

# The discrete-time $H_\infty$ control problem with measurement feedback

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problem with measurement feedback**

A.A. Stoorvogel  
A. Saberi  
B.M. Chen

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The Netherlands

Eindhoven University of Technology  
Department of Mathematics and Computing Science  
Probability theory, statistics, operations research and systems theory  
P.O. Box 513  
5600 MB Eindhoven - The Netherlands

Secretariate: Dommelbuilding 0.03  
Telephone: 040-47 3130

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# The discrete-time $H_\infty$ control problem with measurement feedback

Anton A. Stoorvogel \*

Department of Mathematics  
and Computer Science  
Eindhoven University of Technology  
P.O. Box 513  
5600 MB Eindhoven  
the Netherlands

Ali Saberi †

School of Electrical Engineering  
and Computer Science  
Washington State University  
Pullman, WA 99164-2752  
U.S.A.

Ben M. Chen †

Department of Electrical Engineering  
State University of New York  
at Stony Brook  
Stony Brook, NY 11794-2350  
U.S.A.

## Abstract

This paper is concerned with the discrete-time  $H_\infty$  control problem with measurement feedback. We extend previous results by having weaker assumptions on the system parameters. We also show explicitly the structure of  $H_\infty$  controllers. Finally, we show that it is in certain cases possible, without loss of performance, to reduce the dynamical order of the controllers.

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# 1 Introduction

The  $H_\infty$  control problem has been studied extensively. First in continuous-time (see e.g. [3, 4, 10, 13]) and later in discrete time (see e.g. [1, 8, 6, 14]). For a more extensive reference list we refer to two recent books [2, 15].

The objective of this paper is to present a solution of the general discrete-time  $H_\infty$  control problem. One way to approach this problem is to transform the discrete-time  $H_\infty$  optimal control problem into an equivalent continuous-time  $H_\infty$  control problem via bilinear transformation. Then the continuous-time controllers that are solutions to the auxiliary problem can be obtained and transformed back to their discrete-time equivalent using inverse bilinear transformation. However, in our opinion it is more natural to solve this problem directly in discrete-time setting and in terms of the original system's performance. This approach leaves the possibility of directly observing the effect of certain physical parameters which might otherwise be blurred by the transformation to continuous-time. In view of this, and in accordance with earlier literature [1, 6, 8, 12], we take this direct approach in solving the discrete-time  $H_\infty$  optimal control problem.

Compared to the existing literature, we solve this problem under weaker assumptions. All the existing literature on the discrete-time  $H_\infty$  control problem make the following assumptions on the system:

- The subsystem from the control input to the control output should be left invertible and should not have invariant zeros on the unit circle.
- The subsystem from the disturbance to the measurement output should be right invertible and should not have invariant zeros on the unit circle.

These conditions are the discrete time analogon of what is called regular problems in continuous time  $H_\infty$  control problems. In this paper, we remove the above mentioned left and right invertibility condition.

Moreover, we give a representation of one controller in a suitable form such that it becomes very transparent that this controller is a state *and disturbance* estimator in conjunction with a full-information feedback (i.e. a feedback of both state and disturbance). Such an interpretation was not available before and due to the involved formulas it was not very clear what kind of structure discrete time  $H_\infty$  controllers should have.

Finally, a novel aspect of this paper is that we show that if certain states or disturbances are observed directly, then this yields the possibility of deriving a controller of lower McMillan degree. This result again corresponds to those obtained in continuous-time case (see [16]).

The notation in this paper will be fairly standard. By  $\mathcal{N}$  and  $\mathcal{R}$  we denote the natural numbers and the real numbers, respectively. Moreover by  $\sigma$  we denote the shift:

$$(\sigma x)(k) := x(k+1) \quad \forall k \in \mathcal{N}$$

$\text{rank}_{\mathcal{K}}$  denotes the rank as a matrix with entries in the field  $\mathcal{K}$ . By  $\mathcal{R}(z)$  we denote the field of real rational functions. Moreover, by  $X^\dagger$  we denote the Moore-Penrose inverse of the matrix  $X$ . Finally, by  $\rho(X)$  we denote the spectral radius of the matrix  $X$ .

## 2 Problem formulation and main results

We consider the following time-invariant system:

$$\Sigma : \begin{cases} \sigma x = Ax + Bu + Ew, \\ y = C_1x + D_{12}w, \\ z = C_2x + D_{21}u + D_{22}w, \end{cases} \quad (2.1)$$

where for all  $k \in \mathcal{N}$ ,  $x(k) \in \mathcal{R}^n$  is the state,  $u(k) \in \mathcal{R}^m$  is the control input,  $y(k) \in \mathcal{R}^l$  is the measurement,  $w(k) \in \mathcal{R}^q$  is the unknown disturbance and  $z(k) \in \mathcal{R}^p$  is the output to be controlled.  $A, B, E, C_1, C_2, D_{12}, D_{21}$  and  $D_{22}$  are matrices of appropriate dimension.

If we apply a dynamic feedback law  $u = Fy$  to  $\Sigma$  then the closed loop system with zero initial conditions defines a convolution operator  $\Sigma_{cl,F}$  from  $w$  to  $y$ . We seek a feedback law  $u = Fy$  which is internally stabilizing and which minimizes the  $\mathcal{L}_2$ -induced operator norm of  $\Sigma_{cl,F}$  over all internally stabilizing feedback laws. We will investigate dynamic feedback laws of the form:

$$\Sigma_F : \begin{cases} \sigma p = Kp + Ly, \\ u = Mp + Ny. \end{cases} \quad (2.2)$$

We will say that the dynamic compensator  $\Sigma_F$ , given by (2.2), is internally stabilizing when applied to the system  $\Sigma$ , described by (2.1), if the following matrix is asymptotically stable:

$$\begin{pmatrix} A + BNC_1 & BM \\ LC_1 & K \end{pmatrix}, \quad (2.3)$$

i.e. all its eigenvalues lie in the open unit disc. Denote by  $G_F$  the closed loop transfer matrix. The  $\mathcal{L}_2$ -induced operator norm of the convolution operator  $\Sigma_{cl,F}$  is equal to the  $H_\infty$  norm of the transfer matrix  $G_F$  and is given by:

$$\begin{aligned} \|G_F\|_\infty &:= \sup_{\theta \in [0, 2\pi]} \|G_F(e^{i\theta})\| \\ &= \sup_w \left\{ \frac{\|z\|_2}{\|w\|_2} \mid w \in \mathcal{L}_2^l, w \neq 0 \right\} \end{aligned}$$

where the  $\mathcal{L}_2$ -norm is given by:

$$\|p\|_2 := \left( \sum_{k=0}^{\infty} p^T(k)p(k) \right)^{1/2}$$

and where  $\|\cdot\|$  denotes the largest singular value. We shall refer to the norm  $\|G_F\|_\infty$  as the  $H_\infty$  norm of the closed loop system.

In this paper we will derive necessary and sufficient conditions for the existence of a dynamic compensator  $\Sigma_F$  which is internally stabilizing and which is such that the closed loop transfer matrix  $G_F$  satisfies  $\|G_F\|_\infty < 1$ . By scaling the plant we can thus, in principle, find the infimum of the  $H_\infty$  norm of the closed loop system over all stabilizing controllers. This will involve a search procedure. Furthermore, if a stabilizing  $\Sigma_F$  exists which makes the  $H_\infty$  norm of the closed loop system less than 1, then we derive an explicit formula for one particular  $F$  satisfying these requirements. We also give an alternative non-minimal representation for this controller whose structure makes clear that this controller is the interconnection of a current

state and current disturbance estimator and a static full-information feedback. In section 5 we show that in some cases we can reduce the dynamical order of the estimator and we will derive an explicit method to derive controllers of lower dynamical order.

In the formulation of our main result we will need the concept of invariant zero. Recall that  $z_0$  is called an *invariant zero* of the system  $(A, B, C, D)$  if

$$\text{rank}_{\mathcal{R}} \begin{pmatrix} z_0 I - A & -B \\ C & D \end{pmatrix} < \text{rank}_{\mathcal{R}(z)} \begin{pmatrix} z I - A & -B \\ C & D \end{pmatrix}$$

We can now formulate one of our main results. This is an extension of [1, 8, 14].

**Theorem 2.1 :** *Consider the system (2.1). Assume that the systems  $(A, B, C_2, D_{21})$  and  $(A, E, C_1, D_{12})$  have no invariant zeros on the unit circle. The following statements are equivalent:*

- (i) *There exists a dynamic compensator  $\Sigma_F$  of the form (2.2) such that the resulting closed loop system is internally stable and the closed loop transfer matrix  $G_F$  satisfies  $\|G_F\|_{\infty} < 1$ .*
- (ii) *There exist symmetric matrices  $P \geq 0$  and  $Q \geq 0$  such that*

(a) *We have*

$$R > 0, \tag{2.4}$$

where

$$\begin{aligned} V &:= B^T P B + D_{21}^T D_{21}, \\ R &:= I - D_{22}^T D_{22} - E^T P E + (E^T P B + D_{22}^T D_{21}) V^{\dagger} (B^T P E + D_{21}^T D_{22}). \end{aligned}$$

(b)  *$P$  satisfies the discrete algebraic Riccati equation:*

$$P = A^T P A + C_2^T C_2 - \begin{pmatrix} B^T P A + D_{21}^T C_2 \\ E^T P A + D_{22}^T C_2 \end{pmatrix}^T G(P)^{\dagger} \begin{pmatrix} B^T P A + D_{21}^T C_2 \\ E^T P A + D_{22}^T C_2 \end{pmatrix}. \tag{2.5}$$

where

$$G(P) := \begin{pmatrix} D_{21}^T D_{21} & D_{21}^T D_{22} \\ D_{22}^T D_{21} & D_{22}^T D_{22} - I \end{pmatrix} + \begin{pmatrix} B^T \\ E^T \end{pmatrix} P \begin{pmatrix} B & E \end{pmatrix}. \tag{2.6}$$

(c) *For all  $z \in \mathcal{C}$  with  $|z| \geq 1$ , we have*

$$\begin{aligned} \text{rank}_{\mathcal{R}} \begin{pmatrix} z I - A & -B & -E \\ B^T P A + D_{21}^T C_2 & B^T P B + D_{21}^T D_{21} & B^T P E + D_{21}^T D_{22} \\ E^T P A + D_{22}^T C_2 & E^T P B + D_{22}^T D_{21} & E^T P E + D_{22}^T D_{22} - I \end{pmatrix} \\ = n + q + \text{rank}_{\mathcal{R}(z)} C_2 (z I - A)^{-1} B + D_{21} \end{aligned}$$

(d) We have

$$S > 0, \quad (2.7)$$

where

$$\begin{aligned} W &:= D_{12}D_{12}^T + C_1QC_1^T, \\ S &:= I - D_{22}D_{22}^T - C_2QC_2^T + (C_2QC_1^T + D_{22}D_{12}^T)W^\dagger(C_1QC_2^T + D_{12}D_{22}^T). \end{aligned}$$

(e)  $Q$  satisfies the following discrete algebraic Riccati equation:

$$Q = AQA^T + EE^T - \begin{pmatrix} C_1QA^T + D_{12}E^T \\ C_2QA^T + D_{22}E^T \end{pmatrix}^T H(Q)^\dagger \begin{pmatrix} C_1QA^T + D_{12}E^T \\ C_2QA^T + D_{22}E^T \end{pmatrix}. \quad (2.8)$$

where

$$H(Q) := \begin{pmatrix} D_{12}D_{12}^T & D_{12}D_{22}^T \\ D_{22}D_{12}^T & D_{22}D_{22}^T - I \end{pmatrix} + \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} Q \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}^T. \quad (2.9)$$

(f) For all  $z \in \mathbb{C}$  with  $|z| \geq 1$ , we have

$$\begin{aligned} \text{rank}_{\mathcal{R}} \begin{pmatrix} zI - A & AQC_1^T + ED_{12}^T & AQC_2^T + ED_{22}^T \\ -C_1 & C_1QC_1^T + D_{12}D_{12}^T & C_1QC_2^T + D_{12}D_{22}^T \\ -C_2 & C_2QC_1^T + D_{22}D_{12}^T & C_2QC_2^T + D_{22}D_{22}^T - I \end{pmatrix} \\ = n + q + \text{rank}_{\mathcal{R}(z)} C_1(zI - A)^{-1}E + D_{12} \end{aligned}$$

(g)  $\rho(PQ) < 1$ . □

**Remarks :**

- (i) Necessary and sufficient conditions for the existence of an internally stabilizing feedback compensator which makes the  $H_\infty$  norm of the closed loop system less than some, a priori given, upper bound  $\gamma > 0$  can be easily derived from theorem 2.1 by scaling.
- (ii) In this paper, we only investigate controllers of the form (2.2). This is not an essential restriction, since it can be shown that we can not make the  $H_\infty$  norm less by allowing more general, possibly even non-linear, causal feedbacks.

For the special cases of full-information and state feedback we can dispense with the second Riccati equation. Moreover, in these cases there always exist suitable static controllers. More specifically:

- *Full information case* :  $C_1 = \begin{pmatrix} I \\ 0 \end{pmatrix}$ ,  $D_{12} = \begin{pmatrix} 0 \\ I \end{pmatrix}$ .

In this case we have  $y_1 = x$  and  $y_2 = w$ , i.e. we know both the state and the disturbance of the system at time  $k$ . It is easy to check that  $Q = 0$  satisfies conditions (d)-(f).



Moreover this guarantees that the coupling condition (g) is automatically satisfied. Therefore there exists a stabilizing controller which yields a closed loop system with the  $H_\infty$  norm strictly less than 1 if and only if there exists a positive semi-definite matrix  $P$  satisfying conditions (a)-(c). Moreover in that case we can find static output feedbacks  $u = F_1x + F_2w$  with the desired properties. One particular choice for  $F = (F_1, F_2)$  is given by:

$$F_1 := -V^\dagger (B^T P A + D_{21}^T C_2) + (I - V^\dagger V) F_0 \quad (2.10)$$

$$F_2 := -V^\dagger (B^T P E + D_{21}^T D_{22}) \quad (2.11)$$

where  $F_0$  is an arbitrary matrix such that  $A + B F_1$  is stable.

- *State feedback case* :  $C_1 = I$ ,  $D_{12} = 0$ .

In that case, it is easy to see that a necessary condition for the existence of a positive semi-definite matrix  $Q$  satisfying conditions (d)-(f) is that  $\|D_{22}\| < 1$ . In that case, it is easy to check that

$$Q = E(I - D_{22}D_{22}^T)^{-1}E^T$$

satisfies conditions (d)-(f). Condition (g) then reduces to

$$I - D_{22}D_{22}^T - E^T P E > 0 \quad (2.12)$$

Moreover, condition (2.12) implies that condition (a) is automatically satisfied. Therefore there exists a stabilizing controller which yields a closed loop system with the  $H_\infty$  norm strictly less than 1 if and only if there exists a positive semi-definite matrix  $P$  satisfying conditions (b),(c) and additionally condition (2.12).

In that case we can find a static output feedback  $u = Fx$  with the desired properties. One particular choice for  $F$  is given by:

$$F := -V^\dagger (B^T P A + D_{21}^T C_2 + [B^T P E + D_{21}^T D_{22}]R^{-1}[(E^T P A_x + D_{22}^T C_x]) + (I - V^\dagger V)F_0$$

where  $F_0$  is an arbitrary matrix such that  $A + BF$  is stable (which can be shown to always exist) and

$$A_x := A - BV^\dagger [B^T P A + D_{21}^T C_2], \quad (2.13)$$

$$C_x := C_2 - D_{21}V^\dagger [B^T P A + D_{21}^T C_2]. \quad (2.14)$$

### 3 The proof of theorem 2.1

The proof of theorem 2.1 is divided into three parts. Each part establishes the proof for a certain part of the theorem. Every part is framed up as a subsection with a heading that represents a significant feature of its proof technique or its overall achievement. The rationale for dividing the proof into three parts is mainly due to the length and the complexity of the proof.

### 3.1 The existence of a solution to the Algebraic Riccati equation

In this subsection we assume that part (i) of the theorem 2.1 is satisfied we will show that the existence of  $P$  satisfying conditions (a)-(c) in (ii) of the theorem 2.1 is necessary. We begin with the following definition.

**Definition 3.1 :** *Let a system  $\Sigma = (A, B, C, D)$  be given. The controllability subspace  $\mathcal{R}^*(\Sigma)$  is the largest subspace  $\mathcal{X}$  of  $\mathcal{R}^n$  for which a mapping  $F$  exists such that*

$$\begin{aligned} (A + BF)\mathcal{X} &\subseteq \mathcal{X} \\ (C + DF)\mathcal{X} &= \{0\} \end{aligned}$$

and such that  $(A, B_1)$  is controllable where  $B_1$  is defined by:

$$Im B_1 = \mathcal{X} \cap B Ker D$$

□

We next perform a basis transformation on the state and input spaces of  $\Sigma$ . We decompose the state space  $\mathcal{X} = \mathcal{R}^*(\Sigma_{ci}) \oplus \mathcal{X}_2$  where  $\Sigma_{ci} = (A, B, C_2, D_2)$  and choose a basis adapted to this decomposition. We also decompose the controller input space  $\mathcal{U} = Ker V \oplus \mathcal{U}_2$  where  $V$  is as defined in theorem 2.1. In the new bases, the matrices in the realization of  $\Sigma$  have a special form:

$$\begin{aligned} A &= \begin{pmatrix} A_{11} + B_{12}F & A_{12} \\ B_r F & A_r \end{pmatrix}, & B &= \begin{pmatrix} B_{11} & B_{12} \\ 0 & B_r \end{pmatrix}, & E &= \begin{pmatrix} E_1 \\ E_2 \end{pmatrix}, \\ C_1 &= \begin{pmatrix} C_{11} & C_{12} \end{pmatrix}, & D_{12} &= D_{12} \\ C_2 &= \begin{pmatrix} D_r F & C_r \end{pmatrix}, & D_{21} &= \begin{pmatrix} 0 & D_r \end{pmatrix}, & D_{22} &= D_{22} \end{aligned} \quad (3.1)$$

The above matrices have the following properties:

- $(A_{11}, B_{11})$  is controllable,
- $(A_r, B_r, C_r, D_r)$  is left invertible,
- $(A_r, B_r)$  is stabilizable.

If part (i) of theorem 2.1 holds, i.e., if the measurement feedback problem is solvable, then we also know that the full-information  $H_\infty$  control problem is solvable. Let  $F_0$  be such that  $A_{11} - B_{11}F_0$  is stable. Then it is easy to see that, after the preliminary feedback  $u = -(F \ F_0)x + v$  the subspace  $\mathcal{R}^*(\Sigma_{ci})$  does not affect the output to be controlled and the dynamics restricted to  $\mathcal{R}^*(\Sigma_{ci})$  is stable. Hence the achievable  $H_\infty$  norm using full-information feedback is completely determined by the following subsystem:

$$\Sigma_r : \begin{cases} \sigma x_2 = A_r x_2 + B_r v_2 + E_2 w, \\ z = C_r x_2 + D_r v_2 + D_{22} w, \end{cases} \quad (3.2)$$

However, for this subsystem the operator mapping the input  $v_2$  to the output  $z$  is left invertible. Therefore we can apply the results from [2, 15] to obtain the following result:

**Lemma 3.2 :** Consider the systems  $\Sigma$  and  $\Sigma_r$  defined by (2.1) and (3.2) respectively. Assume that the system  $(A, B, C_2, D_{21})$  has no invariant zeros on the unit circle. Then also the system  $(A_r, B_r, C_r, D_r)$  has no invariant zeros on the unit circle. Moreover, the following statements are equivalent:

- (i) There exists a full information feedback  $u = F_1 x + F_2 w$  for the system  $\Sigma$  such that the resulting closed loop system is internally stable and the closed loop transfer matrix  $G_F$  satisfies  $\|G_F\|_\infty < 1$ .
- (ii) There exists a full information feedback  $u = F_{1,r} x + F_{2,r} w$  for the system  $\Sigma_r$  such that the resulting closed loop system is internally stable and the closed loop transfer matrix  $G_{F,r}$  satisfies  $\|G_{F,r}\|_\infty < 1$ .
- (iii) There exists a symmetric matrix  $P_r \geq 0$  such that

(a) We have

$$V_r > 0, \quad R_r > 0$$

where

$$\begin{aligned} V_r &:= B_r^T P_r B_r + D_r^T D_r, \\ R_r &:= I - D_{22}^T D_{22} - E_r^T P_r E_r + (E_r^T P_r B_r + D_{22}^T D_r) V_r^{-1} (B_r^T P_r E_r + D_r^T D_{22}). \end{aligned}$$

(b)  $P_r$  satisfies the discrete algebraic Riccati equation:

$$P_r = A_r^T P_r A_r + C_r^T C_r - \begin{pmatrix} B_r^T P_r A_r + D_r^T C_r \\ E_r^T P_r A_r + D_{22}^T C_r \end{pmatrix}^T G_r(P_r)^{-1} \begin{pmatrix} B_r^T P_r A_r + D_r^T C_r \\ E_r^T P_r A_r + D_{22}^T C_r \end{pmatrix}.$$

where

$$G_r(P_r) := \begin{pmatrix} D_r^T D_r & D_r^T D_{22} \\ D_{22}^T D_r & D_{22}^T D_{22} - I \end{pmatrix} + \begin{pmatrix} B_r^T \\ E_r^T \end{pmatrix} P_r \begin{pmatrix} B_r & E_r \end{pmatrix}$$

(c) The matrix  $A_{cl,P}$  is asymptotically stable where

$$A_{cl,P} := A - \begin{pmatrix} B_r & -E_r \end{pmatrix} G_r(P_r)^{-1} \begin{pmatrix} B_r^T P_r A_r + D_r^T C_r \\ E_r^T P_r A_r + D_{22}^T C_r \end{pmatrix}$$

□

**Proof :** The implication (ii)  $\Leftrightarrow$  (iii) can be found in [15]. The implication (ii)  $\Rightarrow$  (i) can be easily checked using the arguments given before this lemma.

The implication (i)  $\rightarrow$  (ii) can be derived in the following manner. First note that we can apply, without loss of generality the transformation  $u = F_0 x_1 + v$  where  $F_0$  is such that  $A_{11} + B_1 F_0$  is stable. Suppose a stabilizing feedback,  $v = F_{11} x_1 + F_{12} x_2 + F_2 w$  exists for the system  $\Sigma$  (after our preliminary transformation) which yields a closed loop transfer matrix

$G_F$  satisfying  $\|G_F\|_\infty < 1$ . Then it is easy to check that the following dynamic compensator stabilizes  $\Sigma_r$  and yields the same closed loop transfer matrix  $G_F$ :

$$\Sigma_F : \begin{cases} \sigma x_1 = (A_{11} + B_{12}F_0)x_1 + A_{12}x_2 + E_1w, \\ u = F_{11}x_1 + F_{12}x_2 + F_2w, \end{cases}$$

However,  $\Sigma_r$  has a subsystem from  $v_2$  to  $z$  which is left-invertible and hence, from [14], we know that the existence of a suitable dynamic full-information feedback also guarantees the existence of a static full-information feedback. ■

This lemma yields a solution  $P_r$  of a discrete time Riccati equation for the reduced order system. We can extend this matrix to the original state space by setting it zero on  $\mathcal{R}^*(\Sigma_{ci})$ , i.e. if we define  $P$  by

$$P = \begin{pmatrix} 0 & 0 \\ 0 & P_r \end{pmatrix} \quad (3.3)$$

then  $P_r$  satisfies the conditions of lemma 3.2 if and only if  $P$  satisfies the conditions of (a)-(c) of theorem 2.1. The above can be combined to yield:

**Lemma 3.3 :** *Assume  $(A, B, C_2, D_{21})$  has no invariant zeros on the unit circle. If part (i) of theorem 2.1 is satisfied then there exists a symmetric matrix  $P \geq 0$  satisfying (a)-(c) of part (ii) of theorem 2.1. □*

We also need to know whether any solution  $P$  satisfying conditions (a)-(c) of theorem 2.1 can be connected to a matrix  $P_r$  satisfying the conditions of lemma 3.2. This is done in the following lemma:

**Lemma 3.4 :** *Let  $P \geq 0$  be a matrix satisfying the conditions (a)-(c) of theorem 2.1. Then*

$$\text{Ker } P \supseteq \mathcal{R}^*(\Sigma_{ci}).$$

*Hence, in our new bases,  $P$  will be of the form (3.3) for some matrix  $P_r$ . Moreover  $P_r$  satisfies the conditions in part (iii) of lemma 3.2. □*

**Proof :** First note that condition (b) implies that

$$P \geq A_x^T P A_x + C_x^T C_x$$

where  $A_x$  and  $C_x$  are defined by (2.13) and (2.14) respectively. It is easily seen that this implies that  $\text{Ker } P$  is controlled invariant.

Secondly conditions (a) and (c) imply that

$$\text{rank}(B^T P B + D_{21}^T D_{21}) = \text{rank}_{\mathcal{R}(z)} C_2(zI - A)^{-1} B + D_{21}$$

These two properties, when combined with the decomposition of the state space as introduced in the beginning of this section, yield the desired result. ■

Using  $P_r$ , or equivalently  $P$ , we can also derive explicit formulas for static full-information compensators which achieve the desired objectives in parts (i) or (ii). This is outlined in the following lemma which is an extension of results in [2, 15].

**Lemma 3.5 :** *Let the systems  $\Sigma$  and  $\Sigma_r$  be defined by (2.1) and (3.2) respectively. Assume that a matrix  $P_r \geq 0$  exists satisfying the conditions in part (iii) of lemma 3.2. Moreover, define  $P$  by (3.3).*

- *A controller satisfying the conditions of part (ii) of lemma 3.2 is described by:*

$$\begin{aligned} F_{1,r} &:= -V^{-1}(B_r^T P_r A_r + D_r^T C_r) \\ F_{2,r} &:= -V^{-1}(B_r^T P_r E_r + D_r^T D_{22}) \end{aligned}$$

- *A controller satisfying the conditions of part (i) of lemma 3.2 is described by*

$$\begin{aligned} F_1 &= \begin{pmatrix} -F & F_{1,r} - F_0 \end{pmatrix} \\ F_2 &= \begin{pmatrix} 0 & F_{2,r} \end{pmatrix} \end{aligned}$$

where  $F$  and  $F_0$  are the parameters of the preliminary feedback described before lemma 3.2.

Alternatively, we can also describe a suitable controller for  $\Sigma$  in terms of the original system parameters of  $\Sigma$ :

$$\begin{aligned} F_1 &:= -V^\dagger(B^T P A + D_{21}^T C_2) + (I - V^\dagger V) \tilde{F} \\ F_2 &:= -V^\dagger(B^T P E + D_{21}^T D_{22}) \end{aligned}$$

where  $\tilde{F}$  is an arbitrary matrix such that  $A + B F_1$  is stable.

**Proof :** The first part of this lemma is a direct result of [14]. The second part of this lemma gives two controllers of which it can be easily shown that when applied to the reduced order system they yield the same closed loop transfer matrix as the controller given in the first part of this lemma when applied to the original system. Hence the closed loop system has  $H_\infty$  norm strictly less than 1. Remains to check existence of a suitable  $\tilde{F}$  to yield internal stability of the closed loop system. This is shown by using the decomposition introduced in the beginning of this section together with stability of  $A_r + B_r F_r$  and stabilizability of  $(A_{11} + B_{12} F, B_{11})$ . ■

In the next subsection we show that the part (i) of theorem 2.1 also implies the remaining statements of the part (ii) of theorem 2.1.

### 3.2 A first system transformation

In this subsection we assume that part (i) of theorem 2.1 is satisfied and we show that part (ii) of the theorem 2.1 holds. A central component of the proof in this subsection is to transform the original system (2.1) into a new system. This transformation is designed such that the problem of finding an internally stabilizing feedback which makes the  $H_\infty$  norm of the closed loop system less than 1 for the original system would be equivalent to the problem of finding an internally stabilizing feedback which makes the  $H_\infty$  norm of the closed loop system less than 1 for the new transformed system. Moreover, this new system has some very desirable properties which makes it much easier to work with. In particular, for this new system the disturbance decoupling problem with measurement feedback is solvable. We will perform the transformation in two steps. First we will perform a transformation related to the full-information  $H_\infty$  problem and next a transformation related to the filtering problem. We assume that we have a positive semi-definite matrix  $P$  satisfying conditions (a)-(c) of theorem 2.1. We define the following system:

$$\Sigma_P : \begin{cases} \sigma x_P = A_P x_P + B u_P + E_P w_P, \\ y_P = C_{1,P} x_P + D_{12,P} w_P, \\ z_P = C_{2,P} x_P + D_{21,P} u_P + D_{22,P} w_P, \end{cases} \quad (3.4)$$

where,

$$\begin{aligned} A_P &:= A + ER^{-1}(E^T P A_x + D_{22}^T C_x), \\ E_P &:= ER^{-1/2}, \\ C_{1,P} &:= C_1 + D_{12} R^{-1}(E^T P A_x + D_{22}^T C_x), \\ C_{2,P} &:= (V^{1/2})^\dagger (B^T P A + D_{21}^T C_2 + [B^T P E + D_{21}^T D_{22}] R^{-1} [(E^T P A_x + D_{22}^T C_x)]), \\ D_{12,P} &:= D_{12} R^{-1/2}, \\ D_{21,P} &:= V^{1/2}, \\ D_{22,P} &:= (V^{1/2})^\dagger (B^T P E + D_{21}^T D_{22}) R^{-1/2}, \end{aligned}$$

where the matrix  $P$  satisfies parts (a)-(c) of theorem 2.1 and the matrices  $A_x$  and  $C_x$  are defined by (2.13) and (2.14), respectively.

In order to continue, we need the system to be in the special basis as defined in the previous section. Using lemma 3.4, we know that  $P$  is of the form (3.3) for some matrix  $P_r$ . We can then define the following system:

$$\Sigma_U : \begin{cases} \sigma x_U = A_U x_U + B_U u_U + E_U w, \\ y_U = C_{1,U} x_U + D_{12,U} w, \\ z_U = C_{2,U} x_U + D_{21,U} u_U + D_{22,U} w. \end{cases} \quad (3.5)$$

where

$$\begin{aligned} V_r &:= B_r^T P_r B_r + D_r^T D_r \\ A_U &:= A_r - B_r V_r^{-1} (B_r^T P_r A_r + D_r^T C_r) \\ B_U &:= B_r \begin{pmatrix} 0 & V_r^{-1/2} \end{pmatrix} \\ E_U &:= E_2 - B_r V_r^{-1} (B_r^T P_r E_2 + D_r^T D_{22}) \end{aligned}$$

$$\begin{aligned}
C_{2,u} &:= C_r - D_r V_r^{-1} (B_r^T P_r A_r + D_r^T C_r) \\
C_{1,u} &:= -R^{-1/2} (E_2^T P_r A_u + D_{12}^T C_{2,u}) \\
D_{12,u} &:= R^{1/2} \\
D_{21,u} &:= D_r V_r^{-1/2} \begin{pmatrix} 0 & V_r^{-1/2} \end{pmatrix} \\
D_{22,u} &:= D_{22} - D_r V_r^{-1} (B_r^T P_r E_2 + D_r^T D_{22})
\end{aligned}$$

where  $R$  is as defined in theorem 2.1. We will show that  $\Sigma_U$  has a very nice property. In order to do this, we first recall the definition of the so-called inner systems. Moreover, some of the important properties of inner systems are also recalled in the following two lemmas.

**Definition 3.6 :** *A system is called inner if the system is internally stable, square (i.e. the number of inputs is equal to the number of outputs) and the transfer matrix of the system, denoted by  $G$ , satisfies:*

$$G(z)G^T(z^{-1}) = I \quad (3.6)$$

**Lemma 3.7 :** *Let the following square system be given:*

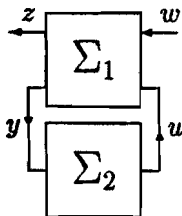
$$\Sigma_{st} : \begin{cases} \sigma x = Ax + Bu, \\ z = Cx + Du. \end{cases} \quad (3.7)$$

*Assume that  $A$  is asymptotically stable. The system  $\Sigma_{st}$  is inner if there exists a matrix  $X$  satisfying:*

- (a)  $X = A^T X A + C^T C$
- (b)  $D^T C + B^T X A = 0$
- (c)  $D^T D + B^T X B = I$  □

**Proof :** See [6, 15].

**Lemma 3.8 :** *Suppose we have the following interconnection of two systems  $\Sigma_1$  and  $\Sigma_2$ , both described by some state space representation:*



$$(3.8)$$

Assume  $\Sigma_1$  is inner. Denote its transfer matrix from  $(w, u)$  to  $(z, y)$  by  $L$ . Moreover, assume that if we decompose  $L$  compatible with the sizes of  $w, u, z$  and  $y$ :

$$L \begin{pmatrix} w \\ u \end{pmatrix} =: \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix} \begin{pmatrix} w \\ u \end{pmatrix} = \begin{pmatrix} z \\ y \end{pmatrix}, \quad (3.9)$$

we have  $L_{21}^{-1} \in H_\infty$  and  $L_{22}$  is strictly proper. Then the following two statements are equivalent:

- (i) The closed loop system (3.8) is internally stable and its closed loop transfer matrix has  $H_\infty$  norm less than 1.
- (ii) The system  $\Sigma_2$  is internally stable and its transfer matrix has  $H_\infty$  norm less than 1.  $\square$

**Proof :** See [9, 13]. ■

Now, we are ready to come back to the system  $\Sigma_u$  and establish some of its properties in the following lemma.

**Lemma 3.9 :** The system  $\Sigma_U$  as defined by (3.5) is inner. Denote the transfer matrix of  $\Sigma_U$  by  $U$ . We decompose  $U$  compatible with the sizes of  $w, u_U, z_U$  and  $y_U$ :

$$U \begin{pmatrix} w \\ u_U \end{pmatrix} =: \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \begin{pmatrix} w \\ u_U \end{pmatrix} = \begin{pmatrix} z_U \\ y_U \end{pmatrix}.$$

Then  $U_{21}$  is invertible and its inverse is in  $H_\infty$ . Moreover  $U_{22}$  is strictly proper. □

**Proof :** It can be easily checked that  $P_r$  satisfies the conditions (i)-(iii) of lemma 3.7. Condition (i) of lemma 3.7 turns out to be equal to the reduced order discrete algebraic Riccati equation as given in lemma 3.2. Conditions (ii) and (iii) follow by simply writing out the equations in terms of the system parameters of system (2.1).

The stable matrix  $A_{cl,P}$ , as defined in lemma 3.2, can be written in the following form:

$$A_{cl,P} = A_U - E_U D_{12,U}^{-1} C_{1,U} \quad (3.10)$$

Next, we show that  $A_U$  is asymptotically stable. We know  $P_r \geq 0$  and

$$P_r = A_U^T P_r A_U + \begin{pmatrix} C_{1,U}^T & C_{2,U}^T \end{pmatrix} \begin{pmatrix} C_{1,U} \\ C_{2,U} \end{pmatrix} \quad (3.11)$$

It can be easily checked that  $x \neq 0$ ,  $A_U x = \lambda x$ ,  $C_{1,U} x = 0$  and  $C_{2,U} x = 0$  implies that  $A_{cl,P} x = \lambda x$ . Since  $A_{cl,P}$  is stable we have  $\text{Re } \lambda < 0$ . Hence the realization (3.5) is detectable. By standard Lyapunov theory the existence of a positive semi definite solution of (3.11) together with detectability guarantee asymptotic stability of  $A_U$ .



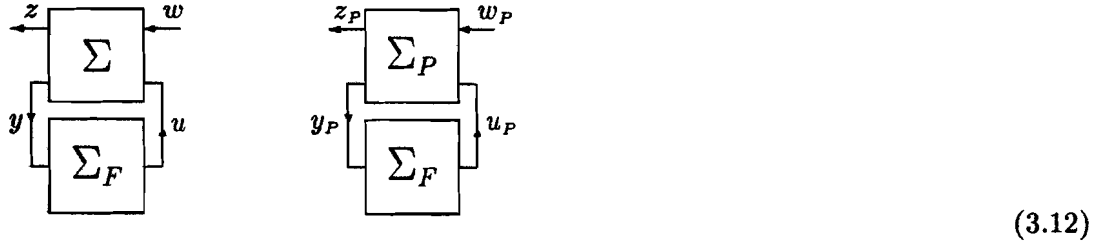
We can immediately write down a realization for  $U_{21}^{-1}$ :

$$\Sigma_{U_{21}^{-1}} : \begin{cases} \sigma x_U = A_{d,P} x_U + E_U D_{12,U}^{-1} w, \\ y_U = -D_{12,U}^{-1} C_{1,U} x_U + D_{12,U}^{-1} w, \end{cases}$$

Since  $A_{d,P}$  is stable we know that  $U_{21}^{-1}$  is an  $H_\infty$  function. Finally, the claim that  $U_{22}$  is strictly proper is trivial to check. ■

We will now formulate our key lemma:

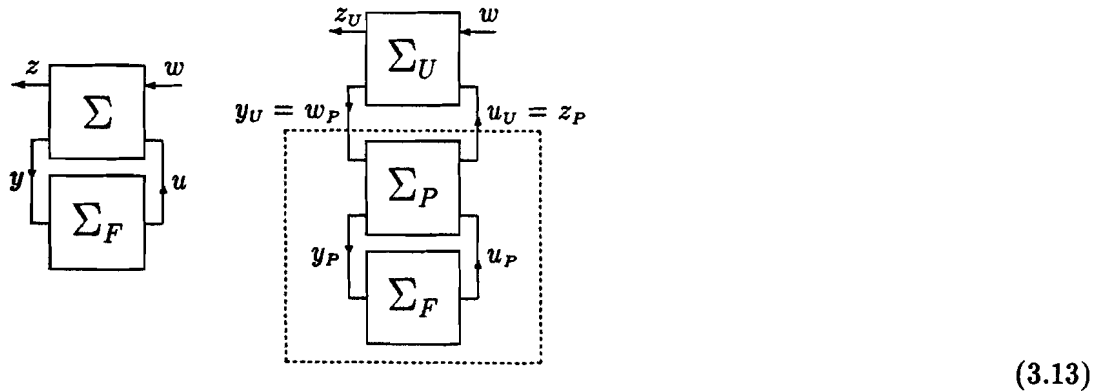
**Lemma 3.10 :** *Let  $P$  satisfy theorem 2.1 part (ii) (a)-(c). Moreover, let  $\Sigma_F$  be an arbitrary linear time-invariant finite-dimensional compensator in the form (2.2). Consider the following two systems, where the system on the left is the interconnection of (2.1) and (2.2) and the system on the right is the interconnection of (3.4) and (2.2):*



Then the following statements are equivalent :

- (i) The system on the left is internally stable and its transfer matrix from  $w$  to  $z$  has  $H_\infty$  norm less than 1.
- (ii) The system on the right is internally stable and its transfer matrix from  $w_P$  to  $z_P$  has  $H_\infty$  norm less than 1. □

**Proof :** We investigate the following systems:



The system on the left is the same as the system on the left in (3.12) and the system on the right is described by the system (3.5) interconnected with the system on the right in (3.12). A realization for the system on the right is given by:

$$\sigma \begin{pmatrix} x_U - x_{2,P} \\ x_P \\ p \end{pmatrix} = \begin{pmatrix} A_{cl,P} & 0 & 0 \\ * & A + BNC_1 & BM \\ * & LC_1 & K \end{pmatrix} \begin{pmatrix} x_U - x_{2,P} \\ x_P \\ p \end{pmatrix} + \begin{pmatrix} 0 \\ E + BND_{12} \\ LD_{12} \end{pmatrix} w$$

$$z_U = \begin{pmatrix} * & C_2 + D_{21}NC_1 & D_{21}M \end{pmatrix} \begin{pmatrix} x_U - x_{2,P} \\ x_P \\ p \end{pmatrix} + (D_{22} + D_{21}ND_{12})w$$

where  $A_{cl,P}$  is defined by (3.10). The \*'s denote matrices which are unimportant for this argument. The system on the right is internally stable if and only if the system described by the above set of equations is internally stable. If we also derive the system equations for the system on the left in (3.13) we immediately see that, since  $A_{cl,P}$  is asymptotically stable, the system on the left is internally stable if and only if the system on the right is internally stable. Moreover, if we take zero initial conditions and both systems have the same input  $w$  then we have  $z = z_U$  i.e. the input-output behaviour of both systems are equivalent. Hence the system on the left has  $H_\infty$  norm less than 1 if and only if the system on the right has  $H_\infty$  norm less than 1.

By lemma 3.9 we may apply lemma 3.8 to the system on the right in (3.13) and hence we find that the closed loop system is internally stable and has  $H_\infty$  norm less than 1 if and only if the dashed system is internally stable and has  $H_\infty$  norm less than 1.

Since the dashed system is exactly the system on the right in (3.12) and the system on the left in (3.13) is exactly equal to the system on the left in (3.12) we have completed the proof. ■

Using the previous lemma, we know that we only have to investigate the system  $\Sigma_P$ . This new system has a nice property which is outlined in the following lemma:

**Lemma 3.11 :** *There exists a matrix  $F_0$  such that if we define:*

$$F_{1,P} = -D_{21,P}^\dagger C_{2,P} + (I - D_{21,P}^\dagger D_{21,P})F_0$$

$$F_{2,P} = -D_{21,P}^\dagger D_{22,P}$$

*then we have:*

- (i)  $A_P + BF_{1,P}$  is stable,
- (ii)  $C_{2,P} + D_{21,P}F_{1,P} = 0$ ,
- (iii)  $D_{22,P} + D_{21,P}F_{2,P} = 0$ .

**Proof :** We first write everything in terms of the new basis introduced in the previous section. Hence the system parameters have the special form described by (3.1). Then it is

easily checked that conditions (ii) and (iii) are always satisfied, independent of the specific choice for  $F_0$ . If we also write the matrix  $F_0$  in the new basis,

$$F_0 = \begin{pmatrix} F_{0,11} & F_{0,12} \\ F_{0,21} & F_{0,22} \end{pmatrix}$$

then we have:

$$A_P + BF_{1,P} = \begin{pmatrix} A_{11} + B_{11}F_{0,11} & * \\ 0 & A_{cl,P} \end{pmatrix}$$

where  $*$  denotes a matrix which is unimportant for our argument. According to lemma 3.2, the matrix  $A_{cl,P}$  is asymptotically stable. Moreover, as noted in the previous section  $(A_{11}, B_{11})$  is controllable. Hence, any matrix  $F_0$  such that  $A_{11} + B_{11}F_{0,11}$  is stable satisfies the conditions of our lemma. Moreover, controllability guarantees the existence of such matrices  $F_0$ . ■

**Remark:** The above lemma implies that the full-information feedback  $u = F_{1,P}x_P + F_{2,P}w_P$  applied to  $\Sigma_P$  yields a stable closed loop system for which the closed loop  $H_\infty$  norm is equal to 0.

Next, we will look at the Riccati equation for the system  $\Sigma_P$ . It can be checked immediately that  $X = 0$  satisfies (a)-(c) of theorem 2.1 for the system  $\Sigma_P$ .

We dualize  $\Sigma_P$ . We know that  $(A, E, C_1, D_{12})$  has no invariant zeros on the unit circle. It can be easily checked that this implies that  $(A_P, E, C_{1,P}, D_{12})$  has no invariant zeros on the unit circle. Hence for the dual of  $\Sigma_P$  we know that  $(A_P^T, C_{1,P}^T, E^T, D_{21}^T)$  has no invariant zeros on the unit circle. If there exists an internally stabilizing feedback for the system  $\Sigma$  which makes the  $H_\infty$  norm of the closed loop system less than 1 then the same feedback is internally stabilizing and makes the  $H_\infty$  norm of the closed loop system less than 1 for the system  $\Sigma_P$ . If we dualize this feedback and apply it to the dual of  $\Sigma_P$  then it is again internally stabilizing and again it makes the  $H_\infty$  norm of the closed loop system less than 1. We can now apply lemma 3.3 which exactly guarantees the existence of a matrix  $Y \geq 0$  satisfying the following conditions

(i)  $Y$  is such that  $S_P > 0$  where

$$\begin{aligned} W_P &:= D_{12,P}D_{12,P}^T + C_{1,P}YC_{1,P}^T, \\ S_P &:= I - D_{22,P}D_{22,P}^T - C_{2,P}YC_{2,P}^T \\ &\quad + (C_{2,P}YC_{1,P}^T + D_{22,P}D_{12,P}^T)W_P^\dagger(C_{1,P}YC_{2,P}^T + D_{12,P}D_{22,P}^T). \end{aligned}$$

(ii)  $Y$  satisfies the following discrete algebraic Riccati equation:

$$Y = A_P Y A_P^T + E_P E_P^T \tag{3.14}$$

$$- \begin{pmatrix} C_{1,P} Y A_P^T + D_{12,P} E_P^T \\ C_{2,P} Y A_P^T + D_{22,P} E_P^T \end{pmatrix}^T H_P(Y)^\dagger \begin{pmatrix} C_{1,P} Y A_P^T + D_{12,P} E_P^T \\ C_{2,P} Y A_P^T + D_{22,P} E_P^T \end{pmatrix}.$$

where

$$H_P(Y) := \begin{pmatrix} D_{12,P}D_{12,P}^T & D_{12,P}D_{22,P}^T \\ D_{22,P}D_{12,P}^T & D_{22,P}D_{22,P}^T - I \end{pmatrix} + \begin{pmatrix} C_{1,P} \\ C_{2,P} \end{pmatrix} Y \begin{pmatrix} C_{1,P} \\ C_{2,P} \end{pmatrix}^T. \tag{3.15}$$

(iii)  $Y$  satisfies a stability condition: for all  $z \in \mathcal{C}$  with  $|z| \geq 1$ , we have

$$\begin{aligned} \text{rank}_{\mathcal{R}} \begin{pmatrix} zI - A & C_{1,P}YA_P^T + D_{12,P}E_P^T & C_{2,P}YA_P^T + D_{22,P}E_P^T \\ -C_{1,P} & C_{1,P}YC_{1,P}^T + D_{12,P}D_{12,P}^T & C_{1,P}YC_{2,P}^T + D_{12,P}D_{22,P}^T \\ -C_{2,P} & C_{2,P}YC_{1,P}^T + D_{22,P}D_{12,P}^T & C_{2,P}YC_{2,P}^T + D_{22,P}D_{22,P}^T - I \end{pmatrix} \\ = n + q + \text{rank}_{\mathcal{R}(z)} C_1(zI - A)^{-1}E + D_{12} \end{aligned}$$

Note that  $Y$  satisfies the conditions (d)-(f) of theorem 2.1 for the system  $\Sigma_P$ .

The following lemma relates the existence and the solution of the above conditions to the conditions in theorem 2.1:

**Lemma 3.12** : *There exist a matrix  $Y \geq 0$  satisfying the above conditions if and only if there exist matrices  $P \geq 0$  and  $Q \geq 0$  satisfying the conditions in part (ii) of theorem 2.1. Moreover, in that case we have:*

$$Y = (I - QP)^{-1}Q \quad \square$$

The above derivation yields the necessity part of theorem 2.1:

**Lemma 3.13** : *Let  $\Sigma$ , described by 2.1), be given with zero initial condition. Assume that  $(A, B, C_2, D_{21})$  and  $(A, E, C_1, D_{12})$  have no invariant zeros on the unit circle. If part (i) of theorem 2.1 is satisfied then there exist matrices  $P$  and  $Y$  satisfying (a)-(f) of part (ii) of theorem 2.1.  $\square$*

This completes the proof (i)  $\Rightarrow$  (ii). In the next section we will prove the reverse implication. Moreover in case the desired compensator  $\Sigma_F$  exists we will derive an explicit formula for one choice for  $\Sigma_F$  which satisfies all requirements.

### 3.3 The transformation into a disturbance decoupling problem with measurement feedback

In this section we assume that there exist matrices  $P$  and  $Q$  satisfying part (ii) of theorem 2.1 for the system (2.1) and we show that the part (i) of theorem 2.1 holds. First we transform our original system  $\Sigma$  into another system  $\Sigma_{P,Y}$ . We will show that a compensator is internally stabilizing and makes the  $H_\infty$  norm of the closed loop system less than 1 for the system  $\Sigma$  if and only if the same compensator is internally stabilizing and makes the  $H_\infty$  norm of the closed loop system less than 1 for our transformed system  $\Sigma_{P,Y}$ . Next we will show that  $\Sigma_{P,Y}$  has a following very special property (see [11]):

There exists an internally stabilizing compensator which makes the closed loop transfer matrix equal to zero, i.e.  $w$  does not have any effect on the output of the system  $z$ . This property of  $\Sigma_{P,Y}$  has a special name: “the Disturbance Decoupling Problem with Measurement feedback and internal Stability (DDPMS) is solvable”.

We know a matrix  $Y := (I - QP)^{-1}Q$  exists satisfying the conditions as outlined in the previous section. Next, we define  $\Sigma_{P,Y}$ . We start by transforming  $\Sigma$  into  $\Sigma_P$ . Then we apply the dual transformation on  $\Sigma_P$  to obtain  $\Sigma_{P,Y}$ :

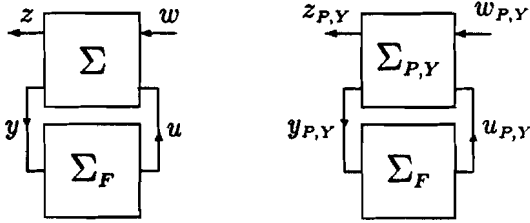
$$\Sigma_{P,Y} : \begin{cases} \sigma x_{P,Y} = A_{P,Y}x_{P,Y} + B_{P,Y}u_{P,Y} + E_{P,Y}w_{P,Y}, \\ y_{P,Y} = C_{1,P}x_{P,Y} + D_{12,P,Y}w_{P,Y}, \\ z_{P,Y} = C_{2,P}x_{P,Y} + D_{21,P,Y}u_{P,Y} + D_{22,P,Y}w_{P,Y}, \end{cases} \quad (3.16)$$

where

$$\begin{aligned} A_{P,Y} &:= A_P - (A_P Y C_{1,P}^T + E_P D_{12,P}^T) W_P^\dagger C_{1,P} \\ E_{P,Y} &:= E_P - (A_P Y C_{1,P}^T + E_P D_{12,P}^T) W_P^\dagger D_{12,P} \\ A_{P,Y} &:= A_P + (A_Y Y C_{2,P}^T + E_Y D_{22,P}^T) S_P^{-1} C_{2,P} \\ C_{2,P,Y} &:= S_P^{-1/2} C_{2,P} \\ B_{P,Y} &:= B + (A_Y Y C_{2,P}^T + E_Y D_{22,P}^T) S_P^{-1} D_{21,P} \\ E_{P,Y} &:= (A_P Y C_{1,P}^T + E_P D_{12,P}^T + [A_Y Y C_{2,P}^T + E_Y D_{22,P}^T] S_P^{-1} [C_{2,P} Y C_{1,P}^T + D_{22,P} D_{12,P}^T]) (W_P^{1/2})^\dagger \\ D_{12,P,Y} &:= W_P^{1/2} \\ D_{21,P,Y} &:= S_P^{-1/2} D_{21,P} \\ D_{22,P,Y} &:= S_P^{-1/2} (C_{2,P} Y C_{1,P}^T + D_{22,P} D_{12,P}^T) (W_P^{1/2})^\dagger \end{aligned}$$

When we first apply lemma 3.10 on the transformation from  $\Sigma$  to  $\Sigma_P$  and then the dual of lemma 3.10 on the transformation from  $\Sigma_P$  to  $\Sigma_{P,Y}$  we find:

**Lemma 3.14 :** *Let  $P$  satisfy theorem 2.1 part (ii) (a)-(c). Moreover let an arbitrary linear time-invariant finite-dimensional compensator  $\Sigma_F$  be given, described by (2.2). Consider the following two systems, where the system on the left is the interconnection of (2.1) and (2.2) and the system on the right is the interconnection of (3.16) and (2.2):*



Then the following statements are equivalent :

- (i) The system on the left is internally stable and its transfer matrix from  $w$  to  $z$  has  $H_\infty$  norm less than 1.
- (ii) The system on the right is internally stable and its transfer matrix from  $w_{P,Y}$  to  $z_{P,Y}$  has  $H_\infty$  norm less than 1.  $\square$

It remains to be shown that for  $\Sigma_{P,Y}$  the (DDPMS) is solvable. We first need the following preliminary lemma

**Lemma 3.15** : *There exist a matrix  $K_0$  such that if we define:*

$$\begin{aligned} K_{1,P,Y} &= -E_{P,Y}D_{12,P,Y}^\dagger + K_0(I - D_{12,P,Y}D_{12,P,Y}^\dagger) \\ K_{2,P,Y} &= -D_{12,P,Y}^\dagger \end{aligned}$$

*then we have:*

- (i)  $A_{P,Y} + K_{1,P,Y}C_{1,P}$  is stable,
- (ii)  $E_{P,Y} + K_{1,P,Y}D_{12,P,Y} = 0$ ,
- (iii)  $D_{22,P,Y} + K_{2,P,Y}D_{12,P,Y} = 0$ .

*Moreover, let  $F_{1,P}$  and  $F_{2,P}$  be as defined in lemma 3.11. If we define*

$$\begin{aligned} F_{1,P,Y} &:= F_{1,P}, \\ F_{2,P,Y} &:= -D_{21,P,Y}^\dagger D_{22,P,Y} \end{aligned}$$

*then we have:*

- (iv)  $A_{P,Y} + BF_{1,P,Y} = A_P + BF_{1,P}$  is stable,
- (v)  $C_{2,P,Y} + D_{21,P,Y}F_{1,P} = 0$ ,
- (vi)  $D_{22,P,Y} + D_{21,P,Y}F_{2,P,Y} = 0$ , □

**Proof** : The construction of a suitable matrix  $K_0$  satisfying conditions (i)-(iii) is dual to the derivation of a suitable  $F_0$  satisfying the conditions of lemma 3.5. Hence details are omitted. Conditions (iv)-(vi) can be checked via straightforward algebraic manipulations. ■

**Remark**: The first part of the lemma is dual to lemma 3.11 and shows that because of the dual transformation we can now observe the states of  $\Sigma_{P,Y}$  perfectly. Surprisingly enough the property that  $\Sigma_P$  could be controlled perfectly is preserved: the second part of the lemma shows that also for  $\Sigma_{P,Y}$  we can find a full-information feedback that stabilizes the system and yields a closed loop system with  $H_\infty$  norm equal to 0.

Now we are ready to show the solvability of (DDPMS) for the system  $\Sigma_{P,Y}$  in the following lemma.

**Lemma 3.16** : *Let  $\Sigma_F$  be given by:*

$$\Sigma_F : \begin{cases} \sigma p = K_{P,Y}p + L_{P,Y}y_{P,Y}, \\ u_{P,Y} = M_{P,Y}p + N_{P,Y}y_{P,Y}, \end{cases} \quad (3.17)$$

*where*

$$\begin{aligned} N_{P,Y} &:= -F_{2,P,Y}K_{2,P,Y} \\ M_{P,Y} &:= F_{1,P,Y} - N_{P,Y}C_{1,P} \\ L_{P,Y} &:= B_{P,Y}N_{P,Y} - K_{1,P,Y} \\ K_{P,Y} &:= A_{P,Y} + B_{P,Y}M_{P,Y} + K_{1,P,Y}C_{1,P} \end{aligned}$$

*The interconnection of  $\Sigma_F$  and  $\Sigma_{P,Y}$  is internally stable and the closed loop transfer matrix from  $w_{P,Y}$  to  $z_{P,Y}$  is zero. □*

**Proof :** We can write out the formulas for a state space representation of the interconnection of  $\Sigma_{P,Y}$  and  $\Sigma_F$ . We then apply the following basis transformation:

$$\begin{pmatrix} x_{P,Y} - p \\ p \end{pmatrix} = \begin{pmatrix} I & -I \\ 0 & I \end{pmatrix} \begin{pmatrix} x_{P,Y} \\ p \end{pmatrix}.$$

After this transformation one immediately sees that the closed loop transfer matrix from  $w_{P,Y}$  to  $z_{P,Y}$  is zero. Moreover the system matrix (2.3) after this transformation is given by:

$$\begin{pmatrix} A_{P,Y} + K_{1,P,Y}C_{1,P} & 0 \\ L_{P,Y}C_{1,P} & A_{P,Y} + B_{P,Y}F_{1,P,Y} \end{pmatrix}$$

Lemma 3.15 guarantees that this matrix is asymptotically stable. Hence  $\Sigma_F$  is internally stabilizing. ■

We know  $\Sigma_F$  is internally stabilizing and the resulting closed loop system has  $H_\infty$  norm less than 1 for the system  $\Sigma_{P,Y}$ . Hence, by applying lemma 3.14, we find that  $\Sigma_F$  satisfies part (i) of theorem 2.1. This completes the proof of (ii)  $\Rightarrow$  (i) of theorem 2.1. We have already shown the reverse implication and hence the proof of theorem 2.1 is completed.

## 4 Controller structure

In the previous section, we found a controller for  $\Sigma$  which satisfies all requirements, but its structure is very cloudy. In this section we define a controller, which also achieves disturbance decoupling when applied to  $\Sigma_{P,Y}$ , but which has a very appealing structure.

We first need to construct a matrix with a desired stability property:

**Lemma 4.1 :** *There exists a matrix  $\bar{K}_0$  such that*

$$\left[ I + \bar{K}_0(I - D_{12,P,Y}D_{12,P,Y}^\dagger)C_{1,P} \right] (A_{P,Y} - E_{P,Y}D_{12,P,Y}^\dagger)$$

*is stable.* □

**Proof :** According to lemma 3.15 there exists a matrix  $K_0$  such that  $A_1 + K_0C_1$  is stable where

$$\begin{aligned} A_1 &= (A_{P,Y} - E_{P,Y}D_{12,P,Y}^\dagger) \\ C_1 &= (I - D_{12,P,Y}D_{12,P,Y}^\dagger)C_{1,P} \end{aligned}$$

Since, for discrete time systems detectability of  $(C_1, A_1)$  implies that the pair  $(C_1A_1, A_1)$  is detectable there exists a matrix  $\bar{K}_0$  such that  $A_1 + \bar{K}_0C_1A_1$  is stable. This implies that  $\bar{K}_0$  satisfies the conditions of the lemma. ■

**Remark:** This lemma might look rather strange but it is essential. If we use one-step-ahead predictors then the estimator is stable if the filter gain  $K$  is such that  $A + KC$  is stable. However, in this section we use current estimators where we also use the measurement

$y(k)$  to estimate  $x(k)$ . In that case the estimator is stable if the filter gain satisfies  $(I + KC)A$ . Intuitively the above lemma tells us that we can find a stable current estimator if we can find a stable one-step-ahead estimator.

Note that an optimal full-information feedback for  $\Sigma_{P,Y}$  is given by:

$$u_{P,Y} = F_{1,P,Y}x_{P,Y} + F_{2,P,Y}w_{P,Y}$$

where we change  $F_{2,P,Y}$  with respect to the previous section into:

$$F_{2,P,Y} := -D_{21,P,Y}^\dagger D_{22,P,Y} + (I - D_{21,P,Y}^\dagger D_{21,P,Y})F_0 Y C_{1,P}^\top (W_P)^\dagger.$$

It can be shown, along the same lines as the proof of lemma 3.16 that the following controller stabilizes  $\Sigma_{P,Y}$  and achieves disturbance decoupling:

$$\begin{aligned} \sigma p &= A_{P,Y}p + B_{P,Y}u_{P,Y} + E_{P,Y}\hat{w} - \bar{K}_0 \Pi_1 (\sigma y - C_{1,P}[A_{P,Y}p + B_{P,Y}u_{P,Y} + E_{P,Y}\hat{w}]) \\ \hat{w} &= D_{12,P,Y}^\dagger (y_{P,Y} - C_{1,P}p) \\ u_{P,Y} &= F_{1,P,Y}\hat{x} + F_{2,P,Y}\hat{w} \end{aligned}$$

where

$$\Pi_1 := I - D_{12,P,Y} D_{12,P,Y}^\dagger = I - W_P W_P^\dagger$$

We are going to apply this controller to the system  $\Sigma$ . However, if we rewrite this controller in terms of the original system parameters it has a very special structure:

$$\begin{aligned} \sigma \hat{x} &= A\hat{x} + Bu + E\hat{w} + \sigma K_1(y - \hat{y}) \\ \sigma \hat{w} &= ER^{-1}(E^\top P A_x + D_{22}^\top C_x)[A\hat{x} + Bu + E\hat{w}] + \sigma K_2(y - \hat{y}) \\ \sigma \hat{y} &= C_1[A\hat{x} + Bu + E\hat{w}] + D_{12}ER^{-1}(E^\top P A_x + D_{22}^\top C_x)[A\hat{x} + Bu + E\hat{w}] \\ u &= F_1\hat{x} + F_2\hat{w} \end{aligned}$$

where

$$\begin{aligned} K_1 &= -\bar{K}_0 \Pi_1 + Y C_{1,P}^\top W_P^\dagger (I + C_{1,P} \bar{K}_0 \Pi_1) \\ K_2 &= D_{12,P}^\top W_P^\dagger (I + C_{1,P} \bar{K}_0 \Pi_1) \end{aligned}$$

while  $F_1$  and  $F_2$  are defined by (2.10) and (2.11) respectively. We see that we have a full-information feedback:

$$u = F_1 x + F_2 w$$

where we replace the state  $x$  and the disturbance  $w$  by their respective estimates  $\hat{x}$  and  $\hat{w}$ . For the state and the disturbance we have build estimators. If we write  $s(k|k)$  for the estimate of the variable  $s$  at time  $k$  using measurements  $y(0), \dots, y(k)$  and  $s(k|k-1)$  for the estimate of the variable  $s$  at time  $k$  using measurements  $y(0), \dots, y(k-1)$  then we can express the structure even clearer. We get the following form:

$$\Sigma_F \begin{cases} x(k+1|k+1) = x(k+1|k) & + K_1 [y(k+1) - y(k+1|k)] \\ w(k+1|k+1) = w(k+1|k) & + K_2 [y(k+1) - y(k+1|k)] \\ x(k+1|k) = Ax(k|k) + Bu(k) + Ew(k|k) \\ w(k+1|k) = ER^{-1}(E^\top P A_x + D_{22}^\top C_x)x(k+1|k) \\ y(k+1|k) = C_1 x(k+1|k) + D_{12}w(k+1|k) \\ u(k) = F_1 x(k|k) + F_2 w(k|k) \end{cases}$$



Note that in the state feedback case we can identify a worst-case response for the disturbance  $w$ :

$$w(k) = ER^{-1}(E^T PA_x + D_{22}^T C_x)x(k) \quad (4.1)$$

In the above controller we have to estimate  $w(k+1|k)$ . Clearly past measurements do not tell us anything. However, this controller expects the worst-case response (4.1) and estimates this worst-case response.

## 5 Reduced order estimator based controller

In this section we show that for the singular  $H_\infty$  optimal control problem satisfying part (i) of theorem 2.1 we can always find a solution which has dynamical order less than that of the plant and is of reduced order observer-based structure. This result is analogous to those obtained in [16] for the continuous-time problems. Without loss of generality, we develop such a reduced order observer-based controller for the system  $\Sigma_{P,Y}$  defined in the previous section. Consider the  $\Sigma_{P,Y}$  defined by (3.16). There exists a constant output prefeedback law  $F_{pre}y_{P,Y}$  such that after applying this prefeedback law, namely setting

$$u_{P,Y} \longrightarrow F_{pre}y_{P,Y} + u_{P,Y}, \quad (5.1)$$

the direct feed-through term from  $w_{P,Y}$  from  $z_{P,Y}$  disappears. Hence without loss of generality, hereafter we assume that  $D_{22,P,Y} = 0$ . Also, there exists a state feedback gain  $F_{P,Y}$  such that

$$(C_{2,P,Y} + D_{21,P}F_{P,Y})(sI - A_{P,Y} - B_{P,Y}F_{P,Y})^{-1}E_{P,Y} \equiv 0.$$

Without loss of generality but for simplicity of presentation, we assume that the matrices  $C_{1,P}$  and  $D_{12,P,Y}$  are already in the form

$$C_{1,P} = \begin{pmatrix} 0 & C_{1,02} \\ I_{p-m_0} & 0 \end{pmatrix} \quad \text{and} \quad D_{12,P,Y} = \begin{pmatrix} D_{12,0} \\ 0 \end{pmatrix}, \quad (5.2)$$

where  $m_0$  is the rank of  $D_{12,P,Y}$  and  $D_{12,0}$  is of full rank. Then the given system  $\Sigma_{P,Y}$  can be written as,

$$\begin{cases} \sigma \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} E_1 \\ E_2 \end{pmatrix} w_{P,Y} + \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} u_{P,Y} \\ \begin{pmatrix} y_0 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 & C_{1,02} \\ I_{p-m_0} & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} D_{12,0} \\ 0 \end{pmatrix} w_{P,Y} \\ z_{P,Y} = C_{2,P,Y} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + D_{21,P,Y} u_{P,Y} \end{cases} \quad (5.3)$$

where  $(x'_1, x'_2)' = x_{P,Y}$  and  $(y'_0, y'_1)' = y_{P,Y}$ . We note that  $y_1 \equiv x_1$ . Thus, one needs to estimate only the state  $x_2$  in the reduced-order estimator. Then following closely the procedure given in [16], we first rewrite the state equation for  $x_1$  in terms of the measured output  $y_1$  and state  $x_2$  as follows,

$$\sigma y_1 = A_{11}y_1 + A_{12}x_2 + E_1 w_{P,Y} + B_1 u_{P,Y}, \quad (5.4)$$

where  $y_1$  and  $u_{P,Y}$  are known. Observation of  $x_2$  is made via  $y_0$  and

$$\tilde{y}_1 = A_{12}x_2 + E_1w_{P,Y} = \sigma y_1 - A_{11}y_1 - B_1u_{P,Y}. \quad (5.5)$$

A reduced-order system for the estimation of state  $x_2$  is given by

$$\begin{cases} \sigma x_2 = A_R x_2 + E_R w_{P,Y} + \begin{pmatrix} A_{21} & B_2 \end{pmatrix} \begin{pmatrix} y_1 \\ u_{P,Y} \end{pmatrix}, \\ y_R = C_R x_2 + D_R w_{P,Y} \end{cases} \quad (5.6)$$

where

$$A_R := A_{22}, \quad E_R := E_2, \quad C_R := \begin{pmatrix} C_{1,02} \\ A_{12} \end{pmatrix}, \quad D_R := \begin{pmatrix} D_{12,0} \\ E_1 \end{pmatrix}. \quad (5.7)$$

Based on (5.6), one can construct a reduced-order observer for  $x_2$  as,

$$\sigma \hat{x}_2 = A_R \hat{x}_2 + \begin{pmatrix} A_{21} & B_2 \end{pmatrix} \begin{pmatrix} y_1 \\ u_{P,Y} \end{pmatrix} + K_R [C_R \hat{x}_2 - y_R], \quad (5.8)$$

where  $K_R$  is the observer gain matrix chosen such that  $A_R + K_R C_R$  is asymptotically stable. For the purpose of implementing (5.8), let us partition  $K_R = (K_{R0}, K_{R1})$  to be compatible with the dimensions of the outputs  $(y'_0, \tilde{y}'_1)'$ , and at the same time define a new variable,

$$v := \hat{x}_2 + K_{R1} \tilde{y}_1.$$

We then obtain the following reduced order estimator based controller,

$$\begin{cases} \sigma v = (A_R + K_R C_R)v + (B_2 + K_{R1} B_1)u_{P,Y} + G_R y_{P,Y}, \\ \hat{x}_{P,Y} = \begin{pmatrix} 0 \\ I_{n-p+m_0} \end{pmatrix} v + \begin{pmatrix} 0 & I \\ 0 & -K_{R1} \end{pmatrix} y_{P,Y}, \\ u_{P,Y} = F_{P,Y} \hat{x}_{P,Y} + F_{pre} y_{P,Y}, \end{cases} \quad (5.9)$$

where

$$G_R = [-K_{R0}, A_{21} + K_{R1} A_{11} - (A_R + K_R C_R) K_{R1}],$$

and  $F_{P,Y}$  is state feedback gain and  $F_{pre}$  is the output prefeedback gain. We know that there exists an output injection such that:

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} + \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix} \begin{pmatrix} 0 & C_{1,02} \\ I_{p-m_0} & 0 \end{pmatrix} \quad (5.10)$$

is stable and

$$\begin{pmatrix} E_1 \\ E_2 \end{pmatrix} + \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix} \begin{pmatrix} D_{12,0} \\ 0 \end{pmatrix} = 0 \quad (5.11)$$

Because the matrix in (5.10) is stable there exists a matrix  $L$  such that

$$A_{22} + K_{21} C_{1,02} + L(A_{12} + K_{11} C_{1,02})$$

is stable. Moreover (5.11) implies that

$$E_2 + K_{21}D_{12,0} + L(E_1 + K_{11}D_{12,0}) = 0.$$

Then it is easy to check that a suitable choice for  $K_R$  is given by:

$$K_R = \begin{pmatrix} K_{R0} & K_{R1} \end{pmatrix} = \begin{pmatrix} K_{21} + LK_{11} & L \end{pmatrix}$$

**Remark :** It is interesting to point out that the state space representation of the reduced order estimator based controller in (5.9) might not be minimal and hence the McMillan degree of this controller might be less than the dynamical order of its state space representation (5.9). This is mainly due to the stable dynamics which are unobservable in the controlled output  $z_{P,Y}$  and they are induced by the output prefeedback law (5.1). A very interesting example is the state feedback case for  $C_1 = I$  and  $D_{12} = 0$ . In this case  $F_{P,Y}$  can be chosen as a zero matrix and the output prefeedback law  $F_{pre}$  in (5.1) is equal to  $F_{1,P}$  given in Lemma 3.11. In view of this, the reduced order estimator based controller (5.9) has McMillan degree equal to zero and it reduces to the static state feedback solution

$$u_{P,Y} = F_{1,P}y.$$

## 6 Conclusion

In this paper, we removed some standard assumptions on the system parameters. Moreover, we specified the structure of discrete time  $H_\infty$  controllers. Finally, we showed how to derive controllers of lower dynamical order without loss of performance. This is done by deriving reduced order observers. Our results are obtained under the assumption that both systems  $(A, B, C_2, D_{21})$  and  $(A, E, C_1, D_{12})$  are free of invariant zeros on unit circle. A most trivial technique to handle invariant zeros on unit circle is to perturb the plant data such that the perturbed plant satisfies our assumptions. However, the resulting criteria for the existence of the solution to the  $H_\infty$  control problem for the perturbed plant are not *algebraic* in the nature. Hence the derivation of *algebraic* criteria directly in discrete domain for this case is an open problem.

Via the bilinear transform and our knowledge about the problems of invariant zeros on the imaginary axis for  $H_\infty$  control problems in continuous time (see [5, 7, 10]), we know that in the case of invariant zeros on the unit circle several problems arise. These are mainly due to the fact that  $H_\infty$  controllers have a tendency of cancelling stable zeros of the system and will try to achieve this approximately if there are zeros on the unit circle. Hence we have poor stability margins. Moreover, the minimal achievable  $H_\infty$  norm may depend discontinuously on the system parameters if there are invariant zeros on the unit circle. Hence we also have numerical difficulties. The main problem in this respect is the nonuniqueness of (sub)optimal  $H_\infty$  controllers. Suppose we want to get closer and closer to the minimal achievable  $H_\infty$  norm. When can we avoid almost pole-zero cancellations near the unit circle? For this question, very little is known. However there are examples where we can get very good stability margins even though there are zeros on the unit circle. Similarly there are examples where we always have bad stability margins near optimality. What is needed is a characterization of the achievable stability margin near optimality.

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## List of COSOR-memoranda - 1992

| Number | Month    | Author   | Title   |
|--------|----------|--|---|
| 92-01  | January  | F.W. Steutel   | On the addition of log-convex functions and sequences   |
| 92-02  | January  | P. v.d. Laan   | Selection constants for Uniform populations   |
| 92-03  | February | E.E.M. v. Berkum<br>H.N. Linssen<br>D.A. Overdijk    | Data reduction in statistical inference   |
| 92-04  | February | H.J.C. Huijberts<br>H. Nijmeijer                     | Strong dynamic input-output decoupling:<br>from linearity to nonlinearity   |
| 92-05  | March    | S.J.L. v. Eijndhoven<br>J.M. Soethoudt               | Introduction to a behavioral approach<br>of continuous-time systems   |
| 92-06  | April    | P.J. Zwietering<br>E.H.L. Aarts<br>J. Wessels        | The minimal number of layers of a perceptron that sorts   |
| 92-07  | April    | F.P.A. Coolen  | Maximum Imprecision Related to Intervals of Measures<br>and Bayesian Inference with Conjugate Imprecise Prior<br>Densities                |
| 92-08  | May      | I.J.B.F. Adan<br>J. Wessels<br>W.H.M. Zijm           | A Note on "The effect of varying routing probability in<br>two parallel queues with dynamic routing under a<br>threshold-type scheduling" |
| 92-09  | May      | I.J.B.F. Adan<br>G.J.J.A.N. v. Houtum<br>J. v.d. Wal | Upper and lower bounds for the waiting time in the<br>symmetric shortest queue system   |
| 92-10  | May      | P. v.d. Laan   | Subset Selection: Robustness and Imprecise Selection  |
| 92-11  | May      | R.J.M. Vacssens<br>E.H.L. Aarts<br>J.K. Lenstra      | A Local Search Template<br>(Extended Abstract)  |
| 92-12  | May      | F.P.A. Coolen  | Elicitation of Expert Knowledge and Assessment of Im-<br>precise Prior Densities for Lifetime Distributions                               |
| 92-13  | May      | M.A. Peters<br>A.A. Stoorvogel                       | Mixed $H_2/H_\infty$ Control in a Stochastic Framework  |

| Number | Month | Author  | Title   |
|--------|-------|---|---|
| 92-14  | June  | P.J. Zwietering<br>E.H.L. Aarts<br>J. Wessels         | The construction of minimal multi-layered perceptrons: a case study for sorting     |
| 92-15  | June  | P. van der Laan                                       | Experiments: Design, Parametric and Nonparametric Analysis, and Selection           |
| 92-16  | June  | J.J.A.M. Brands<br>F.W. Steutel<br>R.J.G. Wilms       | On the number of maxima in a discrete sample  |
| 92-17  | June  | S.J.L. v. Eijndhoven<br>J.M. Soethoudt                | Introduction to a behavioral approach of continuous-time systems part II            |
| 92-18  | June  | J.A. Hoogeveen<br>H. Oosterhout<br>S.L. van der Velde | New lower and upper bounds for scheduling around a small common due date            |
| 92-19  | June  | F.P.A. Coolen   | On Bernoulli Experiments with Imprecise Prior Probabilities                         |
| 92-20  | June  | J.A. Hoogeveen<br>S.L. van de Velde                   | Minimizing Total Inventory Cost on a Single Machine in Just-in-Time Manufacturing   |
| 92-21  | June  | J.A. Hoogeveen<br>S.L. van de Velde                   | Polynomial-time algorithms for single-machine bicriteria scheduling                 |
| 92-22  | June  | P. van der Laan                                       | The best variety or an almost best one? A comparison of subset selection procedures |
| 92-23  | June  | T.J.A. Storcken<br>P.H.M. Ruys                        | Extensions of choice behaviour  |
| 92-24  | July  | L.C.G.J.M. Habets                                     | Characteristic Sets in Commutative Algebra: an overview                             |
| 92-25  | July  | P.J. Zwietering<br>E.H.L. Aarts<br>J. Wessels         | Exact Classification With Two-Layered Perceptrons                                   |
| 92-26  | July  | M.W.P. Savelsbergh                                    | Preprocessing and Probing Techniques for Mixed Integer Programming Problems         |

| Number | Month     | Author   | Title   |
|--------|-----------|--|---|
| 92-27  | July      | I.J.B.F. Adan<br>W.A. van de<br>Waarsenburg<br>J. Wessels            | Analysing $E_k E_r c$ Queues  |
| 92-28  | July      | O.J. Boxma<br>G.J. van Houtum  | The compensation approach applied to a $2 \times 2$ switch                            |
| 92-29  | July      | E.H.L. Aarts<br>P.J.M. van Laarhoven<br>J.K. Lenstra<br>N.L.J. Ulder | Job Shop Scheduling by Local Search   |
| 92-30  | August    | G.A.P. Kindervater<br>M.W.P. Savelsbergh                             | Local Search in Physical Distribution Management                                      |
| 92-31  | August    | M. Makowski<br>M.W.P. Savelsbergh                                    | MP-DIT Mathematical Program data Interchange Tool                                     |
| 92-32  | August    | J.A. Hoogeveen<br>S.L. van de Velde<br>B. Veltman                    | Complexity of scheduling multiprocessor tasks with prespecified processor allocations |
| 92-33  | August    | O.J. Boxma<br>J.A.C. Resing  | Tandem queues with deterministic service times  |
| 92-34  | September | J.H.J. Einmahl   | A Bahadur-Kiefer theorem beyond the largest observation                               |
| 92-35  | September | F.P.A. Coolen  | On non-informativeness in a classical Bayesian inference problem                      |
| 92-36  | September | M.A. Peters  | A Mixed $H_2/H_\infty$ Function for a Discrete Time System                            |
| 92-37  | September | I.J.B.F. Adan<br>J. Wessels  | Product forms as a solution base for queueing systems                                 |
| 92-38  | September | L.C.G.J.M. Habets  | A Reachability Test for Systems over Polynomial Rings using Gröbner Bases             |
| 92-39  | September | G.J. van Houtum<br>I.J.B.F. Adan<br>J. Wessels<br>W.H.M. Zijm        | The compensation approach for three or more dimensional random walks                  |

| Number | Month     | Author  | Title  |
|--------|-----------|---|--|
| 92-40  | September | F.P.A. Coolen                                 | Bounds for expected loss in Bayesian decision theory with imprecise prior probabilities              |
| 92-41  | October   | H.J.C. Huijberts<br>H. Nijmeijer<br>A.C. Ruiz | Nonlinear disturbance decoupling and linearization:<br>a partial interpretation of integral feedback |
| 92-41  | October   | A.A. Stoorvogel<br>A. Saberi<br>B.M. Chen     | The discrete-time $H_\infty$ control problem with measurement feedback                               |