Embedded Markov processes and recurrence

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by

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1. Introduction

Let (X,Σ,m) be a σ-finite measure space. A Markov operator P in $\mathcal{L}_\infty(X,\Sigma,m)$ is a linear operator in $\mathcal{L}_\infty(X,\Sigma,m)$ which satisfies

1) $f \geq 0 \Rightarrow Pf \geq 0$ for all $f \in \mathcal{L}_\infty$,

2) $f = \sum_{n=1}^{\infty} f_n \Rightarrow Pf = \sum_{n=1}^{\infty} Pf_n$ for f and f_n $(n = 1,2,\ldots)$ in $\mathcal{L}_\infty$,

3) $P1 \leq 1$.

Recall that an element of $\mathcal{L}_\infty$ actually is an equivalence class of m-almost equal functions. As usual, we shall make the identification of the equivalence class and any of its representatives. Consequently, in this definition and in the sequel all statements on functions (and sets) have to be interpreted modulo m-null sets. Moreover, all functions and sets are supposed to be Σ-measurable.

Finite or infinite sums of functions have always to be taken pointwise. We shall use the convention to write the operator symbol to the left of the function if we consider the operator as acting in $\mathcal{L}_\infty$, and similarly if we consider an operator in $\mathcal{L}_1$, then the operator symbol is written to the right of the function.

A Markov operator P in $\mathcal{L}_1(X,\Sigma,m)$ is a linear operator in $\mathcal{L}_1(X,\Sigma,m)$ such that

1) $u \geq 0 \Rightarrow uP \geq 0$ for all $u \in \mathcal{L}_1$

2) $\|P\| \leq 1$.

The adjoint operator of a Markov operator in $\mathcal{L}_1$ is a Markov operator in $\mathcal{L}_\infty$, and conversely every Markov operator in $\mathcal{L}_\infty$ is the adjoint operator of a Markov operator in $\mathcal{L}_1$, and the relationship is given by

$$\int u(Pf)\,dm = \int (uP)f\,dm \quad \text{for all } u \in \mathcal{L}_1 \text{ and } f \in \mathcal{L}_\infty .$$

For details the reader is referred to Foguel [2].

It is not difficult to verify that, by means of monotone approximation from below, the domain of both a Markov operator in $\mathcal{L}_1$ and a Markov operator in $\mathcal{L}_\infty$ can be extended to the space $\mathfrak{M}^+(X,\Sigma,m)$ of the (equivalence classes of m-almost equal) nonnegative extended real valued functions. Again, if we consider the extension of a Markov operator in $\mathcal{L}_1$, then the operator symbol is placed at the right of the function, and if we consider the extension of a Markov operator in $\mathcal{L}_\infty$, then the operator symbol is at the left of the function. We then also have

$$\int u(Pf)\,dm = \int (uP)f\,dm \quad \text{for all } u,f \in \mathfrak{M}^+ .$$

A special type of a Markov operator is the operator I_A which is defined by $uI_A = u\,1_A$ for all $u \in \mathcal{L}_1$. Then obviously $I_A f = 1_A f$ for all $f \in \mathcal{L}_\infty$.

Instead of saying P is a Markov operator in $\mathcal{L}_1$, $\mathcal{L}_\infty$ or an extension to $\mathfrak{M}^+(X,\Sigma,m)$, we shall simply say P is a Markov process on (X,Σ,m), and it will be clear from the context as which type of operator P is considered. This terminology is justified by the following interpretation.

For every $A \in \Sigma$, choose a representative $P(\cdot,A)$ for $P1_A$. Then for m-almost all $x \in X$, we have

i) $\quad 0 \le P(x,A) \le 1 \quad$ for all $A \in \Sigma$,

ii) $\quad P(x, \bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} P(x,A_n)$

for every sequence of disjoint sets in Σ.

Hence $P(\cdot,A)$ is "almost" a transition probability, and $P1_A(x)$ can be interpreted as the probability that we enter the set A in one transition from x. Similarly $PI_A P1_B(x)$ can be considered as the probability that from state x the process is after one transition in A, and after the second transition in B.

Our first aim in this note is to obtain a straightforward deduction of the decomposition of space X into a conservative part C and a dissipative part D. The usual way to treat this decomposition due to E. Hopf [3] is to consider P as an operator in $\mathcal{L}_1$, to deduct the maximal ergodic theorem and with the aid of this theorem to obtain a $\mathcal{L}_1$-characterisation of C and D, which then by dualisation can be translated into a $\mathcal{L}_\infty$-characterisation.

Despite the mathematical elegancy of this treatment, especially of Garcia's proof of Hopf's maximal ergodic theorem, there are two disadvantages. The first one is that the probabilistic interpretation of the maximal ergodic theorem is not obvious; the second one that in this way a rather weak description of the dissipative part is obtained. Therefore we prefer to go the other way round. We shall start with the $\mathcal{L}_\infty$-characterisation of C and D and then dualize this characterisation to $\mathcal{L}_1$. Our description of D can already be found in Feldman [1] but his proof is more probabilistic in its nature. The basic tool for our proof of the decomposition theorems will be lemma 1 in the next section. In the course of the proof of this lemma an operator Q will occur. This operator represents what is sometimes called the embedded or induced process. In the third section we shall study the relationship of the conservative parts of X with respect to the processes P and Q, and with the aid of the process Q give a somewhat more detailed description of the dissipative part of X with respect to P.

2. The conservative and dissipative part of a Markov process

Let P be a Markov process on (X,Σ,m). We shall prove the following theorem.

Theorem 1. There exist disjoint sets C and D with $X = C \cup D$ such that

i) for all $A \subset C$ we have $\sum_{n=0}^{\infty} P^n 1_A = \infty$ on A

ii) there exists a partition $D_1, D_2, \ldots$ of D such that

$$\sum_{n=0}^{\infty} P^n 1_{D_i} \in \mathcal{L}_\infty \quad \text{for } i = 1,2,\ldots .$$

The sets C and D are called the conservative part and the dissipative part of X with respect to P, and are mod m uniquely determined by the conditions i) and ii).

This theorem has the following interpretation in terms of recurrence. Since $P^n 1_A(x)$ is the expectation of a visit to the set A at time n, starting in x, condition i) says that for every subset A of the conservative part the expected number of visits to A, starting in A, is infinite, i.e. the strong recurrence property holds for every subset of the conservative part. On the dissipative part the situation is quite different. For every partition element D_i there exists an integer n_i such that the expected number of visits to

D_i is less than n_i, no matter where we start.
The characterisation of the conservative part and the dissipative part by means of the $\mathcal{L}_\infty$-operator as in theorem 1 is given by Feldman [1]. A dualisation of this theorem yields a characterisation by means of the $\mathcal{L}_1$-operator as given by Hopf [3], see also Foguel [2], II.2.3. We shall first give this dualisation and then prove theorem 1.
Let u be a nonnegative integrable function. Then we have for all $A \subset C$

$$\int_A (\sum_{n=0}^{\infty} uP^n)dm = \int u(\sum_{n=0}^{\infty} P^n 1_A)dm = \begin{cases} 0 \\ \infty \end{cases}, \text{ since } \sum_{n=0}^{\infty} P^n 1_A = \begin{cases} 0 \\ \infty \end{cases} \text{ on } X,$$

$$\int_{D_i} (\sum_{n=0}^{\infty} uP^n)dm = \int u(\sum_{n=0}^{\infty} P^n 1_{D_i})dm < \infty \quad \text{for } i = 1,2,\ldots .$$

Hence we obtain the following result.

Theorem 2. There exist disjoint sets C and D with $X = C \cup D$ such that for all nonnegative $u \in \mathcal{L}_1$ we have

$$\sum_{n=0}^{\infty} uP^n = \begin{cases} 0 \text{ or } \infty & \text{on } C \\ < \infty & \text{on } D . \end{cases}$$

The sets C and D are mod m uniquely determined by this condition.

The proof of theorem 1 rests on the next lemma.

Lemma 1. If $\sum_{n=1}^{\infty} P^n 1_A \le M$ on A, then $\sum_{n=1}^{\infty} P^n 1_A \le M+1$ on X.

Intuitively, this lemma is obvious: if the expected number of visits from A to A is at most M, then for any point of X after the first visit to A we expect at most M further visits to A, hence we expect over all at most M + 1 visits to A.

Proof of lemma 1. The formula

$$P^n 1_A = \sum_{k=1}^{n} (PI_{A^c})^{k-1} PI_A P^{n-k} 1_A$$

is easily verified by writing out, and simply says that the probability of reaching A in n transitions equals the sum of the probabilities that A is

is reached for the first time after k transitions, and next A is entered again after n - k transitions. Then we have

$$\sum_{n=1}^{\infty} P^n 1_A = \sum_{n=1}^{\infty} \sum_{k=1}^{n} (PI_{A^c})^{k-1} PI_A P^{n-k} 1_A =$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} (PI_{A^c})^k PI_A P^n 1_A$$

by rearranging terms, which is allowed since all terms are nonnegative. Now define for every $f \in \mathfrak{M}$

$$Qf = \sum_{k=0}^{\infty} (PI_{A^c})^k PI_A f \ .$$

Obviously Q is a σ-additive mapping of $\mathfrak{M}^+$ into itself. By writing out we easily verify

$$\sum_{k=0}^{n} (PI_{A^c})^k P 1_A + (PI_{A^c})^n P 1_{A^c} \leq 1 \ ,$$

and therefore

$$Q1 = \sum_{k=0}^{\infty} (PI_{A^c})^k P 1_A \leq 1 \ .$$

It follows that Q is a Markov process on (X,Σ,m), which satisfies $Qf = QI_A f$ for all $f \in \mathfrak{M}^+$. We obtain

$$\sum_{n=1}^{\infty} P^n 1_A = \sum_{n=0}^{\infty} QP^n 1_A = Q \sum_{n=1}^{\infty} P^n 1_A + Q1_A =$$

$$= Q(I_A \sum_{n=1}^{\infty} P^n 1_A) + Q1_A \leq$$

$$\leq M + 1 \ . \qquad \square$$

Proof of theorem 1. The uniqueness of C and D mod m is obvious. Consider the class

$$F = \{F \mid \sum_{n=1}^{\infty} P^n 1_F \in \mathcal{L}_\infty\} \ .$$

Since subsets of elements of F are again in F, and m is σ-finite, by an exhaustion procedure we can construct a sequence of disjoint sets $D_1, D_2, \ldots$ in F such that if $D = \bigcup_{i=1}^{\infty} D_i$ and $C = X \setminus D$, then all elements of F which are contained in C are m-null sets. The set D satisfies condition ii) of theorem 1; it remains to show that the set C satisfies condition i).
Let A be a subset of C, and put

$$A_k = \{ \sum_{n=1}^{\infty} P^n 1_A \leq k \} \cap A \quad \text{for all } k \in \mathbb{N} .$$

Then

$$\sum_{n=1}^{\infty} P^n 1_{A_k} \leq \sum_{n=1}^{\infty} P^n 1_A \leq k \text{ on } A_k ,$$

and by lemma 1

$$\sum_{n=1}^{\infty} P^n 1_{A_k} \leq k+1 \text{ on } X .$$

Since $A_k \subset C$, we have $m(A_k) = 0$ for every k, and therefore $\sum_{n=1}^{\infty} P^n 1_A = \infty$ m-almost everywhere on A. □

3. The embedded Markov process

Let P be a Markov process on (X, Σ, m). In the proof of lemma 1 the Markov process Q defined by

$$Qf = \sum_{k=0}^{\infty} (PI_{A^c})^k PI_A f$$

for a fixed set $A \in \Sigma$ appeared. For all $B \in \Sigma$ $Q1_B(x)$ can be interpreted as the probability that at the first visit under P to the set A we also are in B. This process Q is sometimes called the embedded or induced process.
Note that because of the property $Qf = QI_A f$ for all $f \in \mathfrak{M}^+$, we can restrict the process Q to the set A. In many cases in the literature the term "embedded process" is used for this restriction.
The next theorem and its corollaries show that there is a close relationship between recurrence properties under P and recurrence properties under Q.

Theorem 3. Let P be a Markov process on (X,Σ,m) and let Q be the embedded process of P with respect to some set $A \in \Sigma$. Then for all $B \in \Sigma$ we have

$$\sum_{n=1}^{\infty} Q^n 1_{A\cap B} = \sum_{n=1}^{\infty} P^n 1_{A\cap B} .$$

Proof. As in the proof of lemma 1, the formula

$$P^n 1_{A\cap B} = \sum_{k=1}^{n} (PI_{A^c})^{k-1} PI_A P^{n-k} 1_{A\cap B}$$

is easily verified by writing out. Hence we have

$$\sum_{n=1}^{\infty} P^n 1_{A\cap B} = \sum_{n=1}^{\infty} \sum_{k=1}^{n} (PI_{A^c})^{k-1} PI_A P^{n-k} 1_{A\cap B} =$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} (PI_{A^c})^{k} PI_A P^{n} 1_{A\cap B} =$$

$$= \sum_{n=0}^{\infty} QP^n 1_{A\cap B} =$$

$$= Q\Big(\sum_{n=1}^{\infty} P^n 1_{A\cap B}\Big) + Q1_{A\cap B} .$$

Therefore by iteration we obtain for every m

$$\sum_{n=1}^{\infty} P^n 1_{A\cap B} = Q^m\Big(\sum_{n=1}^{\infty} P^n 1_{A\cap B}\Big) + \sum_{n=1}^{m} Q^n 1_{A\cap B} \geq \sum_{n=1}^{m} Q^n 1_{A\cap B} ,$$

$$\sum_{n=1}^{\infty} P^n 1_{A\cap B} \geq \sum_{n=1}^{\infty} Q^n 1_{A\cap B} .$$

In order to show that the equality sign holds, we only have to show that for every $N \in \mathbb{N}$ we have

$$\sum_{n=1}^{N} Q^n 1_{A\cap B} \geq \sum_{n=1}^{N} P^n 1_{A\cap B} .$$

From the definition of Q we obtain

$$Q1_{A\cap B} = \sum_{k=0}^{\infty} (PI_{A^c})^{k} P1_{A\cap B} \geq P1_{A\cap B} .$$

Now suppose the formula has been proved for some N.
Then

$$\sum_{n=1}^{N+1} Q^n 1_{A\cap B} = Q\Big(\sum_{n=1}^{N} Q^n 1_{A\cap B} + 1_{A\cap B}\Big) \geq$$

$$\geq Q\Big(\sum_{n=1}^{N} P^n 1_{A\cap B} + 1_{A\cap B}\Big) =$$

$$= \sum_{k=0}^{\infty} (PI_{A^c})^k PI_A \sum_{n=0}^{N} P^n 1_{A\cap B} \geq$$

$$\geq \sum_{n=1}^{N+1} \sum_{k=1}^{n} (PI_{A^c})^{k-1} PI_A P^{n-k} 1_{A\cap B} =$$

$$= \sum_{n=1}^{N+1} P^n 1_{A\cap B} ,$$

hence the formula is also true for N + 1. This completes the proof of the theorem. □

Corollary 1. Let P and Q be as in theorem 3, and let C be the conservative part of X with respect to P. Then the conservative part of X with respect to Q is the set A ∩ C.

Proof. For every B ⊂ A ∩ C we have

$$\sum_{n=1}^{\infty} Q^n 1_B = \sum_{n=1}^{\infty} P^n 1_B = \infty \text{ on } B ,$$

hence A ∩ C belongs to the conservative part of X with respect to Q. Let $D_1, D_2, \ldots$ be the partition of the dissipative part D of X with respect to P as in theorem 1, then

$$\sum_{n=1}^{\infty} Q^n 1_{A\cap D_i} = \sum_{n=1}^{\infty} P^n 1_{A\cap D_i} \leq \sum_{n=1}^{\infty} P^n 1_{D_i} \in \mathcal{L}_\infty ,$$

hence A ∩ D belongs to the dissipative part of X with respect to Q. Finally, since $Q1_{A^c} = 0$, we have $\sum_{n=1}^{\infty} Q^n 1_{A^c} = 0$, and therefore also A^c belongs to the dissipative part of X with respect to Q. □

The next corollary says that if it is certain that almost all points of A return to A under P, then the expected number of visits to A must be infinite.

Corollary 2. If $Q1_A = 1$ on A, then $\sum_{n=1}^{\infty} P^n 1_A = \infty$ on A.

Proof. From $Qf = QI_A f$ for all $f \in \mathfrak{M}^+$ we conclude $Q^n 1_A = 1$ on A, and therefore

$$\sum_{n=1}^{\infty} P^n 1_A = \sum_{n=1}^{\infty} Q^n 1_A = \infty \text{ on } A .$$ □

Corollary 3. If $Q1_A \leq q < 1$ on A, then $\sum_{n=1}^{\infty} P^n 1_A \leq \frac{1}{1-q}$ on X.

Proof. From $Qf = QI_A f$ for all $f \in \mathfrak{M}^+$ we conclude $Q^n 1_A \leq q^n$ on A, hence

$$\sum_{n=1}^{\infty} P^n 1_A = \sum_{n=1}^{\infty} Q^n 1_A \leq \frac{q}{1-q} \text{ on } A .$$

Therefore by lemma 1

$$\sum_{n=1}^{\infty} P^n 1_A \leq \frac{q}{1-q} + 1 = \frac{1}{1-q} \text{ on } X .$$ □

This last corollary has as a consequence that any set A for which almost all points have a probability at most $q < 1$ of returning to A under P most belong to the dissipative part. In some sense the converse of this statement is also true:

Theorem 4. Let P be a Markov process on (X,Σ,m) with conservative and dissipative parts C and D respectively. Then for all $A \subset C$ we have

$$\sum_{k=0}^{\infty} (PI_{A^c})^k P 1_A = 1 \text{ on } A ,$$

and there exists a partition $D_1, D_2, \ldots$ of D and a sequence $q_1, q_2, \ldots$ with $0 \leq q_i < 1$ such that

$$\sum_{k=0}^{\infty} (PI_{D_i^c})^k P1_{D_i} \leq q_i \text{ on } D_i , \quad i = 1,2,\ldots .$$

Note that because of the corollaries 2 and 3 of theorem 3, this theorem is a slight strengthening of theorem 1. On the conservative part it is certain that for every subset A almost all points of A will return to A under P, while there exists a partition of the dissipative part such that for almost all points of a partition element the probability of returning to that partition element is uniformly less than 1.
In the proof of the theorem we need the following, in its interpretation obvious, lemma:

Lemma 2. If $A \subset B$, then

$$\sum_{k=0}^{\infty} (PI_{A^c})^k P1_A \leq \sum_{k=0}^{\infty} (PI_{B^c})^k P1_B .$$

Proof. For every N we have

$$\sum_{n=0}^{N} (PI_{A^c})^n P1_A + \sum_{n=0}^{N} (PI_{A^c})^n (1 - P1) + (PI_{A^c})^N P1_{A^c} = 1 ,$$

$$\sum_{n=0}^{N} (PI_{B^c})^n P1_B + \sum_{n=0}^{N} (PI_{B^c})^n (1 - P1) + (PI_{B^c})^N P1_{B^c} = 1 ,$$

which because of $1_{B^c} \leq 1_{A^c}$ implies

$$\sum_{n=0}^{N} (PI_{A^c})^n P1_A \leq \sum_{n=0}^{N} (PI_{B^c})^n P1_B ,$$

$$\sum_{n=0}^{\infty} (PI_{A^c})^n P1_A \leq \sum_{n=0}^{\infty} (PI_{B^c})^n P1_B . \qquad \square$$

Proof of theorem 4. Consider the class

$$\mathfrak{A} := \{A \mid \exists_{q<1} \sum_{n=0}^{\infty} (PI_{A^c})^n P1_A \leq q \text{ on } A\} .$$

Because of lemma 2 we have that all subsets of an element of $\mathcal{A}$ are in $\mathcal{A}$. Therefore we can construct by an exhaustion procedure a sequence $D_1, D_2, \ldots$ of disjoint elements of $\mathcal{A}$ such that for every element $A \in \mathcal{A}$ with $A \cap \bigcup_{i=1}^{\infty} D_i = \emptyset$ we have $m(A) = 0$. By corollary 3 of theorem 3 every set D_i belongs to the dissipative part of X, and it remains to show that every set A with $A \cap \bigcup_{i=1}^{\infty} D_i = \emptyset$ belongs to the conservative part. To this end it suffices to note that by lemma 2 and the construction of the sets D_i we have

$$\sum_{n=0}^{\infty} (PI_{A^c})^n P 1_A = 1 \text{ on } A ,$$

and therefore by corollary 2 $\sum_{n=1}^{\infty} P^n 1_A = \infty$ on A. □

Literature

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