# Multigrid applied to two-dimensional exponential fitting for drift-diffusion models

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# MULTIGRID APPLIED TO TWO-DIMENSIONAL EXPONENTIAL FITTING FOR DRIFT-DIFFUSION MODELS

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## 1. Introduction

We consider drift-diffusion models of the following type:

$$
(1.1) \quad \begin{cases} \text{Find } u \in H^1(\Omega) \text{ such that} & \\ -\operatorname{div}(\nabla u + u\,\nabla\psi) + cu = f & \text{in } \Omega \subset \mathbb{R}^2 \\ u = g & \text{on } \Gamma_0 \subset \partial\Omega \\ \dfrac{\partial u}{\partial n} + u\,\dfrac{\partial \psi}{\partial n} = 0 & \text{on } \Gamma_1 = \partial\Omega \backslash \Gamma_0 \end{cases}
$$

We assume that $f, g, \psi$ and $c$ are given functions, with $c \geq 0$. The current $\mathbf{J}$ is defined by $\mathbf{J} = \nabla u + u\,\nabla\psi$.
An important application of these equations is in the field of semiconductor device simulation (then $\|\nabla\psi\|$ extremely large in part of the domain). Recently in [5, 6, 8] Brezzi, Marini, Pietra introduced a two-dimensional exponential fitting method for the discretization of (1.1). The method, which results in a linear system with an $M$-matrix, has nice features such as current conservation and good approximation of sharp shapes. The method is derived by using the Slotboom variable, a mixed finite element scheme and Lagrange multipliers. For the case $c \equiv 0$ the well-known Raviart-Thomas mixed finite element scheme is used (see [5, 6]). For the situation $c \not\equiv 0$ new mixed finite element schemes are introduced in [7, 8].
It is well-known that multigrid methods can be very efficient for solving the large sparse systems which result from the discretization of elliptic boundary value problems. In the situation here there is no "standard" multigrid method which can be used. In [9] we introduced a multigrid method for the discretized problem (1.1) if $c \equiv 0$. This method is based on a connection between the discretization resulting from the $2D$ exponential fitting method and a suitable nonconforming linear finite element discretization. In this paper we show that such a connection also exists for the situation $c \not\equiv 0$ (in which the exponential fitting method is essentially different compared with $c \equiv 0$). As in [9] this then leads to a reasonable multigrid method. We describe the resulting algorithm and present some numerical results.

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## 2. Mixed finite element scheme

We start with summarizing some results from [8]. An analysis of properties of the discretization and error estimates can be found in [7, 8].

Using the *Slotboom variable* $\rho := e^{\psi} u$ we can rewrite (1.1) as follows

$$(2.1)\quad \begin{cases} \text{Find } \; u \in H^1(\Omega) \; \text{ such that} & \\ -\operatorname{div}(e^{-\psi} \nabla \rho) + c\, e^{-\psi} \rho = f & \text{in } \Omega \\ \rho = \chi := e^{\psi} g & \text{on } \Gamma_0 \\ \dfrac{\partial \rho}{\partial n} = 0 & \text{on } \Gamma_1 \,. \end{cases}$$

DEFINITIONS 2.2. For ease we assume that $\Omega$ is a polygonal domain. Let $\{T_k\}_{k \geq 0}$ be a regular sequence of decompositions of $\Omega$ into triangles $T$. Let $E_k$ be the set of edges of $T_k$, and $E_k = \{e_i\}_{i \in I_0 \cup I}$ with $I_0$ the index set of edges $e_i \subset \Gamma_0$ and $I$ the index set of edges $e_i \subset \bar{\Omega} \backslash \Gamma_0$. Midpoints of edges are denoted by $m_i$.

For $T \in T_k$ let $\Sigma(T) = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tau^* \right\}$ with $\tau^*$ a given polynomial vector which will be specified below. The finite element spaces are defined as follows:

$$V_k = \{\tau \in (L^2(\Omega))^2 \mid \tau_{|T} \in \Sigma(T) \quad \text{for all} \quad T \in T_k\} \,,$$

$$W_k = \{\varphi \in L^2(\Omega) \mid \varphi_{|T} \in P_0(T) \quad \text{for all} \quad T \in T_k\} \,, \quad \text{and for } \; \zeta \in L^2(\Gamma_0)$$

$$\Lambda_{k,\zeta} = \{\mu \in L^2(E_k) \mid \mu_{|e} \in P_0(e) \text{ for all } e \in E_k, \int_{e_i} (\mu - \zeta)\, ds = 0 \; \forall i \in I_0\} \,.$$

For $h \in L^2(A)$ we use the notation $\bar{h}_{|A} = \frac{1}{|A|} \int_A h(x)\, dx$ (e.g. $\overline{e^{\psi}}_{|T} = \frac{1}{|T|} \int_T e^{\psi(\mathrm{x})}\, d\mathrm{x}$). For a given triangle $T$ let $V_{\mathbf{max}}$ and $V_{\mathbf{min}}$ be the vertices of $T$ where $\psi$ assumes a maximum and minimum value respectively, and let $V_{\mathbf{med}}$ be the third vertex; the edge connecting $V_{\mathbf{max}}$ an $V_{\mathbf{med}}$ is denoted by $\tilde{e}$.

In the discretization on level $k$ (cf. (2.3) below) we assume that $\psi$ is piecewise linear on $T_k$ and $c$ is piecewise constant on $T_k$. The mixed-hybrid discretization of (2.1) is

$$
(2.3)\begin{cases}
\text{Find } \mathbf{J}_k \in V_k\ ,\ \ \rho_k \in W_k\ ,\ \ \lambda_k \in \Lambda_{k,\chi} \ \text{ such that} \\
\sum_T \int_T \overline{e^{\psi}}_{|T}\, \mathbf{J}_k \cdot \boldsymbol{\tau}\, d\mathbf{x} - \sum_T \int_T \operatorname{div} \boldsymbol{\tau}\, \rho_k\, d\mathbf{x} + \sum_T \int_{\partial T} \lambda_k\, \boldsymbol{\tau} \cdot \mathbf{n}\, ds = 0 \quad \forall \boldsymbol{\tau} \in V_k \\
\sum_T \int_T \operatorname{div} \mathbf{J}_k\, \varphi\, d\mathbf{x} + \sum_T \int_T c(\overline{e^{\psi}}_{|\tilde{e}})^{-1}\, \rho_k\, \varphi\, d\mathbf{x} = \int_\Omega f\, \varphi\, d\mathbf{x} \quad \forall \varphi \in W_k \\
\sum_T \int_{\partial T} \mu\, \mathbf{J}_k \cdot \mathbf{n}\, ds = 0 \qquad \forall \mu \in \Lambda_{k,0}\ .
\end{cases}
$$

The Lagrange multiplier $\lambda_k$ can be used as an approximation of $\rho$ at the interelements (cf. [1], [7]).

CHOICE OF $\boldsymbol{\tau}^*$. The choice of $\boldsymbol{\tau}^*$ in $\Sigma(T)$ is crucial. In the classical Raviart-Thomas space, that can be used if $c \equiv 0$, one takes $\boldsymbol{\tau}^* = \begin{pmatrix} x \\ y \end{pmatrix}$. In [8] two examples for a $\boldsymbol{\tau}^*$ which is suitable for the situation $c \geq 0$ are considered. Here we restrict ourselves to the first example. In this example one takes $\boldsymbol{\tau}^* = (\tau_1^*, \tau_2^*)$ as the element of minimum norm among all $\boldsymbol{\tau} = (\tau_1, \tau_2)$ which satisfy

(2.4) $\quad \tau_1, \tau_2 \in P_1(T)$

(2.5) $\quad \int_{\tilde{e}} \boldsymbol{\tau} \cdot \mathbf{n}\, ds = |\tilde{e}|\ ,\ \ \int_e \boldsymbol{\tau} \cdot \mathbf{n}\, ds = 0$ for the edges $e$ of $T$ with $e \neq \tilde{e}$

(2.6) $\quad \int_T \tau_1\, d\mathbf{x} = \int_T \tau_2\, d\mathbf{x} = 0\ .$

In the linear system associated with (2.3) the unknowns corresponding to $\mathbf{J}_k$ and $\rho_k$ can be eliminated by static condensation. This leads to a final system acting on the unknown $\lambda_k$ only, with a symmetric positive definite $M$-matrix, provided the triangulation is of weakly acute type.
Below in Lemma 2.8 we show how this final problem for $\lambda_k$ can be formulated in a variational form. We first introduce some notation.

DEFINITIONS 2.7. Let $\alpha_T := \frac{1}{|T|} \int_T (\tau_1^*)^2 + (\tau_2^*)^2\, d\mathbf{x}$ and
$\beta_T = [1 + \alpha_T\, c_{|T}(\overline{e^{\psi}}_{|T})\ (\overline{e^{\psi}}_{|\tilde{e}})^{-1}\, |T|^2\, |\tilde{e}|^{-2}]^{-1}$. We define symmetric bilinear forms $b_k$, $c_k$ and a linear functional $F_k$ on $L^2(E_k)$ as follows:

$$b_k(\lambda, \mu) = \sum_T\ (|T|\, \overline{e^{\psi}}_{|T})^{-1} \int_{\partial T} \lambda\, \mathbf{n}\, ds \cdot \int_{\partial T} \mu\, \mathbf{n}\, ds$$

$$c_k(\lambda, \mu) = \sum_T\ \beta_T |T|\, |\tilde{e}|^{-2}\, c_{|T}(\overline{e^{\psi}}_{|\tilde{e}})^{-1} \int_{\partial T} \lambda\, \boldsymbol{\tau}^* \cdot \mathbf{n}\, ds \int_{\partial T} \mu\, \boldsymbol{\tau}^* \cdot \mathbf{n}\, ds$$

$$F_k(\mu) = \sum_T\ \bar{f}_{|T}\, \beta_T\, |T|\, |\tilde{e}|^{-1} \int_{\partial T} \mu\, \boldsymbol{\tau}^* \cdot \mathbf{n}\, ds\ .$$

LEMMA 2.8. *The solution* $\lambda_k$ *of* (2.3) *is also the unique solution of the following problem*

$$(2.9)\qquad \begin{cases} \textit{Find } \lambda_k \in \Lambda_{k,\chi} \textit{ such that} \\ b_k(\lambda_k,\mu) + c_k(\lambda_k,\mu) = F_k(\mu) \quad \textit{for all} \quad \mu \in \Lambda_{k,0}\,. \end{cases}$$

*Proof.* First note that $b_k + c_k$ is positive definite on $\Lambda_{k,0}$, so there is a unique solution of (2.9). Now we show that the unique solution $\lambda_k$ of (2.3) satisfies $b_k(\lambda_k,\mu) + c_k(\lambda_k,\mu) = F_k(\mu)$ for all $\mu \in \Lambda_{k,0}$.
Let $\mathbf{J}_k$, $\rho_k$, $\lambda_k$ be the solution of (2.3). Take $T \in \mathcal{T}_k$. Now write $\mathbf{J}_{k|T} = \alpha_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \alpha_3\,\boldsymbol{\tau}^* =: \mathbf{J}_k^{(0)} + \mathbf{J}_k^{(1)}$ with $\mathbf{J}_k^{(0)} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$, $\mathbf{J}_k^{(1)} = \alpha_3\,\boldsymbol{\tau}^*$.
The first equation of (2.3) with $\boldsymbol{\tau} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ (on $T$) yields

$$\overline{e^{\psi}}_{|T} \int_T \mathbf{J}_k^{(0)} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} d\mathbf{x} + \overline{e^{\psi}}_{|T} \int_T \mathbf{J}_k^{(1)} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} d\mathbf{x} + \int_{\partial T} \lambda_k \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \mathbf{n}\, ds = 0\,.$$

Using (2.6) we get

$$\overline{e^{\psi}}_{|T}\, |T|\, \mathbf{J}_k^{(0)} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = -\int_{\partial T} \lambda_k \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \mathbf{n}\, ds\,,$$

so $\mathbf{J}_k^{(0)} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = -(|T|\,\overline{e^{\psi}}_{|T})^{-1} \displaystyle\int_{\partial T} \lambda_k \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \mathbf{n}\, ds\,.$

Combining this with the analogous result for $\boldsymbol{\tau} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ yields

$$\text{(a)}\qquad \mathbf{J}_k^{(0)} = -(|T|\,\overline{e^{\psi}}_{|T})^{-1} \int_{\partial T} \lambda_k\, \mathbf{n}\, ds\,.$$

Now in the first equation of (2.3) we take $\boldsymbol{\tau} = \boldsymbol{\tau}^*$ (on $T$). Then we get:

$$\overline{e^{\psi}}_{|T}\{\alpha_1 \int_T \tau_1^*\, d\mathbf{x} + \alpha_2 \int_T \tau_2^*\, d\mathbf{x} + \alpha_3 \int_T \boldsymbol{\tau}^* \cdot \boldsymbol{\tau}^*\, d\mathbf{x}\} - \rho_{k|T} \int_{\partial T} \boldsymbol{\tau}^* \cdot \mathbf{n}\, ds + \int_{\partial T} \lambda_k\, \boldsymbol{\tau}^* \cdot \mathbf{n}\, ds = 0$$

and thus

$$\text{(b)}\qquad \overline{e^{\psi}}_{|T}\, \alpha_T\, |T|\, \alpha_3 - |\tilde{e}|\, \rho_{k|T} = -\int_{\partial T} \lambda_k\, \boldsymbol{\tau}^* \cdot \mathbf{n}\, ds \qquad \text{(use (2.5), (2.6), 2.7 ).}$$

In the second equation of (2.3) we take $\varphi \equiv 1$ (on $T$); this yields:

$$\int_T \operatorname{div}(\mathbf{J}_k^{(0)} + \mathbf{J}_k^{(1)})\, d\mathbf{x} + \int_T c(\overline{e^{\psi}}_{|\tilde{e}})^{-1} \rho_k \, d\mathbf{x} = |T|\, \bar{f}_{|T}$$

and thus $\alpha_3 \int_{\partial T} \boldsymbol{\tau}^* \cdot \mathbf{n}\, ds + |T|\, c_{|T}(\overline{e^{\psi}}_{|\tilde{e}})^{-1} \rho_{k|T} = |T|\, \bar{f}_{|T}$ .
Using (2.5) this results in

(c) $\qquad |\tilde{e}|\, \alpha_3 + |T|\, c_{|T}(\overline{e^{\psi}}_{|\tilde{e}})^{-1} \rho_{k|T} = |T|\, \bar{f}_{|T}$ .

The equations in (b) and (c) together form a nonsingular system for the unknowns $\alpha_3$ and $\rho_{k|T}$. Solving this system yields

$$\alpha_3 = \beta_T\, |\tilde{e}|^{-1}\, |T|\, \{\bar{f}_{|T} - |\tilde{e}|^{-1} c_{|T}(\overline{e^{\psi}}_{|\tilde{e}})^{-1} \int_{\partial T} \lambda_k\, \boldsymbol{\tau}^* \cdot \mathbf{n}\, ds\} =: \alpha_3^*$$

and thus

(d) $\qquad \mathbf{J}_k^{(1)} = \alpha_3^* \boldsymbol{\tau}^*$ .

Now take a $\mu \in \Lambda_{k,0}$. The third equation of (2.3) combined with (a) and (d) yields

$$-\sum_T \int_{\partial T} \mu(|T|\, \overline{e^{\psi}}_{|T})^{-1} \int_{\partial T} \lambda_k\, \mathbf{n}\, ds \cdot \mathbf{n}\, ds + \sum_T \int_{\partial T} \mu\, \alpha_3^* \boldsymbol{\tau}^* \cdot \mathbf{n}\, ds = 0 .$$

So $\sum_T (|T|\, \overline{e^{\psi}}_{|T})^{-1} \int_{\partial T} \lambda_k\, \mathbf{n}\, ds \cdot \int_{\partial T} \mu\, \mathbf{n}\, ds - \sum_T \alpha_3^* \int_{\partial T} \mu\, \boldsymbol{\tau}^* \cdot \mathbf{n}\, ds = 0$
and thus $b_k(\lambda_k, \mu) - \sum_T \alpha_3^* \int_{\partial T} \mu\, \boldsymbol{\tau}^* \cdot \mathbf{n}\, ds = 0 \qquad$ (use 2.7) .
Using the definition of $\alpha_3^*$ we get

$$\begin{aligned}
-\sum_T \alpha_3^* \int_{\partial T} \mu\, \boldsymbol{\tau}^* \cdot \mathbf{n}\, ds &= -\sum_T \beta_T\, |\tilde{e}|^{-1}\, |T|\, \bar{f}_{|T} \int_{\partial T} \mu\, \boldsymbol{\tau}^* \cdot \mathbf{n}\, ds \\
&\quad + \sum_T \beta_T\, |T|\, |\tilde{e}|^{-2} c_{|T}(\overline{e^{\psi}}_{|\tilde{e}})^{-1} \int_{\partial T} \lambda_k\, \boldsymbol{\tau}^* \cdot \mathbf{n}\, ds \int_{\partial T} \mu\, \boldsymbol{\tau}^* \cdot \mathbf{n}\, ds \\
&= -F_k(\mu) + c_k(\lambda_k, \mu) .
\end{aligned}$$

We conclude that $b_k(\lambda_k, \mu) + c_k(\lambda_k, \mu) = F_k(\mu)$. □

The Lagrange multiplier $\lambda_k$ is an approximation of $\rho = e^{\psi} u$ and is not suited for actual computation if the range of $\psi$ is large (which often happens in semiconductor problems); moreover we are interested in approximating $u$ instead of $\rho$. So we rescale $\lambda_k$ to get an approximation $\tilde{\mu}_k$ of $u$ (at the interelements). We define the isomorphism $Q_k \;:\; \Lambda_{k,g} \to \Lambda_{k,\chi}$ $\;(g, \chi$ as in (2.1)) as follows: Take $\mu \in \Lambda_{k,g}$, then:

$$\begin{array}{ll} \text{for } \ e \subset \Gamma_0 & (Q_k\,\mu)_{|e} = \bar{\chi}_{|e} \\ \text{for } \ e \subset \bar{\Omega}\backslash\Gamma_0 & (Q_k\,\mu)_{|e} = \overline{e^{\psi}}_{|e}\,\mu_{|e} \ . \end{array}$$

Using this isomorphism we can rewrite (2.9) as follows

$$(2.10) \quad \begin{cases} \text{Find } \ \tilde{\mu}_k \in \Lambda_{k,g} \ \text{ such that} \\ \\ b_k(Q_k\,\tilde{\mu}_k,\mu) + c_k(Q_k\,\tilde{\mu}_k,\mu) = F_k(\mu) \quad \text{for all} \quad \mu \in \Lambda_{k,0} \ . \end{cases}$$

The problem (2.10) is the final one, which we actually want to solve. Rewriting (2.10) as a matrix-vector problem using the standard basis $\{\mu_i\}_{i\in I_0\cup I}$ of $\Lambda_k := \{\mu \in L^2(E_k) \mid \mu_{|e} \in P_0(e) \text{ for all } e \in E_k\}$ (cf. 2.2 for $I_0, I$) yields the following system of equations for the unknowns $\{\alpha_j\}_{j\in I}$ with $\tilde{\mu}_k = \sum_{j\in I} \alpha_j\,\mu_j + \sum_{j\in I_0} \bar{g}_{|e_j}\,\mu_j$:

$$(2.11) \quad \sum_{j\in I} \{b_k(Q_k\,\mu_j,\mu_i) + c_k(Q_k\,\mu_j,\mu_i)\}\,\alpha_j =$$

$$- \sum_{j\in I_0} \{b_k(Q_k\,\mu_j,\mu_i) + c_k(Q_k\,\mu_j,\mu_i)\}\,\bar{g}_{|e_j} + F_k(\mu_i) \quad \text{for} \quad i \in I \ .$$

REMARK 2.12. For the local stiffness matrix $m_{ij}^T$ corresponding to (2.11) we take a triangle $T$ and number its edges $e_1, e_2, e_3$ such that $e_1 = \tilde{e}$ (cf. 2.2); the corresponding unit outward normals are denoted by $\mathbf{n}^{(i)}$ $(i = 1,2,3)$ and $\boldsymbol{\nu}^{(i)} := |e_i|\,\mathbf{n}^{(i)}$. Then $m_{ij}^T = b_{k|T}(Q_k\,\mu_j,\mu_i) + c_{k|T}(Q_k\,\mu_j,\mu_i)$ $(i,j \in \{1,2,3\})$. And

$$\begin{aligned} b_{k|T}(Q_k\,\mu_j,\mu_i) &= (|T|\,\overline{e^{\psi}}_{|T})^{-1} \int_{\partial T} \overline{e^{\psi}}_{|e_j}\,\mu_j\,\mathbf{n}\,ds \cdot \int_{\partial T} \mu_i\,\mathbf{n}\,ds \\ &= (\overline{e^{\psi}}_{|T})^{-1}\,\overline{e^{\psi}}_{|e_j}\,\boldsymbol{\nu}^{(j)} \cdot \boldsymbol{\nu}^{(i)} \,/\, |T| \ , \end{aligned}$$

$$\begin{aligned} c_{k|T}(Q_k\,\mu_j,\mu_i) &= \beta_T\,|T|\,|e_1|^{-2}\,c_{|T}(\overline{e^{\psi}}_{|e_1})^{-1} \int_{\partial T} \overline{e^{\psi}}_{|e_j}\,\mu_j\,\boldsymbol{\tau}^*\cdot\mathbf{n}\,ds \int_{\partial T} \mu_i\,\boldsymbol{\tau}^*\cdot\mathbf{n}\,ds \\ &= \beta_T\,|T|\,|e_1|^{-2}\,c_{|T}(\overline{e^{\psi}}_{|e_1})^{-1}\,\overline{e^{\psi}}_{|e_j} \int_{e_j} \boldsymbol{\tau}^*\cdot\mathbf{n}^{(j)}\,ds \int_{e_i} \boldsymbol{\tau}^*\cdot\mathbf{n}^{(i)}\,ds \\ &= \begin{cases} \beta_T\,|T|\,c_{|T} & \text{if } \ i = j = 1 \\ \\ 0 & \text{otherwise} \end{cases} \qquad \text{(use (2.5))} \ . \end{aligned}$$

These formulas for the local stiffness matrix can also be found in [8]. If the triangulation is of weakly acute type, then the resulting (nonsymmetric) stiffness matrix is an $M$-matrix.

## 3. Connection with nonconforming finite elements

The natural procedure of §2 led to the matrix-vector problem in (2.11). In this section we show that the same system (2.11) results from the following procedure: We consider the original problem (1.1) in variational formulation (find $u \in H^1_g(\Omega)$ such that $\int_\Omega e^{-\psi}\, \nabla(e^{\psi}\, u) \cdot \nabla\, v\, dx + \int_\Omega c\, u\, v\, dx = \int_\Omega f\, v\, dx$ for all $v \in H^1_0(\Omega)$ ) and use a (seemingly unnatural) modified discretization in the nonconforming $P1$ Crouzeix-Raviart finite element space. This modified discretization is described in Lemma 3.2 and Remark 3.5 below. In §4 we use this connection with nonconforming finite elements to make a multigrid solver for (2.11).

DEFINITIONS 3.1. The *Crouzeix-Raviart* ($P1$) *space* corresponding to $T_k$ is given by $S_k = \{v \in L^2(\Omega) \mid v_{|T}$ is linear for all $T \in T_k$, $v$ is continuous at midpoints of edges$\}$. The standard basis of $S_k$ is denoted by $\{\varphi_i\}_{i\in I_0 \cup I}$ ($I, I_0$ as in 2.2). For $\zeta \in L^2(\Gamma_0)$ we define $S_{k,\zeta} = \{v \in S_k \mid v(m_i) = \bar{\zeta}_{|e_i}$ for all $i \in I_0\}$.
We define the linear operator $R_k : S_k \to S_k$ by $R_k(\sum \alpha_i\, \varphi_i) = \sum \alpha_i\, \overline{e^{\psi}}_{|e_i}\, \varphi_i$.
For a given $T$ the midpoint of edge $\tilde{e}$ (cf. 2.2) is denoted by $\tilde{m}$.
On a triangle $T$ we define the quadrature rule $K_T$ for approximating $\int_T q(\mathrm{x})\, dx$ by $K_T(q) = |T|\, q(\tilde{m})$.

LEMMA 3.2. *Consider the following problem (with $f, g$ as in* (1.1) *)*

$$(3.3)\quad \begin{cases} \textit{Find } \tilde{\eta}_k \in S_{k,g} \textit{ such that for all } \varphi \in S_{k,0} \\[1ex] \sum\limits_T \int\limits_T (\overline{e^{\psi}}_{|T})^{-1}\, \nabla(R_k\, \tilde{\eta}_k) \cdot \nabla\, \varphi\, dx + \sum\limits_T \beta_T\, K_T(c\, \tilde{\eta}_k\, \varphi) = \sum\limits_T \beta_T\, K_T(\bar{f}_{|T}\, \varphi)\ . \end{cases}$$

*Write $\tilde{\eta}_k = \sum_{j\in I} \alpha_j\, \varphi_j + \sum_{j\in I_0} \bar{g}_{|e_j}\, \varphi_j$. Taking $\varphi = \varphi_i$ $(i \in I)$ in* (3.3) *results in a system of equations for the unknowns $\{\alpha_j\}_{j\in I}$ that is given in* (2.11).

*Proof.* Use the notation $a_k(\eta, \varphi) = \sum\limits_T \int\limits_T (\overline{e^{\psi}}_{|T})^{-1}\, \nabla\, \eta \cdot \nabla\, \varphi\ dx$ , $d_k(\eta, \varphi) = \sum\limits_T \beta_T\, K_T(c\, \eta\, \varphi)$. Taking $\varphi = \varphi_i$ in (3.3) yields

$$(3.4)\quad \sum_{j\in I} \{a_k(R_k\, \varphi_j, \varphi_i) + d_k(\varphi_j, \varphi_i)\}\, \alpha_j =$$

$$- \sum_{j\in I_0} \{a_k(R_k\, \varphi_j, \varphi_i) + d_k(\varphi_j, \varphi_i)\}\, \bar{g}_{|e_j} + \sum_T \beta_T\, K_T(\bar{f}_{|T}\, \varphi_i)\ .$$

By comparing (3.4) with (2.11) it follows that we only have to show:

$$\text{(a)}\qquad b_k(Q_k\, \mu_j, \mu_i) = a_k(R_k\, \varphi_j, \varphi_i)$$

(b) $c_k(Q_k\,\mu_j,\mu_i) = d_k(\varphi_j,\varphi_i)$

(c) $F_k(\mu_i) = \sum\limits_T \beta_T\, K_T(\bar{f}_{|T}\,\varphi_i)$ .

Using the definitions and checking per triangle it follows that it is sufficient to prove

(a') $$|T|^{-1} \int\limits_{\partial T} \mu_j\,\mathbf{n}\,ds \cdot \int\limits_{\partial T} \mu_i\,\mathbf{n}\,ds = \int\limits_T \nabla\varphi_j \cdot \nabla\varphi_i\,d\mathbf{x}$$

(b') $$|\tilde{e}|^{-2}\,(\overline{e^{\psi}}_{|\tilde{e}})^{-1}\,(\overline{e^{\psi}}_{|e_j}) \int\limits_{\partial T} \mu_j\,\boldsymbol{\tau}^*\cdot\mathbf{n}\,ds \int\limits_{\partial T} \mu_i\,\boldsymbol{\tau}^*\cdot\mathbf{n}\,ds = \varphi_j(\tilde{m})\,\varphi_i(\tilde{m})$$

(c') $$|\tilde{e}|^{-1} \int\limits_{\partial T} \mu_i\,\boldsymbol{\tau}^*\cdot\mathbf{n}\,ds = \varphi_i(\tilde{m})\ .$$

We only have to consider $T$, $i$, $j$ with $e_i \subset \partial T$, $e_j \subset \partial T$. We use the notation $\boldsymbol{\nu}^{(k)} := |e_k|\,\mathbf{n}^{(k)}$ , $k \in \{i,j\}$. One easily verifies that $\nabla(\varphi_{j|T}) = |T|^{-1}\,\boldsymbol{\nu}^{(j)}$ holds. Both sides of (a') are equal to $|T|^{-1}\,\boldsymbol{\nu}^{(j)}\cdot\boldsymbol{\nu}^{(i)}$, so (a') holds. With respect to (b') and (c') note that $\varphi_i(\tilde{m}) = 1$ if $e_i = \tilde{e}$ and $\varphi_i(\tilde{m}) = 0$ otherwise. Also $\int\limits_{\partial T} \mu_i\,\boldsymbol{\tau}^*\cdot\mathbf{n}\,ds = |e_i|$ if $e_i = \tilde{e}$ and $\int\limits_{\partial T} \mu_i\,\boldsymbol{\tau}^*\cdot\mathbf{n}\,ds = 0$ otherwise (use (2.5)). So both sides of (b') are equal to 1 if $e_i = e_j = \tilde{e}$ and 0 otherwise. Both sides of (c') are equal to 1 if $e_i = \tilde{e}$ and 0 otherwise. □

REMARK 3.5. In the continuous variational problem we have the bilinear form $(u,v) \rightarrow \int\limits_\Omega e^{-\psi}\,\nabla(e^{\psi}\,u)\cdot\nabla v\,d\mathbf{x} + \int\limits_\Omega c\,u\,v\,d\mathbf{x}$ and the right hand side functional $v \rightarrow \int\limits_\Omega f\,v\,d\mathbf{x}$. From Lemma 3.2 it follows that the system (2.11) results from a discretization in the Crouzeix-Raviart ($P1$) space with the following modifications:

(1) in $\int\limits_T e^{-\psi}\,\nabla(e^{\psi}\,u)\cdot\nabla v\,d\mathbf{x}$ the function $e^{-\psi}$ is replaced by its harmonic average $(\overline{e^{\psi}}_{|T})^{-1}$ and $e^{\psi}\,u = \sum_i \alpha_i\,e^{\psi}\,\varphi_i$ is replaced by $\sum_i \alpha_i\,\overline{e^{\psi}}_{|e_i}\,\varphi_i$.

(2) $\int\limits_T c\,u\,v\,d\mathbf{x}$ is replaced by $\beta_T\,K_T(c\,u\,v)$; so the integral over $T$ is replaced by the quadrature rule $K_T$ and there is an artificial perturbation due to $\beta_T$ (note that $\beta_T - 1 = O(|T|)$ ).

(3) $\int\limits_T f\,v\,d\mathbf{x}$ is replaced by $\beta_T\,K_T(\bar{f}_{|T}\,v)$ (cf. (2)).

## 4. Multigrid method

Through the equivalence between (2.11) and (3.3) we are led to multigrid methods for nonconforming finite elements (as in [2], [3,4]) for solving the system (2.11). Here we only very briefly discuss the components of the multigrid method we use. A more comprehensive explanation can be found in [9].

We assume a regular sequence of triangulations $\{T_k\}_{k\geq 0}$ in which $T_k$ is obtained from $T_{k-1}$ by connecting the midpoints of the edges in the triangles of $T_{k-1}$. We assume that $T_0$ is of weakly acute type. In the experiments in §5 we have $\Omega = [0,1] \times [0,1]$ and we use only right isosceles triangles.
For coarse grid approximation we use the stiffness matrix induced by the discretization (3.3) on the coarse grid.
We use a prolongation $P_k : S_{k-1,0} \rightarrow S_{k,0}$ as in [2]: For $P_k$ we take the orthogonal projection w.r.t. the $L^2$-inner product of $S_{k-1,0}$ on $S_{k,0}$. For this $P_k$ formulas suited for computation can be found in [2].
For smoothing, in the situation where we have a triangulation with right isosceles triangles, we use a variant of Gauss-Seidel as explained in [9]. This variant resembles a collective Gauss-Seidel smoother in which an unknown on a diagonal line is relaxed together with its four neighbouring unknowns (with relatively low costs because in part of the unknowns we only have three point difference stars).
The multigrid algorithm we use is described in [2], [9]. The algorithm is very similar to the standard multigrid algorithm. The only difference is that due to the nonconforming spaces we use the prolongation $P_k$ in the coarse grid correction. The structure of the coarse grid problem is as follows: Find $u_{k-1}^* \in S_{k-1,0}$ such that $a_{k-1}(u_{k-1}^*, \varphi) = f_k(P_k\, \varphi) - a_k(u_k, P_k\, \varphi)$ for all $\varphi \in S_{k-1,0}$.

## 5. Numerical results

In this section we show results of the multigrid method of §4 applied to two model problems. We use $\Omega = [0,1] \times [0,1]$ and uniform triangulations with right isosceles triangles. The meshwidth (= half of the length of the shortest edge of a triangle) is denoted by $h$. We use a coarsest grid with $h = \frac{1}{4}$. On the finest grid we have $k_{\max}$ and $h_{\min}$ related by $h_{\min} = 2^{-(2+k_{\max})}$. We always use one pre- and one post-smoothing. The number of coarse grid corrections is denoted by $\mu$ ($\mu = 0$: no coarse grid corrections; $\mu = 1$: $V$-cycle; $\mu = 2$: $W$-cycle).
We measure the performance of a method by way of the average reduction factor (arf) which results by taking an arbitrary starting vector and then computing the geometric mean of the norm reductions of the defect in a (sufficiently large) number of iterations.

If $\psi \equiv c \equiv 0$ in (1.1) the linear system (2.11) we have to solve corresponds to a modified nonconforming finite element discretization of the Poisson equation (cf. (3.3)). Our algorithm (for the case $\mu = 2$, $\Gamma_0 = \partial\Omega$) with $k_{\max} =$

1, 2, 3, 4, 5 then has average reduction factors of 0.0043, 0.026, 0.058, 0.067, 0.068 respectively.

EXPERIMENT 1. We consider a model problem from [8], however with less convection (then diffusion also plays a role for realistic $h$).
We take $\psi$ such that $\nabla\psi = -20(1, 0.5)$ on $\Omega$, $f \equiv 0$, and $c \equiv 10^{10}$ in the quarter of the circle centered in (1,0) and with radius 0.5, $c \equiv 0$ elsewhere. $\Gamma_0 = \partial\Omega$ and on $\Gamma_0$ $u(x, y) = 0$ if $(x = 1)$ or $((y = 0)$ and $(x \geq 0.5))$ and $u(x, y) = 1$ elsewhere. The solution for $h = 2^{-6}$ is represented on a $32 \times 32$ grid in Fig. 1. The average reduction factors for varying $k_{max}$ and $\mu$ are given in Fig. 2.

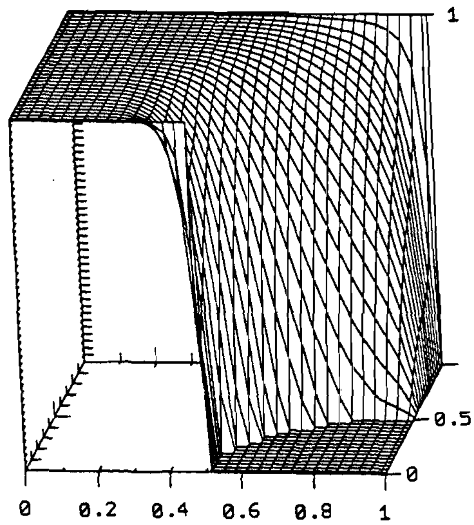


Figure 1

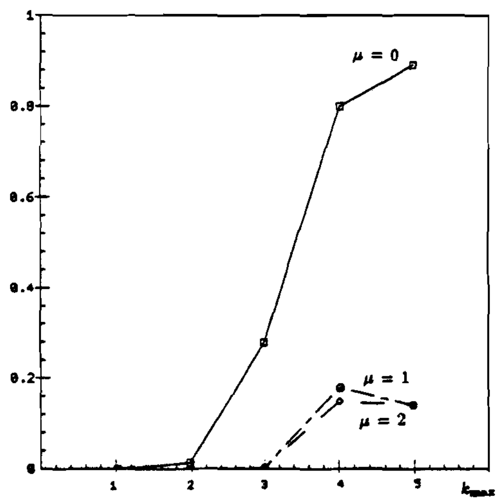


Figure 2

EXPERIMENT 2. We consider a model problem with $||\nabla\psi||$ very large in part of the domain and $c$ very large in (another) part of the domain (cf. [9] for results if $c \equiv 0$ in $\Omega$). For the function $\psi$ we define $\rho := ((x-1)^2 + (y-1)^2)^{\frac{1}{2}}$, $\psi_0(\rho) := 0$ if $0 \leq \rho \leq 0.8$ , $\psi_0(\rho) := \rho - 0.8$ if $0.8 \leq \rho \leq 0.9$ , $\psi_0(\rho) = 0.1$ if $\rho \geq 0.9$, and we take $\psi = 10^3\psi_0$ (so $||\nabla\psi(\rho)||_2 = 0$ if $0 < \rho < 0.8$ or $0.9 < \rho < 1$ and $||\nabla\psi(\rho)||_2 = 10^3$ if $0.8 < \rho < 0.9$). We take $f \equiv 0$, and $c(x, y) = 10^6$ if $3x - 5y \geq 0$, $c(x, y) = 0$ elsewhere. $\Gamma_1 = \{(x, y) \mid ((x = 1)$ and $(y \leq 0.5))$ or $((y = 1)$ and $(x \leq 0.5))\}$; on $\Gamma_0$ we have $u(x, y) = 0$ if $x = 0$ or $y = 0$ and $u(x, y) = 1$ elsewhere. The solution for $h = 2^{-6}$ is represented on a $32 \times 32$ grid in Fig. 3. The average reduction factors for varying $k_{max}$ and $\mu$ are given in Fig. 4.

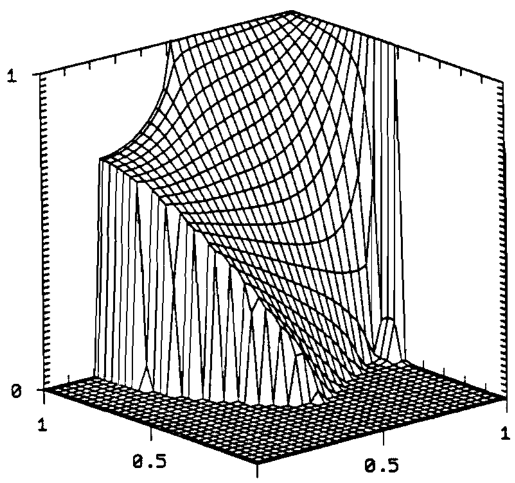


Figure 3

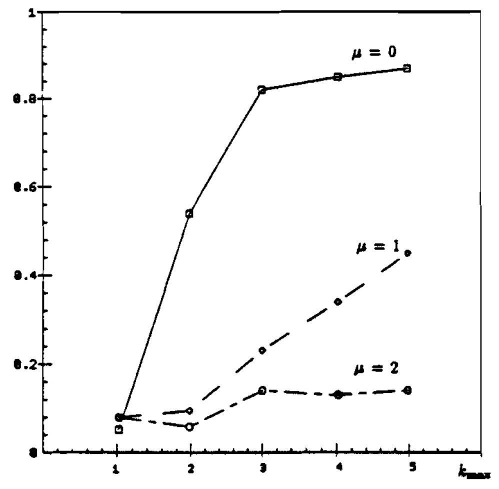


Figure 4

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