# Radiative heat transfer in semi-transparent media

Document status and date:
Published: 01/01/1995

Document Version:
Publisher's PDF, also known as Version of Record (includes final page, issue and volume numbers)

Please check the document version of this publication:

• A submitted manuscript is the version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher's website.
• The final author version and the galley proof are versions of the publication after peer review.
• The final published version features the final layout of the paper including the volume, issue and page numbers.

Link to publication

# Radiative heat transfer in semi-transparent media

## by

## C.J. Aldridge and W. Potze

Authors' affiliations:

C.J. Aldridge, Eindhoven University of Technology, Department of Mathematics and Computing Science, Eindhoven, The Netherlands;

present address: University of Strathclyde, Department of Mathematics, Livingstone Tower, 26 Richmond Street, Glasgow G1 1XH, United Kingdom.

W. Potze, Philips Research Laboratories, Prof. Holstlaan 4, 5656 AA Eindhoven, The Netherlands.

# Radiative heat transfer in semi-transparent media

## by

## C.J. Aldridge & W. Potze

**Abstract**

A mathematical model is proposed to describe the radiative heat transfer in a circular cylindrical tube. The spectrally dependent radiative intensity determining the radiative heat transport in the tube is investigated. The tube is considered to be composed of semi-transparent material in which scattering is assumed to be negligible. At the boundaries of the tube the radiative intensity is specularly reflected. For an infinitely long hollow circular cylindrical tube, the radiative intensity is solved analytically from the radiative transport equation. In the case of a quartz glass tube, results for the radiative intensity are presented and its effect on the temperature is determined.

# Contents

# 1 Introduction

In glass processing the temperature of the glass is in general very high. Glass can be deformed if its temperature is higher than the softening point. The deformations occurring in a manufacturing process depend on the local viscosity of the glass. This viscosity depends strongly on the temperature [1, 2], and it also depends on the applied load [3] or rate of deformation [4]. It decreases significantly with increasing temperatures. Therefore the formability and hence the final shape of a glass product depends strongly on the temperature distribution in the glass part. Temperature gradients in the cooling stage of a production process may introduce thermal stresses [5, 6] and residual stresses in the final produced glass part. These thermal stresses or residual stresses may cause an early failure of the glass product. Therefore, the temperature distribution in a glass product is very important.

Heat transfer in a semi-transparent medium, like glass, occurs by conduction and radiation [7, 8, 9, 10]. At low temperatures conduction is the major means of heat transport, at high temperatures heat radiation also contributes significantly to the heat transport. Heat radiation or thermal radiation may be viewed as transported by electromagnetic waves [8, 10]. Thermal radiation is radiant energy emitted by a medium due solely to the temperature of the medium. This means the temperature of the medium governs the emission of thermal radiation. The range of wavelengths of importance for heat transfer considerations is between 0.1 $\mu m$ (ultraviolet) and 100 $\mu m$ (infrared). Heat radiation may be polarized, because of its electromagnetic wave behaviour. In glass thermal radiation is emitted (cooling), absorbed (heating) and scattered (redirected). The heat radiation depends on the direction of propagation, the position in the medium, the frequency and the time. At the boundaries of the volume of the glass, a part of the heat radiation will be transmitted, another part will be reflected and a part will be scattered, depending on the quality of the surface and on the angle of incidence.

The quantity that determines the heat radiation in the glass is the radiative intensity $I_\nu(\boldsymbol{x}, \boldsymbol{s}, t)$ [8, 9, 10], representing the radiative energy flow in a certain direction per unit

5

time, solid angle, unit area normal to the propagation direction and per unit frequency. Instead of the dependence of the radiative intensity on the frequency $\nu$, the dependence on wavelength or wavenumber is also often used. In this paper the dependence on the frequency is preferred, because frequency is unchanged if radiation passes from one medium to another one with a different refractive index, while wavelength and wavenumber do change. The material parameters that directly influence the heat radiation in a medium are the refractive index and the absorption and scattering coefficients, which are dependent on frequency. Since scattering in glass can be neglected, the only material parameters that influence heat radiation are the refractive index and the absorption coefficient.

In this paper a mathematical model of the heat radiation in an infinitely long hollow circular cylindrical tube is proposed. The spectrally dependent radiative intensity is solved from the radiative transport equation and its contribution to the temperature change is determined. It is assumed that the temperature and the radiative intensity in the tube are axially symmetric. Furthermore, it is assumed that the heat radiation is unpolarized, hence no polarization effects are taken into account. The absorption coefficient and the refractive index in the considered medium are taken to be only dependent on the frequency $\nu$. Therefore the radiative energy in the medium propagates along straight lines. Specular reflection and transmission of the radiation at the inner and outer boundaries occurs. In contrast to [11], where the heat flux emitted from an isothermal infinitely long circular cylindrical tube is determined, here the heat transport in the volume of the tube is determined. Other studies have investigated the radiative heat transfer in an infinitely long circular cylindrical tube with black surfaces [12], with scattering [13, 14] or in a tube of finite length [15, 16].

In the present case of an infinitely long hollow circular cylindrical tube an exact expression for the spectrally dependent radiative intensity is derived. This is probably not possible for a tube of finite length. For a given temperature distribution in the tube, the radiative intensity is calculated and the contribution to the temperature change is determined.

6
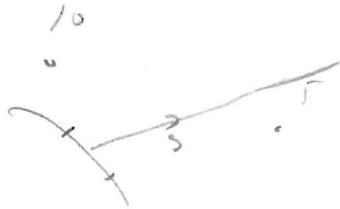
# 2 Model equations

## 2.1 Thermodynamics and radiative heat transfer

Conservation of energy for the material is

$$\rho c_p(T_t + (\boldsymbol{u} \cdot \boldsymbol{\nabla})T) = -\boldsymbol{\nabla} \cdot \boldsymbol{q}, \tag{1}$$

where $\rho$ is the density, $\boldsymbol{u}$ is the velocity, $T$ is the temperature, $c_p$ is the specific heat capacity, and $\boldsymbol{q}$ is the heat flux, and the subscript $t$ represents differentiation with respect to time. The heat flux $\boldsymbol{q}$ consists of a part $\boldsymbol{q_c}$ caused by heat conduction and a part $\boldsymbol{q_r}$ caused by heat radiation. Fourier's law is used for the constitutive equation of the heat flux by conduction, which, in our case of an isotropic glass is seen to be

$$\boldsymbol{q_c} = -k_c\boldsymbol{\nabla}T, \tag{2}$$

where $k_c = k_c(T)$ is the thermal conductivity.

The basic variable that describes radiative heat transfer within the glass is the spectral intensity, $I_\nu$. This is defined as the amount of radiant energy at frequency $\nu$ transported at a point $\boldsymbol{x}$ in the direction $\boldsymbol{s}$ per unit time, frequency, solid angle and normal area. Thus $I_\nu = I_\nu(\boldsymbol{x}, \boldsymbol{s}, t)$, where $|\boldsymbol{s}| = 1$. The spectral intensity $I_\nu$ has the unit $(\mathrm{Ws})/(\mathrm{m^2 sr})$, however we will drop the dimensionless unit steradians $(\mathrm{sr} = \mathrm{m^2/m^2})$ and use only SI base units in the formulae and graphs below. The general equation of radiative heat transport within an absorbing, emitting and scattering volume is [10]

$$\frac{1}{c}\frac{\partial I_\nu}{\partial t} + (\boldsymbol{s} \cdot \boldsymbol{\nabla})I_\nu = \kappa_\nu I_{b,\nu} - (\kappa_\nu + \sigma_{s\nu})I_\nu + \frac{\sigma_{s\nu}}{4\pi}\int_{\Omega'=4\pi}\Phi(\boldsymbol{s}', \boldsymbol{s})I_\nu(\boldsymbol{s}')d\Omega'. \tag{3}$$

Here $c$ is the speed of light in the glass, $\kappa_\nu$ and $\sigma_{s\nu}$ are the absorption and scattering coefficients, $I_{b,\nu}$ is the spectral blackbody intensity, $\Phi(\boldsymbol{s}', \boldsymbol{s})$ is a scattering phase function, and $\Omega$ is the solid angle (the integration is thus taken over all directions). The left-hand side of (3) represents the change of the radiative intensity in the direction of propagation $\boldsymbol{s}$. The right-hand side represents the change in radiative intensity due to emission, absorption, scattering away from the $\boldsymbol{s}$-direction and scattering into the $\boldsymbol{s}$-direction, respectively. The subscript $\nu$ is used throughout this report to indicate a frequency-dependent variable.

7

We could alternatively choose wavelength $\lambda$, or wavenumber $\eta$, as spectral quantities ($\nu = c/\lambda = c\eta$). Wavelength is often the preferred variable in recent literature, e.g. [10, 17, 18], on radiative transfer in semi-transparent media. However frequency is unchanged as radiation passes from one medium to another (with different refractive index), and so appears a sound choice for this study.

The radiative heat flux $\boldsymbol{q_r}$ is defined as

$$\boldsymbol{q_r} = \int_0^\infty \int_{\Omega=4\pi} \boldsymbol{s}\, I_\nu(\boldsymbol{x}, \boldsymbol{s}, t)\, d\Omega\, d\nu. \tag{4}$$

We now detail some of the terms in (3). The spectral absorption coefficient $\kappa_\nu$ of quartz glass typically varies over several orders of magnitude across the spectrum of thermal radiation. For frequencies lower than around $6 \times 10^{13}$ s$^{-1}$, $\kappa_\nu$ is relatively large, while for higher frequencies $\kappa_\nu$ falls sharply to relatively small values [19]. In general the absorption coefficient is also dependent on temperature [20].

The blackbody intensity is given by Planck's law as

$$I_{b,\nu} = \frac{2h\nu^3 n_\nu^2)}{c_0^2(e^{h\nu/kT} - 1)}, \tag{5}$$

where $h = 6.626 \times 10^{-34}$ J s is Planck's constant, $k = 1.3806 \times 10^{-23}$ J K$^{-1}$ is Boltzmann's constant, $n_\nu$ is the refractive index, and $c_0 = 2.998 \times 10^8$ m s$^{-1}$ is the speed of electromagnetic radiation (= speed of light) in a vacuum ($c = c_0/n_\nu$). Since for glass $n_\nu \approx 1.5$, radiative heat transfer occurs on a relatively rapid time scale, and we may neglect the time derivative in (3). The refractive index $n_\nu$ of glass in general is also dependent on the temperature $T$ [21]. Moreover, scattering of radiation is not considered to be significant for this problem [22], and we may take $\sigma_{s\nu} = 0$. Hence the radiative transport equation (3) reduces to

$$(\boldsymbol{s} \cdot \boldsymbol{\nabla})I_\nu = \kappa_\nu(I_{b,\nu} - I_\nu). \tag{6}$$

In Figure 1 the dependence of $I_{b,\nu}$ on frequency for three different temperatures is shown. The total blackbody intensity $I_b$ is given by

$$I_b = \int_0^\infty I_{b,\nu}\, d\nu = (1/\pi)n^2\sigma T^4, \tag{7}$$

8

$\sim \exp \dfrac{h\nu}{kT}$
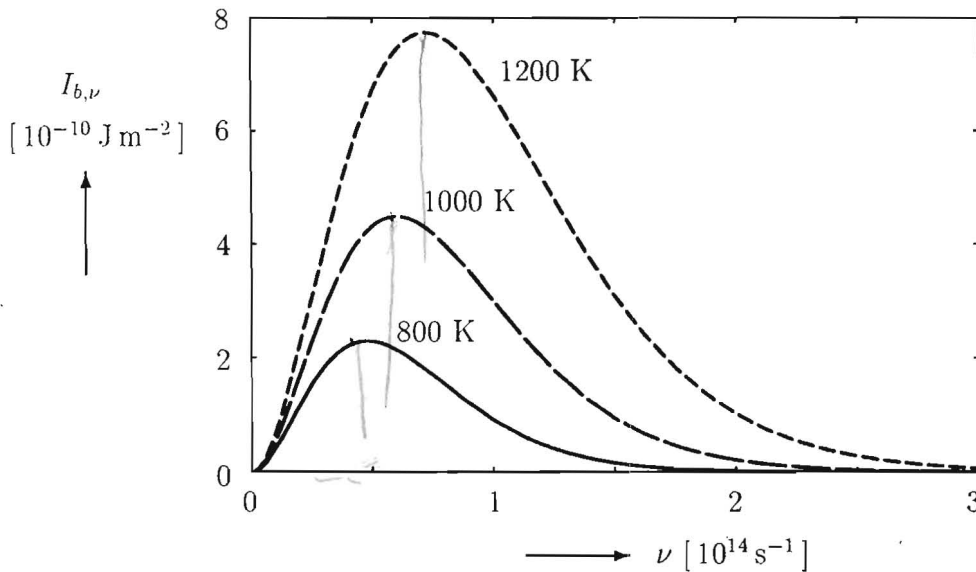


Figure 1: Spectral blackbody intensity, $T = 800,\ 1000,\ 1200$ K, $n_\nu = 1.5$.

(taking $n_\nu = n$ constant); here $\sigma$ is the Stefan-Boltzmann constant

$$\sigma = \frac{2\,\pi^5\,k^4}{15\,c_0^2\,h^3} = 5.670 \times 10^{-8}\,\mathrm{W\,m^{-2}\,K^{-4}}. \tag{8}$$

If a material is considered to be grey, that is, if $\kappa_\nu$ and $n_\nu$ are constant, then the source term in the equation of transfer for the total intensity $I(\boldsymbol{x}, \boldsymbol{s}, t) = \int_0^\infty I_\nu(\boldsymbol{x}, \boldsymbol{s}, t)\, d\nu$ is thus proportional to $T^4$.

The equation of transfer (6) requires a boundary condition specifying the intensities directed into the volume at each point on the boundary. The boundary conditions will be discussed in the next section.

### 2.1.1 Radiative boundary conditions

The boundaries of the volume are considered to be optically smooth. Then no scattering occurs at the boundaries, hence the radiation is reflected and transmitted. Let $\boldsymbol{r}_0$ be a point on the boundary of the volume, $\boldsymbol{n}$ be the outward unit normal at $\boldsymbol{r}_0$, and let $\theta$ be the angle between $\boldsymbol{s}$ and $\boldsymbol{n}$, thus $\cos\theta = \boldsymbol{s} \cdot \boldsymbol{n}$, $\theta \in [0, \pi)$. Then the radiative intensity $I_\nu(\boldsymbol{r}_0, \boldsymbol{s})$ where $\boldsymbol{s} \cdot \boldsymbol{n} < 0$ is composed of two components due to reflection and transmission, as shown in Figure 2. As polarization effects of heat radiation are neglected, we assume the

9
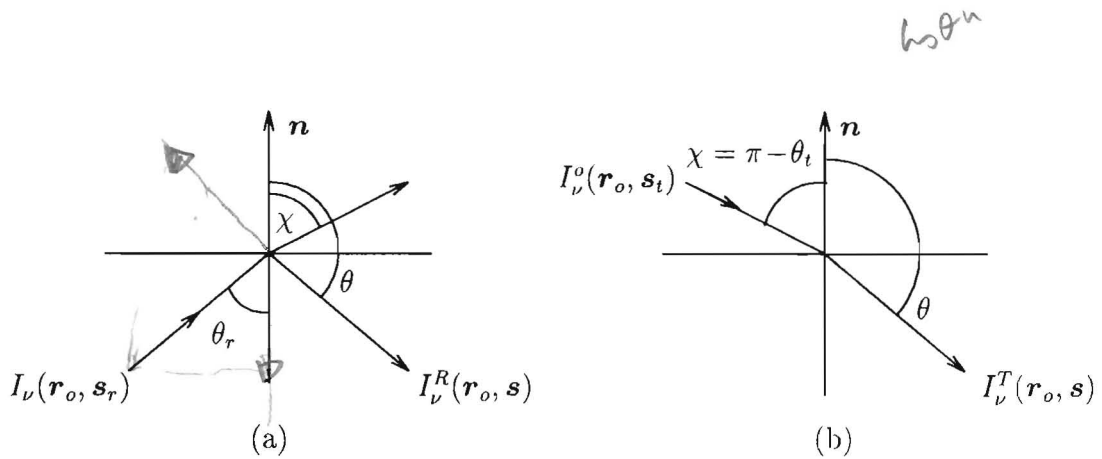
Figure 2: (a) Reflected and (b) transmitted components of boundary intensity.

radiation to be unpolarized. In this case of unpolarized radiation incident on an interface between two perfect (non-attenuating) media (see, for example, [9] p.102), the reflected component is given by

$$I_\nu^R(\boldsymbol{r}_0, \boldsymbol{s}) = \mathcal{R}(\theta_r) I_\nu(\boldsymbol{r}_0, \boldsymbol{s}_r). \tag{9}$$

Here $\boldsymbol{s}_r$ is the direction of the incident radiation, given by

$$\boldsymbol{s}_r = \boldsymbol{s} - 2\cos\theta \, \boldsymbol{n}, \tag{10}$$

$\theta_r \in [0, \pi/2)$ is the angle between $\boldsymbol{s}_r$ and $\boldsymbol{n}$, given by $\theta_r = \pi - \theta$, so $\boldsymbol{s}_r \cdot \boldsymbol{n} = \cos\theta_r$, and $\mathcal{R}(\theta_r)$ is the 'reflectance function'. This is given by Fresnel's equation,

$$\mathcal{R}(\theta_r) = \begin{cases} \dfrac{1}{2} \dfrac{\sin^2(\theta_r - \chi)}{\sin^2(\theta_r + \chi)} \left( 1 + \dfrac{\cos^2(\theta_r + \chi)}{\cos^2(\theta_r - \chi)} \right) & \text{if } n_\nu \sin\theta_r \leq 1, \\ 1 & \text{otherwise.} \end{cases} \tag{11}$$

Here $\chi \in [0, \pi/2]$ is the angle between the refracted ray and $\boldsymbol{n}$, given by Snell's law,

$$\frac{\sin\chi}{\sin\theta_r} = n_\nu. \tag{12}$$

If $\theta_r = 0$, then

$$\mathcal{R}(0) = \left( \frac{n_\nu - 1}{n_\nu + 1} \right)^2, \tag{13}$$

which is the limit of the expression in (11) as $\theta_r \to 0$. Figure 3 shows the reflectance function $\mathcal{R}$ defined by (11) and (13), taking $n_\nu = 1.5$. We see that for small angles
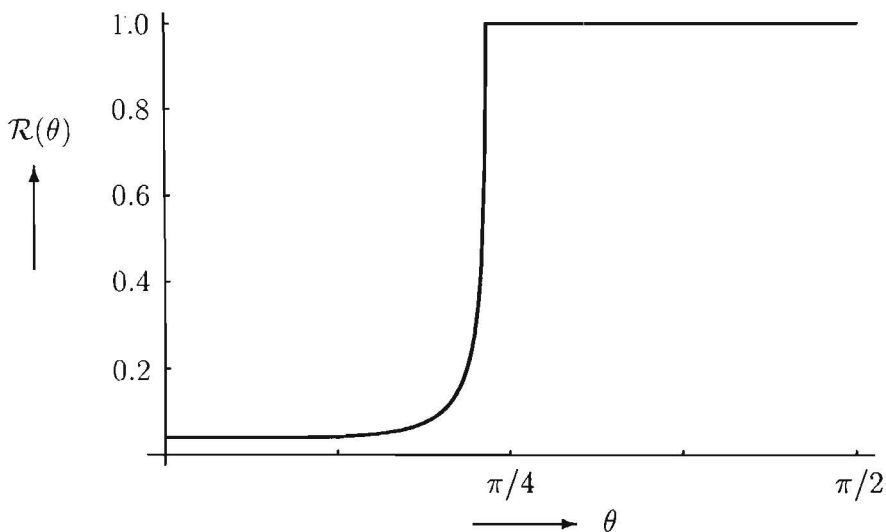
Figure 3: Reflectance function, $n_\nu = 1.5$.

of incidence, $\mathcal{R}$ is very low ($\mathcal{R}(0) = 0.04$), but for larger angles rises sharply to unity ($\mathcal{R}(\theta_r) = 1$ for $\theta_r \geq \sin^{-1} \frac{2}{3} \approx 0.73$ rad), corresponding to total internal reflection.

The transmitted component of $I_\nu(\boldsymbol{r}_0, \boldsymbol{s})$ is

$$I_\nu^T(\boldsymbol{r}_0, \boldsymbol{s}) = \begin{cases} n_\nu^2 \left(1 - \mathcal{R}(\theta_r)\right) I_\nu^o(\boldsymbol{r}_0, \boldsymbol{s}_t) & \text{if } n_\nu \sin \theta_r \leq 1, \\ \\ 0 & \text{otherwise.} \end{cases} \tag{14}$$

Here $I_\nu^o(\boldsymbol{r}_o, \boldsymbol{s}_t)$ is the incident intensity from outside the volume, $\boldsymbol{s}_t$ is given by

$$\boldsymbol{s}_t = \frac{1}{\sin \theta} \left( \sin(\theta_t) \, \boldsymbol{s} + \sin(\theta - \theta_t) \, \boldsymbol{n} \right), \tag{15}$$

and $\theta_t \in (\pi/2, \pi]$ is the angle between $\boldsymbol{s}_t$ and $\boldsymbol{n}$, given by $\sin \theta_t / \sin \theta = n_\nu$, so $\boldsymbol{s}_t \cdot \boldsymbol{n} = \cos \theta_t$.

Hence from (9) and (14) the boundary condition is

$$I_\nu(\boldsymbol{r}_0, \boldsymbol{s}) = \mathcal{R}(\theta_r) I_\nu(\boldsymbol{r}_0, \boldsymbol{s}_r) + n_\nu^2 \left(1 - \mathcal{R}(\theta_r)\right) I_\nu^o(\boldsymbol{r}_0, \boldsymbol{s}_t). \tag{16}$$

We now consider some special cases corresponding to simple geometries.

### 2.1.2 Single slab

Consider the radiative heat transfer in a one-dimensional medium, contained between two parallel planes $x = x_i$ and $x = x_o > x_i$, with no variation in the directions normal to

11

the $x$-axis. Then we may write $I_\nu = I_\nu(x, \theta)$, where $\cos\theta = \boldsymbol{s} \cdot \boldsymbol{e_x}$, and the equation of transfer (6) may be written as

$$\cos\theta \frac{\partial I_\nu}{\partial x} + \kappa_\nu I_\nu = \kappa_\nu I_{b,\nu}. \tag{17}$$

From (16), boundary conditions are

$$I_\nu(x_o, \theta) = \mathcal{R}(\pi - \theta) I_\nu(x_o, \pi - \theta) + n_\nu^2 \left(1 - \mathcal{R}(\pi - \theta)\right) I_\nu^o(x_o, \theta_{t_o}), \quad \pi/2 < \theta \leq \pi, \tag{18}$$

where $\sin\theta_{t_o} = n_\nu \sin\theta$, $\theta_{t_o} \in (\pi/2, \pi]$, and

$$I_\nu(x_i, \theta) = \mathcal{R}(\theta) I_\nu(x_i, \pi - \theta) + n_\nu^2 \left(1 - \mathcal{R}(\theta)\right) I_\nu^i(x_i, \pi - \theta_{t_o}), \quad 0 \leq \theta < \pi/2. \tag{19}$$

Here $I_\nu^o$ and $I_\nu^i$ are the incident intensities from outside the volume at $x = x_o$ and $x = x_i$, respectively.

### 2.1.3 Infinite axisymmetric tube

We now consider radiative transfer in a hollow axisymmetric tube. Position is given in cylindrical polar coordinates $(r, \phi_c, z)$, where the tube axis lies on the $z$-axis and the azimuthal angle $\phi_c$ is measured from some (arbitrary) fixed reference direction, as shown in Figure 4. The inner radius of the tube is $r = r_i$ and the outer radius is $r = r_o$.

Direction $\boldsymbol{s}$ is parameterised by local spherical polars $(\theta, \phi)$, where $\theta$ is the (polar) angle between $\boldsymbol{s}$ and $\boldsymbol{e_z}$, and $\phi$ is the (azimuthal) angle between $\boldsymbol{e_r}$ and the projection of $\boldsymbol{s}$ onto the $(r, \phi_c)$–plane (see Figure 5). Hence

$$\boldsymbol{s} = \sin\theta \cos\phi \, \boldsymbol{e_r} + \sin\theta \sin\phi \, \boldsymbol{e_{\phi_c}} + \cos\theta \, \boldsymbol{e_z}, \tag{20}$$

so

$$\boldsymbol{s} \cdot \boldsymbol{\nabla} = \sin\theta \left( \cos\phi \frac{\partial}{\partial r} + \frac{1}{r} \sin\phi \frac{\partial}{\partial \phi_c} \right) + \cos\theta \frac{\partial}{\partial z}, \tag{21}$$

and since the radiative energy propagates along straight lines, so $\phi + \phi_c = \text{constant}$ along $\boldsymbol{s}$, the equation of transfer (6) gives

$$\sin\theta \left( \cos\phi \frac{\partial I_\nu}{\partial r} - \frac{1}{r} \sin\phi \frac{\partial I_\nu}{\partial \phi} \right) + \cos\theta \frac{\partial I_\nu}{\partial z} + \kappa_\nu I_\nu = \kappa_\nu I_{b,\nu}. \tag{22}$$

12
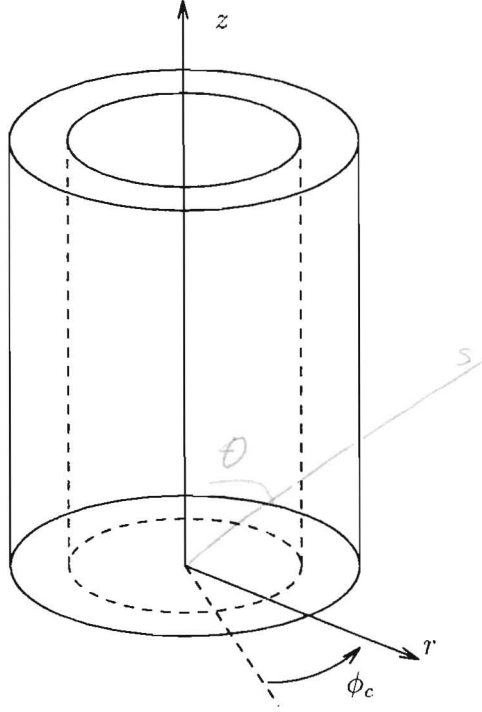
Figure 4: Cylindrical polar position coordinates.

We suppose at present that the problem is independent of $z$, so $\partial I_\nu / \partial z = 0$ in (22), and we may write $I_\nu = I_\nu(r, \theta, \phi)$.

To describe reflection and transmission at the boundaries we introduce a second parameterisation of direction $(\omega, \zeta)$, defined by

$$\cos \omega = \sin \theta \cos \phi, \tag{23}$$

$$\tan \zeta = \tan \theta \sin \phi. \tag{24}$$

Here $\omega$ is the polar angle between $\boldsymbol{s}$ and $\boldsymbol{e_r}$, while $\zeta$ is the azimuthal angle between $\boldsymbol{e_z}$ and the projection of $\boldsymbol{s}$ onto the $(z, \phi_c)$–plane (see Figure 6).

At the outer radius of the tube, the boundary condition is analogous to (18) and is given by

$$I_\nu(r_o, \theta, \phi) = \mathcal{R}(\omega_r) I_\nu(r_o, \theta_r, \phi_r) + n_\nu^2 (1 - \mathcal{R}(\omega_r)) I_\nu^o(r_o, \theta_t, \phi_t),$$

$$0 < \theta < \pi, \ \pi/2 < \phi \leq \pi. \tag{25}$$

13
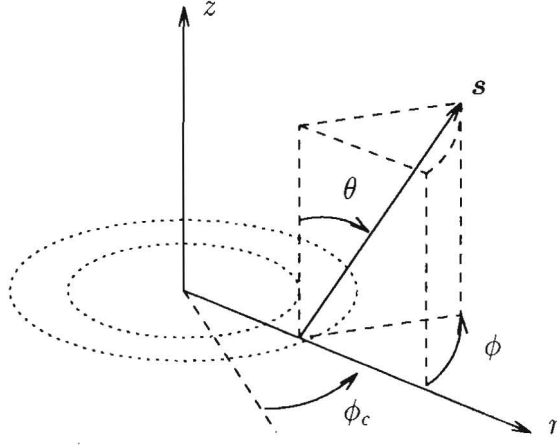
Figure 5: Local spherical polar coordinates for direction.
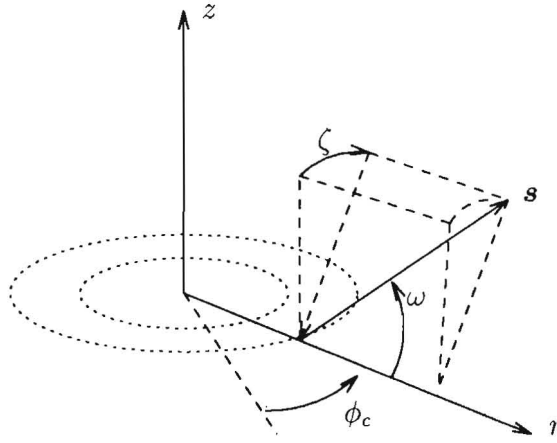


Figure 6: Alternative direction coordinates at boundaries.

Since we assume that the intensities are independent of $z$ and there is no absorption, emission or scattering in the region $r < r_i$, hence no radiative energy is generated or absorbed in the interior of the tube, the boundary condition at the inner radius is equivalent to total internal reflection and is given by

$$I_\nu(r_i, \theta, \phi) = I_\nu(r_i, \theta_r, \phi_r). \quad 0 < \theta < \pi, \ 0 \le \phi < \pi/2, \tag{26}$$

Here the direction $s_r$ of the incident radiation at the boundary from the interior of the semi-transparent medium, which after reflection becomes $s$, is given by $\theta_r = \theta$, $\phi_r = \pi - \phi$,

14

and $\omega_r \in [0, \pi/2)$ is given from above by

$$\cos \omega_r = -\sin \theta \cos \phi. \tag{27}$$

The direction $\boldsymbol{s}_t$ of the incident radiation at the boundary from outside the semi-transparent medium, which after transmission becomes $\boldsymbol{s}$, is given by $\theta_t \in (0, \pi)$, $\phi_t \in (\pi/2, \pi]$, defined by

$$\sin \theta_t \cos \phi_t = \cos \omega_t, \tag{28}$$
$$\tan \theta_t \sin \phi_t = \tan \zeta_t, \tag{29}$$

where $\omega_t \in (\pi/2, \pi]$, $\sin \omega_t = n_\nu \sin \omega$, and $\zeta_t = \zeta$.

### 2.1.4 Finite axisymmetric tube

We now suppose that the problem is $z$-dependent, in which case the intensity may be written as $I_\nu = I_\nu(r, z, \theta, \phi)$. The equation of transfer is again given by (22), and the boundary condition at the outside of the tube remains as (25) (with the inclusion of the independent variable $z$). The boundary condition at the inside of the tube requires some modification.

Let $\theta \in (0, \pi)$, $\phi \in [0, \pi/2)$. The intensity entering the volume in the direction $(\theta, \phi)$ at $(r_i, z)$ is

$$I_\nu(r_i, z, \theta, \phi) = \mathcal{R}(\omega) I_\nu(r_i, z, \theta_r, \phi_r) + n_\nu^2 (1 - \mathcal{R}(\omega)) I_\nu^i(r_i, z, \theta_t, \phi_t). \tag{30}$$

Here $\omega$ is given by (23), $\theta_r = \theta$, $\phi_r = \pi - \phi$, and $\theta_t \in (0, \pi)$, $\phi_t \in [0, \pi/2)$ are given by

$$\sin \theta_t \cos \phi_t = \cos \omega_t, \tag{31}$$
$$\tan \theta_t \sin \phi_t = \tan \zeta_t, \tag{32}$$

where

$$\sin \omega_t = n_\nu \sin \omega, \quad \omega_t \in (0, \pi/2); \quad \zeta_t = \zeta, \tag{33}$$

and $\zeta$ is given by (24). If the transmitted intensity $I_\nu^i$ arrives at $(r_i, z)$ directly from the
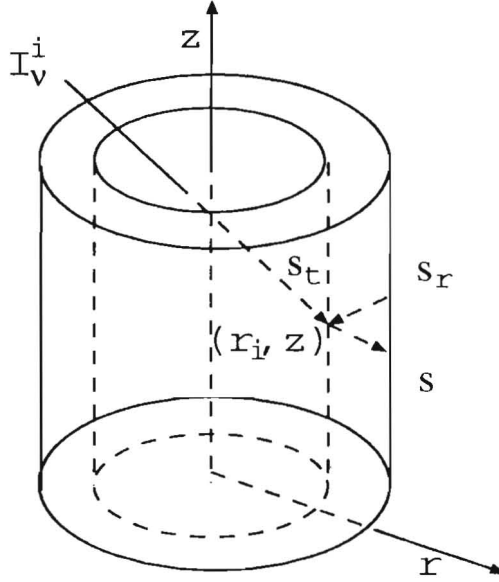
Figure 7: Intensity arriving at the inner boundary of the tube directly from the environment.

environment surrounding the tube, without reflecting first at the inside of the tube, as drawn in Figure 7, then the boundary condition at $r = r_i$ is given by (30). Otherwise, the transmitted intensity is

$$I_\nu^i(r_i, z, \theta_t, \phi_t) = \mathcal{R}(\omega) I_\nu^i(r_i, \tilde{z}, \theta_t, \phi_t) + \frac{1}{n_\nu^2}(1 - \mathcal{R}(\omega)) I_\nu(r_i, \tilde{z}, \theta_r, \phi_r), \qquad (34)$$

where $\tilde{z} = z - \Delta z$, with $\Delta z$ denoting the increment in $z$ associated with the path traversing the interior of the tube, as shown in Figure 8. Since the projection of the path onto the $(r, \phi_c)$–plane has length $l = 2r_i \cos\phi$, and $\tan\theta = l/\Delta z$, the increment is given by

$$\Delta z = \frac{2r_i \cos\phi}{\tan\theta}. \qquad (35)$$

The boundary condition at $r = r_i$ may then be found from (34), using (30) to eliminate $I_\nu^i$ at $z$ and $\tilde{z}$. We find

$$I_\nu(r_i, z, \theta, \phi) - \mathcal{R}(\omega) I_\nu(r_i, z, \theta_r, \phi_r)$$

$$= \mathcal{R}(\omega) I_\nu(r_i, \tilde{z}, \theta, \phi) + (1 - 2\mathcal{R}(\omega)) I_\nu(r_i, \tilde{z}, \theta_r, \phi_r). \qquad (36)$$

16
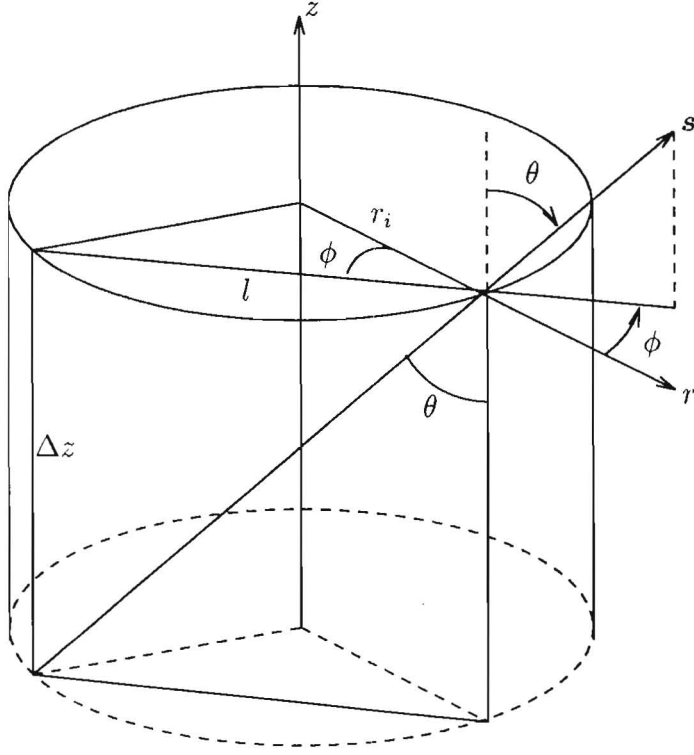
Figure 8: Path traversing interior of tube.

In addition, boundary conditions are required at the end planes of the tube. Suppose that the ends of the tube are at $z = z_1$ and $z = z_2 > z_1$. Then the incoming intensities at the end planes are

$$I_\nu(r, z_1, \theta, \phi), \quad r_i < r < r_o, \ \ 0 < \theta \leq \pi/2, \ \ 0 \leq \phi < 2\pi, \tag{37}$$

$$I_\nu(r, z_2, \theta, \phi), \quad r_i < r < r_o, \ \ \pi/2 \leq \theta < \pi, \ \ 0 \leq \phi < 2\pi. \tag{38}$$

These intensities should be expressed in terms of reflected and transmitted intensities as before. If the tube is infinitely long and the problem is independent of $z$, which was the case in the previous section, the boundary condition (36) at the inside of the tube reduces to (26), by setting $I_\nu(r_i, \tilde{z}, \theta, \phi) = I_\nu(r_i, z, \theta, \phi)$ in (36).

17

# 3  Non-dimensionalisation

We now settle on the geometry of an infinite hollow axisymmetric tube, for which we will compute the intensities as determined by the equation of transfer (22), with $\partial I_\nu / \partial z = 0$. and boundary conditions (26) and (25). Our first step is to introduce a new dependent variable. The radiative term in the energy equation (1) is $-\boldsymbol{\nabla} \cdot \boldsymbol{q_r}$, where the radiative flux $\boldsymbol{q_r}$ is defined by (4). Taking the divergence of (4) and using the equation of transfer (6) we find

$$\boldsymbol{\nabla} \cdot \boldsymbol{q_r} = \int_0^\infty \kappa_\nu \int_{\Omega=4\pi} ( I_{b,\nu} - I_\nu ) \, d\Omega \, d\nu. \tag{39}$$

This motivates the introduction of the 'relative intensity' $i_\nu = I_\nu - I_{b,\nu}$ as a new dependent variable. For an infinite axisymmetric tube, from (22), (25) and (26), $i_\nu$ satisfies

$$\sin\theta \left( \cos\phi \, \frac{\partial i_\nu}{\partial r} - \sin\phi \, \frac{1}{r} \frac{\partial i_\nu}{\partial \phi} \right) + \kappa_\nu i_\nu = -\sin\theta\cos\phi \, \frac{\partial I_{b,\nu}}{\partial r}, \tag{40}$$

$$i_\nu(r_i, \theta, \phi) = i_\nu(r_i, \theta, \pi - \phi), \quad 0 < \theta < \pi, \ 0 \le \phi \le \pi/2, \tag{41}$$

$$i_\nu(r_o, \theta, \phi) = \mathcal{R}(\omega_r) i_\nu(r_o, \theta, \pi - \phi) + (1 - \mathcal{R}(\omega_r))( n_\nu^2 I_\nu^o(r_o, \theta_t, \phi_t) - I_{b,\nu}(r_o) ),$$
$$0 < \theta < \pi, \ \pi/2 < \phi \le \pi. \tag{42}$$

Our next step is to non-dimensionalise the equations. Let $\Delta r = r_o - r_i$ be the thickness of the tube, and let $\overline{T}$ be a typical temperature. We let

$$r = r_i + \Delta r \, r^*, \tag{43}$$

$$T = \overline{T} T^*, \tag{44}$$

where $*$ denotes a dimensionless variable. As we are dealing with spectral quantities, we must define a dimensionless frequency: (5) motivates the choice

$$\frac{h\nu}{kT} = \frac{h\nu}{k\overline{T}} \frac{1}{T^*} = \frac{\nu^*}{T^*}, \tag{45}$$

that is

$$\nu = \frac{k\overline{T}}{h} \nu^*. \tag{46}$$

18

For example, if $\overline{T} = 1000$ K, we find that a unit dimensionless frequency corresponds to $2.1 \times 10^{13}$ s$^{-1}$. Suppose that $n_\nu = n = $ constant. Hence from (5) we let

$$I_{b,\nu} = \overline{I_{b,\nu}} I_{b,\nu}^*, \tag{47}$$

where

$$\overline{I_{b,\nu}} = \frac{2n^2 k^3 \overline{T}^3}{c_0^2 h^2}, \tag{48}$$

$$I_{b,\nu}^* = \frac{\nu^{*\,3}}{e^{\nu^*/T^*} - 1}. \tag{49}$$

For example, if $\overline{T} = 1000$ K and $n = 1.5$, the spectral blackbody intensity scale is $\overline{I_{b,\nu}} = 3.0 \times 10^{-10}$ J m$^{-2}$. We also let

$$i_\nu = \overline{I_{b,\nu}} i_\nu^*. \tag{50}$$

The absorption coefficient is scaled with a typical value, say

$$\kappa_\nu = \overline{\kappa} \kappa_\nu^*. \tag{51}$$

This leads to the dimensionless system below:

$$\sin\theta \left( \cos\phi \frac{\partial i_\nu^*}{\partial r^*} - \sin\phi \frac{\delta}{1 + \delta r^*} \frac{\partial i_\nu^*}{\partial \phi} \right) + \tau_0 \kappa_\nu^* i_\nu^* = -\sin\theta \cos\phi \frac{\partial I_{b,\nu}^*}{\partial r^*}, \tag{52}$$

$$i_\nu^*(0, \theta, \phi) = i_\nu^*(0, \theta, \pi - \phi), \quad 0 < \theta < \pi,\ 0 \le \phi \le \pi/2, \tag{53}$$

$$i_\nu^*(1, \theta, \phi) = \mathcal{R}(\omega_r) i_\nu^*(1, \theta, \pi - \phi) + (1 - \mathcal{R}(\omega_r))( I_\nu^{o*}(1, \theta_t, \phi_t) - I_{b,\nu}^*(1) ),$$
$$0 < \theta < \pi,\ \pi/2 < \phi \le \pi, \tag{54}$$

where

$$\delta = \frac{\Delta r}{r_i}, \quad \tau_0 = \overline{\kappa} \Delta r, \tag{55}$$

and

$$n^2 I_\nu^o = \overline{I_{b,\nu}} I_\nu^{o*}. \tag{56}$$

Since the problem is linear, it is convenient to consider the relative intensity as composed of three components, say

$$i_\nu^* = i_{\nu R}^* + i_{\nu D}^* + i_{\nu E}^*. \tag{57}$$

19

Here $i^*_{\nu R}$ is the relative intensity if the temperature is constant throughout the volume, so $T^* = 1$, and there is no external source of radiation. The second component $i^*_{\nu D}$ is the difference between the relative intensity for non-constant temperature and $i^*_{\nu R}$, again in the absence of external sources. The third component $i^*_{\nu E}$ is the relative intensity due solely to external radiation sources. From (52), (53), and (54), the three components satisfy

$$\sin\theta \left( \cos\phi \frac{\partial i^*_\nu}{\partial r^*} - \sin\phi \frac{\delta}{1+\delta r^*} \frac{\partial i^*_\nu}{\partial \phi} \right) + \tau_0 \kappa^*_\nu i^*_\nu = \mathcal{A}, \tag{58}$$

$$i^*_\nu(0,\theta,\phi) = i^*_\nu(0,\theta,\pi-\phi), \quad 0 < \theta < \pi,\ 0 \le \phi \le \pi/2. \tag{59}$$

$$i^*_\nu(1,\theta,\phi) = \mathcal{R}(\omega_r)i^*_\nu(1,\theta,\pi-\phi) + (1-\mathcal{R}(\omega_r))\mathcal{B}, \quad 0<\theta<\pi,\ \pi/2 < \phi \le \pi, \tag{60}$$

where

$$\mathcal{A} = \begin{cases} 0 & \text{(R)}, \\ -\sin\theta\cos\phi \dfrac{\partial I^*_{b,\nu}}{\partial r^*} & \text{(D)}, \\ 0 & \text{(E)}, \end{cases} \tag{61}$$

$$\mathcal{B} = \begin{cases} -I^*_{b,\nu}|_{T^*=1} & \text{(R)}, \\ -(I^*_{b,\nu}(1) - I^*_{b,\nu}|_{T^*=1}) & \text{(D)}, \\ I^{o*}_\nu(1,\theta_t,\phi_t) & \text{(E)}. \end{cases} \tag{62}$$

# 4   Solution by the method of characteristics

We now formally derive the solution of (52), a first-order hyperbolic partial differential equation. Characteristics of (52) are given by (dropping the $*$'s)

$$\frac{\partial r}{\partial s} = \dot{r} = \sin\theta\cos\phi, \tag{63}$$

$$\dot{\theta} = 0, \tag{64}$$

$$\dot{\phi} = -\sin\theta\sin\phi \frac{\delta}{1+\delta r}, \tag{65}$$

$$\dot{i}_\nu = -\tau_0 \kappa_\nu i_\nu - \sin\theta\cos\phi \frac{\partial I_{b,\nu}}{\partial r}, \tag{66}$$

20

where $s$ is a length parameter along the characteristic. From (64), it follows that the polar angle $\theta$ is constant along a characteristic, hence

$$\theta = \theta_I, \tag{67}$$

where $\theta_I$ is an initial value of $\theta$. Dividing (63) and (65) leads to

$$\int \frac{\delta}{1 + \delta r} \, dr = -\int \frac{\cos \phi}{\sin \phi} \, d\phi,$$

and so

$$\ln(1 + \delta r) = -\ln|\sin \phi| + \ln c, \quad c > 0,$$

hence along a characteristic

$$(1 + \delta r) \sin \phi = (1 + \delta r_I) \sin \phi_I, \tag{68}$$

where $r_I$ and $\phi_I$ are initial values of $r$ and $\phi$ (Cauchy data). This relationship also follows geometrically from the straight line through the path of radiative energy transport. Along this straight line, dimensionally, $r \sin \phi$, indicating the distance from this line to the centre of the tube, has to be constant

If $\phi_I = 0$, then $\phi \equiv 0$ and $r = r_I + s \sin \theta$. If $\phi_I \neq 0$, then from (65) and (68),

$$\dot{\phi} = \frac{-\delta \sin \theta}{(1 + \delta r_I) \sin \phi_I} \sin^2 \phi,$$

and rearranging

$$\int \frac{d\phi}{\sin^2 \phi} = \frac{-\delta \sin \theta}{(1 + \delta r_I) \sin \phi_I} \int ds.$$

Therefore $\phi$ is given as a function of $s$ by

$$\cot \phi = \cot \phi_I + \frac{\delta \sin \theta}{(1 + \delta r_I) \sin \phi_I} s. \tag{69}$$

Hence from (68), $r$ is given as a function of $s$ by

$$\begin{aligned}
(1 + \delta r)^2 &= (1 + \delta r_I)^2 \sin^2 \phi_I \, \mathrm{cosec}^2 \phi \\
&= (1 + \delta r_I)^2 \sin^2 \phi_I \left(1 + \cot^2 \phi\right) \\
&= (1 + \delta r_I)^2 \sin^2 \phi_I + \left((1 + \delta r_I) \cos \phi_I + \delta s \sin \theta\right)^2.
\end{aligned} \tag{70}$$

21

On the other hand, given $r$ and $\phi$, we may determine $s$: from (68) and (69),

$$s = \frac{(1 + \delta r)\cos\phi - (1 + \delta r_I)\cos\phi_I}{\delta \sin\theta}.$$

(71)

Note that the length parameter $s$ is the actual arc length along the characteristic. Physically the characteristic determines the ray of radiation. Finally, from (66), $i_\nu$ is given as a function of $s$ by

$$i_\nu = i_{\nu I}e^{-\tau(s)} - \sin\theta \int_0^s \cos\phi(s')\frac{\partial I_{b,\nu}}{\partial r}(r(s'))\, e^{-(\tau(s)-\tau(s'))}\, ds',$$

(72)

where

$$\tau(s) = \tau_0 \int_0^s \kappa_\nu(r(s'))ds'$$

(73)

is equivalent to the usual optical depth variable.

As mentioned before, in order to solve the first-order differential equations (63)–(66), we need the intensities at the boundaries. These are determined by the implicit boundary conditions (53) and (54) and the differential equations themselves. The determination of these intensities is discussed next.

We now identify the characteristic projections in the $(r, \phi)$– plane. Consider characteristics emanating from $\Gamma_0 = (r_I, \phi_I, i_{\nu I})$, where $r_I = 0$, $\phi_I \in [0, \pi/2]$. These correspond to 'rays' of radiation starting at the inside of the tube. As the rays of radiation are straight lines, it is clear that along such characteristics $\phi$ decreases as $r$ increases, if $\phi_I \in (0, \pi/2)$ (see Figure 9a). The characteristic projection is given from (68) by

$$(1 + \delta r)\sin\phi = \sin\phi_I,$$

(74)

and reaches $r = 1$ with $\sin\phi = \sin\phi_I/(1 + \delta)$. If $\phi_I = 0$, then $\phi \equiv 0$. The 'limiting ray' corresponding to $\phi_I = \pi/2$ is given by

$$(1 + \delta r)\sin\phi = 1,$$

(75)

and reaches $r = 1$ with $\phi = \gamma$, say, where $\gamma \in (0, \pi/2)$, $\sin\gamma = 1/(1 + \delta)$. Hence the characteristic projections emanating from $r = 0$ with $\phi_I \in [0, \pi/2]$ reach $r = 1$ with $\phi \in [0, \gamma]$.

22
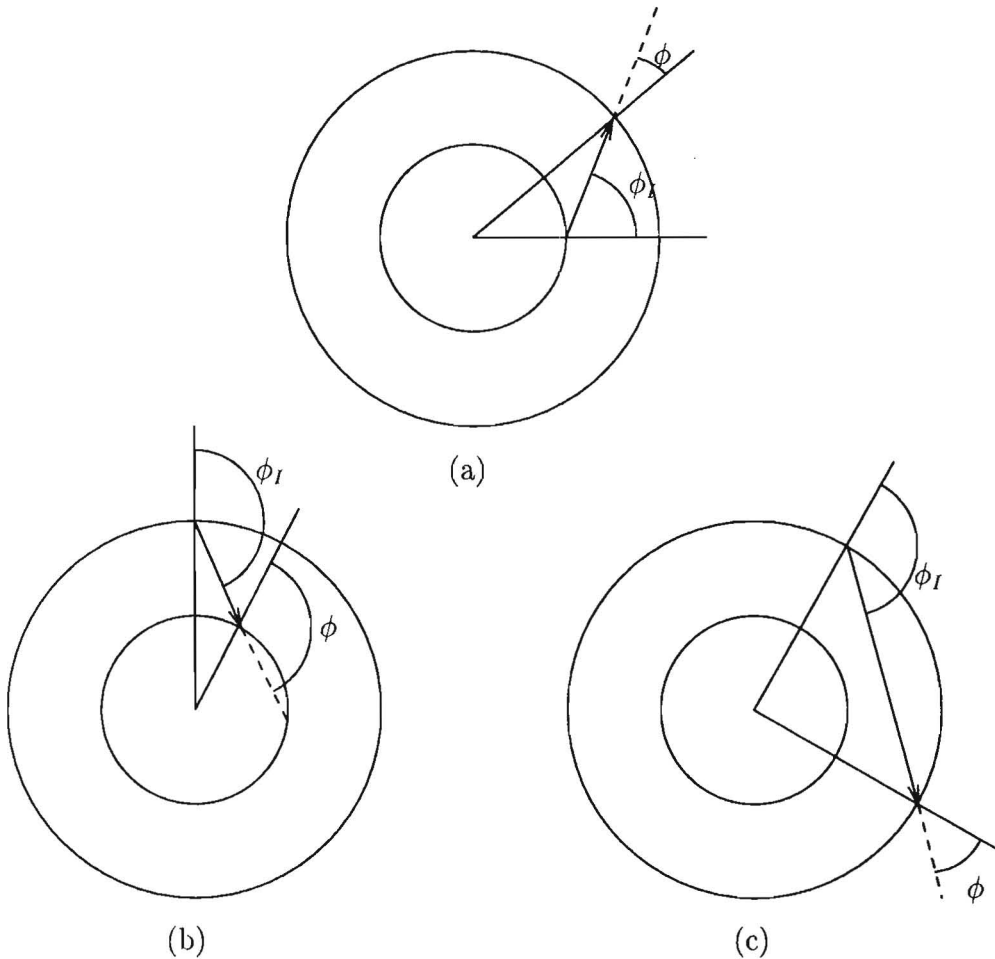
Figure 9: Rays of radiation through a tube (a) from inside to outside, (b) from outside to inside, (c) from outside to outside.

Now consider characteristics emanating from $\Gamma_1 = (r_I, \phi_I, i_{\nu I})$, where $r_I = 1$, $\phi_I \in (\pi/2, \pi]$. The physical picture is again clear: rays leaving the outside of the tube may either (i) reach the inside of the tube if $\phi_I$ is sufficiently large, or (ii) return directly to the outside (see Figure 9b and c). From the discussion above we see that case (i) occurs when $\phi_I \in [\pi - \gamma, \pi]$, and case (ii) if $\phi_I \in (\pi/2, \pi - \gamma)$.

The characteristic projection starting at $r_I = 1$ is given from (68) by

$$(1 + \delta r) \sin \phi = (1 + \delta) \sin \phi_I, \tag{76}$$

and both $\phi$ and $r$ decrease if $\phi_I \in (\pi - \gamma, \pi)$. If $\phi_I = \pi$, then $\phi \equiv \pi$. The limiting case
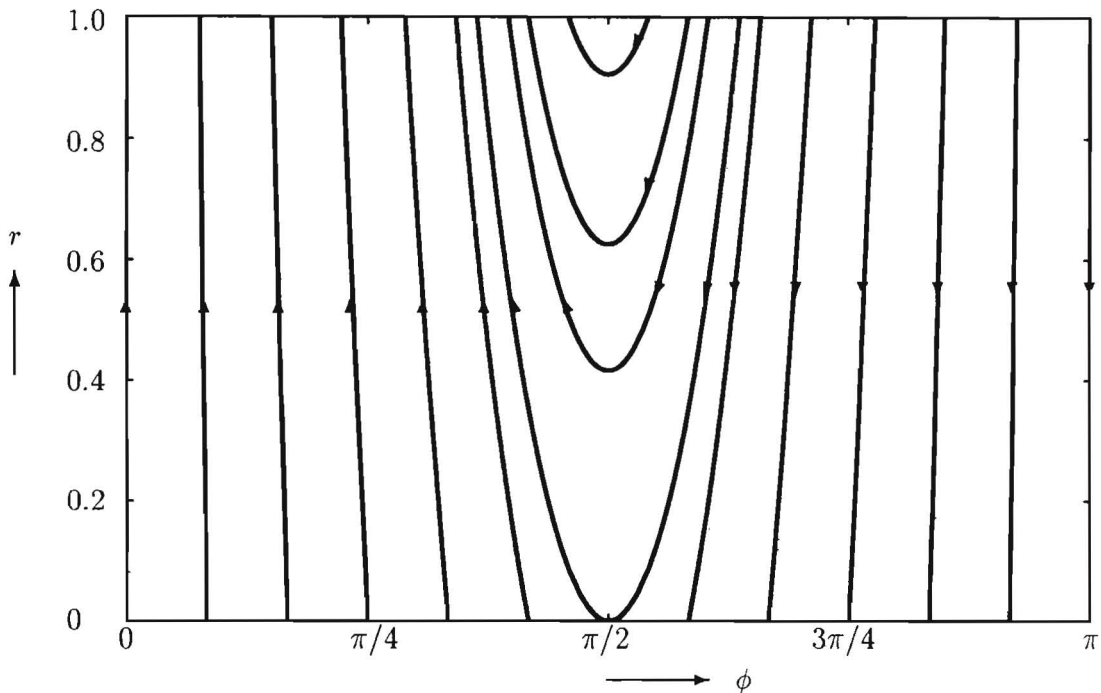
Figure 10: Characteristic projections, $\delta = 0.1$.

$\phi_I = \pi - \gamma$ is given by (75). If $\phi_I \in (\pi/2, \pi - \gamma)$, then $r$ and $\phi$ initially both decrease, before $r$ reaches a minimum value of $\sin\phi_I - (1 - \sin\phi_I)/\delta$ at $\phi = \pi/2$, whereafter $r$ increases again while $\phi$ continues to decrease. The characteristic projections are shown in Figure 10 for the case $\delta = 0.1$.

We may now calculate $i_\nu$ as an integral along characteristics emanating from $\Gamma_0$ and $\Gamma_1$. For a given $r \in [0, 1]$, let $\gamma_r \in [0, \pi/2]$ be such that $(r, \gamma_r)$ lies on the characteristic projection starting at $r = 0, \phi = \pi/2$, as shown in Figure 11. Hence

$$(1 + \delta r)\sin\gamma_r = 1. \tag{77}$$

Then if $\phi \in [0, \gamma_r]$, the characteristic through $(r, \phi, i_\nu)$ emanates from $\Gamma_0$, while if $\varphi \in (\gamma_r, \pi]$, the characteristic emanates from $\Gamma_1$.

Let $r \in [0, 1]$ and $\phi \in [0, \gamma_r]$. Then from (72),

$$i_\nu(r, \theta, \phi) = i_\nu(0, \theta, \phi_0^+)e^{-\tau(s^+)} - \int_0^{s^+} S(s')e^{-(\tau(s^+)-\tau(s'))}\,ds', \tag{78}$$

24

where

$$\sin \phi_0^+ = (1 + \delta r) \sin \phi, \quad \phi_0^+ \in [0, \pi/2], \tag{79}$$

$$s^+ = \frac{1}{\delta \sin \theta} \left( (1 + \delta r) \cos \phi - \cos \phi_0^+ \right), \tag{80}$$

from (71) and (74), and we introduce for convenience

$$S(s) = \sin \theta \, \cos \phi(s) \, \frac{\partial I_{b,\nu}}{\partial r}(r(s)). \tag{81}$$

If $\phi \in (\gamma_r, \pi]$, then

$$i_\nu(r, \theta, \phi) = i_\nu(1, \theta, \phi_1^-) e^{-\tau(s^-)} - \int_0^{s^-} S(s') e^{-(\tau(s^-) - \tau(s'))} \, ds', \tag{82}$$

where

$$(1 + \delta) \sin \phi_1^- = (1 + \delta r) \sin \phi, \quad \phi_1^- \in (\pi/2, \pi], \tag{83}$$

and

$$s^- = \frac{-1}{\delta \sin \theta} \left( (1 + \delta r) \cos \phi - (1 + \delta) \cos \phi_1^- \right). \tag{84}$$

We may now use the characteristic solutions (78) and (82), together with boundary conditions (53) and (54), to calculate the intensities explicitly. Firstly, we consider the characteristics which emanate from or reach $\Gamma_0$.

Let $r \in [0, 1]$ and $\phi \in [0, \gamma_r]$. As shown in Figure 11, the characteristic projection through $(r, \varphi)$ emanates from the point $(0, \phi_0^+)$, and reaches $r = 1$ at $\phi = \phi_1^+$, say, where

$$(1 + \delta) \sin \phi_1^+ = \sin \phi_0^+, \quad \phi_1^+ \in [0, \gamma]. \tag{85}$$

From (78),

$$i_\nu(1, \theta, \phi_1^+) = i_\nu(0, \theta, \phi_0^+) e^{-\tau(s_1^+)} - \int_0^{s_1^+} S(s') e^{-(\tau(s_1^+) - \tau(s'))} \, ds', \tag{86}$$

where $s_1^+$ is given by

$$s_1^+ = \frac{1}{\delta \sin \theta} \left( (1 + \delta) \cos \phi_1^+ - \cos \phi_0^+ \right). \tag{87}$$

Along the characteristic projection, $\phi$ and $r$ are given in terms of the parameter $s$ by

$$\begin{cases} \cot \phi = \cot \phi_0^+ + \dfrac{\delta \sin \theta}{\sin \phi_0^+} s & \text{if } \phi_0^+ \neq 0, \\[2mm] r = s \sin \theta & \text{otherwise,} \end{cases} \tag{88}$$

25
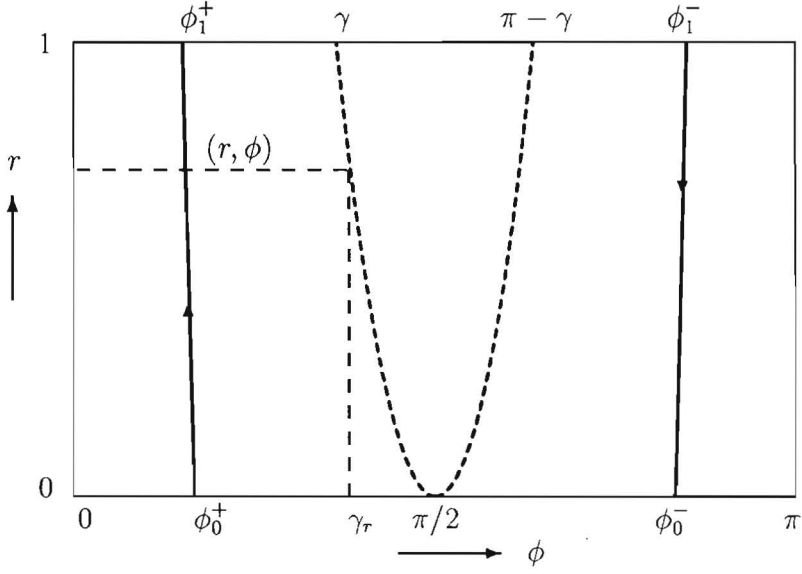
Figure 11: Characteristic projections between $r = 0$ and $r = 1$.

$$(1 + \delta r)\sin\phi = \sin\phi_0^+, \tag{89}$$

from (68) and (69). Now let $\phi_1^- = \pi - \phi_1^+ \in [\pi - \gamma, \pi]$. By symmetry, the characteristic projection emanating from $(1, \phi_1^-)$ reaches $r = 0$ at $\phi = \phi_0^- = \pi - \phi_0^+$. From (82),

$$i_\nu(0, \theta, \phi_0^-) = i_\nu(1, \theta, \phi_1^-)e^{-\tau(s_0^-)} - \int_0^{s_0^-} S(s')e^{-(\tau(s_0^-) - \tau(s'))}\, ds', \tag{90}$$

where $s_0^-$ is given by

$$s_0^- = \frac{1}{\delta \sin\theta}(\cos\phi_0^- - (1 + \delta)\cos\phi_1^-) = s_1^+. \tag{91}$$

Along the characteristic projection,

$$\begin{cases} \cot\phi = \cot\phi_1^- + \dfrac{\delta \sin\theta}{(1 + \delta)\sin\phi_1^-}\, s & \text{if } \phi_1^- \neq \pi, \\[2ex] r = 1 - s\sin\theta & \text{otherwise,} \end{cases} \tag{92}$$

$$(1 + \delta r)\sin\phi = (1 + \delta)\sin\phi_1^-. \tag{93}$$

From boundary conditions (53) and (54),

$$i_\nu(0, \theta, \phi_0^+) = i_\nu(0, \theta, \phi_0^-), \tag{94}$$

26

$$i_\nu(1,\theta,\phi_1^-) = \mathcal{R}(\omega_r)i_\nu(1,\theta,\phi_1^+) + (1 - \mathcal{R}(\omega_r))(\,I_\nu^o(1,\theta_t,\phi_t) - I_{b,\nu}(1)\,), \tag{95}$$

Hence from (86), (90), (94) and (95), the intensities $i_\nu(0,\theta,\phi_0^\pm)$ and $i_\nu(1,\theta,\phi_1^\pm)$ may be calculated. We obtain the following expressions. If $\phi \in [0,\gamma_r]$, then with (78) we have

$$
\begin{aligned}
i_\nu(r,\theta,\phi) \;=\; & \frac{1}{1 - \mathcal{R}(\omega_r)e^{-2\tau(s_1^+)}}\Bigg\{ -\mathcal{R}(\omega_r)\int_{s^+}^{s_1^+} S(s')e^{-\tau(s_1^+)+\tau(s')-\tau(s_0^-)-\tau(s^+)}\,ds' \\
& -\int_0^{s_0^-} S(s')e^{-\tau(s_0^-)+\tau(s')-\tau(s^+)}\,ds' - \int_0^{s^+} S(s')e^{-\tau(s^+)+\tau(s')}\,ds' \\
& + (1 - \mathcal{R}(\omega_r))(I_\nu^o(1,\theta_t,\phi_t) - I_{b,\nu}(1))e^{-\tau(s_0^-)-\tau(s^+)}\Bigg\},
\end{aligned}
\tag{96}
$$

and if $\phi \in [\pi - \gamma_r, \pi]$, then with (82) we have

$$
\begin{aligned}
i_\nu(r,\theta,\phi) \;=\; & \frac{1}{1 - \mathcal{R}(\omega_r)e^{-2\tau(s_0^-)}}\Bigg\{ -\mathcal{R}(\omega_r)\int_{s^-}^{s_0^-} S(s')e^{-\tau(s_0^-)+\tau(s')-\tau(s_1^+)-\tau(s^-)}\,ds' \\
& -\mathcal{R}(\omega_r)\int_0^{s_1^+} S(s')e^{-\tau(s_1^+)+\tau(s')-\tau(s^-)}\,ds' - \int_0^{s^-} S(s')e^{-\tau(s^-)+\tau(s')}\,ds' \\
& + (1 - \mathcal{R}(\omega_r))(I_\nu^o(1,\theta_t,\phi_t) - I_{b,\nu}(1))e^{-\tau(s^-)}\Bigg\}.
\end{aligned}
\tag{97}
$$

Secondly, we consider characteristics that emanate from and return to $\Gamma_1$. Let $r \in [0,1]$ and $\phi \in (\gamma_r, \pi - \gamma_r)$. The characteristic projection through $(r,\phi)$ emanates from the point $(1,\phi_1^-)$, where $\phi_1^-$ is given by (83), and returns to $r = 1$ at $\phi = \phi_1^+ = \pi - \phi_1^-$. From (82)

$$i_\nu(1,\theta,\phi_1^+) = i_\nu(1,\theta,\phi_1^-)e^{-\tau(s_1^-)} - \int_0^{s_1^-} S(s')e^{-(\tau(s_1^-)-\tau(s'))}\,ds', \tag{98}$$

where from (84),

$$
\begin{aligned}
s_1^- \;=\; & \frac{1}{\delta \sin\theta}(\,(1+\delta)\cos\phi_1^+ - (1+\delta)\cos\phi_1^-) \\
\;=\; & \frac{2(1+\delta)}{\delta \sin\theta}\cos\phi_1^+.
\end{aligned}
\tag{99}
$$

The intensity $i_\nu(1,\theta,\phi_1^+)$ may now be found from (98) and boundary condition (54), and the general intensity $i_\nu(r,\theta,\phi)$ is then given by (82). We obtain

$$
\begin{aligned}
i_\nu(r,\theta,\phi) = \frac{1}{1 - \mathcal{R}(\omega_r)e^{-\tau(s_1^-)}}\Bigg\{ & -\mathcal{R}(\omega_r)\int_{s^-}^{s_1^-} S(s')e^{-\tau(s_1^-)+\tau(s')-\tau(s^-)}\,ds' \\
& -\int_0^{s^-} S(s')e^{-\tau(s^-)+\tau(s')}\,ds' + (1 - \mathcal{R}(\omega_r))(I_\nu^o(1,\theta_t,\phi_t) - I_{b,\nu}(1))e^{-\tau(s^-)}\Bigg\}.
\end{aligned}
\tag{100}
$$

# 5 Intensity due to a constant temperature

Suppose the temperature is constant throughout the tube, and no radiation is incident from outside. Then, using (58)–(62), the intensity $i_{\nu R}$ satisfies

$$\sin\theta\left(\cos\phi\,\frac{\partial i_{\nu R}}{\partial r} - \sin\phi\,\frac{\delta}{1+\delta r}\,\frac{\partial i_{\nu R}}{\partial\phi}\right) + \tau_0\kappa_\nu i_{\nu R} = 0, \tag{101}$$

$$i_{\nu R}(0,\theta,\phi) = i_{\nu R}(0,\theta,\pi-\phi), \quad 0 < \theta < \pi,\ 0 \le \phi \le \pi/2, \tag{102}$$

$$i_{\nu R}(1,\theta,\phi) = \mathcal{R}(\omega_r)i_{\nu R}(1,\theta,\pi-\phi) - (1-\mathcal{R}(\omega_r))I_{b,\nu}|_{T=1}.$$
$$0 < \theta < \pi,\ \pi/2 < \phi \le \pi, \tag{103}$$

and the previous expressions for the intensities simplify considerably. Setting $S$ and $I_\nu^o$ to zero yields the following formulae. If $\phi \in [0,\gamma_r]$, then

$$i_{\nu R}(r,\theta,\phi) = \frac{-(1-\mathcal{R}(\omega_r))I_{b,\nu}|_{T=1}e^{-\tau(s_0^-)-\tau(s^+)}}{1-\mathcal{R}(\omega_r)e^{-2\tau(s_1^+)}}, \tag{104}$$

if $\phi \in (\gamma_r, \pi-\gamma_r)$,

$$i_{\nu R}(r,\theta,\phi) = \frac{-(1-\mathcal{R}(\omega_r))I_{b,\nu}|_{T=1}e^{-\tau(s^-)}}{1-\mathcal{R}(\omega_r)e^{-\tau(s_1^-)}}, \tag{105}$$

and if $\phi \in [\pi-\gamma_r, \pi]$,

$$i_{\nu R}(r,\theta,\phi) = \frac{-(1-\mathcal{R}(\omega_r))I_{b,\nu}|_{T=1}e^{-\tau(s^-)}}{1-\mathcal{R}(\omega_r)e^{-2\tau(s_0^-)}}. \tag{106}$$

## 5.1 Total internal reflection

We now consider the extent of total internal reflection at the outside of the tube, which occurs for sufficiently large angles of incidence. Specifically,

$$\mathcal{R}(\omega_r) = 1 \text{ for } \sin\omega_r > 1/n,\ \omega_r \in [0,\pi/2). \tag{107}$$

Hence from (27), $\mathcal{R}(\omega_r) = 1$ when

$$-\sqrt{1-\frac{1}{n^2}} < \sin\theta\cos\phi < 0, \tag{108}$$
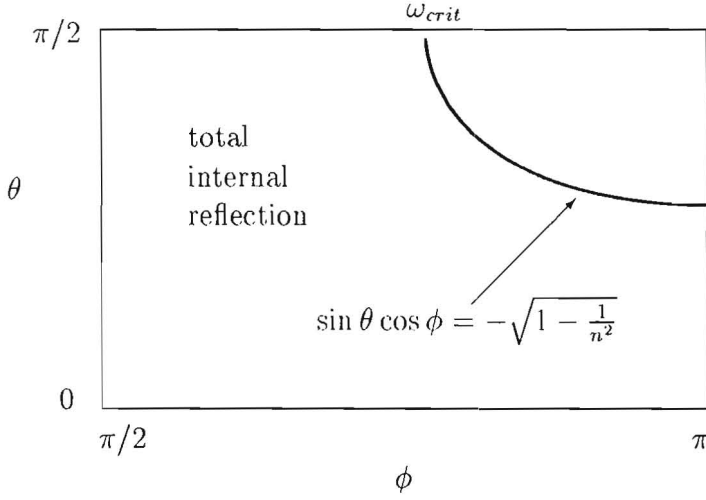
Figure 12: Limits of total internal reflection.

and in this case (103) reduces to

$$i_{\nu R}(1, \theta, \phi) = i_{\nu R}(1, \theta, \pi - \phi). \tag{109}$$

Figure 12 shows the range of $\theta$ and $\phi$ corresponding to total internal reflection. If $\theta = \pi/2$, this occurs when $\phi \in (\pi/2, \omega_{crit})$, where $\sin \omega_{crit} = 1/n$. For $\theta < \pi/2$, total internal reflection occurs when $\phi \in (\pi/2, \phi_{crit})$ say, where $\phi_{crit} \in (\omega_{crit}, \pi]$. Taking the refractive index as $n = 1.5$, we find $\omega_{crit} = 2.41$ rad.

If total internal reflection occurs, (103) reduces to the form (109). In this case, we obtain the trivial solution $i_{\nu R}(r, \theta, \phi) = 0$, as given by (104)–(106). This holds along the characteristic projections $(r, \phi)$ emanating from $(1, \phi_1^-)$, where

$$-\sqrt{1 - \frac{1}{n^2}} < \sin \theta \cos \phi_1^- < 0, \tag{110}$$

that is, say,

$$\phi_1^- < \phi_{crit}. \tag{111}$$

Hence from (83),

$$i_{\nu R}(r, \theta, \phi) = 0 \ \ \text{if} \ \ \frac{(1 + \delta r) \sin \phi}{1 + \delta} > \sin \phi_{crit}. \tag{112}$$

29

In particular, if $\delta = 0.1$ and $n = 1.5$, then $\omega_{crit} > \pi - \gamma \approx 2.00$ rad, and the rays emanating from and returning directly to $r = 1$ make no contribution to the radiative heat flux, assuming that the temperature remains constant. This occurs in general if $\pi - \gamma < \omega_{crit}$, and since $\sin \gamma = 1/(1 + \delta)$, $\sin \omega_{crit} = 1/n$, provided $n > 1 + \delta$.

## 5.2 Results

The intensities given by formulae (104)-(106) were calculated using the 'Mathematica' package [23]. The following data were used:

$$
\begin{array}{ll}
\text{tube inside radius} & r_i = 10\,\text{mm}, \\
\text{outside radius} & r_o = 11\,\text{mm}, \\
\text{refractive index} & n = 1.5, \\
\text{temperature} & T = 1000\,\text{K}.
\end{array}
$$

The value of the absorption coefficient $\kappa_\nu$ was taken from a sequence of experimental values measured over a range of frequencies [19], which are shown in Figure 13. The calculated dimensional intensities are shown in Figures 14, 15 and 16.

Figure 14 shows the intensities at four particular frequencies, at the inside of the tube $r = r_i$, for the range of directions $\theta = \pi/2$, $0 \leq \phi \leq \pi$. The profiles are symmetric about $\phi = \pi/2$ because the boundary condition at the inside radius is equivalent to total internal reflection. Rays that undergo total internal reflection at the outside of the tube reach and emanate from the inside with angles in the range $\phi \in (\tilde{\phi}, \pi - \tilde{\phi})$, where $\sin \tilde{\phi} = (1 + \delta) \sin \omega_{crit} = (1 + \delta)/n$. We find $\tilde{\phi} \approx 0.82$ rad. In this range, the intensities are equal to the blackbody values given by (5). For values of $\phi$ outside this range, the intensities decrease, depending on the size of the absorption coefficient $\kappa_\nu$. At low frequencies, $\kappa_\nu$ is large, and the intensities are close to the blackbody values; at higher frequencies, $\kappa_\nu$ is small, and the intensities are a small fraction of the blackbody values. The first derivative with respect to $\phi$ exhibits a discontinuity, which is caused by the reflectance function being a continuous but not a smooth function of the angle of incidence $\omega_r$. Note that the blackbody intensity depends on the frequency, as already has been shown in Figure 1.
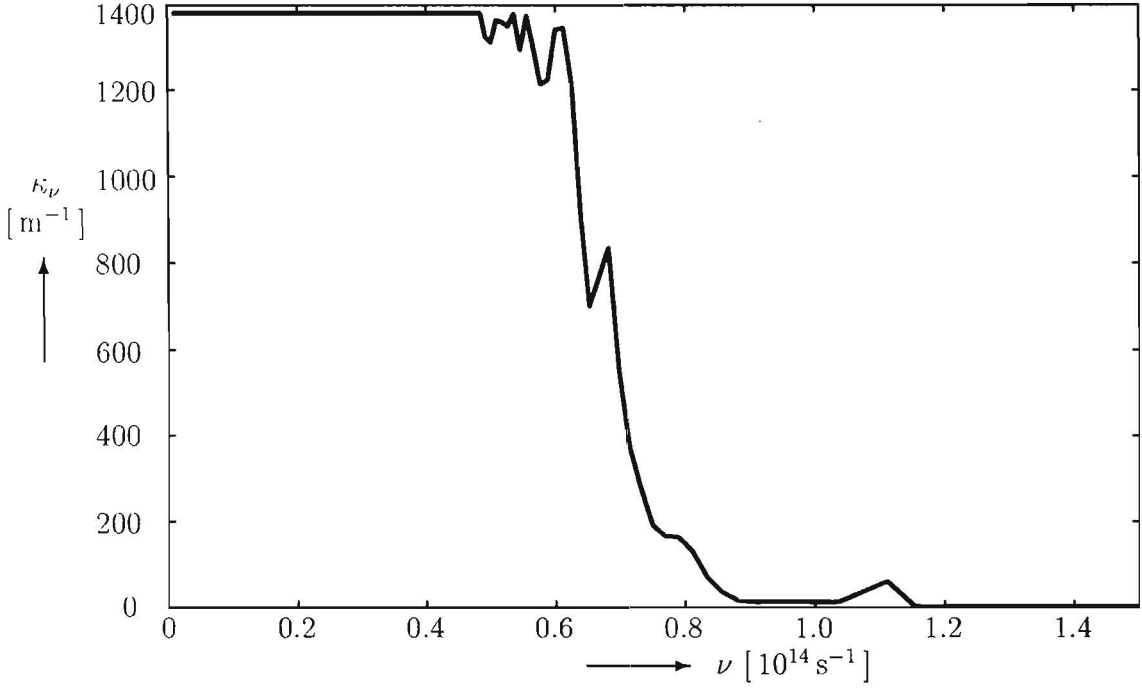
30

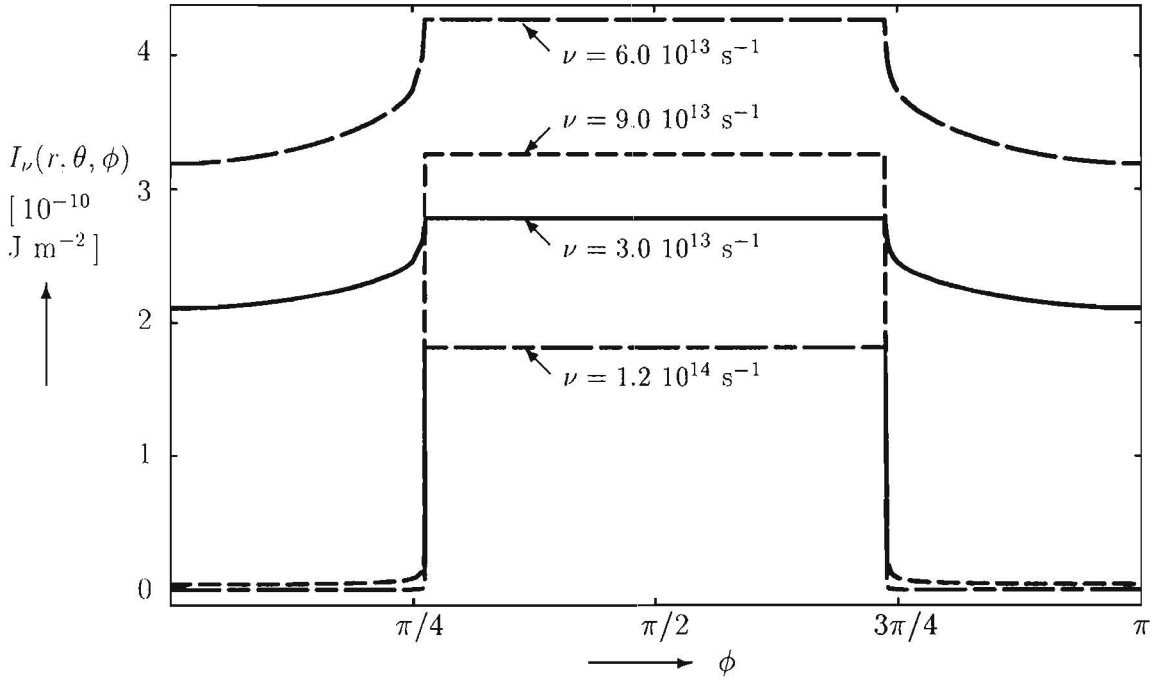Figure 13: Absorption coefficient values.



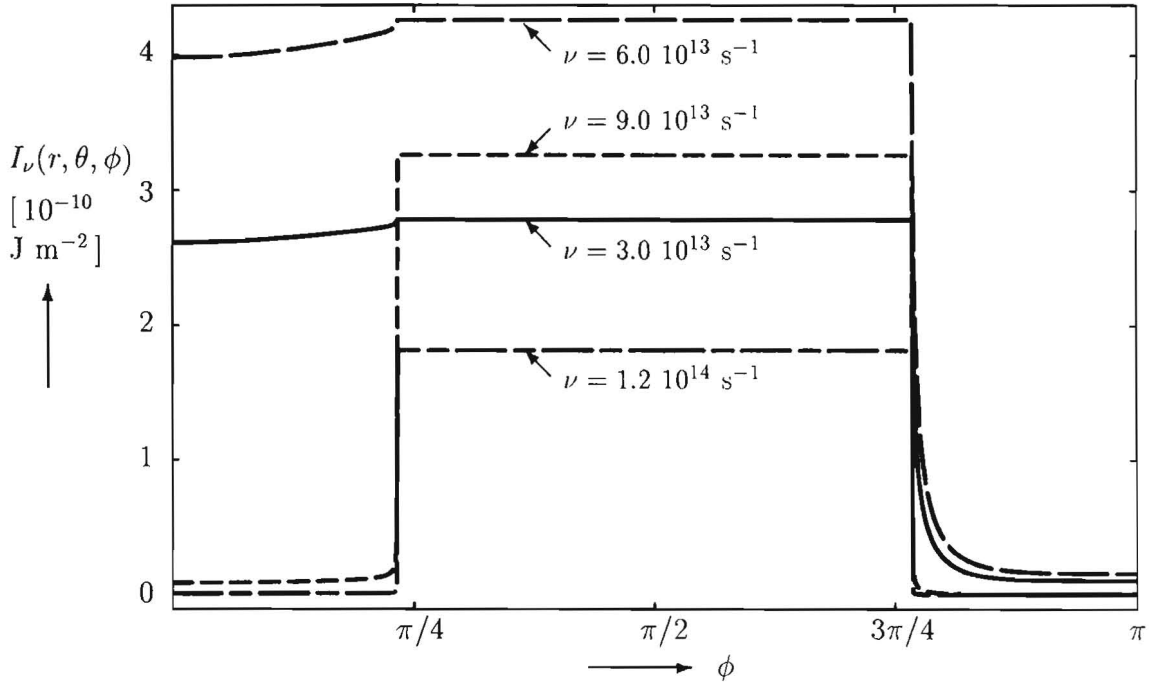Figure 14: Spectral intensities, $r = r_i$, $\delta = 0.1$, $\theta = \pi/2$.

31

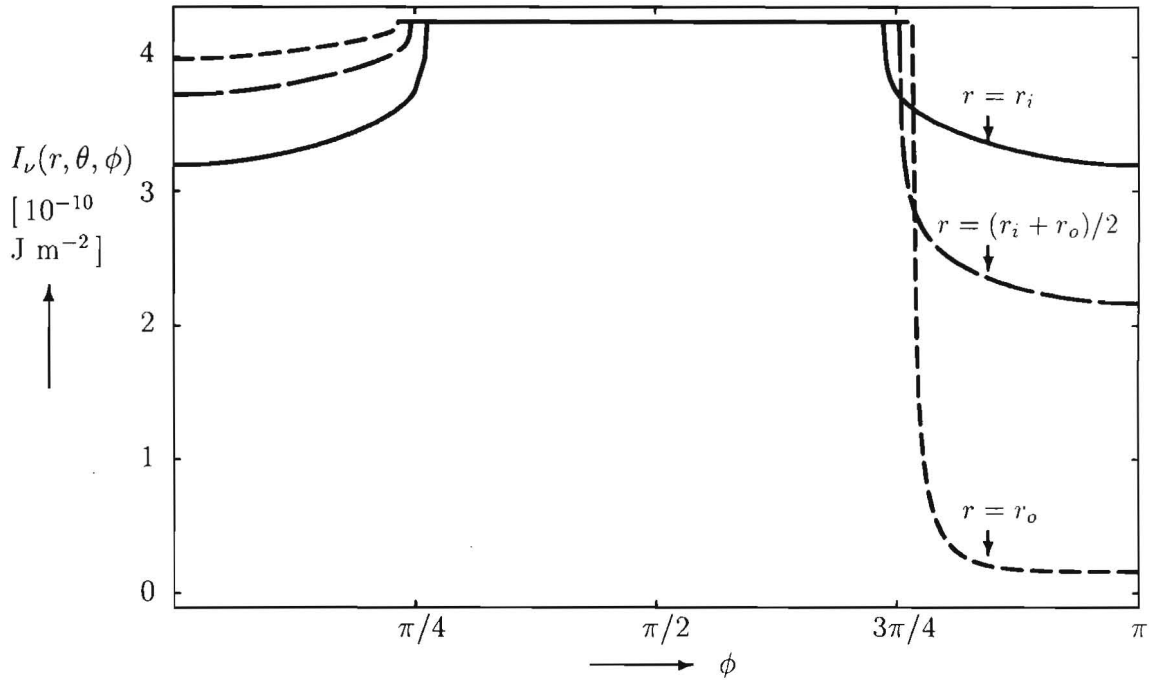Figure 15: Spectral intensities, $r = r_o$, $\delta = 0.1$, $\theta = \pi/2$.



Figure 16: Spectral intensities, $\nu = 6 \times 10^{13}$ s$^{-1}$, $\delta = 0.1$, $\theta = \pi/2$.

32

The corresponding intensities at the outer radius, $r = r_o$, are shown in Figure 15. The intensities are equal to the blackbody values for $\phi \in (\pi - \omega_{crit}, \omega_{crit})$, where $\omega_{crit} \approx 2.41$ rad. The low values of intensity for $\phi > \omega_{crit}$ correspond to low values of the reflectance function for $\phi < \pi - \omega_{crit}$. The intensities for $\phi > \omega_{crit}$ are a fraction $\mathcal{R}(\pi - \phi)$ of those for $\phi^* = \pi - \phi < \pi - \omega_{crit}$. For instance the intensity for $\phi = \pi$ is only $\mathcal{R}(0) \approx 0.04$ of that for $\phi = 0$. As mentioned before, the intensity is closer to that of a blackbody for smaller frequencies, corresponding to larger absorption coefficients. This is shown clearly for azimuthal angles $\phi < \pi/2$.

Figure 16 shows the intensities at the frequency $\nu = 6 \times 10^{13}\,\mathrm{s}^{-1}$ at $r = r_i$, $(r_i + r_o)/2$, and $r_o$. The range of angles where the intensity equals the blackbody value broadens from the inside to the outside of the tube. Ouside this range, the intensity increases towards the blackbody value from $r = r_o$, $\phi > \omega_{crit}$ via $r = r_i$ to $r = r_o$, $\phi < \pi - \omega_{crit}$. Along a characteristic, the intensity increases exponertially to the blackbody value, as follows from (104)–(106), and hence the intensity midway between the inner and outer boundary is closer to that at the outer boundary for angles $\phi \in [0, \pi/2)$ and closer to that at the inner boundary for angles $\phi \in (\pi/2, \pi]$.

The divergence of the radiative heat flux was calculated and is shown in Figure 17. It increases from the inside of the tube towards the outside. This indicates that the rate of cooling due to heat radiation is greater at the outer side than at the inner side. The variation of the divergence of the heat flux with the radius $r$ is very small at the inner side. For an infinite slab with total reflection of radiation at one boundary, the derivative of $\boldsymbol{\nabla} \cdot \boldsymbol{q_r}$ in the thickness direction is zero at that boundary. For a tube $\mathrm{d}(\boldsymbol{\nabla} \cdot \boldsymbol{q_r})/\mathrm{d}r$ at $r = r_i$ does not vanish (see Appendix), however it may become very small depending on $\delta = \Delta r/r_i$. Note that this behaviour was obtained for an isothermal tube. If a temperature gradient along the radius of the tube exists, the behaviour can be different from that presently found.

Calculations were also done for an alternative tube geometry, with inside radius $r_i = 1\,\mathrm{mm}$ and outside radius $r_o = 2\,\mathrm{mm}$ (hence the thickness $\Delta r = 1\,\mathrm{mm}$ as before). In this case, rays entering the volume from the outside of the tube return directly to the outside
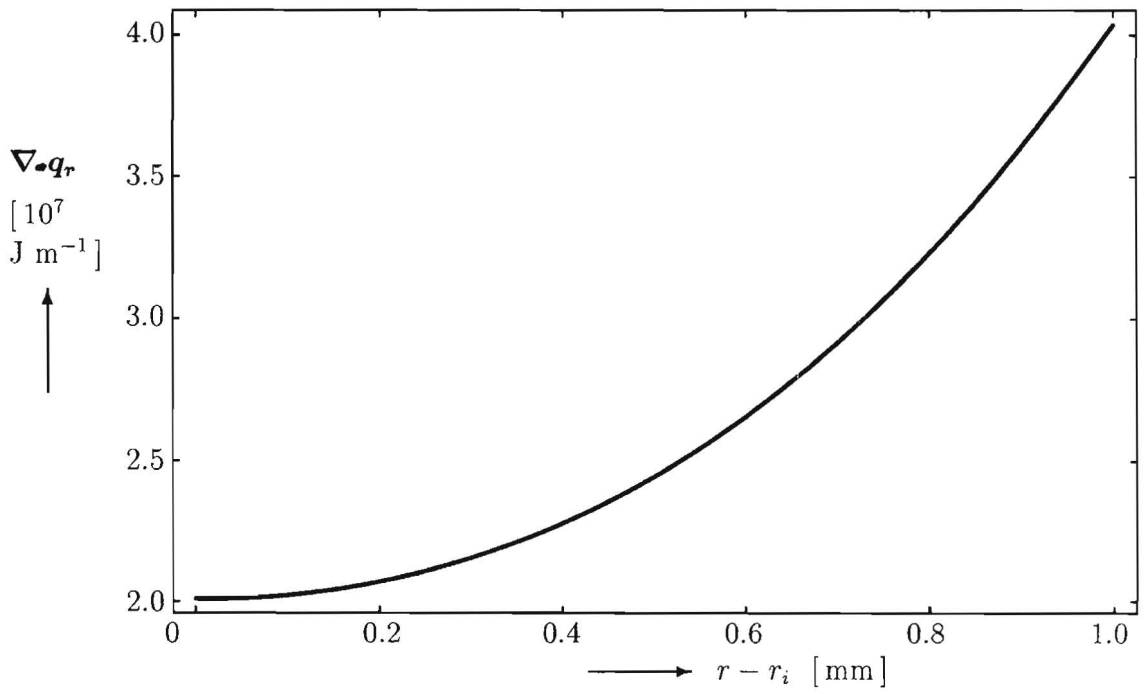
33

Figure 17: Divergence of radiative flux, $T = 1000\,\mathrm{K}$, $\delta = 0.1$.
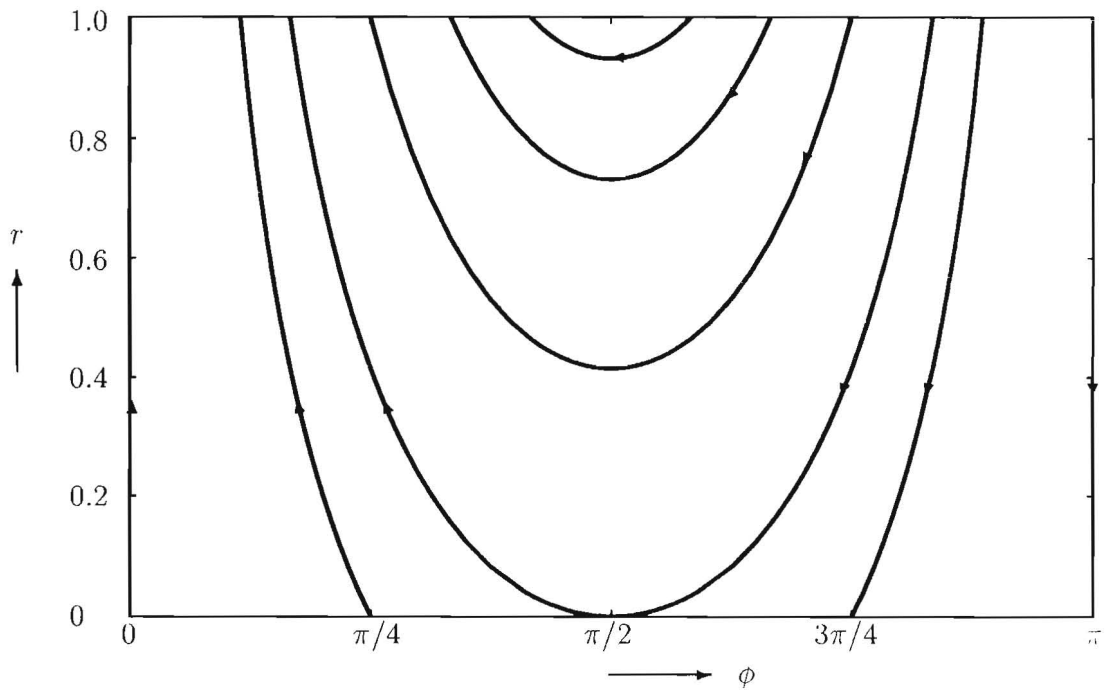


Figure 18: Characteristic projections, $\delta = 1$.

34

over a greater range of initial angle than for the previous geometry. Since $\delta = \Delta r/r_i = 1$, and the limiting angle $\gamma = \sin^{-1}(1/(1+\delta)) = \pi/6$, the characteristic projections emanating from $r = r_o, \varphi \in [\pi/2, 5\pi/6]$ return directly to $r = r_o$ as shown in Figure 18.
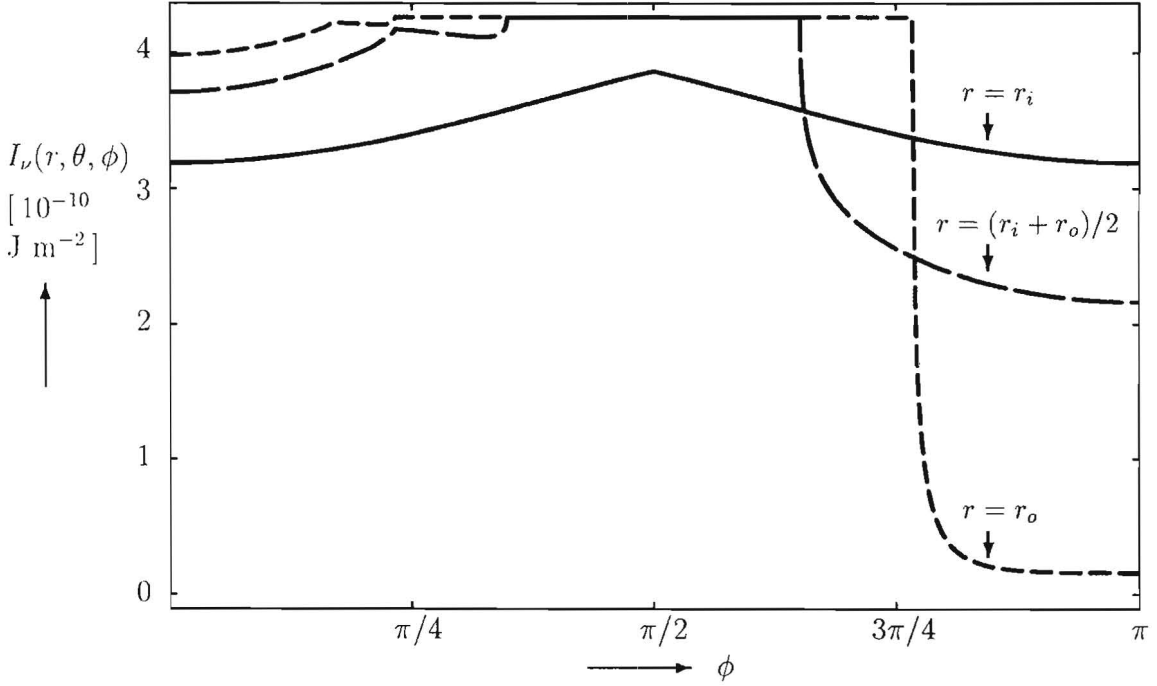


Figure 19: Spectral intensities, $\nu = 6 \times 10^{13}$ s$^{-1}$, $\delta = 1.0$, $\theta = \pi/2$.

Figure 19 shows the intensities at the frequency $\nu = 6 \times 10^{13}$ s$^{-1}$, at $r = r_i$, $(r_i + r_o)/2$, and $r_o$, over the range of directions $\theta = \pi/2$, $\phi \in [0, \pi]$ (c.f. Figure 16). Since $\gamma < \pi - \omega_{crit}$, the rays arriving at and leaving $r = r_i$ do not experience total internal reflection at $r = r_o$, hence the intensity at $r = r_i$ remains below the blackbody value. For $r > r_o/n$ the intensity reaches the blackbody value over a range of $\phi$ which broadens with increasing $r$. The intensity has a discontinuity in the first derivative (with respect to $\phi$) at $\phi = \gamma_r$. This is due to the variation in length along the characteristic projection(s) through $(r, \phi)$ from $r = 1$. Holding $r$ constant and varying $\phi$ from 0 to $\pi/2$, we see that $s_0^- + s^+$ increases, until $\phi = \gamma_r$ where $\partial(s_0^- + s^+)/\partial\phi$ becomes infinite. For $\phi > \gamma_r$, $s^-$ is a decreasing function of $\phi$ (and reaches 0 at $\phi = \pi/2$). Since the intensity is inversely proportional to the exponential of the path length, the curve is not smooth at $\phi = \gamma_r$. Figure 20 shows

35
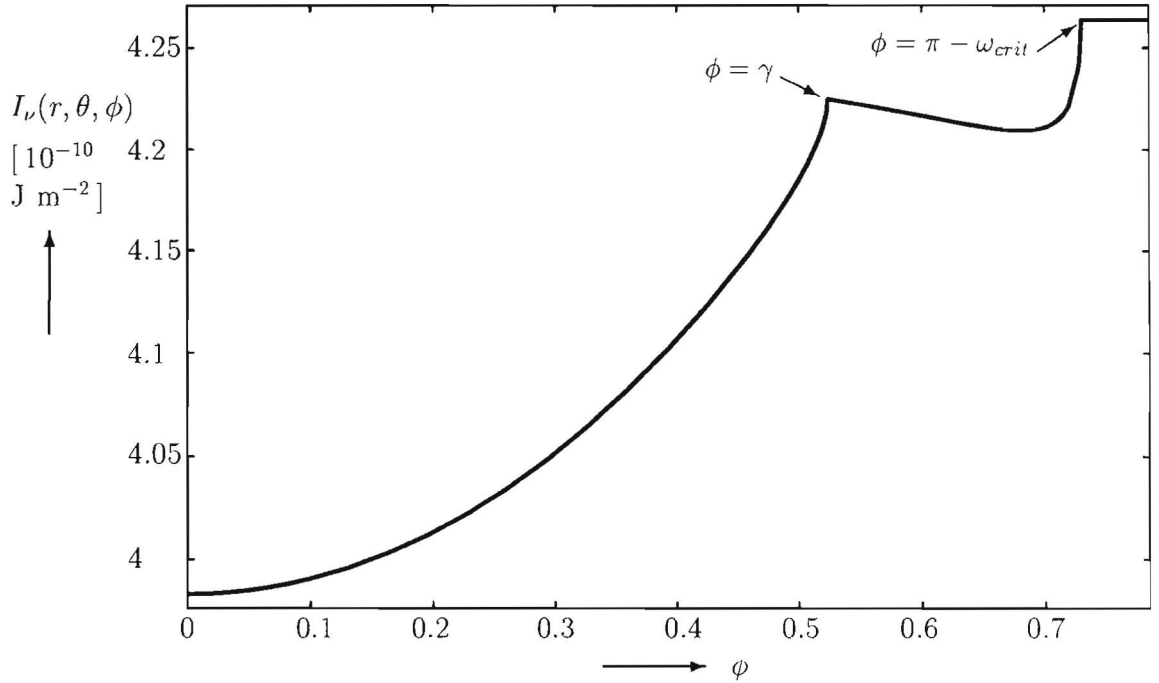
Figure 20: Spectral intensity. $r = r_o$, $\nu = 6 \times 10^{13}$ s$^{-1}$, $\delta = 1.0$, $\theta = \pi/2$.

the situation at $r = r_o$ in more detail. There is a corresponding discontinuity in the first derivative of the intensity at $\phi = \pi - \gamma_r$, however it is not visible in Figure 19.

# 6  Conclusions

A mathematical model is proposed to determine the radiative heat transport in an infinitely long hollow circular cylindrical tube. Neglecting scattering and assuming optically smooth boundaries, an analytical expression for the spectrally dependent radiative intensity is derived from the radiative transport equation. If the temperature in the tube is constant, for those directions of radiation where total reflection at the outer boundary occurs, the intensity equals that of a blackbody. These intensities do not contribute to the heat transport in the tube. The divergence of the heat flux, indicating the cooling rate in the tube, is greatest near the outer boundary and smallest near the inner boundary. If the ratio of the outer and inner radius is larger than the refractive index, the intensity has a second discontinuity in the first derivative with respect to the azimuthal angle.

36

# References

[1] J.H. Simmons, R.K. Mohr and C.J. Montrose, *Non-Newtonian Viscous Flow in Glass*, J. Appl. Physics, vol 53, pp.4075-4080, 1982.

[2] C.T. Moynihan, *Correlation between the Width of the Glass Transition Region and the Temperature Dependence of the Viscosity of High-Tg Glasses*, J. Am. Ceram. Soc., vol 76, pp.1081-1087, 1993.

[3] H. Hessenkemper and R. Brückner, *Load-Dependent Flow Behaviour of Silicate Glass Melts*, Glastech. Ber., vol 61, pp.312-320, 1988.

[4] P. Manns and R. Brückner, *Non-Newtonian Flow Behaviour of a Soda-Lime Silicate Glass at High Deformation Rates*, Glastech. Ber., vol 61, pp.46-56, 1988.

[5] S.P. Timoshenko and J.N. Goodier, *Theory of Elasticity*, McGraw-Hill, Singapore, 1982.

[6] J. Lubliner, *Plasticity Theory*, MacMillan, New York, 1990.

[7] H.S. Carslaw and J.C. Jaeger, *Conduction of Heat in Solids*, Oxford Univ. Press, Oxford, 1959.

[8] E.M. Sparrow and R.D. Cess, *Radiative Heat Transfer*, Brooks/Cole, Belmont, 1966.

[9] R. Siegel and J.R. Howell, *Thermal Radiation Heat Transfer*, McGraw-Hill, New York, 1992. Wens ·FMk 92

[10] M.F. Modest, *Radiative Heat Transfer*, McGraw-Hill, Singapore, 1993.

[11] J.O. Isard, *The Emission of Thermal Radiation from Hot Glass. Part I. Emissivity of Sheets, Spheres, Cylinders, and Tubes*, Glass Techn., vol. 27, pp.24-36, 1986.

[12] D.K. Pandey and A.C. Cogley, *An Integral Solution Procedure for radiative Transfer in Concentric Cylindrical Media*, ASME paper, no. 83-WA/HT-78, 1983.

[13] S.K. Loyalka, *Radiative Heat Transfer between Parallel Plates and Concentric Cylinders*, Int. J. Heat Mass Transfer, vol. 12, pp.1513-1517, 1969.

[14] F.H. Azad and M.F. Modest, *Evaluation of the Radiative Heat Flux in Absorbing, Emitting and Linear-Anisotropical Scattering Cylindrical Media*, Trans. of the ASME, J. of Heat Transfer, vol 103, pp.350-356, 1981.

[15] M.A. Heaslet and R.F. Warming, *Theoretical Predictions of Radiative Transfer in a Homogeneous Cylindrical Medium*, J. Quant. Spectrosc. Radiat. Transfer, vol 6, pp.751-774, 1966.

[16] M.P. Mengüç and R. Viskanta, *Radiative Transfer in Axisymmetric, Finite Cylindrical Enclosures*, Trans. of the ASME, J. of Heat Transfer, vol 108, pp.271-276, 1986.

[17] V. Le Dez and M. Lallemand, *Thermal Emission of Axisymmetric bodies of Semi-Transparant Materials*, J. Quant. Spectrosc. Radiat. Transfer, vol 47, pp.345-352, 1992.

[18] V. Le Dez, J.C. Huclin and M. Lallemand, *Directional Emissivities of Semi-Transparent Media. Cases of Dihedron, Infinite Cylinder and Cone*, Glastech. Ber., vol 66, pp.127-137, 1993.

[19] L. van de Tempel and T. Brandsma, *Personal communications*, Philips Research Laboratories, Eindhoven, the Netherlands , 1994.

[20] L.V. Prikhodko and Kh.S. Bagdasarov, *Temperature Dependence of IR Absorption in Fused Quartz at High Temperatures*, Opt. Spektrosk., vol 34, pp.1210-1211, 1973.

[21] O.V. Lukin and A.N. Magunov, *Temperature Measurement of Glass and Quartz Plates by Laser Interferometry*, Opt. Spektrosk., vol 74, pp.630-633, 1993.

[22] J. Schroeder, Light Scattering of Glass, In *Treatise on Materials Science and Technology, volume 12. Glass I: Interaction with Electromagnetic Radiation* (p.199), eds. M. Tomozawa, & R.H. Doremus, Academic Press, New York, 1977.

[23] S. Wolfram, *Mathematica: A System for doing Mathematics by Computer*, Addison-Wesley, Redwood City, 1988.

# A  Derivative of the divergence of the radiative heat flux at the inside of the tube

*Theorem.*

If the temperature in an infinitely long hollow axisymmetrical tube is constant, the absorption coefficient $\kappa_\nu$ does not depend on the radial position and the tube is such that $n_\nu > r_o/r_i$ for all $\nu \in [0, \infty)$, then the derivative of the divergence of the radiative heat flux with respect to the radius is non-zero (negative) at the inner boundary. If $r_i \to \infty$, while the thickness $\Delta r$ remains finite, hence if $\delta \to 0$ and $\Delta r$ remains finite, the heat transfer problem approaches that of an infinite slab, with total reflection of radiation at one boundary. In that case the derivative of the divergence of the radiative heat flux in the thickness direction is zero at the side with total radiation reflection.

*Proof.*

The divergence of the radiative heat flux is given by (39)

$$\boldsymbol{\nabla} \cdot \boldsymbol{q_r} = \int_0^\infty \kappa_\nu \int_{\Omega = 4\pi} ( I_{b,\nu} - I_\nu ) \, d\Omega \, d\nu \;. \tag{113}$$

Expressed into the dimensionless quantities this becomes

$$\boldsymbol{\nabla} \cdot \boldsymbol{q_r} = -\overline{\kappa I_{b,\nu}} \int_0^\infty \kappa_\nu^* \int_{\Omega = 4\pi} i_\nu^* \, d\Omega \, d\nu \;. \tag{114}$$

In the case of a constant temperature in the tube and no external radiation sources this reduces to, in view of (57) and (110),

$$\boldsymbol{\nabla} \cdot \boldsymbol{q_r} = -4\overline{\kappa I_{b,\nu}} \int_0^\infty \kappa_\nu^* \int_{\sin\theta\cos\phi_1^+ > \sqrt{1 - 1/n_\nu^2}} ( i_{\nu R}^*(r^*, \theta, \phi) + i_{\nu R}^*(r^*, \theta, \pi - \phi) ) \, d\Omega \, d\nu \;, \tag{115}$$

where at a position $r^*$ the range of $\theta, \phi \in [0, \pi/2]$ values are such that

$$\sin\theta \sqrt{1 - \left( \frac{1 + \delta r}{1 + \delta} \sin\phi \right)^2} = \sin\theta \sqrt{1 - \sin^2 \phi_1^+} > \sqrt{1 - \frac{1}{n_\nu^2}} \;. \tag{116}$$

Note that $i_{\nu R}^* = 0$ in those directions for which $\sin\theta\cos\phi_1^+ \leq \sqrt{1 - (1/n_\nu^2)}$, hence the intensities $I_\nu$ equal those $I_{b,\nu}$ of a blackbody. The integrand can be differentiated with

respect to $r$ if $\omega_{crit} > \pi - \gamma$, hence with (11) and (77) if $n_\nu > r_o/r_i$ for all $\nu \in [0, \infty)$, yielding

$$\frac{\mathrm{d}\boldsymbol{\nabla} \cdot \boldsymbol{q}_r}{\mathrm{d}r} = -4\frac{\overline{\kappa I_{b,\nu}}}{r_i \delta} \int_0^\infty \kappa_\nu^* \int_{\sin\theta\cos\phi_1^+ > \sqrt{1-1/n_\nu^2}} \frac{\partial(i_{\nu R}^*(r^*,\theta,\phi) + i_{\nu R}^*(r^*,\theta,\pi-\phi))}{\partial r^*} \sin\theta \, d\theta d\phi \, d\nu .$$

(117)

In the following the '*'s will be dropped and quantities are dimensionless.

The dimensionless intensity $i_{\nu R}(r,\theta,\phi)$ in an infinitely long axisymmetrical tube at constant temperature is given by (104)–(106). At the inner boundary ($r = 0$) the intensity is equal to

$$i_{\nu R}(0,\theta,\phi) = \frac{-(1 - \mathcal{R}(\omega_r))I_{b,\nu}|_{T=1} \, e^{-\tau(s_0^-)}}{1 - \mathcal{R}(\omega_r)e^{-2\tau(s_1^+)}} , \quad \phi \in [0, \gamma_r] \text{ or } \varphi \in [\pi - \gamma_r, \pi] .$$

(118)

Then for $\phi \in [0, \gamma_r]$ we have from (78) with $S(s') = 0$

$$i_{\nu R}(r,\theta,\phi) = i_{\nu R}(0,\theta,\phi_0^+)e^{-\tau(s^+)}$$

(119)

and for $\phi \in [\pi - \gamma_r, \pi]$, using (82) and (90) with $S(s') = 0$

$$i_{\nu R}(r,\theta,\phi) = i_{\nu R}(0,\theta,\pi-\phi_0^+)e^{\tau(s_0^-)-\tau(s^-)} .$$

(120)

From (80), (84) with $\phi$ replaced by $\pi - \phi$ and (91) it is found that

$$s_0^- = s^- + s^+$$

(121)

and therefore, as $\kappa_\nu$ does not depend on the radial position r,

$$\tau(s_0^-) = \tau(s^-) + \tau(s^+) .$$

(122)

Then it follows from (53), (120) and (122) that

$$i_{\nu R}(r,\theta,\pi-\phi) = i_{\nu R}(0,\theta,\phi_0^+)e^{\tau(s^+)} , \quad \phi \in [0, \gamma_r] .$$

(123)

Summing the intensities in (119) and (123) yields

$$i_{\nu R}(r,\theta,\phi) + i_{\nu R}(r,\theta,\pi-\phi) = i_{\nu R}(0,\theta,\phi_0^+)(e^{\tau(s^+)} + e^{-\tau(s^+)})$$

$$= 2i_{\nu R}(0,\theta,\phi_0^+)\cosh(\tau(s^+)) , \quad \phi \in [0, \gamma_r] .$$

(124)

Differentiating the expression in (124) and using the chain rule we have for $\phi \in [0, \gamma_r) \subset [0, \pi/2]$

$$
\begin{aligned}
\frac{\partial}{\partial r}(i_{\nu R}(r, \theta, \phi) + i_{\nu R}(r, \theta, \pi - \phi)) &= 2 \Bigg( \frac{\partial i_{\nu R}(0, \theta, \phi_0^+)}{\partial \phi_0^+} \frac{\partial \phi_0^+}{\partial r} \cosh(\tau(s^+)) \\
&\quad + i_{\nu R}(0, \theta, \phi_0^+) \sinh(\tau(s^+)) \frac{\mathrm{d}\tau(s^+)}{\mathrm{d}s^+} \frac{\partial s^+}{\partial r} \Bigg) \, .
\end{aligned}
\tag{125}
$$

First the second term on the right hand side of (125) is investigated. If the derivatives $\mathrm{d}\tau(s^+)/\mathrm{d}s^+$ and $\partial s^+/\partial r$ are finite, this term is zero at the inner radius $r = 0$, because at $r = 0$ we have $s^+ = 0$ and therefore $\tau(s^+) = 0$. Now it is shown that the derivatives $\mathrm{d}\tau(s^+)/\mathrm{d}s^+$ and $\partial s^+/\partial r$ are finite.

From (73) we have

$$
\frac{\mathrm{d}\tau(s^+)}{\mathrm{d}s^+} = \tau_0 \bar{\bar{\kappa}} \kappa_\nu \, ,
\tag{126}
$$

which is finite. Furthermore from (79) and (80) it follows that

$$
\begin{aligned}
s^+ &= \frac{1}{\delta \sin \theta} \left( (1 + \delta r) \cos \phi - \cos \phi_0^+ \right) \\
&= \frac{1}{\delta \sin \theta} \left( (1 + \delta r) \cos \phi - \sqrt{1 - \sin^2 \phi_0^+} \right) \\
&= \frac{1}{\delta \sin \theta} \left( (1 + \delta r) \cos \phi - \sqrt{1 - ((1 + \delta r) \sin \phi)^2} \right) \, , \quad \phi, \phi_0^+ \in [0, \frac{\pi}{2}) \, ,
\end{aligned}
\tag{127}
$$

therefore

$$
\frac{\partial s^+}{\partial r} = \frac{1}{\sin \theta} \left( \cos \phi + \frac{(1 + \delta r) \sin^2 \phi}{\sqrt{1 - ((1 + \delta r) \sin \phi)^2}} \right) \, .
\tag{128}
$$

At $r = 0$ the expression in (128) simplifies to

$$
\left. \frac{\partial s^+}{\partial r} \right|_{r=0} = \frac{1}{\sin \theta} \left( \cos \phi_0^+ + \sin \phi_0^+ \tan \phi_0^+ \right) \, ,
\tag{129}
$$

which is also finite, since we consider $\theta > 0$ in the region of integration in (117).

Next the first term on the right hand side of (125) is investigated. From (79) it follows that

$$
\phi_0^+ = \arcsin((1 + \delta r) \sin \phi) \, ,
\tag{130}
$$

Therefore, we have

$$
\frac{\partial \phi_0^+}{\partial r} = \frac{\delta \sin \phi}{\sqrt{1 - ((1 + \delta r) \sin \phi)^2}} \, ,
\tag{131}
$$

43

which at the inside of the tube $r = 0$ becomes

$$\left.\frac{\partial \phi_0^+}{\partial r}\right|_{r=0} = \frac{\delta \sin \phi_0^+}{\sqrt{1 - \sin^2 \phi_0^+}} = \delta \tan \phi_0^+ \ . \tag{132}$$

Furthermore, from (118) it follows that

$$\begin{aligned}
\frac{\partial i_{\nu R}(0, \theta, \phi_0^+)}{\partial \phi_0^+} &= \frac{I_{b,\nu}|_{T=1} \, e^{-\tau(s_0^-)}}{1 - \mathcal{R}e^{-2\tau(s_0^-)}} \left\{ \frac{\mathrm{d}\mathcal{R}(\omega_r)}{\mathrm{d}\omega_r} \frac{\partial \omega_r}{\partial \phi_0^+} + (1 - \mathcal{R}) \frac{\mathrm{d}\tau(s_0^-)}{\mathrm{d}s_0^-} \frac{\mathrm{d}s_0^-}{\mathrm{d}\phi_0^+} \right. \\
&\quad \left. + \frac{(1 - \mathcal{R})e^{-2\tau(s_0^-)}}{1 - \mathcal{R}e^{-2\tau(s_0^-)}} \left( -\frac{\mathrm{d}\mathcal{R}}{\mathrm{d}\omega_r} \frac{\partial \omega_r}{\partial \phi_0^+} + 2\mathcal{R}\frac{\mathrm{d}\tau(s_0^-)}{\mathrm{d}s_0^-)} \frac{\mathrm{d}s_0^-}{\mathrm{d}\phi_0^+} \right) \right\} \ ,
\end{aligned} \tag{133}$$

from (27), because at $r = 1$

$$\cos \omega_r = \sin \theta \cos \phi_1^+ \ , \tag{134}$$

and by (85)

$$(1 + \delta) \sin \phi_1^+ = \sin \phi_0^+ \ , \tag{135}$$

hence

$$\omega_r = \arccos(\sin \theta \cos \phi_1^+) \ , \tag{136}$$

with

$$\phi_1^+ = \arcsin\left(\frac{\sin \phi_0^+}{1 + \delta}\right) \ . \tag{137}$$

Because

$$\frac{\mathrm{d}\phi_1^+}{\mathrm{d}\phi_0^+} = \frac{1}{\sqrt{1 - \left(\frac{\sin \phi_0^+}{1+\delta}\right)^2}} \frac{\cos \phi_0^+}{1 + \delta} = \frac{\cos \phi_0^+}{\sqrt{(1 + \delta)^2 - \sin^2 \phi_0^+}} \ , \tag{138}$$

we have

$$\begin{aligned}
\frac{\partial \omega_r}{\partial \phi_0^+} &= \frac{-1}{\sqrt{1 - (\sin \theta \cos \phi_1^+)^2}} (-\sin \theta \sin \phi_1^+) \frac{\mathrm{d}\phi_1^+}{\mathrm{d}\phi_0^+} \\
\\
&= \frac{\sin \theta \sin \phi_0^+ \cos \phi_0^+}{\sqrt{1 - \sin^2 \theta (1 - (\frac{\sin \phi_0^+}{1+\delta})^2)} \sqrt{(1 + \delta)^2 - \sin^2 \phi_0^+} \, (1 + \delta)} \\
\\
&= \frac{\sin \theta \sin \phi_0^+ \cos \phi_0^+}{\sqrt{(1 + \delta)^2 - \sin^2 \theta((1 + \delta)^2 - \sin^2 \phi_0^+)} \sqrt{(1 + \delta)^2 - \sin^2 \phi_0^+}} \ .
\end{aligned} \tag{139}$$

44

Finally from (91), (135) and (138) we have

$$\frac{\mathrm{d}s_0^-}{\mathrm{d}\phi_0^+} = \frac{\sin\phi_0^+}{\delta\sin\theta}\left(1 - \frac{\cos\phi_0^+}{\sqrt{(1+\delta)^2 - \sin^2\phi_0^+}}\right) \tag{140}$$

and from (73) we have

$$\frac{\mathrm{d}\tau(s_0^-)}{\mathrm{d}s_0^-} = \tau_0\overline{\kappa}\kappa_\nu \ . \tag{141}$$

For $\theta, \phi_0^+ \in (0, \pi/2)$ we have from (139) and (140)

$$\frac{\partial\omega_r}{\partial\phi_0^+} > 0 \ , \quad \frac{\mathrm{d}s_0^-}{\mathrm{d}\phi_0^+} > 0 \tag{142}$$

and as $\mathrm{d}\tau(s_0^-)/\mathrm{d}s_0^-$ in (141) and $\mathrm{d}\mathcal{R}(\omega_r)/\mathrm{d}\omega_r$ are positive for $\omega_r < \pi - \omega_{crit}$ we have from (133)

$$\frac{\partial i_{\nu R}(0, \theta, \phi_0^+)}{\partial\phi_0^+} > 0 \ . \tag{143}$$

Furthermore we have for $\phi_0^+ \in (0, \pi/2)$ from (132)

$$\left.\frac{\partial\phi_0^+}{\partial r}\right|_{r=0} > 0 \ . \tag{144}$$

Hence the first term at the right-hand side of (125) is positive at $r = 0$. Therefore, from (117), it follows that

$$\left.\frac{\mathrm{d}(\boldsymbol{\nabla}\cdot\boldsymbol{q}_r)}{\mathrm{d}r}\right|_{r=0} < 0 \ . \tag{145}$$

If $r_i \to \infty$, while $\Delta r$ remains finite, the heat transfer problem in the hollow tube approaches that in an infinite slab with total radiation reflection at one boundary. In that case it follows from (55) that $\delta \to 0$, while $\Delta r$ remains finite. Then from (132) it is found that

$$\left.\frac{\partial\phi_0^+}{\partial r}\right|_{r=0} \to 0 \tag{146}$$

and the right-hand side of (125) tends to zero. Therefore, using (117), the derivative of the radiative heat flux

$$\left.\frac{\mathrm{d}(\boldsymbol{\nabla}\cdot\boldsymbol{q}_r)}{\mathrm{d}r}\right|_{r=0} \to 0 \ . \tag{147}$$

45

# Notation

*Section 2.1*

| | |
|---|---|
| $c$ | speed of elecromagnetic radiation |
| $c_p$ | specific heat capacity |
| $c_0$ | speed of electromagnetic radiation in a vacuum |
| $h$ | Planck's constant |
| $I_b$ | total blackbody intensity |
| $I_{b,\nu}$ | spectral blackbody intensity |
| $I_\nu$ | spectral intensity |
| $k$ | Boltzmann's constant |
| $k_c$ | thermal conductivity |
| $n_\nu$ | refractive index |
| $t$ | time |
| $T$ | temperature |
| $\boldsymbol{q}$ | heat flux |
| $\boldsymbol{q_r}$ | radiative heat flux |
| $\boldsymbol{q_c}$ | conduction heat flux |
| $\boldsymbol{s}$ | direction (unit vector) |
| $\boldsymbol{u}$ | velocity |
| $\boldsymbol{x}$ | position |
| $\eta$ | wavenumber |
| $\kappa_\nu$ | absorption coefficient |
| $\lambda$ | wavelength |
| $\nu$ | frequency |
| $\rho$ | density |
| $\sigma$ | Stefan-Boltzmann constant |
| $\sigma_{s\nu}$ | scattering coefficient |
| $\Phi$ | scattering phase function |

$\Omega$       solid angle

$\nabla$       gradient operator

subscripts

$t$       differentiation with respect to $t$

*Sections 2.1.1, 2.1.2, 2.1.3*

$e_x$       unit vector in $x$–direction

$I_\nu^R$       reflected component of intensity at boundary

$I_\nu^T$       transmitted component of intensity at boundary

$I_\nu^o$       incident intensity from outside the volume

$n$       unit outward normal

$r_0$       position on boundary

$\mathcal{R}(\cdot)$       reflectance function

$s_r$       direction of incident radiation at boundary from inside the medium

$s_t$       direction of incident radiation at boundary from outside the medium

$x$       position (in slab)

$x_i, x_o$       boundaries of slab

$\theta$       angle between $s$ and $n$, $\cos\theta = s \cdot n$

$\chi$       angle between refracted ray and normal

subscripts

$r$       reflected

$t$       transmitted

*Sections 2.1.4, 2.1.5*

$e_r$       unit vector in $r$–direction

$e_z$       unit vector in $z$–direction

$e_{\phi_c}$       unit vector in $\phi_c$–direction

$l$       length of path traversing interior of tube

$r$       radial position coordinate

$z$      axial position coordinate

$\Delta z$      increment in $z$ along path traversing interior of tube

$\zeta$      azimuthal angle of direction at boundary

$\theta$      polar angle of direction

$\phi$      azimuthal angle of direction

$\phi_c$      azimuthal angle position coordinate

$\omega$      polar angle of direction at boundary

subscripts

$i$      inner boundary

$o$      outer boundary

*Section 3*

$i_\nu$      relative spectral intensity

$i_{\nu D}$      relative intensity due to temperature variations

$i_{\nu E}$      relative intensity due to external sources

$i_{\nu R}$      relative intensity at constant temperature

$\delta$      ratio of tube thickness to inside radius

$\Delta r$      thickness of tube

$\tau_0$      optical thickness scale

superscripts

$\overline{\cdot}$      scale

$*$      dimensionless variable

*Section 4*

$s$      length parameter along characteristic

$S$      source term

$\tau$      optical depth

$\gamma$      critical azimuthal angle at $r = 1$ - characteristic projections

$\gamma_r$      critical azimuthal angle for $0 < r < 1$

49

$\Gamma$      initial values $(r_I, \phi_I, i_I)$

subscripts

$I$      initial value

0      inside radius

1      outside radius

superscripts

+      outgoing ray

−      incoming ray

*Section 5*

subscripts

*crit*      critical value associated with total internal reflection