

# The stabilizer of Dye's spread on a hyperbolic quadric in $PG(4n-1,2)$ within the orthogonal group

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**Geometria.** — *The stabilizer of Dye's spread on a hyperbolic quadric in  $PG(4n-1, 2)$  within the orthogonal group.* Nota (\*) di A.M. COHEN e H.A. WILBRINK, presentata dal Socio G. ZAPPA.

RIASSUNTO. — Recentemente R.H. Dye ha costruito vibrazioni come indicato nel titolo. Egli ha determinato i loro stabilizzatori entro il gruppo ortogonale nei casi  $n = 2, 3$ . La presente nota riguarda il caso  $n \geq 3$ . Si fa uso della caratterizzazione di Holt di certi gruppi di permutazioni triplamente transitivi di grado  $2^{2n-1} + 1$ .

## 1. INTRODUCTION

The projective space  $PG(4n-1, 2)$  is viewed in the usual way as the incidence structure of 1- and 2-dimensional subspaces of the vector space  $\mathbf{F}_2^{4n}$ . The hyperbolic quadric  $\Omega$  will be fixed as the set of projective points  $X$  in  $PG(4n-1, 2)$  whose homogenous coordinates  $(X_1, X_2, \dots, X_{4n})$  satisfy

$$q(X) = X_1 X_2 + X_3 X_4 + \dots + X_{4n-1} X_{4n} = 0.$$

The hyperbolic quadratic form  $q$  on  $PG(4n-1, 2)$  admits a symplectic polarity that we shall denote by  $B$ . A spread on the quadric  $\Omega$  is defined to be a partitioning  $\mathcal{S} = \{S_1, \dots, S_{2^{2n-1}+1}\}$  of  $\Omega$  into  $2^{2n-1} + 1$  projective  $(2n-1)$ -dimensional totally isotropic subspaces of  $(PG(4n-1, 2), q)$ .

## 2. CONSTRUCTION OF THE SPREAD

The following construction of a spread on  $\Omega$  is to be found in [3]. Fix a nonisotropic point  $P$  and an isotropic point  $Q$  of  $(PG(4n-1, 2), q)$  such that  $B(P, Q) \neq 0$ . Then the projective space  $H$  underlying  $P^1 \cap Q^1$  is a  $PG(4n-3, 2)$  with symplectic polarity  $B_0$  induced by  $B$ . By means of scalar restriction from the Galois field  $\mathbf{F}_{2^{2n-1}}$  to  $\mathbf{F}_2$ , the projective line  $PG(1, 2^{2n-1})$  with nondegenerate symplectic polarity  $B_1$  can be regarded as a  $PG(4n-3, 2)$  with nondegenerate symplectic polarity  $\text{trace}_{\mathbf{F}_2^{2n-1}|\mathbf{F}_2} \circ B_1$ . Thus  $(H, B_0)$  can be identified with  $(PG(1, 2^{2n-1}), \text{trace}_{\mathbf{F}_2^{2n-1}|\mathbf{F}_2} \circ B_1)$  whenever the latter is viewed as a projective space over  $\mathbf{F}_2$ . Under this identification, the points of  $PG(1, 2^{2n-1})$  correspond to totally isotropic  $(2n-2)$ -dimensional subspaces of  $(H, B_0)$  partitioning  $H$ . Next,  $H$  is mapped bijectively onto  $P^1 \cap \Omega$  by means of projection from  $P$ . Note that totally

(\*) Pervenuta all'Accademia il 7 luglio 1980.

isotropic subspaces of  $(H, B_0)$  map into totally isotropic subspaces of  $(P^1, q|_{P^1})$  inside  $\Omega$ , so that the partitioning of  $(H, B_0)$  maps onto a partitioning of  $P^1 \cap \Omega$  into totally isotropic subspaces. In order to obtain a spread, note that each of these  $(2n-2)$ -dimensional subspaces should be extended to a maximal totally isotropic subspace of  $(PG(4n-1, 2), q)$ . It follows from [2] that this can be done in precisely two different ways such that no two subspaces intersect. The two resulting spreads on  $\Omega$  are mapped into one another by the symmetry with center  $P$ . Moreover, the subspaces belonging to one of these two spreads are all in the same  $\Omega_{4n}^+(2)$ -orbit, where  $\Omega_{4n}^+(2)$  stands for the commutator subgroup of the orthogonal group  $O_{4n}^+(2)$  with respect to  $q$ . Hence, the spread is uniquely determined by the requirement that its elements are maximal totally isotropic subspaces from a fixed  $\Omega_{4n}^+(2)$ -orbit. The spread thus constructed will be denoted  $\mathcal{P}$ .

### 3. THE STABILIZER OF THE SPREAD

Let  $G$  denote the stabilizer of the spread  $\mathcal{P}$  within  $O_{4n}^+(2)$  and let  $G_R$  for  $R$  a point of  $PG(4n-1, 2)$  stand for the subgroup of  $G$  fixing  $R$ . Since  $P\Gamma_2(2^{2n-1})$  is in a canonical way a group of automorphisms of  $(PG(1, 2^{2n-1}), \text{trace}_{\mathbb{F}_2^{2n-1}|\mathbb{F}_2} \circ B_1)$  and thus of  $(H, B_0)$ , it can be embedded uniquely into  $G_P$ . This implies that  $G_P$  contains a subgroup  $K$  isomorphic to  $P\Gamma_2(2^{2n-1})$ . The following lemma summarizes what is known about  $G$  from [3].

LEMMA. (Let  $q, \mathcal{P}, K$  and  $G$  be as above)

- (i)  $K$  acts on  $\mathcal{P}$  as  $P\Gamma_2(2^{2n-1})$  acts on  $PG(1, 2^{2n-1})$ ;
- (ii)  $G_P = K \cong P\Gamma_2(2^{2n-1})$ ;  $G_P$  has three orbits on the set of nonisotropic points of  $(PG(4n-1, 2), q)$  with cardinalities  $1, 2^{4n-2} - 1, 2^{2n-1}(2^{2n-1} - 1)$ ;
- (iii) If  $n = 2$ , then  $G \cong \text{Alt}(9)$ ;
- (iv) If  $n = 3$ , then  $G = G_P \cong P\Gamma_2(2^5)$ .

The proof of (ii) can be found on page 191 in [3] in an argument that is valid in the present situation (though not explicitly stated).

Statement (iv) is demonstrated by use of specific knowledge of the subgroups of  $Sp_6(2)$ .

The theorem which we aim to prove, shows that (iv) is representative for what happens for  $n \geq 3$ .

THEOREM. Let  $n \geq 3$ . Suppose  $P$  is a nonisotropic point and  $Q$  an isotropic point of a nondegenerate hyperbolic space  $(PG(4n-1, 2), q)$  such that  $P + Q$  is a hyperbolic line. Let  $\mathcal{P}$  be the spread constructed in 2 departing from  $P$  and  $Q$ , and let  $G$  be as defined in 3. Then  $G = G_P \cong P\Gamma_2(2^{2n-1})$ .

## 4. PROOF OF THE THEOREM

We proceed in four steps.

(4.1) *G does not possess a normal subgroup which is regular on the set of nonisotropic points of  $(PG(4n-1, 2), q)$ .*

*Proof.* Suppose  $N$  is a counterexample. Then  $G_P$  acts on  $N$  by conjugation as it does on the nonisotropic points. In particular  $N$  has two  $G_P$ -orbits distinct from  $\{1\}$ . Let  $p$  and  $q$  denote the orders of representatives from these two orbits. Then by Cauchy's lemma  $N$  has order  $p^a q^b$  for  $a, b \in \mathbb{N}$ ; moreover  $p$  and  $q$  are prime numbers. On the other hand, the regularity of  $N$  implies that its order is  $2^{2n-1}(2^{2n}-1)$ . The comparison of these two expressions for  $|N|$  yields that  $2^{2n}-1$  is a prime power, which is absurd.  $\square$

(4.2) *If  $N$  is a nontrivial normal subgroup of  $G$ , then  $[G:N] = [G_P:N_P]$  is a divisor of  $2n-1$ .*

*Proof.* If  $G = G_P$ , the statement concerns  $G \cong P\Gamma L_2(2^{2n-1})$  and is known to hold. So we may assume  $G > G_P$  for the rest of the proof. In view of the orbit structure of  $G_P$  described in (ii) of the lemma, this means that  $G$  is primitive on the set of nonisotropic points. So any nontrivial normal subgroup  $N$  of  $G$  is transitive on these  $2^{2n-1}(2^{2n}-1)$  points, so  $[G:N] = [G_P:N_P]$ . Moreover  $N_P$  is normal in  $G_P \cong P\Gamma L_2(2^{2n-1})$ , whence  $N_P = 1$  or we are through. The former possibility, however, is excluded by (4.1).  $\square$

(4.3) *The permutation representation of  $G$  on  $\mathcal{P}$  is faithful.*

*Proof.* Let  $N$  be the kernel of this representation. If  $N$  is nontrivial, then  $[G:N] = [G_P:N_P]$  by (4.2); but (i) of the lemma states that  $N_P = 1$ , whence  $[G:N] = |G_P|$ , contradicting (4.2). The conclusion is that  $N$  is trivial.  $\square$

(4.4) *If  $n \geq 3$ , then  $G = G_P$ .*

*Proof.* By (4.3) the group  $G$  can be regarded as a triply transitive permutation group of degree  $2^{2n-1} + 1$ . Application of a theorem by Holt [4] yields that  $G$  contains a normal subgroup  $N$  isomorphic to either  $\text{Sym}(2^{2n-1} + 1)$ ,  $\text{Alt}(2^{2n-1} + 1)$  or  $\text{PSL}_2(2^{2n-1})$ . Comparing orders with  $|G|$ , we obtain that  $N$  is an isomorph of  $\text{PSL}_2(2^{2n-1})$ . From (4.2) it follows that  $G = G_P$ .  $\square$

*Remarks.* For  $n = 2$ , the arguments of the proof are equally valid. They result in:  $G \cong P\Gamma L_2(2^{2n-1})$  or  $G \cong \text{Alt}(9)$ . Together with the observation that all spreads are in a single  $O_{4n}^+(2)$ -orbit, this reestablishes (iii) of the lemma.

De Clerck, Dye and Thas [1] have shown that any spread leads to a partial geometry with parameters  $(s, t, \alpha) = (2^{2n-1} - 1, 2^{2n-1}, 2^{2n-2})$  on the nonisotropic points of  $\text{PG}(4n-1, q)$ . Using the above theorem, it is not hard to see that  $G$  is the part of the automorphism group of the partial geometry derived from  $\mathcal{P}$  that is contained in  $O_{4n}^+(2)$ .

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