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J. WIJNGAARD

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DR. F. H. SIMONS

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Introduction

Since Howard [5] published his book "Dynamic programming and Markov processes", quite a number of extensions have been worked out for both the construction of numerical methods towards optimal solutions and the proof of the existence of an optimal solution. We shall deal with the latter problem in the average costs case.

A Markovian decision process consists of the following elements:

a) *State space*. At each time $t = 0,1,2,\ldots$ the system is in one of the states $u \in S$. The state at time t is denoted by X_t.
b) *Actions*. For each $u \in S$ there is a set of possible actions $A(u)$. In state u one can choose an arbitrary action $d \in A(u)$. The state u and the action d determine the probability of being in a measurable set $E \subset S$ next time, $P_d(u,E)$, (we assume the existence of a σ-field Σ in S). A *policy* prescribes for each time t which action has to be chosen. If the action depends only on the state, the policy is called stationary.
c) *Costs*. The expected costs of action d in state u are denoted by $c(u,d)$.
d) *Costs-criterion*. Let $C_1,C_2,C_3,\ldots$ be the expected costs in the first, second,... period. We shall concentrate on the *average costs*

$$g := \lim_{n\to\infty} \frac{1}{n} \sum_{\ell=0}^{n-1} C_\ell \ .$$

For the average costs case Ross [12] derived a general result:

if there exists a bounded measurable function f on S and a constant g such that for all $u \in S$

$$g + f(u) = \operatorname*{Min}_{d\in A(u)} \{c(u,d) + \int f(s)P_d(u,ds)\} ,$$

then a stationary policy exists which is average optimal.

We shall consider the problem of the existence of a stationary policy which is optimal in the class of all stationary policies, if this policy is over-all optimal or not. The process is a discrete time Markov process on S when a stationary policy is used. Our problem may be represented by a set of pairs $\{(P_\alpha,r_\alpha)\}$, $\alpha \in A$, where A is the set of all stationary policies, P_α the Markov process under policy α, and the function r_α gives the

one-period costs $r_\alpha(u) = c(u,\alpha(u))$. We have to prove the existence of an $\alpha_0 \in A$ such that $g_{\alpha_0}(u) \leq g_\alpha(u)$ for all $u \in S$ and $\alpha \in A$, ($g_\alpha(u)$ are the average costs under policy α, starting in u).
The most obvious way to tackle this problem is to prove the compactness of A and the continuity of g_α in α. To this end we must introduce a topology in A, (we shall use a metric topology). Then it is not essential that A be the set of stationary policies. We may consider A to represent a set of indices only. We shall show some difficulties arising in proving the continuity of g_α in α.

Example. The state space consists of three elements, $S = \{1,2,3\}$. Once in state 2 or 3 one must stay there, the costs being 0 and 10 each period. In state 1 one of the actions $d \in [0,\frac{1}{2}]$ can be chosen. The probability of a transition to the states 1,2,3 is $1-d-d^2, d, d^2$. The costs of each of these actions d in state 1 are 1. The average costs, starting in state 2 or 3, are 0 and 10, independently of the policy used in state 1. If one uses policy $d = 0$, the average costs starting in state 1, $g_0(1)$, are equal to 1 since the system will never leave state 1. If one uses plicy $d > 0$, the system will certainly leave state 1 and will never return. In this case the average costs starting in state 1, $g_d(1)$, are equal to

$$d.0 + d^2.10 + (1-d-d^2)(d.0+d^2.10) + \ldots = 10\,\frac{d}{1+d}\,.$$

Hence $\inf_{d\in[0,\frac{1}{2}]} \{g_d(1)\} = 0$ but this infimum is not attained since $g_0(1) = 1$.

There is no optimal policy. The average costs as function of d have a discontinuity in $d = 0$. This discontinuity corresponds to a discontinuity in the number of ergodic sets. For $d > 0$ there are two ergodic sets, the sets $\{2\}$ and $\{3\}$, but for $d = 0$ the set $\{1\}$ is also an ergodic set. The eigenvalues of the transition matrix corresponding to the policy d are 1 and $1-d-d^2$. For $d = 0$ the eigenvalues coincide.

These continuity problems can be investigated with the aid of the perturbation theory of linear operators. Each Markov process in a finite state space corresponds to a transition matrix. In a more general state space S each Markov process corresponds to a linear operator in the space of all complex valued bounded measurable functions on S. As in the finite case the point 1 is one of the eigenvalues of the operator. Now let $\{(P_\alpha,r_\alpha)\}$, $\alpha \in A$ be a set of Markov processes with costs and assume that A is a metric space. In the example we had $A = [0,\frac{1}{2}]$. Although the transition matrix is continuous in α and the one-period costs even independent of α, the average costs starting in state 1 have a discontinuity corresponding to a discontinuity in the dimension of the eigenspace of eigenvalue 1. Apart from this discontinuity the average costs are continuous. Using the perturbation theory of linear operators it can be shown that this restricted continuity of $g_\alpha(u)$ holds if

- the cost functions r_α are bounded and continuous in α,
- the Markov processes P_α are *quasi-compact* and continuous in α.

Quasi-compactness of a Markov process is defined in terms of the corresponding linear operator. Essential is that this operator has only a finite number of eigenvalues on the unit circle, each of these a root of unity, and each with a finite dimensional eigenspace.
In chapter 2 we shall introduce quasi-compact Markov processes and we shall investigate the eigenvalues on the unit circle of the corresponding linear operators. As a preliminary we give in chapter 1 some results from spectral and perturbation theory of linear operators in Banach spaces. In section 1 of chapter 4 we use these results to prove the restricted continuity of $g_\alpha(u)$ in α for all u in state space S.
If the eigenspace of eigenvalue 1 of the operator corresponding to P_α is one-dimensional, then $g_\alpha(u)$ is independent of u. If this is true for all $\alpha \in A$, the compactness of A implies the existence of an optimal $\alpha_0 \in A$. The existence of an optimal policy for more general cases is also considered in section 4.1.

A more probabilistic concept, which is equivalent to quasi-compactness, is the *Doeblin-condition*. For a countable state space the Doeblin-condition for a Markov process P is equivalent to the existence of a finite set A, an integer n, and an $\varepsilon > 0$, such that the probability of being in the set A after n transitions $P^{(n)}(u,A) \geq \varepsilon$ for each starting state u. To show how severe this condition and hence quasi-compactness is we consider the following inventory problem:

> At the beginning of each period the inventory level is assumed to be ..... -2,-1,0,1,2,.... . One may order a quantity of at most R units, the delivery is instantaneous. During the period there is a demand for 0,1,2,... units with a probability of $p_0,p_1,p_2,\ldots$. The transition probability under order policy α is $P_\alpha(i,j) = p_{i+\alpha(i)-j}$, ($\alpha(i)$ is the quantity to order in state i).

If R is large enough there are policies α such that for each state i one can find a finite set A, an integer n, and an $\epsilon > 0$ such that $P_\alpha^{(n)}(i,A) \geq \epsilon$. However, if j is more than nR units below the lowest element of A, then $P_\alpha^{(n)}(j,A) = 0$. Hence there is no policy such that the corresponding Markov process satisfies the Doeblin-condition. Such decision processes can be studied by introducing embedded Markov processes. We extend the proof of the continuity of g_α to the case in which there is a subset A of the state space such that the embedded Markov process of P_α on A exists and is quasi-compact for all $\alpha \in A$.
Embedded Markov processes are introduced in section 2 of chapter 2. We derive some properties of Markov processes with a quasi-compact embedded Markov process. Chapter 3 deals with the existence of the average costs for these Markov processes with unbounded cost functions. The continuity of the average costs and the existence of an optimal α for problems $\{(P_\alpha,r_\alpha)\}$, $\alpha \in A$ is worked out in section 4.2.
In the work of de Leve [8] quasi-compact embedded Markov processes play also an important role. De Leve constructs a method for finding optimal solutions, while we investigate the existence of an optimal solution.

As observed before the existence of a constant g and a bounded function $f(\cdot)$ on S satisfying the equation

$$(1)\qquad g + f(u) = \min_{d \in A(u)} \{c(u,d) + \int f(s)P_d(u,ds)\}$$

guarantees the existence of a stationary policy α which is average optimal. For this policy α we have (Ross [12]),

$$g + f(u) = r_\alpha(u) + \int f(s)P_\alpha(u,ds),\ (r_\alpha(u) = c(u,\alpha(u))) ,$$

where the constant g is equal to the average costs $g_\alpha(u)$ for all $u \in S$. Now suppose we have the problem $\{(P_\alpha,r_\alpha)\}$, $\alpha \in A$ with g_α constant on S for each $\alpha \in A$, and assume that the equation

$$(2)\qquad y(u) = r_\alpha(u) - g_\alpha(u) + \int y(s)P_\alpha(u,ds)$$

in y(u) has a bounded solution $f_\alpha(u)$. If $g_\alpha, f_\alpha(\cdot)$ satisfy equation (1), the policy α is optimal. This means that one can use the solutions of (2) to state a condition for optimality. The existence of solutions of the equations (2) is considered in chapter 3. In section 5.3 a condition for optimality for inventory problems is developed. The difficulty of the unboundedness of the cost functions can be overcome by introducing spaces B_ω of functions f such that $\frac{f}{\omega}$ is bounded. In section 5.4 we show how the results of section 5.2, (existence of an optimal ordering policy), and section 5.3 can be used to prove that the optimal ordering policy of a specific inventory problem is of a given structure.

CHAPTER 1. QUASI-COMPACT LINEAR OPERATORS

In this first chapter we shall introduce quasi-compact linear operators and apply some results from spectral theory and perturbation theory to this class of operators. First we shall give some preliminaries. In section 1.2 we state some spectral and spectral decomposition properties. Quasi-compact operators are defined in section 1.3. Quasi-compactness is a slight generalization of compactness. The last section of this chapter is dedicated to perturbation theory of quasi-compact operators.

1.1. *Preliminaries*

Let X and Y be complex Banach spaces. The space L(X,Y) is the space of all bounded linear operators from X to Y. To each $T \in L(X,Y)$ we can adjoin a real number $\|T\| := \sup_{x \in X, \|x\|=1} \|Tx\|$. This function $\|\cdot\|$ is a norm on L(X,Y) and with this norm L(X,Y) is a Banach space.

Let $\mathbb{C}$ be the space of all complex numbers with the absolute value as norm. We shall denote $L(X,\mathbb{C})$ by X^*; X^* is called the *adjoint space* of X. The elements of X^* are usually called *bounded linear functionals* on X. With each operator $T \in L(X,Y)$ there corresponds an *adjoint operator* $T^* \in L(Y^*,X^*)$ defined by $T^*y^* = y^* \circ T$ for all $y^* \in Y^*$. The operators T and T^* have the same norm.

Let R(T) denote the *range* of the operator T and N(T) the *null space*. In the rest of this chapter we shall assume that X = Y. In this case the operators $T^2, T^3, \ldots$ exist and it is easy to see that

$$N(T) \subset N(T^2) \subset N(T^3) \subset \ldots \quad \text{and} \quad R(T) \supset R(T^2) \supset R(T^3) \supset \ldots .$$

If there is a smallest integer $n_0 \geq 1$ such that $N(T^{n_0}) = N(T^{n_0+1})$, then n_0 is called the *index* of T; otherwise the index is said to be infinite. If there is a smallest integer $m_0 \geq 1$ such that $R(T^{m_0}) = R(T^{m_0+1})$, then m_0 is called the *co-index* of T; otherwise we define the co-index to be infinite.

LEMMA 1.1. Let both index and co-index of T be finite, then they are equal and $X = N(T^p) \oplus R(T^p)$, where p is the index of T.

In this statement the symbol $\oplus$ stands for direct sum. For a proof of this lemma, we refer to Zaanen [18], Ch. 11, § 3, Th. 8.

In the following chapters we shall mainly deal with two special Banach spaces. These will be given here as examples.

i) The space $\mathcal{B}(V,\Sigma)$.
Let V be a set and Σ a σ-field of subsets of V.
$\mathcal{B}(V,\Sigma)$, or shortly $\mathcal{B}$, is the space of all complex valued bounded measurable functions on V. Let $\|f\| := \sup_{u\in V} |f(u)|$ for all $f \in \mathcal{B}$.
Then $\|\cdot\|$ is a norm on $\mathcal{B}$ and with this norm, $\mathcal{B}$ is a Banach space.

ii) The space $\mathcal{M}(V,\Sigma)$.
A complex valued σ-additive function on Σ is called a *measure* on Σ. We shall speak of a *signed measure* if the function on Σ is real valued and of a *positive measure* if the function is real valued and nonnegative.
A positive measure μ with $\mu(V) = 1$ is called a *probability*.
Let $\mathcal{M}(V,\Sigma)$, or shortly $\mathcal{M}$, be the space of all measures on Σ. It is easy to see that $\mathcal{M}$ is a linear space over the complex numbers.
Let $\mu \in \mathcal{M}$. By the Hahn-Jordan decomposition theorem there exist positive measures $\mu_1, \mu_2, \mu_3, \mu_4$ such that $\mu = \mu_1 - \mu_2 + i(\mu_3 - \mu_4)$.
For all $E \in \Sigma$ the *total variation* of μ on E, $v_\mu(E)$, is defined by

$$v_\mu(E) := \sup \sum_{i=1}^{n} |\mu(E_i)| \ ,$$

where the supremum is taken over all finite sequences $\{E_i\}_1^n$ of disjoint sets in Σ with $E_i \subset E$, $1 \le i \le n$.
The following relation holds

$$(1) \qquad |\mu(E)| \le v_\mu(E) \le \mu_1(E) + \mu_2(E) + \mu_3(E) + \mu_4(E) \le$$
$$\le \mu_1(V) + \mu_2(V) + \mu_3(V) + \mu_4(V) \ .$$

It is easy to verify that v_μ is a positive measure on Σ.
The definition of v_μ implies

$$v_{\mu+\nu} \le v_\mu + v_\nu \ , \qquad \mu \in M,\ \nu \in M\ ,$$

$$v_{\alpha\mu} = |\alpha| \cdot v_\mu \ , \qquad \mu \in M,\ \alpha \in \mathbb{C}\ .$$

Define $\|\mu\| := v_\mu(V)$, $\mu \in M$. Then $\|\cdot\|$ is a norm on M. Now, let $\{\mu_n\}_1^\infty$ be a Cauchy sequence in M with respect to $\|\cdot\|$. Then, because of the relation (1), we can define the function μ on Σ by

$$\mu(E) := \lim_{n\to\infty} \mu_n(E)\ , \qquad E \in \Sigma\ .$$

Using (1) it turns out that $\mu \in M$ and $\|\mu_n - \mu\| \to 0$.
Hence M with norm $\|\cdot\|$ is a Banach space.

To conclude this section we shall indicate the relationship between the spaces B and M. This relationship will be very important in the sequel. We need some properties of integrals of complex valued functions with respect to complex valued measures.
Let

$$f := \sum_{\ell=1}^{n} \alpha_\ell\, 1_{A_\ell}\ ,$$

where $(A_1,\ldots,A_n)$ is a measurable partition of V, and $\alpha_\ell \in \mathbb{C}$, $1 \le \ell \le n$. Functions of this type are said to be *simple* functions. Obviously the simple functions form a dense linear subspace of B. For every $\mu \in M$ we define

$$\mu f := \int f\, d\mu := \sum_{\ell=1}^{n} \alpha_\ell\, \mu(A_\ell)\ .$$

It is easy to verify that for each $\mu \in M$, $\mu(\cdot)$ is a linear functional on the space of all simple functions on V such that

$$|\mu f| \le \|\mu\|\cdot\|f\|\ .$$

This functional $\mu(\cdot)$ has a unique extension to a linear functional on B, also denoted by $\mu(\cdot)$, satisfying $|\mu f| \le \|\mu\|\cdot\|f\|$.

LEMMA 1.2. $$\sup_{\mu\in M,\|\mu\|=1} |\mu f| = \|f\| \quad \text{for all } f \in B\ ,$$

$$\sup_{f\in B,\|f\|=1} |\mu f| = \|\mu\| \quad \text{for all } \mu \in M\ .$$

PROOF. Use $|\mu f| \le \|\mu\| \cdot \|f\|$ and construct for a fixed $\mu \in M$ a suitable sequence of simple functions $f_1, f_2, \ldots$, respectively, for a fixed $f \in B$ a suitable sequence of probabilities $\mu_1, \mu_2, \ldots$. □

LEMMA 1.3. M is isometrically isomorphic with a closed subspace of B^*, the adjoint space of B, and B is isometrically isomorphic with a closed subspace of M^*.

PROOF. Let the linear mapping φ from M to B^* for all $\mu \in M$ be defined by $(\varphi\mu)(f) := \mu f$, $f \in B$. It is easy to verify that φ is an isomorphism between M and $\varphi(M)$. By lemma 1.2, $\|\varphi\mu\| = \|\mu\|$ and therefore $\varphi(M)$ is closed. This completes the proof of the first statement. The second statement can be shown similarly. □

1.2. *Spectral theory*

In the sequel the spectral decomposition of an operator plays an important role. For convenience of the reader some properties of the spectrum and spectral decomposition are collected in this section. The presentation is mainly based on Dunford-Schwartz [3], VII.3.

Let T be a fixed operator in L(X,X) with $\|T\| > 0$. The *resolvent set* $\rho(T)$ of T is the set of complex numbers λ such that the operator $\lambda I - T$ is 1-1 and onto (I is the identity). If $\lambda \in \rho(T)$, then $R(\lambda;T) := (\lambda I - T)^{-1}$ exists and is bounded.
The complement of $\rho(T)$ in $\mathbb{C}$ is called the *spectrum* of T and will be denoted by $\sigma(T)$. The *spectral radius* $r(T)$ of T is defined by

$$r(T) := \sup_{\lambda \in \sigma(T)} |\lambda| \ .$$

For the proofs of the following properties we refer to [3], VII.3.

i) $\rho(T)$ is open, $\sigma(T)$ is closed and nonempty.

ii) $r(T) = \lim_{n\to\infty} \sqrt[n]{\|T^n\|} \le \|T\|$.

iii) $R(\lambda;T)$ is an operator valued function which is analytic on $\rho(T)$.

iv) $R(\lambda;T) = \sum_{n=0}^{\infty} \frac{T^n}{\lambda^{n+1}}$ for $|\lambda| > r(T)$.

v) $\sigma(T) = \sigma(T^*)$, $R(\lambda;T^*) = R(\lambda;T)^*$ for $\lambda \in \rho(T) = \rho(T^*)$.

The definition of $\sigma(T)$ implies

$$\sigma(T) = \{\lambda \in C \mid N(\lambda I - T) \neq \{o\} \vee R(\lambda I - T) \neq X\} .$$

A point $\lambda \in \sigma(T)$ such that $N(\lambda I - T) \neq \{o\}$ is called an *eigenvalue* of T and $N(\lambda I - T)$ is the corresponding *eigenspace*. The index of an eigenvalue λ will be the index of $\lambda I - T$.

LEMMA 1.4. Let λ with $|\lambda| = \|T\|$ be an eigenvalue of T, then the index of λ is 1.

PROOF. It is sufficient to show:

$$N((\lambda I - T)^2) \subset N(\lambda I - T) .$$

Let $x \in N((\lambda I - T)^2)$ and $y := (\lambda I - T)x$. That means $(\lambda I - T)y = 0$. Using $x = \frac{y}{\lambda} + \frac{T}{\lambda} x$, we get

$$x = \frac{y}{\lambda} + \frac{T}{\lambda} x = \frac{y}{\lambda} + \frac{T}{\lambda} \left(\frac{y}{\lambda} + \frac{T}{\lambda} x\right) = \frac{2y}{\lambda} + \frac{T^2}{\lambda^2} x = \ldots = \frac{ny}{\lambda} + \frac{T^n}{\lambda^n} x ,$$

for all $n \in \mathbb{N}$.

Since $\|\frac{T^n}{\lambda^n}\| \leq 1$ it follows that $y = 0$ and $x \in N(\lambda I - T)$. □

As a preliminary for the spectral decomposition theorem we recall the concept of a function of an operator as given in [3], VII.3.9.

Let f be a complex valued function on $\mathbb{C}$ which is analytic on some neighbourhood of $\sigma(T)$. Let U be an open set whose boundary B consists of a finite number of rectifiable Jordan curves, oriented in the positive sense. Suppose that $U \supset \sigma(T)$ and that $U \cup B$ is contained in the domain of analyticity of f. The operator $f(T)$ is defined by

$$(1) \qquad f(T) := \frac{1}{2\pi i} \int_B f(\lambda) R(\lambda;T) d\lambda .$$

The operator $f(T)$ depends only on the values of f on $\sigma(T)$.

A *spectral set* is a subset of $\sigma(T)$ which is both open and closed in $\sigma(T)$. If α is a spectral set, then $\tilde{\alpha} := \sigma(T) \setminus \alpha$ is also a spectral set. For each spectral set α it is possible to choose a function f satisfying the conditions of the above definition with $f(\lambda) = 1$ on α and $f(\lambda) = 0$ on $\tilde{\alpha}$. For such a function f the operator $f(T)$ is denoted by $E_\alpha(T)$, or shortly by E_α. The range of E_α is denoted by X_α.

The following properties are immediate consequences of [3], VII.3.10.

i) $E_\alpha^2 = E_\alpha$ (E_α is a projection).

ii) $E_\alpha T = TE_\alpha$, hence $Tx \in X_\alpha$ if $x \in X_\alpha$, X_α is invariant under T.
The restriction of T to X_α is denoted by T_α.

iii) $E_\alpha + E_{\tilde\alpha} = I$ and $E_\alpha \cdot E_{\tilde\alpha} = 0$. This implies $X = X_\alpha \oplus X_{\tilde\alpha}$.

If λ is an isolated point of $\sigma(T)$, then the set $\{\lambda\}$ is of course a spectral set. In this case we shall write E_λ and $E_{\tilde\lambda}$, ..., instead of $E_{\{\lambda\}}$ and $E_{\{\tilde\lambda\}}$, ...
A *pole* of T of *order* n is an isolated point of $\sigma(T)$ where the function $R(\cdot;T)$ has a pole of order n.

LEMMA 1.5 (Spectral decomposition theorem). Let $\alpha_1,\ldots,\alpha_n$ be disjoint spectral sets such that $\sigma(T) = \bigcup_{i=1}^{n} \alpha_i$. Then the following properties hold:

i) $(X,T) = (X_{\alpha_1},T_{\alpha_1}) \oplus (X_{\alpha_2},T_{\alpha_2}) \oplus \ldots \oplus (X_{\alpha_n},T_{\alpha_n})$;

ii) $\sigma(T_{\alpha_i}) = \alpha_i$;

iii) λ is a pole of T_{α_i} of order n if and only if $\lambda \in \alpha_i$ and λ is a pole of T of order n.

PROOF. Statement i) is an immediate consequence of [3], VII.3.10.
The statements ii) and iii) are given explicitly in [3], VII.3.20. □

The next lemma shows the relationship between poles of T and eigenvalues of T.

LEMMA 1.6. An isolated point λ of $\sigma(T)$ is a pole of order n if and only if $(\lambda I - T)^n E_\lambda = 0$ and $(\lambda I - T)^{n-1} E_\lambda \neq 0$.
Furthermore, if λ is a pole of T of order n, then λ is an eigenvalue of T with index and co-index equal to n, and $X_\lambda = N((\lambda I - T)^n)$, $X_{\tilde\lambda} = R((\lambda I - T)^n)$.

PROOF. The first statement is part of [3], VII.3.18 (a pole of order 0 is impossible by [3], VII.3.3).
Now let λ be a pole of order n. In [3], VII.3.18 it is also shown that λ is an eigenvalue with index n.
Because of $(\lambda I - T)^n E_\lambda = 0$ we have $X_\lambda \subset N((\lambda I - T)^n)$. Hence for all $x \in X$,

$$(\lambda I - T)^n x = (\lambda I - T)^n (E_\lambda x + E_{\tilde\lambda} x) = (\lambda I - T)^n E_{\tilde\lambda} x = (\lambda I - T_{\tilde\lambda})^n E_{\tilde\lambda} x .$$

Therefore

$$(1) \qquad R((\lambda I - T_{\tilde\lambda})^n) = R((\lambda I - T)^n .$$

Furthermore

$$(2) \qquad (\lambda I - T)^n x = 0 \Rightarrow (\lambda I - T_{\tilde\lambda})^n E_{\tilde\lambda} x = 0 .$$

By lemma 1.5, $\sigma(T_{\tilde\lambda}) = \tilde\lambda$ and hence λ is a point in the resolvent set $\rho(T_{\tilde\lambda})$ of $T_{\tilde\lambda}$. Hence $R((\lambda I - T_{\tilde\lambda})^n) = R(\lambda I - T_{\tilde\lambda}) = X_{\tilde\lambda}$ for all $n \in \mathbb{N}$. Together with (1) this completes the proof of $X_{\tilde\lambda} = R((\lambda I - T)^n)$.
Using (2) we get

$$x \in N((\lambda I - T)^n) \Rightarrow (\lambda I - T_{\tilde\lambda})^n E_{\tilde\lambda} x = 0 \Rightarrow E_{\tilde\lambda} x = 0 \Rightarrow x \in X_\lambda .$$

Therefore, $N((\lambda I - T)^n) \subset X_\lambda$, which completes the proof. □

Now we restrict ourselves to the case $\|T\| = r(T)$. We shall use the decomposition theorem to show the existence of

$$\lim_{n\to\infty} \frac{1}{n} \sum_{\ell=0}^{n-1} \left(\frac{T}{\lambda_i}\right)^\ell ,$$

where λ_i is a pole of T with $|\lambda_i| = r(T)$.

LEMMA 1.7. Let $\|T\| = r(T)$. Asssume that the spectrum of T consists of a finite number of poles $\lambda_1,\ldots,\lambda_q$, on the circle with radius $r(T)$ and of a set α within this circle. Then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{\ell=0}^{n-1} \left(\frac{T}{\lambda_i}\right)^\ell = E_{\lambda_i} , \quad i = 1,\ldots,q .$$

PROOF. By lemma 1.5

$$I = \sum_{j=1}^{q} E_{\lambda_j} + E_\alpha .$$

Hence

$$\frac{1}{n}\sum_{\ell=0}^{n-1}\left(\frac{T}{\lambda_i}\right)^{\ell} = \frac{1}{n}\sum_{\ell=0}^{n-1}\left(\frac{T}{\lambda_i}\right)^{\ell}\left\{\sum_{j=1}^{q} E_{\lambda_j} + E_\alpha\right\} =$$

$$= \sum_{j=1}^{q}\frac{1}{n}\sum_{\ell=0}^{n-1}\left(\frac{T}{\lambda_i}\right)^{\ell} E_{\lambda_j} + \frac{1}{n}\sum_{\ell=0}^{n-1}\left(\frac{T}{\lambda_i}\right)^{\ell} E_\alpha .$$

By lemma 1.6, λ_j is an eigenvalue and by lemma 1.4 the index is 1. Therefore $TE_{\lambda_j} = \lambda_j E_{\lambda_j}$, and

$$(1)\qquad \frac{1}{n}\sum_{\ell=0}^{n-1}\left(\frac{T}{\lambda_i}\right)^{\ell} = \sum_{j=1}^{q}\frac{1}{n}\sum_{\ell=0}^{n-1}\left(\frac{\lambda_j}{\lambda_i}\right)^{\ell} E_{\lambda_j} + \frac{1}{n}\sum_{\ell=0}^{n-1}\left(\frac{T}{\lambda_i}\right)^{\ell} E_\alpha .$$

It is easy to see that

$$(2)\qquad \lim_{n\to\infty}\frac{1}{n}\sum_{\ell=0}^{n-1}\left(\frac{\lambda_j}{\lambda_i}\right)^{\ell} = \begin{cases} 0 & \text{if } j \neq i , \\ 1 & \text{if } j = i . \end{cases}$$

By lemma 1.5, $\sigma(T_\alpha) = \alpha$ and $r(T_\alpha) < r(T) = |\lambda_i|$. Therefore

$$R(\lambda_i;T_\alpha) = \sum_{n=0}^{\infty}\frac{T_\alpha^n}{\lambda_i^{n+1}} \quad \text{and} \quad \lim_{n\to\infty}\frac{1}{n}\sum_{\ell=0}^{n-1}\left(\frac{T}{\lambda_i}\right)^{\ell} E_\alpha = 0 .$$

Together with (1) and (2) this implies

$$E_{\lambda_i} = \lim_{n\to\infty}\frac{1}{n}\sum_{\ell=0}^{n-1}\left(\frac{T}{\lambda_i}\right)^{\ell} . \qquad \square$$

In the last lemma of this section a relationship between poles of T and poles of T^* is stated.

LEMMA 1.8. Let λ be a pole of T of order 1. Then λ is a pole of T^* of order 1. If the dimension of one of the spaces $N(\lambda I - T)$ and $N(\lambda I - T^*)$ is finite, then both are finite and equal to each other.

PROOF. The point λ is an isolated point of $\sigma(T^*)$ since $\sigma(T) = \sigma(T^*)$. We have

$$(\lambda I - T)E_\lambda(T) = E_\lambda(T)(\lambda I - T) = 0 .$$

Hence

$$(E_\lambda(T)(\lambda I - T))^* = (\lambda I - T)^* E_\lambda(T)^* = 0 .$$

By [3], VII.3.10, $E_\lambda(T)^* = E_\lambda(T^*)$, therefore, by lemma 1.6, λ is a pole of T^* of order 1.

It is easy to verify that

(1) $N(\lambda I - T^*) = \{x^* \in X^* \mid x^*x = 0,\ x \in R(\lambda I - T)\}$.

By lemma 1.6, $R(\lambda I - T) = X_\lambda$. Using this and equation (1) we get

$$x^* \in N(\lambda I - T^*) \Leftrightarrow x^*x = x^*(E_\lambda x) , \quad x \in X .$$

Hence $N(\lambda I - T^*)$ is isomorphic with the space of all bounded linear functionals on $N(\lambda I - T) = X_\lambda$, which completes the proof. □

1.3. *Spectral properties of quasi-compact linear operators*

Let X be a complex Banach space. The operator $T \in L(X,X)$ is called *compact* if for each bounded sequence $\{x_i\}_1^\infty$ of elements of X, the sequence $\{Tx_i\}_1^\infty$ has a convergent subsequence.

Obviously, every operator with finite dimensional range is compact, and if T is compact and S bounded, then TS and ST are also compact. Moreover, the operator $T \in L(X,X)$ is compact if and only if the adjoint operator $T^* \in L(X^*,X^*)$ is compact (see [3], VI.5.2).

The spectrum of a compact operator has a very special structure.

LEMMA 1.9. Let $T \in L(X,X)$ be compact. Then its spectrum is at most denumerable and has no points of accumulation, except possibly the point $\lambda = 0$. Every nonzero $\lambda \in \sigma(T)$ is a pole of T and X_λ is finite dimensional.

For the proof we refer to [3], VII.4.5.

A concept related to compactness is quasi-compactness. An operator $T \in L(X,X)$ is said to be *quasi-compact* if there exists a compact operator $K \in L(X,X)$ and a positive integer n such that $\|T^n - K\| < r(T)^n$.

Notice that quasi-compactness of T implies $r(T) > 0$.

REMARK. In other work (e.g. Neveu [9], Yosida [16]), quasi-compactness is defined in a somewhat different way: An operator T is said to be quasi-compact if there exists a sequence $\{K_n\}_1^\infty$ of compact operators such that $\lim_{n\to\infty} \|T^n - K_n\| = 0$, or equivalently, if there exists a compact operator K such that $\|T^n - K\| < 1$ for some $n \in \mathbb{N}$. Our definition agrees with these ones in the case $r(T) = 1$ but not in general. However, in most applications we have $\|T\| = r(T) = 1$.

The advantage of our definition is that quasi-compactness of T is not disturbed by multiplication of T by a constant. This makes it possible to formulate a rather elegant relationship between quasi-compactness of T and the structure of its spectrum.

LEMMA 1.10. An operator $T \in L(X,X)$ is quasi-compact if and only if $\sigma(T) \cap \{\lambda \mid |\lambda| = r(T)\}$ consists of a finite number of poles $\lambda_1,\ldots,\lambda_q$ such that the spaces X_{λ_i}, $i = 1,\ldots,q$, are finite dimensional.

PROOF. Let T be quasi-compact and let the compact operator K and the integer n be such that $\|T^n - K\| < r(T)^n$. Put $T_1 := \frac{T}{r(T)}$, then

$$\|T_1^n - \frac{K}{r(T)^n}\| < 1 .$$

By [3], VIII.8.2, each point $\lambda \in \sigma(T_1)$ with

$$|\lambda|^n > \|T_1^n - \frac{K}{r(T)^n}\| ,$$

in particular each point $\lambda \in \sigma(T_1)$ with $|\lambda| = 1$, is isolated in $\sigma(T_1)$ and X_λ is finite dimensional. Hence each point $\lambda \in \sigma(T)$ with $|\lambda| = r(T)$ is isolated in $\sigma(T)$ and X_λ is finite dimensional. This implies that $\sigma(T)$ contains only a finite number of such points, $\lambda_1,\ldots,\lambda_q$. The space X_{λ_i}, $i = 1,\ldots,q$, is finite dimensional and therefore T_{λ_i} is compact. By lemma 1.9, λ_i is a pole of T_{λ_i} and hence a pole of T (see lemma 1.5), which means that $\sigma(T)$ has a structure as described in the lemma.
Now let $\sigma(T)$ have this structure and put $\alpha := \sigma(T) \setminus \{\lambda_1,\ldots,\lambda_q\}$. By lemma 1.5

$$I = \sum_{i=1}^{q} E_{\lambda_i} + E_\alpha$$

and hence

$$T^\ell = T^\ell \sum_{i=1}^{q} E_{\lambda_i} + T^\ell E_\alpha , \qquad \ell \in \mathbb{N} .$$

Since E_{λ_i} has finite dimensional range it is compact and therefore the operator $K_\ell := T^\ell \sum_{i=1}^{q} E_{\lambda_i}$ is also compact for all $\ell \in \mathbb{N}$. Since $\|T^\ell - K_\ell\| = \|T^\ell E_\alpha\|$ the proof is completed if we can show the existence of an $n \in \mathbb{N}$ such that $\|T^n E_\alpha\| < r(T)^n$. By lemma 1.5, $\sigma(T_\alpha) = \alpha$, hence $r(T_\alpha) < r(T)$.

Let $\beta \in \mathbb{R}$ be such that $r(T_\alpha) < \beta < r(T)$. Using

$$r(T_\alpha) = \lim_{n\to\infty} \sqrt[n]{\|T_\alpha^n\|} \quad \text{and} \quad \left(\frac{\beta}{r(T)}\right)^n \to 0$$

it is easy to see that for sufficiently large n

$$\|T_\alpha^n\| < \beta^n < \frac{1}{\|E_\alpha\|} \cdot r(T)^n .$$

Hence

$$\|T^n E_\alpha\| = \|T_\alpha^n E_\alpha\| \le \|T_\alpha^n\| \cdot \|E_\alpha\| < \beta^n \cdot \|E_\alpha\| < r(T)^n . \qquad \square$$

As a consequence of this result we obtain the following lemma.

LEMMA 1.11. Let $T \in L(X,X)$ be quasi-compact and suppose $\|T\| = r(T)$. Let Y be a closed invariant subspace of X and T_Y the restriction of X to Y. If $r(T_Y) = r(T)$ then T_Y is also quasi-compact.

PROOF. By lemma 1.10, T has a finite number of poles, $\lambda_1,\dots,\lambda_q$, on the circle with radius r(T). By lemma 1.4 the order of these poles is 1. Since by lemma 1.7

$$E_{\lambda_i} = \lim_{n\to\infty} \frac{1}{n} \sum_{\ell=0}^{n-1} \left(\frac{T}{\lambda_i}\right)^\ell ,$$

the subspace Y is also invariant under E_{λ_i}, $i = 1,\dots,q$, and therefore under E_α, where $\alpha := \sigma(T) \setminus \{\lambda_1,\dots,\lambda_q\}$.

Put, as in the proof of lemma 1.10, $K_\ell := T^\ell \sum_{i=1}^{q} E_{\lambda_i}$, then Y is invariant under K_ℓ and for sufficiently large ℓ we have

$$\|T_Y^\ell - K_{\ell Y}\| \le \|T^\ell - K_\ell\| < r(T)^\ell = r(T_Y)^\ell .$$

Since, obviously, $K_{\ell Y}$, the restriction of K_ℓ to Y, is compact, this implies the quasi-compactness of T_Y. $\square$

In the last lemma of this section we consider the case of a quasi-compact operator T with $\|T\| = r(T) = 1$ and with an eigenvalue in the point 1.

LEMMA 1.12. Let $T \in L(X,X)$ be quasi-compact and $\|T\| = r(T) = 1$. Suppose there exists an integer d such that $\lambda_i^d = 1$ for all poles $\lambda_1,\dots,\lambda_q$ of T on the unit circle, and suppose that $\lambda_1 = 1$. Then there exists a real number ρ, $0 < \rho < 1$, and a positive integer N such that

$$(1)\qquad \Big\|\sum_{\ell=m+kd}^{m+kd+d-1} T_{\tilde{1}}^{\ell}\Big\| < \rho^k , \quad k > N .$$

Furthermore, the $\lim\limits_{k\to\infty} \sum\limits_{\ell=0}^{kd-1+m} T_{\tilde{1}}^{\ell}$ exists for $m = 0,1,2,\dots$, and

$$(2)\qquad \frac{1}{d}\sum_{m=0}^{d-1} \lim_{k\to\infty} \sum_{\ell=0}^{kd-1+m} T_{\tilde{1}}^{\ell} = (I - T_{\tilde{1}})^{-1} .$$

PROOF. In the proof we use the restrictions of the operators E_{λ_i}, $i = 2,\dots,q$, and E_α to $X_{\tilde{1}}$. These restrictions are also denoted by E_{λ_i} and E_α. By lemma 1.5,

$$T_{\tilde{1}}^{\ell} = T_{\tilde{1}}^{\ell} \sum_{i=2}^{q} E_{\lambda_i} + T_{\tilde{1}}^{\ell} E_\alpha , \qquad \ell \in \mathbb{N} .$$

Hence

$$\sum_{\ell=m+kd}^{m+kd+d-1} T_{\tilde{1}}^{\ell} = \sum_{i=2}^{q} \Big(\lambda_i^{m+kd} \sum_{\ell=0}^{d-1} \lambda_i^{\ell}\Big) E_{\lambda_i} + \sum_{\ell=m+kd}^{m+kd+d-1} T_{\tilde{1}}^{\ell} E_\alpha .$$

Since $\lambda_i^d = 1$, $i = 2,\dots,q$, we have $\sum\limits_{\ell=0}^{d-1} \lambda_i^{\ell} = 0$ for $i = 2,\dots,q$, and

$$\sum_{\ell=m+kd}^{m+kd+d-1} T_{\tilde{1}}^{\ell} = \sum_{\ell=m+kd}^{m+kd+d-1} T_{\tilde{1}}^{\ell} E_\alpha .$$

Let β be such that $r(T_\alpha) < \beta < 1$ and choose n_0 such that $\|T_\alpha^{\ell}\| < \beta^{\ell}$ for $\ell > n_0$. Then for $k > \frac{n_0}{d}$ and for all $m = 0,1,2,\dots$ we have

$$\Big\|\sum_{\ell=m+kd}^{m+kd+d-1} T_{\tilde{1}}^{\ell} E_\alpha\Big\| = \Big\|\sum_{\ell=m+kd}^{m+kd+d-1} T_\alpha^{\ell} E_\alpha\Big\| < d\cdot\|E_\alpha\|\cdot\beta^{kd} .$$

It is possible to choose a positive ρ with $\beta < \rho < 1$ and an integer $N > \frac{n_0}{d}$ such that $d\cdot\|E_\alpha\|\cdot\beta^{kd} < \rho^k$ for $k > N$.
This completes the proof of (1).

The existence of $\lim_{k\to\infty} \sum_{\ell=0}^{kd-1+m} T_{\tilde{1}}^{\ell}$ for $m = 0,1,2,\ldots$ follows immediately from (1).
Finally,

$$(I - T_{\tilde{1}}) \lim_{k\to\infty} \sum_{\ell=0}^{kd-1+m} T_{\tilde{1}}^{\ell} = I - \lim_{k\to\infty} T_{\tilde{1}}^{kd+m} ,$$

and

$$\lim_{k\to\infty} T_{\tilde{1}}^{kd+m} = \sum_{i=2}^{q} \lambda_i^m E_{\lambda_i} + \lim_{k\to\infty} T_{\tilde{1}}^{kd+m} E_\alpha = \sum_{i=2}^{q} \lambda_i^m E_{\lambda_i} .$$

Hence

$$(I - T_{\tilde{1}}) \left\{\frac{1}{d} \sum_{m=0}^{d-1} \lim_{k\to\infty} \sum_{\ell=0}^{kd-1+m} T_{\tilde{1}}^{\ell}\right\} = I - \frac{1}{d} \sum_{m=0}^{d-1} (\lim_{k\to\infty} T_{\tilde{1}}^{kd+m}) =$$

$$= I - \frac{1}{d} \sum_{m=0}^{d-1} \sum_{i=2}^{q} \lambda_i^m E_{\lambda_i} = I - \frac{1}{d} \sum_{i=2}^{q} E_{\lambda_i} \sum_{m=0}^{d-1} \lambda_i^m = I ,$$

since the sum $\sum_{m=0}^{d-1} \lambda_i^m = 0$ for $i = 2,\ldots,q$. □

1.4. *Perturbation theory*

Let A be a set in the metric space M and let $\varepsilon > 0$. The set $S(A,\varepsilon)$ is defined as the set of all $m \in M$ such that the distance of m to A is less than ε. If A consists of a single print a, we shall write $S(a,\varepsilon)$ instead of $S(\{a\},\varepsilon)$.

In this section A is a metric space with metric ρ, X is a complex Banach space and $T(\alpha)$ is a continuous function on A to $L(X,X)$.
The following two lemmas are consequences of [3], VII.6.3 and 6.7, and the fact that T(.) is continuous on A.

LEMMA 1.13. For each $\varepsilon > 0$ there is a $\delta > 0$ such that $\alpha \in S(\alpha_0,\delta)$ implies $\sigma(T(\alpha)) \subset S(\sigma(T(\alpha_0)),\varepsilon)$ and

$$\|R(\lambda;T(\alpha)) - R(\lambda;T(\alpha_0))\| < \varepsilon \quad \text{if } \lambda \notin S(\sigma(T(\alpha_0)),\varepsilon) .$$

LEMMA 1.14. Let $T(\alpha)$ be a projection for all $\alpha \in A$.
If $R(T(\alpha_0))$ is N-dimensional, there is a $\delta > 0$ such that $R(T(\alpha))$ is N-dimensional for all $\alpha \in S(\alpha_0,\delta)$.

In the next lemma we give some results under the assumption that $T(\alpha)$ is quasi-compact and has the point 1 as an eigenvalue.

LEMMA 1.15. Let for all $\alpha \in A$ the operator $T(\alpha)$ be quasi-compact, $\|T(\alpha)\| = r(T(\alpha)) = 1$, and 1 is an eigenvalue of $T(\alpha)$.

a) Let $\alpha_0 \in A$. There is a $\delta > 0$ such that for all $\alpha \in S(\alpha_0,\delta)$

$$\text{dimension } N(I - T(\alpha)) \leq \text{dimension } N(I - T(\alpha_0)) .$$

b) Let $\{\alpha_n\}_1^\infty$ be a sequence in A converging to $\alpha_0 \in A$, such that $\dim N(I - T(\alpha_n)) = \dim N(I - T(\alpha_0))$ for all $n \in \mathbb{N}$. Then

$$\lim_{n\to\infty} E_1(T(\alpha_n)) = E_1(T(\alpha_0)) .$$

c) Let β be such that $0 < \beta < 1$ and for all $\alpha \in A$ the spectrum of $T(\alpha)$ does not contain points of modulus between β and 1. Then for all $\alpha_0 \in A$ there is a $\delta > 0$ such that for all $\alpha \in S(\alpha_0,\delta)$

$$\dim N(I - T(\alpha)) = \dim N(I - T(\alpha_0)) .$$

PROOF. Let $\alpha_0 \in A$. The quasi-compactness of $T(\alpha_0)$ implies the isolatedness of the point 1 in $\sigma(T(\alpha_0))$, there is an $\varepsilon > 0$ such that $S(1,\varepsilon) \cap \sigma(T(\alpha_0)) = \{1\}$.
By lemma 1.13 there is a $\delta > 0$ such that for all $\alpha \in S(\alpha_0,\delta)$ the spectrum $\sigma(T(\alpha))$ contains no points λ with $\frac{\varepsilon}{3} < |1 - \lambda| < \frac{2\varepsilon}{3}$. The quasi-compactness of $T(\alpha_0)$ implies the existence of a compact operator K and an integer n such that $p := \|T(\alpha_0)^n - K\| < 1$. Because of the continuity of $T(\alpha)$ there is a $\delta_1 > 0$ such that

$$\|T(\alpha)^n - K\| < \frac{1+p}{2} < 1 \quad \text{for all } \alpha \in S(\alpha_0,\delta_1) .$$

Let $\alpha \in S(\alpha_0,\delta_1)$. By [3], VIII.8.2 each point $\lambda \in \sigma(T(\alpha))$ with $|\lambda|^n > \frac{1+p}{2}$ is an isolated point of $\sigma(T(\alpha))$ and $X_\lambda(T(\alpha))$ is finite dimensional. Hence, by lemma 1.9, λ is a pole of $T_\lambda(\alpha)$ and therefore, by lemma 1.5, a pole of $T(\alpha)$.
Now we may assume without loss of generality that for $\alpha \in S(\alpha_0,\delta)$, $S(1,\frac{\varepsilon}{3}) \cap \sigma(T(\alpha))$ contains only poles of $T(\alpha)$.
Let f be a function which is equal to 1 on $S(1,\frac{\varepsilon}{3})$ and equal to 0 on $\mathbb{C} \setminus S(1,\frac{2\varepsilon}{3})$.
Let $\sigma_\alpha := S(1,\frac{\varepsilon}{3}) \cap \sigma(T(\alpha))$ and $\sigma_{\alpha 1} := \sigma_\alpha \setminus \{1\}$.

Then for all $\alpha \in S(\alpha_0,\delta)$

$$f(T(\alpha)) = E_{\sigma_\alpha}(T(\alpha))$$

and

$$(1) \qquad X_{\sigma_\alpha}(T(\alpha)) = X_1(T(\alpha)) \oplus X_{\sigma_{\alpha 1}}(T(\alpha)) .$$

By [3], VII.6.5 and lemma 1.14 there is a δ' with $0 < \delta' < \delta$ such that for all $\alpha \in S(\alpha_0,\delta')$

$$(2) \qquad \dim X_{\sigma_\alpha}(T(\alpha)) = \dim X_{\sigma_{\alpha_0}}(T(\alpha_0)) = \dim X_1(T(\alpha_0)) .$$

By lemma 1.4 the order of the pole 1 of $T(\alpha)$ is 1. Hence, by lemma 1.6,

$$X_1(T(\alpha)) = N(I - T(\alpha)) .$$

Using (1) and (2) we get for all $\alpha \in S(\alpha_0,\delta')$

$$\dim N(I - T(\alpha_0)) = \dim X_1(T(\alpha_0)) = \dim X_{\sigma_\alpha}(T(\alpha)) =$$

$$= \dim X_1(T(\alpha)) + \dim X_{\sigma_{\alpha 1}}(T(\alpha)) =$$

$$= \dim N(I - T(\alpha)) + \dim X_{\sigma_{\alpha 1}}(T(\alpha)) \geq \dim N(I - T(\alpha)) .$$

This completes the proof of a).
If $\dim N(I - T(\alpha)) = \dim N(I - T(\alpha_0))$ for some $\alpha \in S(\alpha_0,\delta')$, then $\sigma_{\alpha 1} = \emptyset$. It follows that

$$\sigma_\alpha = \{1\} \quad \text{and} \quad f(T(\alpha)) = E_{\sigma_\alpha}(T(\alpha)) = E_1(T(\alpha)) .$$

The proof of b) is easily given by application of [3], VII.6.5.
The proof of c) is straightforward by choosing ε such that $1 - \varepsilon > \beta$. □

CHAPTER 2. MARKOV PROCESSES

In this chapter we consider quasi-compact Markov processes (section 2.2) and embedded Markov processes (section 2.3). In the first section we shall give some preliminaries.

2.1. *Sub-Markov processes*

Let V be a set and Σ a σ-field of subsets of V. The spaces $\mathcal{B}(V,\Sigma)$ and $\mathcal{M}(V,\Sigma)$ are defined as in chapter 1. A *sub-transition probability* is a real valued function P on $V \times \Sigma$ such that

i) for all $u \in V$, $P(u,\cdot)$ is a positive measure on Σ with $P(u,V) \leq 1$

ii) for all $A \in \Sigma$, $P(\cdot,A) \in \mathcal{B}(V,\Sigma)$.

A sub-transition probability is called a *transition probability* if $P(u,V) = 1$ for all $u \in V$.

A sub-transition probability P induces operators in $\mathcal{M}$ and $\mathcal{B}$ given by the following definitions:

a) for all $\mu \in \mathcal{M}$ $(\mu P)(\cdot) = \int P(u,\cdot)\mu(du)$

b) for all $f \in \mathcal{B}$ $(Pf)(\cdot) = \int f(v)P(\cdot,dv)$.

The function μP on Σ is an element of $\mathcal{M}$ for all $\mu \in \mathcal{M}$ and the function Pf on V is an element of $\mathcal{B}$ for all $f \in \mathcal{B}$.

The mappings $\mu \to \mu P$ and $f \to Pf$ are linear. In the sequel we shall denote both the (sub-) transition probability and the corresponding operators in $\mathcal{B}$ and $\mathcal{M}$ with the same letter. From the rest of the notation it will be clear in which sense this letter is meant:

$P(\cdot,\cdot)$ is the (sub-) transition probability,

P to the left of a function is the operator in $\mathcal{B}$,

P to the right of a measure is the operator in $\mathcal{M}$.

In each of these cases P is called a *(sub-) Markov process* on (V,Σ).

The operator P has a probabilistic interpretation which can be usefull to understand the meaning of some definitions and lemma's. The remarks referring to this probabilistic interpretation of P are indicated by "remark p.n", n = 1,2,... .

REMARK p.1. Each pair (π,P) with π a probability and P a transition probability defines a discrete time Markov process $X(t)$, $t = 0,1,2,\dots$ with

$$\mathbb{P}\{X(0) \in E\} = \pi(E)$$

and

$$\mathbb{P}\{X(t+1) \in E \mid X(t) = u\} = P(u,E),\ t = 0,1,2,\dots .$$

See for instance Neveu [9], chapter 5.

Now let P be a sub-Markov process on (V,Σ). It is easy to see that $(\mu P)f = \mu(Pf)$ for all $f \in \mathcal{B}$, $\mu \in \mathcal{M}$.
This justifies the notation μPf for both $(\mu P)f$ and $\mu(Pf)$. In lemma 1.3 we proved that $\mathcal{M}$ is isometrically isomorphic with a closed subspace of $\mathcal{B}^*$, the adjoint space of $\mathcal{B}$. The isomorphism was the mapping $\varphi\colon \mathcal{M} \to \mathcal{B}^*$ defined by $(\varphi\mu)(f) = \mu f$, $f \in \mathcal{B}$. Let $P_{\mathcal{B}}$ be the operator P in $\mathcal{B}$ and $P_{\mathcal{M}}$ the operator P in $\mathcal{M}$. Then

$$P_{\mathcal{B}}^*(\varphi\mu)(f) = (\varphi\mu)(P_{\mathcal{B}}f) = \mu(P_{\mathcal{B}}f) = (\mu P_{\mathcal{M}})f = (\varphi(\mu P_{\mathcal{M}}))(f) .$$

This shows that $\varphi(\mathcal{M})$ is invariant under $P_{\mathcal{B}}^*$ and that the restriction of $P_{\mathcal{B}}^*$ to $\varphi(\mathcal{M})$ corresponds to $P_{\mathcal{M}}$. We can prove similarly that the restriction of $P_{\mathcal{M}}^*$ to the subspace of $\mathcal{M}^*$ which is isometrically isomorphic with $\mathcal{B}$ corresponds to $P_{\mathcal{B}}$.
As a consequence of this we get the following lemma.

LEMMA 2.1. Let P be a sub-Markov process on (V,Σ). Then $\|P_{\mathcal{B}}\| = \|P_{\mathcal{M}}\|$ and $\sigma(P_{\mathcal{B}}) = \sigma(P_{\mathcal{M}})$.

PROOF. $\sigma(P_{\mathcal{B}}) = \sigma(P_{\mathcal{B}}^*) \supset \sigma(P_{\mathcal{M}}) = \sigma(P_{\mathcal{M}}^*) \supset \sigma(P_{\mathcal{B}})$ and

$$\|P_{\mathcal{B}}\| = \|P_{\mathcal{B}}^*\| \geq \|P_{\mathcal{M}}\| = \|P_{\mathcal{M}}^*\| \geq \|P_{\mathcal{B}}\| .$$ □

Let A be an element of Σ. A special case of a sub-Markov process which is rather important in the sequel, is the process I_A determined by the sub-transition probability

$$I_A(u,E) := 1_{A\cap E}(u),\ u \in V,\ E \in \Sigma .$$

Application of the corresponding operator in $\mathcal{B}$ is multiplying by the characteristic function of A: $I_A f = 1_A . f$, $f \in \mathcal{B}$, and the corresponding operator in M is given by

$$(\mu I_A)(\cdot) = \mu(A \cap \cdot), \ \mu \in M .$$

Let P and Q be sub-Markov processes on (V,Σ). The sub-Markov process PQ is defined by the sub-transition probability

$$(PQ)(u,E) := (P(Q\ 1_E))(u), \ u \in V, \ E \in \Sigma .$$

The σ-additivity of $(PQ)(u,\cdot)$ follows from the σ-additivity in $\mathcal{B}$ of the operator induced by a sub-transition probability. For the process PQ the operator in $\mathcal{B}$ is given by

$$(PQ)f = P(Qf), \ f \in \mathcal{B}$$

and the operator in M by

$$\mu(PQ) = (\mu P)Q, \ \mu \in M .$$

If R is another sub-Markov process on (V,Σ), then obviously the relation $(PQ)R = P(QR)$ holds.

2.1. *Quasi-compact Markov processes*

In the sections 1.2 and 1.3 we showed that if T is a quasi-compact operator in a complex Banach space X with $\|T\| = r(T)$, then the space X can be decomposed in the subspaces $X_{\lambda_1}, X_{\lambda_2}, \ldots, X_{\lambda_q}$, and X_α where $\lambda_1, \ldots, \lambda_q$ are eigenvalues of T with $|\lambda_i| = \|T\| = r(T)$, α is a spectral set with $\sup_{\lambda\in\alpha} |\lambda| < r(T)$, and $\sigma(T) = \alpha \cup \{\lambda_1, \ldots, \lambda_q\}$.
In this section we assume that P is a Markov process on (V,Σ). Since $P1 = 1$ we have $\|P\| = r(P) = 1$ and 1 is an eigenvalue of P. If the operator P in $\mathcal{B}$ is quasi-compact, the decomposition of $\mathcal{B}$ corresponds to a decomposition of V, which we shall study in this section.
The next lemma makes it possible to speak about quasi-compact Markov processes.

LEMMA 2.2. The operator P in $\mathcal{B}$, $P_{\mathcal{B}}$, is quasi-compact if and only if the operator P in M, P_M, is quasi-compact.

PROOF. The quasi-compactness of the operator $P_{\mathcal{B}}$ implies the existence of an integer n and a compact operator K in $\mathcal{B}$ such that $\| P_{\mathcal{B}}^n - K \| < 1$. Since $\| P_{\mathcal{B}}^n - K \| = \| (P_{\mathcal{B}}^*)^n - K^* \|$ and the operator K^* is also compact, the operator $P_{\mathcal{B}}^*$ is quasi-compact too. The space M is isometrically isomorphic with a closed subspace of $\mathcal{B}^*$ which is invariant under $P_{\mathcal{B}}^*$ and the restriction of $P_{\mathcal{B}}^*$ to this subspace of $\mathcal{B}^*$ corresponds to P_M.
Now the quasi-compactness of P_M is a consequence of lemma 1.11. The proof in the other direction is similar. □

If P is quasi-compact, the point 1 must be a pole of P and $X_1 = N(I - P)$. Using lemma 1.8 we get

$$\dim N(I - P_{\mathcal{B}}) = \dim N(I - P_{\mathcal{B}}^*) \geq \dim N(I - P_M) = \dim N(I - P_M^*) \geq$$
$$\geq \dim N(I - P_{\mathcal{B}}) \ .$$

Hence $\dim N(I - P_{\mathcal{B}}) = \dim N(I - P_M)$.

DEFINITION 2.3. A set $E \in \Sigma$ is *invariant* under P if $(P\, 1_E)(u) = 1$ for $u \in E$. An equivalent definition is: $E \in \Sigma$ is invariant if $(\mu P)(E) = 1$ for all probabilities $\mu \in M$ with $\mu(E) = \mu(V) = 1$.
Notice that $E_1 \cap E_2$ is invariant if E_1 and E_2 are invariant. An element $\mu \in N(I - P)$ is called an invariant measure of P.

LEMMA 2.4. Let μ be an invariant positive measure and let A be an invariant set under P. Then μI_A is invariant.

PROOF. Let $A^c := V \backslash A$. We have

$$(\mu P)(A) = (\mu I_A P)(A) + (\mu I_{A^c} P)(A) = \mu(A) + (\mu I_{A^c} P)(A) = \mu(A) \ .$$

Hence $(\mu I_{A^c} P)(A \cap B) = 0$ for all $B \in \Sigma$. This implies

$$(\mu I_A P)(B) = (\mu I_A P)(A \cap B) = (\mu P)(A \cap B) - (\mu I_{A^c} P)(A \cap B) =$$
$$= (\mu P)(A \cap B) = \mu(A \cap B) = (\mu I_A)(B) \text{ for all } B \in \Sigma \ . \quad □$$

In the next theorem we shall prove that the quasi-compactness of P is coupled with the existence of a finite number of pairwise disjoint invariant sets.

THEOREM 2.5. Let P be quasi-compact and suppose $\dim N(I-P) = n$. Then there exists a unique set probabilities $\{\pi_1,\ldots,\pi_n\}$ such that

i) $\pi_1,\ldots,\pi_n$ are invariant under P

ii) there exist pairwise disjoint invariant sets $E_1,\ldots,E_n \in \Sigma$ such that $\pi_i(E_i) = 1$ for $i = 1,\ldots,n$.

Moreover, if μ is a probability on Σ with $\mu(E_i) = 1$ then

$$\pi_i = \lim_{n\to\infty} \frac{1}{n} \sum_{\ell=0}^{n-1} \mu P^\ell .$$

For the proof of this lemma we need the following two lemma's.

LEMMA 2.6. Let $\mu \in N(I-P)$ be a positive measure on Σ with support F. Then μ has a support $G \subset F$ which is invariant under P.

PROOF. Let A be an arbitrary set in Σ such that $\mu(A) = \mu(V)$. From $\mu(A) = \mu P(A) = \mu(V)$ we conclude that $\mu\{u \in A \mid P(u,A) = 1\} = \mu(V)$. Put

$$G_0 := F,\ G_{k+1} := \{u \in G_k \mid P(u,G_k) = 1\}\ ,\ k = 0,1,2,\ldots\ ,$$

and $G := \bigcap_{k=0}^{\infty} G_k$. Then $\mu(G) = \mu(G_k) = \mu(F) = \mu(V)$ and the invariance of G is a direct consequence of $P(u,G_k) = 1$, $u \in G \subset G_{k+1}$, $k = 0,1,2,\ldots$. □

LEMMA 2.7. Let $\mu \in N(I-P)$ be a (real) signed measure on Σ and let $\mu = \mu^+ - \mu^-$ be the Hahn-Jordan decomposition of μ. Then $\mu^+ \in N(I-P)$ and $\mu^- \in N(I-P)$.

PROOF. Let E be a support of μ^+ such that $E^c := V\backslash E$ is a support of μ^-. Then

$$\mu(E) = (\mu I_E P)(E) + (\mu I_{E^c} P)(E) \le (\mu I_E P)(E) = (\mu^+ P)(E) \le$$

$$\le \mu^+(V) = \mu^+(E) = \mu(E)\ .$$

Hence $(\mu I_E P)(E) = \mu(E)$ and $(\mu I_{E^c} P)(E) = 0$. Therefore

$$(\mu I_E P)(E^c) = (\mu I_E P)(V) - (\mu I_E P)(E) = (\mu I_E)(V) - \mu(E) = 0\ .$$

It follows that for each $F \in \Sigma$ we have

$$\mu^+(F) = \mu(F \cap E) = (\mu I_E P)(F \cap E) + (\mu I_{E^c} P)(F \cap E) = (\mu I_E P)(F \cap E) =$$

$$= (\mu I_E P)(F) \setminus (\mu I_E P)(F \setminus E) = (\mu I_E P)(F) = (\mu^+ P)(F) .$$

This shows the invariance of μ^+. The invariance of μ^- follows from $\mu^- = \mu^+ - \mu$. □

Now we shall give the proof of theorem 2.5.

PROOF OF THEOREM 2.5. Let μ be an arbitrary element of $N(I - P)$. Using the real valuedness of P we see that the real part μ_1 of μ and the imaginary part μ_2 of μ are also elements of $N(I - P)$. By lemma 2.7 the positive measures μ_1^+, μ_1^-, μ_2^+, μ_2^- are elements of $N(I - P)$ too. This implies the existence of n independent probabilities which are a base for $N(I - P)$. Obviously, if the probabilities $\mu_1,\ldots,\mu_k$ have pairwise disjoint supports, they are independent. Let k be the largest number of probabilities in $N(I - P)$ with pairwise disjoint supports. Then $k \le n$. Suppose $k < n$ and let the probabilities $\mu_1,\ldots,\mu_k \in N(I - P)$ have disjoint supports. By lemma 2.6 there are pairwise disjoint supports $E_1,\ldots,E_k$ of $\mu_1,\ldots,\mu_k$, which are invariant sets under P. Since $k < n$ there is a probability $\mu \in N(I - P)$, independent of $\mu_1,\ldots,\mu_k$. Let A be a support of μ which is invariant under P. If $\mu(C) > 0$ with $C := A \setminus \bigcup_{i=1}^{k} E_i$ then μI_C is a nontrivial element of $N(I - P)$ with support C, which yields a contradiction. Hence $\mu(C) = 0$. This implies that for at least one i, $1 \le i \le k$, μI_{E_i} is not a multiple of μ_i. Let j be such an i and let $\pi_j := \frac{1}{\mu(E_j)} \cdot \mu I_{E_j}$. Then the signed measure $\pi_j - \mu_j$ is an element of $N(I - P)$ with nontrivial positive and negative parts. There are disjoint sets E_{j1} and E_{j2} in E_j such that E_{j1} is a support of $(\pi_j - \mu_j)^+$ and E_{j2} is a support of $(\pi_j - \mu_j)^-$. By lemma 2.7 $(\pi_j - \mu_j)^+$ and $(\pi_j - \mu_j)^-$ are elements of $N(I - P)$. This contradicts the maximality of k. Hence $k = n$, there are n probabilities $\pi_1,\ldots,\pi_n$ with pairwise disjoint invariant supports $E_1,\ldots,E_n$, which are a base for $N(I - P)$. Now we have to prove the uniqueness of these probabilities. Let $\{\mu_1,\ldots,\mu_n\}$ be another set of probabilities in $N(I - P)$ with pairwise disjoint supports $F_1,\ldots,F_n$. Each μ_i is a linear combination of $\pi_1,\ldots,\pi_n$: $\mu_i = \sum_{j=1}^{n} \alpha_{ij}\pi_j$. It is easy to verify that $\sum_{j=1}^{n} \alpha_{ij} = 1$ for $i = 1,\ldots,n$ and $\alpha_{ik} \ge 0$. From

$$\mu_i(F_i) = 1 = \sum_{j=1}^{n} \alpha_{ij}\pi_j(F_i) \le \sum_{j=1}^{n} \alpha_{ij} = 1$$

we conclude that $\pi_j(F_i) = 1$ if $\alpha_{ij} > 0$. Therefore, for each $i \in \{1,\dots,n\}$, there is only one j such that $\alpha_{ij} > 0$. It follows that $\alpha_{ij} = 1$ and $\mu_i = \pi_j$. This proves the uniqueness of $\{\pi_1,\dots,\pi_n\}$. Now let μ be a probability on Σ with $\mu(E_i) = 1$ for some i. By lemma 1.7, $\pi := \lim_{n\to\infty} \frac{1}{n} \sum_{\ell=0}^{n-1} \mu P^\ell$ exists and is an element of $N(I - P)$. Since E_i is invariant, $\pi(E_i) = 1$ and therefore $\pi = \pi_i$. □

Now let the conditions of theorem 2.5 be satisfied and let $\pi_1,\dots,\pi_n$ and $E_1,\dots,E_n$ be as in this theorem. By lemma 1.7, $S := \lim_{n\to\infty} \frac{1}{n} \sum_{\ell=0}^{n-1} P^\ell$ exists. It is easy to see that S is a Markov process satisfying $PS = SP = S$. Hence $S(u,\cdot)$ is an invariant probability of P for all $u \in V$. Define the sets $F_1,\dots,F_n$ by $F_i := \{u \in V \mid S(u,\cdot) = \pi_i(\cdot)\}$. The sets F_i are pairwise disjoint and since $F_i \supset E_i$ we have $\pi_i(F_i) = 1$. For $u \in F_i$ we have

$$1 = S(u,E_i) = (PS)(u,E_i) = \int_V P(u,ds)S(s,E_i) .$$

It follows that $S(\cdot,E_i) = 1$, $P(u,\cdot)$-almost everywhere, hence $P(u,F_i) = 1$. This implies that the sets $F_1,\dots,F_n$ are also invariant under P. These sets are called the *maximal invariant sets*.

In the next theorem we shall prove that each eigenvalue λ of P on the unit circle is a root of unity if P is quasi-compact. The proof given here is due to Yosida and Kakutani [17], § 4.5. We need the following lemma.

LEMMA 2.8. Let μ be a probability on Σ and f an element of $\mathcal{B}$ such that $\mu f = 1$ and $|f| = 1$, μ-almost everywhere. Then $f = 1$, μ-almost everywhere.

PROOF. Let g and h be the real and imaginary part of f. Then $\mu f = \mu g + i\mu h = 1$, hence $\mu g = 1$ and $\mu h = 0$. However,

$$1 = \mu g \le \mu(\sqrt{g^2 + h^2}) = \mu 1 = 1$$

which implies that $h = 0$, μ almost everywhere. □

THEOREM 2.9. Let P be quasi-compact and λ an eigenvalue of P on the unit circle. Then λ is a root of unity.

PROOF. Suppose dim $N(I - P) = n$. Let the probabilities $\pi_1,\dots,\pi_n$ and the sets $E_1,\dots,E_n$ be as in theorem 2.5. Put $S := \lim_{n\to\infty} \frac{1}{n} \sum_{\ell=0}^{n-1} P^\ell$ and choose a nonzero element $f \in N(\lambda I - P)$. Then $|f| = |\lambda^\ell f| = |P^\ell f| \le P^\ell |f|$ which implies $|f| \le S|f|$. For each $u \in V$, $S(u,\cdot)$ is an invariant probability and therefore a linear combination of $\pi_1,\dots,\pi_n$. In particular $S(u,\cdot) = \pi_i(\cdot)$ if $u \in E_i$. It follows that for $u \in E_i$

$$|f|(u) \le (S|f|)(u) = \pi_i|f| \le \sup_{u\in E_i} |f|(u) := c_i .$$

Hence $c_i \le \pi_i|f| \le c_i$ which implies that $|f| = c_i$, π_i-almost everywhere for all $i = 1,\dots,n$.

If $|f| = 0$, π_i-almost everywhere for all $i = 1,\dots,n$ then $S|f| = 0$. Since $|f| \le S|f|$ there is at least one $i \in \{1,2,\dots,n\}$ such that $c_i > 0$. Choose $u_0 \in E_i$ such that $|f|(u_0) = c_i$. Define the sets $E_i(\ell)$ for $\ell = 1,2,\dots,$ by $E_i(\ell) := \{u \in E_i \mid f(u) = \lambda^\ell f(u_0)\}$. Then we have

$$\int_V P^\ell(u_0,ds)f(s) = \lambda^\ell f(u_0)$$

and

$$c_i = |\lambda^\ell f(u_0)| = \left|\int_V P^\ell(u_0,ds)f(s)\right| \le \int_V P^\ell(u_0,ds)|f|(s) =$$

$$= \int_{E_i} P^\ell(u_0,ds)|f|(s) \le c_i .$$

Hence $|f| = c_i$, $P^\ell(u_0,\cdot)$-almost everywhere, and by lemma 2.8 $f = \lambda^\ell f(u_0)$, $P^\ell(u_0,\cdot)$-almost everywhere. This means $P^\ell(u_0,E_i(\ell)) = 1$ for $\ell = 1,2,3,\dots$. Suppose that $E_i(\ell) \cap E_i(m) = \emptyset$ for all pairs (ℓ,m) with $\ell \ne m$. Then $P^\ell(u_0,E_i(m)) = 0$ for $m \ne \ell$ and

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} P^k(u_0,E_i(m)) = \pi_i(E_i(m)) = 0 \quad \text{for all } m \in \mathbb{N} .$$

Hence

$$\text{(1)} \qquad \pi_i\Big(\bigcup_{m=1}^{\infty} E_i(m)\Big) = 0 .$$

However,

$$P^{\ell}(u_0, \bigcup_{m=1}^{\infty} E_i(m)) = 1 \quad \text{for all } \ell \in \mathbb{N}$$

and therefore

$$\pi_i(\bigcup_{m=1}^{\infty} E_i(m)) = 1 ,$$

which contradicts (1).
This implies the existence of a pair (ℓ,m) with $\ell \neq m$ and $E_i(\ell) \cap E_i(m) \neq \emptyset$, and therefore $\lambda^{\ell-m} = 1$. □

For later reference we state the following corollary.

COROLLARY 2.10. Let P be quasi-compact and let d be an integer such that $\lambda^d = 1$ for all $\lambda \in \sigma(P)$ with $|\lambda| = 1$. For $f \in \mathcal{B}$ define

$$f_1 := \lim_{k\to\infty} \frac{1}{k} \sum_{\ell=0}^{k-1} P^{\ell} f .$$

Then $g_m := \lim_{k\to\infty} \sum_{\ell=0}^{kd-1+m} P^{\ell}(f - f_1)$ exists for all $m = 0,1,2,\ldots$ and $\frac{1}{d} \sum_{m=0}^{d-1} g_m$ is a solution of the equation $y - Py = f - f_1$.

PROOF. The existence of g_m follows from lemma 1.12. The rest of the proof is a straightforward verification using $Pg_m = g_{m+1} - f + f_1$ and $g_d = g_0$. □

LEMMA 2.11. Let P be quasi-compact and λ an eigenvalue of P on the unit circle. Then $\mu \in N(\lambda I - P) \Rightarrow v_\mu \in N(I - P)$, where v_μ is the total variation of μ.

PROOF. Let $A_1,\ldots,A_n$ be a partition of V. Then for every $f \in \mathcal{B}$ and $\mu \in \mathcal{M}$ we have by lemma 1.2

$$|\mu f| \leq \sum_{i=1}^{n} \left|\int_{A_i} f d\mu\right| \leq \sum_{i=1}^{n} \sup_{A_1} |f| \cdot v_\mu(A_i) ,$$

and therefore $|\mu f| \leq v_\mu |f|$.
Now let $\mu \in N(\lambda I - P)$. Then for all $E \in \Sigma$ we have

$$v_\mu(E) = v_{\lambda\mu}(E) = v_{\mu P}(E) = \sup \sum_i |(\mu P)(E_i)| \leq$$

$$\leq \sup \sum_i (v_\mu P)(E_i) = (v_\mu P)(E) ,$$

where the supremum has to be taken over all finite partitions $\{E_i\}$ of E. It follows that $v_\mu \le v_\mu P$. Since obviously $v_\mu(V) = (v_\mu P)(V)$, we conclude $v_\mu = v_\mu P$ on Σ. □

LEMMA 2.12. Let P be quasi-compact, dim $N(I - P) = n$, and $\pi_1,\dots,\pi_n$ as in theorem 2.5. Then there exists a real number β, $0 < \beta < 1$ and an integer N such that $\|P^\ell f\| \le \beta^\ell \|f\|$ for all $\ell > N$ and for all functions $f \in \mathcal{B}$ which are π_i almost everywhere equal to zero for $i = 1,\dots,n$.

PROOF. Let λ be an eigenvalue of P on the unit circle. For all $u \in V$ the measure μ_u, with

$$\mu_u(E) = \lim_{n\to\infty} \frac{1}{n} \sum_{\ell=0}^{n-1} \frac{P^\ell(u,E)}{\lambda^\ell},$$

is an element of $N(\lambda I - P)$. Hence, by lemma 2.11, $v_{\mu_u} \in N(I - P)$ and each v_{μ_u} is a linear combination of $\pi_1,\dots,\pi_n$.

Let $f \in \mathcal{B}$ be π_i-almost everywhere equal to 0 for $i = 1,\dots,n$. Then $f = 0$, v_{μ_u}-almost everywhere for each u. Since

$$f_\lambda := \lim_{n\to\infty} \frac{1}{n} \sum_{\ell=0}^{n-1} \frac{P^\ell f}{\lambda^\ell}$$

satisfies $f_\lambda(u) = \mu_u f$ for all $u \in V$ we get $|f_\lambda(u)| = |\mu_u f| \le v_{\mu_u}|f| = 0$ for all $u \in V$.

This implies that $f \in X_\alpha$ where α is the subset of $\sigma(P)$ within the unit circle and X_α is the range of $E_\alpha(P)$, (see section 1.1).

Let $\beta > 0$ be such that $\sup_{\lambda\in\alpha} |\lambda| < \beta < 1$ and let P_α be the restriction of P to X_α. Then there is an integer N such that $\|P_\alpha^\ell\| \le \beta^\ell$ for $\ell > N$ and hence $\|P^\ell f\| \le \beta^\ell \|f\|$ for $\ell > N$. □

COROLLARY 2.13. Let P be quasi-compact, dim $N(I - P) = n$, and $\pi_1,\dots,\pi_n$ as in theorem 2.5. If $A \in \Sigma$ is such that $\pi_i(A) = 0$ for $i = 1,\dots,n$ then $r(PI_A) < 1$.

PROOF. Let β be as in lemma 2.12. For each nonnegative function $f \in \mathcal{B}$ we have for sufficiently large n

$$(PI_A)^n f \le P^n I_A f \le \beta^n \|1_A f\| \le \beta^n \|f\| .$$

Then

$$r(PI_A) = \lim_{n\to\infty} \sqrt[n]{\|(PI_A)^n\|} \le \beta \ . \qquad \square$$

A space of some importance in the sequel is the space M_0, the subspace of M with all measures μ on Σ such that $\mu(V) = 0$. It is easy to see that M_0 is closed and invariant under P. The restriction of P to M_0 is denoted by P_0.

LEMMA 2.14. Let P be quasi-compact and dim $N(I - P) = 1$. Then $1 \in \rho(P_0)$. Let d be an integer such that $\lambda_i^d = 1$ for all eigenvalues λ_i of P on the unit circle. Then there is an integer $N > 0$ and a real number β with $0 < \beta < 1$, such that

$$\|\sum_{\ell=nd}^{nd+d-1} P_0^\ell\| < \beta^n \quad \text{for } n > N \ .$$

PROOF. Because of lemma 1.5 and 1.12 it is sufficient to show $M_0 \subset X_{\tilde{1}}$. Since all elements of $N(I - P)$ are multiples of a probability, $N(I - P_0)$ which is a subspace of $N(I - P)$ contains only the zero. Each $\mu \in M_0$ can be written as $\mu_{\tilde{1}} + \mu_1$ where $\mu_{\tilde{1}} \in X_{\tilde{1}}$ and $\mu_1 \in X_1$. Since

$$\mu_1 = \lim_{n\to\infty} \frac{1}{n} \sum_{\ell=0}^{n-1} \mu P^\ell$$

and $\mu \in M_0$ we conclude $\mu_1 \in M_0$, $\mu_1 \in N(I - P_0)$, and $\mu_1 = 0$, which implies $\mu = \mu_{\tilde{1}} \in X_{\tilde{1}}$. $\square$

There is a close relationship between quasi-compactness, the Doeblin condition, and uniform μ-recurrency. This relationship is studied in the rest of this section.

DEFINITION 2.15. A Markov process P on (V,Σ) is said to satisfy the *Doeblin condition* if there exist an integer $n > 0$, two positive numbers η, θ with $0 < \eta, \theta < 1$ and a probability μ on Σ such that $\mu(F) \ge \theta \Rightarrow P^n(u,F) \ge \eta$, (or equivalently $\mu(F) < 1 - \theta \Rightarrow P^n(u,F) < 1 - \eta$).

A Markov process P satisfies the Doeblin condition if and only if it is quasi-compact. The reader is referred to Neveu [9], V.3.
One direction of the coimplication is shown in the proof of lemma 2.17.

DEFINITION 2.16. Let μ be a positive measure on Σ, not equal to 0, and let P be a Markov process on (V,Σ). P is said to be *μ-recurrent* if for each $A \in \Sigma$ with $\mu(A) > 0$ $\lim_{n\to\infty} (PI_B)^n 1_V(v) = 0$ for $B := V\backslash A$ and for all $v \in V$. If for all $A \in \Sigma$ with $\mu(A) > 0$ the convergence is uniform on V, P is said to be *uniformly μ-recurrent.*

REMARK p.2. If $P(u,E)$ is interpreted as $\mathbb{P}\{X(t+1) \in E \mid X(t) = u\}$, then

$$(PI_B)^n(1_V)(v) = \mathbb{P}\{X(1) \in B,\ X(2) \in B,\dots,X(n) \in B \mid X(0) = v\} .$$

Hence $\lim_{n\to\infty} (PI_B)^n(1_V)(v)$ can be interpreted as the probability that starting in v the system is in B at any time. If this limit is 0 the probability that the system will reach the set A is equal to 1.

The relation between quasi-compactness and uniform μ-recurrency is shown in the next two lemma's.

LEMMA 2.17. Let P be a quasi-compact Markov process on (V,Σ) with $\dim N(I - P) = 1$, and let π be the invariant probability. Then P is uniformly μ-recurrent.

PROOF. Let $A \in \Sigma$ be such that $\pi(A) > 0$ and d an integer such that $\lambda_i^d = 1$ for all eigenvalues λ_i of P on the unit circle. By lemma 2.14 there is a real β, $0 < \beta < 1$ and an integer N such that for all probabilities λ on Σ and for all $n > N$

$$(1)\qquad \left\|\frac{1}{d}\sum_{\ell=0}^{d-1} (\lambda - \pi)P^{nd+\ell}\right\| = \left\|\frac{1}{d}\sum_{\ell=0}^{d-1} \lambda P^{nd+\ell} - \pi\right\| < \beta^n .$$

Substitution of $\lambda(\cdot) = P(u,\cdot)$ in (1) yields

$$\left\|\frac{1}{d}\sum_{\ell=0}^{d-1} P^{nd+\ell+1}(u,\cdot) - \pi(\cdot)\right\| < \beta^n \quad \text{for } n > N .$$

Choose $\theta < \pi(A)$ such that $0 < \theta < \frac{1}{2d}$ and $n_0 > N$ such that $\beta^{n_0} < \theta$. Then for $n > n_0$ we have for every $E \in \Sigma$ with $\pi(E) < \theta$

$$(2)\qquad \frac{1}{d}\sum_{\ell=0}^{d-1} P^{nd+\ell+1}(u,E) < 2\theta \quad \text{for all } u \in V$$

and

$$P^{nd+1}(u,E) < 2\theta d \quad \text{for all } u \in V .$$

At this stage of the proof we actually have shown that the Doeblin condition is satisfied with respect to π.
Define the transition probability Q on $V \times \Sigma$ by

$$Q(u,E) = P(u,E) \qquad \text{for } u \in V\backslash A,\ E \in \Sigma\ ,$$

$$Q(u,E) = \frac{\pi(A \cap E)}{\pi(A)} \qquad \text{for } u \in A,\ E \in \Sigma\ .$$

Using (2) it is easy to verify that Q satisfies the Doeblin condition. The set A is an invariant set of Q. Suppose there are two disjoint invariant sets of Q, F_1 and F_2. If $F_1 \cap A = \emptyset$ and $F_2 \cap A = \emptyset$ then F_1 and F_2 are also invariant sets of P which contradicts the fact that $\dim N(I - P) = 1$. Now let $F_1 \cap A \neq \emptyset$. Then by the definition of Q, $\pi(A \cap F_1) = \pi(A)$ and $F_2 \subset B$. Hence F_2 is an invariant set of P. Therefore $\pi(F_2) = 1$ and $\pi(A) = 0$ which yields a contradiction.
This implies that there are no two disjoint invariant sets of Q. Hence $\dim N(I - Q) = 1$. By corollary 2.13, $r(QI_B) < 1$ for $B := V\backslash A$. Therefore $\lim_{n\to\infty} (QI_B)^n(1_V)(v) = 0$, uniform on V. For $v \in B$ we have $(PI_B)^n 1_V(v) = (QI_B)^n 1_V(v)$. Then for all $u \in V$,

$$(PI_B)^{n+1} 1_V(u) = \int_B P(u,ds)((QI_B)^n 1_V)(s)\ ,$$

which tends to 0 uniformly on V. □

LEMMA 2.18. Let μ be a positive measure on Σ and P a uniformly μ-recurrent Markov process on (V,Σ). Then P satisfies the Doeblin condition.

PROOF. Orey [10], 1.7 proved the existence of a probability π, integers d and n_0, and real numbers a, ρ with $a > 0$ and $0 < \rho < 1$, such that

$$\left\| \frac{1}{d} \sum_{\ell=0}^{d-1} \lambda P^{n+\ell} - \pi \right\| < a.\rho^n \qquad \text{for } n > n_0$$

and for all probabilities λ. The rest of the proof is analogous to the first part of the proof of lemma 2.17. □

2.3. *Embedded Markov processes*

In this section we shall define embedded Markov processes and entry Markov processes and we shall discuss some properties of these processes. These properties will be used in the chapters 3 and 4.

As in the preceding section we shall assume that P is a Markov process on (V,Σ). For convenience we shall write P_E instead of PI_E for $E \in \Sigma$.

The next lemma serves only as an introduction to the concept of the embedded Markov process, which will be defined in definition 2.20.

LEMMA 2.19. Let $A \in \Sigma$, $B := V\backslash A$. Define the function Q on $V \times \Sigma$ by

$$Q(u,E) = \sum_{n=0}^{\infty} (P_B^n P_A 1_E)(u) \quad \text{for all } u \in V,\ E \in \Sigma\ .$$

Then Q is a sub-transition probability on $V \times \Sigma$, the operator Q on $\mathcal{B}(V,\Sigma)$ is given by

$$(1) \qquad (Qf)(u) = \sum_{n=0}^{\infty} (P_B^n P_A f)(u) \quad \text{for } u \in V,\ f \in \mathcal{B}(V,\Sigma)\ ,$$

and the operator Q on $\mathcal{M}(V,\Sigma)$ by

$$(2) \qquad (\mu Q)(E) = \sum_{n=0}^{\infty} (\mu P_B^n P_A)(E) \quad \text{for } E \in \Sigma,\ \mu \in \mathcal{M}(V,\Sigma)\ .$$

Furthermore, Q is a Markov process on (V,Σ) if and only if

$$\lim_{n\to\infty} (P_B^n 1_V)(u) = 0 \quad \text{for all } u \in V\ .$$

PROOF. We have $P_A = P - P_B$. Hence

$$\sum_{n=0}^{N} P_B^n P_A 1_V = \sum_{n=0}^{N} P_B^n P 1_V - \sum_{n=0}^{N} P_B^{n+1} 1_V = \sum_{n=0}^{N} P_B^n 1_V - \sum_{n=0}^{N} P_B^{n+1} 1_V =$$

$$= 1_V - P_B^{N+1} 1_V\ ,$$

which implies that $Q(u,E) \le Q(u,V) \le 1$ for $u \in V$, $E \in \Sigma$. The measurability of Q as function of u and the σ-additivity as function of E are easy to verify. Hence Q is a sub-transition probability on $V \times \Sigma$ and a transition probability if and only if $\lim_{n\to\infty} (P_B^n 1_V)(u) = 0$ for all $u \in V$. The equations (1) and (2) are direct consequences of the definition of Q. □

REMARK p.3. If $P(u,E)$ is interpreted as $\mathbb{P}\{X(t+1) \in E \mid X(t) = u\}$ then

$$(P_B^n 1_V)(u) = \mathbb{P}\{X(1) \in B, X(2) \in B, \ldots, X(n) \in B \mid X(0) = u\} .$$

If $\lim_{n\to\infty} (P_B^n 1_V)(u) = 0$ for all $u \in V$, the system enters the set A almost surely for each initial state, i.e., the random variable indicating the time of the first visit to A, starting in u, is finite, almost surely. In this case, $Q(u,E)$ can be interpreted as the probability that the system is in $A \cap E$ when it enters A for the first time under the condition that at time $t = 0$ the system is in state u. This (sub-) transition probability Q is usually called the embedded, (or induced), (sub-) Markov process. Let τ_u be the random variable indicating the time of the first visit to A, starting in u. Then

$$Q(u,E) = \mathbb{P}\{X(\tau_u) \in E \mid X(0) = u\} .$$

Now we can define embedded Markov processes.

DEFINITION 2.20. Let $A \in \Sigma$, $B := V\backslash A$. The sub-Markov process Q on (V,Σ) with sub-transition probability

$$Q(u,E) := \sum_{n=0}^{\infty} (P_B^n P_A 1_E)(u), \quad u \in V, \ E \in \Sigma ,$$

is called the *embedded* sub-Markov process of P on A.

It follows from lemma 2.19 that Q is a Markov process if and only if $\lim_{n\to\infty} (P_B^n 1_V)(u) = 0$ for all $u \in V$. It is clear that the restriction of Q to $A \times \Sigma_A$ is a sub-transition probability on $A \times \Sigma_A$. We shall denote this process on (A,Σ_A) also by Q. If not stated otherwise we shall consider the embedded process Q on A being a process on (V,Σ).

Notice that $Qf_1 = Qf_2$ on V if $f_1 = f_2$ on A and that $(\mu Q)(E) = 0$ for all $E \subset V\backslash A$ and for all $\mu \in M$.

LEMMA 2.21. Let $A \in \Sigma$, $B := V\backslash A$. Assume that $\lim_{n\to\infty} (P_B^n 1_V)(u) = 0$ for all $u \in V$ and let Q be the embedded Markov process of P on A. If $\mu \in M(V,\Sigma)$ and $f \in B(V,\Sigma)$ are invariant under P, then $\mu I_A Q = \mu I_A$ and $Qf = f$. Conversely, if $Qf = f$, then $Pf = f$ and if E is an invariant set under Q then

$$\bar{E} := \{u \mid Q(u,E) = 1\}$$

is an invariant set under P.

PROOF. The proof of the invariance of μI_A and f under Q is straightforward using

$$\mu I_A P_B^n P_A = (\mu - \mu I_B) P_B^n P_A = \mu P_B^n P_A - \mu I_B P_B^n P_A =$$

$$= \mu P I_B P_B^{n-1} P_A - \mu I_B P_B^n P_A = \mu I_B P_B^{n-1} P_A - \mu I_B P_B^n P_A \,,$$

and

$$P_B^n P_A f = P_B^n P f - P_B^{n+1} f = P_B^n f - P_B^{n+1} f \,.$$

Conversely, suppose Qf = f. Then

$$Pf = P_A f + P_B f = P_A f + P_B Q f = P_A f + P_B \sum_{n=0}^{\infty} P_B^n P_A f =$$

$$= \sum_{n=0}^{\infty} P_B^n P_A f = Qf = f \,.$$

Finally, let E be an invariant set under Q and let $\bar{E} := \{u \mid Q(u,E) = 1\}$. From $Q = P_A + P_B Q$ we conclude

$$Q 1_{A\setminus E} = P 1_{A\setminus E} + P_B Q 1_{A\setminus E} \geq P 1_{A\setminus \bar{E}} + P_{B\setminus \bar{E}} Q 1_{A\setminus E} \,.$$

Since on $\bar{E}$ we have $Q 1_{A\setminus E} = 0$, it follows that $P 1_{A\setminus \bar{E}} = 0$ on $\bar{E}$ and $P_{B\setminus \bar{E}} Q 1_{A\setminus E} = 0$ on $\bar{E}$. From $Q 1_A = 1$ and the definition of $\bar{E}$ we infer that on $V\setminus\bar{E}$ and in particular on $B\setminus\bar{E}$ we have $Q 1_{A\setminus E} > 0$. It follows that $P 1_{B\setminus \bar{E}} = 0$ on $\bar{E}$. Therefore $P 1_{V\setminus \bar{E}} = 0$, $P 1_{\bar{E}} = 1$ on $\bar{E}$. □

The following technical result will be used to show that the embedded process on a set $F \subset A$ of the embedded process on A coincides with the embedded process on F.

LEMMA 2.22. Let $A \in \Sigma$, $F \in \Sigma_A$, $C := V\setminus F$. Let Q be the embedded sub-Markov process of P on A. Then

$$\sum_{n=N}^{\infty} (Q_C^n Q_F 1_E)(u) \leq \sum_{n=N}^{\infty} (P_C^n P_F 1_E)(u) \quad \text{for all } u \in V,\ E \in \Sigma \,.$$

For N = 0 the equality holds.

PROOF. Let $D := A\setminus F$. First we note that for all $u \in V$ and $f \in \mathcal{B}$

$$(Q_C f)(u) = \sum_{n=0}^{\infty} (P_B^n P_A I_C) f(u) = \sum_{n=0}^{\infty} (P_B^n P_D f)(u) \,.$$

Hence

$$(1)\qquad \sum_{n=0}^{\infty} (Q_C^n Q_F 1_E)(u) = \sum_{n=0}^{\infty} \Big(\sum_{m=0}^{\infty} P_B^m P_D\Big)^n \sum_{\ell=0}^{\infty} P_B^{\ell} P_F 1_E(u) =$$

$$= \sum_{n=0}^{\infty} \Big(\sum_{m_i \geq 0} P_B^{m_1} P_D P_B^{m_2} P_D \cdots P_B^{m_n} P_D\Big) \sum_{\ell=0}^{\infty} P_B^{\ell} P_F 1_E(u) =$$

$$= \sum P_B^{n_1} P_D P_B^{n_2} P_D \cdots P_D P_B^{n_m} P_F 1_E(u) ,$$

where the sum has to be taken over all finite sequences $(n_1,\ldots,n_m)$ with $n_i \geq 0$. On the other hand

$$P_C^n = (P_B + P_D)^n = \sum P_B^{n_1} P_D P_B^{n_2} P_D \cdots P_D P_B^{n_\ell} ,$$

where the sum has to be taken over all sequences $(n_1,\ldots,n_\ell)$ with $n_i \geq 0$ and $n_1 + 1 + n_2 + 1 + \ldots + 1 + n_\ell = n$. Substituting this in

$$\sum_{n=0}^{\infty} (P_C^n P_F 1_E)(u)$$

we see that

$$\sum_{n=0}^{\infty} (P_C^n P_F 1_E)(u) = \sum_{n=0}^{\infty} (Q_C^n Q_F 1_E)(u) .$$

The sum

$$\sum_{n=N}^{\infty} (Q_C^n Q_F 1_E)(u)$$

consists of all terms

$$P_B^{n_1} P_D P_B^{n_2} \cdots P_D P_B^{n_m} P_F 1_E(u)$$

in (1) with at least N factors P_D.
The sum

$$\sum_{n=N}^{\infty} (P_C^n P_F 1_E)(u)$$

contains all terms

$$P_B^{n_1} P_D P_B^{n_2} \cdots P_D P_B^{n_m} P_F 1_E(u)$$

where $n_1 + 1 + n_2 + \ldots + n_m \geq N$. Therefore

$$\sum_{n=N}^{\infty} (Q_C^n Q_F 1_E)(u) \leq \sum_{n=N}^{\infty} (P_C^n P_F 1_E)(u) \ . \qquad \square$$

The proof of the next lemma is an immediate consequence of the previous one.

LEMMA 2.23. The embedded (sub-) Markov process on $F \subset A$ of the embedded (sub-) Markov process of P on A is identical to the embedded (sub-) Markov process of P on F.

Now we shall show that (uniform) μ-recurrency of P implies (uniform) μ-recurrency of the embedded Markov process of P on A.

LEMMA 2.24. Let P be (uniformly) μ-recurrent. Then for all A with $\mu(A) > 0$ the embedded sub-Markov process of P on A is a Markov process which is (uniformly) μ-recurrent.

PROOF. Let $A \in \Sigma$ be such that $\mu(A) > 0$ and put $B := V\backslash A$. Since P is μ-recurrent we have by definition $\lim_{n\to\infty} (P_B^n 1_V)(u) = 0$ for all $u \in V$. Then by lemma 2.19 the embedded process Q on A is a Markov process. Let $F \in \Sigma_A$ be such that $\mu(F) > 0$ and let $C := V\backslash F$, $D := A\backslash F$. By lemma 2.22

$$\sum_{n=N}^{\infty} (Q_C^n Q_F 1_V)(u) \leq \sum_{n=N}^{\infty} (P_C^n P_F 1_V)(u) \ .$$

Hence

$$1_V(u) - (Q_C^{N+1} 1_V)(u) = \sum_{n=N}^{\infty} (Q_C^n Q_F 1_V)(u) \leq \sum_{n=N}^{\infty} (P_C^n P_F 1_V)(u) =$$

$$= 1_V(u) - (P_C^{N+1} 1_V)(u) \ .$$

The μ-recurrency of P implies $\lim_{n\to\infty} (Q_C^n 1_V)(u) = 0$ for all $u \in V$. If the convergence of $(P_C^n 1_V)(u)$ is uniform on V then the convergence of $(Q_C^n 1_V)(u)$ is also uniform on V. $\square$

LEMMA 2.25. Let $A, D \in \Sigma$, $B := V\backslash A$, $F := V\backslash D$. Assume that $\lim_{n\to\infty} (P_B^n 1_V)(u) = 0$ on V. Let Q and S be the embedded Markov process of P on A and on $A \cup D$. Let μ be a positive measure on Σ such that Q is μ-recurrent and

$$\int_A S(u,D)\mu(du) > 0 .$$

Then the embedded sub-Markov process of P on D is also a Markov process.

PROOF. If $\mu(D) > 0$ the result is a direct consequence of the μ-recurrency of Q and lemma 2.23. Suppose $\mu(D) = 0$. Then

$$\int_{A\backslash D} S(u,D)\mu(du) = \int_A S(u,D)\mu(du) > 0 ,$$

which implies the existence of an $\varepsilon > 0$ and a set $A_\varepsilon \in \Sigma_{A\backslash D}$ such that $\mu(A_\varepsilon) > 0$ and $S(u,D) > \varepsilon$ for $u \in A_\varepsilon$. Since Q is μ-recurrent the embedded Markov process of Q on A_ε is a Markov process, which by lemma 2.23 coincides with the embedded process of P on A_ε. Using lemma 2.19 it is easy to see that the embedded sub-Markov process T of P, and therefore of S, on $A_\varepsilon \cup D$ is Markovian. Let $B_\varepsilon := V\backslash(A_\varepsilon \cup D)$. From

$$T = \sum_{n=0}^{\infty} S_{B_\varepsilon}^n S_{A_\varepsilon \cup D} = S_{A_\varepsilon \cup D} + S_{B_\varepsilon} T$$

we conclude $T1_{A_\varepsilon} \le S1_{A_\varepsilon} + S1_{B_\varepsilon} = S1_F = 1 - S1_D$, hence $T1_{A_\varepsilon} \le 1 - \varepsilon$ on A_ε. It follows that $(T_{A_\varepsilon}^{n+1} 1_V)(u) \le (1-\varepsilon)^n \to 0$ if $n \to \infty$. Hence the embedded process of T on D which coincides with the embedded process of P on D is Markovian. □

Now we shall define entry Markov processes.

DEFINITION 2.26. Let $A \in \Sigma$ and $B := V\backslash A$. Let Q_1 and Q_2 be the embedded sub-Markov processes of P on A and on B. The *entry* sub-Markov process of P on A is the sub-Markov process Q_2Q_1. If both Q_2 and Q_1 are Markov processes, then Q_2Q_1 is called the entry Markov process of P on A.

REMARK p.4. Since $Q_1(u,E)$ can be interpreted as the probability of being in E the first time that A is entered and since Q_2 has a similar interpretation with respect to B, $(Q_2Q_1)(u,E)$ can be interpreted as the probability of being in E at the first visit to A after having visited B, starting in u. More formalistic, let τ_u' be the random variable indicating the time of the first visit to A after a visit to B, starting in u. Then

$$(Q_2Q_1)(u,E) = \mathbb{P}\{X(\tau_u') \in E \mid X(0) = u\} .$$

In the next lemma we shall show that for every invariant set E of the entry Markov process R of P on A we can find an invariant set $\bar{E}$ of P with $\bar{E} \cap A \supset E \cap A$.

LEMMA 2.27. Let $A \in \Sigma$, $B := V\backslash A$. Assume that $\lim_{n\to\infty} (P_B^n 1_V)(u) = 0$ and $\lim_{n\to\infty} (P_A^n 1_V)(u) = 0$ for all $u \in V$. Let $R := Q_2 Q_1$ be as in definition 2.26. Let E be an invariant set of R. Then the set $\bar{E}$ given by

$$\bar{E} := \{u \in B \mid Q_1(u,E) = 1\} \cup \{u \in A \mid R(u,E) = 1\} ,$$

is an invariant set of P with $\bar{E} \cap A \supset E \cap A$.

PROOF. From $Q_2 = P_B + P_A Q_2$ we conclude

$$R1_{A\backslash E} = P_B Q_1 1_{A\backslash E} + P_A R1_{A\backslash E} \geq P_{B\backslash \bar{E}} Q_1 1_{A\backslash E} + P_{A\backslash \bar{E}} R1_{A\backslash E} .$$

Since $R1_{A\backslash E} = 0$ on $A \cap \bar{E}$ we have $P_{B\backslash \bar{E}} Q_1 1_{A\backslash E} = 0$ on $A \cap \bar{E}$ and $P_{A\backslash \bar{E}} R1_{A\backslash E} = 0$ on $A \cap \bar{E}$. By the definition of $\bar{E}$, $Q_1 1_{A\backslash E} > 0$ on $B\backslash \bar{E}$ and $R1_{A\backslash E} > 0$ on $A\backslash \bar{E}$. Therefore both $P1_{B\backslash \bar{E}} = 0$ on $A \cap \bar{E}$ and $P1_{A\backslash \bar{E}} = 0$ on $A \cap \bar{E}$, which implies that $P1_{\bar{E}} = 1$ on $A \cap \bar{E}$. A similar reasoning applied to $Q_1 1_{A\backslash E}$ yields $P1_{\bar{E}} = 1$ on $B \cap \bar{E}$ and hence $P1_{\bar{E}} = 1$ on $\bar{E}$. □

In lemma 2.24 we proved that (uniform) μ-recurrency of P implies (uniform) μ-recurrency of an embedded Markov process of P on some subset A with $\mu(A) > 0$. Now we shall consider the μ-recurrency and uniform μ-recurrency of entry Markov processes.

LEMMA 2.28. Let P be quasi-compact and suppose $\dim N(I - P) = 1$. Let π be the invariant probability of P and let the set $A \in \Sigma$ be such that $\pi(A) > 0$ and $\pi(V\backslash A) > 0$. Define the measure $\bar{\pi}$ on Σ by $\bar{\pi} := \pi I_B P_A$, where $B := V\backslash A$. Then $\bar{\pi}(V) > 0$ and the entry Markov process of P on A is uniformly $\bar{\pi}$-recurrent.

PROOF. We shall first show that $\bar{\pi}(V) > 0$. Suppose $\bar{\pi}(V) = 0$. Then $P1_A = 0$, π-almost everywhere on B. Since $(\pi P)(A) = \pi(A)$ we therefore have $P1_A = 1_A$, π-alomst everywhere and consequently $P1_B = 1_B$, π-almost everywhere. Then

$$\pi I_A = \pi P I_A = \pi I_A P_A + \pi I_B P_A = \pi I_A P_A = \pi I_A P - \pi I_A P_B = \pi I_A P .$$

Hence πI_A is invariant, $\pi I_A = \pi$ and $\pi(A) = 1$, which contradicts the assumption $\pi(V\backslash A) > 0$. It follows that $\bar{\pi}(V) > 0$. Because of $\pi(A) > 0$, $\pi(B) > 0$, and lemma 2.17 the embedded sub-Markov processes of P on A and on B are

Markov processes. Let R be the entry Markov process of P on A. Let $F \in \Sigma$ be such that $\bar{\pi}(F) > 0$. Then there is an $\varepsilon > 0$ and a set $B_\varepsilon \in \Sigma_B$ such that $\pi(B_\varepsilon) > 0$ and $P(u, F \cap A) > \varepsilon$ for $u \in B_\varepsilon$. Let $C := V \setminus B_\varepsilon$ and $D := V \setminus F$. For $f \in \mathcal{B}(V,\Sigma)$ we have

$$(1) \qquad (Rf)(u) = \sum_{n_1 \geq 0, n_2 \geq 0} (P_A^{n_1} P_B^{1+n_2} P_A f)(u) .$$

We shall prove by induction on N that

$$(2) \qquad \sum_{n=0}^{N-1} P_C^n P_{B_\varepsilon} P 1_{F\cap A} \leq \sum_{n=0}^{N} R_D^n R 1_F = \sum_{n=0}^{N} R_D^n R 1_{F\cap A} .$$

For N = 1 this inequality follows immediately from (1).
Now suppose (2) has been proved for N. Then

$$(3) \qquad \sum_{n=0}^{N} P_C^n P_{B_\varepsilon} P 1_{F\cap A} = P_C \sum_{n=0}^{N-1} P_C^n P_{B_\varepsilon} P 1_{F\cap A} + P_{B_\varepsilon} P 1_{F\cap A} \leq$$

$$\leq P_C \sum_{n=0}^{N} R_D^n R 1_{F\cap A} + P_{B_\varepsilon} P 1_{F\cap A} =$$

$$= (P_A + P_{B\setminus B_\varepsilon}) \sum_{n=0}^{N} R_D^n R 1_{F\cap A} + P_{B_\varepsilon} P 1_{F\cap A} .$$

In order to show that this last expression (3) does not exceed

$$(4) \qquad \sum_{n=0}^{N+1} R_D^n R 1_{F\cap A} .$$

We substitute in (3) and (4)

$$R = \sum_{n_1 \geq 0, n_2 \geq 0} P_A^{n_1} P_B^{1+n_2} P_A$$

and note that each term then occuring in (3) is majorized by a corresponding term in (4).
Since $P1_{F\cap A} \geq \varepsilon . 1_{B_\varepsilon}$ it follows that

$$\sum_{n=0}^{N} R_D^n R 1_F \geq \varepsilon . \sum_{n=0}^{N-1} P_C^n P 1_{B_\varepsilon} .$$

Substitution of $R1_F = 1_V - R1_D$ and $P1_{B_\varepsilon} = 1_V - P1_C$ yields

$$R_D^{N+1} 1_V \le 1 - \varepsilon(1 - P_C^N 1_V) .$$

By lemma 2.17, P is uniformly π-recurrent. This implies the existence of an integer N_0 such that $P_C^{N_0} 1_V < \frac{1}{2}$ on V. Hence, $R_D^{N_0+1} 1_V \le 1 - \frac{1}{2}\varepsilon$ and $R_D^{m(N_0+1)} 1_V \le (1 - \frac{1}{2}\varepsilon)^m$. Therefore $\lim_{n\to\infty} R_D^n 1_V = 0$, uniform on V. □

The next two lemma's will be used to derive conditions for the μ-recurrency of entry Markov processes.

LEMMA 2.29. Let $A \in \Sigma$, $B := V\backslash A$. Define

$$P_{m,n} := \sum P_B^{n_1} P_A P_B^{n_2} P_A \ldots P_B^{n_m} P_A P_B^{n_{m+1}} ,$$

where the sum has to be taken over all sequences $n_1, \ldots, n_{m+1}$ with $n_i \ge 0$ and $\sum_{i=1}^{m+1} n_i + m = n$.

Suppose $\lim_{n\to\infty} (P_B^n 1_V)(u) = 0$ on V and the convergence is uniform on some set $A' \supset A$. Then $\lim_{n\to\infty} (P_{m,n} 1_V)(u) = 0$ on V and the convergence is uniform on A'.

PROOF. Let $Q := \sum_{n=0}^{\infty} P_B^n P_A$ be the embedded Markov process of P on A. Here as well as in the following proofs of this section we have to interprete infinite sums of operators in the pointwise convergence. Put $Q_N := \sum_{n=0}^{N} P_B^n P_A$. Then $Q^m - Q_N^m = \sum P_B^{n_1} P_A \ldots P_B^{n_m} P_A$, where the sum has to be taken over all sequences $n_1, n_2, \ldots, n_m$ with at least one $n_i > N$. Now it is easy to verify that for each $p \ge 2$

$$(P_{m,m(N+p)} 1_V)(u) \le (Q^m 1_V)(u) - (Q_N^m 1_V)(u) = Q^{m-1}(Q - Q_N) 1_V(u) +$$

$$+ Q^{m-2}(Q - Q_N) Q_N 1_V(u) + \ldots + (Q - Q_N) Q_N^{m-1} 1_V(u) \le$$

$$\le (Q^{m-1} + Q^{m-2} + \ldots + Q) P_B^{N+1} 1_V(u) + P_B^{N+1} Q_N^{m-1} 1_V(u) \le$$

$$\le (m-1) \cdot \sup_{v\in A} (P_B^{N+1} 1_V)(v) + (P_B^{N+1} 1_V)(u) .$$

Now the required result is a direct consequence of the assumptions on $P_B^n 1_V$. □

REMARK p.5. The expression $(P_{m,n} 1_V)(u)$ can be interpreted as the probability that, starting in u, in the first n steps of the process the set A is visited m times. Lemma 2.29 states that the probability of being m times in A in n steps of the process is small for large n.

This result is extended in the following lemma.

LEMMA 2.30. Let $A \in \Sigma$, $B := V\backslash A$. Suppose $\lim_{n\to\infty} (P_B^n 1_V)(u) = 0$ on V and the convergence is uniform on A. Let Q be the embedded Markov process of P on A and let $D \in \Sigma_A$ be such that $\lim_{n\to\infty} (Q_D^n 1_V)(u) = 0$, uniform on A. Put $F := A\backslash D$ and $C := V\backslash F$. Let

$$P'_{m,n} = \sum P_C^{n_1} P_F P_C^{n_2} P_F \dots P_C^{n_m} P_F P_C^{n_{m+1}} ,$$

where the sum has to be taken over all sequences $n_1,\dots,n_{m+1}$ with $n_i \geq 0$ and $\sum_{i=1}^{m+1} n_i + m = n$. Then

i) $\lim_{n\to\infty} (P_C^n 1_V)(u) = 0$ on V and the convergence is uniform on A
ii) for all $m \in \mathbb{N}$ $\lim_{n\to\infty} (P'_{m,n} 1_V)(u) = 0$ on V and the convergence is uniform on A.

PROOF. The second statement is an immediate consequence of the first one and lemma 2.29. Therefore it suffices to prove i).
Since $Qf = QI_A f$ for $f \in \mathcal{B}$ we have $Q_D = Q_C$. Hence $\lim_{n\to\infty} (Q_C^n 1_V)(u) = 0$, uniform on A. But

$$Q_C^n 1_V = Q_D^n 1_V = Q_D Q_D^{n-1} 1_V \leq \| I_A Q_D^{n-1} 1_V \|$$

and therefore $\lim_{n\to\infty} (Q_C^n 1_V)(u) = 0$, uniform on V. By lemma 2.22,

$$\sum_{n=0}^{\infty} (P_C^n P_F 1_V)(u) = \sum_{n=0}^{\infty} (Q_C^n Q_F 1_V)(u) .$$

Since

$$\sum_{n=0}^{N} (Q_C^n Q_F 1_V)(u) = 1 - (Q_C^{N+1} 1_V)(u)$$

and

$$\sum_{n=0}^{N} (P_C^n P_F 1_V)(u) = 1 - (P_C^{N+1} 1_V)(u) ,$$

this implies $\lim_{n\to\infty} (P_C^n 1_V)(u) = 0$ on V. We have to prove that the convergence is uniform on A. Put

$$P_{Cm,n} = \sum P_B^{n_1} P_D P_B^{n_2} P_D \ldots P_B^{n_m} P_D P_B^{n_{m+1}} ,$$

and

$$P_{m,n} = \sum P_B^{n_1} P_A P_B^{n_2} P_A \ldots P_B^{n_m} P_A P_B^{n_{m+1}} ,$$

where both sums has to be taken over all sequences $n_1,\ldots,n_{m+1}$ with $n_i \geq 0$ and $\sum_{i=1}^{m+1} n_i + m = n$. Obviously

(1) $$(P_{Cm,n} 1_V)(u) \leq (P_{m,n} 1_V)(u)$$

and

(2) $$(P_C^n 1_V)(u) = \sum_{m=0}^{n} (P_{Cm,n} 1_V)(u) .$$

We get

(3) $$\sum_{m=N}^{n} (P_{Cm,n} 1_V)(u) =$$

$$= \sum_{m=N}^{n} \sum_{\ell=m-N}^{n-N} \sum_{n_1+n_2+\ldots+n_N=n-N-\ell} P_B^{n_1} P_D P_B^{n_2} P_D \ldots P_B^{n_N} P_D P_{Cm-N,\ell} 1_V(u) =$$

$$= \sum_{\ell=0}^{n-N} \sum_{n_1+n_2+\ldots+n_N=n-N-\ell} P_B^{n_1} P_D P_B^{n_2} \ldots P_B^{n_N} P_D \sum_{m=N}^{N+\ell} P_{Cm-N,\ell} 1_V(u) =$$

$$= \sum_{\ell=0}^{n-N} \sum_{n_1+n_2+\ldots+n_N=n-N-\ell} P_B^{n_1} P_D P_B^{n_2} \ldots P_B^{n_N} P_D P_C^{\ell} 1_V(u) \leq$$

$$\leq \sum_{\ell=0}^{n-N} \sum_{n_1+n_2+\ldots+n_N=n-N-\ell} P_B^{n_1} P_D \ldots P_B^{n_N} P_D 1_V(u) \leq (Q_D^N 1_V)(u) .$$

Let $\varepsilon > 0$. Choose N_1 such that $(Q_D^n 1_V)(u) < \varepsilon$ for $u \in A$, $n > N_1$. By the previous lemma there is an integer $N_2 > N_1$ such that for all $n > N_2$ we have $(P_{m,n} 1_V)(u) < \frac{\varepsilon}{N_1 + 1}$ for all $u \in A$, $m \leq N_1$. Then by (1), (2) and (3) we have for $n > N_2$ and $u \in A$

$$(P_C^n 1_V)(u) = \sum_{m=0}^{n} (P_{Cm,n} 1_V)(u) = \sum_{m=0}^{N_1} (P_{Cm,n} 1_V)(u) + \sum_{m=N_1+1}^{n} (P_{Cm,n} 1_V)(u) \leq$$

$$\leq \sum_{m=0}^{N_1} (P_{m,n} 1_V)(u) + (Q_D^{N_1+1} 1_V)(u) \leq 2\varepsilon .$$

Hence $\lim_{n\to\infty} (P_C^n 1_V)(u) = 0$, uniform on A. □

LEMMA 2.31. Let $A \in \Sigma$, $B := V\backslash A$. Suppose $\lim_{n\to\infty} (P_B^n 1_V)(u) = 0$ on V and the convergence is uniform on A. Let the embedded Markov process Q of P on A be uniformly π-recurrent. Let C be a set such that $\pi(A\backslash C) > 0$. Put $D := V\backslash(A \cup C)$ and let S be the embedded Markov process of P on $A \cup C$. Define the measure $\tilde{\pi}$ on Σ by $\tilde{\pi} = \pi I_{A\backslash C} S$. If $\tilde{\pi}(C) > 0$ then the entry sub-Markov process R of P on C is Markovian and $\tilde{\pi}$-recurrent.

PROOF. Suppose $\tilde{\pi}(C) > 0$. By lemma 2.25 the embedded sub-Markov process of P on C is a Markov process. Since $\pi(A \backslash C) > 0$ and Q is π-recurrent, the embedded process of Q and therefore of P on $A\backslash C$ is Markovian. It follows that the embedded process of P on $V\backslash C$ is also Markovian. Hence R is a Markov process.

Now let F be an element of Σ_C such that $\tilde{\pi}(F) > 0$. Put $H := C\backslash F$. Then there is an $\varepsilon > 0$ and a set $A_\varepsilon \in \Sigma_{A\backslash C}$ such that $\pi(A_\varepsilon) > 0$ and $S(u,F) > \varepsilon$ for $u \in A_\varepsilon$. We have to prove that $\lim_{n\to\infty} (R_H^n 1_V)(u) = 0$ on V. For P^n we can write $P^n = \sum P_1 P_2 \dots P_n$, where the sum has to be taken over all terms $P_1 P_2 \dots P_n$ with $P_i \in \{P_{V\backslash C}, P_C\}$. The operator which we get if the factors $P_{V\backslash C} P_C$ in these terms are replaced by $P_{V\backslash C} P_H$, is denoted by T_n. Obviously $T_n 1_V \leq P^n 1_V$. Substitution of

$$R_H = \sum_{\ell_1 \geq 0, \ell_2 \geq 0} P_C^{\ell_1} P_{V\backslash C}^{1+\ell_2} P_H$$

in R_H^n yields

$$R_H^n = \sum_{\ell_1 \geq 0, \ell_2 \geq 0, \dots, \ell_{2n} \geq 0} P_C^{\ell_1} P_{V\backslash C}^{1+\ell_2} P_H P_C^{\ell_3} \dots P_C^{\ell_{2n-1}} P_{V\backslash C}^{1+\ell_{2n}} P_H .$$

The first n factors in these terms correspond to terms in T_n. Let $P_1 \dots P_n$ be a term in T_n and define $(P_1 \dots P_n)^*$ as the sum of all terms in R_H^n which start with the factor $P_1 \dots P_n$. Then $R_H^n = \sum (P_1 \dots P_n)^*$, where the sum has to be taken over all terms $P_1 P_2 \dots P_n$ in T_n.

We shall prove that

(1) $$(P_1P_2\dots P_n)^*1_V \le P_1P_2\dots P_n1_V\ .$$

Suppose there are k factors P_H in $P_1\dots P_n$. If $P_n = P_C$ or P_H, then

(2) $$(P_1\dots P_n)^*1_V = P_1\dots P_n\sum P_C^{\ell_1}P_{V\setminus C}^{\ell_2+1}P_H\dots P_H1_V =$$
$$= P_1\dots P_nR_H^{n-k}1_V \le P_1\dots P_n1_V\ .$$

If $P_n = P_{V\setminus C}$, then

(3) $$(P_1\dots P_n)^*1_V = P_1\dots P_n\sum P_{V\setminus C}^{\ell_1}P_HP_C^{\ell_2}P_{V\setminus C}^{1+\ell_3}P_H\dots P_H1_V =$$
$$= P_1\dots P_n\Big(\sum_{\ell_1=0}^{\infty}P_{V\setminus C}^{\ell_1}P_H\Big)R_H^{n-k-1}1_V \le P_1\dots P_n1_V\ .$$

The results (2) and (3) prove (1). Hence

(4) $$R_H^n1_V = \sum(P_1\dots P_n)^*1_V \le \sum P_1\dots P_n1_V = T_n1_V\ .$$

Let $G \supset C$. Substitution of $P_{V\setminus C} = P_{V\setminus G} + P_{G\setminus C}$ in the terms of T_n yields $T_n = \sum P_1P_2\dots P_n$ where the sum has to be taken over all terms $P_1P_2\dots P_n$ with $P_i \in \{P_{V\setminus G}, P_{G\setminus C}, P_C\}$, the factors $P_{V\setminus G}P_C$ and $P_{G\setminus C}P_C$ replaced by $P_{V\setminus G}P_H$ and $P_{G\setminus C}P_H$. For P^n we can also write $P^n = \sum P_1P_2\dots P_n$, where the sum has to be taken over all terms $P_1P_2\dots P_n$ with $P_i \in \{P_{V\setminus G}, P_G\}$. The operator which we get if all factors $P_{V\setminus G}P_G$ are replaced by $P_{V\setminus G}P_{G\setminus F}$ is denoted by T_n'. Substitution of $P_G = P_{G\setminus C} + P_C$ and $P_{G\setminus F} = P_{G\setminus C} + P_H$ yields $T_n' = \sum P_1P_2\dots P_n$, where the sum has to be taken over all terms $P_1P_2\dots P_n$ with $P_i \in \{P_{V\setminus G}, P_{G\setminus C}, P_C\}$, the factors $P_{V\setminus G}P_C$ replaced by $P_{V\setminus G}P_H$. We saw that T_n is equal to the same sum with the factors $P_{G\setminus C}P_C$ also replaced by $P_{G\setminus C}P_H$. Hence

(5) $$T_n1_V \le T_n'1_V \le P^n1_V\ .$$

We can use this result for $G := (A \cup C)\setminus A_\varepsilon$. Substitution of $P_{V\setminus G} = P_{A_\varepsilon} + P_D$ yields $T_n' = \sum P_1\dots P_n$, where the sum has to be taken over all terms $P_1P_2\dots P_n$ with $P_i \in \{P_{A_\varepsilon}, P_D, P_G\}$, the factors $P_{A_\varepsilon}P_G$ and P_DP_G replaced by $P_{A_\varepsilon}P_{G\setminus F}$ and $P_DP_{G\setminus F}$. The operator which we get if we only replace the first factor P_G after a factor P_{A_ε} by $P_{G\setminus F}$ is denoted by T_n^*. Obviously

(6) $$T_n' 1_V \le T_n^* 1_V \le P^n 1_V \; .$$

We know that $P_{V\setminus A_\varepsilon}^n = \sum P_1 P_2 \dots P_n$, where the sum has to be taken over all terms $P_1 P_2 \dots P_n$ with $P_i \in \{P_D, P_G\}$. The operator which we get if the first factor P_G in these terms is replaced by $P_{G\setminus F}$, is denoted by T_n^ε.
Since $\pi(A_\varepsilon) > 0$ and Q is π-recurrent the process $Q_\varepsilon := \sum_{n=0}^{\infty} P_{V\setminus A_\varepsilon}^n P_{A_\varepsilon}$ is a Markov process. The sub-Markov process which we get if the factors $P_{V\setminus A_\varepsilon}^n$ in Q_ε are replaced by T_n^ε, is called Q_ε^*, $(Q_\varepsilon^* = \sum_{n=0}^{\infty} T_n^\varepsilon P_{A_\varepsilon})$.
Let $T_{m,n}^\varepsilon$ be the sum of all terms in T_n^ε with m factors P_G, the factor P_G which is replaced by $P_{G\setminus F}$ included. Then

(7) $$Q_\varepsilon^* = \sum_{n=0}^{\infty} T_n^\varepsilon P_{A_\varepsilon} = \sum_{n=0}^{\infty} \sum_{m=0}^{n} T_{m,n}^\varepsilon P_{A_\varepsilon} = \sum_{n=0}^{\infty} T_{0,n}^\varepsilon P_{A_\varepsilon} + \sum_{n=1}^{\infty} \sum_{m=1}^{n} T_{m,n}^\varepsilon P_{A_\varepsilon} =$$

$$= \sum_{n=0}^{\infty} P_D^n P_{A_\varepsilon} + \sum_{n=1}^{\infty} \sum_{m=1}^{n} T_{m,n}^\varepsilon P_{A_\varepsilon} \; .$$

Let $T_{n,1,\ell}^\varepsilon$ be the sum of all terms in T_n^ε with the factor $P_{G\setminus F}$ on the ℓ-th place. Then

$$\sum_{m=1}^{n} T_{m,n}^\varepsilon = \sum_{\ell=1}^{n} T_{n,1,\ell}^\varepsilon \; ,$$

and for all nonnegative f

$$T_{n,1,\ell}^\varepsilon f \le T_{\ell,1,\ell}^\varepsilon P_{V\setminus A_\varepsilon}^{n-\ell} f \; .$$

Hence

$$\sum_{n=1}^{\infty} \sum_{m=1}^{n} T_{m,n}^\varepsilon P_{A_\varepsilon} 1_V \le \sum_{n=1}^{\infty} \sum_{\ell=1}^{n} T_{\ell,1,\ell}^\varepsilon P_{V\setminus A_\varepsilon}^{n-\ell} P_{A_\varepsilon} 1_V =$$

$$= \sum_{\ell=1}^{\infty} T_{\ell,1,\ell}^\varepsilon \sum_{n=\ell}^{\infty} P_{V\setminus A_\varepsilon}^{n-\ell} P_{A_\varepsilon} 1_V = \sum_{\ell=1}^{\infty} T_{\ell,1,\ell}^\varepsilon Q_\varepsilon 1_V =$$

$$= \sum_{\ell=1}^{\infty} T_{\ell,1,\ell}^\varepsilon 1_V = \sum_{\ell=1}^{\infty} P_D^{\ell-1} P_{G\setminus F} 1_V = \sum_{n=0}^{\infty} P_D^n P_{G\setminus F} 1_V \; .$$

Therefore, by (7),

$$(8) \qquad Q_\varepsilon^* 1_V \le \sum_{n=0}^{\infty} P_D^n P_{A_\varepsilon} 1_V + \sum_{n=0}^{\infty} P_D^n P_{G\setminus F} 1_V = S_{A_\varepsilon} 1_V + S_{G\setminus F} 1_V =$$

$$= S 1_V - S_F 1_V \le 1 - \varepsilon, \text{ on } A_\varepsilon .$$

By (4), (5), (6) we have $R_H^n 1_V \le T_n^* 1_V$. The sum of all terms in T_n^* with m factors P_{A_ε} is denoted by $T_{m,n}^*$. We shall prove that

$$(9) \qquad \sum_{m=k}^{n} T_{m,n}^* 1_V \le Q_\varepsilon (Q_\varepsilon^*)^{k-1} 1_V .$$

Let $T_{n,k,\ell}^*$ be the sum of all terms in T_n^* with the k-th factor P_{A_ε} on the ℓ-th place. Substitution of

$$Q_\varepsilon^* = \sum_{n=0}^{\infty} T_n^\varepsilon P_{A_\varepsilon} \text{ in } Q_\varepsilon (Q_\varepsilon^*)^{k-1}$$

yields

$$\sum_{\ell=k}^{\infty} T_{\ell,k,\ell}^* 1_V \le Q_\varepsilon (Q_\varepsilon^*)^{k-1} 1_V .$$

Since moreover we have $\sum_{m=k}^{n} T_{m,n}^* = \sum_{\ell=k}^{n} T_{n,k,\ell}^*$ and $T_{n,k,\ell}^* 1_V \le T_{\ell,k,\ell}^* 1_V$, it follows that

$$\sum_{m=k}^{n} T_{m,n}^* 1_V = \sum_{\ell=k}^{n} T_{n,k,\ell}^* 1_V \le \sum_{\ell=k}^{n} T_{\ell,k,\ell}^* 1_V \le Q_\varepsilon (Q_\varepsilon^*)^{k-1} 1_V .$$

By (4), (5), (6), and (9), we get

$$(10) \qquad R_H^n 1_V \le T_n^* 1_V = \sum_{m=0}^{n} T_{m,n}^* 1_V = \sum_{m=0}^{k-1} T_{m,n}^* 1_V + \sum_{m=k}^{n} T_{m,n}^* 1_V \le \sum_{m=0}^{k-1} T_{m,n}^* 1_V +$$

$$+ Q_\varepsilon (Q_\varepsilon^*)^{k-1} 1_V .$$

A straightforward application of lemma 2.30 yields

$$(11) \qquad \lim_{n\to\infty} \sum_{m=0}^{k-1} T_{m,n}^* 1_V = 0 \text{ on } V .$$

From (8) we conclude $Q_\varepsilon (Q_\varepsilon^*)^{k-1} 1_V \le (1-\varepsilon)^{k-1}$. Now, using (7), it is easy to verify that $\lim_{n\to\infty} R_H^n 1_V = 0$. □

To conclude this section we shall give an extension of lemma 2.6.

LEMMA 2.32. Let $A,C \in \Sigma$, $B := V\backslash A$, $D := V\backslash C$. Suppose that the embedded processes Q on A and S on $A \cup C$ are Markov processes. If there exists a probability $\pi \in N(I-Q)$ such that $\pi(A \cap C) = 0$ and $S1_C = 0$, π-almost everywhere, then there exists a set $G \subset D$ such that $\pi(G) = 1$ and $P(u,G) = 1$ for all $u \in G$.

PROOF. From $Q1_A = 1$ we conclude that $\pi(A) = 1$. Hence if

$$H' := \{u \in A \cap D \mid S(u,C) = 0\} ,$$

then $\pi(H') = 1$ and by lemma 2.6 there is a set $H \subset H'$ such that $\pi(H) = 1$ and $Q(u,H) = 1$ for $u \in H$. This implies

$$S(u,A\backslash H) = \sum_{n=0}^{\infty} (P^n_{B\cap D} P_{A\cup C} 1_{A\backslash H})(u) \le \sum_{n=0}^{\infty} (P^n_B P_A 1_{A\backslash H})(u) = Q(u,A\backslash H) = 0$$

for $u \in H$. Now we shall prove by induction on n that

$$(1) \qquad (S^n_{A\backslash C} S1_C)(u) = 0 \quad \text{for } u \in H .$$

For $n = 0$ we have $(S^n_{A\backslash C} S1_C)(u) = (S1_C)(u) = S(u,C) = 0$ for $u \in H \subset H'$. Now let it be true for $n = k$. For $n = k+1$ we have

$$(S^{k+1}_{A\backslash C} S1_C)(u) = (S_{(A\backslash C)\backslash H} S^k_{A\backslash C} S1_C)(u) + (S_H S^k_{A\backslash C} S1_C)(u) .$$

The induction assumption implies $(S_H S^k_{A\backslash C} S1_C)(u) = 0$. Hence

$$(S^{k+1}_{A\backslash C} S1_C)(u) \le (S_{(A\backslash C)\backslash H} 1_V)(u) \le S(u,A\backslash H) = 0 \quad \text{for } u \in H .$$

This completes the proof of (1). Therefore $\sum_{n=0}^{\infty} (S^n_{A\backslash C} S1_C)(u) = 0$ on H. Since, by definition $S_{D\backslash A} f = 0$ for all $f \in \mathcal{B}$ we have $S_{A\backslash C} = S_D$ and hence by lemma 2.22

$$\sum_{n=0}^{\infty} (S^n_{A\backslash C} S1_C)(u) = \sum_{n=0}^{\infty} (P^n_D P1_C)(u) \quad \text{for } u \in V .$$

Let

$$G := \{u \in D \mid \sum_{n=0}^{\infty} (P^n_D P1_C)(u) = 0\} .$$

Then $G \supset H$ and therefore $\pi(G) = 1$. Since

$$\sum_{n=0}^{\infty} (P_D^n P 1_C)(u) = P(u,C) + (P_D \sum_{n=0}^{\infty} P_D^n P 1_C)(u)$$

we get $P(u,C) = 0$ for $u \in \underset{\sim}{G}$ and $P(u,D\setminus G) = 0$ for $u \in G$. Hence $P(u,G) = 1$ for $u \in G$. □

CHAPTER 3. MARKOV PROCESSES WITH COSTS

Let P be a Markov process on (V,Σ) and r a nonnegative, not necessarily bounded, measurable function on V. The pair (P,r) is called a *Markov process with costs* on (V,Σ). The function r is the *costfunction*.

In section 2.1 the expression (Pf)(u) for $f \in \mathcal{B}(V,\Sigma)$ and $u \in V$ is defined as the integral $\int P(u,ds)f(s)$. Now we shall also work with unbounded functions. If for a complex valued measurable function f the integral $\int P(u,ds)f(s)$ exists then it is also denoted by (Pf)(u). If (Pf)(u) exists for all $u \in V$ then we shall speak about the function Pf.

If for all $u \in V$ $(P^{\ell}r)(u)$ exists for all $\ell \in \mathbb{N}$ and if $\lim_{n\to\infty} \frac{1}{n} \sum_{\ell=0}^{n-1} (P^{\ell}r)(u)$ exists, this limit is called the *average costs* of (P,r), starting in u, and is denoted by g(u).

REMARK p.6. The name, average costs, is clear by the probabilistic interpretation of P: If

$$P(u,E) = \mathbb{P}\{X(t+1) \in E \mid X(t) = u\}$$

then

$$(P^n r)(u) = \mathbb{E}\{r(X(t+n)) \mid X(t) = u\} .$$

We are interested in the system of equations in x and y:

(1) $\quad x = Px$

(2) $\quad y = r - x + Py$,

where x and y are complex valued measurable functions on V. These equations are called the *(P,r)-equations*.

The next lemma is a direct consequence of corollary 2.10.

LEMMA 3.1. Let P be quasi-compact and r bounded. Let the integer d be such that $\lambda_i^d = 1$ for all eigenvalues λ_i of P on the unit circle. Then the functions

$$x := \lim_{n\to\infty} \frac{1}{n} \sum_{\ell=0}^{n-1} P^{\ell} r$$

$$y := \frac{1}{d} \sum_{m=0}^{d-1} \lim_{k\to\infty} \sum_{\ell=0}^{kd-1+m} P^{\ell}(r - x)$$

are a solution of the (P,r)-equations.

In section 3.1 we shall prove the existence of a solution of the (P,r)-equations under somewhat weaker conditions. The quasi-compactness of P is replaced by the quasi-compactness of the embedded Markov process of P on some set $A \subset V$. The boundedness of r is replaced by the boundedness of $\sum_{\ell=0}^{\infty} P_B^{\ell} r$ on A, $(B := V\backslash A)$. In section 3.2 some properties of the solution will be given. The function x will again turn out to be equal to the average costs.

3.1. *Existence of a solution*

DEFINITION 3.2. Let f be a nonnegative measurable real valued function on V and let A be a measurable set. The Markov process P is said to be (A,f)-*recurrent* if

i) $P_B^m f$ exists for all $m \in \mathbb{N}$, $(B := V\backslash A)$,

ii) the sum $\sum_{m=0}^{\infty} (P_B^m f)(u)$ exists for all $u \in V$,

iii) the convergence of $\sum_{m=0}^{\infty} (P_B^m f)(u)$ is uniform on A and $\sum_{m=0}^{\infty} (P_B^m f)(u)$ is bounded on A.

We assume that A is a fixed measurable set such that P is $(A,1_V)$-recurrent and (A,r)-recurrent and further that the embedded Markov process Q of P on A is quasi-compact, (Q interpreted as a Markov process on (A,Σ_A)). The $(A,1_V)$-recurrency implies that the embedded sub-Markov process of P on A is a Markov process.
Let E_j, $j = 1,\dots,n$ be the maximal invariant sets of Q, $F := \bigcup_{j=1}^{n} E_j$, and $\Delta := A\backslash F$.

LEMMA 3.3. For each $m \in \mathbb{N}$,

$$(1) \qquad Q^m = \sum_{k=0}^{m-1} Q_\Delta^k \sum_{j=1}^{n} Q_{E_j}^{m-k} + Q_\Delta^m .$$

PROOF. It is easy to see that $Q_{E_j} Q_\Delta = 0$ for all $j = 1,2,\dots,n$ and $Q_{E_i} Q_{E_j} = 0$ if $i \neq j$. Substituting this in

$$Q^m = \left(\sum_{j=1}^{n} Q_{E_j} + Q_\Delta\right)^m$$

yields the expression (1). □

LEMMA 3.4. For all $f \in \mathcal{B}(V,\Sigma)$ and for all $m \in \mathbb{N}$ the following relation holds, $(B := V\backslash A)$

$$(P_B^m Qf)(u) = (P_B^m Qf_{E_j}), \quad u \in E_j, \; j = 1,2,\dots,n .$$

PROOF. It is sufficient to prove the assertion for nonnegative functions, namely, each $f \in \mathcal{B}(V,\Sigma)$ can be written as $f = f_1 - f_2 + i(f_3 - f_4)$, where the functions f_1, f_2, f_3 and f_4 are nonnegative elements of $\mathcal{B}$. Now assume that f is a nonnegative function in $\mathcal{B}$. Substitution of $Qf_E = \sum_{\ell=0}^{\infty} P_B^\ell P_A f_E$ in $P_B^m Qf_E$ yields

$$(1) \qquad P_B^m Qf_E = \sum_{\ell=m}^{\infty} P_B^\ell P_A f_E \le \sum_{\ell=0}^{\infty} P_B^\ell P_A f_E = Qf_E \quad \text{for all } E \in \Sigma .$$

Furthermore

$$(2) \qquad (Qf)(u) = (Qf_{E_j})(u) \quad \text{for } u \in E_j, \; j = 1,\dots,n .$$

Let $\bar{E}_j := V\backslash E_j$, then $f = f_{E_j} + f_{\bar{E}_j}$. By (1) and (2)

$$(P_B^m Qf_{\bar{E}_j})(u) \le (Qf_{\bar{E}_j})(u) = 0 \quad \text{for } u \in E_j .$$

Hence

$$(P_B^m Qf)(u) = (P_B^m Qf_{E_j})(u) \quad \text{for } u \in E_j, \; j = 1,\dots,n .$$

□

Now we can prove the existence of a solution of the (P,r)-equations.

THEOREM 3.5. The equations $x = Px$ and $y = r - x + Py$ in x and y, where x and y are complex valued measurable functions on V, have a solution.

PROOF. By corollary 2.13 the spectral radius of Q_Δ, $r(Q_\Delta) < 1$. Hence, each of the equations $x = Q_{E_j} 1_{E_j} + Q_\Delta x$ in $\mathcal{B}(A,\Sigma_A)$ has a unique solution

$$g_j := \sum_{\ell=0}^{\infty} Q_\Delta^\ell Q_{E_j} 1_{E_j} .$$

Using $(Q_\Delta f)(u) = 0$ for all $f \in \mathcal{B}(A,\Sigma_A)$, $u \in E_j$, $j = 1,2,\ldots,n$, we get $g_j(u) = 1$ for $u \in E_j$ and $g_j(u) = 0$ for $u \in E_i$ if $i \neq j$. This means that $Q_{E_j} 1_{E_j} = Q_{E_j} g_j = (Q - Q_\Delta) g_j$ and that g_j is a solution of the equation $x - Qx = 0$ in $\mathcal{B}(A,\Sigma_A)$.
It is possible to extend g_j to a solution g_j^* of the equation $x - Qx = 0$ in $\mathcal{B}(V,\Sigma)$ by defining $g_j^* := Qg_j$, where Q is used as an operator on $\mathcal{B}(A,\Sigma_A)$ to $\mathcal{B}(V,\Sigma)$.
Each function g_j^*, $j = 1,\ldots,n$ is a solution of the equation $x = Px$ since

$$Pg_j^* = P_B g_j^* + P_A g_j^* = P_B Q g_j^* + P_A g_j^* = \sum_{\ell=1}^{\infty} P_B^\ell P_A g_j^* + P_A g_j^* = Qg_j^* = g_j^* .$$

The problem is to choose a linear combination x of the g_j^* such that the equation $y = r - x + Px$ has also a solution. Let Q_j be the restriction of Q to $E_j \times \Sigma_{E_j}$. The (A,r)-recurrency and $(A,1_V)$-recurrency of P imply the boundedness on A of the functions $\sum_{\ell=0}^{\infty} P_B^\ell r$ and $\sum_{\ell=0}^{\infty} P_B^\ell g_j^*$, $j = 1,\ldots,n$. For convenience we shall write Tf instead of $\sum_{\ell=0}^{\infty} P_B^\ell f$ for $f = r$ or f is bounded. Notice that $P_B Tf = \sum_{\ell=1}^{\infty} P_B^\ell f$.
The restrictions of Tr and Tg_j^* to E_j are elements of $\mathcal{B}(E_j,\Sigma_{E_j})$. Therefore both $\lim_{n\to\infty} \frac{1}{n} \sum_{\ell=0}^{n-1} Q_j^\ell Tr$ and $\lim_{n\to\infty} \frac{1}{n} \sum_{\ell=0}^{n-1} Q_j^\ell T g_j^*$ are elements of $N(I - Q_j)$. Since $\dim N(I - Q_j) = 1$ there is a constant c_j such that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{\ell=0}^{n-1} Q_j^\ell (Tr - c_j T g_j^*) = 0 .$$

Define the function $g^* \in \mathcal{B}(V,\Sigma)$ by $g^* := \sum_{j=1}^{n} c_j g_j^*$. We shall show that the equation $y = r - g^* + Py$ has a solution. This can be done by proving that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{\ell=0}^{n-1} Q^{\ell}(Tr - Tg^*) = 0 \text{ on } A .$$

For the moment we assume that this is true.
Let the integer d be such that $\lambda_i^d = 1$ for all eigenvalues λ_i of Q on the unit circle. By corollary 2.10 the function f' on A defined by

$$f' := \frac{1}{d} \sum_{m=0}^{d-1} \lim_{k\to\infty} \sum_{\ell=0}^{kd-1+m} Q^{\ell}(Tr - Tg^*)$$

is a solution of the equation $y = Tr - Tg^* + Qy$ in $\mathcal{B}(A,\Sigma_A)$. The function f' can be extended to a function f on V by defining $f := Tr - Tg^* + Qf'$. The function f is a solution of the equation $y = r - g^* + Py$. This can be seen as follows:

$$Pf = P_B f + P_A f = P_B(Tr - Tg^* + Qf) + P_A f = P_B T(r - g^*) + P_B Qf +$$

$$+ P_A f = P_B T(r - g^*) + \sum_{\ell=1}^{\infty} P_B^{\ell} P_A f + P_A f = P_B T(r - g^*) + Qf =$$

$$= P_B T(r - g^*) + f - T(r - g^*) = f + \sum_{\ell=1}^{\infty} P_B^{\ell}(r - g^*) -$$

$$+ \sum_{\ell=0}^{\infty} P_B^{\ell}(r - g^*) = f - r + g^* .$$

What remains to be proved is

$$\lim_{n\to\infty} \frac{1}{n} \sum_{\ell=0}^{n-1} Q^{\ell}(Tr - Tg^*) = 0 .$$

The constants c_j have been defined such that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{\ell=0}^{n-1} Q_j^{\ell}(Tr - c_j Tg_j^*) = 0 .$$

Hence

$$(1) \qquad \lim_{n\to\infty} \frac{1}{n} \sum_{\ell=0}^{n-1} Q_{E_j}^{\ell}(Tr - c_j Tg_j^*) = 0 .$$

But $Tg_j^* = TQg_j^*$ and, by lemma 3.4,

$$c_j(Tg_j^*)(u) = c_j(TQg_j^*)(u) = c_j(TQ\, 1_{E_j})(u) \text{ for all } u \in E_j, \quad j = 1,\ldots,n .$$

In the same way

$$(Tg^*)(u) = (TQg^*_{E_j})(u) = c_j(TQ\, 1_{E_j})(u) \text{ for } u \in E_j .$$

Therefore

$$(Tg^*)(u) = c_j(Tg^*_j)(u) \text{ for } u \in E_j .$$

This implies by (1) that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{\ell=0}^{n-1} Q^{\ell}_{E_j} (Tr - Tg^*) = 0 .$$

Now we consider

$$\sum_{\ell=0}^{m} Q^{\ell}(Tr - Tg^*) .$$

Using lemma 3.3 we get

$$(2) \qquad \sum_{\ell=0}^{m} Q^{\ell}(Tr-Tg^*) = \sum_{\ell=0}^{m} Q^{\ell}_{\Delta}(Tr-Tg^*) + \sum_{\ell=1}^{m} \sum_{k=0}^{\ell-1} Q^{k}_{\Delta} \sum_{j=1}^{n} Q^{\ell-k}_{E_j}(Tr-Tg^*) =$$

$$= \sum_{\ell=0}^{m} Q^{\ell}_{\Delta}(Tr-Tg^*) + \sum_{j=1}^{n} \sum_{k=0}^{m-1} Q^{k}_{\Delta} \sum_{\ell=1}^{m-k} Q^{\ell}_{E_j} (Tr-Tg^*) .$$

Since the spectral radius of Q_{Δ} is smaller than one, we have

$$(3) \qquad \lim_{m\to\infty} \frac{1}{m} \sum_{\ell=0}^{m-1} Q^{\ell}_{\Delta}(Tr - Tg^*) = 0 .$$

For each $\varepsilon > 0$ there is an $N_{\varepsilon} \in \mathbb{N}$ such that for all $j = 1,\dots,n$

$$\Big\|\frac{1}{m} \sum_{\ell=0}^{m-1} Q^{\ell}_{E_j} (Tr - Tg^*)\Big\| < \varepsilon \qquad \text{for } m \geq N_{\varepsilon} .$$

Put $m > N_{\varepsilon}$. Then for $j = 1,2,\dots,n$ we have

$$\frac{1}{m} \sum_{k=0}^{m-1} Q^{k}_{\Delta} \sum_{\ell=1}^{m-k} Q^{\ell}_{E_j} (Tr-Tg^*) = \sum_{k=0}^{m-N_{\varepsilon}} \frac{m-k}{m} Q^{k}_{\Delta}\Big(\frac{1}{m-k} \sum_{\ell=1}^{m-k} Q^{\ell}_{E_j} (Tr-Tg^*)\Big) +$$

$$+ \sum_{k=m-N_{\varepsilon}+1}^{m-1} \frac{m-k}{m} Q^{k}_{\Delta}\Big(\frac{1}{m-k} \sum_{\ell=1}^{m-k} Q^{\ell}_{E_j} (Tr - Tg^*)\Big) .$$

Hence

$$\left\| \frac{1}{m} \sum_{k=0}^{m-1} Q_\Delta^k \sum_{\ell=1}^{m-k} Q_{E_j}^\ell (Tr - Tg^*) \right\| \le \varepsilon . \left\| \sum_{k=0}^{m-N_\varepsilon} Q_\Delta^k 1_V \right\| +$$

$$+ \sum_{k=m-N_\varepsilon+1}^{m-1} \frac{N_\varepsilon}{m} \| Tr - Tg^* \| = \varepsilon . \left\| \sum_{k=0}^{m-N_\varepsilon} Q_\Delta^k 1_V \right\| + \frac{(N_\varepsilon - 1)N_\varepsilon}{m} \| Tr - Tg^* \| .$$

Using $r(Q_\Delta) < 1$ we get

$$\left\| \frac{1}{m} \sum_{k=0}^{m-1} Q_\Delta^k \sum_{\ell=1}^{m-k} Q_{E_j}^\ell (Tr - Tg^*) \right\| \le \varepsilon . \left\| \sum_{k=0}^{\infty} Q_\Delta^k \right\|$$

for m large enough. Since ε was arbitrary this implies

$$\lim_{m\to\infty} \frac{1}{m} \sum_{k=0}^{m-1} Q_\Delta^k \sum_{\ell=1}^{m-k} Q_{E_j}^\ell (Tr - Tg^*) = 0 \quad \text{for } j = 1,2,\dots,n .$$

Together with (2) and (3) this completes the proof of

$$\lim_{m\to\infty} \frac{1}{m} \sum_{\ell=0}^{m-1} Q^\ell (Tr - Tg^*) = 0 . \qquad \square$$

3.2. *Properties of the solution*

In this section the assumptions and notations are as in the preceding one.

LEMMA 3.6. Let

$$Tr := \sum_{\ell=0}^{\infty} P_B^\ell r \quad \text{and} \quad T\, 1_V := \sum_{\ell=0}^{\infty} P_B^\ell 1_V .$$

Then $P^m Tr$ and $P^m T\, 1_V$ exist for all $m \in \mathbb{N}$ and

$$\lim_{m\to\infty} \frac{1}{m} (P^m Tr)(u) = \lim_{m\to\infty} \frac{1}{m} (P^m T\, 1_V)(u) = 0 \quad \text{for all } u \in V .$$

PROOF. Substitution of $P = P_A + P_B$ in P^{m+1} yields

$$P^{m+1} = P^m P_A + P^m P_B = P^m P_A + P^{m-1} P_A P_B + P^{m-1} P_B^2 = \dots = \sum_{k=0}^{m} P^{m-k} P_A P_B^k + P_B^{m+1} .$$

Hence

$$P^{m+1}Tr = \sum_{k=0}^{m} P^{m-k}P_A \sum_{\ell=k}^{\infty} P_B^{\ell} r + \sum_{\ell=m+1}^{\infty} P_B^{\ell} r \le \sum_{k=0}^{m} P^{m-k}P_A Tr + Tr .$$

The existence of $P^{m+1}Tr$ is implied by the existence of Tr and the boundedness of Tr on A. The existence of $P^{m+1}T\ 1_V$ is proved similarly. For each $\varepsilon > 0$ there is an integer N_ε such that

$$\sum_{\ell=N_\varepsilon}^{\infty} (P_B^{\ell} r)(u) < \varepsilon \text{ for all } u \in A .$$

For $m > N_\varepsilon$ we have

$$P^{m+1}Tr = \sum_{k=0}^{N_\varepsilon} P^{m-k}P_A \sum_{\ell=k}^{\infty} P_B^{\ell} r + \sum_{k=N_\varepsilon+1}^{m} P^{m-k}P_A \sum_{\ell=k}^{\infty} P_B^{\ell} r + \sum_{\ell=m+1}^{\infty} P_B^{\ell} r .$$

Let $\| Tr \|_A := \sup_{u \in A} (Tr)(u)$. Then

$$(P^{m+1}Tr)(u) \le (N_\varepsilon + 1)\| Tr \|_A + (m - N_\varepsilon)\varepsilon + \sum_{\ell=m+1}^{\infty} (P_B^{\ell} r)(u) .$$

Using standard arguments we can show that $\lim_{m\to\infty} \frac{1}{m} (P^m Tr)(u) = 0$ for all $u \in V$. That $\lim_{m\to\infty} \frac{1}{m} (P^m T\ 1_V)(u) = 0$ can be proved similarly. □

In the next lemma we shall give some properties of the solution of the (P,r)-equation as constructed in the proof of theorem 3.5. The uniqueness of this solution is also considered.

LEMMA 3.7. Let the functions g^* and f be as constructed in the proof of theorem 3.5. Then $P^m f$ exists for all $m \in \mathbb{N}$, $\lim_{m\to\infty} \frac{1}{m} P^m f = 0$, and $g^* = g$ (the average costs of (P,r)). Let g_1, f_1 be another solution of the (P,r)-equations, such that $\lim_{m\to\infty} \frac{1}{m} P^m f_1 = 0$, then $g_1 = g$ and $f - f_1 = Q(f - f_1)$.

PROOF. The functions g^* and f on V were defined in the following way:
$g^* := \sum_{j=1}^{n} c_j g_j^*$, where $g_j^* := Qg_j$ and $g_j \in \mathcal{B}(A,\Sigma_A)$;
$f := Tr - Tg^* + Qf'$ where $f' \in \mathcal{B}(A,\Sigma_A)$.
Hence g^* and Qf' are bounded. By lemma 3.6 $P^m f$ exists for all $m \in \mathbb{N}$ and

(1) $$\lim_{m\to\infty} \frac{1}{m} (P^m f) = 0 .$$

Repeated substitution of $f(u) = r(u) - g^*(u) + (Pf)(u)$ in its right-hand side yields

$$f(u) = \sum_{\ell=0}^{m-1} (P^\ell r)(u) - \sum_{\ell=0}^{m-1} (P^\ell g^*)(u) + (P^m f)(u) .$$

Hence

$$\frac{f(u) - (P^m f)(u)}{m} = \frac{1}{m} \sum_{\ell=0}^{m-1} (P^\ell r)(u) - g^*(u)$$

and by (1)

$$g^*(u) = \lim_{m\to\infty} \frac{1}{m} \sum_{\ell=0}^{m-1} (P^\ell r)(u) = g(u) \quad \text{for all } u \in V .$$

Now we consider the solution (g_1,f_1) of the (P,r)-equations. As for the solution (g^*,f) we can prove

$$g_1(u) = \lim_{m\to\infty} \frac{1}{m} \sum_{\ell=0}^{m-1} (P^\ell r)(u) .$$

Further the function $f - f_1$ satisfies $f - f_1 = P(f - f_1)$, hence by lemma 2.21, $f - f_1 = Q(f - f_1)$. □

For $u \in E_j$ it is possible to write the average costs

$$g(u) = \lim_{m\to\infty} \frac{1}{m} \sum_{\ell=0}^{m-1} (P^\ell r)(u)$$

in a somewhat different way.

LEMMA 3.8. For $u \in E_j$, $j = 1,\dots,n$

$$\lim_{m\to\infty} \frac{1}{m} \sum_{\ell=0}^{m-1} (P^\ell r)(u) = \frac{\lim_{m\to\infty} \frac{1}{m} \sum_{\ell=0}^{m-1} (Q^\ell Tr)(u)}{\lim_{m\to\infty} \frac{1}{m} \sum_{\ell=0}^{m-1} (Q^\ell T I_V)(u)} .$$

PROOF. Let g^* be as constructed in the proof of theorem 3.5. Then $g^*(u) = c_j$ for $u \in E_j$, where

$$c_j = \frac{\lim_{m\to\infty} \frac{1}{m} \sum_{\ell=0}^{m-1} (Q_{E_j}^{\ell}\, Tr)(u)}{\lim_{m\to\infty} \frac{1}{m} \sum_{\ell=0}^{m-1} (Q_{E_j}^{\ell}\, Tg_j^*)(u)} .$$

But for $u \in E_j$

$$(Q_{E_j}^{\ell}\, Tr)(u) = (Q^{\ell} Tr)(u)$$

and

$$(Q_{E_j}^{\ell}\, Tg_j^*)(u) = (Q^{\ell} Tg_j^*)(u) .$$

Further

$$Tg_j^* = TQg_j^* \quad \text{and} \quad T1_V = TQ1_V .$$

By lemma 3.4

$$(TQg_j^*)(u) = (TQ1_{E_j})(u) = (TQ1_V)(u) \qquad \text{for } u \in E_j .$$

Hence $(Tg_j^*)(u) = (T1_V)(u)$ for $u \in E_j$. This completes the proof. □

A more general result of this type is proved by de Leve [8], part II, lemma 1.57.

CHAPTER 4. STATIONARY MARKOVIAN DECISION PROBLEMS

A *stationary Markovian decision problem* (SMD) on (V,Σ) is a set of Markov processes with costs $\{(P_\alpha,r_\alpha)\}$, $\alpha \in A$. The elements of A are called strategies. The *average costs* of (P_α,r_α) starting in u, if existing, are denoted by $g_\alpha(u)$. The strategy $\alpha_0 \in A$ is called *optimal* if $g_\alpha(u)$ exists for all $\alpha \in A$, $u \in V$ and if $g_{\alpha_0}(u) \le g_\alpha(u)$ for $\alpha \in A$, $u \in V$. Suppose that μ is a positive measure on Σ such that $\mu g_\alpha := \int g_\alpha(u)\mu(du)$ exists for all $\alpha \in A$, then the strategy α_0 is called *μ-optimal* if $\mu g_{\alpha_0} \le \mu g_\alpha$ for all $\alpha \in A$.

In this chapter we shall investigate the existence of optimal and μ-optimal strategies. In section 4.1 it is assumed that P_α is quasi-compact for all $\alpha \in A$ and r_α is bounded, uniform on A. Under some extra conditions one can prove the existence of optimal or μ-optimal strategies. This case is extended in section 4.2. The quasi-compactness of P_α is replaced by the quasi-compactness of the embedded Markov process of P_α on some set A (A independent of α). The boundedness of r_α, uniform on A, is replaced by the following conditions:

i) For all $\alpha \in A$ the Markov process P_α is (A,r_α)- and $(A,1)$-recurrent.
ii) The boundedness of $\sum_{n=0}^{\infty} P^n_{\alpha B} r_\alpha$ and $\sum_{n=0}^{\infty} P^n_{\alpha B} 1_V$ on A, with $B := V\backslash A$, is uniform on A.

The conditions of section 4.2 for the existence of optimal and μ-optimal strategies are weaker than those of section 4.1. If A = V, the two cases are identical. But for a good understanding of the statements it is necessary to give both sections.

In section 4.3 the results of section 4.2 are applied to the case where V is countable. These results are related to those of some others.

An important property of an SMD, if one is interested in the existence of optimal strategies, is its completeness.

DEFINITION 4.1. Let (P_1,r_1) and (P_2,r_2) be Markov processes with costs on (V,Σ). For each $F \in \Sigma$ the Markov process with costs (P_1FP_2,r_1Fr_2) is defined by:

$$(P_1FP_2)(u,E) = P_1(u,E),\ (r_1Fr_2)(u) = r_1(u) \quad \text{for } u \in F,\ E \in \Sigma\ ,$$

$$(P_1FP_2)(u,E) = P_2(u,E),\ (r_1Fr_2)(u) = r_2(u) \quad \text{for } u \in V\backslash F,\ E \in \Sigma.$$

DEFINITION 4.2. An SMD $\{(P_\alpha,r_\alpha)\}$, $\alpha \in A$ is *complete* if for all $\alpha_1,\alpha_2 \in A$ and for all $F \in \Sigma$ there is an $\alpha \in A$ such that

$$(P_\alpha,r_\alpha) = (P_{\alpha_1}FP_{\alpha_2},r_{\alpha_1}Fr_{\alpha_2})\ .$$

The strategy α is denoted by $\alpha_1F\alpha_2$.

Intuitively speaking an SMD is complete if for all $F \in \Sigma$ and for all $\alpha_1,\alpha_2 \in A$ it is allowed to apply strategy α_1 on F and strategy α_2 on $V\backslash F$.

If μ is a positive measure then the function μg_α, if existing, is a real valued function on A. The most obvious way to prove the existence of a μ-optimal strategy is the following:

i) Introduce a topology on A such that A is compact;
ii) derive conditions for the continuity of μg_α as function on A.

This method will be used in the sequel. For A we use a metric space. It will turn out that μ-optimality is almost identical to optimality for complete SMD's.

4.1. *The quasi-compact and bounded case*

In this section we consider an SMD, $\{(P_\alpha,r_\alpha)\}$, $\alpha \in A$ on (V,Σ) such that P_α is quasi-compact for all $\alpha \in A$ and r_α is bounded on V, uniform on A.

Let for $\alpha \in A$, n_α be the dimension of $N(I - P_\alpha)$ and $E_{\alpha j}$, $j = 1,\dots,n_\alpha$, the maximal invariant sets of P_α. The union $\bigcup_{j=1}^{n_\alpha} E_{\alpha j}$ is denoted by E_α. Let $S_\alpha := \lim_{m\to\infty} \frac{1}{m} \sum_{\ell=0}^{m-1} P_\alpha^\ell$, (the existence is proved in lemma 1.7). For all $\alpha \in A$ and $j = 1,\dots,n_\alpha$ the probability $\pi_{\alpha j}$ is an element of $N(I - P_\alpha)$ with support $E_{\alpha j}$, (see theorem 2.5 for the existence of these probabilities). If μ is a probability on Σ with $\mu(E_{\alpha j}) = 1$ then $\mu S_\alpha = \pi_{\alpha j}$. The average costs of α, g_α, exist and are equal to $S_\alpha r_\alpha$, $g_\alpha \in N(I - P_\alpha)$ and g_α is constant on $E_{\alpha j}$ for $j = 1,\dots,n_\alpha$. This constant is denoted by $g_{\alpha j}$.

We assume the existence of a metric ρ on A such that

$$\lim_{\rho(\alpha,\alpha_0)\to 0} \| P_\alpha - P_{\alpha_0} \| = 0 \text{ for all } \alpha_0 \in A .$$

$$\lim_{\rho(\alpha,\alpha_0)\to 0} |\pi_{\alpha_0 j} r_\alpha - \pi_{\alpha_0 j} r_{\alpha_0}| = 0 \text{ for all } \alpha_0 \in A \text{ and } j = 1,\dots,n_{\alpha_0} .$$

These assumptions are used in subsection 4.1.1 where the continuity of g_α as a function on A is considered. The existence of an optimal strategy and the relationship between optimality and μ-optimality are investigated in the subsections 4.1.2 and 4.1.3.

4.1.1. *The continuity of* g_α

DEFINITION 4.3. Let μ be a positive measure on Σ. The function g_α is μ-*continuous* in α on some subset $\mathcal{D} \subset A$ if

$$\lim_{\rho(\alpha,\alpha_0)\to 0} |\mu g_\alpha - \mu g_{\alpha_0}| = 0 \quad \text{for all } \alpha_0 \in \mathcal{D} .$$

For all $n \in \mathbb{N}$ the subset of A with all α such that $n_\alpha = n$ is denoted by A_n. We shall prove the μ-continuity of g_α on A_n for all $n \in \mathbb{N}$ and all positive measures μ on Σ. Since S_α is continuous as an operator valued function on A_n but generally not on A, (see lemma 1.15), this cannot be extended to μ-continuity on A.

LEMMA 4.4. The function g_α is μ-continuous on A_n for all $n \in \mathbb{N}$ and for all positive measures μ on Σ.

PROOF. Let $\alpha_0 \in A_n$ and μ a positive measure on Σ. We have

$$\mu g_\alpha = \mu(S_\alpha r_\alpha) = (\mu S_\alpha) r_\alpha .$$

Hence

$$|\mu g_\alpha - \mu g_{\alpha_0}| = |(\mu S_\alpha) r_\alpha - (\mu S_{\alpha_0}) r_{\alpha_0}| \le |(\mu S_\alpha - \mu S_{\alpha_0}) r_\alpha| +$$

$$+ |\mu S_{\alpha_0}(r_\alpha - r_{\alpha_0})| .$$

By lemma 1.15b, S_α is continuous in α on A_n. Together with the boundedness of r_α, uniform on A, this implies

$$\lim_{\rho(\alpha,\alpha_0)\to 0} |(\mu S_\alpha - \mu S_{\alpha_0}) r_\alpha| = 0 \quad (\alpha \in A_n) .$$

Since μS_{α_0} is a linear combination of $\pi_{\alpha_0 1},\ldots,\pi_{\alpha_0 n}$, the assumption

$$\lim_{\rho(\alpha,\alpha_0)\to 0} |\pi_{\alpha_0 j} r_\alpha - \pi_{\alpha_0 j} r_{\alpha_0}| = 0$$

implies

$$\lim_{\rho(\alpha,\alpha_0)\to 0} |\mu S_{\alpha_0}(r_\alpha - r_{\alpha_0})| = 0 .$$

This completes the proof. □

REMARK 4.5. This result implies the continuity of $g_{\alpha 1}$ as a function on A_1. However, the condition

$$\lim_{\rho(\alpha,\alpha_0)\to 0} \| P_\alpha - P_{\alpha_0} \| = 0$$

is unnecessarily strong. This condition can be replaced by

$$\lim_{\rho(\alpha,\alpha_0)\to 0} \| P_\alpha^k (P_\alpha - P_{\alpha_0}) \| = 0 \quad \text{for some } k \geq 1 .$$

PROOF. Considering the proof of lemma 4.4 it is clear that it is sufficient to prove the continuity of S_α as an operator valued function on A_1. Let $\alpha_0 \in A_1$ and let d be an integer such that $\lambda^d = 1$ for all eigenvalues λ of P_{α_0} on the unit circle, (see theorem 2.9). We have

$$S_\alpha - S_{\alpha_0} = \lim_{n\to\infty} \frac{1}{n} \sum_{\ell=0}^{n-1} (P_\alpha^\ell - P_{\alpha_0}^\ell) = \lim_{n\to\infty} \frac{1}{nd} \sum_{\ell=0}^{nd-1} (P_\alpha^\ell - P_{\alpha_0}^\ell) ,$$

and

$$P_\alpha^\ell - P_{\alpha_0}^\ell = (P_\alpha - P_{\alpha_0}) P_{\alpha_0}^{\ell-1} + P_\alpha (P_\alpha - P_{\alpha_0}) P_{\alpha_0}^{\ell-2} + \ldots + P_\alpha^{\ell-1} (P_\alpha - P_{\alpha_0}) .$$

Let

$$P_n := \sum_{\ell=n}^{n+d-1} P_{\alpha_0}^\ell , \; n \in \mathbb{N} .$$

Then

$$\sum_{\ell=nd}^{nd+d-1} (P_\alpha^\ell - P_{\alpha_0}^\ell) = (P_\alpha - P_{\alpha_0}) P_{nd-1} + P_\alpha (P_\alpha - P_{\alpha_0}) P_{nd-2} + \ldots +$$

$$\ldots + P_\alpha^{nd-2}(P_\alpha - P_{\alpha_0})P_1 + P_\alpha^{nd-1}\{(P_\alpha - P_{\alpha_0})\sum_{k=0}^{d-1} P_{\alpha_0}^k +$$

$$+ P_\alpha(P_\alpha - P_{\alpha_0})\sum_{k=0}^{d-2} P_{\alpha_0}^k + \ldots + P_\alpha^{d-2}(P_\alpha - P_{\alpha_0})\sum_{k=0}^{1} P_{\alpha_0}^k + P_\alpha^{d-1}(P_\alpha - P_{\alpha_0})\} .$$

Let P_{n0} be the restriction of P_n to M_0, (see lemma 2.14). There is a real number β, $0 < \beta < 1$, and an integer N such that $\| P_{n0} \| \le \beta^n$ for all $n > N$. Since $\mu(P_\alpha - P_{\alpha_0}) \in M_0$ for all $\mu \in M$ this implies

$$\| (P_\alpha - P_{\alpha_0})P_n \| \le 2\beta^n \quad \text{for all } n > N .$$

Choose $\varepsilon > 0$, $m > N$ such that $2 \frac{\beta^m}{1-\beta} < \varepsilon$, and $\delta > 0$ such that

$$\| P_\alpha^k(P_\alpha - P_{\alpha_0}) \| < \frac{\varepsilon}{(m-1)d + \frac{1}{2}d(1+d)} \quad \text{if } \rho(\alpha,\alpha_0) < \delta .$$

For n such that $nd - m > k$ and α such that $\rho(\alpha,\alpha_0) < \delta$ we get

$$\| \sum_{\ell=nd}^{nd+d-1} (P_\alpha^\ell - P_{\alpha_0}^\ell) \| \le \| \sum_{i=0}^{nd-m-1} P_\alpha^i(P_\alpha - P_{\alpha_0})P_{nd-i-1} \| +$$

$$+ \| \sum_{i=nd-m}^{nd-2} P_\alpha^i(P_\alpha - P_{\alpha_0})P_{nd-i-1} + P_\alpha^{nd-1} \sum_{i=0}^{d-1} P_\alpha^i(P_\alpha - P_{\alpha_0}) \sum_{j=0}^{d-1-i} P_{\alpha_0}^j \| \le$$

$$\le 2 \sum_{j=m}^{nd-1} \beta^j + \frac{\varepsilon}{(m-1)d + \frac{1}{2}d(1+d)} \cdot \| \sum_{j=1}^{m-1} P_j + \sum_{j=1}^{d} \sum_{\ell=0}^{d-j} P_{\alpha_0}^\ell \| \le$$

$$\le \frac{2\beta^m}{1-\beta} + \frac{\varepsilon}{(m-1)d + \frac{1}{2}d(1+d)} \cdot \{(m-1)d + \tfrac{1}{2}d(1+d)\} \le 2\varepsilon .$$

Hence

$$\| \lim_{n\to\infty} \frac{1}{nd} \sum_{\ell=0}^{nd-1} (P_\alpha^\ell - P_{\alpha_0}^\ell) \| < \frac{2\varepsilon}{d} \quad \text{if } \rho(\alpha,\alpha_0) < \delta ,$$

which completes the proof. □

4.1.2. *Existence of optimal strategies*

In lemma 4.4 we proved the μ-continuity of g_α on A_n. This implies the continuity of $g_{\alpha 1}$ on A_1. If $\underset{\sim}{A} = A_1$ and A is compact, there is an optimal strategy.

In the next lemma we give a slight generalization of this result.

LEMMA 4.6. Let A be compact and A_n closed in A for all $n \in \mathbb{N}$. If n_α is bounded on A, there is a μ-optimal strategy for all positive measures μ on Σ.

PROOF. The set A_n is compact for each $n \in \mathbb{N}$. Hence the $\inf_{\alpha \in A_n} \mu g_\alpha$ is attained on A_n for all $n \in \mathbb{N}$. The boundedness of n_α completes the proof. □

The following condition for closedness of A_n is a direct consequence of lemma 1.15c.

LEMMA 4.7. If there is a β, $0 < \beta < 1$, such that for all $\alpha \in A$ the spectrum of P_α has no points with absolute value between β and 1, then for all $n \in \mathbb{N}$ the set A_n is closed in A.

If for all $\alpha \in A$ there is an $\alpha_1 \in A_1$ such that $g_{\alpha_1}(u) \le g_\alpha(u)$, $u \in V$, then A_1 is said to *dominate* A. If A_1 dominates A it is sufficient to consider A_1.

We shall give a condition sufficient for A_1 to dominate A.

DEFINITION 4.8. The SMD is called *communicative**) if for all $\alpha \in A$ and $j = 1,\dots,n_\alpha$ there is an $\alpha_1 \in A_1$ such that $\pi_{\alpha_1 1}(E_{\alpha j}) > 0$.

LEMMA 4.9. If the SMD is complete and communicative, A_1 dominates A.

PROOF. Let $\alpha \in A$. Choose j_0 such that $g_{\alpha j_0} = \min_{j=1,2,3,\dots,n_\alpha} \{g_{\alpha j}\}$. The communicativeness of the SMD implies the existence of a strategy $\alpha_1 \in A_1$ such that $\pi_{\alpha_1 1}(E_{\alpha j_0}) > 0$. Let $\alpha_2 := \alpha E_{\alpha j_0} \alpha_1$. Then $\alpha_2 \in A_1$ and $g_{\alpha_2}(u) = g_{\alpha j_0}$, $u \in E_{\alpha j_0}$. Let $A := E_{\alpha j_0}$ and $B := V \backslash A$. By the lemma's 2.17 and 2.19 the embedded sub-Markov process Q_{α_2} of P_{α_2} on A is a Markov process. Now by lemma 2.21,

*) Bather [1] and Hordijk [4] use the term communicating.

$$g_{\alpha_2} = Q_{\alpha_2} g_{\alpha_2} = \sum_{\ell=0}^{\infty} P^{\ell}_{\alpha_2 B} P_{\alpha_2 A} g_{\alpha_2} ,$$

and since $g_{\alpha_2}(u) = g_{\alpha j_0}$ for $u \in A$ this is equal to

$$g_{\alpha j_0} \cdot \sum_{\ell=0}^{\infty} P^{\ell}_{\alpha_2 B} P_{\alpha_2 A} 1_V = g_{\alpha j_0} ,$$

which completes the proof. □

In subsection 4.1.1 the μ-continuity of g_α on A_1 was proved. This is equivalent to the continuity of $g_{\alpha 1}$ on A_1. The completeness and communicativeness of the SMD, and the compactness of A_1 are sufficient for the existence of an optimal strategy (lemma 4.9). We shall prove that the compactness of A_1 may be replaced by the compactness of A. To this end we need the following lemma.

LEMMA 4.10. Let $\{\alpha_k\}_1^\infty$ be a sequence in A_1 converging to $\alpha_0 \in A$, such that $\lim_{k\to\infty} \pi_{\alpha_k 1}(E_{\alpha_0 j})$ exists for all $j = 1,2,\ldots,n_{\alpha_0}$. Let for all k and j with $\pi_{\alpha_k 1}(E_{\alpha_0 j}) > 0$ the measures π_k^j be defined by

$$\pi_k^j(E) := \frac{\pi_{\alpha_k 1}(E)}{\pi_{\alpha_k 1}(E_{\alpha_0 j})} , \; E \in \Sigma .$$

Then for all j with $\lim_{k\to\infty} \pi_{\alpha_k 1}(E_{\alpha_0 j}) > 0$

$$\lim_{k\to\infty} \| \pi_k^j - \pi_{\alpha_0 j} \|_{E_{\alpha_0 j}} = 0 ,$$

(where $\| . \|_{E_{\alpha_0 j}}$ is the norm of the measure restricted to $E_{\alpha_0 j}$).

PROOF. Let j be such that $\lim_{k\to\infty} \pi_{\alpha_k 1}(E_{\alpha_0 j}) > 0$. Put $A := E_{\alpha_0 j}$ and $B := V\backslash A$. The restriction of the transition probability P_{α_0} to $A \times \Sigma_A$ is a transition probability on $A \times \Sigma_A$, but the restriction of P_{α_k} to $A \times \Sigma_A$ is in general only a sub-transition probability. Define for $k \in \mathbb{N}$ the function $R_k(\cdot,\cdot)$ on $A \times \Sigma_A$ by

$$R_k(u,E) := \pi_{\alpha_0 j}(E)(1 - P_{\alpha_k}(u,A)) , \quad u \in A, \; E \in \Sigma_A .$$

Then $P_k := P_{\alpha_k} + R_k$ is a transition probability on $A \times \Sigma_A$. We have

$$\| P_{\alpha_0} - P_k \|_A \le \| P_{\alpha_0} - P_{\alpha_k} \|_A + \| P_{\alpha_k} - P_k \|_A = \| P_{\alpha_0} - P_{\alpha_k} \|_A + \| R_k \|_A$$

and

$$\| R_k \|_A = \sup_{u \in A} (1 - P_{\alpha_k}(u,A)) = \| P_{\alpha_0} 1_A - P_{\alpha_k} 1_A \|_A \le \| P_{\alpha_0} - P_{\alpha_k} \|_A .$$

Hence

$$\lim_{k\to\infty} \| P_{\alpha_0} - P_k \|_A = 0 .$$

This implies the quasi-compactness of P_k for k sufficiently large. The Markov process P_{α_0} on (A,Σ_A) has only one invariant probability. Therefore, by lemma 1.15a, for large k, the Markov process P_k has only one invariant probability λ_k. By lemma 1.15b

(1) $$\lim_{k\to\infty} \| \lambda_k - \pi_{\alpha_0 j} \|_A = 0 .$$

We shall prove that

(2) $$\lim_{k\to\infty} \| \lambda_k - \pi_k^j \|_A = 0 .$$

We have

$$\pi_k^j = \pi_k^j P_{\alpha_k} = \pi_k^j I_A P_{\alpha_k} + \pi_k^j I_B P_{\alpha_k} .$$

Let π_{kA}^j be the restriction of π_k^j to Σ_A, then for $E \in \Sigma_A$

$$\pi_{kA}^j(E) = \pi_{kA}^j(P_k - R_k)(E) + (\pi_k^j I_B P_{\alpha_k})(E) .$$

Hence, for large k,

(3) $$\lambda_k - \pi_{kA}^j = (\lambda_k - \pi_{kA}^j)P_k + \pi_{kA}^j R_k - (\pi_k^j I_B P_{\alpha_k})_A ,$$

where $(\pi_k^j I_B P_{\alpha_k})_A$ is the restriction of $\pi_k^j I_B P_{\alpha_k}$ to Σ_A.

For the measure $\pi_{kA}^j R_k - (\pi_k^j I_B P_{\alpha_k})_A$ on Σ_A we have

$$(\pi^j_{kA}R_k)(A) - (\pi^j_k I_B P_{\alpha_k})_A(A) = \int_A \pi^j_k(du)R_k(u,A) - \int_B \pi^j_k(du)P_{\alpha_k}(u,A) =$$

$$= \int_A \pi^j_k(du)(1 - P_{\alpha_k}(u,A)) - \int_B \pi^j_k(du)P_{\alpha_k}(u,A) =$$

$$= \int_A \pi^j_k(du) - \int_V \pi^j_k(du)P_{\alpha_k}(u,A) = 0 .$$

Therefore the measure $\pi^j_{kA}R_k - (\pi^j_k I_B P_{\alpha_k})_A$ on Σ_A is an element of the subspace $M_0(A,\Sigma_A)$ of $M(A,\Sigma_A)$, (see lemma 2.14). The measure $\lambda_k - \pi^j_{kA}$ is of course also an element of M_0. We already had $\| R_k \|_A \le \| P_{\alpha_0} - P_{\alpha_k} \|_A$. Further it is easy to see that

$$\| (\pi^j_k I_B P_{\alpha_k})_A \|_A = \int_B \pi^j_k(du)P_{\alpha_k}(u,A) = \pi^j_k(A) - \int_A \pi^j_k(du)P_{\alpha_k}(u,A) =$$

$$= \pi^j_k(A) - \int_A \pi^j_k(du)P_{\alpha_0}(u,A) +$$

$$+ \int_A \pi^j_k(du)(P_{\alpha_0}(u,A) - P_{\alpha_k}(u,A)) =$$

$$= \int_A \pi^j_k(du)(1 - P_{\alpha_k}(u,A)) .$$

Hence

$$\lim_{k\to\infty} \| \pi^j_{kA}R_k - (\pi^j_k I_B P_{\alpha_k})_A \|_A = 0 .$$

Using lemma 2.14 and equation (3) we get $\lim_{k\to\infty} \| \lambda_k - \pi^j_{kA} \|_A = 0$. This completes the proof of (2), and together with (1) the proof of the lemma. □

Now we can prove the main result of this subsection.

THEOREM 4.11. Let the SMD be complete and communicative. If A is compact then an optimal strategy exists.

PROOF. Let $g := \inf_{\alpha\in A_1} g_{\alpha 1}$. The compactness of A implies the existence of a sequence $\{\alpha_k\}$ in A_1 such that $\lim_{k\to\infty} g_{\alpha_k 1} = g$ and $\lim_{k\to\infty} \alpha_k = \alpha_0$ for some $\alpha_0 \in A$.

Instead of $E_{\alpha_0 j}$ we shall write E_j. Let $\Delta := V \setminus \bigcup_{j=1}^{n_{\alpha_0}} E_j$. Without loss of generality we may assume that $\lim_{k\to\infty} \pi_{\alpha_k 1}(E_j)$ exists for all $j = 1,2,\ldots,n_{\alpha_0}$. These limits are denoted by π_j. We have

$$g_{\alpha_k 1} = \pi_{\alpha_k 1} g_{\alpha_k} = \pi_{\alpha_k 1} S_{\alpha_k} r_{\alpha_k} = \pi_{\alpha_k 1} r_{\alpha_k} = \int_\Delta r_{\alpha_k}(u)\pi_{\alpha_k 1}(du) +$$

$$+ \sum_{j=1}^{n_{\alpha_0}} \int_{E_j} r_{\alpha_k}(u)\pi_{\alpha_k 1}(du) .$$

Further

$$\pi_{\alpha_k 1} = \pi_{\alpha_k 1} P_{\alpha_k} = \pi_{\alpha_k 1} P_{\alpha_0} + \pi_{\alpha_k 1}(P_{\alpha_k} - P_{\alpha_0}) .$$

The restriction of the measure $\pi_{\alpha_k 1} P_{\alpha_0}$ to Σ_Δ is identical to the restriction of the measure $\pi_{\alpha_k 1} I_\Delta P_{\alpha_0}$ to Σ_Δ. Hence $\pi_{\alpha_k 1} = \pi_{\alpha_k 1} I_\Delta P_{\alpha_0} + \pi_{\alpha_k 1}(P_{\alpha_k} - P_{\alpha_0})$ (as equation in $M(\Delta,\Sigma_\Delta)$). The convergence of α_k to α_0 implies

$$\lim_{k\to\infty} \| \pi_{\alpha_k 1}(P_{\alpha_k} - P_{\alpha_0}) \|_\Delta = 0 .$$

The spectral radius of $I_\Delta P_{\alpha_0}$ is smaller than 1 (see corollary 2.13). Therefore $\lim_{k\to\infty} \| \pi_{\alpha_k 1} \|_\Delta = 0$ and $\lim_{k\to\infty} \int_\Delta r_{\alpha_k}(u)\pi_{\alpha_k 1}(du) = 0$. Now we have to consider

$$\int_{E_j} r_{\alpha_k}(u)\pi_{\alpha_k 1}(du) .$$

If $\pi_j = 0$,

$$\lim_{k\to\infty} \int_{E_j} r_{\alpha_k}(u)\pi_{\alpha_k 1}(du) = 0 .$$

Let $\pi_j > 0$. By lemma 4.10

$$\lim_{k\to\infty} \{ \int_{E_j} r_{\alpha_k}(u)\pi_{\alpha_k 1}(du) - \pi_j \int_{E_j} r_{\alpha_k}(u)\pi_{\alpha_0 j}(du) \} = 0 .$$

Using the assumption

$$\lim_{\rho(\alpha,\alpha_0)\to 0} |\pi_{\alpha_0 j} r_\alpha - \pi_{\alpha_0 j} r_{\alpha_0}| = 0$$

we get

$$\lim_{k\to\infty} \pi_j \{ \int_{E_j} r_{\alpha_k}(u)\pi_{\alpha_0 j}(du) - \int_{E_j} r_{\alpha_0}(u)\pi_{\alpha_0 j}(du)\} = 0$$

and hence

$$\lim_{k\to\infty} \int_{E_j} r_{\alpha_k}(u)\pi_{\alpha_k 1}(du) = \pi_j \int_{E_j} r_{\alpha_0}(u)\pi_{\alpha_0 j}(du) \ .$$

But

$$g_{\alpha_0 j} = \pi_{\alpha_0 j} g_{\alpha_0} = \pi_{\alpha_0 j} S_{\alpha_0} r_{\alpha_0} = \pi_{\alpha_0 j} r_{\alpha_0} = \int_{E_j} r_{\alpha_0}(u)\pi_{\alpha_0 j}(du) \ .$$

Hence

$$\text{(1)} \qquad g = \lim_{k\to\infty} g_{\alpha_k 1} = \sum_{j=1}^{n_{\alpha_0}} \pi_j \cdot g_{\alpha_0 j} \ .$$

We had $\lim_{k\to\infty} \|\pi_{\alpha_k 1}\|_\Delta = 0$, which implies that $\lim_{k\to\infty} \pi_{\alpha_k 1}(\Delta) = 0$ and

$$\sum_{j=1}^{n_{\alpha_0}} \pi_j = \sum_{j=1}^{n_{\alpha_0}} \lim_{k\to\infty} \pi_{\alpha_k 1}(E_j) = 1 \ .$$

Now, by (1),

$$\min_{j=1,2,\ldots,n_{\alpha_0}} \{g_{\alpha_0 j}\} \le g \ .$$

This implies, by lemma 4.9, the existence of a strategy $\alpha_1 \in A_1$ such that

$$g_{\alpha_1 1} \le \min_{j=1,\ldots,n_{\alpha_0}} \{g_{\alpha_0 j}\} \le g \ .$$

This strategy α_1 is optimal. □

4.1.3. *Optimality μ-almost everywhere and μ-optimality*

In this subsection we shall work with a complete SMD. We shall prove that for all $\alpha_1,\alpha_2 \in A$ there is an $\alpha_0 \in A$ such that

$$g_{\alpha_0}(u) \le \min\{g_{\alpha_1}(u), g_{\alpha_2}(u)\} \quad \text{for all } u \in V .$$

Using this property one can show that the μ-optimality of some strategy α_0 implies that for all $\alpha \in A$, $g_{\alpha_0} \le g_\alpha$, μ-almost everywhere on V. The strategy α_0 is said to be optimal, μ-almost everywhere.

We need the following three lemma's.

LEMMA 4.12. Let $\{(P_\alpha, r_\alpha)\}$, $\alpha \in A$ be a complete SMD. Let $\alpha_1,\alpha_2 \in A$, $F \in \Sigma$ and $\alpha := \alpha_1 F \alpha_2$. Assume that $\pi_{\alpha j}(F) = 0$ for all $j = 1,\dots,n_\alpha$. By lemma 2.6 a set $G \subset V\backslash F$ exists such that $\pi_{\alpha j}(G) = 1$ for all $j = 1,\dots,n_\alpha$, and $P_\alpha(u,G) = 1$ for all $u \in G$. By corollary 2.13 the embedded sub-Markov processes Q_1 and Q_2 of P_α, on $V\backslash F$ and on $F \cup G$, are Markov processes. The functions g_n, $n = 1,2,\dots$, on V are defined by

$$g_1(u) = (Q_1 g_{\alpha_2})(u) \qquad \text{for } u \in F ,$$

$$g_1(u) = (Q_{2F} g_1)(u) + (Q_{2G} g_{\alpha_2})(u) \qquad \text{for } u \in V\backslash F ,$$

and for $n = 2,3,4,\dots$

$$g_n(u) = (Q_1 g_{n-1})(u) \qquad \text{for } u \in F ,$$

$$g_n(u) = (Q_{2F} g_n)(u) + (Q_{2G} g_{\alpha_2})(u) \qquad \text{for } u \in V\backslash F .$$

For these functions g_n the following property holds:

$$\lim_{n\to\infty} (g_n(u) - g_\alpha(u)) = 0 \quad \text{for all } u \in V .$$

PROOF. By lemma 2.21, $g_\alpha = Q_1 g_\alpha$ and $g_\alpha = Q_2 g_\alpha$. Considering the definition of α we see that $g_\alpha(u) = g_{\alpha_2}(u)$ for $u \in G$. Hence $g_\alpha = Q_{2F} g_\alpha + Q_{2G} g_{\alpha_2}$. Now it is easy to verify that for $n \ge 1$

$$g_\alpha(u) - g_n(u) = Q_1 (Q_{2F} Q_1)^{n-1} (g_\alpha - g_{\alpha_2})(u) \qquad \text{for } u \in F ,$$

and

$$g_\alpha(u) - g_n(u) = (Q_{2F} Q_1)^n (g_\alpha - g_{\alpha_2})(u) \qquad \text{for } u \in V\backslash F .$$

Using

$$(Q_{2F}Q_1)(u,V) \le P_{V\backslash G}(u,V)$$

and

$$\lim_{n\to\infty} (P^n_{V\backslash G}1_V)(u) = 0$$

we get the required result. □

It is easy to see that even

$$\lim_{n\to\infty} \| g_n - g_\alpha \| = 0 ,$$

but the lemma is formulated in this way to maintain the analogy with section 4.2.

REMARK p.7. The functions g_n in this lemma can be interpreted as the average costs if one applies strategy α until the $(n+1)$-st time the system enters the set $V\backslash F$, and from then on the strategy α_2.

DEFINITION 4.13. For $\alpha_1,\alpha_2 \in A$ the set $H_{\alpha_1\alpha_2}$, or shortly H, is defined by

$$H_{\alpha_1\alpha_2} := \{u \in V \mid g_{\alpha_1}(u) < g_{\alpha_2}(u)\} .$$

LEMMA 4.14. Let the SMD $\{(P_\alpha,r_\alpha)\}$, $\alpha \in A$ be complete. Let $\alpha_1,\alpha_2 \in A$, and $\alpha := \alpha_1 H_{\alpha_1\alpha_2}\alpha_2$. If $\pi_{\alpha_1 j}(V\backslash H) > 0$ for all $j = 1,\dots,n_{\alpha_1}$, then $\pi_{\alpha j}(H) = 0$ for all $j = 1,2,\dots,n_\alpha$.

PROOF. Suppose it is not true, then there is a $j \in \{1,2,\dots,n_\alpha\}$, such that $\pi_{\alpha j}(H) > 0$. Let $E := E_{\alpha j}$, $F := E\backslash H$ and $P_{\alpha j}$ the restriction of P_α to $(E_{\alpha j},\Sigma_{E_{\alpha j}})$.
Since $\pi_{\alpha_1 j}(V\backslash H) > 0$ for all $j = 1,2,\dots,n_{\alpha_1}$, the embedded sub-Markov process Q_1 of $P_{\alpha j}$ on F is a Markov process (see lemma 2.17). In the same way $\pi_{\alpha j}(H) > 0$ implies that the embedded sub-Markov process Q_2 of $P_{\alpha j}$ on $E \cap H$ is also a Markov process. Let R be the entry Markov process of $P_{\alpha j}$ on F. By lemma 2.28, R is uniform μ-recurrent for some positive measure μ, and therefore, by lemma 2.18, quasi-compact.
The Markov process $P_{\alpha j}$ has only one invariant probability and hence no disjoint invariant sets. Each invariant set of R can be extended to an invariant set of $P_{\alpha j}$, (see lemma 2.27), hence R has no disjoint invariant sets and therefore only one invariant probability.

This means that for all $f \in \mathcal{B}(E,\Sigma_E)$ the $\lim_{n\to\infty} \frac{1}{n} \sum_{\ell=0}^{n-1} R^\ell f$ is constant on E. Let Q_1' be the embedded Markov process of P_{α_1} on $V\backslash H$. By lemma 2.21

$$g_{\alpha_1} = Q_1' g_{\alpha_1} = \sum_{n=0}^{\infty} P^n_{\alpha_1 H} P_{\alpha_1 V\backslash H} g_{\alpha_1} .$$

Using the definition of α we get $g_{\alpha_1}(u) = (Q_1 g_{\alpha_1})(u)$ for $u \in E \cap H$. Let Q_2' be the embedded sub-Markov process of P_{α_2} on H. Using the definition of α we get for $u \in F$, $Q_2'(u,E \cap H) = 1$ and

$$g_{\alpha_2}(u) = (Q_2' g_{\alpha_2})(u) = \sum_{n=0}^{\infty} (P^n_{\alpha_2 V\backslash H} P_{\alpha_2 H} g_{\alpha_2})(u) = (Q_2 g_{\alpha_2})(u) .$$

Hence, for $u \in F$,

(1) $$g_{\alpha_2}(u) = (Q_2 g_{\alpha_2})(u) > (Q_2 g_{\alpha_1})(u) = (Q_2 Q_1 g_{\alpha_1})(u) = (R g_{\alpha_1})(u) \geq$$

$$\geq (R g_{\alpha_2})(u) .$$

Let $S := \lim_{n\to\infty} \frac{1}{n} \sum_{\ell=0}^{n-1} R^\ell$ and $c := (S g_{\alpha_2})(u)$, $u \in E$. Then $c \geq \inf_{u\in F} \{g_{\alpha_2}(u)\}$, and if the infimum is not attained then $c > \inf_{u\in F} \{g_{\alpha_2}(u)\}$.
However, by (1), $g_{\alpha_2}(u) > c$ for all $u \in F$, hence $\inf_{u\in F} \{g_{\alpha_2}(u)\} \geq c$. This implies that the $\inf_{u\in F} \{g_{\alpha_2}(u)\}$ is attained on F, say in u_0. By (1), $g_{\alpha_2}(u_0) > (R g_{\alpha_2})(u_0) \geq g_{\alpha_2}(u_0)$, which yields a contradiction. This completes the proof. □

LEMMA 4.15. Let the SMD $\{(P_\alpha, r_\alpha)\}$, $\alpha \in A$ be complete. Let $\alpha_1, \alpha_2 \in A$ and $H := H_{\alpha_1\alpha_2}$. Assume that $\pi_{\alpha_1 j}(V\backslash H) = 0$ for some $j \in \{1,2,\dots,n_{\alpha_1}\}$. By lemma 2.6 a set $G \subset H$ exists such that $\pi_{\alpha_1 j}(G) = 1$ and $P_{\alpha_1}(u,G) = 1$.
Let $\alpha := \alpha_1 G \alpha_2$. Then $g_\alpha(u) = g_{\alpha_1}(u)$ for $u \in G$ and $g_\alpha(u) \leq g_{\alpha_2}(u)$ for $u \in V$.

PROOF. A direct consequence of the definition of α is that $g_\alpha(u) = g_{\alpha_1}(u)$ for $u \in G$. Let $I_1 := \{i \mid \pi_{\alpha_2 i}(G) = 0\}$, $I_2 := \{i \mid \pi_{\alpha_2 i}(G) > 0\}$, and $E_1 := \bigcup_{i\in I_1} E_{\alpha_2 i}$. The embedded sub-Markov process Q of P_{α_2} on $G \cup E_1$ is a

Markov process, and by lemma 2.21 $g_{\alpha_2} = Q_G g_{\alpha_2} + Q_{E_1} g_{\alpha_2}$.
Let Q' be the embedded Markov process of P_α on $G \cup E_1$. Then for $u \in V\backslash G$ and for all $f \in \mathcal{B}$ we have $(Q'f)(u) = (Qf)(u)$. Hence

$$g_\alpha(u) = (Q'g_\alpha)(u) = (Qg_\alpha)(u) = (Q_G g_\alpha)(u) + (Q_{E_1} g_\alpha)(u) \quad \text{for } u \in V\backslash G.$$

Using $g_\alpha(u) = g_{\alpha_1}(u)$ for $u \in G$ and $g_\alpha(u) = g_{\alpha_2}(u)$ for $u \in E_1$, this implies

$$g_\alpha(u) = (Q_G g_{\alpha_1})(u) + (Q_{E_1} g_{\alpha_2})(u) \le (Q_G g_{\alpha_2})(u) + (Q_{E_1} g_{\alpha_2})(u) = g_{\alpha_2}(u)$$

for all $u \in V\backslash G$. This completes the proof. □

Using the lemma's 4.12, 4.14, and 4.15 we can prove the following theorem.

THEOREM 4.16. Let the SMD $\{(P_\alpha, r_\alpha)\}$, $\alpha \in A$ be complete and let $\alpha_1, \alpha_2 \in A$. Then there is an $\alpha \in A$ such that

$$g_\alpha(u) \le \min\{g_{\alpha_1}(u), g_{\alpha_2}(u)\} \quad \text{for all } u \in V.$$

PROOF. Suppose there is a $j \in \{1,2,\ldots,n_{\alpha_1}\}$ such that $\pi_{\alpha_1 j}(V\backslash H_{\alpha_1\alpha_2}) = 0$. Then by application of lemma 4.15 we can construct a strategy $\alpha \in A$ such that $g_\alpha(u) \le g_{\alpha_2}(u)$ for all $u \in V$ and $\pi_{\alpha_1 j}(V\backslash H_{\alpha_1\alpha}) = 1$. Therefore it is possible to construct stepwise a strategy $\alpha_3 \in A$ such that $g_{\alpha_3}(u) \le g_{\alpha_2}(u)$ for all $u \in V$ and $\pi_{\alpha_1 j}(V\backslash H_{\alpha_1\alpha_3}) > 0$ for all $j \in \{1,2,\ldots,n_{\alpha_1}\}$.
Let $\alpha_0 := \alpha_1 H_{\alpha_1\alpha_3} \alpha_3$. Application of lemma 4.14 yields $\pi_{\alpha_0 j}(H_{\alpha_1\alpha_3}) = 0$ for all $j = 1,2,\ldots,n_{\alpha_0}$. Let the functions g_n, $n = 1,2,\ldots$ be defined as in lemma 4.12 with α_3 instead of α_2, $H_{\alpha_1\alpha_3}$ instead of F, and α_0 instead of α. Then it is easy to see by induction that for all $n \in \mathbb{N}$

$$g_n(u) \le \min\{g_{\alpha_1}(u), g_{\alpha_3}(u)\} \quad \text{for all } u \in V.$$

By lemma 4.12 $\lim_{n\to\infty} (g_n(u) - g_{\alpha_0}(u)) = 0$ for all $u \in V$. Hence

$$g_{\alpha_0}(u) \le \min\{g_{\alpha_1}(u), g_{\alpha_3}(u)\} \le \min\{g_{\alpha_1}(u), g_{\alpha_2}(u)\}.$$ □

COROLLARY 4.17. Let μ be a positive measure on Σ. Assume that the complete SMD $\{(P_\alpha, r_\alpha)\}$, $\alpha \in A$ has a μ-optimal strategy α_0. Then α_0 is optimal μ-almost everywhere.

PROOF. Suppose there is an $\alpha_1 \in A$ such that $g_{\alpha_1} < g_{\alpha_0}$ on some set E with $\mu(E) > 0$. By theorem 4.16 there is an $\alpha \in A$ such that

$$g_\alpha(u) \le \min\{g_{\alpha_1}(u), g_{\alpha_0}(u)\} \quad \text{for all } u \in V .$$

Then

$$\int_V g_\alpha(u)\mu(du) \le \int_V (\min\{g_{\alpha_1}(u), g_{\alpha_0}(u)\})\mu(du) < \int_V g_{\alpha_0}(u)\mu(du) ,$$

which contradicts the μ-optimality of α_0. □

4.2. *The embedded quasi-compact case*

The results of the preceding section will be extended to the case where P_α is not necessarily quasi-compact and r_α not bounded. We assume the existence of a measurable set A such that

i) for all $\alpha \in A$ the Markov process P_α is $(A, 1_V)$-recurrent and (A, r_α)-recurrent,

ii) the embedded Markov process Q_α of P_α on A is quasi-compact for all $\alpha \in A$, (Q_α is interpreted as a Markov process on (A, Σ_A)),

iii) the functions $\sum_{n=0}^{\infty} P^n_{\alpha B} 1_V$ and $\sum_{n=0}^{\infty} P^n_{\alpha B} r_\alpha$ on A, with $B = V\backslash A$, are uniformly bounded on A.

Let for all $\alpha \in A$, n_α be the dimension of $N(I - Q_\alpha)$, $E_{\alpha j}$ for $j = 1, \dots, n_\alpha$ the maximal invariant sets of Q_α, and $\pi_{\alpha j}$ the invariant probabilities of Q_α with support $E_{\alpha j}$. Let

$$E_\alpha := \sum_{j=1}^{n_\alpha} E_{\alpha j} \quad \text{and} \quad S_\alpha := \lim_{n\to\infty} \frac{1}{n} \sum_{\ell=0}^{n-1} Q_\alpha^\ell .$$

By lemma 3.7 the average cost of (P_α, r_α) starting in u,

$$g_\alpha(u) := \lim_{n\to\infty} \frac{1}{n} \sum_{\ell=0}^{n-1} (P_\alpha^\ell r_\alpha)(u)$$

exist for all $\alpha \in A$ and $u \in V$. In lemma 3.8 we proved

$$g_\alpha(u) = \frac{(S_\alpha T_\alpha r_\alpha)(u)}{(S_\alpha T_\alpha 1_V)(u)} \quad \text{for } u \in E_\alpha ,$$

where

$$T_\alpha f = \sum_{n=0}^{\infty} P^n_{\alpha B} f \ .$$

Hence g_α is constant on $E_{\alpha j}$ for $j = 1,\ldots,n_\alpha$. These constants are denoted by $g_{\alpha j}$. Let ρ be a metric on A. The continuity assumptions of section 4.1 are replaced by

iv) $$\lim_{\rho(\alpha,\alpha_0)\to 0} \| Q_\alpha - Q_{\alpha_0} \| = 0 \quad \text{for all } \alpha_0 \in A \ ,$$

v) $$\lim_{\rho(\alpha,\alpha_0)\to 0} |\pi_{\alpha_0 j}(T_\alpha r_\alpha) - \pi_{\alpha_0 j}(T_{\alpha_0} r_{\alpha_0})| = 0$$

for all $\alpha_0 \in A$ and $j = 1,\ldots,n_{\alpha_0}$.

$$\lim_{\rho(\alpha,\alpha_0)\to 0} |\pi_{\alpha_0 j}(T_\alpha 1_V) - \pi_{\alpha_0 j}(T_{\alpha_0} 1_V)| = 0$$

for all $\alpha_0 \in A$ and $j = 1,\ldots,n_{\alpha_0}$.

For $A = V$ these assumptions are identical to the assumptions made in section 4.1.

4.2.1. *The continuity of g_α*

In this subsection we consider the continuity of g_α for the embedded quasi-compact case.

Let A_n be defined as in subsection 4.1.1 with P_α replaced by Q_α.

LEMMA 4.18. Let $n \in \mathbb{N}$ and $\alpha_0 \in A_n$. Then there is a $\delta > 0$ such that for all $\alpha \in A_n$ with $\rho(\alpha,\alpha_0) < \delta$ and for all $i = 1,2,\ldots,n$, $\pi_{\alpha_0 i}(E_{\alpha j}) > 0$ for precisely one $j \in \{1,2,\ldots,n\}$.

Consider the set $A_{n\delta} := \{\alpha \in A_n \mid \rho(\alpha,\alpha_0) < \delta\}$. Let for all $\alpha \in A_{n\delta}$ and $i \in \{1,2,\ldots,n\}$ the integer i_α be defined by $\pi_{\alpha_0 i}(E_{\alpha i_\alpha}) > 0$. Then for all $i = 1,2,\ldots,n$ the functions $\pi_{\alpha_0 i}(E_{\alpha_0 i} \backslash E_{\alpha i_\alpha})$, $\|\pi_{\alpha_0 i} - \pi_{\alpha i_\alpha}\|$, and $|g_{\alpha_0 i} - g_{\alpha i_\alpha}|$ on A_n converge to 0 if $\rho(\alpha,\alpha_0)$ converges to 0.

PROOF. Lemma 1.15b implies the continuity of S_α as an operator valued function on A_n. Let $\delta > 0$ be such that $\| S_\alpha - S_{\alpha_0} \| < \frac{1}{4}$ for all $\alpha \in A_n$ with $\rho(\alpha,\alpha_0) < \delta$.

Let $i \in \{1,2,\ldots,n\}$ and let ν_i be some probability on Σ_A with $\nu_i(E_{\alpha_0 i}) = 1$, then by theorem 2.5, $\pi_{\alpha_0 i}(E_{\alpha_0 i}) = (\nu_i S_{\alpha_0})(E_{\alpha_0 i}) = 1$. Hence $(\nu_i S_\alpha)(E_{\alpha_0 i}) > \frac{3}{4}$ for all $\alpha \in A_n$ with $\rho(\alpha,\alpha_0) < \delta$. But

$$(\nu_i S_\alpha)(E_{\alpha_0 i}) = (\nu_i S_\alpha)(E_{\alpha_0 i} \cap (\bigcup_{j=1}^{n} E_{\alpha j}))$$

and therefore

$$(\nu_i S_{\alpha_0})(E_{\alpha_0 i} \cap (\bigcup_{j=1}^{n} E_{\alpha j})) > \tfrac{1}{2}$$

for all $\alpha \in A_n$ with $\rho(\alpha,\alpha_0) < \delta$. This implies the existence of at least one $j \in \{1,2,\ldots,n\}$ such that $(\nu_i S_{\alpha_0})(E_{\alpha j}) = \pi_{\alpha_0 i}(E_{\alpha j}) > 0$ for $\alpha \in A_n$ with $\rho(\alpha,\alpha_0) < \delta$.

Suppose that for some $\alpha \in A_n$ with $\rho(\alpha,\alpha_0) < \delta$ there are two j's, j_1 and j_2 such that $\pi_{\alpha_0 i}(E_{\alpha j}) > 0$. Let the probabilities ν_{ij_1} and ν_{ij_2} on Σ_A be given by

$$\nu_{ij_1}(E) = \frac{\pi_{\alpha_0 i}(E \cap E_{\alpha j_1})}{\pi_{\alpha_0 i}(E_{\alpha j_1})} \quad \text{and} \quad \nu_{ij_2}(E) = \frac{\pi_{\alpha_0 i}(E \cap E_{\alpha j_2})}{\pi_{\alpha_0 i}(E_{\alpha j_2})} .$$

Then $\nu_{ij_1} S_{\alpha_0} = \nu_{ij_2} S_{\alpha_0} = \pi_{\alpha_0 i}$. Using $(\nu_{ij_1} S_\alpha)(E_{\alpha j_1}) = 1$ and $(\nu_{ij_2} S_\alpha)(E_{\alpha j_2}) = 1$ it is easy to see that

$$\pi_{\alpha_0 i}(E_{\alpha j_1}) = (\nu_{ij_1} S_{\alpha_0})(E_{\alpha j_1}) > \tfrac{3}{4}$$

and

$$\pi_{\alpha_0 i}(E_{\alpha j_2}) = (\nu_{ij_2} S_{\alpha_0})(E_{\alpha j_2}) > \tfrac{3}{4} .$$

The disjunctness of $E_{\alpha j_1}$ and $E_{\alpha j_2}$ implies $\pi_{\alpha_0 i}(E_{\alpha j_1} \cup E_{\alpha j_2}) > 1\frac{1}{2}$ which contradicts the fact that $\pi_{\alpha_0 i}$ is a probability. This completes the proof of the first part of the lemma.

Now let for all $\alpha \in A_{n\delta}$ and $i \in \{1,2,\ldots,n\}$ the integers i_α be such that $\pi_{\alpha_0 i}(E_{\alpha i_\alpha}) > 0$. The probability ν_{ii_α} on Σ is given by

$$\nu_{ii_\alpha}(E) := \frac{\pi_{\alpha_0 i}(E \cap E_{\alpha i_\alpha})}{\pi_{\alpha_0 i}(E_{\alpha i_\alpha})} .$$

Then $\pi_{\alpha i_\alpha} = \nu_{ii_\alpha} S_\alpha$ and $\pi_{\alpha_0 i} = \nu_{ii_\alpha} S_{\alpha_0}$. Hence $\| \pi_{\alpha i_\alpha} - \pi_{\alpha_0 i} \| \le \| S_\alpha - S_{\alpha_0} \|$ and therefore

(1) $$\lim_{\rho(\alpha,\alpha_0)\to 0} \|\pi_{\alpha i_\alpha} - \pi_{\alpha_0 i}\| = 0 .$$

Furthermore $\pi_{\alpha i_\alpha}(E_{\alpha_0 i}\backslash E_{\alpha i_\alpha}) = 0$ and hence

$$\lim_{\rho(\alpha,\alpha_0)\to 0} \pi_{\alpha_0 i}(E_{\alpha_0 i}\backslash E_{\alpha i_\alpha}) = 0 .$$

For $j = 1,\dots,n$ we have

$$g_{\alpha j} = \frac{(S_\alpha T_\alpha r_\alpha)(u)}{(S_\alpha T_\alpha 1_V)(u)} \quad \text{for } u \in E_{\alpha j} ,$$

$(S_\alpha T_\alpha r_\alpha)(u) = \pi_{\alpha j}(T_\alpha r_\alpha)$ for $u \in E_{\alpha j}$, and $(S_\alpha T_\alpha 1_V)(u) = \pi_{\alpha j}(T_\alpha 1_V)$. But

$$|\pi_{\alpha i_\alpha}(T_\alpha r_\alpha) - \pi_{\alpha_0 i}(T_{\alpha_0} r_{\alpha_0})| \le |(\pi_{\alpha i_\alpha} - \pi_{\alpha_0 i})T_\alpha r_\alpha| +$$

$$+ |\pi_{\alpha_0 i}(T_\alpha r_\alpha - T_{\alpha_0} r_{\alpha_0})|$$

and

$$|\pi_{\alpha i_\alpha}(T_\alpha 1_V) - \pi_{\alpha_0 i}(T_{\alpha_0} 1_V)| \le |(\pi_{\alpha i_\alpha} - \pi_{\alpha_0 i})T_\alpha 1_V| +$$

$$+ |\pi_{\alpha_0 i}(T_\alpha 1_V - T_{\alpha_0} 1_V)| .$$

Using (1), the uniform boundedness of $T_\alpha r_\alpha$ and $T_\alpha 1_V$, and the continuity assumptions made at the beginning of this section, we get

$$\lim_{\rho(\alpha,\alpha_0)\to 0} |g_{\alpha i_\alpha} - g_{\alpha_0 i}| = 0 . \qquad \square$$

REMARK 4.19. This result implies the continuity of $g_{\alpha 1}$ on A_1. However, the condition

$$\lim_{\rho(\alpha,\alpha_0)\to 0} \|Q_\alpha - Q_{\alpha_0}\| = 0$$

is unnecessarily strong. It can be replaced by

$$\lim_{\rho(\alpha,\alpha_0)\to 0} \|Q_\alpha^k(Q_\alpha - Q_{\alpha_0})\| = 0 \quad \text{for some } k \ge 1 .$$

Compare remark 4.5.

Now we consider the μ-continuity of g_α.

LEMMA 4.20. Let μ be a positive measure on Σ_A. Then the function g_α is μ-continuous on A_n for all $n \in \mathbb{N}$.

PROOF. We have $g_\alpha = P_\alpha g_\alpha$, hence by lemma 2.21, $g_\alpha = Q_\alpha g_\alpha$. Therefore

$$\int_A g_\alpha(u)\mu(du) = \int_A (Q_\alpha g_\alpha)(u)\mu(du) = \int_A (S_\alpha g_\alpha)\mu(du) =$$

$$= \int_A g_\alpha(u)(\mu S_\alpha)(du) \ .$$

The measure μS_α is a linear combination of the $\pi_{\alpha j}$, $j = 1,\ldots,n_\alpha$. So $(\mu S_\alpha)(A\backslash E_\alpha) = 0$ and

$$\int_A g_\alpha(u)(\mu S_\alpha)(du) = \int_{E_\alpha} g_\alpha(u)(\mu S_\alpha)(du) \ .$$

Let $n \in \mathbb{N}$ and $\alpha_0,\alpha \in A_n$. Then

$$\int_A g_\alpha(u)\mu(du) - \int_A g_{\alpha_0}(u)\mu(du) = \int_{E_\alpha} g_\alpha(u)\mu(S_\alpha - S_{\alpha_0})(du) +$$

$$+ \int_{E_\alpha} (g_\alpha(u) - g_{\alpha_0}(u))(\mu S_{\alpha_0})(du) + \int_{E_\alpha} g_{\alpha_0}(u)(\mu S_{\alpha_0})(du) -$$

$$- \int_{E_{\alpha_0}} g_{\alpha_0}(u)(\mu S_{\alpha_0})(du) \ .$$

The continuity of S_α as an operator valued function on A_n and the uniform boundedness of g_α imply

$$\lim_{\rho(\alpha,\alpha_0)\to 0} \left| \int_{E_\alpha} g_\alpha(u)\mu(S_\alpha - S_{\alpha_0})(du)\right| = 0 \ .$$

Using $(\mu S_{\alpha_0})(V\backslash E_{\alpha_0}) = 0$ we get

$$\int_{E_\alpha} (g_\alpha(u) - g_{\alpha_0}(u))(\mu S_{\alpha_0})(du) = \int_{E_\alpha \cap E_{\alpha_0}} (g_\alpha(u) - g_{\alpha_0}(u))(\mu S_{\alpha_0})(du)$$

and

$$\int_{E_\alpha} g_{\alpha_0}(u)(\mu S_{\alpha_0})(du) = \int_{E_\alpha \cap E_{\alpha_0}} g_{\alpha_0}(u)(\mu S_{\alpha_0})(du) .$$

Hence

(1) $$\int_{E_\alpha} (g_\alpha(u) - g_{\alpha_0}(u))(\mu S_{\alpha_0})(du) = \sum_{j=1}^{n} \int_{E_{\alpha_0 j} \cap E_\alpha} (g_\alpha(u) - g_{\alpha_0}(u))(\mu S_{\alpha_0})(du)$$

and

(2) $$\int_{E_\alpha} g_{\alpha_0}(u)(\mu S_{\alpha_0})(du) - \int_{E_{\alpha_0}} g_{\alpha_0}(u)(\mu S_{\alpha_0})(du) = - \int_{E_{\alpha_0} \setminus E_\alpha} g_{\alpha_0}(u)(\mu S_{\alpha_0})(du).$$

We complete the proof by application of lemma 4.18 on (1) and (2), using that μS_{α_0} is a linear combination of the $\pi_{\alpha_0 j}$. □

4.2.2. *Existence of an optimal strategy*

The proofs of the following two properties are analogous to the proofs of the lemma's 4.6 and 4.7.

i) Let μ be a positive measure on Σ_A. Let A be compact and A_n closed in A for all $n \in \mathbb{N}$. If n_α is bounded on A, the

$$\inf_{\alpha \in A} \{ \int_A \mu(du) g_\alpha(u) \}$$

is attained.

ii) If there is a real β, $0 < \beta < 1$, such that for all $\alpha \in A$ the spectrum of Q_α has no points with absolute value between β and 1, then for all $n \in \mathbb{N}$ the set A_n is closed in A.

In lemma 4.18 we proved the continuity of $g_{\alpha 1}$ on A_1. So, if $A = A_1$ and A is compact then an optimal strategy exists. We shall give conditions under which the set A is dominated by the set A_1 (where dominating is defined as in section 4.1). We need some new concepts.

DEFINITION 4.21. The SMD is called A-*communicative* if for all $\alpha \in A$ and $j = 1,\dots,n_\alpha$ there is an $\alpha_1 \in A_1$ such that $\pi_{\alpha_1 1}(E_{\alpha j}) > 0$.

Notice that A-communicativeness is equivalent with communicativeness if $A = V$.

DEFINITION 4.22. Let $\alpha \in A$ and $i \in \{1,2,\dots,n_\alpha\}$. The set

$$\bar{E}_{\alpha i} := \{u \in V \mid Q_\alpha(u,E_{\alpha i}) = 1\}$$

is called the *extension* of $E_{\alpha i}$.

Notice that $E_{\alpha i} \subset \bar{E}_{\alpha i}$ and that, by lemma 2.21, $\bar{E}_{\alpha i}$ is an invariant set of P_α.

LEMMA 4.23. Let the SMD be complete and A-communicative. Then A_1 dominates A.

PROOF. Let $\alpha \in A$. Choose j_0 such that

$$g_{\alpha j_0} = \min_{j=1,2,\dots,n_\alpha} \{g_{\alpha j}\} .$$

The A-communicativeness of the SMD implies the existence of a strategy $\alpha_1 \in A_1$ such that $\pi_{\alpha_1 1}(E_{\alpha j_0}) > 0$.
Let $C := \bar{E}_{\alpha j_0}$ and $\alpha_2 := \alpha C \alpha_1$. Using lemma 2.17 and lemma 2.23 it is easy to see that the embedded sub-Markov process Q'_{α_2} of P_{α_2} on C is a Markov process. This implies $\alpha_2 \in A_1$. By lemma 2.21, $g_{\alpha_2} = Q'_{\alpha_2} g_{\alpha_2}$. The invariance of $C = \bar{E}_{\alpha j_0}$ under P_α implies $g_{\alpha_2}(u) = g_{\alpha j_0}$ for $u \in C$. Hence

$$g_{\alpha_2}(u) = g_{\alpha j_0} \le g_\alpha(u) \quad \text{for all } u \in V . \qquad \square$$

The following theorem is analogous to theorem 4.11.

THEOREM 4.24. Let the SMD be complete and A-communicative. If A is compact then an optimal strategy exists.

PROOF. Let $g := \inf_{\alpha \in A_1} g_{\alpha 1}$. The compactness of A implies the existence of a sequence $\{\alpha_k\}$ in A_1 converging to $\alpha_0 \in A$ such that $\lim_{k\to\infty} g_{\alpha_k 1} = g$. Without loss of generality we may assume that $\pi_j := \lim_{k\to\infty} \pi_{\alpha_k 1}(E_{\alpha_0 j})$ exists for all $j = 1,\dots,n_{\alpha_0}$. We have

$$g_{\alpha_k 1} = \frac{\pi_{\alpha_k 1}(T_{\alpha_k} r_{\alpha_k})}{\pi_{\alpha_k 1}(T_{\alpha_k} 1_V)} \quad \text{for all } k = 1,2,3,\dots .$$

As in the proof of theorem 4.11 we can show that

$$\lim_{k\to\infty} \pi_{\alpha_k 1}(T_{\alpha_k} r_{\alpha_k}) = \sum_{j=1}^{n_{\alpha_0}} \pi_j . r_j ,$$

where $r_j := \pi_{\alpha_0 j}(T_{\alpha_0} r_{\alpha_0})$ and

$$\lim_{k\to\infty} \pi_{\alpha_k 1}(T_{\alpha_k} 1_V) = \sum_{j=1}^{n_{\alpha_0}} \pi_j . t_j ,$$

where $t_j := \pi_{\alpha_0 j}(T_{\alpha_0} 1_V)$. Hence

$$g = \frac{\sum_{j=1}^{n_{\alpha_0}} \pi_j . r_j}{\sum_{j=1}^{n_{\alpha_0}} \pi_j . t_j}$$

and therefore

$$\min_{j=1,\dots,n_{\alpha_0}} \{\frac{r_j}{t_j}\} \leq g .$$

But $g_{\alpha_0 j} = \frac{r_j}{t_j}$ for $j = 1,2,\dots,n_{\alpha_0}$, which implies that

$$\min_{j=1,\dots,n_{\alpha_0}} \{g_{\alpha_0 j}\} \leq g .$$

By lemma 4.23 there is an $\alpha \in A_1$ such that

$$g_\alpha(u) \leq \min_{j=1,\dots,n_{\alpha_0}} \{g_{\alpha_0 j}\} \quad \text{for all } u \in V .$$

The strategy α is optimal. □

4.2.3. *Optimality μ-almost everywhere and μ-optimality*

As in subsection 4.1.3 we assume that the SMD is complete. We shall prove that for all $\alpha_1,\alpha_2 \in \mathring{A}$ there is an $\alpha_0 \in \mathring{A}$ such that

$$g_{\alpha_0}(u) \le \min\{g_{\alpha_1}(u), g_{\alpha_2}(u)\} \quad \text{for all } u \in V .$$

Then we can show as in subsection 4.1.3 that μ-optimality implies optimality μ-almost everywhere.

The next lemma is analogous to lemma 4.12.

LEMMA 4.25. Let the SMD $\{(P_\alpha, r_\alpha)\}$, $\alpha \in \mathring{A}$ be complete. Let $\alpha_1,\alpha_2 \in \mathring{A}$, $F \in \Sigma$ and $\alpha := \alpha_1 F \alpha_2$. The embedded Markov process of P_α on $F \cup A$ is denoted by Q. Suppose $\pi_{\alpha j}(F \cap A) = 0$ and

$$\int_A Q(u,F)\pi_{\alpha j}(du) = 0 \quad \text{for } j = 1,\dots,n_\alpha .$$

By lemma 2.32 there is a set $G \subset V \backslash F$ such that $\pi_{\alpha j}(G \cap A) = 1$ for $j = 1,\dots,n_\alpha$ and $P_\alpha(u,G) = 1$ for $u \in G$. By corollary 2.13 and lemma 2.23 the embedded sub-Markov processes Q_1 and Q_2 of P_α on $V\backslash F$ and on $F \cup G$ are Markov processes.

The functions g_n, $n = 1,2,\dots$, on V are defined by

$$g_1(u) = (Q_1 g_{\alpha_2})(u) \qquad \text{for } u \in F ,$$

$$g_1(u) = (Q_{2F} g_1)(u) + (Q_{2G} g_{\alpha_2})(u) \qquad \text{for } u \in V\backslash F ,$$

and for $n = 2,3,4,\dots$

$$g_n(u) = (Q_1 g_{n-1})(u) \qquad \text{for } u \in F ,$$

$$g_n(u) = (Q_{2F} g_n)(u) + (Q_{2G} g_{\alpha_2})(u) \qquad \text{for } u \in V\backslash F .$$

For these functions g_n the following property holds

$$\lim_{n\to\infty} (g_n(u) - g_\alpha(u)) = 0 \quad \text{for all } u \in V .$$

The proof is similar to the proof of lemma 4.12.

LEMMA 4.26. Let the SMD $\{(P_\alpha,r_\alpha)\}$, $\alpha \in A$ be complete. Let $\alpha_1,\alpha_2 \in A$, and $\alpha := \alpha_1 H \alpha_2$ where $H := H_{\alpha_1\alpha_2}$ is defined as in 4.13. Put $C := H\backslash A$, $B := V\backslash A$, $D := V\backslash(A \cup H)$. Let Q_1 be the embedded Markov process of P_{α_1} on $A \cup D$ and Q_2 the embedded Markov process of P_α on $A \cup H$. Assume that for all $j = 1,2,\ldots,n_{\alpha_1}$ $\pi_{\alpha_1 j}(A\backslash H) > 0$ and/or

$$\int_A (Q_1 1_{V\backslash H})(u)\pi_{\alpha_1 j}(du) > 0 .$$

Then for all $j = 1,\ldots,n_\alpha$ $\pi_{\alpha j}(A \cap H) = 0$ and

$$\int_A (Q_2 1_H)(u)\pi_{\alpha j}(du) = 0 .$$

PROOF. The assumptions imply that the embedded sub-Markov process S_1 of P_{α_1} on $V\backslash H$ is a Markov process (see lemma 2.25 and the lemma's 2.17 and 2.23). Let S be the embedded sub-Markov process of P_α on $V\backslash H$. Since $S(u,E) = S_1(u,E)$ for $u \in H$ and $E \in \Sigma$, S is also a Markov process. Suppose that for some j $\pi_{\alpha j}(A\backslash H) > 0$ and

$$\int_{A\backslash H} (Q_2 1_H)(u)\pi_{\alpha j}(du) > 0 .$$

Put $E := \tilde{E}_{\alpha j}$ (see definition 4.22), and interprete P_α as the restriction of P_α to (E,Σ_E). For $G \in \Sigma$ we denote $G \cap E$ by G'.
By lemma 2.31 the entry sub-Markov process R of P_α on H' is a Markov process and is $\tilde{\pi}$-recurrent, where $\tilde{\pi}$ is the measure on Σ_E defined by

$$\tilde{\pi}(G) = \int_{A'\backslash H'} (Q_2 1_G)(u)\pi_{\alpha j}(du),\ G \in \Sigma_E .$$

We can choose an $\varepsilon > 0$ and a set $H_\varepsilon \in \Sigma_{H'}$ such that $\tilde{\pi}(H_\varepsilon) > 0$ and $g_{\alpha_2}(u) \geq g_{\alpha_1}(u) + \varepsilon$ for $u \in H_\varepsilon$. For $u \in H'$ we get

$$g_{\alpha_1}(u) = (Sg_{\alpha_1})(u) \geq (Sg_{\alpha_2})(u) = (STg_{\alpha_2})(u) ,$$

where T is the embedded Markov process of P_α on H'.

Hence

$$g_{\alpha_1}(u) \geq (STg_{\alpha_2})(u) = (Rg_{\alpha_2})(u) = (R_{H'\setminus H_\varepsilon} g_{\alpha_2})(u) + (R_{H_\varepsilon} g_{\alpha_2})(u) \geq$$

$$\geq (R_{H_\varepsilon} g_{\alpha_2})(u) + (R_{H'\setminus H_\varepsilon} g_{\alpha_1})(u) \geq (R_{H_\varepsilon} g_{\alpha_2})(u) + (R_{H'\setminus H_\varepsilon} R_{H_\varepsilon} g_{\alpha_2})(u) +$$

$$+ (R_{H'\setminus H_\varepsilon} R_{H'\setminus H_\varepsilon} g_{\alpha_1})(u) \geq \ldots \geq$$

$$\geq \sum_{n=0}^{k-1} (R^n_{H'\setminus H_\varepsilon} R_{H_\varepsilon} g_{\alpha_2})(u) + (R^k_{H'\setminus H_\varepsilon} g_{\alpha_1})(u) .$$

Since R is $\tilde{\pi}$-recurrent $\lim_{k\to\infty} (R^k_{H'\setminus H_\varepsilon} g_{\alpha_1})(u) = 0$ and

$$\sum_{n=0}^{\infty} (R^n_{H'\setminus H_\varepsilon} R_{H_\varepsilon} 1_V)(u) = 1 .$$

Hence

$$g_{\alpha_1}(u) \geq \sum_{n=0}^{\infty} (R^n_{H'\setminus H_\varepsilon} R_{H_\varepsilon} g_{\alpha_2})(u) \geq \varepsilon + \sum_{n=0}^{\infty} (R^n_{H'\setminus H_\varepsilon} R_{H_\varepsilon} g_{\alpha_1})(u) \geq$$

$$\geq \varepsilon + \inf_{u\in H_\varepsilon} g_{\alpha_1}(u) \quad \text{for all } u \in H' .$$

This yields a contradiction.

This means that for all $j = 1,\ldots,n_\alpha$

(1) $$\pi_{\alpha j}(A\setminus H) = 0 \text{ and/or } \int_{A\setminus H} (Q_2 1_H)(u)\pi_{\alpha j}(du) = 0 .$$

Now suppose

$$\int_{A\setminus H} (Q_2 1_H)(u)\pi_{\alpha j}(du) = 0 \quad \text{for some } j .$$

Since

$$(Q_2 1_H)(u) = \sum_{n=0}^{\infty} (P^n_{\alpha D} P_\alpha 1_H)(u)$$

this implies $P_\alpha(u,H) = 0$, $\pi_{\alpha j}$-almost everywhere on $A\setminus H$ and therefore

$$P_\alpha(u,B) = P_\alpha(u,C) + P_\alpha(u,D) = P_\alpha(u,D) ,$$

$\pi_{\alpha j}$-almost everywhere on $A\setminus H$. Hence

$$(Q_\alpha 1_H)(u) = \sum_{n=0}^{\infty} (P^n_{\alpha B} P_\alpha 1_H)(u) = P_\alpha(u,H) + \sum_{n=1}^{\infty} (P^n_{\alpha B} P_\alpha 1_H)(u) =$$

$$= P_\alpha(u,H) + \sum_{n=1}^{\infty} (P^n_{\alpha D} P_\alpha 1_H)(u) = (Q_2 1_H)(u) = 0 ,$$

$\pi_{\alpha j}$-almost everywhere on $A\backslash H$. Therefore

$$\pi_{\alpha j}(A \cap H) = \int_A \pi_{\alpha j}(du) Q_\alpha(u, A \cap H) = \int_{A\cap H} \pi_{\alpha j}(du) Q_\alpha(u, A \cap H)$$

and $Q_\alpha(u, A \cap H) = 1$, $\pi_{\alpha j}$-almost everywhere on $A \cap H$. This implies for $G \in \Sigma_A$

$$\pi_{\alpha j}(G \cap H) = \int_A \pi_{\alpha j}(du) Q_\alpha(u, G \cap H) = \int_{A\cap H} \pi_{\alpha j}(du) Q_\alpha(u, G \cap H) =$$

$$= \int_{A\cap H} \pi_{\alpha j}(du)\{Q_\alpha(u,G) - Q_\alpha(u, G\backslash H)\} = \int_{A\cap H} \pi_{\alpha j}(du) Q_\alpha(u,G).$$

Hence $\pi_{\alpha j} I_H$ is invariant under Q_α and therefore $\pi_{\alpha j}(A \cap H) = 0$ or $\pi_{\alpha j}(A \cap H) = 1$. Now the result (1) implies that $\pi_{\alpha j}(A \cap H) = 0$ or 1 for all $j = 1,\ldots,n_\alpha$. Suppose $\pi_{\alpha j}(A \cap H) = 1$ for some $j \in \{1,\ldots,n_\alpha\}$.
Let Q be the embedded Markov process of P_α on $A \cup D$. If

$$\int_{A\cap H} \pi_{\alpha j}(du) Q(u, V\backslash H) = 0 ,$$

then by lemma 2.32 there is a set $G \subset H$ which is invariant under P_α. This contradicts the fact that the embedded sub-Markov process of P_α on $V\backslash H$ is a Markov process. Hence

$$\int_{A\cap H} \pi_{\alpha j}(du) Q(u, V\backslash H) > 0 .$$

Let $E := \bar{\bar{E}}_{\alpha j}$ and interprete P_α as the restriction of P_α to (E, Σ_E). For $G \in \Sigma$ we denote $G \cap E$ by G'.
By lemma 2.31 the entry sub-Markov process R of P_α on $E\backslash H'$ is a Markov process and is $\tilde{\pi}$-recurrent, where $\tilde{\pi}$ is given by

$$\tilde{\pi}(G) = \int_{A'\cap H'} \pi_{\alpha j}(du) Q(u,G), \; G \in \Sigma_E .$$

Let

$$H_\varepsilon := \{u \in H' \mid g_{\alpha_2}(u) \geq g_{\alpha_1}(u) + \varepsilon\} .$$

Let the measure π^* on Σ_E be defined by

$$\pi^*(G) = \int_{E\setminus H'} \tilde{\pi}(du)T(u,G), \; G \in \Sigma_E ,$$

where T is the embedded Markov process of P_α on H'. Since $\pi^*(H') > 0$ there is an $\eta > 0$ and an $\varepsilon > 0$ such that $\pi^*(H_\varepsilon) > \eta$ and therefore a set $G_\varepsilon \subset E\setminus H'$ such that $\tilde{\pi}(G_\varepsilon) > 0$ and $T(u,H_\varepsilon) \geq \eta$ for $u \in G_\varepsilon$. For $u \in E\setminus H'$ we have

$$g_{\alpha_2}(u) = (Tg_{\alpha_2})(u) > (Tg_{\alpha_1})(u) = (TS)g_{\alpha_1}(u) = (Rg_{\alpha_1})(u) \geq (Rg_{\alpha_2})(u) =$$

$$= (R_{E\setminus(G_\varepsilon \cup H')}g_{\alpha_2})(u) + (R_{G_\varepsilon}g_{\alpha_2})(u) \geq \dots \geq$$

$$\geq \sum_{n=0}^{k-1} (R^n_{E\setminus(G_\varepsilon \cup H')}R_{G_\varepsilon}g_{\alpha_2})(u) + (R^k_{E\setminus(G_\varepsilon \cup H')}g_{\alpha_2})(u) \geq$$

$$\geq \sum_{n=0}^{\infty} (R^n_{E\setminus(G_\varepsilon \cup H')}R_{G_\varepsilon}g_{\alpha_2})(u)$$

since R is $\tilde{\pi}$-recurrent and $\tilde{\pi}(G_\varepsilon) > 0$. For $u \in G_\varepsilon$ we have

$$g_{\alpha_2}(u) = (Tg_{\alpha_2})(u) = (T_{H_\varepsilon}g_{\alpha_2})(u) + (T_{H'\setminus H_\varepsilon}g_{\alpha_2})(u) \geq (T_{H_\varepsilon}g_{\alpha_1})(u) +$$

$$+ T(u,H_\varepsilon).\varepsilon + (T_{H'\setminus H_\varepsilon}g_{\alpha_1})(u) \geq (Tg_{\alpha_1})(u) + \eta.\varepsilon \geq (Rg_{\alpha_2})(u) +$$

$$+ \eta.\varepsilon .$$

Hence, for $u \in E\setminus H'$,

$$g_{\alpha_2}(u) \geq \sum_{n=0}^{\infty} (R^n_{E\setminus(G_\varepsilon \cup H')}R_{G_\varepsilon}Rg_{\alpha_2})(u) + \eta.\varepsilon \geq \eta.\varepsilon + \inf_{u\in E\setminus H'} g_{\alpha_2}(u) ,$$

which yields a contradiction. Hence $\pi_{\alpha j}(A \cap H) = 0$ for all $j = 1,\dots,n_\alpha$. By (1) this implies

$$\int_{A\setminus H} (Q_2 1_H)(u)\pi_{\alpha j}(du) = \int_A (Q_2 1_H)(u)\pi_{\alpha j}(du) = 0 . \qquad \square$$

In the next lemma we consider the case where the conditions of lemma 4.26 are not satisfied.

LEMMA 4.27. Let the SMD $\{(P_\alpha, r_\alpha)\}$, $\alpha \in A$ be complete. Let $\alpha_1, \alpha_2 \in A$, and $H := H_{\alpha_1\alpha_2}$. Put $C := H\backslash A$. Suppose that for some $j \in \{1,2,\dots,n_{\alpha_1}\}$ $\pi_{\alpha_1 j}(A\backslash H) = 0$ and

$$\int_A (Q_1 1_{V\backslash H})(u)\pi_{\alpha_1 j}(du) = 0 ,$$

where Q_1 is the embedded Markov process of P_{α_1} on $V\backslash C$. By lemma 2.32 there is a set $G \subset H$ such that $\pi_{\alpha_1 j}(A \cap G) = 1$ and $P_{\alpha_1}(u,G) = 1$ for $u \in G$. Let $\alpha := \alpha_1 G \alpha_2$, then $g_\alpha = g_{\alpha_1}$ on G and $g_\alpha \le g_{\alpha_2}$ on V.

PROOF. The definition of α implies $g_\alpha = g_{\alpha_1}$ on G. Consider the sets $\bar{E}_{\alpha_2 i}$, $i = 1,\dots,n_{\alpha_2}$. Let $D := V\backslash(A \cup G)$ and Q the embedded Markov process of P_α on $A \cup G$. If $\pi_{\alpha_2 i}(A \cap G) > 0$ or

$$\int_A (Q 1_G)(u)\pi_{\alpha_2 i}(du) > 0 ,$$

then the embedded sub-Markov process on $G \cap \bar{E}_{\alpha_2 i}$ of the restriction of P_{α_2} to $(\bar{E}_{\alpha_2 i}, \Sigma_{\bar{E}_{\alpha_2 i}})$ is a Markov process on $(\bar{E}_{\alpha_2 i}, \Sigma_{\bar{E}_{\alpha_2 i}})$. Let I_2 be the set of all such i and $I_1 := \{1,\dots,n_{\alpha_2}\}\backslash I_2$. By lemma 2.32 there is a set $F \subset V\backslash G$ such that $\pi_{\alpha_2 i}(A \cap F) = 1$ for all $i \in I_1$ and $P_{\alpha_2}(u,F) = 1$ for $u \in F$. The embedded sub-Markov process of P_{α_2} on $G \cup F$ is a Markov process. The rest of the proof is analogous to the last part of the proof of lemma 4.15. □

We can use the lemma's 4.25, 4.26, 4.27 to prove the following result.

THEOREM 4.28. Let the SMD $\{(P_\alpha, r_\alpha)\}$, $\alpha \in A$ be complete and let $\alpha_1, \alpha_2 \in A$. Then there is an $\alpha \in A$ such that

$$g_\alpha(u) \le \min\{g_{\alpha_1}(u), g_{\alpha_2}(u)\} \quad \text{for all } u \in V .$$

The proof of this theorem is completely analogous to the proof of theorem 4.16. The lemma's 4.12, 4.14 and 4.15 are replaced by the lemma's 4.25, 4.26, 4.27.

The proof of the following corollary of theorem 4.28 is analogous to the proof of corollary 4.17.

COROLLARY 4.29. Let μ be a positive measure on Σ. Suppose that the complete SMD $\{(P_\alpha, r_\alpha)\}$, $\alpha \in A$ has a μ-optimal strategy α_0. Then α_0 is optimal μ-almost everywhere.

4.3. *Countable state space*

In this section some results of section 4.2 are applied to the case where V is countable and Σ is the σ-field of all subsets of V.
We shall relate our results to those of some others (Derman [2], Ross [13], Hordijk [4]).

In the next lemma it will be shown that the conditions i), ii), iii), iv), and v), stated at the beginning of section 4.2, are implied by some simpler ones.

LEMMA 4.30. Let the following conditions be satisfied.

a) The functions r_α are bounded on V for all $\alpha \in A$ and the boundedness is uniform on A.
b) There is a metric ρ on A such that $P_\alpha(u,v)$ and $r_\alpha(u)$ are continuous in α for all $u,v \in V$. (Instead of $P_\alpha(u,\{v\})$ we write $P_\alpha(u,v)$.)
c) There is a finite subset A of V such that the sum

$$\sum_{n=0}^{\infty} (P^n_{\alpha B} 1_V)(u)$$

with $B := V \backslash A$, exists for all $u \in V$, $\alpha \in A$, and the convergence is uniform on A for all $u \in A$.

Then the conditions i), ii), iii), iv), and v) are satisfied.

For the proof of this lemma we need the following result.

LEMMA 4.31. Let ρ be a metric on A such that $P_\alpha(u,v)$ is continuous as function on A, for all $u,v \in V$. Let $\{f_\alpha\}$, $\alpha \in A$ be a set of complex valued functions, bounded on V uniform on A and let $f_\alpha(u)$ be continuous in α for all $u \in V$. Then $(P_{\alpha G} f_\alpha)(u)$ is continuous in α for all $u \in V$, $G \in \Sigma$.

PROOF. Choose $u \in V$, $G \in \Sigma$, $\alpha_0 \in A$. Let $\varepsilon > 0$. There is a finite set F_ε such that $P_{\alpha_0}(u,F_\varepsilon) > 1 - \varepsilon$. The continuity of $P_\alpha(u,F_\varepsilon)$ implies the existence of a $\delta > 0$ such that $P_\alpha(u,V\backslash F_\varepsilon) < 2\varepsilon$ for all $\alpha \in A$ with $\rho(\alpha,\alpha_0) < \delta$. We have

$$(P_{\alpha G}f_\alpha)(u) = \int_{G\backslash F_\varepsilon} P_\alpha(u,ds)f_\alpha(s) + \int_{F_\varepsilon \cap G} P_\alpha(u,ds)f_\alpha(s)$$

and

$$(P_{\alpha G}f_\alpha)(u) - (P_{\alpha_0 G}f_{\alpha_0})(u) = \int_{G\backslash F_\varepsilon} P_\alpha(u,ds)f_\alpha(s) - \int_{G\backslash F_\varepsilon} P_{\alpha_0}(u,ds)f_{\alpha_0}(s) +$$

$$+ \int_{F_\varepsilon \cap G} (P_\alpha(u,ds) - P_{\alpha_0}(u,ds))f_\alpha(s) + \int_{F_\varepsilon \cap G} P_{\alpha_0}(u,ds)(f_\alpha(s) - f_{\alpha_0}(s)).$$

The rest of the proof is obvious. □

Now we can give the proof of lemma 4.30.

PROOF OF LEMMA 4.30. The conditions i) and iii) are direct consequences of the conditions a) and c), condition ii) is implied by the finiteness of the set A (Q_α is even compact). To prove iv) it is sufficient to prove the continuity of $Q_\alpha(u,E)$ in α for all $u \in A$, $E \in \Sigma_A$. This is easily done by using the expression

$$Q_\alpha(u,E) = \sum_{n=0}^{\infty} (P^n_{\alpha B}P_{\alpha A}1_E)(u) \ .$$

Namely, condition c) implies that for all $\varepsilon > 0$ there is an integer N such that

$$\sum_{n=N}^{\infty} (P^n_{\alpha B}P_{\alpha A}1_E)(u) < \varepsilon$$

for all $u \in A$, $\alpha \in A$, $E \in \Sigma_A$. The continuity of

$$\sum_{n=0}^{N-1} (P^n_{\alpha B}P_{\alpha A}1_E)(u)$$

in α follows from lemma 4.31. The rest of the proof of iv) is straightforward. That condition v) is also satisfied can be shown similarly, using the continuity of $r_\alpha(u)$ in α. □

The following theorem is a direct consequence of lemma 4.30 and theorem 4.24.

THEOREM 4.32. Let the conditions a), b), and c) of lemma 4.30 be satisfied and let the SMD be complete and A-communicative. If $\mathcal{A}$ is compact then an optimal strategy exists.

Comment

For the case of a countable V we shall relate our results for stationary Markovian decision problems to some results in Markov decision processes with a countable state space. First we have to give the relationship between stationary Markovian decision problems and Markov decision processes.

A Markov decision process consists of the following elements (see Ross [13]).

a) *State space*. At each time $t = 0,1,2,\ldots$ the system is in one of the states $s \in S$. The state at time t is denoted by X_t.
b) *Actions*. For each $u \in S$ there is a set $A(u)$ of admissable actions. In u one can choose an arbitrary action $d \in A(u)$. The state u and the action d determine the probability $\mathbb{P}(E \mid u,d)$ of being in a set E next time.
c) *Costs*. The expected costs of using action d in state u are $c(u,d)$.

An important concept in Markov decision processes is the concept *policy*. A policy prescribes for each time t a probability distribution*) over $A(u)$. This probability distribution can depend on the whole history X_0, d_0, X_1, d_1, $X_2,d_2,\ldots, X_{t-1},d_{t+1},X_t$, where d_i is the action chosen at time i. If the probability distributions only depend on t and X_t, the policy is called Markovian, if the probability distributions only depend on X_t, the policy is called stationary. A policy is called deterministic if the probability distributions are concentrated in one point**). A policy which minimizes the average costs over a certain class of policies C is called average optimal in C.

Now let us consider an SMD such that

- each $\alpha \in \mathcal{A}$ is a function on V,
- for all $u \in V$, $r_\alpha(u)$ and the function $P_\alpha(u,\cdot)$ only depend on α by $\alpha(u)$,
- the SMD is complete.

*) We assume the existence of a σ-field in $A(u)$ such that the point sets $\{d\}$ are elements of this σ-field.

**) Notice that each policy can be made deterministic by enlarging the sets $A(u)$.

In this case one can interprete the strategies α as the stationary deterministic policies of a Markov decision process. The set V corresponds to the state space S of the Markov decision process and the set $\{\alpha(u), \alpha \in A\}$ to the set A(u). The completeness is required to make it possible that one may choose an action in u independently of what is chosen in the other states. An optimal strategy of the SMD, if existing, corresponds to a stationary deterministic policy which is average optimal in the class of all stationary deterministic policies. Derman [2], Ross [13], and Hordijk [4] investigate the existence of a stationary deterministic policy which is average optimal in the class of all Markov policies (Hordijk), or in the class of all policies (Derman and Ross). It is important to be conscious of this fact in relating our results to their results.

We shall give the conditions of Hordijk and Ross for the existence of an average optimal stationary deterministic policy. These are weaker than the conditions of Derman. For a discussion of the results of Derman (and others) we refer to Hordijk [4], section 12. Ross [13] as well as Hordijk require a countable state space.

The conditions of Hordijk, in our terminology, are as follows:

1) the functions $r_\alpha(\cdot)$ are bounded on V, uniform on A;
2) the simultaneous Doeblin condition is satisfied: there is a finite set A, a pos. number c, and an integer n such that $P^n_\alpha(u,A) \geq c$ for all $u \in V$, $\alpha \in A$;
3) there is a metric ρ on A such that A is compact and
4) for all $u,v \in V$ the functions $r_\alpha(u)$ and $P_\alpha(u,v)$ are continuous in α;
5) the SMD is communicative. (The simultaneous Doeblin condition implies the quasi-compactness of P_α for all $\alpha \in A$, so we may speak indeed about communicativeness.)

The most striking difference with the conditions of theorem 4.32 is the simultaneous Doeblin condition. Instead of this condition we require condition c) of lemma 4.30: there is a finite set $A \subset V$ such that the sum

$$\sum_{n=0}^{\infty} (P^n_{\alpha B} 1_V)(u) \ ,$$

where $B := V\backslash A$, exists for all $u \in V$ and $\alpha \in A$, and the convergence is uniform on A for all $u \in A$.

The simultaneous Doeblin condition implies the convergence of

$$\sum_{n=0}^{\infty} (P^n_{\alpha B} 1_V)(u), \text{ uniform on } V \times A \ .$$

Ross [13] gives the following conditions:

- for all $u \in V$ the set $A(u)$ of all possible actions in u is finite;
- the functions $r_\alpha(\cdot)$ are bounded on V, uniform on A;
- there exists a state $v \in V$, an integer $N > 0$, and a sequence of discount factors $\{\beta_n\}$, $0 < \beta_n < 1$, such that $\lim_{n\to\infty} \beta_n = 1$ and $M_{uv}(R_{\beta_n}) < N$ for all $u \in V$, $n \in \mathbb{N}$, where $M_{uv}(R_{\beta_n})$ is the mean time to go from state u to state v when using the β_n-discounted optimal policy R_{β_n}.

The finiteness of $A(u)$ makes the compactness and continuity conditions superfluous. We can see this as follows:

Let $u_1, u_2, u_3, \ldots$ be the elements of V. The set A of all stationary deterministic policies is the Cartesian product $\prod_{i=1}^{\infty} A(u_i)$. Let ρ_n be a metric on $A(u_n)$ and define the metric ρ on A by

$$\rho(\alpha_1, \alpha_2) := \sum_{n=1}^{\infty} 2^{-n} \cdot \rho_n(\alpha_1(u_n), \alpha_2(u_n)) .$$

The metric topology on A is the product topology, (Kelley [7], 4.14) and A is compact in this topology (Tychonov). Now let

$$m_n := \min_{d_1, d_2 \in A(u_n)} \{\rho_n(d_1, d_2)\} .$$

If $\rho(\alpha_1, \alpha_2) < m_n \cdot 2^{-n}$ then $\rho_n(\alpha_1(u_n), \alpha_2(u_n)) < m_n$ and hence $\alpha_1(u_n) = \alpha_2(u_n)$. This implies the continuity of $P_\alpha(u,v)$ and $r_\alpha(u)$ in α.

The last condition of Ross states a very strong recurrency, (recurrency to a point $v \in V$) for a subset $\{R_{\beta_n}\}$, $n = 1,2,\ldots$ of the set of all stationary deterministic policies. This condition quarantees the quasi-compactness of the Markov process under policy R_{β_n} and also $R_{\beta_n} \in A_1$ (only one invariant probability). In condition c) of lemma 4.30 a weaker recurrency is stated (recurrency to a set A), but for all strategies $\alpha \in A$. The A-communicativeness assumed in theorem 4.32 implies that A_1 dominates A.

In a set of conditions different from the just mentioned one Hordijk [4] also requires recurrency to a point. This set of conditions is more directly related to the conditions i) - v) of section 4.2 with V countable and A consisting of one point. The boundedness of r_α and the quasi-compactness of P_α is not required.

CHAPTER 5. INVENTORY PROBLEMS

In this chapter we shall deal with inventory problems. Inventory problems are defined as a special class of stationary Markovian decision problems. In section 5.1 we give assumptions and definitions. The existence of an optimal strategy is investigated in section 5.2. Using the solutions of the (P_α, r_α)-equations one can formulate conditions for optimality and non-optimality of a strategy. This is done in section 5.3. For some classes of inventory problems it is possible to prove that the optimal strategy is of a specific structure. This problem is considered in section 5.4.

5.1. *Preliminaries*

Throughout this chapter we assume that V is the real line and Σ the σ-field of Borel sets on V.

Let the function ω on V be defined by $\omega(u) = e^{|u|}$, $u \in V$. The space of all complex valued functions f on V such that $f_\omega := \frac{f}{\omega} \in \mathcal{B}$ is denoted by $\mathcal{B}_\omega$. The space of all measures μ such that the measure μ_ω, defined by

$$\mu_\omega(E) = \int_E \omega(u)\mu(du) \quad \text{for } E \in \Sigma ,$$

is an element of M, is denoted by M_ω.

Let $\|f\|_\omega := \|f_\omega\|$ for $f \in \mathcal{B}_\omega$ and $\|\mu\|_\omega := \|\mu_\omega\|$ for $\mu \in M_\omega$. Then $\|f\|_\omega$ and $\|\mu\|_\omega$ are norms in $\mathcal{B}_\omega$ and M_ω.

LEMMA 5.1. The spaces $\mathcal{B}_\omega$ and M_ω with norms $\|f\|_\omega$ and $\|\mu\|_\omega$ are Banach spaces.

PROOF. Let $\{f_n\}_1^\infty$ be a Cauchy sequence in $\mathcal{B}_\omega$. Then $\{f_{n\omega}\}$ is a Cauchy sequence in $\mathcal{B}$ which has a limit $f_0 \in \mathcal{B}$. But $f_0.\omega \in \mathcal{B}_\omega$ and

$$\lim_{n\to\infty} \| f_0.\omega - f_n\|_\omega = \lim_{n\to\infty} \| f_0 - f_{n\omega}\| = 0 .$$

Hence $\mathcal{B}_\omega$ is a Banach space.

That M_ω is also a Banach space can be proved similarly. □

If P is a transition probability on $(V \times \Sigma)$ such that

$$\int_V P(\cdot, ds)\omega(s)$$

is an element of $\mathcal{B}_\omega$, P can be interpreted as a linear operator in $\mathcal{B}_\omega$. We will not use a new notation for this operator. On which space P acts will be clear from the context or explicitly stated.

In the next lemma we shall show the similarity of the spaces $\mathcal{B}$ and $\mathcal{M}$ to the spaces $\mathcal{B}_\omega$ and $\mathcal{M}_\omega$.

LEMMA 5.2.

a) The integral $\mu f := \int_V \mu(du) f(u)$ exists for all $\mu \in \mathcal{M}_\omega$, $f \in \mathcal{B}_\omega$, and $|\mu f| \le \|\mu\|_\omega . \|f\|_\omega$.

b) If P is a transition probability such that $P\omega \in \mathcal{B}_\omega$, then $\mu P \in \mathcal{M}_\omega$ for all $\mu \in \mathcal{M}_\omega$ and $(\mu P) f = \mu(Pf)$ for $f \in \mathcal{B}_\omega$.

PROOF. It is sufficient to prove these statements for positive μ and f. This can be done by using the analogous properties of the spaces $\mathcal{M}$ and $\mathcal{B}$, (see the preliminaries of the chapters 1 and 2), and the monotone convergence theorem. □

Now we shall define inventory problems.

DEFINITION 5.3. An *inventory problem* is an SMD, $\{(P_\alpha, r_\alpha)\}$, $\alpha \in A$ on (V,Σ) with the following properties:

i) A is a subset of the set of all nonnegative measurable functions on V.

ii) There is a probability distribution function F on V with $F(a) = 0$ for $a < 0$, $F(0) \neq 1$, and $\int_{0^-}^{\infty} e^x dF(x) < \infty$, such that for all $u \in V$, $\alpha \in A$, and intervals $[a,b)$

$$P_\alpha(u,[a,b)) = -\int_a^b dF(u + \alpha(u) - v) = \int_{u+\alpha(u)-b}^{u+\alpha(u)-a} dF(x) .$$

iii) There are nonnegative measurable functions $r_1, r_2 \in \mathcal{B}_\omega$ such that $r_\alpha(u) = r_1(\alpha(u)) + r_2(u + \alpha(u))$ for all $u \in V$, $\alpha \in A$.

An inventory problem as defined here can be interpreted as a one-point inventory problem with leadtime 0 and backlogging. The distribution function F is the distribution function of the demand per period. The functions r_1 and r_2 give the ordering and inventory costs. For $u \in V$, $\alpha(u)$ is the quantity to order under strategy α.

The following theorem makes it possible to apply the results of chapter 4 to inventory problems.

THEOREM 5.4. Let m, M, R be real numbers such that $m < 0 < M$, $R \leq M - m$, and

$$\int_{0^-}^{\infty} e^x dF(x) < e^R .$$

Let $\{(P_\alpha, r_\alpha)\}$, $\alpha \in A$ be an inventory problem such that for all $\alpha \in A$:

$$\alpha(u) \geq R \qquad \text{for } u \leq m ,$$

$$u + \alpha(u) \leq M \qquad \text{for } u \leq M ,$$

$$\alpha(u) = 0 \qquad \text{for } u > M .$$

Let $A := [m,M]$ and $B := V \backslash A$.
Then there exists an $a > 0$ and a ρ, $0 < \rho < 1$ such that $\| P^n_{\alpha B} \|_\omega < a.\rho^n$ for all $\alpha \in A$ and $n \in \mathbb{N}$.

PROOF. First we have to proof that $P_{\alpha B} f \in B_\omega$ for all $f \in B_\omega$ and $\alpha \in A$. Let $\alpha \in A$ and $f \in B_\omega$.
For $u \leq m$ we have

$$(P_{\alpha B} f)(u) = -\int_{-\infty}^{m} f(v)\, dF(u+\alpha(u)-v) = \int_{u+\alpha(u)-m}^{\infty} f(u+\alpha(u)-x)\, dF(x).$$

Hence

$$\frac{|(P_{\alpha B} f)(u)|}{e^{|u|}} = \frac{1}{e^{-u}} \cdot \left| \int_{u+\alpha(u)-m}^{\infty} f(u + \alpha(u) - x) dF(x) \right| \leq$$

$$\leq \overset{m}{\underset{-\infty}{\| f \|_\omega}} \cdot \int_{u+\alpha(u)-m}^{\infty} e^{x-\alpha(u)} dF(x) ,$$

where

$$\overset{m}{\underset{-\infty}{\|f\|_\omega}} := \sup_{-\infty<u\le m} \left\{\frac{f(u)}{e^{|u|}}\right\} .$$

So

$$(1) \qquad \frac{|(P_{\alpha B}f)(u)|}{e^{|u|}} \le \overset{m}{\underset{-\infty}{\|f\|_\omega}} \cdot \frac{1}{e^{\alpha(u)}} \int_{0^-}^{\infty} e^x dF(x) \le \overset{m}{\underset{-\infty}{\|f\|_\omega}} \cdot e^{-R} \int_{0^-}^{\infty} e^x dF(x) .$$

For $m \le u \le M$ we have

$$(P_{\alpha B}f)(u) = \int_{u+\alpha(u)-m}^{\infty} f(u + \alpha(u) - x) dF(x) .$$

Hence

$$(2) \qquad \frac{|(P_{\alpha B}f)(u)|}{e^{|u|}} \le \overset{m}{\underset{-\infty}{\|f\|_\omega}} \cdot \frac{1}{e^{-u}} \cdot \int_{u+\alpha(u)-m}^{\infty} e^{-u-\alpha(u)+x} dF(x) \le$$

$$\le \overset{m}{\underset{-\infty}{\|f\|_\omega}} \cdot \int_{0^-}^{\infty} e^{x-\alpha(u)} dF(x) \le \overset{m}{\underset{-\infty}{\|f\|_\omega}} \cdot \int_{0^-}^{\infty} e^x dF(x) .$$

For $u > M$ we have

$$(P_{\alpha B}f)(u) = \int_{u-m}^{\infty} f(u - x) dF(x) + \int_{-\infty}^{u-M} f(u - x) dF(x) .$$

Hence

$$(3) \qquad \frac{|(P_{\alpha B}f)(u)|}{e^{|u|}} \le \overset{m}{\underset{-\infty}{\|f\|_\omega}} \cdot \int_{u-m}^{\infty} \frac{e^{x-u}}{e^u} dF(x) + \overset{\infty}{\underset{M}{\|f\|_\omega}} \cdot \int_{-\infty}^{u-M} \frac{e^{u-x}}{e^u} dF(x) \le$$

$$\le \overset{m}{\underset{-\infty}{\|f\|_\omega}} \cdot \int_{0^-}^{\infty} e^x dF(x) + \overset{\infty}{\underset{M}{\|f\|_\omega}} \cdot \int_{0^-}^{\infty} e^{-x} dF(x) .$$

The relations (1), (2), (3) imply that $P_{\alpha B}f \in \mathcal{B}_\omega$ for $f \in \mathcal{B}_\omega$ and for all $\alpha \in A$. Now we have to consider $P^n_{\alpha B}f$. Let

$$r := e^{-R} \int_{0^-}^{\infty} e^x dF(x) \quad \text{and} \quad q := \int_{0^-}^{\infty} e^{-x} dF(x) .$$

Then by (1)

$$(4) \qquad \frac{|(P^n_{\alpha B} f)(u)|}{e^{|u|}} \le r^n \cdot \overset{m}{\underset{-\infty}{\|f\|_\omega}} \qquad \text{for } u \le m$$

and by (2)

$$(5) \qquad \frac{|(P^n_{\alpha B} f)(u)|}{e^{|u|}} \le r^n \cdot e^R \cdot \overset{m}{\underset{-\infty}{\|f\|_\omega}} \qquad \text{for } m \le u \le M\ .$$

For u > M we have

$$(6) \qquad \frac{|(P^n_{\alpha B} f)(u)|}{e^{|u|}} \le r \cdot e^R \cdot \overset{m}{\underset{-\infty}{\|P^{n-1}_{\alpha B} f\|_\omega}} + q \cdot \overset{\infty}{\underset{M}{\|P^{n-1}_{\alpha B} f\|_\omega}} \le \dots \le$$

$$\le (r^n + qr^{n-1} + \dots + q^{n-1} r) e^R \cdot \overset{m}{\underset{-\infty}{\|f\|_\omega}} + q^n \overset{\infty}{\underset{M}{\|f\|_\omega}} \le$$

$$\le (r^n + qr^{n-1} + \dots + q^{n-1} r + q^n) e^R \cdot \|f\|_\omega =$$

$$= \frac{r^{n+1} - q^{n+1}}{r - q} \cdot e^R \cdot \|f\|_\omega\ .$$

The relations (4), (5), (6) complete the proof. □

An inventory problem as in theorem 5.4 is called an (m,M,R)-problem. We shall show that an (m,M,R)-problem satisfies the conditions i) and iii) of section 4.2.

LEMMA 5.5. Let $\{(P_\alpha, r_\alpha)\}$, $\alpha \in A$ be an (m,M,R)-problem and let A := [m,M] and B := V\A. Then for all $\alpha \in A$ the Markov process is (A,1)-recurrent and (A, r_α)-recurrent, and the boundedness on A of the functions

$$\sum_{n=0}^{\infty} P^n_{\alpha B} r_\alpha \quad \text{and} \quad \sum_{n=0}^{\infty} P^n_{\alpha B} 1_V$$

is uniform on A.

PROOF. By theorem 5.4 it is sufficient to prove that $1_V \in B_\omega$, that $r_\alpha \in B_\omega$ for all $\alpha \in A$ and that $\|r_\alpha\|_\omega$ is bounded on A. The boundedness of $\frac{1}{\omega}$ implies that $1_V \in B_\omega$. Further we have $r_\alpha(u) = r_1(\alpha(u)) + r_2(u + \alpha(u))$ and $r_1, r_2 \in B_\omega$. Hence

$$r_\alpha(u) \le \|r_1\|_\omega \cdot e^{\alpha(u)} + \|r_2\|_\omega \cdot e^{|u+\alpha(u)|} .$$

Using that $\{(P_\alpha, r_\alpha)\}$, $\alpha \in A$ is an (m,M,R)-problem it is easy to show that $r_\alpha \in B_\omega$ for $\alpha \in A$ and that $\|r_\alpha\|_\omega$ is bounded on A. □

We defined $\omega(u) := e^{|u|}$. But it is possible to use the same reasoning for $\omega(u) := e^{c|u|}$, where c is some positive constant. Then we can apply the results to the case where the distribution function F has a negative exponential tail.

5.2. *Existence of optimal strategies*

In this section we consider an (m,M,R)-problem $\{(P_\alpha, r_\alpha)\}$, $\alpha \in A$. Using chapter 4 we shall give conditions for the existence of an optimal strategy.

By lemma 5.5 we know that the conditions i) and iii) of section 4.2 are satisfied for $A := [m,M]$. Let, for all $\alpha \in A$, Q_α be the embedded Markov process of P_α on A. In the next lemma we shall give a condition for compactness of Q_α^2.

LEMMA 5.6. If F has a bounded density φ, Q_α^2 is compact for $\alpha \in A$.

PROOF. Let $\varphi_{\alpha v}(u) := \varphi(u + \alpha(u) - v)$ for $\alpha \in A$, $v \in V$, $u \in V$. Then

$$Q_\alpha(u,E) = \int_E q_\alpha(u,v)\,dv ,$$

where

$$q_\alpha(u,v) := \sum_{n=0}^{\infty} (P_{\alpha B}^n \varphi_{\alpha v})(u) .$$

By theorem 5.4 the boundedness of φ implies the boundedness of $q_\alpha(\cdot,\cdot)$ on $A \times A$ for all $\alpha \in A$.

Now let λ be the Lebesgue measure on A. It is easy to show that for all $\alpha \in A$ $\lim_{\lambda(E)\to 0} (\mu Q_\alpha)(E) = 0$ uniformly for all measures μ on Σ_A with $\|\mu\| \le 1$.

Using Dunford-Schwartz [3], IV.9.2 and VI.4.1 we infer that Q_α is weakly compact for all $\alpha \in A$. This implies the compactness of Q_α^2, (see [3], VI.8.13 and the remarks at the end of VI.8). □

Since compactness of Q_α^2 implies quasi-compactness of Q_α, (see section 1.2), this lemma yields a condition sufficient for the condition ii) of section 4.2.

In the rest of this section we assume that F has a bounded density φ and also that, for all $\alpha \in A$, Q_α has only one invariant probability. The results of sub-section 4.2.1 for this case can be summarized in the following way.

LEMMA 5.7. Let ρ be a metric on A such that

a) for $\alpha_0 \in A$,

$$\lim_{\rho(\alpha,\alpha_0)\to 0} \| Q_\alpha^k (Q_\alpha - Q_{\alpha_0}) \| = 0 \quad \text{for some } k \, ;$$

b)

$$\lim_{\rho(\alpha,\alpha_0)\to 0} |\pi_{\alpha_0 1}(T_\alpha r_\alpha) - \pi_{\alpha_0 1}(T_{\alpha_0} r_{\alpha_0})| = 0 \quad \text{for } \alpha_0 \in A \, ,$$

$$\lim_{\rho(\alpha,\alpha_0)\to 0} |\pi_{\alpha_0 1}(T_\alpha 1_V) - \pi_{\alpha_0 1}(T_{\alpha_0} 1_V)| = 0 \quad \text{for } \alpha_0 \in A \, ,$$

$$(T_\alpha := \sum_{n=0}^{\infty} P_{\alpha B}^n) \, .$$

Then $g_{\alpha 1}$ is continuous as function on A.

In the next lemma we shall show that condition a) of lemma 5.7 can be replaced by a continuity condition on P_α.

LEMMA 5.8. Let ρ be a metric on A and α_0 an element of A such that for all $n = 0,1,2,\ldots$

(1) $$\lim_{\rho(\alpha,\alpha_0)\to 0} \| P_{\alpha A} P_{\alpha B}^n (P_{\alpha A} - P_{\alpha_0 A}) \| = 0$$

(2) $$\lim_{\rho(\alpha,\alpha_0)\to 0} \| P_{\alpha A} P_{\alpha B}^n (P_{\alpha B} - P_{\alpha_0 B}) \| = 0 \, .$$

Then

$$\lim_{\rho(\alpha,\alpha_0)\to 0} \| Q_\alpha (Q_\alpha - Q_{\alpha_0}) \| = 0 \, .$$

PROOF. We have $Q_\alpha f = \sum_{n=0}^{\infty} P^n_{\alpha B} P_{\alpha A} f$ for all $\alpha \in \mathcal{A}$, $f \in \mathcal{B}$. Let

$$\| f \|_A := \sup_{u \in A} |f(u)| \quad \text{for } f \in \mathcal{B}$$

and

$$\| \omega \|_A := \sup_{u \in A} |\omega(u)| \ .$$

By theorem 5.4, for each $\varepsilon > 0$ there is an integer N_ε such that for all $\alpha \in \mathcal{A}$ and $f \in \mathcal{B}$

$$\| \sum_{n=N_\varepsilon}^{\infty} P^n_{\alpha B} P_{\alpha A} f \|_\omega \le \varepsilon . \| P_{\alpha A} f \|_\omega \ .$$

Hence

$$| \sum_{n=N_\varepsilon}^{\infty} (P^n_{\alpha B} P_{\alpha A} f)(u) | \le \varepsilon . \| P_{\alpha A} f \|_\omega . \| \omega \|_A \quad \text{for } u \in A \ ,$$

and

$$\| Q_\alpha \{ \sum_{n=N_\varepsilon}^{\infty} P^n_{\alpha B} P_{\alpha A} f - \sum_{n=N_\varepsilon}^{\infty} P^n_{\alpha_0 B} P_{\alpha_0 A} f \} \| \le 2\varepsilon . \| P_{\alpha A} f \|_\omega . \| \omega \|_A \ .$$

But

$$\| P_{\alpha A} f \|_\omega \le \| P_{\alpha A} f \| \le \| f \|_A \le \| f \| \ .$$

Therefore it is sufficient to prove for all $n = 0,1,2,\ldots$

$$\lim_{\rho(\alpha,\alpha_0) \to 0} \| Q_\alpha (P^n_{\alpha B} P_{\alpha A} - P^n_{\alpha_0 B} P_{\alpha_0 A}) \| = 0 \ . \tag{3}$$

It is easy to show that

$$P^n_{\alpha B} P_{\alpha A} - P^n_{\alpha_0 B} P_{\alpha_0 A} = \sum_{k=0}^{n-1} P^k_{\alpha B} (P_{\alpha B} - P_{\alpha_0 B}) P^{n-1-k}_{\alpha_0 B} P_{\alpha_0 A} + $$
$$+ P^n_{\alpha B} (P_{\alpha A} - P_{\alpha_0 A}) \ .$$

Hence, by the assumptions (1) and (2)

$$\lim_{\rho(\alpha,\alpha_0) \to 0} \| P_{\alpha A} (P^n_{\alpha B} P_{\alpha A} - P^n_{\alpha_0 B} P_{\alpha_0 A}) \| = 0 \quad \text{for } n = 0,1,2,\ldots \ . \tag{4}$$

By theorem 5.4 $\| \sum_{n=0}^{\infty} P^n_{\alpha B} \|_\omega$ is bounded on A. Let K be an upperbound. Then for all $f \in B$ we get

$$\| Q_\alpha f \|_A \le \| Q_\alpha f \|_\omega . \| \omega \|_A = \| \sum_{n=0}^{\infty} P^n_{\alpha B} P_{\alpha A} f \|_\omega . \| \omega \|_A \le$$

$$\le \| \sum_{n=0}^{\infty} P^n_{\alpha B} \|_\omega . \| P_{\alpha A} f \|_\omega . \| \omega \|_A \le K . \| f \| . \| \omega \|_A \ .$$

Together with (4) this implies (3). □

Now we consider the condition b) of lemma 5.7.

LEMMA 5.9. Let $\alpha_0 \in A$ and let ρ be a metric on A such that for each $\mu \in M_\omega$ which is continuous with respect to the Lebesgue measure λ, the following properties hold

(1) $$\lim_{\rho(\alpha,\alpha_0)\to 0} \int_V |r_\alpha(u) - r_{\alpha_0}(u)| \mu(du) = 0$$

(2) $$\lim_{\rho(\alpha,\alpha_0)\to 0} \int_V |(P_{\alpha B} f)(u) - (P_{\alpha_0 B} f)(u)| \mu(du) = 0 \text{ for all } f \in B_\omega \ .$$

Then condition b) of lemma 5.7 is satisfied for α_0.

PROOF. First we shall show by induction that for all n = 0,1,2,...

(3) $$\lim_{\rho(\alpha,\alpha_0)\to 0} \int_V |(P^n_{\alpha B} r_\alpha)(u) - (P^n_{\alpha_0 B} r_{\alpha_0})(u)| \mu(du) = 0 \text{ for all } \mu \in M_\omega$$

which are continuous with respect to λ.
For n = 0 (3) is a direct consequence of assumption (1). Now let it be true for n = k. We have

$$\int_V |(P^{k+1}_{\alpha B} r_\alpha)(u) - (P^{k+1}_{\alpha_0 B} r_{\alpha_0})(u)| \mu(du) \le$$

$$\le \int_V \int_B P_\alpha(u,ds) |(P^k_{\alpha B} r_\alpha)(s) - (P^k_{\alpha_0 B} r_{\alpha_0})(s)| \mu(du) +$$

$$+ \int_V |(P_{\alpha B} - P_{\alpha_0 B})(P^k_{\alpha_0 B} r_{\alpha_0})(u)| \mu(du) \ .$$

By assumption (2)

$$(4)\qquad \lim_{\rho(\alpha,\alpha_0)\to 0} \int_V |(P_{\alpha B} - P_{\alpha_0 B})(P^k_{\alpha_0 B} r_{\alpha_0})(u)|\mu(du) = 0 .$$

Further

$$\int_V \int_B P_\alpha(u,ds)|(P^k_{\alpha B} r_\alpha)(s) - (P^k_{\alpha_0 B} r_{\alpha_0})(s)|\mu(du) =$$

$$= \int_B (\mu P_\alpha)(ds)|(P^k_{\alpha B} r_\alpha)(s) - (P^k_{\alpha_0 B} r_{\alpha_0})(s)| ,$$

where μP_α is an element of M_ω, (see lemma 5.2). μP_α is continuous with respect to λ. By (4) and the induction assumption we see that (3) is true for $n = k + 1$. Hence (3) is true for all $n = 0,1,2,\ldots$.

For each $\varepsilon > 0$, theorem 5.4 and the boundedness of $\| r_\alpha \|_\omega$ on A, (see the proof of lemma 5.5), imply the existence of an integer N_ε such that

$$\| \sum_{n=N_\varepsilon}^{\infty} P^n_{\alpha B} r_\alpha \|_\omega < \varepsilon \qquad \text{for all } \alpha \in A .$$

Hence

$$\int_V |\sum_{N_\varepsilon}^{\infty} P^n_{\alpha B} r_\alpha - \sum_{N_\varepsilon}^{\infty} P^n_{\alpha_0 B} r_{\alpha_0}|(u)\mu(du) \le \|\mu\|_\omega . 2\varepsilon \text{ for all } \mu \in M_\omega .$$

Using this and (3) we get for all λ-continuous $\mu \in M_\omega$

$$(5)\qquad \lim_{\rho(\alpha,\alpha_0)\to 0} \int_V |(T_\alpha r_\alpha)(u) - (T_{\alpha_0} r_{\alpha_0})(u)|\mu(du) = 0 .$$

Similarly we can prove

$$(6)\qquad \lim_{\rho(\alpha,\alpha_0)\to 0} \int_V |(T_\alpha 1_V)(u) - (T_{\alpha_0} 1_V)(u)|\mu(du) = 0 .$$

Let $\pi'_{\alpha 1}$ for $\alpha \in A$ be the measure on Σ defined by

$$\pi'_{\alpha 1}(E) = \pi_{\alpha 1}(A \cap E),\ E \in \Sigma .$$

Then $\pi'_{\alpha 1} \in M_\omega$. The existence of a bounded density of F implies the λ-continuity of $\pi'_{\alpha 1}$. Substitution of $\mu := \pi'_{\alpha 1}$ in (5) and (6) completes the proof. □

The next problem is the introduction of a metric ρ on A such that the continuity conditions of lemma 5.7 are satisfied and such that A is compact. Since F has a bounded density $g_{\alpha_1 1} = g_{\alpha_2 1}$ if α_1 and α_2 are λ-almost everywhere equal, (λ is the Lebesgue measure). This makes it possible to interprete each element $\alpha \in A$ as a class of functions which are λ-almost everywhere equal to each other.

Since $\{(P_\alpha, r_\alpha)\}$, $\alpha \in A$ is an (m,M,R)-problem the integral $\int_V \frac{\alpha(u)}{\omega(u)}\,du$ is finite for all $\alpha \in A$. The metric ρ on A defined by

$$\rho(\alpha_1,\alpha_2) := \int_V \frac{|\alpha_1(u) - \alpha_2(u)|}{\omega(u)}\,du$$

is called the *ω-metric*.

Let $A_\omega := \{\frac{\alpha}{\omega}, \alpha \in A\}$. A_ω with the L_1-metric is a subspace of $L_1(V,\Sigma,\lambda)$ which is isometrically isomorphic with the metric space A. Hence compactness of A_ω implies compactness of A.

LEMMA 5.10. Let A be such that

$$\lim_{x \to 0} \int_V \left|\frac{\alpha(u + x)}{\omega(u + x)} - \frac{\alpha(u)}{\omega(u)}\right| du = 0 \text{ , uniform on } A \text{ .}$$

Then the closure of A_ω in L_1 is compact.

PROOF. Using that $\{(P_\alpha, r_\alpha)\}$, $\alpha \in A$ is an (m,M,R)-problem, we can see that $\int_V \frac{\alpha(u)}{\omega(u)}\,du$ is bounded on A, and

$$\lim_{a \to +\infty} \left\{ \int_{-\infty}^{-a} \frac{\alpha(u)}{\omega(u)}\,du + \int_{+a}^{+\infty} \frac{\alpha(u)}{\omega(u)}\,du \right\} = 0, \text{ uniform on } A \text{ .}$$

By [3], IV.8.20 this implies that the closure of A_ω is compact. □

Now we shall consider the continuity conditions of lemma 5.7 with ρ equal to the ω-metric on A.

LEMMA 5.11. If the density φ of F has a bounded derivative φ', the conditions (1) and (2) of lemma 5.8 and condition (2) of lemma 5.9 are satisfied for all $\alpha_0 \in A$ and for ρ equal to the ω-metric on A.

PROOF. First we shall prove that for all $a \le m$, $\varepsilon > 0$ and for each $n \in \mathbb{N}$ there are n finite intervals $B_i := [a_i, M]$, $i = 1,\ldots,n$ such that for all $\alpha \in A$ and all $u \in [a,M]$

$$(1) \qquad |(P^n_{\alpha B}\omega)(u) - (P_{\alpha B_1} P_{\alpha B_2} \cdots P_{\alpha B_n}\omega)(u)| \le \varepsilon .$$

For all $a \le m$ and all $\varepsilon > 0$ it is possible to choose an $a_\varepsilon \le m$ such that for $u \in [a,M]$ and for all $\alpha \in A$

$$\int_{-\infty}^{a_\varepsilon} \varphi(u + \alpha(u) - v)\omega(v)dv < \varepsilon .$$

This proves (1) for $n = 1$.
Now let it be true for $n = k$.
By theorem 5.4 $P_{\alpha B}\omega \in B_\omega$ and $\|P_{\alpha B}\omega\|_\omega$ is bounded on A. Hence, the induction assumption implies for each $a \le m$ and each $\varepsilon > 0$ the existence of finite intervals $B_i := [a_i, M]$, $i = 1,\ldots,k$, such that

$$|(P^{k+1}_{\alpha B}\omega)(u) - (P_{\alpha B_1} P_{\alpha B_2} \cdots P_{\alpha B_k} P_{\alpha B}\omega)(u)| < \varepsilon \text{ for } u \in [a,M],\ \alpha \in A.$$

Let the interval $B_{k+1} := [a_{k+1}, M]$ be such that

$$|(P_{\alpha B}\omega)(u) - (P_{\alpha B_{k+1}}\omega)(u)| < \frac{\varepsilon}{M - a_k} \quad \text{for } u \in [a_k, M],\ \alpha \in A .$$

Then for all $u \in [a,M]$ and $\alpha \in A$ we get

$$|(P^{k+1}_{\alpha B}\omega)(u) - (P_{\alpha B_1} P_{\alpha B_2} \cdots P_{\alpha B_k} P_{\alpha B_{k+1}}\omega)(u)| < \varepsilon + (M - a_k)\frac{\varepsilon}{M - a_k} = 2\varepsilon,$$

which shows that (1) is true for $n = k + 1$ and hence for all $n \in \mathbb{N}$. We can use this result to show for each $\varepsilon > 0$ the existence of intervals $B_i := [a_i, M]$, $i = 1,\ldots,n$, such that for all $\alpha, \alpha_0 \in A$

$$(2) \qquad \|P_{\alpha A}P^n_{\alpha B}(P_{\alpha A} - P_{\alpha_0 A}) - P_{\alpha A}P_{\alpha B_1}P_{\alpha B_2} \cdots P_{\alpha B_n}(P_{\alpha A} - P_{\alpha_0 A})\| < \varepsilon ,$$

and the existence of intervals $B_i := [a_i, M]$, $i = 1,\ldots,n+1$, such that for all $\alpha, \alpha_0 \in A$

$$(3) \qquad \|P_{\alpha A}P^n_{\alpha B}(P_{\alpha B} - P_{\alpha_0 B}) - P_{\alpha A}P_{\alpha B_1} \cdots P_{\alpha B_n}(P_{\alpha B_{n+1}} - P_{\alpha_0 B_{n+1}})\| < \varepsilon .$$

Now let α_0 be an arbitrary element of A, $\Delta_\alpha := |\alpha - \alpha_0|$ for $\alpha \in A$, $\|\varphi'\|$ is the supremum of $\varphi'(u)$, and $C := [a,b]$ is an arbitrary finite interval. Then

$$|(P_{\alpha C} - P_{\alpha_0 C})f(u)| = \left| \int_C \{\varphi(u+\alpha(u)-v) - \varphi(u+\alpha_0(u)-v\} f(v)\,dv \right| \leq$$

$$\leq \Delta_\alpha(u).\|\varphi'\|.\int_C |f(v)|\,dv \leq \Delta_\alpha(u).\|\varphi'\|.(b-a).\|f\| \text{ for all } f \in \mathcal{B} .$$

Let $C_i := [a_i,b_i]$, $i = 1,\dots,n$ be arbitrary finite intervals. Then

$$|P_{\alpha A}P_{\alpha C_1} \dots P_{\alpha C_n}(P_{\alpha C} - P_{\alpha_0 C})f(u)| \leq$$

$$\leq (b-a).\|\varphi'\|.\|f\|.(P_{\alpha A}P_{\alpha C_1} \dots P_{\alpha C_n}\Delta_\alpha)(u) \text{ for } u \in V,\ f \in \mathcal{B} .$$

But

$$(P_{\alpha C_n}\Delta_\alpha)(u) = \int_C \varphi(u+\alpha(u)-v)\Delta_\alpha(v)\,dv \leq \|\omega\|_{C_n}.\|\varphi\|.\int_{C_n} \frac{\Delta_\alpha(v)}{\omega(v)}\,dv \leq$$

$$\leq \|\omega\|_{C_n}.\|\varphi\|.\rho(\alpha,\alpha_0) ,$$

where $\|\varphi\| = \sup_{u \in V} \varphi(u)$. Hence

$$\lim_{\rho(\alpha,\alpha_0)\to 0} \|P_{\alpha A}P_{\alpha C_1} \dots P_{\alpha C_n}(P_{\alpha C} - P_{\alpha_0 C})\| = 0 .$$

Using the results (1) and (2), this proves the conditions (1) and (2) of lemma 5.8 for all $\alpha_0 \in A$.

Now we have to prove condition (2) of lemma 5.9 for all $\alpha_0 \in A$. Using the fact that $\{(P_\alpha, r_\alpha)\}$, $\alpha \in A$ is an (m,M,R)-problem, we see that

$$\int_V |(P_{\alpha B}f)(u) - (P_{\alpha_0 B}f)(u)|\,\mu(du) = \int_{-\infty}^{M} |(P_{\alpha B}f)(u) - (P_{\alpha_0 B}f)(u)|\,\mu(du).$$

For each $\varepsilon > 0$ and each $\mu \in M_\omega$ there is an $a_\varepsilon < m$ such that

$$\int_{-\infty}^{a_\varepsilon} |(P_{\alpha B}f)(u) - (P_{\alpha_0 B}f)(u)|\,\mu(du) < \varepsilon \text{ for all } \alpha,\alpha_0 \in A .$$

Hence, it is sufficient to prove for all $\mu \in M_\omega$ and $\alpha_0 \in A$

$$\lim_{\rho(\alpha,\alpha_0)\to 0} \int_C |(P_{\alpha B}f)(u) - (P_{\alpha_0 B}f)(u)|\mu(du) = 0$$

for all finite intervals C. This can be done by approximating B by a finite interval, as in the first part of the proof. □

Now we have the following result.

THEOREM 5.12. Let $\{(P_\alpha, r_\alpha)\}$, $\alpha \in A$ be an (m,M,R)-problem such that

i) F has a bounded density with bounded derivative;

ii) The embedded Markov process on [m,M], Q_α, has only one invariant probability;

iii) $$\lim_{x\to 0} \int_{-\infty}^{+\infty} \left|\frac{\alpha(u+x)}{\omega(u+x)} - \frac{\alpha(u)}{\omega(u)}\right| du = 0, \text{ uniform on } A\ ;$$

iv) A_ω is closed in $L_1(V,\Sigma,\lambda)$;

v) For each $\mu \in M_\omega$ which is continuous with respect to the Lebesgue measure λ and for ρ the ω-metric

$$\lim_{\rho(\alpha,\alpha_0)\to 0} \int_V |r_\alpha(u) - r_{\alpha_0}(u)|\mu(du) = 0 \text{ for all } \alpha_0 \in A\ .$$

In this case an optimal strategy exists.

5.3. *Criteria for optimality*

Using the (P,r)-equations introduced in chapter 3, we shall give criteria for optimality and nonoptimality. We consider an (m,M,R)-problem such that for all $\alpha \in A$ the embedded Markov process Q_α of P_α on [m,M] is quasi-compact and has only one invariant probability.

The (P_α, r_α)-equations are the equations

$$x = P_\alpha x$$

$$y = r_\alpha - x + P_\alpha y$$

in the complex valued measurable functions x,y on V.

The conditions of section 3.1, where we investigated these equations, are satisfied for all $\alpha \in A$ with $A := [m,M]$. Hence, there is a solution of the following type

$$x = g_\alpha$$
$$y = T_\alpha r_\alpha - T_\alpha g_\alpha + Q_\alpha f' ,$$

where f' is a bounded measurable function and $T_\alpha f = \sum_{n=0}^{\infty} P_{\alpha B}^n f$ for $f = r_\alpha, g_\alpha$. By theorem 5.4, this guarantees the existence of a solution of the equation $y = r_\alpha - g_\alpha + P_\alpha y$ in B_ω.
Let f_α be such a solution. Since Q_α has only one invariant probability g_α is constant on V. In the next lemma we show that for all $u \in V$ and $\alpha \in A$, $(P_\alpha^n \omega)(u)$ is bounded in n. This implies that

$$\lim_{n\to\infty} \frac{1}{n} (P_\alpha^n f)(u) = 0 \quad \text{for } \alpha \in A,\ f \in B_\omega,\ u \in V .$$

Hence, by lemma 3.7, the solution f_α is unique upto a constant.

LEMMA 5.13. For all $u \in V$, $\alpha \in A$ the function $(P_\alpha^n \omega)(u)$ on $\mathbb{N}$ is bounded.

PROOF. Substitution of $P_\alpha = P_{\alpha A} + P_{\alpha B}$ in P_α^{m+1} gives

$$P_\alpha^{m+1} = P_\alpha^m P_{\alpha A} + P_\alpha^m P_{\alpha B} = P_\alpha^m P_{\alpha A} + P_\alpha^{m-1} P_{\alpha A} P_{\alpha B} + P_\alpha^{m-1} P_{\alpha B}^2 = \dots =$$
$$= \sum_{k=0}^{m} P_\alpha^{m-k} P_{\alpha A} P_{\alpha B}^k + P_{\alpha B}^{m+1} .$$

Hence

$$(P_\alpha^{m+1}\omega)(u) \le \sum_{k=0}^{m} \| P_{\alpha A} P_{\alpha B}^k \omega \| + (P_{\alpha B}^{m+1}\omega)(u) \le \sum_{k=0}^{m} \| P_{\alpha B}^k \omega \|_A +$$
$$+ (P_{\alpha B}^{m+1}\omega)(u) \le \|\omega\|_A \sum_{k=0}^{m} \| P_{\alpha B}^k \|_\omega + (P_{\alpha B}^{m+1}\omega)(u) .$$

The boundedness of $(P_\alpha^n \omega)(u)$ in n is a direct consequence of theorem 5.4. □

Now we can give the criteria for optimality and nonoptimality. The structure of these criteria is well known, (see Ross [12]).

LEMMA 5.14. Let $\alpha_0 \in A$.

a) If for all $\alpha \in A$

(1) $$f_{\alpha_0} \leq r_\alpha - g_{\alpha_0} + P_\alpha f_{\alpha_0} ,$$

then the strategy α_0 is optimal.

b) If for some $\alpha \in A$ there is a positive measurable function Δ_α on V such that

$$S_\alpha \Delta_\alpha := \lim_{n\to\infty} \frac{1}{n} \sum_{\ell=0}^{n-1} P_\alpha^\ell \Delta_\alpha > 0$$

and

(2) $$f_{\alpha_0} \geq r_\alpha - g_{\alpha_0} + P_\alpha f_{\alpha_0} + \Delta_\alpha ,$$

then

$$g_\alpha \leq g_{\alpha_0} - S_\alpha \Delta_\alpha < g_{\alpha_0} .$$

The proofs can be given by repeated substitution of f_{α_0}, in the right-hand sides of the inequalities (1) and (2), by the complete right-hand sides, using that

$$\lim_{n\to\infty} \frac{1}{n} (P_\alpha^n f_{\alpha_0})(u) = 0 \quad \text{for all } u \in V .$$

The existence of $S_\alpha \Delta_\alpha$ is a consequence of the fact that $\Delta_\alpha \in B_\omega$ since $\Delta_\alpha \leq f_{\alpha_0} - r_\alpha + g_{\alpha_0} - P_\alpha f_{\alpha_0}$. Further, Q_α has only one invariant probability and therefore $S_\alpha \Delta_\alpha$ is constant on V. □

The inventory structure of the problem makes it possible to formulate the criteria in another, more applicable way.

Define the function J_α on V for all $\alpha \in A$ by

(1) $$J_\alpha(u) = r_2(u) - \int_{-\infty}^{+\infty} f_\alpha(v)\,dF(u-v) = r_2(u) + \int_{-\infty}^{+\infty} f_\alpha(u-x)\,dF(x), \quad u \in V .$$

The function J_α is an element of B_ω. Since f_α is a solution of

(2) $$y = r_\alpha - g_\alpha + P_\alpha y ,$$

the function J_α is a solution of the equation

$$(3) \qquad z(\cdot) = r_2(\cdot) - \int_{-\infty}^{+\infty} \{r_1(\alpha(v)) - g_\alpha(v)\} dF(\cdot - v) - \int_{-\infty}^{+\infty} z(v + \alpha(v)) dF(\cdot - v)$$

in $z(\cdot)$.

But if $z_\alpha \in \mathcal{B}_\omega$ is a solution of (3) then the function y_α on V defined by

$$y_\alpha(u) = r_1(\alpha(u)) + z_\alpha(u + \alpha(u)), \; u \in V ,$$

is an element of $\mathcal{B}_\omega$ and a solution of (2). By the lemma's 5.13 and 3.7 the functions y_α and f_α differ only a constant. Further, if $z_{\alpha 1}$ and $z_{\alpha 2}$ are two solutions of (3), then $z_{\alpha 1}(\cdot + \alpha(\cdot))$ and $z_{\alpha 2}(\cdot + \alpha(\cdot))$ differ only a constant and hence $z_{\alpha 1}$ and $z_{\alpha 2}$ differ only a constant. Therefore the solution of (3) in $\mathcal{B}_\omega$ is also unique upto a constant.

Now we can formulate the criteria a) and b) of lemma 5.14 in J_α instead of in f_α, where J_α is defined as in (1) or as the solution of equation (3) in $\mathcal{B}_\omega$.

LEMMA 5.15. Let $\alpha_0 \in A$. The strategy α_0 is optimal if

$$r_1(\alpha_0(u)) + J_{\alpha_0}(u + \alpha_0(u)) \le r_1(\alpha(u)) + J_{\alpha_0}(u + \alpha(u))$$

for all $u \in V$, $\alpha \in A$. If for some $\alpha \in A$ there is a positive measurable function Δ_α on V such that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{\ell=0}^{n-1} P_\alpha^\ell \Delta_\alpha > 0$$

and

$$r_1(\alpha_0(u)) + J_{\alpha_0}(u + \alpha_0(u)) \ge r_1(\alpha(u)) + J_{\alpha_0}(u + \alpha(u)) + \Delta_\alpha(u) \text{ for } u \in V$$

then $g_\alpha < g_{\alpha_0}$.

REMARK 5.16. Considering this section we see that the assumption of an (m,M,R)-problem is somewhat to strong. It is sufficient to require for each $\alpha \in A$ the existence of some $(m_\alpha, M_\alpha, R_\alpha)$ with $m_\alpha < 0 < M_\alpha$, $R_\alpha \le M_\alpha - m_\alpha$ and $\int_{0^-} e^x dF(x) < e^{R_\alpha}$ such that

$$\alpha(u) \ge R_\alpha \qquad \text{for } u \le m_\alpha ,$$

$$u + \alpha(u) \le M_\alpha \quad \text{for } u \le M_\alpha ,$$

$$\alpha(u) = 0 \qquad \text{for } u > M_\alpha .$$

5.4. *Structure of optimal strategies*

In this section we shall consider the structure of optimal strategies. The following class of inventory problems is well known.

$$r_1(x) = K.\delta(x) + c.x,\ K > 0,\ c > 0,\ (\delta(x) = 0 \text{ if } x = 0,\ \delta(x) = 1 \text{ if } x > 0) .$$

A is the set of all positive measurable functions on V.

In this case one can prove under rather general conditions that the optimal strategy α_0 is of the (s,S)-type. This means that for some pair (s,S)

$$\alpha_0(u) = 0 \qquad \text{for } u \ge s ,$$

$$\alpha_0(u) = S - u \quad \text{for } u < s .$$

For the average costs case this has been proved by Johnson [6] and Tijms [14]. Both consider a discrete state space, but the proof of Tijms can also be used for the continuous case. His proof consists of the following two steps:

i) the existence of a strategy which is optimal in the class of all (s,S)-strategies;
ii) the optimality of this sub-optimal strategy.

In this proof it is essential that under an (s,S)-strategy the process has a renewal point. If one orders, one starts again in S. This makes it possible to derive explicit expressions for g_α, the average costs under some (s,S)-strategy α, and for f_α the solution of the equation $y = r_\alpha - g_\alpha + P_\alpha y$, (see section 5.3).

However, in cases where the process under the optimal strategy has no renewal point, it is possible to carry out the same two steps. Instead of using explicit expressions for g_α and f_α one can work directly with the (P_α, r_α)-equations.

We shall show this with aid of the following example:

A is the set of all measurable functions on V with

$\alpha(u) = 0$ or C, $(C > 0)$,

$r_1(x) = 0$ if $x = 0$ and $r_1(x) = K$ if $x = C$, $(K > 0)$.

We shall prove under certain conditions that the optimal strategy is of the following type: $\alpha(u) = 0$ if $u \geq s$, $\alpha(u) = C$ if $u < s$.

The same method can be used for the cases:

a) A is the set of all measurable functions α on V with $0 \leq \alpha(u) \leq C$. $r_1(x) = c.x$, $c > 0$.

   The optimal strategy is of the following type, (see [15]): $\alpha(u) = 0$, $u \geq s$ and $\alpha(u) = \min\{s - u, C\}$, $u < s$.

b) A is the set of all positive measurable functions on V. $r_1(x) = c(x)$ with $c''(x) \leq 0$.

   The optimal strategy is of the following type: $\alpha(u) = 0$ if $u \geq s$, $\alpha(u) > 0$ if $u < s$ and $u + \alpha(u)$ is nonincreasing for $u < s$.

The last case is considered by Porteus [11] for the discounted costs criterion.

5.4.1. *Example*

In this subsection we consider an inventory problem with A the set of all measurable functions α on V such that $\alpha(u) = 0$ or C, $(C > 0)$ and $r_1(x) = K$ if $x = C$, $r_1(x) = 0$ if $x = 0$ $(K > 0)$.
We shall prove under certain conditions that the optimal strategy is of the (s)-type: $\alpha(u) = 0$ if $u \geq s$ and $\alpha(u) = C$ if $u < s$.

We make the following assumptions:

(1) F has a bounded density φ with $\varphi(x) > 0$ if $x > 0$, and φ has a bounded derivative φ';

(2) $$\int_0^\infty e^x \varphi(x)\,dx < e^C \;;$$

(3) r_2 is differentiable and $r_2'(u + C) - r_2'(u) \geq 0$ for all $u \in V$.

Each strategy (s) satisfies the conditions stated in remark 5.16 with

$$m_{(s)} := s,\ M_{(s)} := s + C,\ R_{(s)} := C\ .$$

Further, $\varphi(x) > 0$ for $x > 0$, implies that the embedded Markov process $Q_{(s)}$ of $P_{(s)}$ on $[s, s+C]$ has only one invariant probability. Hence the average costs under strategy (s) exist for each s and are constant on V, and the equation $y = r_{(s)} - g_{(s)} + P_{(s)}y$ in $\mathcal{B}_\omega$ has a solution $f_{(s)}$ which is unique upto a constant, (see remark 5.16).

Let

$$J_{(s)}(u) := r_2(u) + \int_{-\infty}^{+\infty} \varphi(u - v) f_{(s)}(v)dv,\ u \in V,\ s \in V\ .$$

We can use lemma 5.15 to compare the strategies (s) and (t).

LEMMA 5.17. Let $t > s$.

a) If $K + J_{(s)}(u + C) < J_{(s)}(u)$ for $u \in [s,t]$, then $g_{(t)} < g_{(s)}$.

b) If $J_{(t)}(u) < J_{(t)}(u + C) + K$ for $u \in [s,t]$, then $g_{(s)} < g_{(t)}$.

PROOF. We shall only prove statement a), statement b) can be proved in the same way. Let

$$\Delta_{(t)} := \begin{cases} J_{(s)}(u) - K - J_{(s)}(u + C) & \text{for } u \in [s,t] \\ 0 & \text{for } u \notin [s,t]\ . \end{cases}$$

Using lemma 5.15 with $\alpha_0 := (s)$ we infer that it is sufficient to prove

$$\lim_{n\to\infty} \frac{1}{n} \sum_{\ell=0}^{n-1} P^{\ell}_{(t)} \Delta_{(t)} > 0\ .$$

Let $Q_{(t)}$ be the embedded Markov process of $P_{(t)}$ on $[t, t+C]$ and let $T := \sum_{n=0}^{\infty} P^{n}_{(t)B}$ with $B := V \backslash [t, t+C]$. Then, by lemma 3.8,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{\ell=0}^{n-1} P^{\ell}_{(t)} \Delta_{(t)} = \frac{\lim_{n\to\infty} \frac{1}{n} \sum_{0}^{n-1} Q^{\ell}_{(t)} T\Delta_{(t)}}{\lim_{n\to\infty} \frac{1}{n} \sum_{0}^{n-1} Q^{\ell}_{(t)} T 1_V}\ .$$

But, $\varphi(x) > 0$ if $x > 0$, implies $T\Delta_{(t)} > 0$ on $[t, t+C]$, which completes the proof. □

We shall use this lemma to show that $g_{(s)}$, as function of (s), is decreasing in a neighbourhood of $-\infty$ and increasing in a neighbourhood of $+\infty$. To this end we have to consider the functions $J_{(s)}$.
For each s we have for all $u \in V$

$$J_{(s)}(u) = r_2(u) - g_{(s)} + \int_{-\infty}^{s} \varphi(u-v)\{K + J_{(s)}(v+C)\}dv + $$

$$+ \int_{s}^{\infty} \varphi(u - v)J_{(s)}(v)dv \ .$$

By lemma 5.17 the functions $D_{(s)}$, given by

$$D_{(s)}(u) := K + J_{(s)}(u + C) - J_{(s)}(u) \ ,$$

are important in relating $g_{(s)}$ to $g_{(t)}$. Let $\Delta_r(u) := r_2(u + C) - r_2(u)$ for all $u \in V$. An easy calculation shows that

$$\text{(4)} \qquad D_{(s)}(u) = \Delta_r(u) + \int_{-\infty}^{s} \varphi(u + C - v)D_{(s)}(v)dv + $$

$$+ \int_{s}^{\infty} \varphi(u - v)D_{(s)}(v)dv \ .$$

For $u \leq s$ we have

$$\text{(5)} \qquad D_{(s)}(u) = \Delta_r(u) + \int_{-\infty}^{s} \varphi(u + C - v)D_{(s)}(v)dv \ .$$

Let $D^*_{(s)}$ be the function on $[0,\infty)$ defined by $D^*_{(s)}(y) := D_{(s)}(s - y)$ for $y \geq 0$, and Δ^*_{rs} the function on $[0,\infty)$ defined by $\Delta^*_{rs}(y) = \Delta_r(s - y)$ for $y \geq 0$. Then, by (5),

$$\text{(6)} \qquad D^*_{(s)}(y) = \Delta^*_{rs}(y) + \int_{0}^{\infty} \varphi(C - y + v)D^*_{(s)}(v)dv \ .$$

Now let $B_{0\omega}$ be the space of all complex valued measurable functions f on $[0,\infty)$ such that $\frac{f}{\omega}$ is bounded. With the norm $\|f\|_{0\omega} := \sup_{u\geq 0} \frac{|f(u)|}{|\omega(u)|}$, this space is a Banach space.

Using the assumption $\int_0^\infty e^x\varphi(x)dx < e^C$ we can verify that the integral $\int_0^\infty \varphi(C - y + v)f(v)dv$ exists for all $f \in B_{0\omega}$ and $y \in [0,\infty)$, and that this integral as function of y is an element of $B_{0\omega}$.
Let S_C be the operator in $B_{0\omega}$ given by

$$(S_Cf)(y) = \int_0^\infty \varphi(C - y + v)f(v)dv \quad \text{for } f \in B_{0\omega},\ y \in [0,\infty) .$$

The norm of this operator is smaller than 1 since

$$\frac{1}{e^y} (S_Cf)(y) \le \int_0^\infty \varphi(C - y + v).\|f\|_{0\omega}.e^{v-y}dv =$$

$$= \|f\|_{0\omega}.\frac{1}{e^C}. \int_0^\infty \varphi(C-y+v)e^{v-y+C}dv \le \|f\|_{0\omega}.e^{-C}. \int_0^\infty \varphi(x)e^x dx .$$

Therefore the equation $x = f + S_Cx$ in $B_{0\omega}$ has a unique solution given by

$$x = \sum_{n=0}^\infty S_C^n f .$$

Application of this result to equation (6) yields

$$D^*_{(s)} = \sum_{n=0}^\infty S_C^n \Delta^*_{rs} .$$

Now we can prove the next lemma.

LEMMA 5.18. Let the real numbers a,b and $\varepsilon > 0$ be such that $r_2(u + C) - r_2(u) < 0$ for $u < a$ and $r_2'(u + C) - r_2'(u) > \varepsilon$ for $u > b$. Then there are real numbers c,d such that $g_{(s)}$ is decreasing in s for $s < c$ and increasing in s for $s > d$.

PROOF. By equation (4), for all s the function $D_{(s)}(\cdot)$ is continuous. Hence, as a consequence of lemma 5.17, it is sufficient to prove the existence of real numbers c,d such that

$$D_{(s)}(s) < 0 \text{ for } s < c \text{ and } D_{(s)}(s) > 0 \text{ for } s > d, \text{ or}$$

$$D^*_{(s)}(0) < 0 \text{ for } s < c \text{ and } D^*_{(s)}(0) > 0 \text{ for } s > d \; .$$

Using $\Delta^*_{rs}(u) = \Delta_r(s - u) = r_2(s - u + C) - r_2(s - u) < 0$ for $s < a$, $u \geq 0$, we infer that

$$D^*_{(s)}(0) = \sum_{n=0}^{\infty} (S^n_C \Delta^*_{rs})(0) < 0 \quad \text{for } s < a \; .$$

Now let $s_2 > s_1 > b$. Then

$$D^*_{(s_2)}(0) - D^*_{(s_1)}(0) = \sum_{n=0}^{\infty} S^n_C(\Delta^*_{rs_2} - \Delta^*_{rs_1})(0) \geq \Delta^*_{rs_2}(0) - \Delta^*_{rs_1}(0) =$$

$$= r_2(s_2 + C) - r_2(s_2) - \{r_2(s_1 + C) - r_2(s_1)\} \geq (s_2 - s_1)\varepsilon \; .$$

This implies the existence of a real number d such that $D^*_{(s)}(0) > 0$ for $s > d$, which completes the proof. □

If we can prove the continuity of $g_{(s)}$ in s then the conditions of lemma 5.18 are sufficient for the existence of a minimum of $g_{(s)}$ on $(-\infty,+\infty)$.

LEMMA 5.19. The function $g_{(s)}$ is continuous in s.

PROOF. Let $a < b$ and let $A_{a,b}$ be the set of all s-strategies with $a \leq s \leq b$. The inventory problem with strategy set $A_{a,b}$ is an (m,M,R)-problem with $m := a$, $M := b + C$, $R := C$. The topology on $A_{a,b}$ generated by the ω-metric is equivalent with the usual topology on the interval [a,b].
The conditions of theorem 5.12 are easily verified. Hence $g_{(s)}$ is continuous on [a,b] for all $a < b$ and therefore on $(-\infty,+\infty)$. □

Now we shall show that a strategy which is optimal in the class of all s-strategies is also optimal in a wider class of strategies. The set of all strategies $\alpha \in A$ such that real numbers b_α, c_α exist with $\alpha(u) = C$ for $u < b_\alpha$ and $\alpha(u) = 0$ for $u > c_\alpha$ is denoted by A_r. Notice that each $\alpha \in A_r$ satisfies the conditions stated in remark 5.16.

LEMMA 5.20. Let $\Delta_r(u) = r_2(u + C) - r_2(u)$ for $u \in V$ and

$$\Delta_{rh}(u) = \frac{\Delta_r(u + h) - \Delta_r(u)}{h} .$$

Assume that

$$\lim_{h\to 0} \| \Delta_r' - \Delta_{rh} \|_\omega = 0 .$$

If $g_{(s)}$ attains a minimum in s_0, the strategy (s_0) is optimal in A_r.

PROOF. By lemma 5.15 we have to prove

(7) $$K + J_{(s_0)}(u + C) \le J_{(s_0)}(u) \quad \text{for } u < s_0 ;$$

(8) $$J_{(s_0)}(u) \le K + J_{(s_0)}(u + C) \quad \text{for } u \ge s_0 .$$

The continuity of $D_{(s_0)}$ and the minimality of $g_{(s)}$ in s_0 imply by lemma 5.17 that $D_{(s_0)}(s_0) = 0$.

Let P_{s_0} be the operator in B_ω given by

$$(P_{s_0} f)(u) = \int_{-\infty}^{s_0} \varphi(u + C + v) f(v) dv + \int_{s_0}^{\infty} \varphi(u - v) f(v) dv \text{ for } u \in V, \ f \in B_\omega .$$

As in theorem 5.4 we can prove that $\sum_{n=0}^{\infty} P_{s_0}^n$ converges. Therefore the equation $x = f + P_{s_0} x$ in B_ω has a unique solution $x = \sum_{n=0}^{\infty} P_{s_0}^n f$. Let for h real the function D_h on V be given by

$$D_h(u) = \frac{D_{(s_0)}(u + h) - D_{(s_0)}(u)}{h} \quad \text{for } u \in V .$$

Using equation (4) we get

$$D_h(u) = \Delta_{rh}(u) + (P_{s_0} D_h)(u) + \int_{s-h}^{s} \{\varphi(u - v) - \varphi(u + C - v)\} \frac{D_{(s_0)}(v + h)}{h} dv .$$

Since $\sum_{n=0}^{\infty} P_{s_0}^n$ converges and $\lim_{h\to 0} \| \Delta_r' - \Delta_{rh} \|_\omega = 0$, this implies the convergence of D_h in B_ω for $h \to 0$.

Hence $D'_{(s_0)}$ exists and is an element of $\mathcal{B}_\omega$ and

$$D'_{(s_0)}(u) = \Delta'_r(u) + (P_{s_0} D'_{(s_0)})(u) \quad \text{for all } u \in V .$$

Therefore

$$D'_{(s_0)} = \sum_{n=0}^{\infty} P^n_{s_0} \Delta'_r .$$

Since $\Delta'_r \geq 0$ we get $D'_{(s_0)} \geq 0$. Together with $D_{(s_0)}(s_0) = 0$ this implies the inequalities (7) and (8). $\square$

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SAMENVATTING

Het in dit proefschrift behandelde onderwerp hoort thuis in de theorie van de Markov beslissingsprocessen.
Een Markov beslissingsproces kan als volgt beschreven worden: Op de tijdstippen t = 0,1,2,... verkeert het systeem in één van de toestanden uit één of andere toestandsruimte S. Op elk tijdstip kan men uit een aantal mogelijke akties één kiezen. Deze aktie bepaalt in welke toestand het systeem de volgende keer zal zijn en ook welke kosten men tot dan zal oplopen. Een strategie is een voorschrift dat op elk tijdstip aangeeft hoe de aktie gekozen dient te worden. Strategieën waarvoor geldt dat de te kiezen aktie alleen maar afhangt van de toestand waarin het systeem verkeert noemt men stationair. Onder elke stationaire strategie is het proces een Markov proces.
Bij Markov beslissingsprocessen gaat het om de beste strategie. In dit proefschrift wordt als maat voor de kwaliteit van een strategie de bijbehorende gemiddelde kosten gebruikt. Onderzocht wordt of er een stationaire strategie is die optimaal is in de verzameling van alle stationaire strategieën.
De gemiddelde kosten bij een stationaire strategie worden bepaald door de begintoestand en de overgangswaarschijnlijkheid behorend bij die stationaire strategie. Bij vaste begintoestand zijn de gemiddelde kosten een funktie van de overgangswaarschijnlijkheid. Het gaat dus om het bestaan van een minimum van die funktie.
De meest voor de hand liggende manier om kondities aan te geven voor het bestaan van een dergelijk minimum bestaat uit de volgende stappen:

1) voer een topologie in op de verzameling van overgangswaarschijnlijkheden;
2) ga na onder welke voorwaarden de topologische ruimte compact is en
3) onder welke voorwaarden de gemiddelde kosten kontinu afhangen van de overgangswaarschijnlijkheid.

Deze methode is hier toegepast. Voor de topologie wordt een metrische topologie gebruikt.

Elke overgangswaarschijnlijkheid komt evereen met een Markov operator, dat is een lineaire operator op de Banach ruimte van alle begrensde, meetbare, komplexwaardige funkties op de toestandsruimte. De norm en de spektraal-straal van een Markov operator zijn gelijk aan 1 en het punt 1 is in ieder geval een eigenwaarde.
Een belangrijke rol in dit proefschrift speelt het begrip quasi-compact-heid. Een Markov operator is dan en slechts dan quasi-compact als het korresponderende Markov proces voldoet aan de Doeblin konditie.
Met behulp van perturbatietheorie van lineaire operatoren worden in sektie 4.1 enkele kondities afgeleid voor de kontinuïteit van de gemiddelde kosten voor het geval de Markov operatoren, overeenkomend met elk van de overgangswaarschijnlijkheden, quasi-compact zijn.
In hoofdstuk 1 en hoofdstuk 2, de sekties 1 en 2, worden ter voorbereiding hiervan enkele resultaten gegeven uit de spektraal- en perturbatietheorie van lineaire operatoren en uit de theorie van de Markov processen met diskrete tijdsparameter.
Gebruikmakend van de kondities voor kontinuïteit worden voorwaarden geformuleerd voor het bestaan van een optimale strategie.
De eis dat de Markov operatoren behorend bij elk van de overgangswaarschijnlijkheden quasi-compact zijn is nogal streng. In sektie 4.2 wordt de quasi-compactheid niet vereist voor de Markov processen zelf maar voor de ingebedde Markov processen op een vaste deelverzameling van de toestandsruimte. De resultaten uit sektie 4.1 kunnen gegeneraliseerd worden naar dit geval. Men gebruikt daarvoor bepaalde terugkeereigenschappen van dergelijke Markov processen. Deze worden afgeleid in sektie 2.2.
In hoofdstuk 3 wordt ingegaan op het bestaan van de gemiddelde kosten, (de één-periode kosten hoeven niet noodzakelijkerwijs begrensd te zijn).
De resultaten uit hoofdstuk 4 worden in hoofdstuk 5 toegepast op voorraadproblemen. In de laatste sektie van dit hoofdstuk wordt getoond hoe je zonder gebruikmaking van resultaten voor het verdiskonteerde geval kunt bewijzen dat de gemiddeld optimale strategie van een bepaalde struktuur is. Er wordt een voorbeeld gegeven van een voorraadprobleem waarbij men aan het begin van elke periode alleen maar een vaste hoeveelheid kan bestellen of niets.

CURRICULUM VITAE

De schrijver van dit proefschrift werd op 10 juni 1944 geboren te Arum (Fr). In 1961 behaalde hij het diploma H.B.S.-B aan het Baudartius Lyceum te Zutphen. Van 1961-1968 studeerde hij Wis- en Natuurkunde aan de Vrije Universiteit te Amsterdam. Sinds 1968 is hij Wetenschappelijk Medewerker bij de afdeling Bedrijfskunde van de Technische Hogeschool Eindhoven.

STELLINGEN

I

Beschouw de volgende situatie: Een persoon P loopt op een regenachtige dag van A naar B. Hoewel hij geen paraplu bij zich heeft wil hij toch zo weinig mogelijk water vangen. De duur van de buien is negatief exponentieel verdeeld met gemiddelde $\frac{1}{\lambda}$, de duur van de droge perioden is negatief exponentieel verdeeld met gemiddelde $\frac{1}{\mu}$. Er is geen wind.
Zij α de oppervlakte van P's voorkant (in m^2) en β de oppervlakte van zijn bovenkant. De maximale snelheid van P is w m/sec.
Als geldt dat $(\alpha + \beta/w)\frac{\mu}{\lambda + \mu} < \alpha$ heeft de vergelijking

$$\frac{\beta + \alpha w}{\lambda + \mu}\left(\frac{\mu}{w} + \frac{\lambda}{w}e^{-\frac{\lambda+\mu}{w}x}\right) = \alpha$$

een niet-negatieve oplossing x_k.
De optimale strategie voor P is dan als volgt:
Als de afstand tot B groter is dan x_k-meter moet P zo hard mogelijk lopen als het droog is en stil blijven staan als het regent, als de afstand tot B kleiner is dan x_k-meter moet P zo hard mogelijk lopen, of het nu regent of droog is.

J. Wijngaard, Een regenachtige geschiedenis, Rapport Bdk/OR/74-01, augustus 1974.

II

Het opnemen in de bedrijfskundestudie van eenvoudige modellen van productieproblemen, wachttijdproblemen, spelproblemen, kan men niet verdedigen op grond van de directe practische bruikbaarheid.
De structuur van dergelijke modellen vindt men wel terug in de werkelijkheid. Behandeling ervan zal dus bijdragen in de vorming van het referentiekader van de student en verdient op grond dáárvan een plaats binnen bedrijfskunde.

III

In het algemeen doen bedrijven er verstandiger aan planners te ontwikkelen dan planningssystemen.

IV

Zij P een Markov proces met kosten op een aftelbare toestandsruimte, continu in de tijd. Vanuit toestand n zijn in één stap slechts de toestanden $1,2,3,\ldots,n-1,n,n+1$ bereikbaar. Vanuit elke toestand wordt met zekerheid ooit toestand 1 bereikt en de verwachte kosten tot dan worden eindig verondersteld (c_n). Stel $y_n = c_n - c_{n-1}$, $n > 2$.
Als de drift naar toestand 1 sterk genoeg is zal y_n begrensd zijn of althans niet al te snel divergeren. Gebruikmakend hiervan kan men c_1 gemakkelijk benaderen door y_n voor grote n gelijk te stellen aan 0.

J. Wijngaard en E.G.F. van Winkel, Average number of back orders in a continuous review (s,S) inventory system with exponentially distributed lead time, presented at Euro I, Brussel, januari 1975.

V

Dat de som $\sum_{n=1}^{\infty} e^{-\lambda n} \frac{(\lambda n)^n}{n!}$ voor $0 < \lambda < 1$ gelijk is aan $\frac{\lambda}{1-\lambda}$ en voor $\lambda > 1$ aan $\frac{\lambda w}{1 - \lambda w}$, waarin w bepaald is door $w = e^{\lambda(w-1)}$, kan men vinden door gebruikmaking van het feit dat in het Poissonproces N(t) met parameter λ het verwachte aantal keren dat $N(t) = t$ $(t \geq 1)$ zal optreden, juist gelijk is aan deze som.

VI

Zij $\{(P_\alpha, r_\alpha)\}$, $\alpha \in A$ een stationair Markov beslissingsprobleem op (V,Σ), als gedefinieerd in § 4.2 van dit proefschrift. Laat V aftelbaar zijn en Σ de σ-algebra van alle deelverzamelingen van V. Neem aan dat er een positieve functie w op V is met $\inf_{u \in V} w(u) > 0$, zodanig dat $r_\alpha(u) \geq w(u)$, $u \in V$, $\alpha \in A$. Definieer A_g voor $g > 0$ als de deelverzameling van A met alle α zodanig dat de gemiddelde kosten g_α bestaan en er geldt $g_\alpha(u) \leq g$, $u \in V$. Dan voldoet het beslissingsprobleem $\{(P_\alpha, r_\alpha)\}$, $\alpha \in A_g$ voor elke g aan de condities i), ii), iii) van § 4.2.

VII

In "Quality control under Markovian deterioration" behandelt Ross een inspectie-revisie probleem. Hij beschouwt een productiesysteem dat in goede of slechte staat verkeert. Ross definieert als toestandsruimte het interval $[0,1]$. Het systeem is in toestand $p \in [0,1]$ als de kans dat het productieapparaat in slechte staat verkeert gelijk is aan p. Een natuurlijker toestandsruimte, die de resultaten helderder gemaakt zou hebben was hier geweest de ruimte van de natuurlijke getallen. Het systeem is in toestand n als het n tijdseenheden gedraaid heeft sinds voor het laatste is vastgesteld dat het in goede staat verkeerde.

S.M. Ross, Quality control under Markovian deteroriation, Management Science, 17 (1971), 587-596.

VIII

De interne competitie bij veel schaak- en damclubs wordt gespeeld volgens het Keizersysteem. Daarbij wordt de rangorde bepaald op grond van gewogen wedstrijdpunten. Winst op nummer n van de ranglijst levert $A - n$ punten op, A is een vrij willekeurig getal groter dan het aantal deelnemers. Wil men echter meer recht doen aan de verschillen in puntentotalen dan kan men de gewichten evenredig aan die puntentotalen kiezen. De rangorde wordt dan bepaald door de eigenvector horend bij de grootste eigenwaarde van de uitslagenmatrix.

Keizer, Het systeem Keizer, Planeta, Enschede, 1956.

IX

Beschouw een voorraadprobleem met één voorraadpunt, vast bestelkosten en naleverplicht. De bestelcapaciteit is beperkt (R). Onder de bestelstrategie (s,S) bestelt men niet als de voorraad $u \geq s$ en men bestelt $\min\{S - u, R\}$ als de voorraad $u < s$.
In het algemeen is de gemiddeld optimale bestelstrategie niet van dit (s,S)-type. Echter als de vraag negatief exponentieel verdeeld is en de voorraad- en buiten voorraadkosten zijn lineair en als voor de beste (s,S)-strategie geldt dat $s > 0$, dan is de optimale strategie wel van het (s,S)-type.

J. Wijngaard, An inventory problem with constrained ordercapacity, T.H.-Report 72-WSK-03, augustus 1972.

X

Een linkse stemmer die de belasting ontduikt is vergelijkbaar met een zondagmiddag wandelaar die de hoekjes afsnijdt.

Eindhoven, 29 april 1975. J. Wijngaard