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MEAN VALUE APPROXIMATION FOR CLOSED QUEUEING NETWORKS WITH MULTI SERVER STATIONS

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Abstract
We formulate an approximate recursive relation for the sojourn time, queuelength and throughput of a multi server station, embedded in a closed queueing network. Based on that relation, we derive a mean value approximation and formulate a Schweitzer approximation for solving large networks. Numerical examples show that the approximate mean value algorithm yields accurate results.

1. Introduction

In 1980 Reiser and Lavenberg [5] introduced the Mean Value Algorithm (MVA) for the performance evaluation of closed queueing networks with stations with a fixed service rate. In Reiser [6] an exact, but more complex scheme is given to extend the MVA scheme for queue dependent servers, including multi server stations. This MVA works with marginal probabilities and suffers from a numerical instability problem. In this note we derive a numerically stable approximated MVA (AMVA) which is computationally equivalent to the original fixed rate MVA, thus more attractive.
In Section 2 the AMVA is given. An iterative variant of this algorithm, the Schweitzer method, is presented in Section 3. In Section 4 some numerical examples are discussed. Section 5 is devoted to conclusions and extensions to mixed open/closed networks and multi servers with general service time distributions.

2. Approximate Mean Value Algorithm

Consider a closed product form network with K clients a stations $1, 2, ..., N$. The mean service time in station n is w_n and f_n is the visiting frequency to station is n, $n = 1, ..., N$. Consider a fixed station n and suppose that it is a multi server station with m_n servers and the FCFS service discipline. This service discipline imposes the restriction of exponential servers at station n. Below we formulate recursive relations for the sojourn time $S_n(K)$, the queuelength $L_n(K)$, the throughput $\Lambda_n(K)$ and the probability on queueing $Q_n(K)$.

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The sojourn time consists of the waiting time and the service time. For waiting clients the service unit at station n works with rate m_n / w_n. Therefore the mean sojourn time S_n is given by

$$S_n = \{\textit{probability on queueing}\} w_n / m_n$$
$$+ \{\textit{expected number of waiting clients at arrival}\} w_n / m_n + w_n .$$

By the arrival theorem (Lavenberg and Reiser [3] and Sevcik and Mitrani [9]), this equation becomes

$$S_n(K) = Q_n(K-1)\frac{w_n}{m_n} + (L_n(K-1) - \Lambda_n(K-1)w_n)\frac{w_n}{m_n} + w_n . \tag{1}$$

By Little's rule (Little [4]),

$$\Lambda_n(K) = \frac{K f_n}{\sum_{j=1}^{N} f_j S_j(K)}, \tag{2}$$

$$L_n(K) = \Lambda_n(K) S_n(K). \tag{3}$$

If $K \le m_n$, then $Q_n(K) = 0$, but if $K > m_n$, however, an exact relation for $Q_n(K)$ requires explicit knowledge of the marginal probabilities (Reiser [6]). Instead, we approximate in that case $Q_n(K)$ by the probability on queueing for an $M/M/m$-queue with arrival rate $\Lambda_n(K)$ and mean service time w_n,

$$Q_n(K) = \frac{\dfrac{(\Lambda_n(K)w_n)^{m_n}}{m_n!}}{\dfrac{(\Lambda_n(K)w_n)^{m_n}}{m_n!} + (1 - \dfrac{\Lambda_n(K)w_n}{m_n}) \sum_{l=0}^{m_n-1} \dfrac{(\Lambda_n(K)w_n)^l}{l!}} . \tag{4}$$

Formula (4) is known as Erlang's delay formula (see e.g. Kleinrock [2], formula 3.40).
The exact arrival rate at station n is decreasing in the number of clients at station n. By approximating station n by a $M/M/m_n$-queue, the arrival rate is smoothed and thereby the marginal distribution. Numerical experiments indicated that the smoothing has little effect on the probability on queueing.
The approximate Mean Value Algorithm uses the recursive relations (1), (2), (3) and (4) to compute mean values for multi server stations. These relations are exact for single and infinite server stations. In case of a single server $Q_n(k)$ reduces to $\Lambda_n(k)w_n$ and thus $S_n(k) = L_n(k-1)w_n + w_n$. In case of an infinite server $Q_n(k) = 0$ and $L_n(k) = \Lambda_n(k)$, and thus $S_n(k) = w_n$.

3. Schweitzer Method

The number of recursion steps grows with the number of stations and the population. Due to the computational complexity, approximations are required for large networks. The Schweitzer method (Schweitzer [7] and Schweitzer, Seidmann and Shalev-Oren [8]) is a well known approximation for closed queueing networks with fixed rate servers. The method avoids the recursion in MVA, which causes the computational problems, and iteratively solves the mean value relations for the desired population K. To remove the recursion in AMVA, we express $L_n(K-1)$, $\Lambda_n(K-1)$ and $Q_n(K-1)$ in (1) as functions of those quantities with population K. We anticipate that removing one client from the system results in approximately the same throughput, thus we assume that

$$\Lambda_n(K-1) = \Lambda_n(K), \tag{5}$$

and with (4),

$$Q_n(K-1) = Q_n(K). \tag{6}$$

For the number of waiting clients at each station we assume that this decreases proportionally by removing one client, so

$$L_n(K-1) - \Lambda_n(K-1)w_n = L_n(K) - \Lambda_n(K)w_n - \frac{L_n(K) - \Lambda_n(K)w_n}{\sum_{j=1}^{N} (L_j(K) - \Lambda_j(K)w_j)}. \tag{7}$$

By inserting (5), (6) and (7) into (1), we obtain, together with (2), (3) and (4) a set of equations for the performance measures, which can be solved iteratively.

A refinement of the Schweitzer method is Schweitzer-FODI (First Order Depth Improvement, Van Doremalen, Wessels, Wijbrands [1]). In this method the iterative algorithm is applied for population $K-1$ and then the last recursion step in AMVA is applied to obtain the performance measures for K.

4. Numerical Examples

In this section we compare the results of AMVA with the results of the exact Normalized Convolution Algorithm (NCA) of Reiser [6].

In all examples there are 5 stations with respectively 1, 1, 2, 2 and 4 servers, a service rate of 3, 4, 2, 3, and 1 and relative visiting frequencies of 6, 7, 7, 13 and 9. The examples illustrate the quality of AMVA under low load ($K = 8$), moderate load ($K = 15$) and heavy load ($K = 25$).

	example 1: $K = 8$				example 2: $K = 15$				example 3: $K = 25$			
	Λ	Q	L	S	Λ	Q	L	S	Λ	Q	L	S
NCA	1.760	0.587	1.165	0.662	2.244	0.748	2.437	1.086	2.478	0.826	4.055	1.636
	2.054	0.513	0.920	0.448	2.617	0.654	1.707	0.652	2.891	0.723	2.456	0.850
	2.054	0.338	1.264	0.616	2.617	0.514	2.104	0.804	2.891	0.605	2.876	0.995
	3.814	0.481	1.750	0.459	4.861	0.721	3.570	0.734	5.369	0.844	6.460	1.203
	2.640	0.326	2.902	1.099	3.365	0.658	5.183	1.540	3.717	0.843	9.153	2.462
AMVA	1.745	0.582	1.144	0.656	2.239	0.746	2.415	1.079	2.477	0.826	4.041	1.631
	2.036	0.509	0.905	0.444	2.612	0.653	1.695	0.649	2.890	0.722	2.451	0.848
	2.036	0.343	1.261	0.619	2.612	0.516	2.101	0.804	2.890	0.606	2.875	0.995
	3.780	0.487	1.747	0.462	4.851	0.723	3.558	0.734	5.367	0.845	6.446	1.201
	2.617	0.361	2.943	1.124	3.358	0.669	5.231	1.558	3.715	0.847	9.188	2.473

Table 1: Performance measures for NCA and AMVA

For the above examples we also tested the Schweitzer algorithm of Section 3. In Table 2 we list the average and maximum errors found for all examples with both approximations.

	error (%)	example 1: $K = 8$ Λ	Q	L	S	example 2: $K = 15$ Λ	Q	L	S	example 3: $K = 25$ Λ	Q	L	S
AMVA	average	0.87	3.02	1.04	1.06	0.21	0.53	0.60	0.49	0.05	0.15	0.25	0.22
	maximum	0.87	10.56	1.75	2.30	0.21	1.69	0.93	1.13	0.05	0.46	0.39	0.43
Schweitzer	average	0.78	5.07	3.65	3.78	0.14	0.58	2.95	2.87	0.36	0.35	4.19	3.98
	maximum	0.78	15.41	7.39	8.11	0.14	1.86	4.88	4.75	0.36	0.39	5.97	6.00

Table 2: Relative errors compared to NCA (%)

Table 2 indicates that AMVA yields accurate results, especially for heavy load.
An argument for the excellent performance of AMVA for heavy load networks, is that the system bottle neck behaves as a Poisson source for the rest of the network, which resembles an open network. For open networks relation (4) is exact.
The complexity of the exact MVA is roughly $O(KN\overline{m})$ and the complexity of the NCA is roughly $O(KN^2\overline{m})$, whereas the complexity of AMVA is only $O(KN)$. Here $\overline{m}$ is the average number of servers per station. Furthermore, the exact MVA suffers from numerical instability, which can be solved at the expense of a nested scheme of so-called complementary networks. The NCA suffers from floating-point underflow problems (cf. Section 6 in Reiser [6]). These numerical problems does not occur in AMVA. Finally, AMVA can be easily used as a basis for other approximations, such as the Schweitzer method.
The error percentages for the Schweitzer method in Table 2 fit with the error percentages for the original Schweitzer method for fixed rate servers.

5. Conclusions and Extensions

In this note we derive an excellent approximation for multi server stations in closed queueing networks with one type of clients. An extended Schweitzer method is applied to the approximate MVA.

Most extensions and approximations of MVA for fixed rate servers also apply to AMVA for multi servers. We already showed this for the Schweitzer approximation, and mention here mixed open/closed networks for which an extension is straightforward and a multi server with a general service time distribution.
In a mixed open/closed network with a population vector $\underline{K}$, and with $\underline{e}_r$ a vector with only a 1 on the r-th position, we have for a client of type r that

$$S_{n,r}(\underline{K}) = Q_n(\underline{K}-\underline{e}_r)\frac{w_n}{m_n} + \left[\sum_s (L_{n,s}(\underline{K}-\underline{e}_r) - \Lambda_{n,s}(\underline{K}-\underline{e}_r)w_n)\right]\frac{w_n}{m_n} + w_n .$$

The throughput $\Lambda_{n,s}(\underline{K})$ and queuelength $L_{n,s}(\underline{K})$ follow from (2) and (3). For the probability on queueing in (4) the throughput $\Lambda_n(\underline{K})$ is the aggregated throughput over all client types, that is,

$$\Lambda_n(\underline{K}) = \sum_s \Lambda_{n,s}(\underline{K}) .$$

In case of a general service time distribution the following three extra assumptions are made to obtain a recursive relation for mean values. First the arrival theorem is applied. Secondly, if the service unit is saturated, the service rate is m_n / w_n. Finally, if on arrival the service unit is saturated, the mean time, R, elapsed to the next completion is the minimum of the remaining service times for each server. These remaining service times are i.i.d. random variables, distributed as the residual life in a renewal process with the service times as the interoccurrence times. This yields

$$S_{n,r}(\underline{K}) = Q_n(\underline{K}-\underline{e}_r)R + \left[\sum_s (L_{n,s}(\underline{K}-\underline{e}_r) - \Lambda_{n,s}(\underline{K}-\underline{e}_r)w_n) \right] \frac{w_n}{m_n} + w_n .$$

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