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Hamiltonian four fold 1:1 resonance with two rotational symmetries

Dedicated to Richard Cushman on the occasion of his 65th birthday.

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Abstract

This paper deals with the analysis of Hamiltonian Hopf bifurcations in 4-DOF systems defined by perturbed isotropic oscillators (1-1-1-1 resonance), in the presence of two quadratic symmetries I_1 and I_2. The model is a generalization of the classical models obtained from regularized Kepler systems describing the parallel Stark and Zeeman effects. After normalization the truncated normal form gives rise to an integrable system which is analyzed using reduction to a one degree of freedom system. The Hamiltonian Hopf bifurcations are found using the 'geometric method' set up by one of the authors.

1   Introduction

In this paper we will consider on $\mathbb{R}^8$ with the standard symplectic form the Hamiltonian system with Hamiltonian

$$
\begin{aligned}
H = {} & \tfrac{1}{2}\left(Q_1^2 + Q_2^2 + Q_3^2 + Q_4^2\right) + \tfrac{1}{2}\,\omega^2(q_1^2 + q_2^2 + q_3^2 + q_4^2) \\
& + \varepsilon\left[\lambda\,(q_1^2 + q_2^2 + q_3^2 + q_4^2)(-q_1^2 - q_2^2 + q_3^2 + q_4^2)\right. \\
& \left. \quad + 4\,(q_1^2 + q_2^2 + q_3^2 + q_4^2)\{(q_1q_4 - q_2q_3)^2 + (q_1q_3 + q_2q_4)^2\}\right],
\end{aligned}
\tag{1}
$$

which admits the integrals

$$I_1 = (q_1Q_2 - Q_1q_2) + (q_3Q_4 - Q_3q_4) , \qquad (2)$$

$$I_2 = -(q_1Q_2 - Q_1q_2) + (q_3Q_4 - Q_3q_4) . \qquad (3)$$

We are interested in enlarging the studies done in relation to the 1-1-1 resonance in [4, 9, 10, 11, 12, 13, 14, 15]. Especially we will determine te presence of Hamiltonian Hopf bifurcations following the methods of [13, 14, 15].

In section 2 we will put the system (1) into normal form with respect to H_2. After truncation at order six this system is reduced with respect to the S^1-action given by the flow of the Hamiltonian vector field corresponding to H_2. In 1970 Moser [21] showed that the reduced phase space in this case will be $\mathbb{CP}^3$.

In section 3 a second reduction is performed with respect to the S^1-action given by the flow of the Hamiltonian vector field corresponding to I_1. We show that the reduced phase space is $S^2 \times S^2$ which, when $I_1 = 0$, is, with its Poisson structure, precisely the same as the reduced phase space obtained for normalized perturbed Keplerian systems that have been immersed in 4-D through the Kustaanheimo-Stiefel [17] or Moser [21] regularization transformation [1, 19, 20, 2, 5, 6]. The Hamiltonian (1) is chosen in such a way that it includes the parallel Stark-Zeeman Hamiltonians modeling the hydrogen atom.

In section 4 we will perform a further reduction with respect to the third integral I_2 obtaining a one-degree-of-freedom sytem on a two dimensional reduced phase space.

In the final section we will show that Hamiltonian Hopf bifurcations are present by investigating the tangency of the level surfaces of the reduced Hamiltonian at the conic singular points of the reduced phase space.

2   Normalization and reduction with respect to the oscillatory symmetry.

Let us consider the system defined by (1).

There are 16 invariants for the action corresponding to H_2:

$$\begin{array}{lll}
\pi_1 = Q_1^2 + q_1^2 , & \pi_2 = Q_2^2 + q_2^2 , & \pi_3 = Q_3^2 + q_3^2 , \\
\pi_4 = Q_4^2 + q_4^2 , & \pi_5 = Q_1Q_2 + q_1q_2 , & \pi_6 = Q_1Q_3 + q_1q_3 , \\
\pi_7 = Q_1Q_4 + q_1q_4 , & \pi_8 = Q_2Q_3 + q_2q_3 , & \pi_9 = Q_2Q_4 + q_2q_4 , \\
\pi_{10} = Q_3Q_4 + q_3q_4 , & \pi_{11} = -Q_1q_2 + q_1Q_2 , & \pi_{12} = -Q_1q_3 + q_1Q_3 , \\
\pi_{13} = -Q_1q_4 + q_1Q_4 , & \pi_{14} = -Q_2q_3 + q_2Q_3 , & \pi_{15} = -Q_2q_4 + q_2Q_4 , \\
\pi_{16} = -Q_3q_4 + q_3Q_4 & . &
\end{array}$$

These invariants can be easily derived using complex conjugate co-ordinates. The normal form of H with respect to H_2 can now be expressed in these invariants and are given by

$$
\begin{aligned}
\bar{H}^\lambda &= \frac{1}{16}(\lambda\left(12\,n\left(-\pi_1 - \pi_2 + \pi_3 + \pi_4\right) + 8\left(\pi_{11} - \pi_{16}\right)\left(\pi_{11} + \pi_{16}\right)\right)) \\
&+ \frac{1}{16}(n\left(-6\left(\pi_1 + \pi_2 - \pi_3 - \pi_4\right)^2 + 16\left(\pi_7 - \pi_8\right)^2 + 16\left(\pi_6 + \pi_9\right)^2 - 16\left(\pi_{13} - \pi_{14}\right)^2\right. \\
&\quad - 16\left(\pi_{12} + \pi_{15}\right)^2 - 8\left(\pi_{11} - \pi_{16}\right)^2 - 8\left(\pi_{11} + \pi_{16}\right)^2)) \\
&+ \frac{1}{2}((\pi_1 + \pi_2 - \pi_3 - \pi_4)\left(\pi_{11} - \pi_{16}\right)\left(\pi_{11} + \pi_{16}\right))
\end{aligned} \tag{4}
$$

The invariants are subjected to the following relations, defining the first reduced phase space:

$$
\begin{aligned}
&\pi_1\pi_2 = \pi_5^2 + \pi_{11}^2, &\quad &\pi_1\pi_3 = \pi_6^2 + \pi_{12}^2, &\quad &\pi_1\pi_4 = \pi_7^2 + \pi_{13}^2, \\
&\pi_2\pi_3 = \pi_8^2 + \pi_{14}^2, &\quad &\pi_2\pi_4 = \pi_9^2 + \pi_{15}^2, &\quad &\pi_3\pi_4 = \pi_{10}^2 + \pi_{16}^2, \\
&\pi_1\pi_8 = \pi_5\pi_6 + \pi_{11}\pi_{12} &\quad &\pi_1\pi_{14} = \pi_5\pi_{12} - \pi_6\pi_{11} \\
&\pi_1\pi_9 = \pi_5\pi_7 + \pi_{11}\pi_{13} &\quad &\pi_1\pi_{15} = \pi_5\pi_{13} - \pi_7\pi_{11} \\
&\pi_1\pi_{10} = \pi_6\pi_7 + \pi_{12}\pi_{13} &\quad &\pi_1\pi_{16} = \pi_6\pi_{13} - \pi_7\pi_{12} \\
&\pi_2\pi_6 = \pi_5\pi_8 - \pi_{11}\pi_{14} &\quad &\pi_2\pi_{12} = \pi_5\pi_{14} + \pi_8\pi_{11} \\
&\pi_2\pi_7 = \pi_5\pi_9 - \pi_{11}\pi_{15} &\quad &\pi_2\pi_{13} = \pi_5\pi_{15} + \pi_9\pi_{11} \\
&\pi_2\pi_{10} = \pi_8\pi_9 + \pi_{14}\pi_{15} &\quad &\pi_2\pi_{16} = \pi_8\pi_{15} - \pi_9\pi_{14} \\
&\pi_3\pi_5 = \pi_6\pi_8 + \pi_{12}\pi_{14} &\quad &\pi_3\pi_{11} = \pi_8\pi_{12} - \pi_6\pi_{14} \\
&\pi_3\pi_7 = \pi_6\pi_{10} - \pi_{12}\pi_{16} &\quad &\pi_3\pi_{13} = \pi_{10}\pi_{12} + \pi_6\pi_{16} \\
&\pi_3\pi_9 = \pi_8\pi_{10} - \pi_{14}\pi_{16} &\quad &\pi_3\pi_{15} = \pi_{10}\pi_{14} + \pi_8\pi_{16} \\
&\pi_4\pi_5 = \pi_7\pi_9 + \pi_{13}\pi_{15} &\quad &\pi_4\pi_{11} = \pi_9\pi_{13} - \pi_7\pi_{15} \\
&\pi_4\pi_6 = \pi_7\pi_{10} + \pi_{13}\pi_{16} &\quad &\pi_4\pi_{12} = \pi_{10}\pi_{13} - \pi_7\pi_{16} \\
&\pi_4\pi_8 = \pi_9\pi_{10} + \pi_{15}\pi_{16} &\quad &\pi_4\pi_{14} = \pi_{10}\pi_{15} - \pi_9\pi_{16} \\
&\pi_5\pi_{10} = \pi_7\pi_8 + \left(\pi_{13}\pi_{14} + \pi_{11}\pi_{16}\right) &\quad &\pi_{10}\pi_{11} + \pi_5\pi_{16} = \pi_8\pi_{13} - \pi_7\pi_{14} \\
&\pi_6\pi_9 = \pi_7\pi_8 - \left(\pi_{13}\pi_{14} - \pi_{12}\pi_{15}\right) &\quad &\pi_9\pi_{12} + \pi_6\pi_{15} = \pi_7\pi_{14} + \pi_8\pi_{13} \\
&\pi_1 + \pi_2 + \pi_3 + \pi_4 = 2n
\end{aligned}
$$

$$\tag{5}$$

This provides an orbit mapping for the S^1-action generated by H_2

$$
\rho_1 : \mathbb{R}^8 \to \mathbb{R}^{16}; (q_1, q_2, q_3, q_4, Q_1, Q_2, Q_3, Q_4) \to (\pi_1, \cdots, \pi_{16})
$$

The relations define a 6 dimensional reduced phase space in the target space of this orbit mapping which is isomorphic to $\mathbb{CP}^3$. See [21], [3] for a presentation of the n dimensional case. Because the reduction is regular the above relations define $\mathbb{CP}^3$ as a smooth submanifold of $\mathbb{R}^{16}$.

3

3 Reduction with respect to I_1. The space $S^2_{n+\zeta} \times S^2_{n-\zeta}$

To further reduce from $\mathbb{CP}^3$ to $S^2_{n+\zeta} \times S^2_{n-\zeta}$ one will have to fix $I_1 = \zeta$ and divide out the S^1-action generated by I_1. This can be done by expressing everything in the 8 invariants of the I_1-action on $\mathbb{R}^8$:

$$
\begin{aligned}
H_2 &= \tfrac{1}{2}\left(\pi_1 + \pi_2 + \pi_3 + \pi_4\right), & I_1 &= \pi_{11} + \pi_{16}, \\
l_1 = I_2 &= -\pi_{11} + \pi_{16}, & l_2 &= \pi_{12} + \pi_{15}, \\
l_3 &= -\pi_{13} + \pi_{14}, & k_1 &= \tfrac{1}{2}\left(-\pi_1 - \pi_2 + \pi_3 + \pi_4\right), \\
k_2 &= -\pi_7 + \pi_8, & k_3 &= -\pi_6 - \pi_9.
\end{aligned}
$$

This provides an orbit mapping

$$
\rho_2 : \mathbb{R}^{16} \to \mathbb{R}^8; (\pi_1, \cdots, \pi_{16}) \to (k_1, k_2, k_3, l_1, l_2, l_3, H_2, I_1)
$$

There are 2+2 relations defining the second reduced phase space:

$$
\begin{aligned}
k_1^2 + k_2^2 + k_3^2 + l_1^2 + l_2^2 + l_3^2 &= H_2^2 + I_1^2, \\
k_1 l_1 + k_2 l_2 + k_3 l_3 &= H_2 I_1, \\
I_1 = \zeta, \ H_2 &= n,
\end{aligned}
$$

with $n \geq 0$ and $n \geq \zeta$. Note that

$$
H_2 - I_1 = \tfrac{1}{2}\left((q_1 - Q_2)^2 + (q_2 + Q_1)^2 + (q_3 - Q_4)^2 + (q_4 + Q_3)^2\right) \geq 0.
$$

The relations define a 4 dimensional reduced phase space of $(k_1, k_2, k_3, l_1, l_2, l_3)$-space given by

$$
\begin{aligned}
k_1^2 + k_2^2 + k_3^2 + l_1^2 + l_2^2 + l_3^2 &= n^2 + \zeta^2, \\
k_1 l_1 + k_2 l_2 + k_3 l_3 &= n\zeta,
\end{aligned}
$$

If we define coordinates $(\sigma_1, \sigma_2, \sigma_3, \delta_1, \delta_2, \delta_3)$ by $\sigma_i = l_1 + k_1$ and $\delta_i = l_i - k_i$ with $i = 1, 2, 3$, then we have

$$
\begin{aligned}
\sigma_1^2 + \sigma_2^2 + \sigma_3^2 &= (n + \zeta)^2 \\
\delta_1^2 + \delta_2^2 + \delta_3^2 &= (n - \zeta)^2
\end{aligned}
$$

Thus we see that the reduced phase space is isomorphic to $S^2_{n+\zeta} \times S^2_{n-\zeta}$. The brackets for the invariants $(k_1, k_2, k_3, l_1, l_2, l_3)$ which define the Poisson structure on the orbit space and the symplectic structure on the reduced phase space are given Table 1.

From (4) the twice reduced Hamiltonian, modulo constants, is given by

$$
\bar{\bar{H}}^\lambda = \lambda(\frac{3}{2}nk_1 - \frac{1}{2}\zeta l_1) + n(-\frac{3}{2}k_1^2 + k_2^2 + k_3^2 - \frac{1}{2}l_1^2 - l_2^2 - l_3^2) + \zeta k_1 l_1. \tag{6}
$$

$\{,\}$	k_1	k_2	k_3	l_1	l_2	l_3
k_1	0	$-2l_3$	$2l_2$	0	$-2k_3$	$2k_2$
k_2	$2l_3$	0	$-2l_1$	$2k_3$	0	$-2k_1$
k_3	$-2l_2$	$2l_1$	0	$-2k_2$	$2k_1$	0
l_1	0	$-2k_3$	$2k_2$	0	$-2l_3$	$2l_2$
l_2	$2k_3$	0	$-2k_1$	$2l_3$	0	$-2l_1$
l_3	$-2k_2$	$2k_1$	0	$-2l_2$	$2l_1$	0

Table 1: The bracket relations for $(k_1, k_2, k_3, l_1, l_2, l_3)$

The reduced system is a two-degree-of-freedom system on a four-dimensional phase space. In addition the system has remaining integral $I_2 = l_1$. We will reduce to a one-degree-of-freedom system using the action generated by I_2 which is now renamed as l_1.

Note that when $\zeta = 0$ the reduced phase space in the coordinates $(\sigma, \ \delta)$ is $S_n^2 \times S_n^2$. Together with the Poisson bracket for σ and δ this is precisely the phase space which is obtained for perturbed Keplerian systems [20], [2], [5].

4   Third reduction with respect to l_1. The space $V_{n\,\zeta\,a}$.

To further reduce from $S_{n+\zeta}^2 \times S_{n-\zeta}^2$ to $V_{n\zeta a}$ one divides out the S^1-action generated by l_1 and fixes $l_1 = a$. The 8 invariants for the l_1 action on $\mathbb{R}^8$ are:

H_2 , I_1 , $I_2 = l_1$, $K = k_1$,
$X = \frac{1}{2}\left(k_2^2 + k_3^2\right)$, $Y = \frac{1}{2}\left(l_2^2 + l_3^2\right)$, $Z = k_2l_2 + k_3l_3$, $S = k_2l_3 - k_3l_2$.

There are 3+3 relations defining the third reduced phase space:

$$K^2 + 2X + l_1^2 + 2Y = H_2^2 + I_1^2,$$
$$Kl_1 + Z = H_2I_1,$$
$$4XY = Z^2 + S^2,$$
$$H_2 = n, \ I_1 = \zeta, \ l_1 = a.$$

However, it is more convenient to use the following invariants and relations

H_2, I_1, $I_2 = l_1$, $K = k_1$, $M = X + Y$, $N = X - Y$, $Z = k_2l_2 + k_3l_3$, $S = k_2l_3 - k_3l_2$.

5

$\{,\}$	M	N	Z	S	K	l_1
M	0	$4KS$	0	$-4KN$	0	0
N	$-4KS$	0	$-4l_1S$	$-4(KM-l_1Z)$	$4S$	0
Z	0	$4l_1S$	0	$-4l_1N$	0	0
S	$4KN$	$4(KM-l_1Z)$	$4l_1N$	0	$-4N$	0
K	0	$-4S$	0	$4N$	0	0
l_1	0	0	0	0	0	0

Table 2: The bracket relations for M, N, Z, S, K, l_1.

With relations

$$
\begin{aligned}
&K^2 + l_1^2 + 2M = H_2^2 + I_1^2 \,, \\
&Kl_1 + Z = H_2 I_1 \,, \\
&M^2 - N^2 = Z^2 + S^2 \,, \\
&l_1 = a, \ I_1 = \zeta, \ H_2 = n.
\end{aligned}
\tag{7}
$$

This provides an orbit mapping

$$
\rho_2 : \mathbb{R}^8 \to \mathbb{R}^8; (k_1, k_2, k_3, l_1, l_2, l_3, H_2, I_1) \to (M, N, Z, S, K, l_1, H_2, I_1)
$$

The relations define a 2 dimensional reduced phase space of $(M, N, Z, S, K, l_1, H_2, I_1)$-space given by

$$
\begin{aligned}
&K^2 + 2M = n^2 + \zeta^2 - a^2 \,, \\
&aK + Z = n\zeta \,, \\
&M^2 - N^2 = Z^2 + S^2 \,.
\end{aligned}
\tag{8}
$$

Consequently we may represent the third reduced phase space $V_{n\zeta a}$ in (K, N, S)-space by the equation

$$
(n^2 + \zeta^2 - a^2 - K^2)^2 - 4(n\zeta - aK)^2 = 4N^2 + 4S^2.
\tag{9}
$$

If we set

$$
f(K) = (n^2 + \zeta^2 - a^2 - K^2)^2 - 4(n\zeta - aK)^2 = [(n+\zeta)^2 - (K+a)^2][(n-\zeta)^2 - (K-a)^2],
$$

then our reduced phase space is a surface of revolution obtained by rotating $\phi(K) = \sqrt{f(K)}$ around the K-axis.

The Poisson structure for M, N, Z, S, K, l_1 is given by Table 2. The Hamiltonian on the

third reduced phase space (modulo constants) is:

$$\mathcal{H} = 2nN + \left(\frac{3}{2}n\lambda + a\zeta\right)K - \frac{3}{2}nK^2 \tag{10}$$

In (K, N, S)-space the energy surfaces are parabolic cylinders. The intersection with the reduced phase space give the trajectories of the reduced system. Tangency with the reduce phase spaces gives relative equilibria that generically will correspond to three dimensional tori in the original phase space.

The reduced phase spaces as well as the Hamiltonian are invariant under the discrete symmetry $S \to -S$. Furthermore the reduced phase space is invariant under the discrete symmetry $N \to -N$. We choose not to further reduce our reduced phase space with respect to these discrete symmetries like in [16] because the three dimensional picture makes it easy to access information about the reduced orbits and this way one does not introduce additional critical points (fixed points) which need special attention. We will make use of the fact that all the critical point will be in the plane $S = 0$

Note that one can combine the three successive reductions into one. The composition of the three orbit maps gives an orbit map from $\mathbb{R}^8 \to \mathbb{R}^8$, which is an orbit map for the three-torus action generated by the rotational flows of the three integrals H_2, I_1, I_2, which are independent and commute. Consequently the generic relative equilibrium (i.e. stationary point on the reduced phase space) will correspond to a T^3. Due to the shape of the reduced phase spaces the intersection of the Hamiltonian and these spaces will be circles in general. Thus the generic fibre of the energy momentum map will be a T^4. There will of course also be fibres that are a point (the origin which is a stationary point of the original system and a fixed point for all circle symmetries), a circle (two of the circle symmetries will have a fixed point) or a T^2 (one of the circle symmetries will have a fixed point). The rank of the energy momentum map $\mathbb{R}^8 \to (H, H_2, I_1, I_2)$ will correspond to the dimension of the fibre.

The shape of the reduced phase space is determined by the positive part of $f(K)$. $f(K)$ can be written as

$$f(K) = (K + n + \zeta + a)(K - n - \zeta + a)(K - n + \zeta - a)(K + n - \zeta - a),$$

thus, the four zeroes of $f(K)$ are given by

$$K_1 = -a - n - \zeta \ , \ K_2 = a + n - \zeta \ , \ K_3 = a - n + \zeta \ , \ K_4 = -a + n + \zeta \ .$$
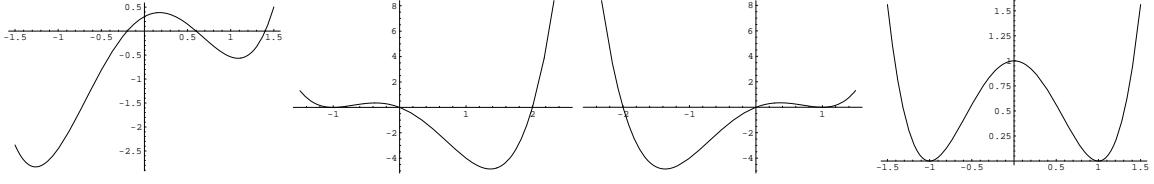
Figure 1: Graph of $f(K)$. From left to right: $a \neq \zeta \neq n$, $a = -\zeta$, $a = \zeta$, $a = \zeta = 0$

So $f(K)$ is positive (or zero) in the subsequent intervals of K:

$$
\begin{array}{lll}
a > \zeta, & K \in [a - n + \zeta, -a + n + \zeta] \\
a < \zeta, & K \in [a - n + \zeta, a + n - \zeta] \\
a = \zeta > 0, & K \in [2\zeta - n, n] \\
a = \zeta < 0, & K \in [-2\zeta - n, n] \\
a = -\zeta < 0, & K \in [-n, n - 2\zeta] \\
a = -\zeta > 0, & K \in [-n, n + 2\zeta] \\
a = \zeta = 0, & K \in [-n, n]
\end{array}
$$

$$
\begin{array}{lll}
a = \pm n, & K = \pm \zeta \\
\zeta = \pm n, & K = \pm a \\
|a| = |\pm \zeta| = n, & K = \pm n \\
a = \zeta = n = 0, & K = 0
\end{array}
$$

(11)

See figure (1).

When we have a simple root of $f(K)$ which belongs to one of the above intervals, we have that the intersection of the reduced phase space with the K-axis is smooth. $f(K)$ has four different roots in the following two cases: (i) $a \neq \zeta$ and $\zeta, a \neq 0$; (ii) $a \neq \zeta$ and $\zeta = 0$ or $a = 0$. In these cases the reduced phase space is diffeomorphic to a sphere. A point on this sphere corresponds to a three-torus in original phase space.

To find the the double zeroes of $f(K)$ we compute the discriminant of $f(K) = 0$. It is

$$(a - n)^2(a + n)^2(a - \zeta)^2(a + \zeta)^2(n - \zeta)^2(n + \zeta)^2 .$$

Thus there are double zeroes at $a = \pm n$, $a = \pm \zeta$ and $\zeta = \pm n$. If we have just one double zero the reduced phase space is a sphere with one cone-like singularity at the intersection point given by the double root ($a = \pm \zeta \neq 0$). If we have two double zeroes the reduced phase space is a sphere with two cone-like singularities at the intersection points given by the double roots ($a = \zeta = 0$). In the other cases the reduced phase space is just one singular point. The singular points correspond to two-tori in original phase space.

Triple zeroes occur when $|a| = |\zeta| = n$. The reduced phase space is just a point which corresponds to a circle in original phase space.
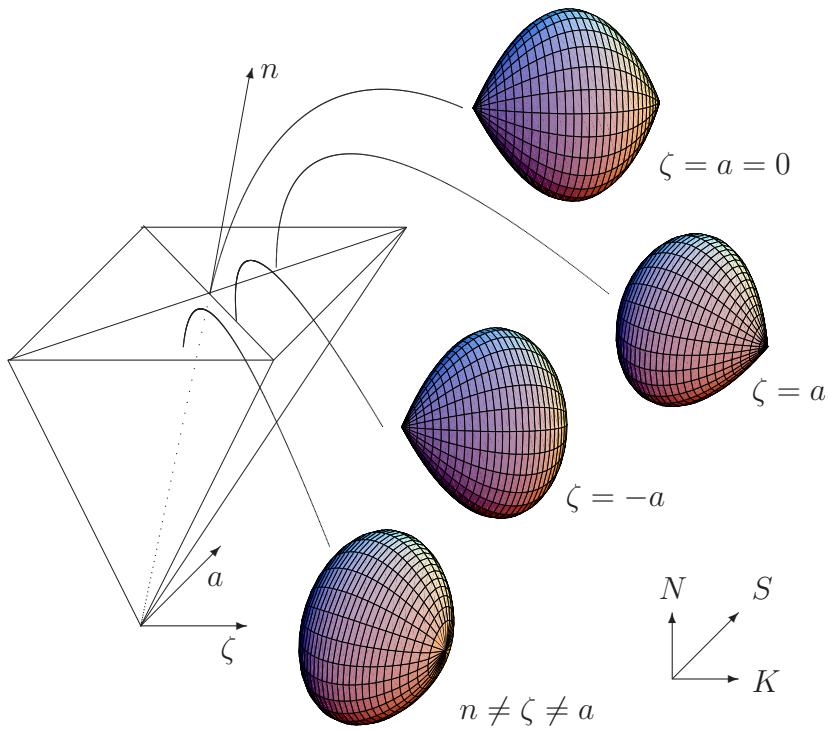
Figure 2: The thrice reduced phase space over the parameter space. K is the symmetry axis of each surface.

Quadruple zeroes only occur when $a = n = \zeta = 0$, which corresponds to the origin in original phase space and is a stationary point. See figure 2. More details on this analysis can be found in [7].

The cone-like singularities of the reduced phase space are candidates for the occurrence of Hamiltonian Hopf bifurcations, therefore in the following we restrict ourselves to the case $a = \zeta$ in which case we have a cone-like point at $K = n$.

5   Hamiltonian Hopf bifurcations at $K = n$ when $a = \zeta$

Let us consider the vector field given by the second reduced hamiltonian $\bar{\bar{H}}^\lambda$, that is the vector field in the second reduced phase space, that is isomorphic to $S^2 \times S^2$:

$$
\begin{aligned}
\dot{n} &= 0, \\
\dot{k}_1 &= 8\,(k_3\,l_2 - k_2\,l_3)\,n, \\
\dot{k}_2 &= l_3\,(-2\,k_1\,n + 2\,l_1\,\zeta + 3\,n\,\lambda) - k_3\,(6\,l_1\,n + \zeta\,(-2\,k_1 + \lambda)), \\
\dot{k}_3 &= l_2\,(2\,k_1\,n - 2\,l_1\,\zeta - 3\,n\,\lambda) + k_2\,(6\,l_1\,n + \zeta\,(-2\,k_1 + \lambda)), \\
\dot{\zeta} &= 0, \\
\dot{l}_1 &= 0, \\
\dot{l}_2 &= -10\,k_1\,k_3\,n + 2\,l_1\,l_3\,n + 2\,k_3\,l_1\,\zeta + 2\,k_1\,l_3\,\zeta + 3\,k_3\,n\,\lambda - l_3\,\zeta\,\lambda, \\
\dot{l}_3 &= -2\,l_1\,(l_2\,n + k_2\,\zeta) + 2\,k_1\,(5\,k_2\,n - l_2\,\zeta) - 3\,k_2\,n\,\lambda + l_2\,\zeta\,\lambda \}
\end{aligned}
$$

On the second reduced phase space with coordinates $\{k_1, k_2, k_3, l_1, l_2, l_3\}$, consider the point that corresponds to $K = n$, $S = N = 0$. This is the point $\{n, 0, 0, \zeta, 0, 0\}$. The linearization of the above vector field at this point is given by

$$
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\zeta\,(4n+\lambda) & 0 & 0 & 0 & -\Delta_1 \\
0 & 0 & \zeta\,(4n+\lambda) & 0 & 0 & 0 & \Delta_1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\Delta_2 & 0 & 0 & 0 & \zeta\,(4n-\lambda) \\
0 & 0 & \Delta_2 & 0 & 0 & 0 & \zeta\,(-4n+\lambda) & 0
\end{pmatrix},
\tag{12}
$$

with $\Delta_1 = 2n^2 - 3\lambda n - 2\zeta^2$ and $\Delta_2 = 10n^2 - 3\lambda n - 2\zeta^2$. The eigenvalues are given by

$$
\pm\sqrt{-\Theta \pm 2\zeta\lambda\sqrt{\Theta} - \zeta^2\lambda^2}
\tag{13}
$$

with

$$
\Theta = 20\,n^4 - 36\,\lambda\,n^3 + (9\,\lambda^2 - 8\,\zeta^2)\,n^2 + 12\,\zeta^2\,\lambda\,n + 4\,\zeta^4.
\tag{14}
$$

When $\Theta = 0$ we have two purely imaginary eigenvalues
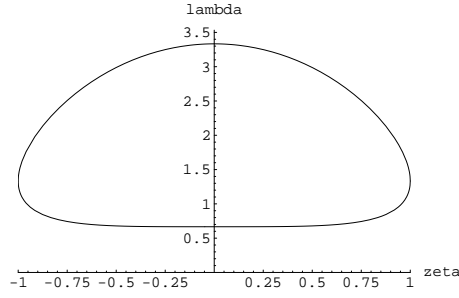
$$
\pm i\lambda\zeta
\tag{15}
$$

Figure 3: The bifurcation curve $\Theta = 0$ in the (ζ, λ)-plane

If $\Theta < 0$ we have two pairs of complex eigenvalues

$$\pm\sqrt{-(\Theta + \lambda^2\zeta^2) \pm i\, 2\lambda\zeta\sqrt{|\Theta|}}, \tag{16}$$

and if $\Theta > 0$ we have two pairs of imaginary eigenvalues

$$\pm\sqrt{-(\sqrt{\Theta} \pm \lambda\zeta)^2} \tag{17}$$

 So when we cross the bifurcation curve given given by $\Theta = 0$ (see figure (3)) we are in the scenario of a Hamiltonian Hopf bifurcation. This curve pulls into the origin if n goes to zero.

The eigenvalue behavior becomes degenerate when $\zeta = 0$, especially when one crosses the bifurcation curve along the line $\zeta = 0$. In this case the linearized system becomes nilpotent if $\lambda = \frac{2}{3}n$ and $\lambda = \frac{10}{3}n$. The eigenvalues go from a double pair of purely imaginary eigenvalues through a quadruple zero eigenvalue to two pairs of real eigenvalues. Because of the presence of the S^1 symmetry corresponding to I_2 this nevertheless gives rise to a Hamiltonian Hopf bifurcation (see the remark in [13])because the nilpotent part of the quadratic Hamiltonian can be embedded into a Lie algebra isomorphic to $sl(2, \mathbb{R})$ which commutes with the semisimple Hamiltonian generating the S^1-symmetry. In this case the singularity theory allows to obtain the standard form also. Note that the most reduced standard form for the Hamiltonian representing a Hamiltonian Hopf bifurcation obtained in [18] also has a purely nilpotent quadratic part. Remains to check the geometric conditions necessary for the presence of a Hamiltonian Hopf bifurcation as formulated in [13].

Consider

$$g(K) = 2N = \frac{3}{2}(K^2 - \lambda K) - \frac{a\,\zeta}{n}K + \frac{h}{n}.$$

representing the Hamiltonian energy in the $(K, 2N)$-plane. Furthermore consider

$$\phi_+(K) = \sqrt{f(K)} = \sqrt{(n-K)^2(n+K+2\zeta)(n+K-2\zeta)}\,,$$

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and

$$\phi_-(K) = -\sqrt{f(K)} = -\sqrt{(n-K)^2(n+K+2\zeta)(n+K-2\zeta)} \, .$$

representing the upper arc, respectively the lower arc, of the reduced phase space with $-n \leq K \leq n$.

Note that we want the reduced phase space to have a cone-like singularity at $K = n$, that is, $a = \zeta$ and $\Theta = 0$. From $\Theta = 0$ we obtain

$$\lambda = 2n - \frac{2\zeta^2}{3n} \pm \frac{4}{3}\sqrt{n^2 - \zeta^2} \, .$$

Furthermore we want g to pass through $(n, 0)$ which gives

$$h = \frac{3}{2}n^3 + 2n^2\sqrt{n^2 - \zeta^2} \text{ for } \lambda = 2n - \frac{2\zeta^2}{3n} + \frac{4}{3}\sqrt{n^2 - \zeta^2} \, ,$$

in which case we consider the upper half of the curve $\Theta = 0$, and g is tangent to the upper arc of the reduced phase space. Furthermore

$$h = \frac{3}{2}n^3 - 2n^2\sqrt{n^2 - \zeta^2} \text{ for } \lambda = 2n - \frac{2\zeta^2}{3n} - \frac{4}{3}\sqrt{n^2 - \zeta^2} \, ,$$

in which case we consider the lower half of the curve $\Theta = 0$, and g is tangent to the lower arc of the reduced phase space.

The first condition to check is wether the reduced energy surface moves in the right way through the cone-like singularity of the reduced phase space. Actually this is a transversality condition for the unfolding of the linear system. We have

$$\frac{d}{d\lambda}g'_\lambda(n) = -\frac{3}{2}n$$

This is nonzero. Consequently the transversality condition is fulfilled.

The second condition is the nondegeneracy condition on the higher order terms of the Hamiltonian. We have to check wether the Hamiltonian has second order contact with the reduced phase space at the vertex of the cone and wether it is tangent to the cone from the inside or the outside. Let

$$u_+(K) = g(K) - \phi_+(K) \text{ and } u_-(K) = g(K) - \phi_-(K) \, .$$

We have

$$u''_+(n) = 3 + \frac{2n}{\sqrt{n^2 - \zeta^2}} \text{ and } u''_-(n) = 3 - \frac{2n}{\sqrt{n^2 - \zeta^2}} \, .$$

At the upper arc of the reduced phase space we will always have tangency from outside the cone. Consequently we have a supercritical Hamiltonian Hopf bifurcation. At the

lower arc of the reduced phase space we have quadratic tangency from inside the cone for $-\frac{\sqrt{5}}{3}n < \zeta < \frac{\sqrt{5}}{3}n$ in which case we have a subcritical Hamiltonian Hopf bifurcation. For $-n \leq \zeta < -\frac{\sqrt{5}}{3}n$ and $\frac{\sqrt{5}}{3}n < \zeta \leq n$ we have quadratic tangency from outside the cone and a supercritical Hamiltonian Hopf bifurcation. At $\zeta = \pm\frac{\sqrt{5}}{3}n$ we have a degenerate Hamiltonian Hopf bifurcation.

Along the curve $\Theta = 0$ there is also a degeneracy at $(\zeta, \lambda) = (\pm n, \frac{4}{3}n)$. At these points the reduced phase space reduces to a point.

These Hamiltonian Hopf bifurcations established for the truncated normalized system will also be present in the original system 1 with possible higher order perturbations that commute with I_1 and I_2. This is a consequence of the fact that the system 1 can be reduced to a two-degree-of-freedom system using the integrals I_1 and I_2. The truncated normal form of this system will then be the same as the one obtained above in the invariants for the three-torus action, and will give rise to the same reduced one-degree-of-freedom system. Consequently the arguments in [18] apply to the two-degree-of-freedom system and consequently we have Hamiltonian Hopf bifurcations of three-tori emanating from a family of two-tori.

Many other aspects of this system including the connection with perturbed Keplerian systems will be considered in a forthcoming paper [8].

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