The stabilizer of Dye's spread on a hyperbolic quadric in $PG(4n-1,2)$ within the orthogonal group

Geometria. — *The stabilizer of Dye's spread on a hyperbolic quadric in* PG $(4n-1, 2)$ *within the orthogonal group.* Nota (*) di

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RIASSUNTO. — Recentemente R. H. Dye ha costruito vibrazioni come indicato nel titolo. Egli ha determinato i loro stabilizzatori entro il gruppo ortogonale nei casi $n = 2, 3$. La presente nota riguarda il caso $n \geq 3$. Si fa uso della caratterizzazione di Holt di certi gruppi di permutazioni triplamente transitivi di grado $2^{2n-1} + 1$.

1. INTRODUCTION

The projective space PG $(4n-1, 2)$ is viewed in the usual way as the incidence structure of 1- and 2-dimensional subspaces of the vector space $\mathbf{F}_2^{4n}$. The hyperbolic quadric Ω will be fixed as the set of projective points X in PG $(4n-1, 2)$ whose homogenous coordinates $(X_1, X_2, \cdots, X_{4n})$ satisfy

$$q(X) = X_1 X_2 + X_3 X_4 + \cdots + X_{4n-1} X_{4n} = 0 .$$

The hyperbolic quadratic form q on PG $(4n-1, 2)$ admits a symplectic polarity that we shall denote by B. A spread on the quadric Ω is defined to be a partitioning $\mathcal{S} = \{S_1, \cdots, S_{2^{2n-1}+1}\}$ of Ω into $2^{2n-1}+1$ projective $(2n-1)$-dimensional totally isotropic subspaces of $(\mathrm{PG}(4n-1, 2), q)$.

2. CONSTRUCTION OF THE SPREAD

The following construction of a spread on Ω is to be found in [3]. Fix a nonisotropic point P and an isotropic point Q of $(\mathrm{PG}(4n-1, 2), q)$ such that $B(P, Q) \neq 0$. Then the projective space H underlying $P^{\perp} \cap Q^{\perp}$ is a PG $(4n-3, 2)$ with symplectic polarity B_0 induced by B. By means of scalar restriction from the Galois field $\mathbf{F}_{2^{2n-1}}$ to $\mathbf{F}_2$, the projective line PG $(1, 2^{2n-1})$ with nondegenerate symplectic polarity B_1 can be regarded as a PG$(4n-3, 2)$ with nondegenerate symplectic polarity $\mathrm{trace}_{\mathbf{F}_{2^{2n-1}} | \mathbf{F}_2} \circ B_1$. Thus (H, B_0) can be identified with $(\mathrm{PG}(1, 2^{2n-1}), \mathrm{trace}_{\mathbf{F}_{2^{2n-1}} | \mathbf{F}_2} \circ B_1)$ whenever the latter is viewed as a projective space over $\mathbf{F}_2$. Under this identification, the points of PG $(1, 2^{2n-1})$ correspond to totally isotropic $(2n-2)$-dimensional subspaces of (H, B_0) partitioning H. Next, H is mapped bijectively onto $P^{\perp} \cap \Omega$ by means of projection from P. Note that totally

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isotropic subspaces of (H, B_0) map into totally isotropic subspaces of $(P^\perp, q|_{P^\perp})$ inside Ω, so that the partitioning of (H, B_0) maps onto a partitioning of $P^\perp \cap \Omega$ into totally isotropic subspaces. In order to obtain a spread, note that each of these $(2n-2)$-dimensional subspaces should be extended to a maximal totally isotropic subspace of $(PG(4n-1, 2), q)$. It follows from [2] that this can be done in precisely two different ways such that no two subspaces intersect. The two resulting spreads on Ω are mapped into one another by the symmetry with center P. Moreover, the subspaces belonging to one of these two spreads are all in the same $\Omega_{4n}^+(2)$-orbit, where $\Omega_{4n}^+(2)$ stands for the commutator subgroup of the orthogonal group $O_{4n}^+(2)$ with respect to q. Hence, the spread is uniquely determined by the requirement that its elements are maximal totally isotropic subspaces from a fixed $\Omega_{4n}^+(2)$-orbit. The spread thus constructed will be denoted $\mathcal{P}$.

3. THE STABILIZER OF THE SPREAD

Let G denote the stabilizer of the spread $\mathcal{P}$ within $O_{4n}^+(2)$ and let G_R for R a point of $PG(4n-1, 2)$ stand for the subgroup of G fixing R. Since $P\Gamma L_2(2^{2n-1})$ is in a canonical way a group of automorphisms of $(PG(1, 2^{2n-1}), \mathrm{trace}_{F_{2^{2n-1}}|F_2} \circ B_1)$ and thus of (H, B_0), it can be embedded uniquely into G_P. This implies that G_P contains a subgroup K isomorphic to $P\Gamma L_2(2^{2n-1})$. The following lemma summarizes what is known about G from [3].

LEMMA. *(Let q, $\mathcal{P}$, K and G be as above)*

(i) K *acts on* $\mathcal{P}$ *as* $P\Gamma L_2(2^{2n-1})$ *acts on* $PG(1, 2^{2n-1})$;

(ii) $G_P = K \cong P\Gamma L_2(2^{2n-1})$; G_P *has three orbits on the set of nonisotropic points of* $(PG(4n-1, 2), q)$ *with cardinalities* $1, 2^{4n-2}-1, 2^{2n-1}(2^{2n-1}-1)$;

(iii) *If* $n = 2$, *then* $G \cong \mathrm{Alt}(9)$;

(iv) *If* $n = 3$, *then* $G = G_P \cong P\Gamma L_2(2^5)$.

The proof of (ii) can be found on page 191 in [3] in an argument that is valid in the present situation (though not explicity stated).

Statement (iv) is demonstrated by use of specific knowledge of the subgroups of $Sp_6(2)$.

The theorem which we aim to prove, shows that (iv) is representative for what happens for $n \geq 3$.

THEOREM. *Let $n \geq 3$. Suppose* P *is a nonisotropic point and* Q *an isotropic point of a nondegenerate hyperbolic space* $(PG(4n-1, 2), q)$ *such that* P + Q *is a hyperbolic line. Let* $\mathcal{P}$ *be the spread constructed in* 2 *departing from* P *and* Q, *and let* G *be as defined in* 3. *Then* $G = G_P \cong P\Gamma L_2(2^{2n-1})$.

4. Proof of the theorem

We proceed in four steps.

(4.1) G *does not possess a normal subgroup which is regular on the set of nonisotropic points of* $(\mathrm{PG}(4n-1, 2), q)$.

Proof. Suppose N is a counterexample. Then G_P acts on N by conjugation as it does on the nonisotropic points. In particular N has two G_P-orbits distinct from $\{1\}$. Let p and q denote the orders of representatives from these two orbits. Then by Cauchy's lemma N has order $p^a q^b$ for $a, b \in \mathbf{N}$; moreover p and q are prime numbers. On the other hand, the regularity of N implies that its order is $2^{2n-1}(2^{2n}-1)$. The comparison of these two expressions for $|N|$ yields that $2^{2n}-1$ is a prime power, which is absurd. □

(4.2) *If* N *is a nontrivial normal subgroup of* G, *then* $[G:N]=[G_P:N_P]$ *is a divisor of* $2n-1$.

Proof. If $G = G_P$, the statement concerns $G \cong P\Gamma L_2(2^{2n-1})$ and is known to hold. So we may assume $G > G_P$ for the rest of the proof. In view of the orbit structure of G_P described in (ii) of the lemma, this means that G is primitive on the set of nonisotropic points. So any nontrivial normal subgroup N of G is transitive on these $2^{2n-1}(2^{2n}-1)$ points, so $[G:N]=[G_P:N_P]$. Moreover N_P is normal in $G_P \cong P\Gamma L_2(2^{2n-1})$, whence $N_P = 1$ or we are through. The former possibility, however, is excluded by (4.1). □

(4.3) *The permutation representation of* G *on* $\mathscr{P}$ *is faithful.*

Proof. Let N be the kernel of this representation. If N is nontrivial, then $[G:N]=[G_P:N_P]$ by (4.2); but (i) of the lemma states that $N_P = 1$, whence $[G:N]=|G_P|$, contradicting (4.2). The conclusion is that N is trivial. □

(4.4) *If* $n \geq 3$, *then* $G = G_P$.

Proof. By (4.3) the group G can be regarded as a triply transitive permutation group of degree $2^{2n-1}+1$. Application of a theorem by Holt [4] yields that G contains a normal subgroup N isomorphic to either $\mathrm{Sym}(2^{2n-1}+1)$, $\mathrm{Alt}(2^{2n-1}+1)$ or $PSL_2(2^{2n-1})$. Comparing orders with $|G|$, we obtain that N is an isomorph of $PSL_2(2^{2n-1})$. From (4.2) it follows that $G = G_P$. □

Remarks. For $n = 2$, the arguments of the proof are equally valid. They result in: $G \cong P\Gamma L_2(2^{2n-1})$ or $G \cong \mathrm{Alt}(9)$. Together with the observation that all spreads are in a single $O_{4n}^{+}(2)$-orbit, this reestablishes (iii) of the lemma.

De Clerck, Dye and Thas [1] have shown that any spread leads to a partial geometry with parameters $(s, t, \alpha) = (2^{2n-1} - 1, 2^{2n-1}, 2^{2n-2})$ on the nonisotropic points of PG $(4n - 1, q)$. Using the above theorem, it is not hard to see that G is the part of the automorphism group of the partial geometry derived from $\mathscr{P}$ that is contained in $O_{4n}^{+}(2)$.

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