P}_k(\mathcal{I} - \mathcal{P})h\|^2 \\ &= \epsilon^{-1/2} \left\| \sum_k \lambda_k \mathcal{P}_k(\mathcal{I} - \mathcal{P})h \right\|^2 \leq \epsilon^{-1/2} 4\epsilon^2 = 4\epsilon\sqrt{\epsilon}. \end{aligned}$$

By the triangle inequality,

$$\begin{aligned} \left\| \mathcal{P}h - \sum_{\Delta_\epsilon} \lambda_k \mathcal{P}_k \mathcal{P}h \right\| &\leq \left\| \mathcal{P}h - \sum_k \lambda_k \mathcal{P}_k \mathcal{P}h \right\| + \left\| \sum_{\Delta_\epsilon^c} \lambda_k \mathcal{P}_k \mathcal{P}h \right\| \\ &\leq \epsilon + \epsilon^{1/4}\|h\|. \end{aligned}$$

Hence, by the triangle inequality,

$$\begin{aligned} \|\mathcal{P}h - \sum_{\Delta_\epsilon} \mathcal{P}_k h\| &\leq \|\mathcal{P}h - \sum_{\Delta_\epsilon} \lambda_k \mathcal{P}_k \mathcal{P}h\| + \|\sum_{\Delta_\epsilon} \lambda_k \mathcal{P}_k \mathcal{P}h - \sum_{\Delta_\epsilon} \mathcal{P}_k h\| \\ &\leq \epsilon + \epsilon^{1/4} \|h\| + \|\sum_{\Delta_\epsilon} \lambda_k \mathcal{P}_k \mathcal{P}h - \sum_{\Delta_\epsilon} \mathcal{P}_k \mathcal{P}h\| + \|\sum_{\Delta_\epsilon} \mathcal{P}_k (\mathcal{I} - \mathcal{P})h\| \\ &\leq \epsilon + \epsilon^{1/4} \|h\| + \sqrt{(2\|h\| + \epsilon)\epsilon} + 2\sqrt{\epsilon}\epsilon^{1/4}. \end{aligned}$$

□

LEMMA 197. *Let \mathbf{X} be a C^* -algebra and let \mathbf{Y} be a normed space. Let $f: \mathbf{X} \rightarrow \mathbf{Y}$ be a linear function satisfying $\|f[x]\| \leq \|x\|$ for $x \geq 0$. Then $\|f[x]\| \leq 8\|x\|$ for every $x \in \mathbf{X}$.*

PROOF. Let $a, b, c, d \in \mathbf{X}$. We have

$$|a + b|^2 \leq |a + b|^2 + |a - b|^2 = 2(|a|^2 + |b|^2)$$

and

$$\|a + b\|^2 = \| |a + b|^2 \| \leq 2\| |a|^2 + |b|^2 \|\|.$$

Hence

$$\|a + b + c + d\|^2 \leq 2\| |a + b|^2 + |c + d|^2 \| \leq 4\| |a|^2 + |b|^2 + |c|^2 + |d|^2 \|\|.$$

Every $x \in \mathbf{X}$ can be written as $x = x_1 - x_2 + i(x_3 - x_4)$ with $x_k \geq 0$ and $x_1 x_2 = x_3 x_4 = 0$. We have

$$\begin{aligned} \left\| \sum_{k=1}^4 x_k^2 \right\| &= \|(x_1 - x_2)^2 + (x_3 - x_4)^2\| = \frac{1}{2} \|x^* x + x x^*\| \leq \frac{1}{2} (\|x^* x\| + \|x x^*\|) \\ &= \frac{1}{2} (\|x\|^2 + \|x^*\|^2) = \|x\|^2 \end{aligned}$$

Hence

$$\begin{aligned} \|f[x]\| &\leq \sum_k \|f[x_k]\| \leq \sum_k \|x_k\| \leq 4 \sup_k \|x_k\| = 4 \sup_k x_k \\ &\leq 4 \left\| \sum_k x_k \right\| \leq 8 \sqrt{\left\| \sum_k x_k^2 \right\|} \leq 8\|x\|. \end{aligned}$$

□

LEMMA 198. *Let (Ω, Σ, μ) be a finite measure space. Let \mathbf{K} be a separable closed linear subspace of $\mathbf{L}_2(\Omega, \Sigma, \mu)$. There is a $\omega \in \Sigma$ such that*

$$(78) \quad \int_{\Delta} |\varphi(x)|^2 \mu(dx) = 0 \quad \forall \varphi \in \mathbf{K} \quad \Leftrightarrow \quad \mu|_{\omega}(\Delta) = 0$$

for $\Delta \in \Sigma$.

LEMMA 199. *Let \mathbf{H} be a separable Hilbert space and let (Ω, Σ, μ) be a finite measure space. Let $M: \Sigma \rightarrow \mathbf{B}_+(\mathbf{H})$ be an FPOVM. Let $\mathbf{V}: \mathbf{H} \rightarrow \mathbf{L}_2(\Omega, \Sigma, \mu)$ be an isometric operator such that*

$$M_h(\Delta) = \int_{\Delta} |\mathbf{V}[h](x)|^2 \mu(dx) \quad \forall h \in \mathbf{H}, \Delta \in \Sigma.$$

There exists an $\omega \in \Sigma$ such that the contraction $\mu|_{\omega}$ of μ to ω satisfies $\mu|_{\omega} \ll M$ and

$$M_h(\Delta) = \int_{\Delta} |\mathbf{V}[h](x)|^2 \mu|_{\omega}(dx) \quad \forall h \in \mathbf{H}, \Delta \in \Sigma.$$

PROOF. This follows from Lemma 198. □

1. Measurable covers and partitions

LEMMA 200. *Let (Ω, Σ) be a measurable space. Let \mathbb{I} be a countable set. Let $(\Delta_n)_{n \in \mathbb{I}}$ be a family of sets in Σ that covers Ω . There exists a measurable partition ω_k , $k \in \mathbb{I}$ of Ω satisfying $\omega_n \subset \Delta_n$ for all n .*

If $\omega_k^{(1)}$, $k \in \mathbb{I}$ and $\omega_k^{(2)}$, $k \in \mathbb{J}$ are two measurable partitions of Ω , then $\omega_i^{(1)} \cap \omega_j^{(2)}$, $(i, j) \in \mathbb{I} \times \mathbb{J}$ is also a measurable partition of Ω .

2. Hilbert-Schmidt operators

The Hilbert space of Hilbert-Schmidt operators on a Hilbert space \mathbf{H} is denoted by $\mathbf{B}_2(\mathbf{H})$. The Hilbert-Schmidt norm is denoted and defined by

$$\|\mathcal{A}\|_2 = \sqrt{\text{Tr}(\mathcal{A}^* \mathcal{A})} = \sqrt{\text{Tr}(\mathcal{A} \mathcal{A}^*)}.$$

PROPOSITION 201. *Let \mathbf{H} be a separable Hilbert space. Let $\mathcal{A} \in \mathbf{B}_2(\mathbf{H})$. Then*

$$\|\mathcal{A}h\| \leq \|\mathcal{A}\|_2 \|h\| \quad \forall h \in \mathbf{H}.$$

Let $\mathcal{A} \in \mathbf{B}_2(\mathbf{H})$ and $\mathcal{B} \in \mathbf{B}_\infty(\mathbf{H})$. Then

$$\text{Tr}(\mathcal{A}^* \mathcal{B} \mathcal{A}) \leq \|\mathcal{B}\|_\infty \text{Tr}(\mathcal{A}^* \mathcal{A}).$$

3. Trace-class operators

The Banach space of trace-class operators on a Hilbert space \mathbf{H} is denoted by $\mathbf{B}_1(\mathbf{H})$ and consists of compositions $\mathcal{A}\mathcal{B}$ of Hilbert-Schmidt operators \mathcal{A} and \mathcal{B} . The trace-class norm is denoted and defined by

$$\|\mathcal{T}\|_1 = \text{Tr}(\sqrt{\mathcal{T}^* \mathcal{T}}).$$

PROPOSITION 202. *Let \mathbf{H} be a separable Hilbert space and let $\mathcal{T} \in \mathbf{B}_1(\mathbf{H})$. Then $\|\mathcal{T}\|_2 \leq \|\mathcal{T}\|_1$.*

PROOF. There are $\mathcal{A}, \mathcal{B} \in \mathbf{B}_2(\mathbf{H})$ such that $\mathcal{T} = \mathcal{A}\mathcal{B}$. We have to prove that

$$\sqrt{\text{Tr}(\mathcal{B}^* \mathcal{A}^* \mathcal{A} \mathcal{B})} \leq \text{Tr}(\sqrt{\mathcal{B}^* \mathcal{A}^* \mathcal{A} \mathcal{B}}).$$

In terms of the eigenvalues (λ_k) of the non-negative compact operator $\sqrt{\mathcal{B}^* \mathcal{A}^* \mathcal{A} \mathcal{B}}$, this can be expressed as

$$\sqrt{\sum_k \lambda_k^2} \leq \sum_k \lambda_k.$$

By squaring both sides, this is easily seen to be satisfied. □

4. Convergence and σ -additivity of measures

PROPOSITION 203. *Let Σ be a σ -field of subsets of a set Ω and let μ_n , $n \in \mathbb{N}$ be a sequence of real-valued measures on Σ such that limits*

$$\mu(\Delta) = \lim_{n \rightarrow \infty} \mu_n(\Delta), \quad \Delta \in \Sigma$$

exists in \mathbb{R} . Then μ is a measure on Σ .

PROOF. This is a part of the Nikodym convergence theorem. (Related results, such as the Nikodym boundedness theorem, can be found in [42] and [35].) □

LEMMA 204. *Let \mathbf{H} be a complex separable Hilbert space. Let Σ be a σ -field of subsets of a set Ω . Let $M: \Sigma \rightarrow \mathbf{B}_\infty(\mathbf{H})$ be a function. Let D be a dense subset of Hilbert space \mathbf{H} . The following three conditions are equivalent*

(a) $\Delta \mapsto (h, M(\Delta)h)$ is σ -additive for all $h \in D$;

- (b) $\Delta \mapsto (h, M(\Delta)h)$ is σ -additive for all $h \in \mathbf{H}$;
(c) $\Delta \mapsto M(\Delta)h$ is σ -additive for all $h \in \mathbf{H}$.

PROOF. (a) implies (b): Trivial.

(a) implies (b): Let $h \in \mathbf{H}$. There are $h_n \in D$ such that $\lim_{n \rightarrow \infty} h_n = h$. Using the triangle inequality and the Cauchy-Bunyakovskii-Schwarz inequality we see that $(h, M(\Delta)h) = \lim_{n \rightarrow \infty} (h_n, M(\Delta)h_n)$ for every $\Delta \in \Sigma$. By Proposition 203, $\Delta \mapsto (h, M(\Delta)h)$ is σ -additive.

(b) implies (c): $\Delta \mapsto M(\Delta)h$ is weakly σ -additive by (b) and the polarization formula. The Orlics-Pettis theorem (see e.g. Chapter I of [35]) says that this implies (c). \square

5. Monotone convergence theorem and a partial converse

THEOREM 205. Let (Ω, Σ, μ) be a measure space, and let f and f_1, f_2, \dots be $[0, \infty]$ -valued Σ -measurable functions on Ω satisfying

$$(79) \quad f_1(x) \leq f_2(x) \leq \dots \quad \text{for } \mu\text{-almost all } x.$$

Consider the following conditions:

(a) $f(x) = \lim_{n \rightarrow \infty} f_n(x)$ for μ -almost all x .

(b) $\int_{\Omega} f(x) \mu(dx) = \lim_{n \rightarrow \infty} \int_{\Omega} f_n(x) \mu(dx)$.

Condition (a) implies (b). If f is μ -integrable and $f_n(x) \leq f(x)$ for all n and μ -almost all x , then (b) implies (a).

PROOF. (a) implies (b): Monotone convergence theorem.

(b) implies (a): Let $g(x) = \sup_n f_n(x)$. Then $g(x) \leq f(x)$ for μ -almost all x , and

$$\int_{\Omega} f(x) \mu(dx) = \lim_{n \rightarrow \infty} \int_{\Omega} f_n(x) \mu(dx) = \int_{\Omega} g(x) \mu(dx),$$

where the monotone convergence theorem is used for the second equality. Hence $\int_{\Omega} f(x) - g(x) \mu(dx) = 0$. Hence $f(x) = g(x)$ for μ -almost all x . \square

6. Radon-Nikodym theorem

THEOREM 206 ([23]). If (Ω, Σ, μ) is a σ -finite positive measure space and ν is a complex-valued measure on (Ω, Σ) such that $\nu \ll \mu$, then there is a unique complex-valued function $f \in L_1(\Omega, \Sigma, \mu)$ such that $\nu(\Delta) = \int_{\Delta} f(x) \mu(dx)$ for every $\Delta \in \Sigma$.

This implies in particular that $\nu(\Sigma)$ is a bounded subset of \mathbb{C} . For a measure $\nu: \Sigma \rightarrow \mathbb{C}$ we define the total variation measure $|\nu|: \Sigma \rightarrow [0, \infty)$ by

$$|\nu|(\Delta) = \sup \sum_{k=1}^{\infty} |\nu(\Delta_k)|,$$

the supremum being taken over all measurable partitions $\{\Delta_k\}$ of Δ . This is a positive measure (see e.g. [93]).

THEOREM 207 ([93], Theorem 6.13). If (Ω, Σ, μ) is a σ -finite positive measure space and $f \in L_1(\Omega, \Sigma, \mu)$, and

$$\lambda(\Delta) = \int_{\Delta} f(x) \mu(dx),$$

then

$$|\lambda|(\Delta) = \int_{\Delta} |f(x)| \mu(dx).$$

In particular $\|f\|_1 = |\lambda|(\Omega)$.

7. Strong topological dual of $L_1(\Omega, \Sigma, \mu)$

The following well-known theorem can be found e.g. in appendix B of [23].

THEOREM 208. *Let (Ω, Σ, μ) be a σ -finite measure space. For $\varphi \in L_\infty(\mu)$, define $L_\varphi: L_1(\mu) \rightarrow \mathbb{C}$ by*

$$L_\varphi(f) = \int_{\Omega} \varphi(x)f(x) \mu(dx)$$

The map $\varphi \mapsto L_\varphi$ is an isometric (and linear and topological) isomorphism of $L_\infty(\mu)$ onto the strong topological dual of $L_1(\mu)$ equipped with its dual norm

$$L \mapsto \sup\{|L(f)| : f \in \text{ball}(L_1(\mu))\}.$$

APPENDIX B

A criteria for the density of subspaces

Let \mathbf{X}' be the topological dual of a normed space \mathbf{X} . For a subspace V of \mathbf{X} let V^\perp be the following subspace of \mathbf{X}' :

$$V^\perp = \{x' \in \mathbf{X}' : x'(x) = 0 \forall x \in V\}.$$

For a subspace W of \mathbf{X}' let ${}^\perp W$ be the following subspace of \mathbf{X} :

$${}^\perp W = \{x \in \mathbf{X} : x'(x) = 0 \forall x' \in W\}.$$

PROPOSITION 209. *Let \mathbf{X}' be the topological dual of a normed space \mathbf{X} and let V be a subspace of \mathbf{X} and let W be a subspace of \mathbf{X}' . Then*

- (a) V^\perp is a weak-star closed linear subspace of \mathbf{X}' .
- (b) ${}^\perp W$ is a weakly closed linear subspace of \mathbf{X} .
- (c) The closure of V , the weak closure of V and ${}^\perp(V^\perp)$ are equal.
- (d) $({}^\perp W)^\perp$ is the weak-star closure of W in \mathbf{X}' .

Consequently, the following conditions for V are equivalent

- V is dense in \mathbf{X} .
- V is weakly dense in \mathbf{X} .
- $V^\perp = \{0\}$.

and the following conditions for W are equivalent

- W is weak-star dense in \mathbf{X}' .
- ${}^\perp W = \{0\}$.

References: Chapter 4 in [94], Section 6.3 of Chapter II in [12]

APPENDIX C

Weak-star topology on dual Banach spaces

The topological dual of a Banach space is called a dual Banach space.

DEFINITION 210. Let X be a normed space. Let Y be a closed linear subspace of X . The quotient norm on X/Y is defined by

$$\|x + Y\| = \inf\{\|x + y\| : y \in Y\}.$$

Let X' be the topological dual of X . The dual norm on X' is defined by

$$\|x'\| = \sup\{|x'(x)| : x \in \text{ball}(X)\},$$

where $\text{ball}(X)$ is the closed unit ball of X .

In Section III.4 of [23], it is shown that the quotient norm is a norm, and that X/Y , equipped with the quotient norm, is a Banach space, if X is a Banach space.

In Section III.5 of [23], it is shown that X' , equipped with the dual norm, is a Banach space.

In Section III.10 of [23], it is shown that, if $p: X \rightarrow X/Y$ is the natural map, composition with p provides a linear isometry from $(X/Y)'$ into X' with range

$$Y^\perp = \{x' \in X' : Y \subset \ker(x')\}.$$

Thus $(X/Y)'$ and Y^\perp can be identified.

The following proposition says in particular that a weak-star closed subspace of a dual Banach space is again a dual Banach space.

PROPOSITION 211. *Let X be a normed space with topological dual X' . Let Z be a closed linear subspace of (X', weak^*) , and let $Y = X/({}^\perp Z)$. Then*

- (i) $Y' = Z$ as sets.
- (ii) The dual norm on Y' equals the restriction to Z of the dual norm on X' .
- (iii) The weak-star topology on Y' equals the topology on Z induced by the weak-star topology on X' .

PROOF. This follows from $({}^\perp Z)^\perp = Z$ and Theorem 2.2 of Section V.2 in [23]. □

PROPOSITION 212. *Let X be a Banach space with topological dual X' . Then (X', weak^*) is sequentially complete.*

PROOF. Let $L_n, n \in \mathbb{N}$ be a Cauchy sequence in (X', weak^*) . Because \mathbb{C} is complete, $\lim_{n \rightarrow \infty} L_n(f)$ exists for every $f \in X$. By the Banach-Steinhaus theorem, there exists an $L \in X'$ such that

$$L(f) = \lim_{n \rightarrow \infty} L_n(f) \quad \forall f \in X.$$

This means that $L_n \rightarrow L$ in (X', weak^*) . □

PROPOSITION 213. *Let X be a Banach space with dual X' . For a sequence $L_n, n \in \mathbb{N}$ in X' to converge in (X', weak^*) , it is necessary and sufficient that*

- the sequence $(\|L_n\|)$ is bounded, and
- the limit $\lim_{n \rightarrow \infty} L_n(f)$ exists for all f in a dense subset \mathfrak{D} of X .

PROOF. The necessity of the first condition is a direct consequence of the uniform boundedness principle. The necessity of the second condition is obvious. Let $M = \sup\{\|L_n\| : n \in \mathbb{N}\}$. For $f \in \mathbf{X}$ and $g \in \mathfrak{D}$,

$$(80) \quad \begin{aligned} |L_n(f) - L_m(f)| &= |L_n(f) - L_n(g)| \\ &\quad + |L_n(g) - L_m(g)| + |L_m(g) - L_m(f)| \\ &\leq 2M\|f - g\| + |L_n(g) - L_m(g)|. \end{aligned}$$

Because \mathfrak{D} is a dense subset of \mathbf{X} and $L_n(g)$ is a Cauchy sequence for every $g \in \mathfrak{D}$, (80) implies that $L_n(f)$ is a Cauchy sequence for every $f \in \mathbf{X}$. Because \mathbb{C} is complete, this implies that $\lim_{n \rightarrow \infty} L_n(f)$ exists for every $f \in \mathbf{X}$. By Proposition 212, (L_n) converges in $(\mathbf{X}', \text{weak}^*)$. \square

PROPOSITION 214. *Let \mathbf{X}, \mathbf{Y} be Banach spaces with duals \mathbf{X}' and \mathbf{Y}' respectively. If \mathbf{Y} is separable and $\rho: \mathbf{X}' \rightarrow \mathbf{Y}'$ is a linear isometry and $\rho: (\mathbf{X}', \text{weak}^*) \rightarrow (\mathbf{Y}', \text{weak}^*)$ is continuous, then $\text{range}(\rho)$ is closed in $(\mathbf{Y}', \text{weak}^*)$.*

PROOF. Let (L_n) be a sequence in \mathbf{X}' such that $(\rho(L_n))$ converges in $(\mathbf{Y}', \text{weak}^*)$. By the uniform boundedness principle, the sequence $(\|\rho(L_n)\|)$ is bounded. Because ρ is isometric, the sequence $(\|L_n\|)$ is bounded. For each $y \in \mathbf{Y}$, the linear form $L \mapsto \rho(L)(y)$ on $(\mathbf{X}', \text{weak}^*)$ is continuous. Therefore a linear mapping $\rho': \mathbf{Y} \rightarrow \mathbf{X}$, satisfying $L(\rho'(y)) = \rho(L)(y)$ for all $L \in \mathbf{X}'$ and $y \in \mathbf{Y}$, exists. If $L \in \mathbf{X}'$ is zero on the subspace $\text{range}(\rho')$ of \mathbf{X} , then $\rho(L) = 0$; hence $L = 0$. By Proposition 209, $\text{range}(\rho')$ is a dense subspace of \mathbf{X} . For each $y \in \mathbf{Y}$, the sequence $(\rho(L_n)(y))$ converges. Hence the sequence $(L_n(\rho'(y)))$ converges; i.e. $(L_n(x))$ converges for $x \in \text{range}(\rho')$. By proposition 213, this implies that (L_n) converges in $(\mathbf{X}', \text{weak}^*)$. Hence $\text{range}(\rho)$ is sequentially closed in $(\mathbf{Y}', \text{weak}^*)$. The proof is completed by the following corollary of the Krein-Smulian theorem: Corollary 12.7 of Chapter V in [23] says: A convex subset of the topological dual of a separable Banach space is weak-star sequentially closed if and only if it is weak-star closed. \square

LEMMA 215. *Let \mathbf{X}, \mathbf{Y} be Banach spaces with topological duals \mathbf{X}' and \mathbf{Y}' respectively. Let $\rho: \mathbf{X}' \rightarrow \mathbf{Y}'$ be linear. The following conditions are equivalent:*

- (a) ρ is weak-star continuous.
- (b) $\rho = L'$ for some bounded operator $L: \mathbf{Y} \rightarrow \mathbf{X}$.

PROOF. (a) implies (b): Let $y \in \mathbf{Y}$. The linear form $x' \mapsto \rho(x')(y)$ on \mathbf{X}' is weak-star continuous. Hence there is an $L(y) \in \mathbf{X}$ such that $\rho(x')(y) = x'(L(y))$ for all $x' \in \mathbf{X}'$. The linearity of $y \mapsto \rho(x')(y)$ implies the linearity of $y \mapsto L(y)$. The weak continuity of $y \mapsto \rho(x')(y)$ implies the weak continuity of $y \mapsto L(y)$. By Theorem VI.1.1 in [23], every weakly continuous map between two Banach spaces is bounded. Hence L is bounded. (b) implies (a): If $x'_i \rightarrow x' \in (\mathbf{X}', \text{weak}^*)$ then $x'_i(L(y)) \rightarrow x'(L(y))$ for every $y \in \mathbf{Y}$. Hence $\rho(x'_i)(y) \rightarrow \rho(x')(y)$ for every $y \in \mathbf{Y}$. I.e. $\rho(x'_i) \rightarrow \rho(x')$ with respect to the weak-star topology of \mathbf{Y}' . Hence ρ is weak-star continuous. \square

PROPOSITION 216. *Let \mathbf{X}, \mathbf{Y} be Banach spaces with topological duals \mathbf{X}' and \mathbf{Y}' respectively. Let $\rho: \mathbf{X}' \rightarrow \mathbf{Y}'$ be linear and weak-star continuous. Then ρ is bounded. If, moreover, ρ is injective and has weak-star closed range, then $\rho: \mathbf{X}' \rightarrow \text{range}(\rho)$ has a bounded and weak-star continuous inverse.*

PROOF. By Lemma 215, ρ is the dual of some bounded linear map $L: \mathbf{Y} \rightarrow \mathbf{X}$. From $\rho(x') = x' \circ L$ for $x' \in \mathbf{X}'$, it follows that ρ is bounded.

Assume now that ρ is injective and has weak-star closed range. The injectivity of ρ implies that $\text{range}(L)$ is dense in \mathbf{X} . By Theorem VI.1.10 in [23], $\text{range}(L)$ is closed.

Hence $\text{range}(L) = \mathsf{X}$. By Proposition VI.1.8 in [23],

$$\ker(L) = {}^\perp(\text{range}(\rho))$$

By the inverse mapping theorem (Theorem III.15.5 in [23]), $L: \mathsf{Y}/\ker(L) \rightarrow \mathsf{X}$ has a bounded linear inverse. By Proposition VI.1.4 in [23], ρ has a bounded linear inverse, and $\rho^{-1} = (L^{-1})'$. By Proposition 211,

$$\text{range}(\rho) = (\mathsf{Y}/\ker(L))'$$

By Lemma 215, $\rho^{-1}: \text{range}(\rho) \rightarrow \mathsf{X}'$ is weak-star continuous. □

APPENDIX D

Bounded sesquilinear forms on Hilbert spaces

Let \mathbf{H}, \mathbf{K} be Banach spaces. Let $s: \mathbf{H} \times \mathbf{K} \rightarrow \mathbb{C}$ be a sesquilinear form. The following three conditions are equivalent:

- There exists an $M > 0$ such that $|s(h, k)| \leq M\|h\|\|k\|$ for all h, k .
- $s: \mathbf{H} \times \mathbf{K} \rightarrow \mathbb{C}$ is continuous.
- s is continuous in $(0, 0)$.

If these conditions are satisfied then s is called bounded and M is called a bound for s .

PROPOSITION 217. *Let \mathbf{H}, \mathbf{K} be Banach spaces. A sesquilinear form $s: \mathbf{H} \times \mathbf{K} \rightarrow \mathbb{C}$ is bounded if, and only if, the following conditions are satisfied:*

- For each $h \in \mathbf{H}$, the linear form s_h on \mathbf{K} , defined by $s_h(k) = s(h, k)$, is bounded;
- For each $k \in \mathbf{K}$, the linear form s^k , on \mathbf{H} , defined by $s^k(h) = \overline{s(k, h)}$, is bounded.

PROOF. Assume that s_h, s^k are bounded for each h, k . Equivalently: For each $h \in \mathbf{H}$, the linear form s_h on \mathbf{K} is continuous, and the set of continuous linear forms $\{s_h : h \in \text{ball}(\mathbf{H})\}$ is simply (i.e. pointwise) bounded. By the uniform boundedness principle, every simply bounded set of continuous linear forms on a Banach space is bounded uniformly on every bounded set. Thus, $\{s_h : h \in \text{ball}(\mathbf{H})\}$ is bounded uniformly on $\text{ball}(\mathbf{K})$. There exists an $M > 0$ such that $|s(h, k)| \leq M\|h\|\|k\|$ for all h, k with $\|h\| = \|k\| = 1$. Hence $|s(h, k)| \leq M\|h\|\|k\|$ for all h, k . \square

The Riesz representation theorem can be used to prove the following theorem (see e.g. [23]):

THEOREM 218. *If \mathbf{H}, \mathbf{K} are Hilbert spaces and $u: \mathbf{H} \times \mathbf{K} \rightarrow \mathbb{C}$ is a bounded sesquilinear form with bound M , then there are unique bounded operators $\mathcal{A}: \mathbf{H} \rightarrow \mathbf{K}$ and $\mathcal{B}: \mathbf{K} \rightarrow \mathbf{H}$ such that*

$$s(h, k) = (\mathcal{A}h, k) = (h, \mathcal{B}k) \quad \forall h \in \mathbf{H}, k \in \mathbf{K}$$

and $\|\mathcal{A}\| = \|\mathcal{B}\| \leq M$.

APPENDIX E

Hilbertian semi-norms

P. Jordan and J. von Neumann proved in [60] that a norm $\|\cdot\|$ on a vector space over \mathbb{C} is generated by an inner-product if and only if it satisfies the parallelogram law

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2 \quad \forall x, y.$$

Consequently, a semi-norm p on a vector space over \mathbb{C} is generated by a semi inner-product if and only if it satisfies the parallelogram law

$$p(x + y)^2 + p(x - y)^2 = 2p(x)^2 + 2p(y)^2 \quad \forall x, y.$$

We prove that this is still true if p is not assumed (a priori) to satisfy the triangle inequality; the triangle inequality is implied by the remaining properties,

$$p(x) \geq 0 \quad \text{and} \quad p(\lambda x) = |\lambda|p(x) \quad \text{for all vectors } x \text{ and scalars } \lambda,$$

satisfied by all semi-norms, together with the parallelogram law.

REMARK 219. A quadratic form on a vector space V is a function of the form $x \mapsto b(x, x)$, where b is a bilinear form on V . It is known [50] that a function $q: V \rightarrow \mathbb{R}$ satisfying

$$q(x + y) + q(x - y) = 2q(x) + 2q(y), \quad q(\lambda x) = |\lambda|^2 q(x) \quad \forall x, y \in V, \lambda \in \mathbb{C}$$

is not necessarily a quadratic form.

LEMMA 220. *Let \mathbf{X} be a vector space over \mathbb{R} . If $\ell: \mathbf{X} \rightarrow \mathbb{R}$ satisfies $\ell(x+y) = \ell(x)+\ell(y)$ for all $x, y \in \mathbf{X}$, then $\ell(cx) = c\ell(x)$ for all $x \in \mathbf{X}$ and $c \in \mathbb{Q}$.*

PROOF. The assumptions imply that $\ell(0) = 0$ and hence $\ell(-x) = -\ell(x)$ for all $x \in \mathbf{X}$. Let $n \in \mathbb{N}$ and $x \in \mathbf{X}$. The assumptions imply $\ell(nx) = n\ell(x)$ and $n\ell(x/n) = \ell(x)$. The lemma is proven by combining these results. \square

LEMMA 221. *Let \mathbf{X} be a vector space over \mathbb{C} . If $q: \mathbf{X} \rightarrow \mathbb{C}$ satisfies*

- $q(x) \in \mathbb{R}$ for all $x \in \mathbf{X}$;
- $q(ix) = q(x)$ for all $x \in \mathbf{X}$;
- $q(x + y) + q(x - y) = 2q(x) + 2q(y)$ for all $x, y \in \mathbf{X}$,

then $s: \mathbf{X} \times \mathbf{X} \rightarrow \mathbb{C}$, defined by

$$s(y, x) = \frac{1}{4}\{q(x + y) - q(x - y) + iq(x + iy) - iq(x - iy)\},$$

satisfies

- $q(x) = s(x, x)$
- $s(x, y) = \overline{s(y, x)}$
- $s(x, y + z) = s(x, y) + s(x, z)$
- $s(x, cy) = cs(x, y)$

for all $x, y, z \in \mathbf{X}$ and $c \in \mathbb{Q} + i\mathbb{Q}$.

PROOF. From $q(x) \in \mathbb{R}$ follows $\operatorname{Re} s(y, x) = \frac{1}{4}\{q(x+y) - q(x-y)\}$. By Theorem 0.1 of [50], $\operatorname{Re} s$ is symmetric, biadditive and satisfies $q(x) = \operatorname{Re} s(x, x)$. From the parallelogram law follows $q(0) = 0$ and $q(-y) = q(y)$. From $q(ix) = q(x)$ and $q(-x) = q(x)$ follows

$s(x, y) = \overline{s(y, x)}$. Hence $s(x, x) \in \mathbb{R}$. Hence $q(x) = s(x, x)$. The rest follows from Lemma 220 \square

LEMMA 222. *If $a, b \geq 0$ then*

$$2ab = \inf\{(ca)^2 + (\frac{1}{c}b)^2 : c \in \mathbb{Q}\}.$$

PROOF. $a^2 + b^2 - 2ab = (a - b)^2 \geq 0$. Hence $2ab \leq a^2 + b^2$ for all $a, b \in \mathbb{R}$. Hence

$$2ab \leq ca^2 + \frac{1}{c}b^2 \quad \forall a, b \in \mathbb{R}, c > 0.$$

First we consider the case $ab = 0$: If $a = 0$, then we let c approach ∞ and if $b = 0$, we let c approach 0.

Now assume that $ab \neq 0$. We have

$$2ab = \min\{(ca)^2 + (\frac{1}{c}b)^2 : c > 0\};$$

the minimum is attained at $c = \sqrt{b/a}$. This implies the result. \square

DEFINITION 223. A semi-norm p on a vector space \mathbf{X} over is said to be Hilbertian if there exists a positive sesquilinear form s on $\mathbf{X} \times \mathbf{X}$ such that $p(x) = \sqrt{s(x, x)}$ for all $x \in \mathbf{X}$.

THEOREM 224. *Let \mathbf{X} be a vector space over \mathbb{C} . If $q: \mathbf{X} \rightarrow \mathbb{C}$ satisfies*

- (i) $q(x) \geq 0$ for all $x \in \mathbf{X}$;
 - (ii) $q(cx) = |c|^2 q(x)$ for all $x \in \mathbf{X}$ and $c \in \mathbb{C}$;
 - (iii) $q(x + y) + q(x - y) = 2q(x) + 2q(y)$ for all $x, y \in \mathbf{X}$,
- then $p: \mathbf{X} \rightarrow [0, \infty)$, defined by $p(x) = \sqrt{q(x)}$, is a Hilbertian semi-norm.

PROOF. By squaring the triangle inequality

$$p(x + y) \leq p(x) + p(y)$$

we get the equivalent inequality

$$q(x + y) \leq q(x) + q(y) + 2p(x)p(y).$$

By Lemma 222, this inequality is satisfied if, and only if,

$$(81) \quad q(x + y) - q(x) - q(y) \leq c^2 q(x) + \frac{1}{c^2} q(y)$$

for all $c \in \mathbb{Q}$. By (iii), the left hand side of (81) is

$$\frac{1}{2}\{q(x + y) - q(x - y)\}$$

and by (ii) and (iii) the right hand side of (81) is

$$q(cx) + q(\frac{1}{c}y) = \frac{1}{2}\{q(cx + \frac{1}{c}y) + q(cx - \frac{1}{c}y)\}.$$

Hence p satisfies the triangle inequality if, and only if,

$$(82) \quad q(x + y) - q(x - y) \leq q(cx + \frac{1}{c}y) + q(cx - \frac{1}{c}y) \quad \forall x, y \in \mathbf{X}, c \in \mathbb{Q}.$$

By Lemma 221,

$$q(x + y) - q(x - y) = q(cx + \frac{1}{c}y) - q(cx - \frac{1}{c}y) \quad \forall x, y \in \mathbf{X}, c \in \mathbb{Q}.$$

Together with (i), this shows that (82) is satisfied. Hence p satisfies the triangle inequality. Hence also

$$|p(x) - p(y)| \leq p(x - y) \quad \forall x, y \in \mathbf{X}.$$

In particular,

$$|p(x + \alpha y) - p(x + \beta y)| \leq |\alpha - \beta| p(y)$$

for $x, y \in \mathbf{X}$ and $\alpha, \beta \in \mathbb{C}$. This can be used to prove that s , defined in Lemma 221, satisfies $s(x, cy) = cs(x, y)$ for all $c > 0$. It is easily seen that s is a sesquilinear form otherwise. \square

THEOREM 225. *Let \mathbf{X} be an inner-product space over \mathbb{C} . The inner-product is denoted by (\cdot, \cdot) . If $q: \mathbf{X} \rightarrow \mathbb{C}$ satisfies*

- (i) $q(x) \geq 0$, for all $x \in \mathbf{X}$;
 - (ii) $q(x_n) \rightarrow 0$ for every null-sequence (x_n) in \mathbf{X} ;
 - (iii) $q(cx) = |c|^2 q(x)$ for all $x \in \mathbf{X}$ and $c \in \mathbb{C}$;
 - (iv) $q(x + y) + q(x - y) = 2q(x) + 2q(y)$ for all $x, y \in \mathbf{X}$,
- then there is an $\mathcal{A} \in \mathbf{B}_+(\mathbf{H})$ such that $q(x) = (x, \mathcal{A}x)$ for all $x \in \mathbf{X}$. If, moreover, $q(x) \leq M(x, x)$ for all $x \in \mathbf{X}$ then $\|\mathcal{A}\|_\infty \leq M$.

PROOF. By Theorem 224, there exists a positive sesquilinear form s on $\mathbf{X} \times \mathbf{X}$ such that $q(x) = s(x, x)$ for all $x \in \mathbf{X}$. We will show that for each $x \in \mathbf{X}$, the linear form s_x on \mathbf{X} , defined by $s_x(y) = s(x, y)$, is continuous in 0. Let $p(x) = \sqrt{q(x)}$. Then p satisfied the triangle inequality. Hence

$$|p(x \pm y_n) - p(x)| \leq p(y_n), \quad |p(x \pm iy_n) - p(x)| \leq p(y_n).$$

From this, together with condition (ii), it follows that $q(x \pm y_n) \rightarrow q(x)$ and $q(x \pm iy_n) \rightarrow q(x)$ for all x and null-sequences (y_n) in \mathbf{X} . From this, together with

$$\begin{aligned} \operatorname{Re} s_x(y) &= \frac{1}{4}\{q(x + y) - q(x - y)\}, \\ \operatorname{Im} s_x(y) &= \frac{1}{4}\{q(x + iy) - q(x - iy)\}, \end{aligned}$$

it follows that $s_x(y_n) \rightarrow 0$. Hence s_x is continuous in 0 for all x . By Proposition 217, this implies that s is bounded. Let \mathbf{H} be the Hilbert space completion of \mathbf{X} and let s^e be the unique extension of s to a bounded sesquilinear form on $\mathbf{H} \times \mathbf{H}$. By Theorem 218, there exists an $\mathcal{A} \in \mathbf{B}(\mathbf{H})$ such that $s^e(x, y) = (x, \mathcal{A}y)$. Hence $q(x) = (x, \mathcal{A}x)$ for all $x \in \mathbf{X}$.

Assume now that $q(x) \leq M(x, x)$ for all $x \in \mathbf{X}$. In e.g. [54], it is shown that a sesquilinear form is bounded if and only if the associated quadratic form is bounded and that for symmetric sesquilinear forms the two norms coincide. Hence s and s^e are bounded with norm $\leq M$. By Theorem 218, there exists an $\mathcal{A} \in \mathbf{B}(\mathbf{H})$ with norm $\leq M$ such that $s^e(x, y) = (x, \mathcal{A}y)$. \square

APPENDIX F

Bargmann space

Define measure μ on \mathbb{C} by $\mu(dz) = \pi^{-1}e^{-|z|^2}d\operatorname{Re}(z)d\operatorname{Im}(z)$. The Bargmann space \mathbf{H}_B is the Hilbert space of entire analytic functions that are square-integrable with respect to μ . This space is introduced in [9]. It is a functional Hilbert space; it contains elements e_z such that $(e_z, \varphi) = \varphi(z)$ for all $z \in \mathbb{C}$ and $\varphi \in \mathbf{H}_B$. Convergence in \mathbf{H}_B implies uniform convergence on compacta.

The family of functions $u_n(z) = \frac{z^n}{\sqrt{n!}}$, $n \in \mathbb{N}_0$ is an orthonormal basis of \mathbf{H}_B . The linear transformation $\varphi \mapsto [\varphi]$, the μ -equivalence class of φ , maps \mathbf{H}_B isometrically onto the closure of $\operatorname{span}\{u_n : n \in \mathbb{N}_0\}$ in $L_2(\mathbb{C}, \mu)$. The equivalence class of e_z is given by

$$[e_z] = \sum_{n=0}^{\infty} \overline{u_n(z)} u_n.$$

It is easily seen that the entire analytic representant is $w \mapsto e^{\bar{z}w}$. Hence

$$e_z(w) = e^{\bar{z}w} \quad \forall z, w \in \mathbb{C}.$$

The integral transform

$$f \mapsto (\mathbf{U}_B f)(z) = \int_{-\infty}^{\infty} A(z, q) f(q) dq,$$

with

$$A(z, q) = \pi^{-1/4} \exp\{-\frac{1}{2}(z^2 + q^2) + \sqrt{2}zq\},$$

establishes a unitary mapping from $L_2(\mathbb{R})$ onto \mathbf{H}_B . If φ_n is the n 'th Hermite basis function then $\mathbf{U}_B[\varphi_n] = u_n$.

If \mathcal{A} and \mathcal{A}^* are the creation and annihilation operators on $L_2(\mathbb{R})$, then

$$\mathbf{U}_B \mathcal{A} \mathbf{U}_B^* = \frac{d}{dz} \quad \text{and} \quad \mathbf{U}_B \mathcal{A}^* \mathbf{U}_B^* = \mathcal{Z},$$

where \mathcal{Z} denotes the operator of multiplication with the identity function. Hence $\mathcal{Z}^* = \frac{d}{dz}$ and $\mathcal{Z}^* e_w = \bar{w} e_w$.

APPENDIX G

C^* -algebras

A $*$ -subalgebra of $B_\infty(H)$ is a linear subspace A of $B_\infty(H)$ with the following properties:

- $\mathcal{A}, \mathcal{B} \in A$ implies $\mathcal{A}\mathcal{B} \in A$,
- $\mathcal{A} \in A$ implies $\mathcal{A}^* \in A$.

A unital $*$ -subalgebra is a $*$ -subalgebra that contains the identity operator. A (unital) C^* -subalgebra is a norm-closed (unital) $*$ -subalgebra.

1. Von Neumann algebras

A von Neumann algebra is a SOT-closed unital C^* -subalgebra of $B_\infty(H)$. The von Neumann algebra generated by a subset $A \subset B_\infty(H)$ is the smallest von Neumann algebra contained in $B_\infty(H)$ and containing A .

PROPOSITION 226 (Proposition 4.8 of Chapter IX in [23]). *A von Neumann algebra is the norm closed linear span of its projections.*

A C^* -subalgebra A of $B_\infty(H)$ is called non-degenerate if for every non-zero $f \in H$ there exists an $\mathcal{A} \in A$ such that $\mathcal{A}f \neq 0$. Example: Unital C^* -algebras are non-degenerate.

THEOREM 227 (von Neumann's double commutant theorem). *A non-degenerate C^* -subalgebra A of $B_\infty(H)$ is SOT-closed if and only if $A'' = A$.*

Consequences are:

- The SOT-closure of a non-degenerate C^* -subalgebra A of $B_\infty(H)$ is A'' .
- A C^* -subalgebra A of $B_\infty(H)$ is a von Neumann algebra if and only if $A'' = A$.
- The commutant A' of a subset $A \subset B_\infty(H)$ is a von Neumann algebra.
- The von Neumann algebra generated by a self-adjoint subset $A \subset B_\infty(H)$ equals A'' .

The intersection of the closed unit ball of a Banach space with a subset A is denoted by $\text{ball}(A)$. The following result is a part of Kaplansky's density theorem.

THEOREM 228. *Let A be a non-degenerate C^* -subalgebra of $B_\infty(H)$. The SOT-closure of $\text{ball}(A)$ is $\text{ball}(A'')$.*

THEOREM 229 (Theorem 45.6 in [25]). *A C^* -subalgebra of $B_\infty(H)$ is weak-star closed if, and only if, it contains the supremum of every norm-bounded increasing net of self-adjoint operators contained in the algebra.*

2. Commutative von Neumann algebras

PROPOSITION 230 ([100], Chapter III, Proposition 1.21). *Every commutative von Neumann algebra of operators on a separable Hilbert space is generated by a single bounded self-adjoint operator.*

3. Maximal commutative von Neumann algebras

A commutative subalgebra is said to be maximal if it is not contained in any other commutative subalgebra.

PROPOSITION 231 ([84], Section 7 of Chapter II). *Every commutative subalgebra is contained in a maximal commutative subalgebra.*

PROPOSITION 232 ([112], Proposition 21.2). *A C^* -subalgebra A of $B_\infty(H)$ is a maximal commutative von Neumann algebra if and only if $A = A'$.*

DEFINITION 233. Let (Ω, Σ, μ) be a σ -finite measure space. Let $A_\mu \subset B_\infty(L_2(\Omega, \mu))$ be defined by

$$A_\mu = \{\mathcal{M}_\varphi : \varphi \in L_\infty(\Omega, \mu)\},$$

where \mathcal{M}_φ is the operator of multiplication with φ .

PROPOSITION 234 ([25]). *Let (Ω, Σ, μ) be a σ -finite measure space. A_μ is a maximal commutative von Neumann algebra:*

$$A_\mu = A'_\mu = A''_\mu.$$

DEFINITION 235. Let H, K be two Hilbert spaces. A subset A_1 of $B_\infty(H)$ and a subset A_2 of $B_\infty(K)$ are called spatially isomorphic if there is a unitary operator $U: H \rightarrow K$ such that $A_2 = UA_1U^*$.

THEOREM 236 (Theorem 14.5 in [25]). *Let H be a separable Hilbert space. Every maximal commutative von Neumann algebra contained in $B_\infty(H)$ is spatially isomorphic to an algebra A_μ for a finite positive regular Borel measure μ on a compact metric space K with $\text{supp}(\mu) = K$.*

LEMMA 237. *Let \mathcal{A} be a closed densely defined operator on Hilbert space H . Then:*

- (i.) $\mathcal{D}(\mathcal{A})$ is a Hilbert space with inner product $(g, h)_\mathcal{A} = (g, h) + (\mathcal{A}g, \mathcal{A}h)$.
- (ii.) $\mathcal{I} + \mathcal{A}^*\mathcal{A}$ has a bounded linear inverse, and $\mathcal{A}^*\mathcal{A}$ is self-adjoint. We have $\mathcal{D}(\mathcal{A}^*\mathcal{A}) = \{(\mathcal{I} + \mathcal{A}^*\mathcal{A})^{-1}h : h \in H\}$.
- (iii.) $\mathcal{D}(\mathcal{A}) = \mathcal{D}(|\mathcal{A}|) = \mathcal{D}((\mathcal{I} + \mathcal{A}^*\mathcal{A})^{1/2})$.
- (iv.) $\ker(\mathcal{A}) = \ker(|\mathcal{A}|) = \ker(\mathcal{A}^*\mathcal{A})$, where $|\mathcal{A}| = \sqrt{\mathcal{A}^*\mathcal{A}}$.

PROOF. (i): We will show that $\mathcal{D}(\mathcal{A})$ is complete: If (h_n) is a Cauchy sequence in $\mathcal{D}(\mathcal{A})$ with respect to $(\cdot, \cdot)_\mathcal{A}$ then $(\{h_n, \mathcal{A}h_n\})$ is a Cauchy sequence in $H \times H$. Because \mathcal{A} is closed, the sequence has a limit $(h, \mathcal{A}h)$ with $h \in \mathcal{D}(\mathcal{A})$. This implies that $(\|h_n - h\|_\mathcal{A})$ converges to 0.

(ii): This is part of Theorem 2 of Section 3 in Chapter VII of [111].

(iii): The first equality follows from $\mathcal{A}^*\mathcal{A} = |\mathcal{A}|^2$ and Theorem 5.40 in [108]. The second equality follows from the spectral theorem.

(iv): The first equality is part of Theorem 5.39 in [108]. The second equality follows from $\mathcal{A}^*\mathcal{A} = |\mathcal{A}|^2$. \square

APPENDIX H

Operator ranges

In [44] a survey of the theory of operator ranges (the ranges of bounded operators in Hilbert space) is given. We use the following result:

THEOREM 238 ([38]). *Let \mathcal{A} and \mathcal{B} be bounded operators on Hilbert space \mathbf{H} . The following conditions are equivalent:*

- (a) $\text{range}(\mathcal{A}) \subset \text{range}(\mathcal{B})$.
- (b) $\mathcal{A}^*\mathcal{A} \leq \lambda^2\mathcal{B}^*\mathcal{B}$ for some $\lambda > 0$.
- (c) $\mathcal{A} = \mathcal{B}\mathcal{C}$ for some bounded operator \mathcal{C} on \mathbf{H} .

COROLLARY 239. *If \mathcal{P} is an operator of orthogonal projection on Hilbert space \mathbf{H} and \mathcal{A} is a bounded operator on \mathbf{H} such that $|\mathcal{A}| \leq \lambda\mathcal{P}$ for some $\lambda > 0$, then $\text{range}(\mathcal{A}) \subset \text{range}(\mathcal{P})$, or equivalently, $\mathcal{A} = \mathcal{P}\mathcal{A}$. If $\text{range}(\mathcal{P})$ is one-dimensional and \mathcal{A} is self-adjoint, this implies that $\mathcal{A} = c\mathcal{P}$ for some scalar c .*

PROOF. We have $|\mathcal{A}|^2 = \mathcal{A}^*\mathcal{A}$. From $|\mathcal{A}| \leq \lambda\mathcal{P}$ follows $|\mathcal{A}| \leq \lambda\mathcal{I}$ and hence that

$$\mathcal{A}^*\mathcal{A} = \sqrt{|\mathcal{A}|}|\mathcal{A}|\sqrt{|\mathcal{A}|} \leq \lambda\sqrt{|\mathcal{A}|}\mathcal{I}\sqrt{|\mathcal{A}|} = \lambda\|\mathcal{A}\| \leq \lambda^2\mathcal{P} = \lambda^2\mathcal{P}^*\mathcal{P}.$$

The rest follows from Theorem 238. □

APPENDIX I

Measurable Banach space-valued functions

Let (Ω, Σ, μ) be a finite measure space and X a Banach space. A function $\Psi: \Omega \rightarrow \mathsf{X}$ is μ -measurable if there is a sequence of simple functions (Ψ_n) with

$$\lim_{n \rightarrow \infty} \|\Psi_n(x) - \Psi(x)\| = 0$$

for μ -almost all x . A function $\Psi: \Omega \rightarrow \mathsf{X}$ is weakly μ -measurable if, for every continuous linear form $\ell: \mathsf{X} \rightarrow \mathbb{C}$, the composition $\ell \circ \Psi$ is μ -measurable. Every μ -measurable function is weakly μ -measurable.

PROPOSITION 240 ([35]). *Let (Ω, Σ, μ) be a measure space and X a separable Banach space. Every weakly μ -measurable function $\Psi: \Omega \rightarrow \mathsf{X}$ is μ -measurable.*

PROPOSITION 241 ([35]). *Let (Ω, Σ, μ) be a finite measure space, let X be a Banach space and let $\Psi: \Omega \rightarrow \mathsf{X}$ be a μ -measurable function. There exists a sequence of simple functions $\Psi_n: \Omega \rightarrow \mathsf{X}$ such that*

- $\Psi_n(x) \rightarrow \Psi(x)$ for μ -almost all $x \in \Omega$,
- $\|\Psi_n(x)\| \leq 2\|\Psi(x)\|$ for all x and n .

Let (Ω, Σ) be a measurable space and X a Banach space. A function $\Psi: \Omega \rightarrow \mathsf{X}$ is Σ -measurable if $\Psi^{-1}(\Delta) \in \Sigma$ for every Borel subset Δ of X . We have the following relation between these concepts.

PROPOSITION 242. *Let (Ω, Σ, μ) be a finite measure space, let X be a Banach space. Every μ -measurable function is equal μ -almost everywhere to a Σ -measurable function.*

PROOF. Let $\Psi: \Omega \rightarrow \mathsf{X}$ be a μ -measurable function. There is a sequence (Ψ_n) of simple functions and a μ -null set $\mathfrak{N} \in \Sigma$ such that $\lim_{n \rightarrow \infty} \|\Psi(x) - \Psi_n(x)\| = 0$ for all $x \in \Omega \setminus \mathfrak{N}$. Each Ψ_n is Σ -measurable. Hence $x \mapsto \Psi_n(x)1_{\Omega \setminus \mathfrak{N}}(x)$ is Σ -measurable. Let $\Phi(x) = \lim_{n \rightarrow \infty} \Psi_n(x)1_{\Omega \setminus \mathfrak{N}}(x)$. The pointwise limit of a sequence of Σ -measurable functions is Σ -measurable; hence Φ is Σ -measurable. We have $\Psi(x) = \Phi(x)$ for μ -almost all x . \square

LEMMA 243. *Let H be a separable complex Hilbert space and let (Ω, μ) be a measure space. Let $\Phi: \Omega \rightarrow \mathsf{B}_+(\mathsf{H})$ be a function. The following conditions are equivalent*

- (i) $(h, \Phi(\cdot)h)$ is μ -measurable for every $h \in \mathsf{H}$.
- (ii) $\Phi(\cdot)h: \Omega \rightarrow \mathbb{R}$ is weakly μ -measurable for every $h \in \mathsf{H}$.
- (iii) $\Phi(\cdot)h: \Omega \rightarrow \mathbb{R}$ is μ -measurable for every $h \in \mathsf{H}$.

Function Φ is called SOT μ -measurable if these conditions are satisfied.

PROPOSITION 244. *Let (Ω, Σ, μ) be a finite measure space, let H be a separable Hilbert space and let $\Psi, \Phi: \Omega \rightarrow \mathsf{B}_\infty(\mathsf{H})$ be (SOT) μ -measurable functions. The function $\Xi: \Omega \rightarrow \mathsf{B}_\infty(\mathsf{H})$, defined by*

$$\Xi(x) = \Phi(x)\Psi(x),$$

is (SOT) μ -measurable.

PROOF. Assume first that Φ and Ψ are SOT μ -measurable. Let (φ_n) be an orthonormal basis of \mathbf{H} . Then

$$(h, \Xi(x)h) = \sum_n (h, \Phi(x)\varphi_n)(\varphi_n, \Psi(x)h).$$

Hence Ξ is SOT μ -measurable.

Assume now that Φ and Ψ are μ -measurable. Assume first that Φ, Ψ are simple functions. Then Ξ is also a simple function and measurable by definition. Now assume that Φ, Ψ are not simple. Let $\Phi_n, \Psi_n: \Omega \rightarrow \mathbf{B}_\infty(\mathbf{H})$, $n \in \mathbb{N}$ be sequences of integrable simple functions such that $\|\Phi_n(x) - \Phi(x)\| \rightarrow 0$ and $\|\Psi_n(x) - \Psi(x)\| \rightarrow 0$ for μ -almost all x . Define a sequence of measurable functions $\Xi_n: \Omega \rightarrow \mathbf{B}_\infty(\mathbf{H})$ by

$$\Xi_n(x) = \Phi_n(x)\Psi_n(x).$$

We have

$$\begin{aligned} \Xi(x) - \Xi_n(x) &= (\Phi(x) - \Phi_n(x))(\Psi(x) - \Psi_n(x)) \\ &\quad + (\Phi(x) - \Phi_n(x))\Psi_n(x) + \Phi_n(x)(\Psi(x) - \Psi_n(x)). \end{aligned}$$

Hence

$$\begin{aligned} \|\Xi(x) - \Xi_n(x)\| &\leq \|\Phi(x) - \Phi_n(x)\| \|\Psi(x) - \Psi_n(x)\| \\ &\quad + \|\Phi(x) - \Phi_n(x)\| \|\Psi_n(x)\| + \|\Phi_n(x)\| \|\Psi(x) - \Psi_n(x)\|. \end{aligned}$$

Hence $\Xi_n(x) \rightarrow \Xi(x)$ for μ -almost all x . Hence Ξ is μ -measurable. \square

LEMMA 245. *Let (Ω, Σ, μ) be a finite measure space, let \mathbf{H} be a separable Hilbert space and let $\Psi: \Omega \rightarrow \mathbf{B}_\infty(\mathbf{H})$ be a SOT μ -measurable function with self-adjoint values such that $0 \leq \Psi(x) \leq \mathcal{I}$ for all x . The function $\Xi: \Omega \rightarrow \mathbf{B}_+(\mathbf{H})$, defined by*

$$\Xi(x) = \sqrt{\Psi(x)},$$

is SOT μ -measurable.

PROOF. There is a sequence of real polynomials p_n on \mathbb{R} such that

$$0 \leq p_1(t) \leq p_2(t) \leq \cdots \leq p_n(t) \leq \sqrt{t} \quad \forall t \in [0, 1]$$

and $\lim_{n \rightarrow \infty} p_n(t) = \sqrt{t}$ uniformly for $t \in [0, 1]$. For each n , let the SOT μ -measurable function $\Xi_n: \Omega \rightarrow \mathbf{B}_\infty(\mathbf{H})$ be defined by

$$\Xi_n(x) = p_n(\Psi(x)).$$

Then $\|\Xi(x) - \Xi_n(x)\| = \sup\{\sqrt{t} - p_n(t) : 0 \leq t \leq 1\}$, and this approaches 0 as $n \rightarrow \infty$. Hence Ξ is SOT μ -measurable. \square

For $\mathcal{A} \in \mathbf{B}_\infty(\mathbf{H})$ let $\mathcal{P}_\mathcal{A}$ be the operator of orthogonal projection onto $\text{cl}(\text{range}(\mathcal{A}))$.

PROPOSITION 246. *Let (Ω, Σ, μ) be a finite measure space, let \mathbf{H} be a separable Hilbert space and let $\Psi: \Omega \rightarrow \mathbf{B}_\infty(\mathbf{H})$ be a SOT μ -measurable function such that $0 \leq \Psi(x) \leq \mathcal{I}$ for all x . The function $\Xi: \Omega \rightarrow \mathbf{B}_p(\mathbf{H})$, defined by*

$$\Xi(x) = \mathcal{P}_{\Psi(x)},$$

is SOT μ -measurable.

PROOF. For each n , let the measurable function $\Xi_n: \Omega \rightarrow \mathbf{B}_\infty(\mathbf{H})$ be defined by

$$\Xi_n(x) = (\Psi(x))^{1/2^n}.$$

The continuous monotone increasing function $\sqrt{\cdot}$ maps the interval $[0, 1]$ onto itself and has the following property: Every sequence of points obtained by applying $\sqrt{\cdot}$ repeatedly, beginning with a point in $(0, 1]$, is monotone increasing and approaches 1. By the

monotone convergence theorem, $\lim_{n \rightarrow \infty} (h, \Xi_n(x)h) = (h, \Xi(x)h)$ for every $h \in \mathbf{H}$ and all x . Hence Ξ is SOT μ -measurable. \square

PROPOSITION 247. *Let (Ω, Σ, μ) be a σ -finite measure space, let \mathbf{H} be a separable Hilbert space and let $\Psi: \Omega \rightarrow \mathbf{B}_\infty(\mathbf{H})$ be a SOT μ -measurable function such that $0 \leq \Psi(x) \leq \mathcal{I}$ for all x . The function $\psi: \Omega \rightarrow [0, \infty]$, defined by*

$$\psi(x) = \dim \text{range}(\Psi(x)),$$

is μ -measurable.

PROOF. Let $\Xi(x) = \mathcal{P}_{\Psi(x)}$. By Proposition 246, $\Xi: \Omega \rightarrow \mathbf{B}_\infty(\mathbf{H})$ is SOT μ -measurable. We have $\psi(x) = \text{Tr}(\Xi(x))$. It is easily seen that ψ is the pointwise limit of a sequence of μ -measurable functions. Hence ψ is μ -measurable. \square

Carleman operators

Let \mathbf{H} be a Hilbert space. A linear operator $\mathcal{R}: \mathbf{H} \rightarrow \mathbf{L}_2(M, \mu)$ is called a Carleman operator if there exists a function $k: M \rightarrow \mathbf{H}$ such that for all $f \in \mathcal{D}(\mathcal{R})$,

$$(83) \quad \mathcal{R}[f](x) = (k(x), f)_{\mathbf{H}} \quad \text{for } \mu\text{-almost all } x \in M.$$

The function k is called the inducing function of \mathcal{R} . Every Carleman operator is closable. Hence $\mathcal{D}(\mathcal{R}) = \mathbf{H}$ implies that \mathcal{R} is bounded. Every Hilbert-Schmidt operator from a Hilbert space \mathbf{H} to $\mathbf{L}_2(M, \mu)$ is a bounded Carleman operator. A Carleman operator is a Hilbert-Schmidt operator if and only if $x \mapsto \|k(x)\|^2$ is μ -integrable.

Some results about Carleman operators are presented in [108]. In [52], bounded integral operators on $\mathbf{L}_2(\mathbb{R})$ (of which bounded Carleman operators are special cases) are investigated.

1. Generating vectors

LEMMA 248 (Lemma II.4.4 in [104]). *Let \mathbf{H} be a separable Hilbert space. Let (Ω, d) be a metric space. Let μ be a regular Borel measure on Ω with the following properties:*

- *Bounded Borel sets have finite μ -measure.*
- *For every function $f: \Omega \rightarrow \mathbb{C}$ which is integrable on bounded Borel sets, there exists a μ -null set \mathfrak{N} such that*
 - *For all $r > 0$ and all $x \in \Omega \setminus \mathfrak{N}$, the closed ball $\text{ball}(x, r)$ with radius r and center x has positive μ -measure.*
 - *For all $x \in \Omega \setminus \mathfrak{N}$, the limits*

$$\lim_{r \downarrow 0} \frac{1}{\mu(\text{ball}(x, r))} \int_{\text{ball}(x, r)} f(y) \mu(dy), \quad x \in \Omega \setminus \mathfrak{N}$$

exist.

- *The function defined by the limits is μ -almost everywhere equal to f .*

Let $\mathcal{R}: \mathbf{H} \rightarrow \mathbf{L}_2(\Omega, \mu)$ be a Carleman operator with inducing function k such that $x \mapsto \|k(x)\|^2$ is μ -integrable on bounded Borel sets. Let (v_k) be an orthonormal basis of \mathbf{H} and let

$$m_k(x, r) = \frac{1}{\mu(\text{ball}(x, r))} \int_{\text{ball}(x, r)} \mathcal{R}[v_k](y) \mu(dy), \quad r > 0.$$

There exists a μ -null set \mathfrak{N} such that for $x \in \Omega \setminus \mathfrak{N}$:

- (a) *For all k , the limit $\varphi_k(x) = \lim_{r \downarrow 0} m_k(x, r)$ exists.*
- (b) *$e_x = \sum_k \overline{\varphi_k(x)} v_k$ and $e_x(r) = \sum_k \overline{m_k(x, r)} v_k$, with $r > 0$, converge in \mathbf{H} .*
- (c) *$\lim_{r \downarrow 0} \|e_x(r) - e_x\| = 0$.*

THEOREM 249 ([104]). *Assume that the conditions of Lemma 248 are satisfied. For each $h \in \mathbf{H}$ there is a representant $\widetilde{\mathcal{R}[h]}$ in $\mathcal{R}[h]$ such that for $x \in \Omega \setminus \mathfrak{N}$,*

- (a) *$\widetilde{\mathcal{R}[h]}(x) = \sum_k (v_k, h) \varphi_k(x)$.*
- (b) *$\widetilde{\mathcal{R}[h]}(x) = (e_x, h)$.*

$$(c) \widetilde{\mathcal{R}}[h](x) = \lim_{r \downarrow 0} \frac{1}{\text{ball}(x, r)} \int_{\text{ball}(x, r)} \mathcal{R}[h](y) \mu(dy).$$

APPENDIX K

Application of Gauss's theorem

1. On the real line

PROPOSITION 250. Let f be entire analytic function and let $d \in \mathbb{R}$. If for all $\tau > 0$ and $y \in \mathbb{R}$, $|f(x + iy)| = O(e^{\tau x^2})$ as $x \rightarrow \infty$, then

$$\int_{-\infty}^{\infty} f(x + id) dx = \int_{-\infty}^{\infty} f(x) dx$$

if both integrands are integrable.

PROOF. We will use the following special case of Gauss's theorem: For an entire analytic function g and $a, b \in \mathbb{R}$,

$$\begin{aligned} & \int_a^b g(x) dx + \int_0^d g(b + iy) dy + \int_b^a g(x + id) dx \\ & + \int_d^0 g(a + iy) dy = 0. \end{aligned}$$

Let $c = \int_0^d f(iy) dy$. If

$$\lim_{a \rightarrow -\infty} \int_0^d f(a + iy) dy = 0 \quad \text{and} \quad \lim_{b \rightarrow \infty} \int_0^d f(b + iy) dy = 0$$

then

$$\lim_{a \rightarrow -\infty} \int_a^0 f(x) dx - \int_a^0 f(x + id) dx = c$$

and

$$\lim_{b \rightarrow \infty} \int_0^b f(x) dx - \int_0^b f(x + id) dx = -c.$$

Hence if

$$\lim_{x \rightarrow \pm\infty} f(x + iy) = 0$$

uniform for y in compact subsets of \mathbb{R} then

$$\int_{-\infty}^{\infty} f(x + id) dx = \int_{-\infty}^{\infty} f(x) dx.$$

Let $\tau > 0$. Define the entire analytic function f_τ on \mathbb{C} by

$$f_\tau(z) = f(z) \exp(-\tau z^2).$$

Then

$$|f_\tau(z)| = |f(z)| \exp\{-\tau \operatorname{Re}(z)^2 + \tau \operatorname{Im}(z)^2\}.$$

Hence

$$\int_{-\infty}^{\infty} f_\tau(x + id) dx = \int_{-\infty}^{\infty} f_\tau(x) dx \quad \forall \tau > 0.$$

By the dominated convergence theorem,

$$\int_{-\infty}^{\infty} f(x + id) dx = \int_{-\infty}^{\infty} f(x) dx.$$

□

An entire analytic function f is completely determined by its restriction to the real line. By the following consequence of Proposition 252, certain integrals over $f(x + iy)$ can be expressed as repeated integrals involving $f(x)$.

COROLLARY 251. *Let $\omega: \mathbb{R} \rightarrow [0, \infty)$ be a Lebesgue measurable function and let f, g be entire analytic functions satisfying*

(a) $\bar{g}f$ is $\omega(y)dxdy$ -integrable;

(b) $x \mapsto \overline{g(x + iy)}f(x)$ is integrable for every $y \in \mathbb{R}$;

(c) $|g(x + iy)f(x)| = O(e^{\tau x^2})$ for every $y \in \mathbb{R}$ and $\tau > 0$.

Then

$$\iint_{\mathbb{R}^2} \overline{g(x + iy)}f(x + iy) \omega(y)dxdy = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \overline{g(x + 2iy)}f(x) dx \right) \omega(y)dy.$$

2. On the complex plane

PROPOSITION 252. *Let φ be an entire analytic function of two complex variables and let $a, b \in \mathbb{C}$. If for all $\tau > 0$ and $w \in \mathbb{C}$, $|\varphi(z - w, \bar{z} + \bar{w})| = O(e^{\tau|z|^2})$ as $z \rightarrow \infty$, then*

$$\int_{\mathbb{C}} \varphi(z + a, \bar{z} + b) dz = \int_{\mathbb{C}} \varphi(z, \bar{z}) dz$$

if both integrands are integrable.

PROOF. We will use the following special case of Gauss's theorem: For an entire analytic function g and $c, d \in \mathbb{R}$,

$$(84) \quad \begin{aligned} & \int_{-c}^c g(x) dx - \int_{-c}^c g(x + id) dx \\ &= \int_0^d g(-c + iy) dy - \int_0^d g(c + iy) dy. \end{aligned}$$

We will apply this to the entire analytic function f on \mathbb{C}^2 , defined by

$$f(z_1, z_2) = \varphi(z_1 + iz_2, z_1 - iz_2).$$

For $z, w \in \mathbb{C}$,

$$\begin{aligned} \varphi(z, \bar{z}) &= f(\operatorname{Re}(z), \operatorname{Im}(z)), \\ \varphi(z - w, \bar{z} + \bar{w}) &= f(\operatorname{Re}(z) - i \operatorname{Im}(w), \operatorname{Im}(z) + i \operatorname{Re}(w)). \end{aligned}$$

Hence

$$\begin{aligned} & \int_{-c_1}^{c_1} \int_{-c_2}^{c_2} \varphi(z, \bar{z}) d \operatorname{Re}(z) d \operatorname{Im}(z) - \int_{-c_1}^{c_1} \int_{-c_2}^{c_2} \varphi(z - w, \bar{z} + \bar{w}) d \operatorname{Re}(z) d \operatorname{Im}(z) \\ &= \int_{-c_1}^{c_1} D(x_1, c_2) dx_1 + \int_{-c_2}^{c_2} E(c_1, x_2 + i \operatorname{Re}(w)) dx_2. \end{aligned}$$

where

$$\begin{aligned} D(x_1, c_2) &= \int_{-c_2}^{c_2} f(x_1, x_2) dx_2 - \int_{-c_2}^{c_2} f(x_1, x_2 + i \operatorname{Re}(w)) dx_2, \\ E(c_1, z) &= \int_{-c_1}^{c_1} f(x_1, z) dx_1 - \int_{-c_1}^{c_1} f(x_1 - i \operatorname{Im}(w), z) dx_1. \end{aligned}$$

By (84),

$$D(x_1, c_2) = \int_0^{\operatorname{Re}(w)} f(x_1, -c_2 + iy_2) dy_2 - \int_0^{\operatorname{Re}(w)} f(x_1, c_2 + iy_2) dy_2,$$

$$E(c_1, z) = \int_0^{-\operatorname{Im}(w)} f(-c_1 + iy_1, z) dy_1 - \int_0^{-\operatorname{Im}(w)} f(c_1 + iy_1, z) dy_1.$$

Hence if

$$\lim_{\operatorname{Im}(z) \rightarrow \pm\infty} \int_{\mathbb{R}} \varphi(z - a, \bar{z} + \bar{a}) d\operatorname{Re}(z) = 0,$$

$$\lim_{\operatorname{Re}(z) \rightarrow \pm\infty} \int_{\mathbb{R}} \varphi(z - a, \bar{z} + \bar{a}) d\operatorname{Im}(z) = 0$$

uniform for a in compact subsets of \mathbb{C} , then

$$\int_{\mathbb{C}} \varphi(z, \bar{z}) dz = \int_{\mathbb{C}} \varphi(z - w, \bar{z} + \bar{w}) dz$$

Let $\tau > 0$. Define the entire analytic function φ_τ on \mathbb{C}^2 by

$$\varphi_\tau(z_1, z_2) = \varphi(z_1, z_2) \exp(-\tau z_1 z_2).$$

Then

$$|\varphi_\tau(z - a, \bar{z} + \bar{a})| = |\varphi(z - a, \bar{z} + \bar{a})| \exp(-\tau|z|^2 + \tau|a|^2) \quad \forall a \in \mathbb{C}.$$

Let $w = \bar{b} - a$. Then

$$\int_{\mathbb{C}} \varphi_\tau(z, \bar{z}) dz = \int_{\mathbb{C}} \varphi_\tau(z - w, \bar{z} + \bar{w}) dz$$

By the dominated convergence theorem,

$$\int_{\mathbb{C}} \varphi(z, \bar{z}) dz = \int_{\mathbb{C}} \varphi(z - w, \bar{z} + \bar{w}) dz.$$

This implies the result. □

3. Gaussian convolution on $L_p(\mathbb{R}^m)$

Let $m \in \mathbb{N}$. Let $p, q \in [1, \infty]$ be related by $\frac{1}{p} + \frac{1}{q} = 1$. We will denote $L_p(\mathbb{R}^m)$ by L_p and $L_q(\mathbb{R}^m)$ by L_q . Let $s > 0$. For $z \in \mathbb{C}$ and $f \in L_1(\mathbb{R}^m, e^{-(z-x)^2/(2s)} dx)$, let

$$\mathcal{G}_s[f](z) = (2\pi s)^{-m/2} \int_{\mathbb{R}^m} e^{-(z-x)^2/(2s)} f(x) dx.$$

For $z, w \in \mathbb{C}^m$ let $z \cdot w = \sum_{j=1}^m z_j w_j$ and $z^2 = z \cdot z$.

For $g \in L_q$ define $g^* \in L_q$ by $g^*(x) = \overline{g(-x)}$. Clearly $\|g^*\| = \|g\|$. For $f \in L_p$ and $g \in L_\infty$, the convolution product $g * f$ is the function

$$g * f(x) = \int_{\mathbb{R}^m} g(x-y)f(y) dy.$$

By Hölder's inequality, $g * f: \mathbb{R}^m \rightarrow \mathbb{C}$ is a bounded function, and

$$(85) \quad \|g * f\|_\infty \leq \|g\|_q \|f\|_p.$$

If $p < \infty$, every $f \in L_p$ can be approximated in L_p by a sequence (f_n) of infinitely differentiable functions with compact support. Together with the triangle inequality, (85) implies that $g * f$ can be approximated uniformly by the infinitely differentiable functions $g * f_n$, $n \in \mathbb{N}$. Hence $g * f$ is continuous. Because $g * f = f * g$, this is also true if $p = \infty$:

$$(86) \quad g * f \in C_b(\mathbb{R}^m) \quad \forall g \in L_q, f \in L_p,$$

where $C_b(\mathbb{R}^m)$ denotes the set of bounded continuous functions on \mathbb{R}^m .

LEMMA 253. Let $x, y \in \mathbb{R}$ and $g \in L_1(\mathbb{R}^m, e^{-(x-u)^2/(2s)} du)$. Then

$$|\mathcal{G}_s[g](x+iy)| \leq (2s\pi)^{-m/2} e^{y^2/(2s)} \int_{\mathbb{R}^m} |g(x+u)| e^{-u^2/(2s)} du.$$

PROOF. This follows easily from

$$(87) \quad \begin{aligned} \mathcal{G}_s[g](x+iy) &= \int_{\mathbb{R}^m} \frac{e^{-(x+iy-u)^2/(2s)}}{(2s\pi)^{m/2}} g(u) du \\ &= s^{m/2} e^{y^2/(2s)} \int_{\mathbb{R}^m} \frac{e^{iy \cdot u}}{(2\pi)^{m/2}} e^{-su^2/2} g(x+su) du \end{aligned}$$

□

LEMMA 254. Let $x \in \mathbb{R}^m$ and let $f, g \in L_1(\mathbb{R}^m, e^{-|u-x|^2/(2s)} du)$. Let \mathfrak{c} be an integrable function on \mathbb{R}^m and let

$$(88) \quad \mathfrak{d}(x) = \int_{\mathbb{R}^m} \mathfrak{c}(y) \frac{e^{iy \cdot x}}{(2\pi)^m} dy.$$

Then

$$(89) \quad \begin{aligned} &\int_{\mathbb{R}^m} \mathfrak{c}(y) \overline{\mathcal{G}_s[g](x+iy)} \mathcal{G}_s[f](x+iy) \frac{\exp\{-y^2/s\}}{(s\pi)^{m/2}} dy \\ &= \iint_{\mathbb{R}^{2m}} \frac{e^{-(u-x)^2/s}}{(s\pi)^{m/2}} \left(\frac{2}{s}\right)^m \mathfrak{d}\left(\frac{2}{s}v\right) \overline{g(u-v)} f(u+v) e^{-v^2/s} dv du. \end{aligned}$$

PROOF. By (87),

$$(90) \quad \begin{aligned} & \overline{\mathcal{G}_s[g](x+iy)} \mathcal{G}_s[f](x+iy) \frac{\exp\{-y^2/s\}}{(s\pi)^{m/2}} \\ &= (s/\pi)^{m/2} \int_{\mathbb{R}^{2m}} \frac{e^{-iy(v-u)}}{(2\pi)^m} \overline{g(x+sv)} f(x+su) e^{-s(u^2+v^2)/2} dvdu. \end{aligned}$$

Because \mathbf{c} is integrable, Fubini's theorem can be used to get

$$\begin{aligned} & \int_{\mathbb{R}^m} \mathbf{c}(y) \overline{\mathcal{G}_s[g](x+iy)} \mathcal{G}_s[f](x+iy) \frac{\exp\{-y^2/s\}}{(s\pi)^{m/2}} dy \\ &= (s/\pi)^{m/2} \iint_{\mathbb{R}^{2m}} \mathfrak{d}(u-v) \overline{g(x+sv)} f(x+su) e^{-s(u^2+v^2)/2} dvdu \\ &= (s\pi)^{-m/2} \iint_{\mathbb{R}^{2m}} \left(\frac{2}{s}\right)^m \mathfrak{d}\left(\frac{2}{s} \frac{u-v}{\sqrt{2}}\right) \overline{g(x+\sqrt{2}v)} f(x+\sqrt{2}u) e^{-(u^2+v^2)/s} dvdu. \end{aligned}$$

For (89) we use

$$u^2 + v^2 = \left(\frac{u+v}{\sqrt{2}}\right)^2 + \left(\frac{u-v}{\sqrt{2}}\right)^2$$

and the following: Let $F: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be defined by

$$F(u, v) = (F_1(u, v), F_2(u, v)) = \left(\frac{u-v}{\sqrt{2}}, \frac{u+v}{\sqrt{2}}\right).$$

Let $\partial_k = \frac{\partial}{\partial x_k}$. We have $[\partial_k F]_{k,\ell=1}^2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$. The Jacobian of transformation F is 1. Hence

$$\begin{aligned} & \int_{\mathbb{R}} \mathbf{c}(y) \overline{\mathcal{G}_s[g](x+iy)} \mathcal{G}_s[f](x+iy) \frac{\exp\{-y^2/s\}}{(s\pi)^{m/2}} dy \\ &= (s\pi)^{-m/2} \iint_{\mathbb{R}^{2m}} \left(\frac{2}{s}\right)^m \mathfrak{d}\left(-\frac{2}{s}v\right) \overline{g(x+u+v)} f(x+u-v) e^{-(u^2+v^2)/s} dvdu. \end{aligned}$$

□

LEMMA 255. Let $f \in L_p$ and $g \in L_q$. Let \mathbf{c} be an integrable function on \mathbb{R}^m and let \mathfrak{d} be as in (88). Then

$$(91) \quad \begin{aligned} & \int_{\mathbb{R}^m} \mathbf{c}(y) \left(\int_{\mathbb{R}^m} \overline{\mathcal{G}_s[g](x+2iy)} \mathcal{G}_s[f](x) dx \right) \frac{\exp\{-y^2/s\}}{(s\pi)^{m/2}} dy \\ &= \int_{\mathbb{R}^m} \left(\frac{1}{s}\right)^m \mathfrak{d}\left(\frac{1}{s}v\right) g^* * f(v) e^{-v^2/(4s)} dv. \end{aligned}$$

If \mathbf{c} has compact support then

$$(92) \quad \int_{\mathbb{R}^m} \overline{G_{\mathbf{c}}(x)} \mathcal{G}_s[f](x) dx = \int_{\mathbb{R}^m} \left(\frac{1}{s}\right)^m \mathfrak{d}\left(\frac{1}{s}v\right) g^* * f(v) e^{-v^2/(4s)} dv,$$

where $G_{\mathbf{c}}(x) = \int_{\mathbb{R}^m} \mathbf{c}(y) \overline{\mathcal{G}_s[g](x+2iy)} \frac{\exp\{-y^2/s\}}{(s\pi)^{m/2}} dy$.

PROOF. By (90) and Hölder's inequality,

$$\begin{aligned} & \int_{\mathbb{R}^m} |\overline{\mathcal{G}_s[g](x+iy)} \mathcal{G}_s[f](x+iy)| \frac{\exp\{-y^2/s\}}{(s\pi)^{m/2}} dx \\ & \leq \|f\|_p \|g\|_q (s/\pi)^{m/2} (2\pi)^{-m} \iint_{\mathbb{R}^{2m}} e^{-s(u^2+v^2)/2} dvdu = \|f\|_p \|g\|_q (s\pi)^{-m/2}. \end{aligned}$$

From Lemma 254 follows

$$\begin{aligned} & \iint_{\mathbb{R}^{2m}} \mathbf{c}(y) \overline{\mathcal{G}_s[g](x+iy)} \mathcal{G}_s[f](x+iy) \frac{\exp\{-y^2/s\}}{(s\pi)^{m/2}} dx dy \\ &= \iint_{\mathbb{R}^{2m}} \left(\frac{1}{s}\right)^m \mathfrak{d}\left(\frac{1}{s}v\right) \overline{g(u-v)} f(u) e^{-v^2/(4s)} dudv. \end{aligned}$$

(91) follows from Corollary 251.

From (87) follows

$$|\mathcal{G}_s[g](x+2iy)| \leq s^{m/2} e^{2y^2/s} (2\pi)^{-m/2} \int_{\mathbb{R}^m} e^{-su^2/2} |g(x+su)| du.$$

This, together with (91), (86) and Fubini's theorem implies (92). \square

THEOREM 256. *Let $f \in \mathbf{L}_p$ and $g \in \mathbf{L}_q$. Let \mathbf{c} be an infinitely differentiable function on \mathbb{R}^m with compact support such that $\mathbf{c}(0) = 1$. Then*

$$(93) \quad \int_{\mathbb{R}^m} \overline{g(x)} f(x) dx = \lim_{n \rightarrow \infty} \int_{\mathbb{R}^m} \overline{G_n(x)} \mathcal{G}_s[f](x) dx,$$

where $G_n(x) = \int_{\mathbb{R}^m} \mathbf{c}(y/n) \mathcal{G}_s[g](x+2iy) \frac{\exp\{-y^2/s\}}{(s\pi)^{m/2}} dy$.

PROOF. Let \mathfrak{d} be as in (88). By Lemma 255,

$$\begin{aligned} \int_{\mathbb{R}^m} \overline{G_n(x)} \mathcal{G}_s[f](x) dx &= \int_{\mathbb{R}^m} \left(\frac{n}{s}\right)^m \mathfrak{d}\left(\frac{n}{s}v\right) g^* * f(v) e^{-v^2/(4s)} dv \\ &= \int_{\mathbb{R}^m} \left(\frac{1}{s}\right)^m \mathfrak{d}\left(\frac{1}{s}v\right) g^* * f(v/n) e^{-v^2/(4sn^2)} dv. \end{aligned}$$

Let $\mathbf{C}_{00}^\infty(\mathbb{R}^m)$ be the space of infinitely differentiable functions on \mathbb{R}^m with compact support and let $\mathcal{S}(\mathbb{R}^m)$ be the Schwartz space of infinitely differentiable functions on \mathbb{R}^m with derivatives that decay rapidly at infinity. It is well-known that

$$\mathbf{C}_{00}^\infty(\mathbb{R}^m) \subset \mathcal{S}(\mathbb{R}^m) \subset \mathbf{L}_1(\mathbb{R}^m)$$

and that $\mathcal{S}(\mathbb{R}^m)$ is invariant under Fourier transformation. Hence $\mathbf{c} \in \mathbf{C}_{00}^\infty(\mathbb{R}^m)$ implies $\mathfrak{d} \in \mathcal{S}(\mathbb{R}^m)$. Hence \mathfrak{d} is integrable. By the dominated convergence theorem and the fact that $g^* * f \in \mathbf{C}_b(\mathbb{R}^m)$, we have

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^m} \overline{G_n(x)} \mathcal{G}_s[f](x) dx = \int_{\mathbb{R}^m} \left(\frac{1}{s}\right)^m \mathfrak{d}\left(\frac{1}{s}v\right) g^* * f(0) dv.$$

This is equal to $\int_{\mathbb{R}^m} \mathfrak{d}(v) dv g^* * f(0) = \mathbf{c}(0) g^* * f(0) = g^* * f(0)$. \square

APPENDIX L

Integral operators with a Gaussian kernel

Let φ_n be the n 'th Hermite basis function in $L_2(\mathbb{R})$:

$$\varphi_n(x) = \frac{H_n(x)}{\sqrt{2^n n!}} \pi^{-1/4} e^{-x^2/2}.$$

1. Calculations

LEMMA 257. *If $n \in \mathbb{N}$, $z \in \mathbb{C}$, $p, r > 0$ and $q \in \mathbb{R}$ then*

$$\sqrt{\frac{|q|}{2\pi}} \int_{\mathbb{R}} e^{-\frac{1}{2}(pz^2+ry^2+2qzy)} r^{1/4} \varphi_n(\sqrt{r}y) dy = u_n(z)$$

where

$$u_n(z) = ac(z) \frac{(bz)^n}{\sqrt{n!}}$$

with $a = (2\pi)^{-1/4} (\frac{q^2}{2r})^{1/4}$, $b = \frac{-q}{\sqrt{2r}}$ and $c(z) = \exp\{(\frac{q^2}{2r} - p) \frac{z^2}{2}\}$.

PROOF. From

$$\frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-(z-y)^2} H_n(y) dy = (2z)^n$$

follows

$$\sqrt{\frac{|q|}{2\pi}} \int_{\mathbb{R}} e^{-ry^2+qzy} (r/\pi)^{1/4} H_n(\sqrt{r}y) dy = ae^{q^2 z^2 / (4r)} (qz/\sqrt{r})^n.$$

□

LEMMA 258. *For $z \in \mathbb{C}$, $p, r > 0$ and $q \in \mathbb{R}$ let*

$$(94) \quad \rho(z) = \frac{|q|}{\sqrt{\pi r}} \exp\left\{\frac{pr - q^2}{r} \operatorname{Re}(z)^2 - p \operatorname{Im}(z)^2\right\}.$$

Then (u_n) is an orthonormal family in $L_2(\mathbb{C}, \rho(z)dz)$ and

$$\sum_{n=0}^{\infty} |u_n(z)|^2 = \frac{q^2}{2\pi r \rho(z)}.$$

PROOF. Let a and b be as in Lemma 257. Then

$$\rho(z) = \frac{b^2}{\pi a^2} \exp\left\{\frac{pr - q^2}{r} \operatorname{Re}(z)^2 - p \operatorname{Im}(z)^2\right\}.$$

Hence

$$(95) \quad |c(z)|^2 \rho(z) = \frac{b^2}{\pi a^2} \exp\{-|bz|^2\}.$$

Hence

$$u_k(\bar{z}) u_\ell(z) \rho(z) = \frac{b^2}{\pi} \frac{(b\bar{z})^k (bz)^\ell}{\sqrt{k! \ell!}}.$$

Hence

$$\int_{\mathbb{C}} u_k(\bar{z}) u_\ell(z) \rho(z) dz = \frac{1}{\pi} \int_{\mathbb{C}} \frac{\bar{z}^k z^\ell}{\sqrt{\ell! k!}} e^{-|z|^2} dz = \delta_{k\ell}.$$

We have

$$\sum_{n=0}^{\infty} |u_n(z)|^2 = a^2 |c(z)|^2 \exp(|bz|^2)$$

By (95) this is $b^2/(\pi\rho(z))$. □

2. The integral operators \mathcal{S}_{pqr}

For $z, w \in \mathbb{C}^m$ let

- (a) $z \cdot w = \sum_{j=1}^m z_j w_j$, and $z^2 = z \cdot z$,
 (b) $\operatorname{Re}(z) = (\operatorname{Re}(z_j)) \in \mathbb{R}^m$, and $\operatorname{Im}(z) = (\operatorname{Im}(z_j)) \in \mathbb{R}^m$.

Let $p, r > 0$ and $q \in \mathbb{R}$.

DEFINITION 259. For $f \in L_2(\mathbb{R}^m)$ let

$$\mathcal{S}_{pqr}[f](z) = \left(\frac{|q|}{2\pi}\right)^{m/2} \int_{\mathbb{R}^m} e^{-\frac{1}{2}(pz^2 + ry^2 + 2qz \cdot y)} f(y) dy.$$

For $z \in \mathbb{C}^m$ let

$$\rho(z) = \prod_{j=1}^m \rho(z_j),$$

where $\rho(z_j)$ is defined by (94).

LEMMA 260. $\mathcal{S}_{pqr}: L_2(\mathbb{R}^m) \rightarrow L_2(\mathbb{C}^m, \rho(z) dz)$ is a linear isometry and

$$(96) \quad |\mathcal{S}_{pqr}[f](z)|^2 \leq \|f\|^2 (2\pi)^{-m/2} \left(\frac{q^2}{2r}\right)^{m/2} \exp\left\{-\frac{pr - q^2}{r} \operatorname{Re}(z)^2 + p \operatorname{Im}(z)^2\right\}.$$

If, conversely, an entire analytic function $\varphi: \mathbb{C}^m \rightarrow \mathbb{C}$ satisfies

$$(97) \quad |\varphi(z)|^2 \leq M \exp\left\{-\frac{pr - q^2}{r} \operatorname{Re}(z)^2 + p \operatorname{Im}(z)^2\right\} \quad \forall z \in \mathbb{C}$$

for some $M > 0$, then $\varphi \in \operatorname{range}(\mathcal{S}_{p',q',r'})$ where $r' > 0$ and $q' \in \mathbb{R}$ and

$$p' > p \quad \text{and} \quad p'r' - (q')^2 < \frac{r'}{r} (pr - q^2).$$

PROOF. Everything can be reduced to the case $m = 1$. (96) follows from Lemma 258 and the following estimate:

$$\begin{aligned} \left| \sum_{n=0}^{\infty} (u_n, \mathcal{S}_{pqr}[f]) u_n(z) \right|^2 &\leq \left(\sum_{n=0}^{\infty} |(u_n, \mathcal{S}_{pqr}[f])| |u_n(z)| \right)^2 \\ &\leq \sum_{n=0}^{\infty} |(u_n, \mathcal{S}_{pqr}[f])|^2 \sum_{n=1}^{\infty} |u_n(z)|^2. \end{aligned}$$

Assume that φ satisfies (97) and let $\rho'(z)$ be related to p', q', r' by (94). It is easily seen that $\varphi \in L_2(\mathbb{C}, \rho')$ and that $\operatorname{range}(\mathcal{S}_{p',q',r'})$ consists of the entire analytic functions in $L_2(\mathbb{C}, \rho')$. Hence $\varphi \in \operatorname{range}(\mathcal{S}_{p',q',r'})$. □

2.1. Gaussian convolution. Let $s > 0$ and

$$\mathcal{G}_s[f](z) = (2\pi s)^{-m/2} \int_{\mathbb{R}^m} e^{-(z-y)^2/(2s)} f(y) dy, \quad z \in \mathbb{C}^m.$$

Let

$$\rho_s(z) = \frac{1}{\sqrt{s\pi}} \exp\{-\operatorname{Im}(z)^2/s\}.$$

LEMMA 261. $\mathcal{G}_s: \mathbf{L}_2(\mathbb{R}^m) \rightarrow \mathbf{L}_2(\mathbb{C}^m, \rho_s(z)dz)$ is a linear isometry and

$$|\mathcal{G}_s[f](z)|^2 \leq \|f\|^2 (2\sqrt{s\pi})^{-m} \exp\{\operatorname{Im}(z)^2/(2s)\}.$$

If, conversely, an entire analytic function $\varphi: \mathbb{C}^m \rightarrow \mathbb{C}$ satisfies

$$|\varphi(z)|^2 \leq M \exp\{\operatorname{Im}(z)^2/(2s)\} \quad \forall z \in \mathbb{C}^m$$

for some $M > 0$, then $\varphi \in \operatorname{range}(\mathcal{G}_{s'})$ where $0 < s' < s$.

PROOF. Let $p = r = 1/s$ and $q = -1/s$. Then $pr - q^2 = 0$ and $\mathcal{G}_s = \mathcal{S}_{p,q,r}$. \square

2.2. Harmonic oscillator. Let $\tau > 0$ and

$$\mathcal{N}_\tau[f](z) = (2\pi \sinh \tau)^{-m/2} \int_{\mathbb{R}^m} \exp\left\{\frac{-1}{2 \sinh \tau} (\cosh \tau (x^2 + y^2) - 2x \cdot y)\right\} f(y) dy.$$

If $m = 1$, then $\mathcal{N}_\tau[\varphi_n] = \exp\{-(n + \frac{1}{2})\tau\}\varphi_n$.

Let

$$\rho_\tau(z) = (\pi \sinh \tau \cosh \tau)^{-m/2} \exp\{\tanh(\tau) \operatorname{Re}(z)^2 - \coth(\tau) \operatorname{Im}(z)^2\}.$$

LEMMA 262. $\mathcal{N}_\tau: \mathbf{L}_2(\mathbb{R}^m) \rightarrow \mathbf{L}_2(\mathbb{C}^m, \rho_\tau)$ is a linear isometry and

$$|\mathcal{N}_\tau[f](z)|^2 \leq \|f\|^2 (2\pi \sinh(2\tau))^{-m/2} \exp\{-\tanh \tau \operatorname{Re}(z)^2 + \coth \tau \operatorname{Im}(z)^2\}.$$

If, conversely, an entire analytic function $\varphi: \mathbb{C}^m \rightarrow \mathbb{C}$ satisfies

$$|\varphi(z)|^2 \leq M \exp\{-\tanh \tau \operatorname{Re}(z)^2 + \coth \tau \operatorname{Im}(z)^2\} \quad \forall z \in \mathbb{C}^m$$

for some $M > 0$, then $\varphi \in \operatorname{range}(\mathcal{N}_{\tau'})$ where $0 < \tau' < \tau$.

PROOF. Let $p = r = \coth \tau$ and $q = -1/\sinh \tau$. Then $pr - q^2 = 1$ and $\mathcal{N}_\tau = \mathcal{S}_{p,q,r}$. \square

REMARK 263. Lemma 262 follows from the results in [105].

2.3. Combination. For $t \in (-\min(s, \frac{1}{s}), 1)$ let

$$\rho_{s,t}(z) = \frac{\exp\left\{\frac{t \operatorname{Re}(z)^2}{st+1} - \frac{\operatorname{Im}(z)^2}{s+t}\right\}}{(\pi(st+1)(s+t))^{m/2}}.$$

LEMMA 264. Let $s \geq 0$ and $z \in \mathbb{C}^m$. For every $f \in \mathbf{L}_2(\mathbb{R}^m)$, the function $t \mapsto \mathcal{G}_s \mathcal{N}_{\operatorname{atanh}(t)}[f](z)$ on $(0, 1)$ has an analytic continuation to

$$\{t \in \mathbb{C} : -\min(s, \frac{1}{s}) < \operatorname{Re}(t) < 1\},$$

where $\frac{1}{s} = +\infty$ if $s = 0$.

Let $t \in (-\min(s, \frac{1}{s}), 1)$. Then $\mathcal{G}_s \mathcal{N}_{\operatorname{atanh}(t)}: \mathbf{L}_2(\mathbb{R}^m) \rightarrow \mathbf{L}_2(\mathbb{C}^m, \rho_{s,t}(z)dz)$ is a linear isometry and

$$|\mathcal{G}_s \mathcal{N}_{\operatorname{atanh}(t)}[f](z)|^2 \leq \|f\|^2 \sqrt{1-t^2} \frac{\exp\left\{-\frac{t \operatorname{Re}(z)^2}{st+1} + \frac{\operatorname{Im}(z)^2}{s+t}\right\}}{2^m (\pi(st+1)(s+t))^{m/2}}.$$

If, conversely, an entire analytic function $\varphi: \mathbb{C} \rightarrow \mathbb{C}$ satisfies

$$|\varphi(z)|^2 \leq M \exp\left\{-\frac{t \operatorname{Re}(z)^2}{st+1} + \frac{\operatorname{Im}(z)^2}{s+t}\right\}$$

for some $M > 0$, then $\varphi \in \operatorname{range}(\mathcal{G}_{s'} \mathcal{N}_{\operatorname{atanh}(t')})$ where $t' \in (-\min(s', \frac{1}{s'}), 1)$ and

$$s' + t' < s + t, \quad \frac{t'}{s't' + 1} < \frac{t}{st + 1}.$$

REMARK 265. The function $t \mapsto t/(st+1)$ on $(-\min(s, \frac{1}{s}), 1)$ is monotone increasing. The function $t \mapsto 1/(s+t)$ on $(-\min(s, \frac{1}{s}), 1)$ is monotone decreasing and positive.

PROOF. If $t > 0$ then $\mathcal{G}_s \mathcal{N}_{\text{atanh}(t)} = \mathcal{S}_{p,q,r}$ where

$$p = \frac{1}{s+t}, \quad r = \frac{st+1}{s+t}, \quad q = \frac{-\sqrt{1-t^2}}{s+t}.$$

Condition $-\min(s, \frac{1}{s}) < \text{Re}(t) < 1$ implies that $\text{Re}(s+t) > 0$ and that $\text{Re}(st+1) > 0$ and hence that p, r and q depend analytically on t . We have

$$pr - q^2 = \frac{t}{s+t}.$$

The assumptions on s and t imply that $p, r, -q > 0$. □

COROLLARY 266. *Let $s \geq 0$. Then*

$$|\mathcal{G}_s[\varphi_n](z)|^2 \leq e^{-(n+1/2)\text{atanht}} M_{s,t}(z) \quad \forall n \in \mathbb{N}, t \in [0, \min(s, \frac{1}{s})]$$

with $M_{s,t}(z) = \sqrt{1-t^2} \frac{\exp\{-\frac{t \text{Re}(z)^2}{st+1} + \frac{\text{Im}(z)^2}{s+t}\}}{2^m (\pi(st+1)(s+t))^{m/2}}$.

Consequently, we can define \mathcal{G}_s on a Hermite series $\sum_{n=0}^{\infty} c_n \varphi_n$ with $(c_n) = O(e^{n \text{atanht}})$ and $t \in [0, \min(s, \frac{1}{s})]$ by

$$\mathcal{G}_s\left[\sum_{n \in \mathbb{N}_0^m} c_n \varphi_n\right](z) = \sum_{n \in \mathbb{N}_0^m} c_n \mathcal{G}_s[\varphi_n](z).$$

The sum on the right-hand side converges pointwise.

PROOF. This follows from Lemma 264 and

$$\mathcal{G}_s[\varphi_n](z) = e^{-(n+1/2)\text{atanht}} \mathcal{G}_s[\mathcal{N}_{-\text{atanht}} \varphi_n](z).$$

□

APPENDIX M

Gelfand-Shilov space $\mathcal{S}_{1/2}^{1/2}$.

1. Introduction

The Gelfand-Shilov space $\mathcal{S}_{1/2}^{1/2}(\mathbb{R}^m)$ is the space of functions on \mathbb{R}^m that have continuations to entire analytic functions φ satisfying

$$|\varphi(z)| \leq M \exp\{-A \operatorname{Re}(z)^2 + B \operatorname{Im}(z)^2\} \quad \forall z \in \mathbb{C}^m$$

for certain $M, A, B > 0$. It was introduced by Gelfand and Shilov in [49]. $\mathcal{S}_{1/2}^{1/2}(\mathbb{R}^m)$ can be identified with a subspace of $L_2(\mathbb{R}^m)$:

$$\mathcal{S}_{1/2}^{1/2}(\mathbb{R}^m) = \{h \in L_2(\mathbb{R}^m) : \exists t > 0 \forall n \in \mathbb{N}_0^m : |(\varphi_n, h)| = O(e^{-tn})\},$$

where (φ_n) is the Hermite basis of $L_2(\mathbb{R}^m)$. In [32], $\mathcal{S}_{1/2}^{1/2}(\mathbb{R}^m)$, with $m = 1$, is used as a test space for a theory of generalized functions. This theory is based on the semigroup properties of a particular one-parameter family \mathcal{N}_τ , $\tau > 0$ of operators on $L_2(\mathbb{R}^m)$ which are called smoothing operators. The operators \mathcal{N}_τ are characterized in terms of their action on the Hermite basis by

$$(98) \quad \mathcal{N}_\tau \varphi_n = e^{-(n+1/2)\tau} \varphi_n, \quad n \in \mathbb{N}_0^m.$$

This can be generalized to $m \in \mathbb{N}$. The test space $\mathcal{S} = \mathcal{S}_{1/2}^{1/2}(\mathbb{R}^m)$ can be expressed in terms of the ranges of the smoothing operators as

$$\mathcal{S} = \bigcup_{\tau > 0} \mathcal{N}_\tau(\mathcal{H}).$$

The space of generalized functions \mathcal{T} is formed by the solutions $u: (0, \infty) \rightarrow L_2(\mathbb{R}^m)$ of the evolution equation

$$\frac{du}{dt} = -\mathcal{N}u,$$

where $-\mathcal{N}$ is the infinitesimal generator of the semigroup (\mathcal{N}_τ) (See [33]). We write $\mathcal{N}_\tau[u] = u(\tau)$ for $\tau > 0$. The generalized functions act on \mathcal{S} through a sesquilinear form $\langle \cdot, \cdot \rangle: \mathcal{T} \times \mathcal{S}$, defined by

$$(99) \quad \langle F, h \rangle = (\mathcal{N}_\tau[F], \mathcal{N}_{-\tau}[h]),$$

for $\tau > 0$ such that $h \in \mathcal{N}_\tau(\mathcal{H})$. The semi-group properties of \mathcal{N}_τ imply that this does not depend on the particular value of $\tau > 0$.

2. The spaces \mathcal{H}_+ and \mathcal{H}_- .

We use a new notation for the spaces \mathcal{S} and \mathcal{T} introduced in Section 1: Let $\mathcal{H} = L_2(\mathbb{R}^m)$, $\mathcal{H}_+ = \mathcal{S}$ and $\mathcal{H}_- = \mathcal{T}$. Elements of \mathcal{H}_- are considered as linear forms on \mathcal{H}_+ through (99). We write $\mathcal{H}_+(\mathbb{R}^m)$ and $\mathcal{H}_-(\mathbb{R}^m)$ if we want to make the dependence on m explicit.

It is easily seen that

$$(100) \quad \mathcal{H}_+ = \bigcup \{\mathcal{H}_+(M, \tau) : M > 0, \tau > 0\},$$

where

$$H_+(M, \tau) = \{f \in \text{range}(\mathcal{N}_\tau) : \|\mathcal{N}_\tau^{-1}f\| \leq M\}.$$

For $A, B \in \mathbb{R}$ and $M > 0$ let $A(M, A, B)$ be the set of entire analytic functions φ on \mathbb{C}^m satisfying

$$|\varphi(z)|^2 \leq M \exp\{-A \operatorname{Re}(z)^2 + B \operatorname{Im}(z)^2\} \quad \forall z \in \mathbb{C}^m.$$

It is easily seen that

$$\cup_{\tau>0} A(M, \tanh \tau, \coth \tau) = \cup_{A,B>0} A(M, A, B) \quad \forall M > 0.$$

From Lemma 262 follows:

$$(101) \quad \begin{aligned} \forall M, \tau > 0 \quad \exists M', A, B > 0 \quad \text{such that} \quad A(M', A, B) \subset H_+(M, \tau); \\ \forall M, A, B > 0 \quad \exists M', \tau > 0 \quad \text{such that} \quad H_+(M', \tau) \subset A(M, A, B). \end{aligned}$$

This, together with (100), implies that

$$H_+ = \bigcup \{A(M, A, B) : M, A, B > 0\}.$$

We can get a similar representation of H_- : Denote the set of linear forms on H_+ by H_+^* . Then

$$(102) \quad H_- = \bigcap \{H_-(M, \tau) : M, \tau > 0\} = \bigcap \{H_-(M, A, B) : M, A, B > 0\},$$

where

$$H_-(M, \tau) = \{L \in H_+^* : L \text{ is bounded on } H_+(M, \tau)\}$$

and

$$H_-(M, A, B) = \{L \in H_+^* : L \text{ is bounded on } H_+(M, A, B)\}.$$

3. The spaces $H_+^{(s)}$ and $H_-^{(s)}$

Let $s > 0$. The Gaussian convolution operator \mathcal{G}_s is defined on $L_2(\mathbb{R}^m)$ by (48). Define \mathcal{G}_s on $H_-(\mathbb{R}^m)$ by

$$(103) \quad \mathcal{G}_s \left[\sum_{n \in \mathbb{N}_0^m} c_n \varphi_n \right] = \sum_{n \in \mathbb{N}_0^m} c_n \mathcal{G}_s[\varphi_n], \quad \text{where } (c_n) = O(e^{tn}) \quad \forall t > 0.$$

By Corollary 266, the sum on the right-hand side converges pointwise. Let

$$H_+^{(s)} = \mathcal{G}_s(H_+(\mathbb{R}^m)), \quad H_-^{(s)} = \mathcal{G}_s(H_-(\mathbb{R}^m)).$$

We write $H_+^{(s)}(\mathbb{R}^m)$ and $H_-^{(s)}(\mathbb{R}^m)$ if we want to make the dependence on m explicit.

THEOREM 267.

$$\begin{aligned} H_+^{(s)} &= \bigcup_{M>0} \{ \bigcup_{M>0} A(M, A, B) : A > 0, B > 0, sB < 1 \} \quad \forall s \geq 0 \\ H_-^{(s)} &= \bigcap_{M>0} \{ \bigcup_{M>0} A(M, A, B) : A < 0, sB > 1 \} \quad \forall s > 0. \end{aligned}$$

PROOF. If $M > 0$ and $A \geq A' \in \mathbb{R}$ and $B \leq B' \in \mathbb{R}$, then $A(M, A, B) \subset A(M, A', B')$. It is easily seen that for $s \geq 0$,

$$\bigcup_{t>0} A\left(M, \frac{t}{st+1}, \frac{1}{s+t}\right) = \bigcup_{\substack{A>0 \\ B>0 \wedge sB<1}} A(M, A, B) \quad \forall M > 0.$$

From Lemma 264 it follows that

$$(104) \quad H_+^{(s)} = \bigcup \left\{ A\left(M, \frac{t}{st+1}, \frac{1}{s+t}\right) : M > 0, t > 0 \right\}.$$

It is easily seen that for $s > 0$,

$$\bigcup_{-\min(s, 1/s) < t < 0} A(M, \frac{t}{st+1}, \frac{1}{s+t}) = \bigcup_{\substack{A > 0 \\ sB > 1}} A(M, A, B) \quad \forall M > 0.$$

From Lemma 264 it follows that

$$H_-^{(s)} = \bigcap_{M > 0} \left\{ \bigcup_{-\min(s, 1/s) < t < 0} A(M, \frac{t}{st+1}, \frac{1}{s+t}) \right\}.$$

□

4. Duality between $H_-^{(s)}$ and $H_+^{(s)}$

LEMMA 268. *Let $F \in H_-$. Then*

$$\lim_{\tau \downarrow 0} \mathcal{G}_s \mathcal{N}_\tau[F](x) = \mathcal{G}_s[F](x) \quad \forall x \in \mathbb{R}^m.$$

There are $\epsilon, M > 0$, such that

$$\mathcal{G}_s \mathcal{N}_\tau[F] \in A(M, A, B) \quad \forall \tau \in [0, \epsilon), A < 0, B > 1/s$$

PROOF. The first part follows from (103) and (98). The second part follows from Lemma 264. □

THEOREM 269. *Let $\varphi \in H_+^{(s)}$ and $\psi \in H_-^{(s)}$. We have*

$$\langle \mathcal{G}_s^{-1} \psi, \mathcal{G}_s^{-1} \varphi \rangle = \int_{\mathbb{C}} \overline{\psi(z)} \varphi(z) \rho_s(z) dz.$$

PROOF. There is an $F \in H_-$ such that $\mathcal{G}_s[F] = \psi$. Let $f_\tau = \mathcal{N}_\tau[F]$. Let $\psi_\tau = \mathcal{G}_s[f_\tau]$. It is easily seen that

$$\langle \mathcal{G}_s^{-1} \psi_\tau, \mathcal{G}_s^{-1} \varphi \rangle = \int_{\mathbb{C}} \overline{\psi_\tau(z)} \varphi(z) \rho_s(z) dz$$

for every $\tau > 0$, and that

$$\langle \mathcal{G}_s^{-1} \psi, \mathcal{G}_s^{-1} \varphi \rangle = \lim_{\tau \downarrow 0} \langle \mathcal{G}_s^{-1} \psi_\tau, \mathcal{G}_s^{-1} \varphi \rangle.$$

There are $M, A > 0$ and $B < 1/s$ such that $\varphi \in A(M, A, B)$. By Lemma 268 and the dominated convergence theorem, this implies that

$$\lim_{\tau \downarrow 0} \int_{\mathbb{C}} \overline{\psi_\tau(z)} \varphi(z) \rho_s(z) dz = \int_{\mathbb{C}} \overline{\psi(z)} \varphi(z) \rho_s(z) dz.$$

□

5. Approximation in H_-

Let \mathbf{c} be a bounded function on \mathbb{R}^m which has a compact support, is continuous in 0, and satisfies $\mathbf{c}(0) = 1$.

DEFINITION 270. For $F \in H_-$ define $\mathcal{C}_n[F] \in H_-$ by

$$\langle \mathcal{C}_n[F], h \rangle = \int_{\mathbb{C}^m} \mathbf{c}(\text{Im}(z)/n) \overline{\mathcal{G}_s[F](z)} \mathcal{G}_s[h](z) \rho_s(z) dz \quad \forall h \in H_+.$$

PROPOSITION 271. *If $h \in H_+$ and $F \in H_-$ then*

$$\langle \mathcal{C}_n[F], h \rangle = \int_{\mathbb{R}^m} \overline{f_n(x)} \mathcal{G}_s[h](x) dx,$$

where $f_n(x) = \int_{\mathbb{R}^m} \mathbf{c}(y/n) \mathcal{G}_s[F](x + 2iy) \rho_s(iy) dy$.

PROOF. This follows from Proposition 250, Theorem 267, and Fubini's theorem. \square

REMARK 272. \mathcal{C}_n is a non-negative convolution operator on \mathbf{H} . (This is easily seen if we take $F(x) = e^{iyx}$ with $y \in \mathbb{R}$ in Proposition 271.)

THEOREM 273. Let $F \in \mathbf{H}_-$ and $h \in \mathbf{H}_+$. Then

$$\langle F, h \rangle = \lim_{n \rightarrow \infty} \langle \mathcal{C}_n[F], h \rangle.$$

The convergence is uniform for h in subsets $\mathbf{H}_+(M, A, B)$ with $M, A, B > 0$.

PROOF. By Theorem 269,

$$\langle F, h \rangle = \int_{\mathbb{C}} \overline{\mathcal{G}_s[F](z)} \mathcal{G}_s[h](z) \rho_s(z) dz.$$

By definition,

$$\langle \mathcal{C}_n[F], h \rangle = \int_{\mathbb{C}^m} \mathbf{c}(\text{Im}(z)/n) \overline{\mathcal{G}_s[F](z)} \mathcal{G}_s[h](z) \rho_s(z) dz.$$

It suffices to prove that for every $\psi \in \mathbf{H}_-^{(s)}$,

$$(105) \quad \lim_{n \rightarrow \infty} \int_{\mathbb{C}^m} (1 - \mathbf{c}(\text{Im}(z)/n)) |\overline{\psi(z)} \varphi(z)| \rho_s(z) dz = 0$$

uniform for $\varphi \in \mathcal{G}_s(H_+(M, \tau))$, for every $M, \tau > 0$. From (104) it follows that it suffices to prove that for every $\psi \in \mathbf{H}_-^{(s)}$, limit (105) converges uniformly for $\varphi \in \mathbf{A}(M, A, B)$, for every $M, A > 0$ and $B < 1/s$. Let such ψ, M, A and B be given. By Theorem 267, there are $M', A' > 0$ such that

$$|\overline{\psi(z)} \varphi(z)| \rho_s(z) \leq M' \exp\{-A'|z|^2\} \quad \forall z \in \mathbb{C}, \varphi \in \mathbf{A}(M, A, B).$$

Because the function in the right-hand-side of this estimate is integrable, the dominated convergence theorem can be used to prove that limit (105) converges uniformly for $\varphi \in \mathbf{A}(M, A, B)$. \square

APPENDIX N

Estimates

1. Estimates for Bessel functions on $(0, \infty)$

We prove in this section that for every $n \in \mathbb{N}_0$ the n 'th Bessel function of the first kind J_n satisfies

$$|J_n(x)| \leq 3x^{-1/3} \quad \forall x > 0.$$

First we will prove some preparatory results.

LEMMA 274. *Let $-\infty \leq a < b \leq \infty$, and let $\varphi: (a, b) \rightarrow (c, d)$ be a monotone increasing diffeomorphism with a monotone increasing derivative. Assume that $p \in (0, \infty)$ and $n \in \mathbb{N}$ are such that $d - c = np$. Let $f: (c, d) \rightarrow \mathbb{R}$ be a continuous function satisfying $f(x + p) = -f(x)$ for all $x \in (c, d - p)$ and $f(c + t) \geq 0$ for $t \in (0, p)$. Then*

$$\int_a^b f(\varphi(x)) dx \leq \int_a^{\varphi^{-1}(c+p)} f(\varphi(x)) dx.$$

PROOF. Let $I = \int_a^b f(\varphi(x)) dx$. For $k \in \{0, 1, \dots, n - 1\}$ let

$$I_k = \int_{\varphi^{-1}(c+kp)}^{\varphi^{-1}(c+(k+1)p)} f(\varphi(x)) dx.$$

Note that the sign of the integrand is constant for each k . By the integral transformation theorem and the well-know formula for the derivative of the inverse of a function,

$$I_k = \int_{c+kp}^{c+(k+1)p} f(y) \frac{1}{\varphi'(\varphi^{-1}(y))} dy.$$

From

$$|f(c + (k + 1)p + t)| = |f(c + kp + t)| \quad \forall t \in (0, p), k \in \{0, \dots, n - 1\}$$

and the fact that $1/\varphi'(\varphi^{-1}(x))$ is a monotone decreasing function, it follows that $|I_{k+1}| \leq |I_k|$ for all k . Hence $I = I_0 + \sum_{k=1}^{n-1} I_k \leq I_0$. \square

LEMMA 275. *Let $-\infty \leq a < b \leq \infty$, and let $\varphi: (a, b) \rightarrow (c, d)$ be a monotone increasing diffeomorphism with a monotone increasing derivative. Assume that $p \in (0, \infty)$ and $n \in \mathbb{N}$ are such that $d - c = 2np$. Let $f: (c, d) \rightarrow \mathbb{R}$ be a continuous function satisfying $f(x + p) = -f(x)$ for all $x \in (c, d - p)$ and $0 \leq f(c + t) = -f(c + p - t)$ for $t \in (0, \frac{p}{2})$. Then*

$$\int_a^b f(\varphi(x)) dx \leq \int_a^{\varphi^{-1}(c+\frac{p}{2})} f(\varphi(x)) dx.$$

PROOF. Let $I = \int_a^b f(\varphi(x)) dx$. For $k \in \{0, 1, \dots, 2n - 1\}$ let

$$I_k = \int_{\varphi^{-1}(c+k\frac{p}{2})}^{\varphi^{-1}(c+(k+1)\frac{p}{2})} f(\varphi(x)) dx.$$

Note that the sign of the integrand is constant for each k . By the integral transformation theorem and the well-know formula for the derivative of the inverse of a function,

$$I_k = \int_{c+k\frac{p}{2}}^{c+(k+1)\frac{p}{2}} f(y) \frac{1}{\varphi'(\varphi^{-1}(y))} dy.$$

From

$$|f(c + (k + 1)\frac{p}{2} + t)| = |f(c + k\frac{p}{2} + (\frac{p}{2} - t))| \quad \forall t \in (0, \frac{p}{2}), k \in \{0, \dots, 2n - 1\}$$

and the fact that $1/\varphi'(\varphi^{-1}(x))$ is a monotone decreasing function, it follows that $|I_{k+1}| \leq |I_k|$ for all k . Hence $I = I_0 + \sum_{k=1}^{2n-1} I_k \leq I_0$. \square

LEMMA 276. *Let $n \in \mathbb{N}$. Then $J_n(n) \leq n^{-1/3}$.*

REMARK 277. Formula 9.1.61 in [75] provides a sharper estimate. However no proof and no analogue estimates for the Bessel functions of the second kind are given.

PROOF. The following integral representations for J_n , the n 'th Bessel function of the first kind is well known:

$$J_n(z) = \frac{1}{\pi} \int_0^\pi \cos(n\theta - z \sin \theta) d\theta.$$

We use this formula to prove that $J_n(n) \leq n^{-1/3}$. It is known that the first positive zero j_ν of J_ν on $[0, \infty)$ satisfies $j_\nu > \nu$ and that $J_\nu(x)$ is positive for $x \in (0, j_\nu)$.

For $\nu > 0$ define diffeomorphism $\varphi_\nu: (0, \pi) \rightarrow (0, \nu\pi)$ by

$$\varphi_\nu(\theta) = \nu(\theta - \sin \theta).$$

Then

$$J_n(n) = \frac{1}{\pi} \int_0^\pi \cos(\varphi_n(\theta)) d\theta.$$

From Lemma 275 it follows that

$$\begin{aligned} J_n(n) &\leq \frac{1}{\pi} \int_0^{\varphi_n^{-1}(\frac{\pi}{2})} \cos(\varphi_n(\theta)) d\theta \\ &= n^{-1/3} \frac{1}{\pi} \int_0^{n^{1/3}\varphi_n^{-1}(\frac{\pi}{2})} \cos(\varphi_n(n^{-1/3}\theta)) d\theta. \end{aligned}$$

We will show that $n^{1/3}\varphi_n^{-1}(\frac{\pi}{2}) \leq \pi$, or equivalently, that $\frac{\pi}{2} \leq \varphi_n(n^{-1/3}\pi)$. We show that even

$$(106) \quad \varphi_\nu(\nu^{-1/3}\pi) \geq \pi \quad \forall \nu \geq 1.$$

Note that we have equality for $\nu = 1$. We have

$$\frac{d}{d\nu} \varphi_\nu(\nu^{-1/3}\pi) = \nu^{-1/3}\pi \left(\frac{2}{3} + \frac{1}{3} \cos(\nu^{-1/3}\pi) \right) - \sin(\nu^{-1/3}\pi).$$

This is ≥ 0 because

$$\frac{2}{3} + \frac{1}{3} \cos \theta \geq \frac{\sin \theta}{\theta} \quad \forall \theta \in [0, \pi].$$

Hence (106). Hence $J_n(n) \leq n^{-1/3}$. \square

LEMMA 278. *Let $n \in \mathbb{N}$. Then $|Y_n(n)| \leq (2.2)n^{-1/3}$.*

PROOF. The following integral representations for Y_n , the n 'th Bessel function of the second kind is known: If $\operatorname{Re}(z) > 0$ then

$$-Y_n(z) = \frac{1}{\pi} \int_0^\pi \sin(n\theta - z \sin \theta) d\theta + \frac{1}{\pi} \int_0^\infty (e^{nt} + e^{-nt} \cos(n\pi)) e^{-z \sinh t} dt.$$

It is known that the first positive zero y_ν of Y_ν on $[0, \infty)$ satisfies $y_\nu > \nu$ and that $Y_\nu(x)$ is negative for $x \in (0, y_\nu)$. An argument similar to the one used in the proof of Lemma 276 shows that the first integral in the above representation of $-Y_n(n)$ is $\leq n^{-1/3}$. Using

$$\sinh t \geq t + t^3/6 \quad \forall t \geq 0,$$

we see that

$$(e^{nt} + e^{-nt} \cos(n\pi)) e^{-z \sinh t} \leq 2e^{nt} e^{-n \sinh t} \leq 2e^{nt} e^{-n(t+t^3/6)} = 2e^{-nt^3/6}.$$

Because

$$\int_0^\infty e^{-t^3} dt = \frac{1}{3} \int_0^\infty e^{-x} x^{-2/3} dx = \frac{1}{3} \Gamma\left(\frac{1}{3}\right),$$

this implies that

$$-Y_n(n) \leq n^{-1/3} + \frac{2}{\pi} \int_0^\infty e^{-nt^3/6} dt = n^{-1/3} + \frac{2}{\pi} 6^{1/3} \frac{1}{3} \Gamma\left(\frac{1}{3}\right) n^{-1/3} \leq (2.2)n^{-1/3}.$$

□

LEMMA 279. $J_0(x) \leq 2x^{-1/2}$ for $x > 0$.

PROOF. The following integral representations for J_0 is well known:

$$J_0(x) = \frac{2}{\pi} \int_0^\infty \sin(x \cosh t) dt.$$

Hence

$$J_0(x) = x^{-1/2} \frac{2}{\pi} \int_0^\infty \sin(x \cosh(x^{-1/2}t)) dt.$$

Because $\cosh(y) \geq 1 + y^2/2$ we have for $t \geq \sqrt{2\pi}$,

$$x \cosh(x^{-1/2}t) \geq x + \frac{1}{2}t^2 \geq \pi \quad \forall x > 0.$$

The function $x \mapsto x \cosh(x^{-1/2}t)$ is positive and monotone increasing on $(0, \infty)$. By Lemma 274,

$$|J_0(x)| \leq x^{-1/2} \frac{2}{\pi} \int_0^{\sqrt{2\pi}} |\sin(x \cosh(x^{-1/2}t))| dt.$$

Hence

$$|J_0(x)| \leq x^{-1/2} \frac{2\sqrt{2}}{\sqrt{\pi}} \leq 2x^{-1/2} \quad \forall x > 0.$$

□

LEMMA 280. $|J_n(x)| \leq 3x^{-1/3}$ for all $n \in \mathbb{Z}$ and $x > 0$.

PROOF. Since $J_{-n}(z) = (-1)^n J_n(z)$, we can assume without loss of generality that $n \geq 0$. For $n = 0$, the estimate follows from $x^{1/2}|J_0(x)| \leq 2$.

Let $n \in \mathbb{N}$. Let $M_n(x) = \sqrt{J_n(x)^2 + Y_n(x)^2}$. In [107] (page 446) we find that $x \mapsto x^{1/2}M_n(x)$ is monotonic decreasing in $(0, \infty)$ when $n > 1/2$. Hence $x \mapsto x^{1/3}M_n(x)$ is monotonic decreasing in $(0, \infty)$. The results above Lemma 280 imply that $n^{1/3}M_n(n) \leq 3$. Combining these results we see that $x^{1/3}M_n(x) \leq 3$ for all $x \geq n$. Hence $x^{1/3}|J_n(x)| \leq 3$ for all $x \geq n$.

For $n \geq 1$, $J_n(x)$ is non-negative and increasing on $0 \leq x \leq n$. (Proof: For $n \geq 1$, $J'_n(0) \geq 0$ and $j'_n > n$, where j'_n is the first positive zero of J'_n . See [107]) Hence $0 \leq x^{1/3} J_n(x) \leq n^{1/3} J_n(n)$ when $0 \leq x \leq n$. Hence $x^{1/3} |J_n(x)| \leq 3$ for all $x \geq 0$ and $n \geq 1$. \square

2. Estimates for the Gamma function

PROPOSITION 281 ([66]). *If $k \geq 0$ and $0 < \lambda < 1$ or $\lambda > 2$ then*

$$\frac{\Gamma(k + \lambda)}{\Gamma(k + 1)} < (k + \lambda/2)^{\lambda-1}.$$

REMARK 282. This generalization (to non-integer values of k) of Lorch's improvement of Gautschi's inequality was obtained by Laforgia in [66].

PROPOSITION 283 ([51]). *If $\delta, \alpha + \delta > 0$ then*

$$\frac{\Gamma^2(\delta + \alpha)}{\Gamma(\delta)\Gamma(\delta + 2\alpha)} \leq \frac{\delta}{\delta + \alpha^2}.$$

REMARK 284. This is Gurland's inequality. It is proved in [51] (and in [91]) by application of the Rao-Cramer inequality (in estimation theory) to the gamma distribution. It is proved without an appeal to statistical arguments in [13] by using Gauss's formula for $F(a; b; c; a)$.

We have equality if, and only if, $\alpha \in \{0, 1\}$. An alternative form is

$$\frac{\Gamma^2\left(\frac{x+y}{2}\right)}{\Gamma(x)\Gamma(y)} \leq \frac{\min(x, y)}{\min(x, y) + (x - y)^2/4}$$

for $x, x + y > 0$. Here, the two appearances of $\min(x, y)$ can be replaced (simultaneously) by anything larger; e.g. by x, y or $(x + y)/2$. Yet another form of Gurland's inequality is the first inequality in

$$\frac{\Gamma^2(j)}{\Gamma(j + m)\Gamma(j - m)} \leq \frac{j - |m|}{j - |m| + m^2} \leq \frac{j}{j + m^2}$$

for $j > 0$ and $m > -j$.

3. Estimates for Laguerre polynomials on $(0, \infty)$

The following integral representations for L_n^m , the n 'th generalized Laguerre polynomial with parameter m is well known: For $n, m \in \mathbb{N}_0$ and $r > 0$,

$$(107) \quad n! r^{m/2} L_n^m(r) = e^r \int_0^\infty e^{-y} y^{n+m/2} J_m(2\sqrt{yr}) dy.$$

DEFINITION 285. For $n, m \in \mathbb{N}_0$, let

$$\ell_{m,n}(r) = (-1)^n \sqrt{\frac{n!}{(n+m)!}} r^{m/2} L_n^m(r).$$

PROPOSITION 286. *For $n, m \in \mathbb{N}_0$ and $r > 0$,*

$$|\ell_{m,n}(r)| \leq 3r^{-1/6} e^r \frac{1 + \min(n, n+m)}{1 + \min(n, n+m) + m^2/4} (2n + m + 1)^{-1/6}.$$

In particular

$$|L_n(r)| \leq 3r^{-1/6} e^r (2n + 1)^{-1/6} \quad \forall r > 0, n \in \mathbb{N}_0.$$

PROOF. By (107) and Lemma 280,

$$\begin{aligned} |\ell_{m,n}(r)| &\leq \frac{3r^{-1/6}e^r}{2^{1/3}\sqrt{n!(n+m)!}} \int_0^\infty e^{-y}y^{n+m/2-1/6} dy \\ &= \frac{3r^{-1/6}e^r}{2^{1/3}\sqrt{n!(n+m)!}} \Gamma\left(n + \frac{m}{2} + \frac{5}{6}\right). \end{aligned}$$

By Proposition 281,

$$|\ell_{m,n}(r)| \leq 3(2r)^{-1/6}e^r \frac{\Gamma\left(n + \frac{m}{2} + 1\right)}{\sqrt{n!(n+m)!}} \left(2n + m + \frac{5}{6}\right)^{-1/6}.$$

This, together with Proposition 283 and $2n + m + \frac{5}{6} \geq \frac{5}{6}(2n + m + 1)$, implies the result. \square

Cambell-Baker-Hausdorff formulas

1. Introduction

The first Cambell-Baker-Hausdorff formula is

$$(108) \quad e^{\mathcal{A}}\mathcal{B}e^{-\mathcal{A}} = e^{[\mathcal{A}, \cdot]}\mathcal{B} = \mathcal{B} + [\mathcal{A}, \mathcal{B}] + \frac{1}{2!}[\mathcal{A}, [\mathcal{A}, \mathcal{B}]] + \dots$$

The second Cambell-Baker-Hausdorff formula is $e^{\mathcal{A}}e^{\mathcal{B}} = e^{\mathcal{C}}$ where

$$\mathcal{C} = \mathcal{A} + \mathcal{B} + \frac{1}{2}[\mathcal{A}, \mathcal{B}] + \frac{1}{12}\{[\mathcal{A}, [\mathcal{A}, \mathcal{B}]] + [\mathcal{B}, [\mathcal{B}, \mathcal{A}]]\} + \dots$$

The Lee-Trotter product formula is:

$$(109) \quad e^{\mathcal{A}+\mathcal{B}} = \lim_{n \rightarrow \infty} \left(e^{\frac{1}{n}\mathcal{A}} e^{\frac{1}{n}\mathcal{B}} \right)^n.$$

Formulas (108) and (109) hold for all operators \mathcal{A} and \mathcal{B} on a finite dimensional Hilbert space.

References: [65], [62], [95], [81]

Special cases of the Cambell-Baker-Hausdorff formulas are often used even in the infinite dimensional case without justification or references. In this appendix we proof (108) under the assumption that \mathcal{A} is nilpotent, and we proof the second Cambell-Baker-Hausdorff formula in case \mathcal{A} and \mathcal{B} are self-adjoint and commute with $[\mathcal{A}, \mathcal{B}]$ and satisfy some technical conditions.

EXAMPLE 287. Using (108) and the Taylor series around 0 of the (hyperbolic) sin and cos functions respectively, we get: If $[\mathcal{C}, \mathcal{A}] = \mathcal{B}$ and $[\mathcal{C}, \mathcal{B}] = \mathcal{A}$ then

$$e^{s\mathcal{C}}\mathcal{A}e^{-s\mathcal{C}} = \cosh(s)\mathcal{A} + \sinh(s)\mathcal{B}, \quad s \in \mathbb{R}$$

If $[\mathcal{C}, \mathcal{A}] = \mathcal{B}$ and $[\mathcal{C}, \mathcal{B}] = -\mathcal{A}$ then

$$e^{s\mathcal{C}}\mathcal{A}e^{-s\mathcal{C}} = \cos(s)\mathcal{A} + \sin(s)\mathcal{B}, \quad s \in \mathbb{R}.$$

2. $e^{[\mathcal{A}, \cdot]}$ if \mathcal{A} is nilpotent

Let \mathbb{V} be a vector space and let \mathcal{A} and \mathcal{B} be linear transformations on \mathbb{V} .

PROPOSITION 288. *If for every $v \in \mathbb{V}$ there is an $N_v \in \mathbb{N}$ such that $\mathcal{A}^n v = 0$ for all $n \geq N_v$, then there is, for every $v \in \mathbb{V}$, an $\tilde{N}_v \in \mathbb{N}$ such that $[\mathcal{A}, \cdot]^n(\mathcal{B})v = 0$ for all $n \geq \tilde{N}_v$. In that case*

$$e^{\mathcal{A}}\mathcal{B}e^{-\mathcal{A}} = e^{[\mathcal{A}, \cdot]}(\mathcal{B})$$

where for $e^{\pm\mathcal{A}}$ is defined on \mathbb{V} by

$$e^{\pm\mathcal{A}}v = \sum_{n=0}^{\infty} \frac{(\pm 1)^n}{n!} \mathcal{A}^n v$$

and $e^{[\mathcal{A}, \cdot]}(\mathcal{B})$ is defined on \mathbb{V} by

$$e^{[\mathcal{A}, \cdot]}(\mathcal{B})v = \sum_{n=0}^{\infty} \frac{1}{n!} [\mathcal{A}, \cdot]^n(\mathcal{B})v.$$

PROOF. $[\mathcal{A}, \cdot]^n(\mathcal{B}) = \sum_{k=1}^{2^n} \mathcal{B}_k^{(n)}$, where $\mathcal{B}_k^{(n)}$ is the composition, in some order, of n times the operator \mathcal{A} and one time the operator \mathcal{B} . Consequently $[\mathcal{A}, \cdot]^n(\mathcal{B})v = 0$ for

$$n \geq \max(N_{\mathcal{B}v}, N_{\mathcal{B}\mathcal{A}v} + 1, N_{\mathcal{B}\mathcal{A}^2v} + 2, \dots, N_{\mathcal{B}\mathcal{A}^{n-1}v} + N_v - 1).$$

The coefficient of $\mathcal{A}^{n-k}\mathcal{B}\mathcal{A}^k$ in the expansion of $e^{\mathcal{A}}\mathcal{B}e^{-\mathcal{A}}$ is $\frac{(-1)^k}{(n-k)!k!}$. All we have to do is to prove that

$$(110) \quad [\mathcal{A}, \cdot]^n(\mathcal{B})v = \sum_{k=0}^n \binom{n}{k} (-1)^k \mathcal{A}^{n-k} \mathcal{B} \mathcal{A}^k v.$$

Using $[\mathcal{A}, \cdot]^n(\mathcal{B}) = [\mathcal{A}, [\mathcal{A}, \cdot]^{n-1}(\mathcal{B})]$ and mathematical induction, this can be reduced to

$$\begin{aligned} & \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} \mathcal{A}^{n-k} \mathcal{B} \mathcal{A}^k v - \sum_{k=1}^n (-1)^{k-1} \binom{n-1}{k-1} \mathcal{A}^{n-k} \mathcal{B} \mathcal{A}^k v \\ &= \sum_{k=0}^n (-1)^k \binom{n}{k} \mathcal{A}^{n-k} \mathcal{B} \mathcal{A}^k v, \end{aligned}$$

which follows from $\binom{n-1}{k} + \binom{n-1}{k-1} = \binom{n}{k}$. \square

3. A generalization

PROPOSITION 289. *Let \mathbf{H} be a Hilbert space and let \mathcal{A} be a (possibly unbounded) operator on \mathbf{H} . Let \mathbf{V} be a subspace of \mathbf{H} consisting of vectors v in the domain of \mathcal{A} such that $\mathcal{A}v \subset \mathbf{V}$, and*

$$(111) \quad \sum_{n=0}^{\infty} \frac{1}{n!} \|\mathcal{A}^n v\| < \infty.$$

Let $v \in \mathbf{V}$ and let \mathcal{B} be a linear transformation on \mathbf{V} such that

$$(112) \quad \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^n \binom{n}{k} \|\mathcal{A}^{n-k} \mathcal{B} \mathcal{A}^k v\| < \infty.$$

Then

$$e^{\mathcal{A}} \mathcal{B} e^{-\mathcal{A}} v = e^{[\mathcal{A}, \cdot]}(\mathcal{B})v,$$

where $e^{\pm \mathcal{A}} v = \sum_{n=0}^{\infty} \frac{(\pm 1)^n}{n!} \mathcal{A}^n v$ for v satisfying (111), and $e^{[\mathcal{A}, \cdot]}(\mathcal{B})v = \sum_{n=0}^{\infty} \frac{1}{n!} [\mathcal{A}, \cdot]^n(\mathcal{B})v$ for v satisfying (112).

PROOF. By (a vector-valued version of) Fubini's theorem,

$$e^{\mathcal{A}} \mathcal{B} e^{-\mathcal{A}} v = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^n \binom{n}{k} (-1)^k \mathcal{A}^{n-k} \mathcal{B} \mathcal{A}^k v.$$

The result follows from (110). \square

REMARK 290. (112) is satisfied if there are $M_n > 0$ such that $\|\mathcal{A}^{n-k} \mathcal{B} \mathcal{A}^k v\| \leq M_n$ for all n, k , and $\sum_{n=0}^{\infty} \frac{2^n}{n!} M_n < \infty$. Let, for example $\mathbf{H} = \mathbf{L}_2(\mathbb{C})$. Let

$$\mathbf{V} = \text{span}\{(r, \theta) \mapsto e^{-r^2/2} r^n e^{im\theta} : n \in \mathbb{N}_0, m \in \mathbb{Z}\}.$$

Define the unbounded operators \mathcal{A} and \mathcal{B} by $\mathcal{A}[f](r, \theta) = \frac{\partial}{\partial \theta} f(r, \theta)$ and $\mathcal{B}[f](r, \theta) = r \cos(\theta) f(r, \theta)$. We have

$$\|\mathcal{A}^{n-k} \mathcal{B} \mathcal{A}^k f\|^2 \leq 2^{2(n-k)} \int_0^{\infty} \int_0^{2\pi} |r (\frac{\partial}{\partial \theta})^n f(r, \theta)|^2 r dr d\theta$$

If $s \in \mathbb{N}_0$ and $m \in \mathbb{Z}$ and $f(r, \theta) = r^s e^{im\theta}$, then

$$\|\mathcal{A}^{n-k} \mathcal{B} \mathcal{A}^k f\| \leq (2|m|)^n \|(r, \theta) \mapsto r f(r, \theta)\| \quad \forall n, k.$$

Consequently, (112) is satisfied for all $v \in \mathbb{V}$. Let $\mathcal{C}f(r, \theta) = r \sin(\theta) f(r, \theta)$. We have $[\mathcal{A}, \mathcal{B}]f(r, \theta) = -\mathcal{C}f(r, \theta)$ and $[\mathcal{A}, \mathcal{C}]f(r, \theta) = \mathcal{B}f(r, \theta)$. Hence

$$\begin{aligned} e^{\alpha \frac{\partial}{\partial \theta}} r \cos(\theta) e^{-\alpha \frac{\partial}{\partial \theta}} f(r, \theta) &= (\cos(\alpha) r \cos(\theta) - \sin(\alpha) r \sin(\theta)) f(r, \theta) \\ &= r \cos(\theta + \alpha) f(r, \theta). \end{aligned}$$

4. If operators \mathcal{A} and \mathcal{B} are skew-adjoint

PROPOSITION 291 ([108], Theorem 8.35). *Let \mathcal{A} be a skew-adjoint operator on Hilbert space \mathbf{H} . Then*

$$e^{[\mathcal{A}, \cdot]} = \exp\{\mathcal{A} \otimes \mathcal{I} + \mathcal{I} \otimes \mathcal{A}\} = \exp\{\mathcal{A}\} \otimes \exp\{\mathcal{A}\}$$

on the space of Hilbert-Schmidt operators on \mathbf{H} . This means that $e^{[\mathcal{A}, \cdot]} \mathcal{B} = e^{\mathcal{A}} \mathcal{B} e^{-\mathcal{A}}$ for a Hilbert-Schmidt operator \mathcal{B} .

THEOREM 292 (Trotter product formula, Theorem 5 of Section 8 in [85], Theorem X.51 in [92]). *Let \mathcal{A}, \mathcal{B} and $\overline{\mathcal{A} + \mathcal{B}}$ be the infinitesimal generators of strongly continuous contraction semigroups P^t, Q^t and R^t on a Banach space \mathbf{X} . Then for all $u \in \mathbf{X}$,*

$$R^t u = \lim_{n \rightarrow \infty} \left(P^{\frac{t}{n}} Q^{\frac{t}{n}} \right)^n u$$

uniformly for t in any compact subset of $[0, \infty)$.

THEOREM 293 (Theorem 7 in [85]). *Let \mathcal{A}, \mathcal{B} be skew-adjoint operators on a Hilbert space \mathbf{H} , and suppose that the restriction of $[\mathcal{A}, \mathcal{B}]$ to $\mathfrak{D}(\mathcal{A}\mathcal{B}) \cap \mathfrak{D}(\mathcal{B}\mathcal{A}) \cap \mathfrak{D}(\mathcal{A}^2) \cap \mathfrak{D}(\mathcal{B}^2)$ is essentially skew-adjoint. Then for all $u \in \mathbf{H}$,*

$$e^{t[\overline{\mathcal{A}, \mathcal{B}}]} u = \lim_{n \rightarrow \infty} \left(e^{-\sqrt{\frac{t}{n}} \mathcal{A}} e^{-\sqrt{\frac{t}{n}} \mathcal{B}} e^{\sqrt{\frac{t}{n}} \mathcal{A}} e^{\sqrt{\frac{t}{n}} \mathcal{B}} \right)^n u$$

uniformly for t in any compact subset of $[0, \infty)$.

5. If operators \mathcal{A} and \mathcal{B} commute with $[\mathcal{A}, \mathcal{B}]$.

PROPOSITION 294. *Let \mathcal{A}, \mathcal{B} be skew-adjoint operators on a Hilbert space \mathbf{H} , and suppose that the restriction of $[\mathcal{A}, \mathcal{B}]$ to $\mathfrak{D}(\mathcal{A}\mathcal{B}) \cap \mathfrak{D}(\mathcal{B}\mathcal{A}) \cap \mathfrak{D}(\mathcal{A}^2) \cap \mathfrak{D}(\mathcal{B}^2)$ is essentially skew-adjoint. If $e^{t\mathcal{A}}$ and $e^{t\mathcal{B}}$ commute with $e^{-t\mathcal{A}} e^{-t\mathcal{B}} e^{t\mathcal{A}} e^{t\mathcal{B}}$ for every t then*

$$(113) \quad e^{t^2[\overline{\mathcal{A}, \mathcal{B}}]} = e^{-t\mathcal{A}} e^{-t\mathcal{B}} e^{t\mathcal{A}} e^{t\mathcal{B}} \quad \forall t \in \mathbb{R}.$$

If, moreover, $\mathcal{A} + \mathcal{B}$ is essentially skew-adjoint then

$$(114) \quad e^{t\overline{\mathcal{A} + \mathcal{B}}} = e^{t\mathcal{A}} e^{t\mathcal{B}} e^{-\frac{1}{2}t^2[\overline{\mathcal{A}, \mathcal{B}}]} \quad \forall t \in \mathbb{R}.$$

PROOF. Let

$$F(t) = e^{-t\mathcal{A}} e^{-t\mathcal{B}} e^{t\mathcal{A}} e^{t\mathcal{B}}.$$

We have

$$F(t)^2 = e^{-t\mathcal{A}} F(t) e^{t\mathcal{A}} F(t) = e^{-2t\mathcal{A}} e^{-2t\mathcal{B}} e^{2t\mathcal{A}} e^{2t\mathcal{B}} \quad \forall t.$$

Hence

$$F(t)^4 = F(t)^2 e^{-t\mathcal{B}} F(t)^2 e^{t\mathcal{B}} = e^{-2t\mathcal{A}} e^{-2t\mathcal{B}} e^{2t\mathcal{A}} e^{2t\mathcal{B}} = F(2t) \quad \forall t.$$

Hence

$$(F(t))^{4^n} = F(2^n t) \quad \forall t.$$

Hence

$$(F(\sqrt{\frac{t}{4^n}}))^{4^n} = F(\sqrt{t}) \quad \forall t.$$

By Theorem 293,

$$e^{t\overline{[\mathcal{A}, \mathcal{B}]}}u = \lim_{n \rightarrow \infty} (F(\sqrt{\frac{t}{4^n}}))^{(4^n)} = F(\sqrt{t}).$$

Hence (113). Using mathematical induction and (113), we get

$$(e^{t\mathcal{A}}e^{t\mathcal{B}})^n = e^{tn\mathcal{A}}e^{tn\mathcal{B}}e^{-\frac{1}{2}t^2n(n-1)\overline{[\mathcal{A}, \mathcal{B}]}} \quad \forall n \in \mathbb{N}, t \in \mathbb{R}.$$

Hence

$$(e^{\frac{t}{n}\mathcal{A}}e^{\frac{t}{n}\mathcal{B}})^n = e^{t\mathcal{A}}e^{t\mathcal{B}}e^{-\frac{1}{2}t^2(1-\frac{1}{n})\overline{[\mathcal{A}, \mathcal{B}]}} \quad \forall n \in \mathbb{N}, t \in \mathbb{R}.$$

This, together with Theorem 292, implies (114). □

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Index

- $(\cdot)'$, 14
- $(\cdot)_p$, 14
- $A\Delta B$, 18
- C^* -subalgebra, 119
- H_n , 89
- J_n , 144
- L_n^m , 91, 146
- $P^{(M)}$, 60
- S , 46
- V_M , 8
- $[\rho]_M$, 7
- $A(M, A, B)$, 140
- A_μ , 120
- $B_+(\mathbb{H})$, 14
- $B_0(\mathbb{H})$, 14
- $B_1(\mathbb{H})$, 14
- $B_2(\mathbb{H})$, 14
- $B_\infty(\mathbb{H})$, 14
- $B_{00}(\mathbb{H})$, 14
- $M^{(\text{Bargmann})}$, 46
- $\mathbb{1}$, 20
- $\mathbb{1}^{(\mathbb{H})}$, 21
- \mathcal{C}_n , 141
- Δ_x , 32
- \mathcal{F}_θ , 69
- $\mathbf{G}_{u,v}$, 71
- \mathcal{G}_s , 72, 136, 140
- $\mathcal{G}_s \mathcal{N}_{\text{atanh}(t)}$, 137
- \mathbb{H} , 17
- \mathbb{H}_+ , 139
- $\mathbb{H}_+^{(s)}$, 140
- \mathbb{H}_- , 139
- $\mathbb{H}_-^{(s)}$, 140
- $\mathbb{H}_1 \hat{\otimes} \mathbb{H}_2$, 13
- \mathbf{K} , 14
- $L_\infty(\Omega, \Sigma, M)$, 26
- $M(\Sigma_1; \Omega_2, \Sigma_2, M_2)$, 29
- \mathcal{N}_τ , 137, 139
- Ω , 17
- $\Psi_{\ell,k}$, 92
- \mathbf{R}_φ , 70
- $S_{1/2}^{1/2}(\mathbb{R}^m)$, 139
- \mathcal{S}_{pqr} , 136
- Σ , 17
- $\Sigma_1 \times \Sigma_2$, 32
- \mathbf{U} , 95
- \mathcal{U}_z , 95
- \mathbf{W}_s , 72
- \mathcal{Z}_λ , 70
- $\text{co}(M(\Sigma))$, 35
- \forall_M , 18
- \forall_μ , 13
- \leftarrow , 29
- \leftrightarrow , 29
- $\text{span}(\cdot)$, 13
- $\varphi_n^{(q,p)}$, 73
- ρ_M , 26
- ρ'_M , 26
- $\rho_M(z)$, 56
- ρ_s , 136
- $\rho_{s,t}$, 137
- σ -additive set function, 17
- σ -field, 17
- $\tilde{\mathbf{G}}_{u,v}$, 35, 71
- Δ , 71, 89
- $e^{[\mathcal{A}, \cdot]}$, 149
- g_z , 46, 72
- SOT, 14
- WOT, 14
- weak*, 14
- absolutely continuous, 13
- additive set function, 17
- algebra, 13
- algebra of subsets, 17
- approximation of POVMs by PVMs, 58
- Bargmann POVM, 46
- Bargmann space, 117
- Bessel function of the first kind, 143
- Borel σ -field, 13
- Borel subsets, 13
- bounded operator density, 23
- Cambell-Baker-Hausdorff formula, 149
- Carleman operator, 127
- characterization of maximality, 43
- commutative multiplicity theorem, 53
- commutative von Neumann algebras, 119
- complete set of commuting self-adjoint operators, 37
- complete set of observables, 37
- complex translation, 129
- conditional confidence measure, 32
- cone generated by a POVM, 35
- contraction of a measure, 13
- convex hull of the range of a POVM, 35

- De Bruijn, 139
- density of subalgebras, 48, 119
- density of subspaces, 105
- density operator, 69
- Dirac, 37
- dominance of POVMs, 29
- double commutant, 119

- equivalent POVMs, 29
- essential supremum, 26
- essentially bounded, 26
- estimate of Bessel functions, 143
- estimate of Laguerre polynomials, 146

- Federer metric space, 61
- field of subsets, 17
- finitely additive set function, 17
- FPOVM, 18

- Gamma function, 146
- Gauss's theorem, 129
- Gaussian convolution, 136
- Gautschi's inequality, 146
- Gelfand-Shilov space, 139
- generalized functions, 139
- generalized Laguerre polynomial, 91, 146
- Gurland's inequality, 146

- harmonic oscillator, 137
- Hermite basis function, 135
- Hermite polynomial, 89
- Hermite series, 63, 138, 140
- hyponormal operators, 55

- image measure, 35
- injectivity of a POVM, 28
- isometry, 13
- isomorphic POVMs, 34

- Kaplansky's density theorem, 119
- kernel, 32

- Lee-Trotter product formula, 149
- linear isometry, 13

- maximal, 29
- maximal commutative von Neumann algebra, 37, 119
- measurable function, 123
- measurable partition, 13
- measurable space, 17
- measure, 17
- measure space, 17
- Minimal Naimark extension, 19
- minimal normal extension, 55

- Naimark extension, 19
- non-degenerate, 119
- normal extension, 55

- operational approach, 7
- operator, 13
- operator density, 23
- operator ranges, 121

- Pegg-Barnett phase PVM, 60
- positive operator-valued measure, 18
- POVM, 18

- quantum mechanical expectation value, 79

- restriction algebra, 56

- section of a measurable set, 32
- sesquilinear form, 13
- smearing, 57
- smoothing operators, 139
- spatially isomorphic, 120
- spectral inclusion theorem, 55
- spectral radius, 55
- strong operator topology, 14
- subalgebra, 13
- subnormal operators, 55
- Susskind-Glogower phase POVM, 46

- Trotter product formula, 149, 151

- uniform multiplicity, 51
- unital C^* -subalgebra, 119
- unitary, 13

- von Neumann algebra, 119

- weak operator topology, 14
- weak-star topology, 14
- weakly measurable, 39
- weakly measurable function, 123

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Samenvatting

In de, aan de quantum mechanica gerelateerde, meettheorie worden positieve operator waardige maten gebruikt als wiskundig model voor metingen. De wiskundige eigenschappen hiervan worden onderzocht waarbij vooral wordt gekeken naar dominantie en het hieraan gerelateerde begrip maximaliteit. De wiskundige methoden die worden toegepast bij deze onderzoeken komen uit de hoek van de functionaal analyse, maattheorie en operator theorie.

Eveneens aan de quantum mechanica gerelateerd zijn de zogenaamde fase-ruimte representaties. Hieraan ten grondslag ligt de keuze van een lineaire afbeelding van de vector ruimte die wordt bepaald door de verzameling van dichtheidsoperatoren, welke model staan voor de quantum mechanische toestanden, naar een vector ruimte van functies op het zogenaamde fasevlak. Onderzocht worden, in het speciale geval van een familie van faseruimte representaties interpolerende tussen de Wigner en Husimi representaties, de wiskundige eigenschappen van de bijbehorende afbeeldingen. Hierbij wordt gebruik gemaakt van methoden uit de functionaal analyse en de complexe functietheorie van een enkele variabele.

Curriculum Vitae

Robbert Beukema werd op 6 januari 1975 geboren te Groningen. Na de middelbare school (Augustinuscollege Groningen) studeerde hij vanaf 1993 wiskunde aan de Rijksuniversiteit Groningen, alwaar hij in 1998 afstudeerde. Vanaf 1998 was hij als OIO werkzaam bij de toegepaste analyse groep van de faculteit Wiskunde en Informatica aan de Technische Universiteit Eindhoven.