The discrete time H_∞ control problem with measurement feedback

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Abstract

This paper is concerned with the discrete time H_∞ control problem with measurement feedback. It turns out that, as in the continuous time case, the existence of an internally stabilizing controller which makes the H_∞ norm strictly less than 1 is related to the existence of stabilizing solutions to two algebraic Riccati equations. However in the discrete time case the solutions of these algebraic Riccati equations have to satisfy extra conditions.

Keywords : H_∞ control, Discrete time, Algebraic Riccati equation, Measurement feedback.

1 Introduction

In recent years a considerable amount of papers have appeared about the by now well-known H_∞ optimal control problem (e.g. [1], [2], [3], [7], [8], [14], [15], [17], [20]). However, all these papers discuss the continuous time case. In this paper we will, in contrast with the above papers, we will discuss the discrete time case.

One way to tackle this problem is to transform the system into a continuous time system, to derive controllers for the latter system and then transform back to discrete time. In our opinion however it is more natural to have the formulas directly available in terms of the original parameters. This leaves the possibility to directly see the effect of certain physical parameters. This possibility might otherwise be blurred by the transformation to the continuous time.

In the above papers several methods were used to solve the H_∞ control problem, e.g. the frequency domain approach, the polynomial aproach and the time domain approach. Recently, a paper appeared solving the discrete time H_∞ control problem using frequency domain techniques ([6]). Another paper approaches the problem using time-domain techniques and differential games ([19]). However, the latter paper only discusses the full-information case.

In correspondence with [19], we will use time-domain techniques and differential games. The present paper has a lot of familiarities with the papers [15, 17] which deal with the continuous time case. It is an extension of a previous paper [16], which deals with the full-information case.

Compared with [17, 19] we have weaker assumptions. Firstly we do not assume that the system matrix A is invertible. Secondly we weaken the assumptions from [6, 19] on the direct feedthrough matrices to the assumption that two particular transfer matrices are left and right invertible respectively. The only other assumption we have to make is that two subsystems have no invariant zeros on the unit circle. Our assumptions are exactly the discrete time analogues of the assumptions in [4].

As in the continuous time case, the necessary and sufficient conditions for the existence of suitable controllers involve positive semi-definite stabilizing solutions of two algebraic Riccati equations. As in the continuous time case the quadratic term in these algebraic Riccati equations is indefinite. However, compared to the continuous time case, the solutions of these equations have to satisfy another assumption: matrices depending on these solutions should be positive definite.

The outline of this paper is as follows. In section 2 we will formulate the problem and give our main results. In section 3 we will derive the existence of a stabilizing solution of the first algebraic Riccati equation starting from the assumption that there exists an internally stabilizing feedback which makes the H_∞ norm less than 1. In section 4, we will show the existence of a stabilizing solution of the second algebraic Riccati equation and complete the proof that our conditions are necessary. This is done by transforming the original system into a new system with the property that a controller "works" for the new system if and only if it "works" for the original system. In section 5 it is shown that our conditions are also sufficient. It turns out that the system transformation of section 4 repeated in a dual form exactly gives the desired results. We will end with some concluding remarks in section 6.

2 Problem formulation and main results

We consider the following time-invariant system:

$$\Sigma : \begin{cases} x(k+1) &= Ax(k) &+ & Bu(k) &+ & Ew(k), \\ y(k) &= C_1x(k) &+ & &+ & D_{12}w(k), \\ z(k) &= C_2x(k) &+ & D_{21}u(k) &+ & D_{22}w(k), \end{cases} \tag{2.1}$$

where $x(k) \in \mathcal{R}^n$ is the state, $u(k) \in \mathcal{R}^m$ is the control input, $y(k) \in \mathcal{R}^l$ is the measurement, $w(k) \in \mathcal{R}^l$ the unknown disturbance and $z(k) \in \mathcal{R}^p$ the output to be controlled. $A, B, E, C_1, C_2, D_{12}, D_{21}$ and D_{22} are matrices of appropriate dimension. If we apply a dynamic feedback law $u = Fy$ to Σ then the closed loop system with zero initial conditions defines a convolution operator $\Sigma_{cl,F}$ from w to y. We seek a feedback law $u = Fy$ which is internally stabilizing and which minimizes the ℓ_2-induced operator norm of $\Sigma_{cl,F}$ over all internally stabilizing feedback laws. We will investigate dynamic feedback laws of the form:

$$\Sigma_F : \begin{cases} p(k+1) &= K_c p(k) + L_c y(k), \\ u(k) &= M_c p(k) + N_c y(k). \end{cases} \tag{2.2}$$

We will say that the dynamic compensator Σ_F, given by (2.2), is internally stabilizing if the following matrix is asymptotically stable:

$$\begin{pmatrix} A + BN_cC_1 & BM_c \\ L_cC_1 & K_c \end{pmatrix}, \tag{2.3}$$

i.e. all its eigenvalues lie in the open unit disc. Denote by G_F the closed loop transfer matrix. The ℓ_2 induced operator norm of the convolution operator $\Sigma_{cl,F}$ is equal to the H_∞ norm of the transfer matrix G_F and is given by:

$$\begin{aligned} \|G_F\|_\infty &:= \sup_w \left\{ \frac{\|z\|_2}{\|w\|_2} \;\middle|\; w \in \ell_2^l,\ w \neq 0 \right\} \\ &:= \sup_{\theta \in [0,2\pi]} \|G_F(e^{i\theta})\| \end{aligned}$$

where the ℓ_2-norm is given by:

$$\|p\|_2 := \left(\sum_{k=0}^{\infty} p^{\mathrm{T}}(k) p(k) \right)^{1/2}$$

and where $\|.\|$ denotes the largest singular value. In this paper we will derive necessary and sufficient conditions for the existence of a dynamic compensator Σ_F which is internally stabilizing and which is such that the closed loop transfer matrix G_F satisfies $\|G_F\|_\infty < 1$. By scaling the plant we can thus, in principle, find the infimum of the closed loop H_∞ norm over all stabilizing controllers. This will involve a search procedure. Furthermore if a stabilizing Σ_F exists which makes the H_∞ norm less than 1, we derive an explicit formula for one particular F satisfying these requirements.
In the formulation of our main result we will need the concept of invariant zero: z_0 is called an *invariant zero* of the system (A, B, C, D) if

$$\operatorname{rank}_{\mathcal{R}} \begin{pmatrix} z_0 I - A & -B \\ C & D \end{pmatrix} < \operatorname{rank}_{\mathcal{R}(z)} \begin{pmatrix} zI - A & -B \\ C & D \end{pmatrix}$$

where $\operatorname{rank}_{\mathcal{K}}$ denotes the rank as a matrix with entries in the field $\mathcal{K}$. By $\mathcal{R}(z)$ we denote the field of real rational functions. The system (A, B, C, D) is called *left (right) invertible* if the transfer matrix $C(zI - A)^{-1}B + D$ is left (right) invertible as a matrix with entries in the field of real rational functions. We can now formulate our main result:

Theorem 2.1 : *Consider the system (2.1). Assume that $(A,\ B,\ C_2,\ D_{21})$ has no invariant zeros on the unit circle and is left invertible. Moreover, assume that $(A,\ E,\ C_1,\ D_{12})$ has no invariant zeros on the unit circle and is right invertible. The following statements are equivalent:*

(*i*) *There exists a dynamic compensator Σ_F of the form (2.2) such that the resulting closed loop transfer matrix G_F satisfies $\|G_F\|_\infty < 1$ and the closed loop system is internally stable.*

(*ii*) *There exist symmetric matrices $P \geq 0$ and $Y \geq 0$ such that*

(*a*) *We have*

$$V > 0, \qquad R > 0, \tag{2.4}$$

where

$$\begin{aligned} V &:= B^{\mathrm{T}}PB + D_{21}^{\mathrm{T}}D_{21}, \\ R &:= I - D_{22}^{\mathrm{T}}D_{22} - E^{\mathrm{T}}PE + \left(E^{\mathrm{T}}PB + D_{22}^{\mathrm{T}}D_{21}\right)V^{-1}\left(B^{\mathrm{T}}PE + D_{21}^{\mathrm{T}}D_{22}\right). \end{aligned}$$

This implies that the matrix $G(P)$ is invertible where:

$$G(P) := \left[\begin{pmatrix} D_{21}^{\mathrm{T}}D_{21} & D_{21}^{\mathrm{T}}D_{22} \\ D_{22}^{\mathrm{T}}D_{21} & D_{22}^{\mathrm{T}}D_{22} - I \end{pmatrix} + \begin{pmatrix} B^{\mathrm{T}} \\ E^{\mathrm{T}} \end{pmatrix} P \begin{pmatrix} B & E \end{pmatrix}\right]. \tag{2.5}$$

(*b*) *P satisfies the discrete algebraic Riccati equation:*

$$P = A^{\mathrm{T}}PA + C_2^{\mathrm{T}}C_2 - \begin{pmatrix} B^{\mathrm{T}}PA + D_{21}^{\mathrm{T}}C_2 \\ E^{\mathrm{T}}PA + D_{22}^{\mathrm{T}}C_2 \end{pmatrix}^{\mathrm{T}} G(P)^{-1} \begin{pmatrix} B^{\mathrm{T}}PA + D_{21}^{\mathrm{T}}C_2 \\ E^{\mathrm{T}}PA + D_{22}^{\mathrm{T}}C_2 \end{pmatrix}. \tag{2.6}$$

(*c*) *The matrix $A_{cl,P}$ is asymptotically stable where:*

$$A_{cl,P} := A - \begin{pmatrix} B & E \end{pmatrix} G(P)^{-1} \begin{pmatrix} B^{\mathrm{T}}PA + D_{21}^{\mathrm{T}}C_2 \\ E^{\mathrm{T}}PA + D_{22}^{\mathrm{T}}C_2 \end{pmatrix}. \tag{2.7}$$

Moreover if, given the matrix P satisfying (a)-(c), we define the following matrices:

$$\begin{aligned} H &:= E^{\mathrm{T}}PA + D_{22}^{\mathrm{T}}C_2 - \left[E^{\mathrm{T}}PB + D_{22}^{\mathrm{T}}D_{21}\right]V^{-1}\left[B^{\mathrm{T}}PA + D_{21}^{\mathrm{T}}C_2\right], \\ A_P &:= A + ER^{-1}H, \\ E_P &:= ER^{-1/2}, \\ C_{1,P} &:= C_1 + D_{12}R^{-1}H, \\ C_{2,P} &:= V^{-1/2}\left(B^{\mathrm{T}}PA + D_{21}^{\mathrm{T}}C_2\right) + V^{-1/2}\left[B^{\mathrm{T}}PE + D_{21}^{\mathrm{T}}D_{22}\right]R^{-1}H, \\ D_{12,P} &:= D_{12}R^{-1/2}, \\ D_{21,P} &:= V^{1/2}, \\ D_{22,P} &:= V^{-1/2}\left(B^{\mathrm{T}}PE + D_{21}^{\mathrm{T}}D_{22}\right)R^{-1/2}, \end{aligned}$$

then the matrix Y should satisfy:

(*d*) *We have*

$$W > 0, \qquad S > 0, \tag{2.8}$$

where

$$W := D_{12,P}D_{12,P}^{\mathrm{T}} + C_{1,P}YC_{1,P}^{\mathrm{T}},$$

$$S := I - D_{22,P}D_{22,P}^{\mathrm{T}} - C_{2,P}YC_{2,P}^{\mathrm{T}}\left(C_{2,P}YC_{1,P}^{\mathrm{T}} + D_{22,P}D_{12,P}^{\mathrm{T}}\right)W^{-1}\left(C_{1,P}YC_{2,P}^{\mathrm{T}} + D_{12,P}D_{22,P}^{\mathrm{T}}\right).$$

This implies that the matrix $H_P(Y)$ is invertible where:

$$H_P(Y) := \begin{pmatrix} D_{12,P}D_{12,P}^{\mathrm{T}} & D_{12,P}D_{22,P}^{\mathrm{T}} \\ D_{22,P}D_{12,P}^{\mathrm{T}} & D_{22,P}D_{22,P}^{\mathrm{T}} - I \end{pmatrix} + \begin{pmatrix} C_{1,P} \\ C_{2,P} \end{pmatrix} Y \begin{pmatrix} C_{1,P}^{\mathrm{T}} & C_{2,P}^{\mathrm{T}} \end{pmatrix}. \tag{2.9}$$

(*e*) *Y satisfies the following discrete algebraic Riccati equation:*

$$Y = A_P Y A_P^{\mathrm{T}} + E_P E_P^{\mathrm{T}} - \begin{pmatrix} C_{1,P} Y A_P^{\mathrm{T}} + D_{12,P} E_P^{\mathrm{T}} \\ C_{2,P} Y A_P^{\mathrm{T}} + D_{22,P} E_P^{\mathrm{T}} \end{pmatrix}^{\mathrm{T}} H_P(Y)^{-1} \begin{pmatrix} C_{1,P} Y A_P^{\mathrm{T}} + D_{12,P} E_P^{\mathrm{T}} \\ C_{2,P} Y A_P^{\mathrm{T}} + D_{22,P} E_P^{\mathrm{T}} \end{pmatrix}. \tag{2.10}$$

(*f*) *The matrix $A_{cl,P,Y}$ is asymptotically stable where:*

$$A_{cl,P,Y} := A_P - \begin{pmatrix} C_{1,P} Y A_P^{\mathrm{T}} + D_{12,P} E_P^{\mathrm{T}} \\ C_{2,P} Y A_P^{\mathrm{T}} + D_{22,P} E_P^{\mathrm{T}} \end{pmatrix}^{\mathrm{T}} H_P(Y)^{-1} \begin{pmatrix} C_{1,P} \\ C_{2,P} \end{pmatrix}. \tag{2.11}$$

In case there exist $P \geq 0$ and $Y \geq 0$ satisfying (ii) then a controller of the form (2.2) satisfying the requirements in (i) is given by:

$$\begin{aligned} N_c &:= -D_{21,P}^{-1} \left(C_{2,P} Y C_{1,P}^{\mathrm{T}} + D_{22,P} D_{12,P}^{\mathrm{T}} \right) W^{-1}, \\ M_c &:= -(D_{21,P}^{-1} C_{2,P} + N C_{1,P}), \\ L_c &:= BN + \left(A_P Y C_{1,P}^{\mathrm{T}} + E_P D_{12,P}^{\mathrm{T}} \right) W^{-1}, \\ K_c &:= A_{cl,P} - L C_{1,P}, \end{aligned}$$

□

Remark :

(*i*) Necessary and sufficient conditions for the existence of an internally stablizing feedback compensator which makes the H_∞ norm less than some a priori given upper bound $\gamma > 0$ can be easily derived from theorem 2.1 by scaling.

(*ii*) If we compare these conditions with the conditions for the continuous time case (see [2, 15]) we note that conditions (2.4) and (2.8) are added. A simple example showing that simply the assumption $G(P)$ invertible is not sufficient is given by the system:

$$\left\{ \begin{array}{lcl} x(k+1) & = & u(k) + 2w(k) \\ y(k) & = & \begin{pmatrix} 1 \\ 0 \end{pmatrix} x(k) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(k) \\ z(k) & = & \begin{pmatrix} 1 \\ 0 \end{pmatrix} x(k) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(k) \end{array} \right. \tag{2.12}$$

There doesn't exist a dynamic compensator satisfying the requirements of part (i) of theorem 2.1 but there does exist a positive semidefinite matrix P satisfying (2.6) such that the matrix (2.7) is asymptotically stable, namely $P = 1$. However for this P we have $R = -1$. Therefore matrices like E_P are ill-defined and we can not even look for a matrix Y satisfying (2.8)-(2.11).

(*iii*) Since our starting point of the proof of (i) $\Rightarrow$ (ii) will not be part (i) of theorem 2.1 but condition 3.2, it can be seen that we can not make the H_∞ norm less by allowing more general, possibly even non-linear, causal feedbacks.

The proof of the existence of a stabilizing solution of the Riccati equation will be reminiscent of the proof given in [17] for the continuous time case. However due to our weaker assumptions and the conditions (2.4) and (2.8) there are quite a number of extra intricacies. The remainder of the proof is based on [15].
Another interesting case was discussed in [16]. However the latter reference only gives the general outline of the proof. In contrast, the present paper will give much more details. [16] discusses the so called full information case:

Full information case : $C_1 = \begin{pmatrix} I \\ 0 \end{pmatrix}, \quad D_{12} = \begin{pmatrix} 0 \\ I \end{pmatrix}.$

In this case we have $y_1 = x$ and $y_2 = w$, i.e. we know both the state and the disturbance of the system at time k of the system. However we can not apply theorem 2.1 to this case since the system (A, E, C_1, D_{12}) is not right invertible. Nevertheless following the proof for this special case it can be shown that there exists a feedback satisfying part (i) of theorem 2.1 if and only if there exist a symmetric matrix $P \geq 0$ satisfying conditions (a)-(c) of part (ii) of theorem 2.1. Moreover in that case we can find static output feedbacks $u = F_1 x + F_2 w$ with the desired properties. One particular choice for $F = (F_1, F_2)$ is given by:

$$F_1 := -(D_{21}^{\mathrm{T}} D_{21} + B^{\mathrm{T}} P B)^{-1} (B^{\mathrm{T}} P A + D_{21}^{\mathrm{T}} C_2) \tag{2.13}$$
$$F_2 := -(D_{21}^{\mathrm{T}} D_{21} + B^{\mathrm{T}} P B)^{-1} (B^{\mathrm{T}} P E + D_{21}^{\mathrm{T}} D_{22}) \tag{2.14}$$

3 Existence of stabilizing solutions of the Riccati equations

In this section we assume that part (i) of theorem 2.1 is satisfied. We will show that the existence of P satisfying conditions (a)-(c) in (ii) is necessary. Consider system (2.1). For given disturbance w and control input u let $x_{u,w,\xi}$ and $z_{u,w,\xi}$ denote the resulting state and output respectively for initial state $x(0) = \xi$. If $\xi = 0$ we will simply write $x_{u,w}$ and $z_{u,w}$. We first give a definition:

Definition 3.1 : *An operator $f : \ell_2 \to \ell_2$, $w \to f(w)$ is called causal if for any $w_1, w_2 \in \ell_2$ and $k \in \mathbb{N}$:*

$$w_1|_{[0,k]} = w_2|_{[0,k]} \quad \Rightarrow \quad f(w_1)|_{[0,k]} = f(w_2)|_{[0,k]}.$$

f is called strictly causal if for any $w_1, w_2 \in \ell_2$ and $k \in \mathbb{N}$ we have

$$w_1|_{[0,k-1]} = w_2|_{[0,k-1]} \quad \Rightarrow \quad f(w_1)|_{[0,k]} = f(w_2)|_{[0,k]}.$$

□

A controller of the form (2.2) always defines a causal operator. In case $N = 0$ this operator is strictly causal. We will label the following condition:

Condition 3.2 : *(A, B) stabilizable and for the system (2.1) there exists causal $f : \ell_2^l \to \ell_2^m$ and $\delta > 0$ such that for all $w \in \ell_2^l$ if $u = f(w)$ we have $x_{u,w} \in \ell_2^n$ and $\|z_{u,w}\|_2^2 \leq (1 - \delta^2)\|w\|_2^2$.* □

If there exists a dynamic compensator Σ_F such that $\|G_F\|_\infty < 1$ and the closed loop system is internally stable, then condition 3.2 is satisfied. Hence if the requirements of part (i) of theorem 2.1 are satisfied then condition 3.2 holds. Note that condition 3.2 is equivalent to the requirement that there exists a causal operator f such that the feedback $u = f(x, w)$ satisfies condition 3.2. This follows from the fact that, after applying the feedback, there exists a causal operator g mapping w to x and therefore we could have started with the causal operator $u = f(g(w), w)$ in the first place. Conversely if we have the feedback $u = f(w)$ then we define $f_1(x, w) := f(w)$ which then satisfies the requirements of the reformulated condition 3.2.
We will show that the existence of such causal f and $\delta > 0$ of condition 3.2 already implies that there exist a positive semi definite solution of the discrete algebraic Riccati equation (2.6) such that (2.7) is asymptotically stable and (2.4) is satisfied. We will assume

$$D_{21}^{\mathrm{T}}[\ C_2 \quad D_{22}\ ] = 0$$

for the time being and we will derive the more general statement later. In order to prove the existence of the desired P we will investigate the following sup-inf problem:

$$C^*(\xi) := \sup_{w \in \ell_2^l} \inf_u \ \left\{ \ \|z_{u,w,\xi}\|_2^2 - \|w\|_2^2 \ \mid \ u \in \ell_2^m \text{ such that } x_{u,w,\xi} \in \ell_2^n \ \right\} \tag{3.1}$$

for arbitrary initial state ξ. It turns out that if condition 3.2 holds then this "sup-inf" problem is bounded from above for all initial states. It will turn out that there exists a $P \geq 0$ such that $C^*(\xi) = \xi^T P \xi$. It can then be shown that this P exactly satisfies conditions (a)-(c) of theorem 2.1. We will first infimize, for given $w \in \ell_2$ and $\xi \in \mathcal{R}^n$, the function $\|z_{u,w,\xi}\|_2^2 - \|w\|_2^2$ over all $u \in \ell_2$ for which $x_{u,w,\xi} \in \ell_2$. After that we will maximize over $w \in \ell_2$
As a tool we will use Pontryagin's maximum principle. This is only defined for the finite horizon case and only gives necessary conditions for optimality. However in [9] a sufficient condition for optimality is derived over a finite horizon. We will use the ideas from [9], together with our stability requirement, $x_{u,w,\xi} \in \ell_2$, to adapt the proof to the infinite horizon case.
Let L be such that $D_{21}^{\mathrm{T}} D_{21} + B^{\mathrm{T}} L B$ is invertible and such that L is the positive semi-definite solution of the following discrete algebraic Riccati equation:

$$L = A^{\mathrm{T}} L A + C_2^{\mathrm{T}} C_2 - A^{\mathrm{T}} L B \left(D_{21}^{\mathrm{T}} D_{21} + B^{\mathrm{T}} L B\right)^{-1} B^{\mathrm{T}} L A \tag{3.2}$$

for which

$$A_L := A - B \left(D_{21}^{\mathrm{T}} D_{21} + B^{\mathrm{T}} L B\right)^{-1} B^{\mathrm{T}} L A \tag{3.3}$$

is asymptotically stable. The existence of such L is guaranteed under the assumption that (A, B, C_2, D_{21}) has no invariant zeros on the unit circle, is left invertible and moreover (A, B) is stabilizable (see [13]). We define

$$r(k) := -\sum_{i=k}^{\infty} [X_1 A^{\mathrm{T}}]^{i-k} X_1 \left(L E w(i) + C_2^{\mathrm{T}} D_{22} w(i+1)\right) \tag{3.4}$$

where

$$X_1 := I - L B \left(D_{21}^{\mathrm{T}} D_{21} + B^{\mathrm{T}} L B\right)^{-1} B^{\mathrm{T}} \tag{3.5}$$

Note that r is well-defined since $A_L = X_1^{\mathrm{T}} A$ asymptotically stable implies that $X_1 A^{\mathrm{T}}$ is asymptotically stable. Next we define

$$\begin{aligned}
y(k) &:= M^{-1}B^{\mathrm{T}}\left[A^{\mathrm{T}}r(k+1) - LEw(k) - C_2^{\mathrm{T}}D_{22}w(k+1)\right] && (3.6)\\
\tilde{x}(k+1) &:= A_L\tilde{x}(k) + By(k) + Ew(k), \qquad \tilde{x}(0) = \xi && (3.7)\\
\eta(k) &:= -X_1LA\tilde{x}(k) + r(k) && (3.8)
\end{aligned}$$

for $k = 0, 1, 2, \ldots$ where $M := D_{21}^{\mathrm{T}}D_{21} + B^{\mathrm{T}}LB$. Since X_1A^{T} is asymptotically stable, it can be checked straightforwardly that, given $\xi \in \mathcal{R}^n$ and $w \in \ell_2^l$, we have $r, \tilde{x}, \eta \in \ell_2$. Moreover $\tilde{x}, r, y$ and η are, for given $\xi \in \mathcal{R}^n$ and $w \in \ell_2^l$, the unique solutions of the following boundary value problem:

$$\left\{\begin{array}{lcll}
\tilde{x}(k+1) &=& A_L\tilde{x}(k) + By(k) + Ew(k), & \tilde{x}(0) = \xi\\
r(k-1) &=& X_1\left[A^{\mathrm{T}}r(k) - LEw(k-1) - C_2^{\mathrm{T}}D_{22}w(k)\right] & \lim_{k\to\infty} r(k) = 0\\
y(k) &=& M^{-1}B^{\mathrm{T}}\left[A^{\mathrm{T}}r(k+1) - LEw(k) - C_2^{\mathrm{T}}D_{22}w(k+1)\right] &\\
\eta(k) &=& -X_1LA\tilde{x}(k) + r(k) &
\end{array}\right. \qquad (3.9)$$

for $k = 0, 1, 2, \ldots$. Uniqueness and existence stems from the fact that the two difference equations are not coupled and the matrix X_1A^{T} is asymptotically stable. Therefore, after some calculations, we find the following lemma:

Lemma 3.3 : *Let $\xi \in \mathcal{R}$ and $w \in \ell_2^l$ be given. The functions $r, \tilde{x}, \eta, y \in \ell_2$ are the unique solutions of the following boundary value problem:*

$$\left\{\begin{array}{lcll}
\tilde{x}(k+1) &=& A_L\tilde{x}(k) + By(k) + Ew(k), & \tilde{x}(0) = \xi\\
\eta(k-1) &=& A^{\mathrm{T}}\eta(k) - C_2^{\mathrm{T}}C_2\tilde{x}(k) - C_2^{\mathrm{T}}D_{22}w(k) & \lim_{k\to\infty} \eta(k) = 0\\
y(k) &=& M^{-1}B^{\mathrm{T}}\left[\eta(k) + L\tilde{x}(k+1) - LEw(k)\right] &\\
r(k) &=& X_1LA\tilde{x}(k) + \eta(k) &
\end{array}\right. \qquad (3.10)$$

for $k = 0, 1, 2, \ldots$. □

In the statement of Pontryagin's Maximum Principle the second equation is the so-called "adjoint Hamilton-Jacobi equation" and η is called the "adjoint state variable". We have constructed a solution to this equation and we will show that this η yields indeed a minimizing u. The proof is adapted from [9, Theorem 5.5]:

Lemma 3.4 : *Let the system (2.1) be given. Moreover let w and ξ be fixed. Then*

$$\tilde{u} := -\left(D_{21}^{\mathrm{T}}D_{21} + B^{\mathrm{T}}LB\right)^{-1}B^{\mathrm{T}}LA\tilde{x} + y = \arg\inf_u \left\{ \|z_{u,w,\xi}\|_2 \mid u \in \ell_2^m \text{ such that } x \in \ell_2^n \right\}$$

□

Proof : It can be easily checked that $\tilde{x} = x_{\tilde{u},w,\xi}$. Define

$$\mathcal{J}_T(u) := \sum_{i=0}^{T} \|C_2 x_{u,w,\xi}(i) + D_{21}u(i) + D_{22}w(i)\|^2. \tag{3.11}$$

Let $u \in \ell_2^m$ be an arbitrary control input such that $x_{u,w,\xi} \in \ell_2^n$. We find

$$\begin{aligned}\mathcal{J}_T(u) - \mathcal{J}_{T-1}(u) - 2\eta^{\mathrm{T}}(T)x(T+1) + 2\eta^{\mathrm{T}}(T-1)x(T) = \\ \|C_2 x(T)\|^2 + \left[D_{21}^{\mathrm{T}} D_{21} u(T) - 2B^{\mathrm{T}}\eta(T)\right]^{\mathrm{T}} u(T) \\ - 2\eta^{\mathrm{T}}(T)Ew(T) - 2x^{\mathrm{T}}(T)C_2^{\mathrm{T}} C_2 \tilde{x}(T)\end{aligned} \tag{3.12}$$

We also find

$$\begin{aligned}\mathcal{J}_T(\tilde{u}) - \mathcal{J}_{T-1}(\tilde{u}) - 2\eta^{\mathrm{T}}(T)\tilde{x}(T+1) + 2\eta^{\mathrm{T}}(T-1)\tilde{x}(T) = \\ - \|C_2\tilde{x}(T)\|^2 + \left[D_{21}^{\mathrm{T}} D_{21}\tilde{u}(T) - 2B^{\mathrm{T}}\eta(T)\right]^{\mathrm{T}} \tilde{u}(T) - 2\eta^{\mathrm{T}}(T)Ew(T)\end{aligned} \tag{3.13}$$

It can be seen that we have $\lim_{T\to\infty} \mathcal{J}_T(u) = \|z_{u,w,\xi}\|^2$. Moreover $\lim_{T\to\infty} \mathcal{J}_T(\tilde{u}) = \|z_{\tilde{u},w,\xi}\|^2$. Hence if we sum (3.12) and (3.13) from zero to infinity ($\mathcal{J}_{-1}(u) = 0$ and $\eta(-1)$ defined by (3.10) for $k = 0$) and subtract from each other we find: (Note that $x(0) = \tilde{x}(0) = \xi$)

$$\begin{aligned}\|z_{\tilde{u},w,\xi}\|_2^2 - \|z_{u,w,\xi}\|_2^2 = \sum_{i=0}^{\infty} - \|C_2\left(x(i) - \tilde{x}(i)\right)\|^2 + \\ + \sum_{i=0}^{\infty} \left[D_{21}^{\mathrm{T}} D_{21}\tilde{u}(i) - 2B^{\mathrm{T}}\eta(i)\right]^{\mathrm{T}} \tilde{u}(i) - \left[D_{21}^{\mathrm{T}} D_{21} u(i) - 2B^{\mathrm{T}}\eta(i)\right]^{\mathrm{T}} u(i)\end{aligned} \tag{3.14}$$

It can easily be checked that $B^{\mathrm{T}}\eta(i) = D_{21}^{\mathrm{T}} D_{21}\tilde{u}(i)$ for all i. Therefore we have

$$\left[D_{21}^{\mathrm{T}} D_{21}\tilde{u}(i) - 2B^{\mathrm{T}}\eta(i)\right]^{\mathrm{T}} \tilde{u}(i) = \inf_u \left[D_{21}^{\mathrm{T}} D_{21} u - 2B^{\mathrm{T}}\eta(i)\right]^{\mathrm{T}} u \tag{3.15}$$

(3.14) and (3.15) together imply that:

$$\|z_{\tilde{u},w,\xi}\|_2^2 \leq \|z_{u,w,\xi}\|_2^2 \tag{3.16}$$

which is exactly what we had to prove. Since (A, B, C_2, D_{21}) is left invertible it can easily be shown that the minimizing u is unique. ∎

We are now going to maximize over $w \in \ell_2$. This will then yield $\mathcal{C}^*(\xi)$. Define $\mathcal{F}(\xi, w) := (\tilde{x}, \tilde{u}, \eta)$ and $\mathcal{G}(\xi, w) := z_{\tilde{u},w,\xi} = C_2\tilde{x} + D_{21}\tilde{u} + D_{22}w$. It is clear from the previous lemma that $\mathcal{F}$ and $\mathcal{G}$ are bounded linear operators. Define

$$\mathcal{C}(\xi, w) := \|\mathcal{G}(\xi, w)\|_2^2 - \|w\|_2^2 \tag{3.17}$$

$$\|w\|_C := \left(-\mathcal{C}(0, w)\right)^{1/2} \tag{3.18}$$

It can be easily shown that $\|.\|_C$ defines a norm on ℓ_2^l. Using our condition 3.2 it can be shown straightforwardly that

$$\|w\|_2 \geq \|w\|_C \geq \delta\|w\|_2 \tag{3.19}$$

where δ is such that condition 3.2 is satisfied. Hence $\|.\|_C$ and $\|.\|_2$ are equivalent norms. We have

$$\mathcal{C}^*(\xi) = \sup_{w \in \ell_2^l} \mathcal{C}(\xi, w) \tag{3.20}$$

We can derive the following properties of $\mathcal{C}^*$:

Lemma 3.5 :

(i) *For all $\xi \in \mathcal{R}^n$ we have*

$$0 \leq \xi^{\mathrm{T}} L \xi \leq \mathcal{C}^*(\xi) \leq \frac{\xi^{\mathrm{T}} L \xi}{\delta^2} \tag{3.21}$$

where δ is such that (3.19) is satisfied.

(ii) *For all $\xi \in \mathcal{R}^n$ there exists an unique $w_* \in \ell_2^l$ such that $\mathcal{C}^*(\xi) = \mathcal{C}(\xi, w_*)$.* □

Proof : Part (i): It is well known that L, as the stabilizing solution of the discrete time algebraic Riccati equation (3.2), is the cost of the discrete-time, linear quadratic problem with internal stability (see [13]). Hence $\|\mathcal{G}(\xi,0)\|_2^2 = \mathcal{C}(\xi,0) = \xi^{\mathrm{T}} L \xi$. Therefore we have $0 \leq \xi^{\mathrm{T}} L \xi \leq \mathcal{C}^*(\xi)$. Moreover

$$\begin{aligned}
\mathcal{C}(\xi, w) &= \|\mathcal{G}(\xi, w)\|_2^2 - \|w\|_2^2 \\
&\leq (\|\mathcal{G}(\xi,0)\|_2 + \|\mathcal{G}(0,w)\|_2)^2 - \|w\|_2^2 \\
&\leq \left(\sqrt{\xi^{\mathrm{T}} L \xi} + \sqrt{1-\delta^2}\,\|w\|_2\right)^2 - \|w\|_2^2 \\
&\leq \frac{\xi^{\mathrm{T}} L \xi}{\delta^2}
\end{aligned}$$

Part (ii) can be proven in the same way as in [17]. First show that $\|.\|_C$ satisfies:

$$\|w_\alpha - w_\beta\|_C^2 = 2\,\mathcal{C}(\xi, w_\alpha) + 2\,\mathcal{C}(\xi, w_\beta) - 4\,\mathcal{C}\,(\xi, 1/2\,(w_\alpha + w_\beta)) \tag{3.22}$$

for arbitrary $\xi \in \mathcal{R}^n$. Then it can be shown that a maximizing sequence of $\mathcal{C}(\xi, w)$ is a Cauchy sequence with respect to the $\|.\|_C$-norm and hence, since $\|.\|_C$ and $\|.\|_2$ are equivalent norms, there exists a maximizing ℓ_2 function w_*. It is straightforward to show uniqueness using (3.22). ■

Define $\mathcal{H} : \mathcal{R}^n \to \ell_2^l,\ \xi \to w_*$. Unlike the explicit expression for $\tilde{u}$ we can only derive an implicit formula for w_*. We can however show that w_* is the unique solution of a linear equation:

Lemma 3.6 : *Let $\xi \in \mathcal{R}^n$ be given. $w_* = \mathcal{H}\xi$ is the unique ℓ_2-function w satisfying:*

$$(I - D_{22}^{\mathrm{T}} D_{22})\, w = -E^{\mathrm{T}} \eta + D_{22}^{\mathrm{T}} C_2 x \tag{3.23}$$

where $(x, u, \eta) = \mathcal{F}(\xi, w)$. □

Proof : Define $(x_*, u_*, \eta_*) = \mathcal{F}(\xi, w_*)$. Moreover define $w_0 := -E^{\mathrm{T}} \eta(w_*) + D_{22}^{\mathrm{T}} D_{22} w_* + D_{22}^{\mathrm{T}} C_2 x_*$ and $(x_0, u_0, \eta_0) := \mathcal{F}(\xi, w_0)$. We find:

$$\|z_{u_0,w_0,\xi}(T)\|^2 - \|w_0(T)\|^2 - 2\eta_*^{\mathrm{T}}(T)x_0(T+1) + 2\eta_*(T-1)^{\mathrm{T}}x_0(T) =$$
$$\|z_{u_*,w_*,\xi}(T) - z_{u_0,w_0,\xi}(T)\|^2 - \|z_{u_*,w_*,\xi}(T)\|^2 + \|w_0(T)\|^2 \quad (3.24)$$

Here we used that $D_{21}^{\mathrm{T}}D_{21}u_*(i) = B^{\mathrm{T}}\eta_*(i)$ for all i. We also find:

$$\|z_{u_*,w_*,\xi}(T)\|^2 - \|w_*(T)\|^2 - 2\eta_*^{\mathrm{T}}(T)x_*(T+1) + 2\eta_*(T-1)^{\mathrm{T}}x_*(T) =$$
$$2w_0^{\mathrm{T}}(T)w_*(T) - \|z_{u_*,w_*,\xi}(T)\|^2 - \|w_*(T)\|^2 \quad (3.25)$$

Summing (3.24) and (3.25) from zero to infinity and subtracting from each other gives us:

$$\mathcal{C}(\xi, w_*) = \mathcal{C}(\xi, w_0) - \|w_0 - w_*\|_2^2 - \|z_{u_0,w_0,\xi} - z_{u_*,w_*,\xi}\|_2^2 \quad (3.26)$$

Since w_* maximizes $\mathcal{C}(\xi, w)$ over all w, this implies $w_0 = w_*$. That w_* is the unique solution of the equation $w = -E^{\mathrm{T}}\eta(w)$ can be shown in a similar way. Assume that, apart from w_*, also w_1 satisfies (3.23). Let $(x_1, u_1, \eta_1) := \mathcal{F}(\xi, w_1)$. We find from (3.25):

$$\|z_{u_*,w_*,\xi}(T)\|^2 - \|w_*(T)\|^2 - 2\eta_*^{\mathrm{T}}(T)x_*(T+1) + 2\eta_*(T-1)^{\mathrm{T}}x_*(T) =$$
$$\|w_*(T)\|^2 - \|z_{u_*,w_*,\xi}(T)\|^2 \quad (3.27)$$

We also find:

$$\|z_{u_1,w_1,\xi}(T)\|^2 - \|w_1(T)\|^2 - 2\eta_*^{\mathrm{T}}(T)x_1(T+1) + 2\eta_*(T-1)^{\mathrm{T}}x_1(T) =$$
$$\|z_{u_1,w_1,\xi}(T)\|^2 - \|w_1(T)\|^2 + 2w_*^{\mathrm{T}}(T)w_1(T) - 2z_{u_*,w_*,\xi}^{\mathrm{T}}(T)z_{u_1,w_1,\xi}(T) \quad (3.28)$$

Summing (3.27) and (3.28) from 0 to ∞ and subtracting from each other gives us:

$$\mathcal{C}(\xi, w_*) = \mathcal{C}(\xi, w_1) - \|w_* - w_1\|_C^2 \quad (3.29)$$

Since w_* was maximizing we find $\|w_* - w_1\|_C = 0$ and hence $w_* = w_1$. q.e.d. ■

We will now show that $\mathcal{C}^*(\xi) = \xi^{\mathrm{T}}P\xi$ for some matrix P. In order to do that we first show that u_*, η_* and w_* are linear functions of x_*:

Lemma 3.7 *There exist constant matrices K_1, K_2 and K_3 such that*

$$u_* = K_1 x_*, \quad (3.30)$$
$$\eta_* = K_2 x_*, \quad (3.31)$$
$$w_* = K_3 x_*. \quad (3.32)$$

□

Proof : We will first look at time 0. By lemma 3.6 it is easily seen that $\mathcal{H} : \xi \to w_*$ is linear. Hence also the mapping from ξ to $w_*(0)$ is linear. This implies the existence of a matrix K_3 such that $w_*(0) = K_3\xi$. From (3.10) and lemma 3.4 it is easily seen that u_* and η_* are linear functions of ξ and

w_*. This implies, since w_* is a linear function of ξ, that $u_*(0)$ and $\eta_*(0)$ are linear functions of ξ and hence there exist K_1 and K_2 such that $u_*(0) = K_1\xi$ and $\eta_*(0) = K_2\xi$.
We will now look at time t. The sup-inf problem starting at time t with initial value $x(t)$ can now be solved. Due to time invariance we see that w_* restricted to $[t,\infty)$ satisfies $w_* = -E^{\mathrm{T}}\eta(w_*)$ and hence for this problem the optimal x and η are x_* and η_*. But since t is the initial time for this optimization problem, which is exactly equal to the original optimization problem, we find equations (3.30)-(3.32) at time t with the same matrices K_1, K_2 and K_3 as at time 0. Since t was arbitrary this completes the proof. ∎

Lemma 3.8 : *There exists a $P \geq 0$ such that $\eta_*(k) = -Px_*(k+1)$ $k = 0,1,2,\ldots$ Moreover for this P we find*

$$\mathcal{C}^*(\xi) = \xi^{\mathrm{T}} P \xi. \tag{3.33}$$

□

Proof : We have

$$\begin{aligned} \eta_*(k) &= A^{\mathrm{T}}\eta_*(k+1) - C_2^{\mathrm{T}}C_2x_*(k+1) - C_2^{\mathrm{T}}D_{22}w_*(k+1) \\ &= (A^{\mathrm{T}}K_2 - C_2^{\mathrm{T}}C_2 - C_2^{\mathrm{T}}D_{22}K_3)\,x_*(k+1), \qquad k = 0,1,2,\ldots \end{aligned}$$

We define $P := -(A^{\mathrm{T}}K_2 - C_2^{\mathrm{T}}C_2 - C_2^{\mathrm{T}}D_{22}K_3)$. We will prove that this P satisfies (3.33). We sum equation (3.27) from zero to infinity. Since $\lim_{T\to\infty}\eta_*(T) = 0$ and $\lim_{T\to\infty} x_*(T) = 0$ we find

$$\mathcal{C}(\xi, w_*) + 2\eta_*^{\mathrm{T}}(-1)x_*(0) = -\mathcal{C}(\xi, w_*)$$

Since $\mathcal{C}(\xi, w_*) = \mathcal{C}^*(\xi)$ and $\eta_*(-1) = -P\xi$ we find (3.33). ∎

We will now show that this matrix P satisfies condition (a)-(c) of theorem 2.1. We first show part (a). Since we do not know yet if P is symmetric we have to be a little bit careful. This essential step in our derivation is new compared to the method used in [17]:

Lemma 3.9 : *Let P be given by lemma 3.8. The matrices V and R as defined in the statement of theorem 2.1 part (ii) (a) satisfy:*

$$\begin{aligned} &(V + V^{\mathrm{T}}) > 0 \\ &(R + R^{\mathrm{T}}) > 0 \end{aligned}$$

□

Proof : By lemma 3.5 and lemma 3.8, we have $(P + P^{\mathrm{T}})/2 \geq L$ and therefore $(V + V^{\mathrm{T}})/2 \geq D_{21}^{\mathrm{T}}D_{21} + B^{\mathrm{T}}LB$. The latter matrix is positive definite and hence $(V + V^{\mathrm{T}})/2$ is positive definite, i.e. $V + V^{\mathrm{T}} > 0$.
We will now look at the following "sup-inf-sup-inf"-problem for arbitrary initial condition:

$$\mathcal{J}(\xi) := \sup_{w(0)} \inf_{u(0)} \sup_{w+} \inf_{u+} \; \|z_{u,w,\xi}\|^2 - \|w\|^2 \tag{3.34}$$

where $w^+ := w|_{[1,\infty)}$ and $u^+ := u|_{[1,\infty)}$. Since condition 3.2 holds we know there exists a causal function g which makes the ℓ_2-induced operator norm strictly less than 1. In (3.34) we may set $u = g(w)$ since by causality we know that $u(0)$ only depends on $w(0)$ and u^+ depends on the whole function w. Thus we get:

$$\mathcal{J}(\xi) = \sup_{w(0)} \inf_{u(0)} \sup_{w^+} \inf_{u^+} \|z_{u,w,\xi}\|_2^2 - \|w\|_2^2 \;\leq\; \sup_{w} \|z_{g(w),w,\xi}\|_2^2 - \|w\|_2^2 \tag{3.35}$$

$$\leq \frac{\xi^{\mathrm{T}} L \xi}{\delta^2} \tag{3.36}$$

where δ as defined by condition 3.2. The last inequality can be proven in the same way as lemma 3.5. Since, by lemma 3.8, we have:

$$\sup_{w^+} \inf_{u^+} \|z_{u^+,w^+,x(1)}\|_2^2 - \|w^+\|_2^2 = x(1)^{\mathrm{T}} P x(1), \tag{3.37}$$

we can reduce (3.34) to the following "sup-inf" problem:

$$\mathcal{J}(\xi) = \sup_{w(0)} \inf_{u(0)} \begin{pmatrix} \xi \\ u(0) \\ w(0) \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} A^{\mathrm{T}}PA + C^{\mathrm{T}}C & A^{\mathrm{T}}PB + C^{\mathrm{T}}D_{21} & A^{\mathrm{T}}PE + C^{\mathrm{T}}D_{22} \\ B^{\mathrm{T}}PA + D_{21}^{\mathrm{T}}C & V & B^{\mathrm{T}}PE + D_{21}^{\mathrm{T}}D_{22} \\ E^{\mathrm{T}}PA + D_{21}^{\mathrm{T}}C & E^{\mathrm{T}}PB + D_{22}^{\mathrm{T}}D_{21} & E^{\mathrm{T}}PE + D_{22}^{\mathrm{T}}D_{22} - I \end{pmatrix} \begin{pmatrix} \xi \\ u(0) \\ w(0) \end{pmatrix}$$

Define

$$\tilde{w}(0) = w(0) - (E^{\mathrm{T}}PB + D_{22}^{\mathrm{T}}D_{21})\, V^{-1}\xi$$

then we get

$$\mathcal{J}(\xi) = \sup_{\tilde{w}(0)} \inf_{u(0)} \begin{pmatrix} \xi \\ u(0) \\ \tilde{w}(0) \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} * & * & * \\ * & V & 0 \\ * & 0 & -R \end{pmatrix} \begin{pmatrix} \xi \\ u(0) \\ \tilde{w}(0) \end{pmatrix} \tag{3.38}$$

where $*$ denotes a matrix whose exact form is not important for this argument. Since, by (3.36), the above sup-inf problem is bounded from above, we immediately find that a necessary condition is $R + R^{\mathrm{T}} \geq 0$. Assume $R + R^{\mathrm{T}}$ is not invertible. Then there exists a $v \neq 0$ such that $v^{\mathrm{T}} R v = 0$. Let $\xi = 0$ and let $w^+(u(0))$ be the ℓ_2-function which attains the optimum in the optimization (3.37) with initial state $x(1) = Bu(0) + Ev$. Define the function w by

$$[w(u(0))](t) := \begin{cases} v & \text{if } t = 0 \\ [w^+(u(0))]\,(t) & \text{otherwise} \end{cases} \tag{3.39}$$

By (3.35) and (3.38) we find that, for this particular choice for w:

$$\inf_{u} \|z_{u,w(u(0)),0}\|_2^2 - \|w\|_2^2 > 0 \tag{3.40}$$

Assume δ and g are such that condition 3.2 is satisfied. Fix u by $u = g(w)$. Note that the map from u to w, defined by (3.39), is strictly causal and g is causal. Therefore u is uniquely defined by $u = g(w(u(0))$. In order to prove this we note that $u(0)$ only depends on $w(u(0)) = v$ and hence w^+ as a function of $u(0)$ is well defined which, in turn, yields u. Denote u and w chosen in this way by u_1 and w_1. Using condition 3.2 we find:

$$\|z_{u_1,w_1,0}\|_2^2 - \|w_1\|_2^2 < -\delta^2\|w_1\|_2^2$$

Combined with (3.40) this implies that $w_1 = 0$. However $w_1(0) = v \neq 0$. Therefore we have a contradiction and hence our assumtion that $R + R^{\mathrm{T}}$ is not invertible was incorrect. Together with $R + R^{\mathrm{T}} \geq 0$ this yields $R + R^{\mathrm{T}} > 0$. ■

Lemma 3.10 : *Assume $(A,\ B,\ C_2,\ D_{21})$ has no invariant zeros on the unit circle and is left invertible. Moreover, assume that $D_{21}^{\mathrm{T}}[C_2 \quad D_{22}] = 0$. If the statement in part (i) of theorem 2.1 is satisfied then there exists a symmetric matrix $P \geq 0$ satisfying (a)-(c) of part (ii) of theorem 2.1.* □

Proof : By combining (3.9),lemma 3.4 and lemma 3.6 and rewriting the equations we find that u_*, w_* and x_* are uniquely determined by the following set of equations:

$$\begin{cases} x_*(k+1) = Ax_*(k) + Bu_*(k) + Ew_*(k), & x_*(0) = \xi \\ r_*(k-1) = X_1\left[A^{\mathrm{T}}r_*(k) - LEw_*(k-1) - C_2^{\mathrm{T}}D_{22}w_*(k)\right] & \lim_{k\to\infty} r(k) = 0 \\ u_*(k) = M^{-1}B^{\mathrm{T}}\left[A^{\mathrm{T}}r_*(k+1) - LEw_*(k) - C_2^{\mathrm{T}}D_{22}w_*(k+1) - LAx_*(k)\right] & \\ Zw_*(k) = E^{\mathrm{T}}X_1\left(LAx_*(k) - A^{\mathrm{T}}r_*(k+1) - C_2D_{22}w_*(k+1)\right) & \end{cases} \tag{3.41}$$

for $k = 0,1,2,\dots$ where $M := D_{21}^{\mathrm{T}}D_{21} + B^{\mathrm{T}}LB$ and $Z := I - D_{22}^{\mathrm{T}}D_{22} - E^{\mathrm{T}}X_1LE$.
We know that $-(R + R^{\mathrm{T}})/2$ is the Schur complement of $(V + V^{\mathrm{T}})/2$ in $G((P + P^{\mathrm{T}})/2)$. By lemma 3.9 we now that $R + R^{\mathrm{T}} > 0$ and $V + V^{\mathrm{T}} > 0$. Therefore $G((P + P^{\mathrm{T}})/2)$ has m eigenvalues on the positive real axis and l eigenvalues on the negative real axis. We know $G((P + P^{\mathrm{T}})/2) - G(L) \geq 0$ since $(P + P^{\mathrm{T}})/2 \geq L$. An easy consequence of the theorem of Courant-Fischer then tells us that $G(L)$ has at least l eigenvalues on the negative real axis. Since $-Z$ is the Schur complement of $M > 0$ in $G(L)$ this implies that $Z < 0$.
By lemma 3.8 we have $\eta_*(k) = -Px_*(k+1)$ $k = 0,1,2,\dots$. Using this after some tedious calculations we find that:

$$\begin{aligned} w_*(k) &= Z^{-1}\left\{E^{\mathrm{T}}X_1(P-L)x_*(k+1) + (D_{22}^{\mathrm{T}}C_2 + E^{\mathrm{T}}X_1LA)x_*(k)\right\} \\ u_*(k) &= M^{-1}B^{\mathrm{T}}\left\{(P-L)x_*(k+1) + LAx_*(k) + LEw_*(k)\right\} \end{aligned}$$

Thus we get

$$\begin{aligned} &\left\{I + \left[BM^{-1}B^{\mathrm{T}} - X_1^{\mathrm{T}}EZ^{-1}E^{\mathrm{T}}X_1\right](P-L)\right\}x_*(k+1) = \\ &\qquad\qquad X_1^{\mathrm{T}}\left\{A + EZ^{-1}E^{\mathrm{T}}X_1LA + EZ^{-1}D_{22}^{\mathrm{T}}C_2\right\}x_*(k) \end{aligned} \tag{3.42}$$

Since, by lemma 3.9, R as defined in theorem 2.1 is invertible, it can be shown that the matrix on the left is invertible and hence (3.42) uniquely defines $x_*(k+1)$ as a function of $x_*(k)$. It turns out that (3.42) can be rewritten in the form $x_*(k+1) = A_{cl,P}x_*(k)$ with $A_{cl,P}$ as defined by (2.7). Since $x_* \in \ell_2^n$

for every initial state ξ we know that $A_{cl,P}$ is asymptotically stable. Next we show that P satisfies the discrete algebraic Riccati equation (2.6). From the backwards difference equation in (3.10) combined with lemma 3.8 and the formula given above for w_* we find:

$$P = A^{\mathrm{T}} P A_{cl,P} + C_2^{\mathrm{T}} C_2 + C_2^{\mathrm{T}} D_{22} Z^{-1} \left\{ E^{\mathrm{T}} X_1 (P - L) A_{cl,P} + D_{22}^{\mathrm{T}} C_2 + E^{\mathrm{T}} X_1 L A \right\} \tag{3.43}$$

By some extensive calculations this equation turns out to be equivalent to the discrete algebraic Riccati equation (2.6). Next we show that P is symmetric. Note that both P and P^{T} satisfy the discrete algebraic Riccati equation. Using this we find that:

$$(P - P^{\mathrm{T}}) = A_{cl,P}^{\mathrm{T}} (P - P^{\mathrm{T}}) A_{cl,P}$$

Since $A_{cl,P}$ is asymptotically stable this implies that $P = P^{\mathrm{T}}$. P can be shown to be positive semi definite by combining lemma 3.5 and (3.33). It remains to be shown that P satisfies (2.4). Since P is symmetric we know that V and R are symmetric. (2.4) is then an immediate consequence of lemma 3.9. ■

Corollary 3.11 : *Assume $(A,\ B,\ C,\ D_1)$ has no invariant zeros on the unit circle and is left invertible. If part (i) of theorem 2.1 is satisfied then there exists a symmetric matrix $P \geq 0$ satisfying (a)-(c) of part (ii) of theorem 2.1.* □

Proof : We first apply a preliminary feedback $u = \tilde{F}_1 x + \tilde{F}_2 w + v$ such that $D_{21}^{\mathrm{T}}(C_2 + D_{21}\tilde{F}_1) = 0$ and $D_{21}^{\mathrm{T}}(D_{22} + D_{21}\tilde{F}_2) = 0$. Denote the new A, C_2, D_{22} and E by $\tilde{A}, \tilde{C}_2, \tilde{D}_2$ and $\tilde{E}$. For this new system part (i) of theorem 2.1 is satisfied. Hence, since for this new system $D_{21}^{\mathrm{T}}[\tilde{C}_2 \quad \tilde{D}_2] = 0$, we find conditions in terms of the new parameters. Rewriting in terms of the original parameters gives the desired conditions (a)-(c) as given in part (ii) of theorem 2.1. ■

4 A first system transformation

In order to proceed with the proof of theorem 2.1, (i) $\Rightarrow$ (ii), in this section we will transform our original system (2.1) into a new system. The problem of finding an internally stabilizing feedback which makes the H_∞ norm less than 1 for the original system is equivalent to the problem of finding an internally stabilizing feedback which makes the H_∞ norm less than 1 for the new transformed system. However, this new system has some very desirable properties which make it is much easier to work with. In particular, for this new system the disturbance decoupling problem with measurement feedback is solvable. We will perform the transformation in two steps. First we will perform a transformation related to the full-information H_∞ problem and next a transformation related to the filtering problem.

We assume that we have a positive semi-definite matrix P satisfying conditions (a)-(c) of theorem 2.1. By the results of the previous section this matrix exists in case part (i) of theorem 2.1 is satisfied. We define the following system:

$$\Sigma_P : \begin{cases} x_P(k+1) &= A_P x_P(k) + B u_P(k) + E_P w_P(k), \\ y_P(k) &= C_{1,P} x_P(k) + \quad\quad + D_{12,P} w_P(k), \\ z_P(k) &= C_{2,P} x_P(k) + D_{21,P} u_P(k) + D_{22,P} w_P(k), \end{cases} \tag{4.1}$$

where the matrices are as defined in the statement of theorem 2.1. Furthermore, we define the following system

$$\Sigma_U : \begin{cases} x_U(k+1) &= A_U x_U(k) + B_U u_U(k) + E_U w(k), \\ y_U(k) &= C_{1,U} x_U(k) + \qquad + D_{12,U} w(k), \\ z_U(k) &= C_{2,U} x_U(k) + D_{21,U} u_U(k) + D_{22,U} w(k), \end{cases} \tag{4.2}$$

where

$$\begin{aligned} A_U &:= A - BV^{-1}(B^T PA + D_{21}^T C_2) \\ B_U &:= BV^{-1/2} \\ E_U &:= E - BV^{-1}(B^T PE + D_{21}^T D_{22}) \\ C_{1,U} &:= -R^{-1/2} H \\ C_{2,U} &:= C_2 - D_{21} V^{-1}(B^T PA + D_{21}^T C_2) \\ D_{12,U} &:= R^{1/2} \\ D_{21,U} &:= D_{21} V^{-1/2} \\ D_{22,U} &:= D_{22} - D_{21} V^{-1}(B^T PE + D_{21}^T D_{22}) \end{aligned}$$

and V,R and H are as defined in theorem 2.1. We will show that Σ_U has a very nice property. In order to do this, we will first give a definition and some results we will need in the sequel. A system is called *inner* if the transfer matrix of the system, denoted by G, satisfies:

$$G(z)G^T(z^{-1}) = I \tag{4.3}$$

We will now formulate a generalization of [6, lemma 5] to the case that $G(z)$ may have poles in zero. The proof is slightly more complicated than the one given in [6] since if G has a pole in zero then $G^T(z^{-1})$ is not proper any more. Nevertheless a proof can be given by simply writing out (4.3).

Lemma 4.1 : *Assume we have a system*

$$\Sigma_{st} : \begin{cases} x(k+1) &= Ax(k) + Bu(k) \\ z(k) &= Cx(k) + Du(k) \end{cases} \tag{4.4}$$

Assume A is stable. The system Σ_{st} is inner if there exists a matrix X satisfying:

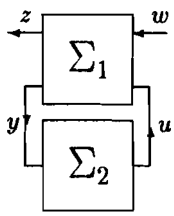
(*a*) $X = A^T XA + C^T C$

(*b*) $D^T C + B^T XA = 0$

(*c*) $D^T D + B^T XB = I$ □

We have the following important property of inner systems (see [10, 15]:

Lemma 4.2 : *Suppose we have the following interconnection of two systems Σ_1 and Σ_2, both described by some state space representation:*



(4.5)

Assume Σ_1 is internally stable and inner. Denote its transfer matrix from (w, u) to (z, y) by L. Moreover, assume that if we decompose L compatible with the sizes of w, u, z and y:

$$L\begin{pmatrix} w \\ u \end{pmatrix} =: \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix}\begin{pmatrix} w \\ u \end{pmatrix} = \begin{pmatrix} z \\ y \end{pmatrix}, \tag{4.6}$$

we have $L_{21}^{-1} \in H_\infty$ and $\lim_{z\to\infty} L_{22}(z) = 0$. Then the following two statements are equivalent:

(i) The closed loop system (4.5) is internally stable and its closed loop transfer matrix has H_∞ norm less than 1.

(ii) The system Σ_2 is internally stable and its transfer matrix has H_∞ norm less than 1. □

Lemma 4.3 : *The system Σ_U as defined by (4.2) is internally stable and inner. Denote the transfer matrix of Σ_U by U. We decompose U compatible with the sizes of w, u_U, z_U and y_U.:*

$$U\begin{pmatrix} w \\ u_U \end{pmatrix} =: \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}\begin{pmatrix} w \\ u_U \end{pmatrix} = \begin{pmatrix} z_U \\ y_U \end{pmatrix}.$$

Then U_{21} is invertible and its inverse is in H_∞ . Moreover $\lim_{z\to\infty} U_{22}(z) = 0$. □

Proof : It can be easily checked that P as defined by theorem 2.1 (a)-(c) satisfies the conditions (a)-(c) of lemma 4.1. (a) of lemma 4.1 turns out to be equal to the discrete algebraic Riccati equation (2.6). (b) and (c) follow by simply writing out the equations in terms of the original system parameters of system (2.1).
Next we show that A_U is asymptotically stable. We know $P \geq 0$ and

$$P = A_U^{\mathrm{T}} P A_U + \begin{pmatrix} C_{1,U}^{\mathrm{T}} & C_{2,U}^{\mathrm{T}} \end{pmatrix}\begin{pmatrix} C_{1,U} \\ C_{2,U} \end{pmatrix} \tag{4.7}$$

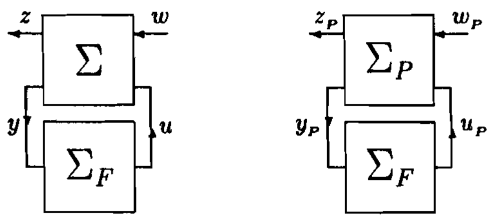
It can be easily checked that $x \neq 0$, $A_U x = \lambda x$, $C_{1,U}x = 0$ and $C_{2,U}x = 0$ implies that $A_{cl,P}x = \lambda x$ where $A_{cl,P}$ is defined by (2.7). Since $A_{cl,P}$ is asymptotically stable we have Re $\lambda < 0$. Hence the realization (4.2) is detectable. By standard Lyapunov theory the existence of a positive semi definite solution of (4.7) together with detectability guarantee asymptotic stability of A_U.
We can immediately write down a realization for U_{21}^{-1}:

$$\Sigma_{U_{21}^{-1}} : \begin{cases} x_U(k+1) &= \left(A_U - E_U D_{12,U}^{-1} C_{1,U}\right) x_U(k) + E_U D_{12,U}^{-1} w(k), \\ y_U(k) &= -D_{12,U}^{-1} C_{1,U} x_U(k) + D_{12,U}^{-1} w(k), \end{cases}$$

Since $A_{cl,P} = A_U - E_U D_{12,U}^{-1} C_{1,U}$ we know that U_{21}^{-1} is an H_∞ function.
Finally, the claim that $\lim_{s\to\infty} U_{22}(s) = 0$ is trivially to check. This completes the proof. ∎

We will now formulate our key lemma:

Lemma 4.4 : *Let P satisfy theorem 2.1 part (ii) (a)-(c). Moreover, let Σ_F be an arbitrary linear time-invariant finite-dimensional compensator in the form (2.2). Consider the following two systems, where the system on the left is the interconnection of (2.1) and (2.2) and the system on the right is the interconnection of (4.1) and (2.2):*



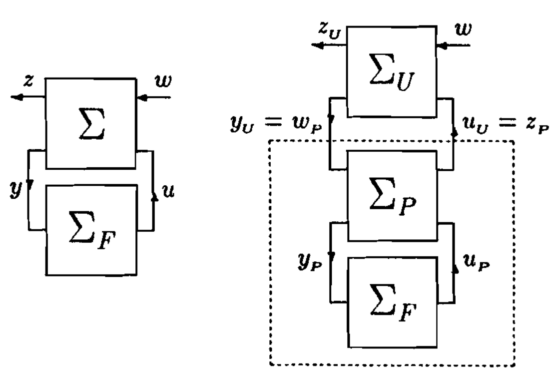
(4.8)

Then the following statements are equivalent :

(*i*) *The system on the left is internally stable and its transfer matrix from w to z has H_∞ norm less than 1.*

(*ii*) *The system on the right is internally stable and its transfer matrix from w_P to z_P has H_∞ norm less than 1.* □

Proof : We investigate the following systems:



(4.9)

The system on the left is the same as the system on the left in (4.8) and the system on the right is described by the system (4.2) interconnected with the system on the right in (4.8). A realization for the system on the right is given by:

$$\begin{cases} \begin{pmatrix} x_U - x_{1,P} \\ x_P \\ p \end{pmatrix}(k+1) = \begin{pmatrix} A_{cl,P} & 0 & 0 \\ * & A+BNC_1 & BM \\ * & LC_1 & K \end{pmatrix} \begin{pmatrix} x_U - x_{1,P} \\ x_P \\ p \end{pmatrix}(k) + \begin{pmatrix} 0 \\ E+BND_1 \\ LD_1 \end{pmatrix} w(k) \\ z_U(k) = \begin{pmatrix} * & C_2+D_2NC_1 & D_2M \end{pmatrix} \begin{pmatrix} x_U - x_{1,P} \\ x_P \\ p \end{pmatrix}(k) \end{cases}$$

where $A_{cl,P}$ is defined by (2.7). The $*$'s denote matrices which are unimportant for this argument. The system on the right is internally stable if and only if the system described by the above set of equations is internally stable. If we also derive the system equations for the system on the left in (4.9) we immediately see that, since $A_{cl,P}$ is asymptotically stable, the system on the left is internally stable if and only if the system on the right is internally stable. Moreover, if we take zero initial conditions and both systems have the same input w then we have $z = z_U$ i.e. the input-output behaviour of both systems are equivalent. Hence the system on the left has H_∞ norm less than 1 if and only if the system on the right has H_∞ norm less than 1.
By lemma 4.3 we may apply lemma 4.2 to the system on the right in (4.9) and hence we find that the closed loop system is internally stable and has H_∞ norm less than 1 if and only if the dashed system is internally stable and has H_∞ norm less than 1.
Since the dashed system is exactly the system on the right in (4.8) and the system on the left in (4.9) is exactly equal to the system on the left in (4.8) we have completed the proof. ■

Using the previous lemma, we know that we only have to investigate the system Σ_P. This new system has some very nice properties which we will use. First we will look at the Riccati equation for the system Σ_P. It can be checked immediately that $X = 0$ satisfies (a)-(c) of theorem 2.1 for the system Σ_P.
We now dualize Σ_P. We know that (A, E, C_1, D_{12}) is right-invertible and has no invariant zeros on the unit circle. It can be easily checked that this implies that $(A_P, E, C_{1,P}, D_{12})$ is right-invertible and has no invariant zeros on the unit circle. Hence for the dual of Σ_P we know that $(A_P^{\mathrm{T}}, C_{1,P}^{\mathrm{T}}, E^{\mathrm{T}}, D_{21}^{\mathrm{T}})$ is left-invertible and has no invariant zeros on the unit circle. If there exists an internally stabilizing feedback for the system Σ which makes the H_∞ norm less than 1 then the same feedback is internally stabilizing and makes the H_∞ norm less than 1 for the system Σ_P. If we dualize this feedback and apply it to the dual of Σ_P then it is again internally stabilizing and again it makes the H_∞ norm less than 1. We can now apply corollary 3.11 which exactly guarantees the existence of a matrix Y satisfying conditions (d)-(f) of theorem 2.1. Thus we derived the following lemma which gives the necessity part of theorem 2.1:

Lemma 4.5 : *Let the system (2.1) be given with zero initial state. Assume that (A, B, C_2, D_{21}) has no invariant zeros on the unit circle and is left invertible. Moreover assume that (A, E, C_1, D_{12}) has no invariant zeros on the unit circle and is right invertible. If part (i) of theorem 2.1 is satisfied then there exist matrices P and Y satisfying (a)-(f) of part (ii) of theorem 2.1.* □

This completes the proof (i) $\Rightarrow$ (ii). In the next section we will proof the reverse implication. Moreover in case the desired F exists we will derive an explicit formula for one choice for F which satisfies all requirements.

5 The transformation into a disturbance decoupling problem with measurement feedback

In this section we will assume that there exist matrices P and Y satisfying part (ii) of theorem 2.1 for the system (2.1). We will transform our original system Σ into another system $\Sigma_{P,Y}$. We will show that a compensator is internally stabilizing and makes the H_∞ norm less than 1 for the system Σ if and only if the same compensator is internally stabilizing and makes the H_∞ norm less than 1 for our transformed system $\Sigma_{P,Y}$. After that we will show that $\Sigma_{P,Y}$ has a very special property (see [12]):

> There exists an internally stablizing compensator which makes the closed loop transfer matrix equal to zero, i.e. w does not have any effect on the output of the system z. This property of $\Sigma_{P,Y}$ has a special name: "the Disturbance Decoupling Problem with Measurement feedback and internal Stability (DDPMS) is solvable".

We first define $\Sigma_{P,Y}$. First transform Σ into Σ_P. Then we apply the dual transformation on Σ_P to obtain $\Sigma_{P,Y}$:

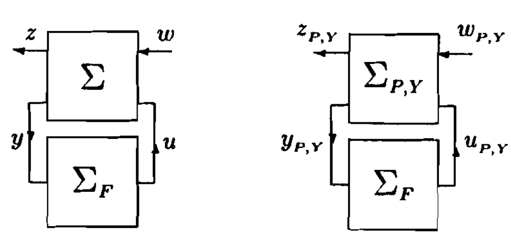
$$\Sigma_{P,Y}: \left\{ \begin{array}{lclclcl} x_{P,Y}(k+1) & = & A_{P,Y} x_{P,Y}(k) & + & B_{P,Y} u_{P,Y}(k) & + & E_{P,Y} w_{P,Y}(k), \\ y_{P,Y}(k) & = & C_{1,P} x_{P,Y}(k) & + & & + & D_{12,P,Y} w_{P,Y}(k), \\ z_{P,Y}(k) & = & C_{2,P,Y} x_{P,Y}(k) & + & D_{21,P,Y} u_{P,Y}(k) & + & D_{22,P,Y} w_{P,Y}(k), \end{array} \right. \tag{5.1}$$

where

$$\begin{array}{lcl} \tilde{H} & := & A_P Y C_{2,P}^{\mathrm{T}} + E_P D_{22,P}^{\mathrm{T}} - \left(A_P Y C_{1,P}^{\mathrm{T}} + E_P D_{12,P}^{\mathrm{T}}\right) W^{-1} \left(C_{1,P} Y C_{2,P}^{\mathrm{T}} + D_{12,P} D_{22,P}^{\mathrm{T}}\right) \\ A_{P,Y} & := & A_P + \tilde{H} S^{-1} C_{2,P} \\ C_{2,P,Y} & := & S^{-1/2} C_{2,P} \\ B_{P,Y} & := & B + \tilde{H} S^{-1} D_{21,P} \\ E_{P,Y} & := & \left(A_P Y C_{1,P}^{\mathrm{T}} + E_P D_{12,P}^{\mathrm{T}}\right) W^{-1/2} + \tilde{H} S^{-1} \left(C_{2,P} Y C_{1,P}^{\mathrm{T}} + D_{22,P} D_{12,P}^{\mathrm{T}}\right) W^{-1/2} \\ D_{12,P,Y} & := & W^{1/2} \\ D_{21,P,Y} & := & S^{-1/2} D_{21,P} \\ D_{22,P,Y} & := & S^{-1/2} \left(C_{2,P} Y C_{1,P}^{\mathrm{T}} + D_{22,P} D_{12,P}^{\mathrm{T}}\right) W^{-1/2} \end{array}$$

When we first apply lemma 4.4 on the transformation from Σ to Σ_P and then the dual of lemma 4.4 on the transformation from Σ_P to $\Sigma_{P,Y}$ we find:

Lemma 5.1 : *Let P satisfy theorem 2.1 part (ii) (a)-(c). Moreover let an arbitrary linear time-invariant finite-dimensional compensator Σ_F be given, described by (2.2). Consider the following two systems, where the system on the left is the interconnection of (2.1) and (2.2) and the system on the right is the interconnection of (5.1) and (2.2):*



Then the following statements are equivalent :

(i) *The system on the left is internally stable and its transfer matrix from w to z has H_∞ norm less than 1.*

(ii) *The system on the right is internally stable and its transfer matrix from $w_{P,Y}$ to $z_{P,Y}$ has H_∞ norm less than 1.* □

It remains to be shown that for $\Sigma_{P,Y}$ the Almost Disturbance Decoupling Problem with internal Stability and Measurement feedback is solvable:

Lemma 5.2 : *Let Σ_F be given by:*

$$\Sigma_F : \begin{cases} p(k+1) &= K_{P,Y}p(k) + L_{P,Y}y_{P,Y}(k), \\ u_{P,Y}(k) &= M_{P,Y}p(k) + N_{P,Y}y_{P,Y}(k), \end{cases} \tag{5.2}$$

where

$$\begin{aligned} N_{P,Y} &:= -D^{-1}_{21,P,Y}D_{22,P,Y}D^{-1}_{12,P,Y} \\ M_{P,Y} &:= -\left(D^{-1}_{21,P,Y}C_{2,P,Y} + N_{P,Y}C_{1,P}\right) \\ L_{P,Y} &:= B_{P,Y}N_{P,Y} + E_{P,Y}D^{-1}_{12,P,Y} \\ K_{P,Y} &:= A_{P,Y} + B_{P,Y}M_{P,Y} - E_{P,Y}D^{-1}_{12,P,Y}C_{1,P} \end{aligned}$$

The interconnection of Σ_F and $\Sigma_{P,Y}$ is internally stable and the closed loop transfer matrix from $w_{P,Y}$ to $z_{P,Y}$ is zero. □

Proof : We can write out the formulas for a state space representation of the interconnection of $\Sigma_{P,Y}$ and Σ_F. We then apply the following basis transformation:

$$\begin{pmatrix} x_{P,Y} - p \\ p \end{pmatrix} = \begin{pmatrix} I & -I \\ 0 & I \end{pmatrix} \begin{pmatrix} x_{P,Y} \\ p \end{pmatrix}.$$

After this transformation one immediately sees that the closed loop transfer matrix from $w_{P,Y}$ to $z_{P,Y}$ is zero. Moreover the system matrix (2.3) after this transformation is given by:

$$\begin{pmatrix} A_{cl,P,Y} & 0 \\ L_{P,Y}C_{1,P} & A_{cl,P} \end{pmatrix}$$

Since $A_{cl,P,Y}$ and $A_{cl,P}$ are asymptotically stable matrices, this implies that indeed Σ_F is internally stabilizing. ■

This controller is the same as the controller described in the statement of theorem 2.1. We know Σ_F is internally stabilizing and the resulting closed loop system has H_∞ norm less than 1 for the system $\Sigma_{P,Y}$. Hence, by applying lemma 5.1, we find that Σ_F satisfies part (i) of theorem 2.1. This completes the proof of (ii) $\Rightarrow$ (i) of theorem 2.1. We have already shown the reverse implication and hence the proof of theorem 2.1 is completed.

6 Conclusions

In this paper we have solved the discrete time H_∞ problem with measurement feedback. It is shown that the techniques for the continuous time case can be applied to the discrete time case. Unfortunately the formulas are much more complex but it is still possible to give an explicit formula for one controller satisfying all requirements. It would however be interesting to generalize this work and find a characterization of all controllers satisfying the requirements. Another interesting open problem is to derive recursive formulas to calculate the solutions to these algebraic Riccati equations. It would also be interesting to find two dual Riccati equations and a coupling conditions as in [4]. Nevertheless the results presented in this paper show that it is very well possible to solve discrete time H_∞ problems directly, instead of transforming them to continuous time. The assumption of left-invertibility is not very restrictive. It implies that there are several inputs which have the same effect on on the output and this non-uniqueness can be factored out. (see for a continuous time treatment [11]) The assumption of right invertibility can be removed by dualizing this reasoning. However at this moment it is unclear how to remove the assumptions concerninig zeros on the unit-circle. Finally an interesting extension would be the finite horizon discrete time case. (see for a continuous time treatment [18])

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