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ABSTRACT

In this paper, a stochastic model of the operation of a carousel system is developed. Such systems are widely used in automated warehousing to store small to medium sized products with a moderate demand rate. Important performance indicators for a carousel system are its throughput rate and the mean response time. Approximate analytic expressions are derived for the distribution of the time needed to pick one complete order, consisting of a number of different items. Based on these expressions. the carousel can be treated as an M/G/1 queueing system, permitting the derivation of various queueing characteristics such as the mean response time and the waiting time when orders arrive randomly. Comparisons of analytic and simulation results are also provided.

1 INTRODUCTION

A carousel is a computer controlled warehousing system that is used for storage and order picking of small to medium sized products. Such a system may hold many different products stored to bins that rotate in a closed loop. The system is operated by an order picker (either human or robotic) which occupies a fixed position in front of the carousel. To retrieve a product, the carousel system automatically rotates the bin with the requested product to the position of the order picker. Accordingly, it is referred to as a *product-to-picker* system. The order picker may effectively use the rotation time of the carrousel for activities such as sorting, packaging or labeling of the retrieved goods.

Two types of carousel systems are used in practice: the horizontal carousel and the vertical carousel. The horizontal carousel consists of a large number of carriers driven by a powered chain that can rotate around a narrow closed loop. Each carrier consists of multiple bin positions. The vertical carousel or paternoster consists of a number of shelves connected at the ends to powered chains that rotate vertically. Each shelf carries multiple bins. Most vertical carousels are contained in a closed metal cage with only one opening at the position of the order picker. This allows for protection of the stored products. Most human-operated horizontal carousels are not higher than two meters. The length of a carousel can be much longer, up to 50 meters, say.

The design process of any integrated warehousing system may be viewed as a three stage program (*cf.* Medeiros [3]). In the first stage, several design alternatives for warehousing subsystems are generated based on product and order characteristics. In a second stage, the designs for the subsystems are linked together and evaluated by means of analytic studies. Finally, in the third stage, the selected overall design may be evaluated with the aid of computer simulation. Several attempts have been made to implement the first and second stage into a decision support system. The model we present in this paper may be used in the second stage.

A carousel system is used for order picking. An *order* lists the items that must be picked together, *e.g.* for the same customer. An order contains multiple *line orders*, each defining the number of items requested of a particular product. We may either pick the orders one after the other (*single order picking*) or pick multiple orders in parallel (*batch picking*). For batch picking we may distinguish the sort-while-pick strategy and the pick-and-sort strategy. Under the sort-while-pick strategy the picker is provided with several totes for storing the orders separately. Under the pick-and-sort strategy the orders are picked together and sorted afterwards.

Carousel systems have received considerable attention in the literature. Bartoldi and Platzman [1] and Stern [8] consider the single order pick sequencing problem. They assume that the time to move between bins within the same carrier or shelf is negligible. Accordingly, they reduce the problem to finding the shortest Hamiltonian path on a circle. They present a linear time algorithm that finds an optimal solution. The algorithm selects the shortest route from $2n$ candidate routes, when n is the number of pick locations that have to be visited for one order.

Van den Berg [2] and Ghosh and Wells [4] present dynamic programming algorithms that find an optimum sequence of the picks when multiple orders are given that have to be processed in a specified sequence, *e.g.*, first come first served. Bartoldi and Platzman [1] and Van den Berg [2] also consider the pick sequencing problem for multiple orders, when orders may be processed in any arbitrary sequence. Sharp *et al.* [7] derive a formula for the minimum order pick time of a carousel system when this system only considers two specific routes of the $2n$ candidates in Bartoldi and Platzman [1] and Stern [8].

In this paper, the carousel is modeled as a single server in an M/G/I queueing system, with the orders as the customers to be served. The special feature is that the service time, *i.e.* the order pick time, depends on the order pick strategy chosen for a particular order. Clearly, for an order consisting of n line orders (for which n locations have to be visited) $2n$ routes are in principle possible. It follows immediately from Pollaczek-Khintchine's formula that the average response time (waiting plus service time) of the system is minimized if for each order a service strategy is chosen that minimizes the first and second moment of the order pick time of that order. Therefore, the essential part of the analysis presented in this paper consists of deriving the distribution and corresponding moments of the order pick time for an arbitrary order.

In section 2, we introduce a model for the carousel system. Section 3 is devoted to an approximation of the distribution of the order pick time. Based on the results in this section, standard results from M/G/1 queueing systems are next applied to determine the minimum response time experienced by an order. Section 5 concludes the paper.

2 MODELLING THE CAROUSEL SYSTEM

In this section we first list some assumptions, after which we model the carousel system as an M/G/1 queueing system. Next, we define the order pick strategies which provide the basis for the approximate order pick time distributions.

2.1 Assumptions

The time that is required to pick an order consists of several elements. First, time is needed to let the carousel rotate the positions from which products have to be retrieved, the *pick locations*, in front of the picker. Next, some time is required to search for the product, to retrieve the product and to count the items. Finally, time may be needed to sort and document the retrieved product. The length of the path that the carousel rotates is called the *rotation distance*, which may be reduced by finding an effective sequence for visiting the pick locations.

We consider a bi-directional horizontal or vertical carousel with one order picker. We make the following assumptions.

1 We do not consider the replenishment of the carousel.

2 Orders arrive according to a Poisson process.

3 The orders are picked one by one (single order picking) and according to a First Come First Served routine.

4 The number of line orders per order is random.

5 The time needed by the order picker to move between bins within the same carrier or shelf is negligible.

6 If a product is stored at multiple locations, then the oldest bin in storage is selected for order picking, *i.e.*, products are picked according to the first in first out rule.

7 The time for extracting items is equally distributed for all products.

8 The pick locations form a continuous circle, instead of a discrete set of locations.

9 For each line order, the pick location is uniformly distributed over the carousel.

10 The carousel rotates at a constant speed. The acceleration/deceleration time can be included in the extraction time.

11 An order is picked along the shortest route.

12 The time for putting away items after picking is negligible.

2.2 A single server queueing system

As mentioned in the introduction, the carousel system is naturally modeled as an $M/G/1$ queueing system with a controllable service rate, due to different order pick strategies. Orders are assumed to arrive according to a Poisson process with an intensity λ. For all orders, the number of line orders is random and equally distributed among the possible locations. The random variable N denotes the number of line orders of an arbitrary order, the index set $I \subset \mathbb{N}$ denotes the value that can be taken by N, and for each $n \subset I$ the probability that $N = n$ is given by p_n. The carousel represents a single server and it serves the orders in a FCFS service discipline. The time needed to serve one order is called the order pick time; this time depends on the number and locations of the line order items and on the sequence in which these locations are rotated to the order picker. Some reflection shows that the analysis will not change if we describe one complete order pick cycle by an order picker visiting n locations in a non-rotating circular storage system, instead of by a carousel rotating these n locations to a static order picker. Accordingly, a pick sequence for items in one order is often referred to as a route. The next subsection shows that in principle $2n$ routes have to be considered to pick n line orders belonging to one order. In order to minimize the mean order pick time and hence the mean response time of the system, we should find the shortest route for each particular order. Therefore, the main emphasis of this paper is on deriving exact or approximate distributions for the minimal service time, *i.e.* the order pick time corresponding to the shortest route.

2.3 Routing Strategies.

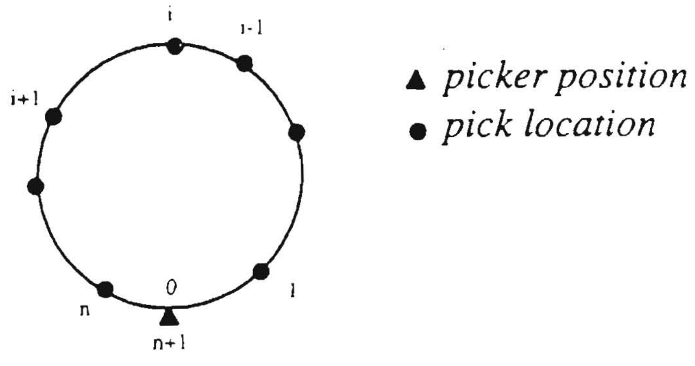

Figure 1. Picklocations in a carousel.

We model the carousel as a circle with nodes representing the locations to be visited for one order. We define the length of one complete rotation to be equal to 1. Consider one order with n line orders ($n \in I$). We introduce one node on the circle representing the starting position and assign number 0 or $n+1$ to this node. Furthermore, we introduce n nodes representing the pick locations corresponding to the order. We number these nodes $1, \ldots, n$ as we encounter them when traveling counter-clockwise along the circle starting from node 0.

A *strategy* defines a route along all nodes. The optimal strategy is among $2n$ different strategies (see also Bartoldi and Platzman [1] and Stern [8]). We denote these strategies as follows:

- Strategy 1: Travel in the clockwise direction until all nodes have been visited.
- Strategy 2: Travel in the counter-clockwise direction until all nodes have been visited. This strategy is symmetric to strategy 1.
- Strategy $2i+1$, for $i = 1, \ldots, n-1$: Travel counter-clockwise to node i and then travel clockwise to node $i+1$.
- Strategy $2i+2$, for $i = 1, \ldots, n-1$: Travel clockwise to node $n+1-i$ and then travel counter-clockwise to node $n-i$. This strategy is symmetric to strategy $2i+1$.

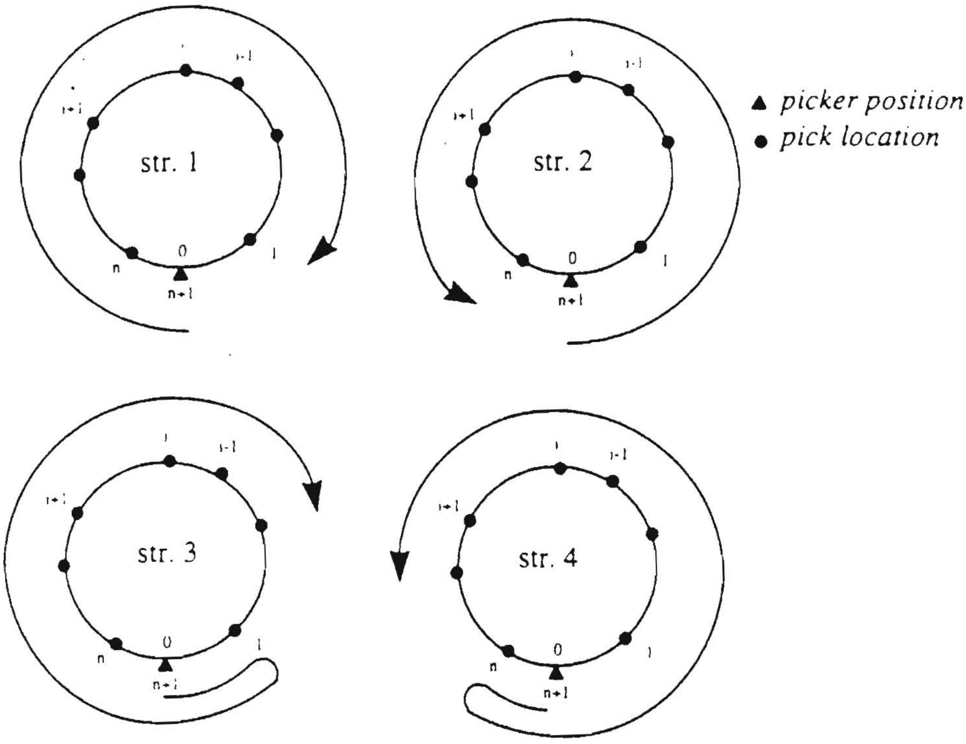

Figure 2. The routes defined by the strategies 1-4.

For each strategy i, the corresponding rotation distance is denoted by the random variable D_n^i. Furthermore we define for $i = 1, ..., 2n$ the strategy i^*. This strategy represents the strategy among strategies $1, ..i$, that defines the shortest route. The resulting rotation distance is defined as $D_n^{i^*}$. By definition, strategy $(2n)^*$ is the optimal route. The resulting rotation distance is denoted by $D_n^{2n^*}$, or simply D_n. We define the functions $F_n^i(x)$ and $F_n^{i^*}(x)$ that represent the cumulative density function (cdf) of D_n^i and $D_n^{i^*}$, respectively. The probability density functions (pdf's) are denoted by $f_n^i(x)$ and $f_n^{i^*}(x)$, respectively. The functions $F_n^{(2n)^*}(x)$ and $f_n^{(2n)^*}(x)$ are also denoted as $F_n(x)$ and $f_n(x)$, respectively.

In the next section we derive approximations for the distribution and the first three moments of the rotation distance D_n.

3 ANALYSIS OF THE ORDER PICK TIME

In this section we present tight upper bounds for the order pick time required for an order with a given number of line orders. The tightest upper bound will be exploited in the next section to approximate the first three moments of the order pick time of an arbitrary order, which are needed to determine the response time of an arbitrary order.

Consider an order with $n \geq 1$ line orders and let its order pick time be denoted by S_n. This time S_n consists of n extraction times and the time to travel along the n pick locations. Let the random variable T_{ext} represent the time required for one extraction. then the total extraction time is equal to $\sum_{i=1}^{n} T_{ext,i}$. where $T_{ext,1}, \ldots, T_{ext,n}$ have the same distribution as T_{ext} and are mutually independent. The second part of S_n is equal to the length D_n of the shortest path along the n pick locations multiplied by the (deterministic) time t_{rot} that is needed for one full rotation.

Hence we find for $n \geq 1$:

$$S_n = \sum_{i=1}^{n} T_{ext,i} + t_{rot} \cdot D_n. \tag{1}$$

Note that this relation is based on the assumptions 1,5, 7, 10 and 12 in the previous section.

From (1). we immediately find

$$\mathbb{E}S_n = n \cdot \mathbb{E}T_{ext} + t_{rot} \cdot \mathbb{E}D_n, \qquad n \geq 1, \tag{2}$$

while similar expressions are obtained for the higher moments of S_n. For the analysis in the next section, we need the first three moments of S_n for all relevant $n \geq 1$, and hence we need the first three moments of T_{ext} and D_n. The distribution function of T_{ext} is assumed to be known (e.g. based on historical data). The rotation distances $D_n, n \geq 1$, require special attention. Below, we present upper bounds for the (distribution and) moments of these distances. Obviously, from (2) the corresponding upper bounds for the mean order pick times $\mathbb{E}S_n$ are obtained, and similarly for $\mathbb{E}S_n^2$ and $\mathbb{E}S_n^3$.

The remaining part of this section consists of four subsections. The first three subsections discuss the upper bounds. In the fourth subsection, the tightness of the upper bounds is investigated by comparing these bounds to simulation results for the rotation distances D_n.

3.1 Upper Bound 1

For an order with $n \geq 1$ line orders, D_n represents the length of the shortest route along the n positions where products have to be picked. This shortest route is selected from the set of routes defined by the strategies $1, ..., 2n$. Strategy 2^* selects the best route defined by the strategies 1 and 2, and may have a length already close to the length of the shortest route. Hence, we are interested in the length $D_n^{2^*}$ of the route chosen by this strategy 2^*.

Obviously, $D_n^{2^*}$ constitutes an upper bound for D_n. Mathematically, this means that $D_n^{2^*}$ is stochastically larger than D_n, *i.e.* $F_n^{2^*}(x) \leq F_n(x)$ for all $x \geq 0$. The latter relation implies that $\mathbb{E}(D_n^{2^*})^k \geq \mathbb{E}(D_n)^k$ for all $k \geq 1$. Whenever we write upper bound in the sequel, we mean "stochastically larger" when random variables are concerned. Contrary to the distribution function $F_n(x)$ of D_n, the distribution function $F_n^{2^*}(x)$ of $D_n^{2^*}$ can be determined rather easily.

Lemma 1

$$F_n^{2^*} = \begin{cases} 2x^n - 0, & 0 \leq x \leq \frac{1}{2} \\ 2x^n - (2x-1)^n, & \frac{1}{2} < x \leq 1. \end{cases}$$

Proof. We present the proof for the case $n \geq 2$. The proof for the case $n = 1$ is easy and therefore left to the reader.

Hence, let $n \geq 2$ and let X_k denote the distance between node $k-1$ and k ($k = 1, .., n+1$). Consider the circle with the $n+1$ nodes as depicted in Figure 1. The length of the route defined by strategy 1 is equal to $1 - X_1$. Similarly, the length of the route defined by strategy 2 is equal to $1 - X_{n+1}$. So, we find that

$$D_n^{2^*} = \min\{1 - X_1, 1 - X_{n+1}\}. \tag{3}$$

To obtain the distribution function of $D_n^{2^*}$, we first derive preliminary results for X_1 and for X_{n+1} conditioned to X_1. Node 0 may be considered as a fixed point on the circle, while each of the other n nodes constitutes an arbitrarily chosen point on the circle. Hence,

$$P(X_1 \geq x) = (1-x)^n, \; 0 \leq x \leq 1 \tag{4}$$

(the circle is equivalent to an interval $[0,1]$, and $X_1 \geq x$ if and only if each of the

n nodes, representing the n line orders, is lying in $[x, 1]$). The distribution function $F_{X_1}(x)$ and the probability density function $f_{X_1}(x)$ are thus equal to:

$$
\begin{aligned}
F_{X_1}(x) &= 1-(1-x)^n,\ 0 \le x \le 1, \\
f_{X_1}(x) &= n(1-x)^{n-1},\ 0 \le x \le 1.
\end{aligned}
$$

Because of the symmetry, X_1 and X_{n+1} are equally distributed, and therefore $F_{X_1}(x)$ and $f_{X_1}(x)$ are also the distribution function and probability density function for X_{n+1}. Similar to (4), we find that for each $0 \le y \le 1$,

$$
P(X_{n+1} \ge x | X_1 = y) = (\frac{1-y-x}{1-y})^{n-1},\ 0 \le x \le 1-y. \tag{5}
$$

Of course, this expression also holds with X_1 and X_{n+1} interchanged.

We now can determine the distribution function $F_n^{2*}(x)$ of D_n^{2*}, using (3).

$$
\begin{aligned}
F_n^{2*}(x) &= P(D_n^{2*} \le x) = P(1 - X_1 \le x \cup 1 - X_{n+1} \le x) \\
&= P(X_1 \ge 1-x \cup X_{n+1} \ge 1-x) \\
&= P(X_1 \ge 1-x) + P(X_{n+1} \ge 1-x) \\
&\quad -P(X_1 \ge 1-x \cap X_{n+1} \ge 1-x),\ 0 \le x \le 1.
\end{aligned} \tag{6}
$$

Next, we distinguish two cases. For $0 \le x < \frac{1}{2}$, it holds that $1-x > \frac{1}{2}$ and the third probability in (6) is equal to 0. So, in that case

$$
F_n^{2*}(x) = P(X_1 \ge 1-x) + P(X_{n+1} \ge 1-x) = 2x^n,
$$

For $\frac{1}{2} \le x \le 1$, we find (use (5))

$$
\begin{aligned}
F_n^{2*}(x) &= 2x^n - \int_{1-x}^{1} P(X_{n+1} \ge 1-x | X_1 = y) \cdot f_{X_1}(y) dy \\
&= 2x^n - \int_{1-x}^{x} (\frac{1-y-(1-x)}{1-y})^{n-1} \cdot n(1-y)^{n-1} dy \\
&= 2x^n - \int_{1-x}^{x} n(x-y)^{n-1} dy \\
&= 2x^n - (2x-1)^n.
\end{aligned}
$$

This completes the proof. $\square$

From the distribution function $F_n^{2^*}(x)$, we find the following moments:

$$\begin{aligned}
\mathbb{E}D_n^{2^*} &= \frac{n-\frac{1}{2}}{n+1}, \\
\mathbb{E}(D_n^{2^*})^2 &= \frac{n-\frac{1}{2}}{(n+1)(n+2)}, \\
\mathbb{E}(D_n^{2^*})^3 &= \frac{4n^3+6n^2-3n-3}{4(n+3)(n+2)(n+1)}.
\end{aligned}$$

From now on, the upper bound $D_n^{2^*}$ for D_n will be referred to by D_n^{upp1}. Finally, it is noted that D_n^{upp1} coincides with D_n for $n = 1$ (strategy 2^* is optimal in that case).

3.2 Upper Bound 2

An improved upper bound D_n^{upp2} for D_n is obtained by the length $D_n^{4^*}$ of the route defined by strategy 4^*. For $n = 1$, this strategy does not exist, and hence D_n^{upp2} is kept equal to $D_n^{upp1} = D_n^{2^*}$ for this case. So,

$$D_n^{upp2} = \begin{cases} D_n^{2^*} & if \quad n = 1, \\ D_n^{4^*} & if \quad n \geq 2. \end{cases}$$

Expressions for the distribution function and moments of $D_n^{4^*}$ can be obtained by an analysis that is more extensive, but in essence similar to the analysis performed for $D_n^{2^*}$. Here, it suffices to present the main results; for the proofs of these results, see Rouwenhorst [6]. For $n = 2$, strategy 4^* is optimal, and we find

$$F_n^{4^*}(x) = \begin{cases} \frac{8}{3}x^2, & 0 \leq x \leq \frac{1}{2}, \\ \frac{8}{3}x^2 - 2(2x-1)^2, & \frac{1}{2} < x \leq \frac{3}{4}. \end{cases}$$

and $\mathbb{E}D_2^{4^*} = \mathbb{E}D_2 = \frac{5}{12}$, $\mathbb{E}(D_2^{4^*})^2 = \mathbb{E}(D_2)^2 = \frac{19}{96}$, $\mathbb{E}(D_2^{4^*})^3 = \mathbb{E}(D_2)^3 = \frac{13}{128}$. For $n \geq 3$, we have

Lemma 2

$$F_n^{4^*}(x) = \begin{cases} 3x^n, & 0 \le x \le \frac{1}{2} \\ 3x^n - \frac{9}{4}(2x-1)^n, & \frac{1}{2} < x \le \frac{2}{3} \\ 3x^n - \frac{9}{4}(2x-1)^n - (3x-2)^n, & \frac{2}{3} < x \le \frac{3}{4} \\ 3x^n - \frac{9}{4}(2x-1)^n - (3x-2)^n + \frac{3}{2}(4x-3)^n, & \frac{3}{4} < x \le \frac{5}{6} \\ 3x^n - \frac{9}{4}(2x-1)^n - (3x-2)^n + \frac{3}{2}(4x-3)^n - \frac{1}{4}(6x-5)^n, & \frac{5}{6} < x \le 1 \end{cases}$$

and

$$\begin{aligned} \mathbb{E}D_n^{4^*} &= \frac{n - \frac{7}{8}}{n+1}, \\ \mathbb{E}(D_n^{4^*})^2 &= \frac{144n^2 - 108n - 97}{144(n+2)(n+1)}, \\ \mathbb{E}(D_n^{4^*})^3 &= \frac{192n^3 + 72n^2 - 508n - 103}{192(n+3)(n+2)(n+1)}. \end{aligned}$$

3.3 Upper Bound 3

A further improved upper bound for D_n is given by

$$D_n^{upp3} = \begin{cases} D_n^{2^*} & if \quad n = 1; \\ D_n^{4^*} & if \quad 2 \le n \le 4; \\ D_n^{6^*} & if \quad n \ge 5. \end{cases}$$

This upper bound is kept equal to D_n^{upp2} for $n = 1$ and $n = 2$, since strategy 6^* does not exist for these cases, and also for $n = 3$ and $n = 4$, since strategy 6^* has not been analyzed for these cases (up to now). For $n \ge 5$, it holds that

Lemma 3

$$\begin{aligned} \mathbb{E}D_n^{6^*} &= \frac{4480n - 4593}{4480(n+1)} \approx \frac{n - 1.025}{n+1}, \\ \mathbb{E}(D_n^{6^*})^2 &\approx \frac{n^2 - 1.05n - 0.68}{(n+2)(n+1)}, \\ \mathbb{E}(D_n^{6^*})^3 &\approx \frac{n^3 - 0.0757n^2 - 3.12n - 0.247}{(n+3)(n+2)(n+1)}. \end{aligned}$$

A proof of these results can be found in Rouwenhorst [6].

3.4 On the Tightness of the Upper Bounds

In Table 1, 2 and 3 we have listed the first three moments produced by the upper bounds D_n^{upp1}, D_n^{upp2} and D_n^{upp3} and the first three moments of D_n, for different values of n. The results for D_n have been obtained by simulation (with 30.000 orders for each n, which appeared to be sufficient to obtain accurate results). In Figure 3 the first moments are depicted graphically.

Table 1. The First Moment of the Rotation Distance.

n	D_n^{upp1}	D_n^{upp2}	D_n^{upp3}	D_n
1	0.250	0.250	0.250	0.251
2	0.500	0.417	0.417	0.417
3	0.625	0.531	0.531	0.526
4	0.700	0.625	0.625	0.603
5	0.750	0.688	0.663	0.659
6	0.786	0.732	0.713	0.703
10	0.864	0.830	0.816	0.805
20	0.929	0.911	0.904	0.897

Table 2. The Second Moment of the Rotation Distance.

n	D_n^{upp1}	D_n^{upp2}	D_n^{upp3}	D_n
1	0.083	0.083	0.083	0.084
2	0.198	0.198	0.198	0.198
3	0.425	0.304	0.304	0.298
4	0.517	0.411	0.411	0.382
5	0.583	0.490	0.454	0.450
6	0.634	0.550	0.518	0.506
10	0.754	0.690	0.673	0.655
20	0.865	0.832	0.819	0.807

Table 3. The Third Moment of the Rotation Distance.

n	D_n^{upp1}	D_n^{upp2}	D_n^{upp3}	D_n
1	0.031	0.031	0.031	0.031
2	0.102	0.102	0.102	0.101
3	0.306	0.183	0.183	0.178
4	0.396	0.280	0.280	0.250
5	0.467	0.359	0.319	0.315
6	0.522	0.423	0.386	0.372
10	0.664	0.589	0.560	0.538
20	0.807	0.762	0.744	0.728

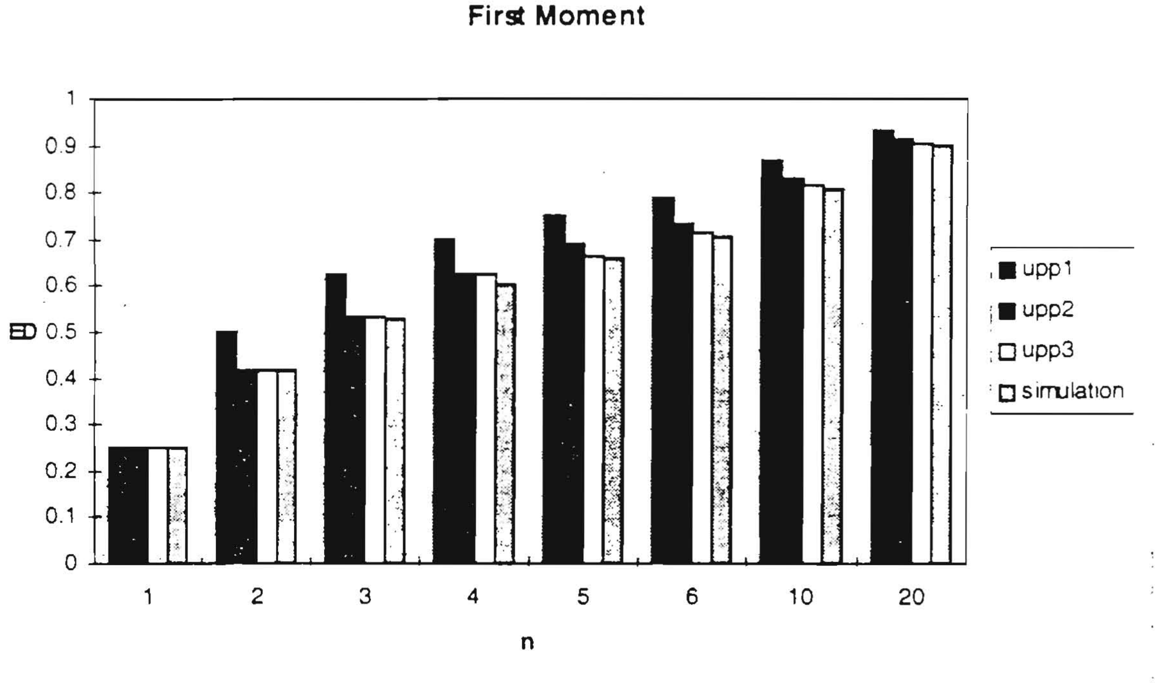

Figure 3. Graphical representation of the first moment of the upper bounds D_n^{upp1}, D_n^{upp2} and D_n^{upp3}, and for D_n itself, for different values of n.

For $n = 1$, the results for the upper bounds and for D_n should coincide. For $n = 2$, the results for D_n^{upp2} and D_n^{upp3} on the one hand and for D_n on the other hand should also coincide. The small differences that still are observed in Table 1 are caused by a small inaccuracy in the simulation results for D_n.

The results show that the upper bound D_n^{upp1} is only tight for $n = 1$ and larger values of n. The upper bound D_n^{upp2} is tight for all values of n. Nevertheless, the upper bound D_n^{upp3} still gives a significant improvement. The upper bound D_n^{upp3} is very tight for all values of n. Hence, we may conclude that D_n^{upp3} is very appropriate to serve as an approximation for D_n.

4. MODEL ANALYSIS

The previous section has been devoted to the order pick time of an order with a given number of line orders. In this section, first the order pick time of an arbitrary order is determined. This will complete the description of the M/G/1 queueing system, as introduced in Section 2. After that, the maximum throughput and response times are considered.

The order pick time and the rotation distance of an arbitrary order are represented by the random variables S and D, respectively. Then $S = S_n$ with probability p_n, $n \in I$, and $\mathbb{E}S^k = \sum_{n\in I}(p_n \cdot \mathbb{E}S_n^k)$ for all $k \in \mathbb{N}$. Similarly, $D = D_n$ with probability $p_n, n \in I$, and $\mathbb{E}D^k = \sum_{n\in I}(p_n \cdot \mathbb{E}D_n^k)$ for all $k \in \mathbb{N}$. Using (2), we find

$$\begin{aligned} \mathbb{E}S &= \sum_{n\in I} p_n(n\mathbb{E}T_{ext} + t_{rot}\mathbb{E}D_n) \\ &= \mathbb{E}N\mathbb{E}T_{ext} + t_{rot}\mathbb{E}D_n. \end{aligned} \tag{7}$$

Here N represents the number of line orders for an arbitrary order; hence, $\mathbb{E}N^k = \sum_{n\in I}(n^k \cdot p_n)$ for all $k \in \mathbb{N}$. Explicit expressions for the higher moments of S can be derived in a similar way as for the first moment. For the second and third moment, we obtain

$$\mathbb{E}S^2 = \mathbb{E}N^2(\mathbb{E}T_{ext})^2 + \mathbb{E}N \cdot [\mathbb{E}T_{ext}^2 - (\mathbb{E}T_{ext})^2] + 2t_{rot}\mathbb{E}T_{ext}\sum_{n\in I} p_n n\mathbb{E}D_n + t_{rot}^2 ED^2, \tag{8}$$

$$\begin{aligned} \mathbb{E}S^3 = {} & \mathbb{E}N^3(\mathbb{E}T_{ext})^3 + 3\mathbb{E}N^2\mathbb{E}T_{ext}[\mathbb{E}T_{ext}^2 - (\mathbb{E}T_{ext})^2] \\ & + \mathbb{E}N[\mathbb{E}T_{ext}^3 - 3\mathbb{E}T_{ext}\mathbb{E}T_{ext}^2 + 2(\mathbb{E}T_{ext})^3] \\ & + 3t_{rot}([\mathbb{E}T_{ext}^2 - (\mathbb{E}T_{ext})^2]\sum_{n\in I} p_n n\mathbb{E}D_n + (\mathbb{E}T_{ext})^2\sum_{n\in I} p_n n^2\mathbb{E}D_n) \\ & + 3t_{rot}^2\mathbb{E}T_{ext}\sum_{n\in I} p_n n\mathbb{E}D_n^2 + t_{rot}^3\mathbb{E}D^3. \end{aligned} \tag{9}$$

Note that, in the (8) and (9), the numerical evaluation of the sums over $n \in I$ is straightforward if the index set I is finite. If I is infinite, then the probabilities p_n are forced to become very small for large values of n and an efficient numerical evaluation of these sums is still possible.

The first moment of the order pick time S is sufficient to determine the maximum throughput $\lambda_{\max}$,

$$\lambda_{\max} = \frac{1}{\mathbb{E}S}. \tag{10}$$

The first three moments of S are sufficient to determine the first two moments of the response time T of an arbitrary order. This time T is equal to the sum of the waiting time W of an arbitrary order and the order pick time S. The first two moments of W are given by Pollaczek-Khintchine's formula (see *e.g.* Kleinrock [5], page 200):

$$\mathbb{E}W = \frac{\lambda \mathbb{E}S^2}{2(1-\rho)}. \tag{11}$$

$$\mathbb{E}W^2 = \frac{\lambda \mathbb{E}S^3}{3(1-\rho)} + 2(\mathbb{E}W)^2. \tag{12}$$

where ρ represents the work load, *i.e.* $\rho = \lambda \mathbb{E}S$.

For T, we find

$$\mathbb{E}T = \mathbb{E}W + \mathbb{E}S. \tag{13}$$

$$\mathbb{E}T^2 = \mathbb{E}W^2 + 2\mathbb{E}S\mathbb{E}W + \mathbb{E}S^2. \tag{14}$$

Note that the first two moments of the response time of an order with $n \in I$ line orders may be obtained by substituting S_n for S on the right-hand sides of (13) and (14) (but not in (11) and (12)).

Formulas (7)-(14) are exact. However, an exact computation of the maximum throughput $\lambda_{\max}$ and the first two moments of the response time is not possible, since we do not have exact formulae for the rotation distances D_n. Nevertheless, accurate approximations can be obtained. We approximate the rotation distances by the tight upper bound D_n^{upp3} as described in Subsection 3.3. This results in a tight lower bound for λ_{max} and tight upper bounds for $\mathbb{E}T$ and $\mathbb{E}T^2$.

The tight upper bounds for $\mathbb{E}T$ and $\mathbb{E}T^2$ can be further exploited to obtain also good approximations for excess probabilities of the response time T. These approximations can be obtained by fitting a mixture of Erlang distributions on the first two moments of T. Appropriate mixtures are the $E_{k-1,k}$ distribution and the hyper-exponential distribution (*cf.* Tijms [9]).

5. CONCLUSIONS AND FURTHER RESEARCH

In this paper a model for the performance analysis of a carousel system is presented. Using a stochastic framework, highly accurate estimates for the maximum throughput of the system and the minimum response time for an arbitrary order are obtained.

Further research will concentrate on the consequences of relaxing some of the assumptions listed in Section 2.1. as well as the analysis of systems in which several carousels are linked to a single order picker. Also, the influence of different (class based) storage policies on the system throughput will be investigated.

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