Characterization theorems in finite geometry

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CHARACTERIZATION THEOREMS IN FINITE GEOMETRY

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PROEFSCHRIFT

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PROMOTOR: Prof. dr. J.H. van Lint

CO-PROMOTOR: Prof. dr. J.J. Seidel

PREFACE

Apart from an introductory chapter this thesis consists of the following five papers.

Nearaffine planes, Geom. Dedicata 12 (1), 53-62.

Finite Minkowski planes, Geom. Dedicata 12 (2), 119-129.

Two-transitive Minkowski planes, Geom. Dedicata 12 (4), 383- 395.

A characterization of the classical unitals, in: Finite geometries, N.L. Johnson, M.J. Kallaher & C.T. Long eds., Marcel Dekker, Lecture notes in pure and applied mathematics 82, New York, 1983.

A characterization of two classes of semi-partial geometries by their parameters, to appear in Simon Stevin.

This last paper was written together with Andries Brouwer. The way we worked together on this paper makes it impossible for me to decide what part of the paper is his and what part is mine.

I would like to express my gratitude to the publishers D. Reidel of Geometriae Dedicata, Marcel Dekker of Finite geometries and J.A. Thas of Simon Stevin for their permission to include these papers in this thesis. I would also like to thank my thesis supervisors Prof. dr. J.H. van Lint and Prof. dr. J.J. Seidel for introducing me to combinatorcs and finite geometry, and for more or less forcing me to write this thesis (I still wonder how they did it). Finally, I have to thank the Mathematical Centre and in particular Andries Brouwer and Arjeh Cohen, for their support and interest in my work during the four fine years I spent there in which period all five papers were written.

CONTENTS

INTRODUCTION

It is the purpose of this first chapter to introduce the nonexpert mathematician to some of the results and techniques from finite geometry in general, and to each of the five papers which constitute the main part of this thesis, in particular. In each of these five papers a characterization of a finite "incidence structure" is given. However, if one wants to fully understand and appreciate a characterization of any object, it is first necessary to get acquainted with the most basic properties of that object. This is what we shall try to achieve here for the objects discussed in the papers. In addition to this we shall take the opportunity to say something about other theorems characterizing geometries of which ours can be viewed as low dimensional cases.

Basically, characterization theorems in finite geometry fall into four classes. First of all there are the purely geometric characterizations such as the theorem of Veblen and Young characterizing the projective spaces (see section 2), or the Buekenhout-Shult theorem on polar spaces (see section 3). Secondly, there are theorems which use some kind of assumption on the automorphism group of the object in question. The Ostrom-Wagner theorem which we shall discuss in section 1, is a good example of this. Thirdly, there are the characterizations with the help of a combinatorial property as is the case, for example, in the Dembowski-Wagner theorem which we shall prove in section 2. Finally, it is sometimes possible to characterize geometries if one knows that they are embedded in another geometry (see for example the theorem by Buekenhout-Lefèvre in [6]).

Before we start our discussion a word of warning: the geometries that we shall consider are always assumed to be finite (although for some of the results that we shall state this is really not essential).

1. PROJECTIVE PLANES AND AFFINE PLANES

Perhaps the most extensively studied objects in finite geometry are the projective planes. There are several ways to give a definition of a projective plane. Here we shall adopt one which excludes the degenerate cases and which is easy to generalize to a definition for projective spaces of arbitrary dimension.

DEFINITION. Let X be a set of *points* and $\mathcal{L}$ a collection of distinguished subsets of X called *lines*. Then $(X,\mathcal{L})$ is called a *projective plane* if $|\mathcal{L}| \geq 2$ and the following axioms are satisfied :

(P1) If x and y are distinct points, then there is a unique line $L = xy$ such that $x,y \in L$;

(P2) If L_1 and L_2 are distinct lines, then they meet in a unique point;

(P3) Every line contains at least 3 points.

The classical models of projective planes are obtained as follows. Let V be a 3-dimensional vector space over $\mathbb{F}_q$, the field of q elements. For X take the set of all 1-dimensional subspaces of V and for $\mathcal{L}$ the set of all 2-dimensional subspaces of V (more precisely, since we have defined lines to be subsets of X, a line is not a 2-dimensional subspace but the set of all 1-dimensional subspaces contained in a 2-dimensional subspace). It is easy to check that now (P1),(P2) and (P3) are satisfied. Indeed, two distinct 1-spaces span a unique 2-space; two distinct 2-spaces in a 3-space meet nontrivially and a 2-space over $\mathbb{F}_q$ contains $(q^2-1)/(q-1)=q+1\geq 3$ 1-spaces. The question we are interested in is: are these the only examples of projective planes ? The answer is no. In fact so many different kinds of projective planes are known (see e.g. [8]) that a complete classification seems hopeless. Here we shall content ourselves with one example of a class of projective planes which cannot be obtained from a 3-dimensional vector space. To describe these planes it will be more convenient to work with affine planes. By definition an *affine plane* is an incidence structure of points and lines satisfying (P1) and

(A2) For every point x and line L such that $x \notin L$, there is exactly one line through x which does not meet L;

(A3) There exist three noncollinear points.

It is easy to establish the well-known correspondence between affine planes and projective planes: deleting a line L_∞ from a projective plane gives an affine plane and conversely every affine plane can be extended to a projective plane by adding a line "at infinity". If we follow this procedure for the projective plane associated with the 3-dimensional vector space $(\mathbb{F}_q)^3$ and with L_∞ defined by $z=0$, say, then every point not on L_∞ has a unique representation $\langle(x,y,1)\rangle$ and can therefore be identified with $(x,y) \in (\mathbb{F}_q)^2$. This gives us the familiar affine planes with point set $(\mathbb{F}_q)^2$ and with the lines given by an equation $y=ax+b$ or $x=c$. Now it is possible in the above construction to replace the field $\mathbb{F}_q$ by other algebraic structures. For example a quasifield will do as well. Here, a (finite) *quasifield* is a set Q with two binary operations, $+$ and $\cdot$ say, such that

1) $(Q,+)$ is a group with identity 0,
2) $(Q\setminus\{0\},\cdot)$ is a loop with identity 1,
3) $x\cdot(y+z) = x\cdot y + x\cdot z$ for all $x,y,z \in Q$,
4) $0\cdot x = 0$ for all $x \in Q$.

It is not hard to show that every quasifield Q yields an affine plane with point set Q^2 and lines given by an equation $y=a\cdot x+b$ or $x=c$. We shall describe a class of quasifields known as the *André quasifields*. For the set Q take $\mathbb{F}_{q^n}$ (as a set) and define addition in Q as in $\mathbb{F}_{q^n}$. Let A be the group of field automorphisms of $\mathbb{F}_{q^n}$ fixing the subfield $\mathbb{F}_q$ of $\mathbb{F}_{q^n}$ elementwise, and let $N: \mathbb{F}^*_{q^n} \to \mathbb{F}^*_q$ be the norm map defined by

$$N(x) = \prod_{\alpha\in A} x^\alpha, \; x \in \mathbb{F}^*_{q^n} .$$

If μ is any map from $\mathbb{F}^*_q$ into A with $\mu(1)=1$, then we can define a multiplication $\cdot$ on Q to make Q into a quasifield as follows

$$x\cdot y = xy^{\mu(N(x))} \quad (x,y \in Q),$$

where on the RHS multiplication is in $\mathbb{F}_{q^n}$ of course.

We shall now give some of the properties which characterize the projective planes associated with a 3-dimensional vector space. The first one is probably the best known.

<u>THEOREM 1</u>. *A projective plane is isomorphic to a projective plane associated with a 3-dimensional vector space if and only if the following condition holds:*

(Desargues' theorem) If a_1, a_2, a_3 *and* b_1, b_2, b_3 *are two triangles such that the lines* a_1b_1, a_2b_2 *and* a_3b_3 *are concurrent, then the points* $a_1a_2 \cap b_1b_2$, $a_1a_3 \cap b_1b_3$ *and* $a_2a_3 \cap b_2b_3$ *are collinear (see Figure 1).*

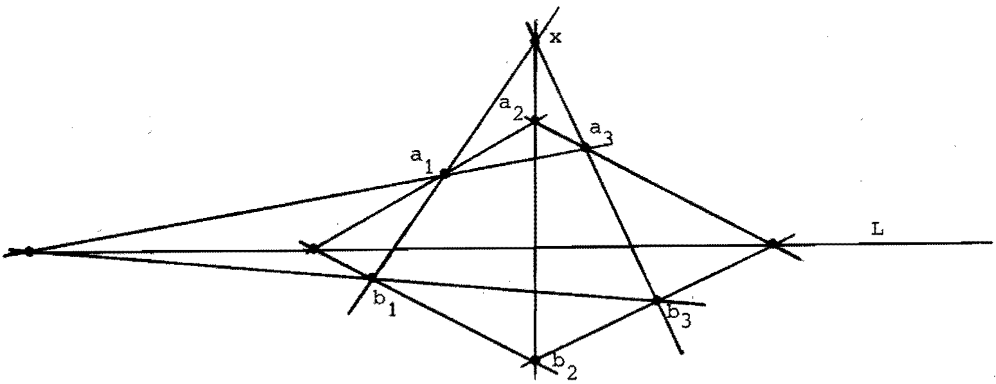


Figure 1.

We shall only indicate how Theorem 1 can be proved (for details see e.g. [10] or [16]). The basic idea behind the proof of Theorem 1 is that Desargues' theorem is equivalent to the existence of certain automorphisms of the projective plane (an *automorphism* of a projective plane is a permutation of the points which induces a permutation of the lines). For example, consider Figure 1 and suppose σ is an automorphism fixing x and all the points on L. Clearly, since every line through x intersects L, all lines through x are also fixed. If σ maps a_1 to b_1, then apparently a_2 is mapped onto b_2 and a_3 is mapped onto b_3; in fact we can determine the image of any point. It is easy to see that Desargues' theorem is equivalent to the existence of this type of automorphisms. Now we have already an algebraic structure associated with our projective plane, namely the group generated by these automorphisms. The special properties of these automorphisms allow us to reconstruct a field F and a 3-dimensional vector space V over F from this group in such a way that the projective plane we started with is isomorphic to the projective plane associated with V. A projective plane in which Desargues' theorem holds is called a *Desarguesian* projective plane.

Let us now look at a typical group theoretic characterization of the Desarguesian projective planes.

THEOREM 2. (Ostrom & Wagner [11]) *Let* $P = (X,\mathcal{L})$ *be a projective plane. If the automorphism group* Γ *of* P *is 2-transitve on* X, *then* P *is a Desarguesian projective plane.*

Here, 2-transitivity means that for all $x_1,x_2,y_1,y_2 \in X$, $x_1 \neq x_2$, $y_1 \neq y_2$, there is a $\gamma \in \Gamma$ such that $x_i^{\gamma}=y_i$, i=1,2. Again we only explain the main ideas of the proof. The trick here is to look at *involutions*, i.e., automorphisms of order 2. By the 2-transitivity, the even number $|X|(|X|-1)$ divides the order of Γ so there exist elements of order 2 in Γ (notice that finiteness is really essential here). Let σ be an involution. If $x \in X$ and if x is nonfixed, i.e., if $x^{\sigma} \neq x$, then the line xx^{σ} is fixed for $(xx^{\sigma})^{\sigma}=x^{\sigma}x^{\sigma^2}=x^{\sigma}x$. Dually, if L is a nonfixed line, then $L \cap L^{\sigma}$ is a fixed point. From these considerations it follows that the configuration of fixed points and lines of σ is either

a) a subplane, or

b) σ fixes all points on a line L and all lines through a point x.

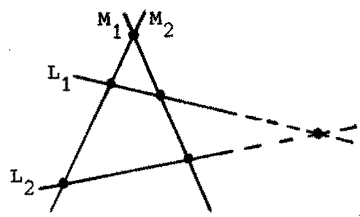
The easy part of the proof is case b), since here σ is one of the automorphisms whose existence is equivalent to Desargues' theorem (the only problem here is to show that there are sufficiently many of these automorphisms). The hard part is case a). Suffice it to say that here an induction argument can be used to finish the proof.

We shall see later on that this technique of looking at involutions can also be used to characterize the 2-transitive Minkowski planes.

2. PROJECTIVE SPACES

Let V be a vector space of arbitrary dimension. Again we shall use the projective terminology and call the 1-dimensional subspaces *points* and the 2-dimensional subspaces *lines*. Clearly, the points and lines satisfy the axioms (P1) and (P3) of the previous section but (P2) is only satisfied for those lines L_1 and L_2 which are contained in a plane (a 3-dimensional subspace). In terms of points and lines only, this is expressed in (P4).

(P4) (Pasch's axiom) If M_1 and M_2 are lines meeting in a point x and L_1 and L_2 are lines both meeting M_1 and M_2 not in x, then L_1 and L_2 meet.



DEFINITION. Let X be a set of *points* and $\mathcal{L}$ a collection of distinguished subsets of X called *lines*. Then $(X,\mathcal{L})$ is called a *projective space* if (P1), (P3) and (P4) are satisfied.

Clearly, every projective plane is a projective space. The following theorem, due to Veblen & Young, shows that for higher dimensions there is no analogue to the "nondesarguesian" planes.

THEOREM 3. *Let* $(X,\mathcal{L})$ *be a projective space containing two nonintersecting lines. Then* $(X,\mathcal{L})$ *is isomorphic to the geometry of 1- and 2- dimensional subspaces of a vector space.*

We explain the main steps in the proof of this theorem. Let $(X,\mathcal{L})$ be a projective space. A subset $Y \subset X$ is called a *subspace* if every line which meets Y in at least two points, is completely contained in Y. Clearly, every subspace together with the lines it contains is also a projective space. It is also easy to prove that if Y is any subspace and x is any point not contained in Y, then the set Z of all points on lines through x which meet Y (i.e., $Z = \bigcup_{y \in Y} xy$) is also a subspace. If we take for Y a line, the resulting Z is easily seen to be a projective plane. Now look at Figure 1, not as a configuration in the plane but with x not in the plane generated by a_1, a_2 and a_3, say. The points $a_ia_j \cap b_ib_j$, $1 \leq i < j \leq 3$, are all on the intersection line L of the planes generated by a_1, a_2, a_3 and b_1, b_2, b_3, so Desargues' theorem holds in this case. In fact, Desargues' theorem holds in all cases for, if x, a_1, a_2, a_3 happen to be in a plane, we can always view the configuration as the projection of a nonplanar configuration from a point onto the plane generated by x, a_1, a_2, a_3. By Theorem 1 we now know already that all projective planes which are properly contained in a projective space are isomorphic to a projective plane associated with a

3-dimensional vector space (this result is an example of a characterization using an embeddability property). The rest of the proof consists in glueing together these 3-dimensional vector spaces to one big vector space (see e.g. [3],[12] or [16]for more details).

As a typical application of Theorem 3 we shall prove the Dembowski-Wagner theorem which is a combinatorial characterization of projective spaces in terms of points and hyperplanes. For this we need some terminology which will also be useful later on. A t-*design* with parameters v, k, λ (or a t-(v,k,λ) *design*) is a pair $(X,\mathcal{B})$ where $\mathcal{B}$ is a collection of k-subsets (called *blocks*) of a set X of v *points* such that every t-subset of X is contained in exactly λ blocks. For any two points x and y in a 2-design we define the *line* through x and y as the intersection of the blocks containing x and y. Notice that every two distinct points in a 2-design are on a unique line. For example, let V be a vector space of dimension n over $\mathbb{F}_q$, X the set of all points of the projective space associated with V and let $\mathcal{B}$ be the set of all hyperplanes of V. Then $(X,\mathcal{B})$ is a $2\text{-}\left(\frac{q^n-1}{q-1},\frac{q^{n-1}-1}{q-1},\frac{q^{n-2}-1}{q-1}\right)$ design and the lines in the 2-design sense are precisely the lines in the projective space sense. This design has the property that the total number of blocks is equal to the total number of points. A 2-design with this property is called *symmetric* or *projective*.

THEOREM 4. (Dembowski-Wagner) *Let* $(X,\mathcal{B})$ *be a symmetric* 2-(v,k,λ)*design. Then* $(X,\mathcal{B})$ *is the design of points and hyperplanes of a projective space if and only if every line has at least* $(v-\lambda)/(k-\lambda)$ *points.*

PROOF. Since $|\mathcal{B}|=v$, every point is on k blocks. Let L be any line. Since L is contained in λ blocks, every point x on L is on $k-\lambda$ blocks B such that $L \cap B = \{x\}$. Therefore $v-(\lambda+|L|(k-\lambda))$ blocks do not meet L. From our hypothesis it follows that $|L| = (v-\lambda)/(k-\lambda)$ and that every line meets every block. Let x be any point not on L and suppose that ρ blocks contain L and x. Then $k-\rho$ blocks contain x but not L. This number also equals $|L|(\lambda-\rho)$ (for each $y \in L$ there are $\lambda-\rho$ blocks B on x and y such that $L \cap B = \{y\}$). Therefore $k-\rho = |L|(\lambda-\rho)$ and so ρ is a constant. Define *planes* as the intersection of all blocks containing three noncollinear points. Any three noncollinear points now determine a unique plane. Let L and M be two distinct lines in a plane E. Let B be a block containing L but not M. Then $L = B \cap E$, so $L \cap M = (B \cap E) \cap M = B \cap (E \cap M) = B \cap M \neq \emptyset$, i.e. any two lines in a plane meet. This proves Pasch's axiom.

3. SYMPLECTIC, UNITARY AND ORTHOGONAL GEOMETRY

We shall now turn to certain substructures of projective spaces for which there is a characterization quite similar to the characterization of Veblen & Young for projective spaces. Let us start with an analytic description of these substructures. Suppose V is a vector space of dimension n over $\mathbb{F}_q$ and let σ be an automorphism of $\mathbb{F}_q$. We shall often write $\overline{\lambda} = \lambda^\sigma$ for $\lambda \in \mathbb{F}_q$. A (σ-*sesquilinear*) *form* f on V is a map $f: V \times V \to \mathbb{F}_q$ satifying

i) $f(\lambda x,y) = \overline{\lambda} f(x,y)$ and $f(x,\lambda y) = \lambda f(x,y)$, $x,y \in V$, $\lambda \in \mathbb{F}_q$;

ii) $f(x,y+z) = f(x,y) + f(x,z)$ and $f(x+y,z) = f(x,z) + f(y,z)$, $x,y,z \in V$.

The form f is called *reflexive* if for all $x,y \in V$, $f(x,y) = 0 \Leftrightarrow f(y,x) = 0$ and f is called *nondegenerate* if $f(x,y) = 0$ for all $x \in V \Rightarrow y = 0$. If $n \geq 2$ and f is a nondegenerate reflexive form on V, then there are only a few possibilities for f (see e.g. [2]) :

i) $\sigma = 1$ and $f(x,x) = 0$ for all $x \in V$.
In this case f is called a *symplectic* form and it is possible to show that n has to be even and that w.r.t. to a suitable basis $v_1, v_2, \cdots, v_n$ of V,

$$f(x,y) = \xi_1\eta_2 - \xi_2\eta_1 + \xi_3\eta_4 - \xi_4\eta_3 + \cdots + \xi_{n-1}\eta_n - \xi_n\eta_{n-1},\ x=\Sigma\xi_i v_i, y=\Sigma\eta_i v_i.$$

ii) $\sigma^2 = 1$, $\sigma \neq 1$ and for some $\lambda_0 \in \mathbb{F}_q$, $\lambda_0 f(x,y) = \overline{\lambda_0 f(y,x)}$ for all $x,y \in V$.
In this case $\lambda_0 f$ is called *hermitian* and w.r.t. a suitable basis $v_1, v_2, \cdots, v_n$ of V

$$f(x,y) = \Sigma\overline{\xi}_i\eta_i,\ x = \Sigma\xi_i v_i, y = \Sigma\eta_i v_i.$$

iii) $\sigma = 1$ and $f(x,y) = f(y,x)$ for all $x,y \in V$.
In this case f is called *symmetric*. For even q, symmetric forms are not very interesting and for odd q, symmetric forms are equivalent with quadratic forms which we shall now discuss.

A *quadratic form* Q on V is a map $Q: V \to \mathbb{F}_q$ such that

a) $Q(\lambda x) = \lambda^2 Q(x)$ for all $\lambda \in \mathbb{F}_q$, $x \in V$, and

b) $f(x,y) := Q(x+y) - Q(x) - Q(y)$ defines a bilinear form on V.

Notice that f is symmetric and that $f(x,x) = Q(2x) - 2Q(x) = 2Q(x)$. Conversely if q is odd and f is any symmetric form on V, then $Q(x) := \frac{1}{2} f(x,x)$ is a quadratic form with associated bilinear form f, so for q odd, f and Q determine each other. A quadratic form Q is called *nondegenerate* if $Q(x) \neq 0$ for all $x \in V \setminus \{0\}$ which satisfy $f(x,y) = 0$ for all $y \in V$ (for odd q this is equivalent to f is nondegenerate, but if q is even f can be degenerate whereas Q is not (see type (I) below)). The standard forms for a non-degenerate quadratic form w.r.t. a suitable basis are as follows. If n is odd there is essentially one type:

(I) $Q(x) = \xi_1\xi_2 + \xi_3\xi_4 + \cdots + \xi_{n-2}\xi_{n-1} + \alpha\xi_n^2$, for some $\alpha \in \mathbb{F}_q$.

If n is even there are two types:

(II) $Q(x) = \xi_1\xi_2 + \xi_3\xi_4 + \cdots + \xi_{n-1}\xi_n$, or

(III) $Q(x) = \xi_1\xi_2 + \xi_3\xi_4 + \cdots + \xi_{n-3}\xi_{n-2} + \xi_{n-1}^2 + \alpha\xi_{n-1}\xi_n + \beta\xi_n^2$,

where $X^2 + \alpha X + \beta$ is irreducible over $\mathbb{F}_q$.

Suppose f is a reflexive form on V. If $f(x,y) = 0$ we write $x \perp y$ and say that x and y are *orthogonal*. Since f is reflexive, $\perp$ is a symmetric relation. For $X \subset V$ we set

$$X^\perp := \{v \in V \mid v \perp x \text{ for all } x \in X\}.$$

A subspace X of V is called *totally isotropic* if $X \subset X^\perp$, i.e. if $f(x,y) = 0$ for all $x,y \in X$. Similarly, if Q is a quadratic form on V, then any sub-space X with $Q(X) = 0$ is called *totally singular*. (If q is odd, then X is totally singular if and only if X is totally isotropic w.r.t. the bilinear form f associated with Q.) A vector space V equipped with a nondegenerate symplectic, hermitian or quadratic form is called a *symplectic*, *unitary* or *orthogonal geometry*. Especially the set of all totally isotropic (singular) points in symplectic, unitary and orthogonal geometry gives us all kinds of interesting configurations. For example, take a quadratic form of type (III) with $n = 4$ and work over $\mathbb{R}$ for the moment with $Q(x) = \xi_1\xi_2 + \xi_3^2 + \xi_4^2$.

The set of totally singular points here is a sphere (put $\xi_1 = \eta_1 + \eta_2$, $\xi_2 = \eta_1 - \eta_2$ and look in the affine 3-space defined by $\eta_2 = 1$), so any three totally singular points determine a plane which will intersect the sphere in a conic. Precisely the same is true over a finite field: let X be the set of totally singular points and $\mathcal{B} = \{X \cap E \mid E \text{ a plane with } |X \cap E| \geq 3\}$, then $(X, \mathcal{B})$ is a 3-design. Keeping the picture of the sphere in mind it is easy to compute the parameters of the design. If P is any totally singular point, then P is on $q+1$ tangent lines (all the lines in the plane tangent to the sphere passing through P) which carry no other points of the sphere, and therefore on $(q^2+q+1)-(q+1) = q^2$ lines which intersect the sphere in one other point. Hence $|X| = q^2+1$, and a similar argument in the plane shows that every conic contains $q+1$ points. Thus $(X, \mathcal{B})$ is a 3-$(q^2+1, q+1, 1)$ design. A *Möbius plane* is by definition a 3-$(n^2+1, n+1, 1)$ design. The Möbius planes that we have just constructed are characterized by the fact that they satisfy the Theorem of Miquel (see [18]). They play a role similar to that of the Desarguesian planes in the theory of projective planes. Here also, "nonmiquelian" Möbius planes are known to exist (although not as many as nondesarguesian projective planes). A similar story can be told by starting off with a quadratic form $Q(x) = \xi_1\xi_2 + \xi_3\xi_4$ of type (II). We then arrive at the so-called Minkowski planes which we shall discuss in greater detail in the next section.

There is a very satisfactory characterization of the symplectic, unitary and orthogonal geometries which have totally isotropic or totally singular subspaces of dimension at least three, known as the Buekenhout-Shult theorem, which we shall now formulate.

DEFINITION. Let X be a set of *points* and $\mathcal{L}$ a collection of distinguished subsets of X called *lines* such that

i) the set of lines is nonempty and each line has at least three points,
ii) no point is collinear with all remaining points,
iii) for every point x and every line L not containing x, x is collinear with either one or all points of L.

Then $(X, \mathcal{L})$ is called a *polar space*.

Every symplectic, unitary or orthogonal geometry containing totally isotropic (totally singular) lines yields a polar space in the following way: points are the totally isotropic (singular) points, lines are the totally isotropic (singular) lines. Let us check iii) for a symplectic or unitary space V. Let $\langle x \rangle$ be a totally isotropic point and L a totally isotropic line. Since $\langle x \rangle^{\perp} = \{y \mid f(x,y) = 0\}$ is a hyperplane of V, the 2-dimensional subspace L intersects $\langle x \rangle^{\perp}$ nontrivially. If $y \in L \cap \langle x \rangle^{\perp}$, $y \neq 0$, then $f(\lambda x + \mu y, \rho x + \sigma y) = 0$ since $f(x,x) = f(y,y) = f(x,y) = 0$, so the line $\langle x,y \rangle$ is totally isotropic. If $L \not\subset \langle x \rangle^{\perp}$, then $\langle x \rangle$ is collinear (in the polar space sense) with exactly one point of L, if $L \subset \langle x \rangle^{\perp}$, then $\langle x \rangle$ is collinear with all points of L.

<u>THEOREM 5</u>. *Let* $(X,\mathcal{L})$ *be a polar space. Then*

a) $(X,\mathcal{L})$ *is isomorphic to the geometry of all totally isotropic or totally singular points and lines of a symplectic, unitary or orthogonal geometry, or*

b) $(X,\mathcal{L})$ *satisfies the following stronger version of iii):*

iv) for every point x *and every line* L *not containing* x, x *is collinear with exactly one point of* L.

The first characterization of polar spaces was obtained by Veldkamp [17] who used a more complicated set of axioms. This set of axioms was later simplified by Tits (see [15]) and Buekenhout and Shult (see [5]).
A polar space which satisfies iv) is called a *generalized quadrangle*. Here the generalized quadrangles play a role similar to that of the projective planes in the theory of projective spaces. Again many generalized quadrangles are known which are not isomorphic to the geometry of totally isotropic or totally singular points and lines of a symplectic, unitary or orthogonal geometry. For example, the following geometry of ab points and a + b lines as shown in Figure 2 is a generalized quadrangle.

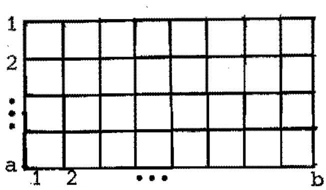


Figure 2.

However, since lines in a projective space over $\mathbb{F}_q$ carry $q+1$ points, this can only be a geometry of totally isotropic or totally singular points and lines if $a=b=q+1$ for some prime power q. The orthogonal geometry over $\mathbb{F}_q$ of type (II) for $n=4$ belonging to the quadratic form $Q(x) = \xi_1\xi_2 + \xi_3\xi_4$ yields a generalized quadrangle of this type with $a=b=$ $=q+1$; the two sets of $q+1$ mutually disjoint lines correspond to the two sets of rulings on the hyperboloid $\xi_1\xi_2 + \xi_3\xi_4 = 0$. Additional axioms are necessary to characterize the classical generalized quadrangles. For example, there is a theorem by Buekenhout & Lefèvre (see [6]) which says that a generalized quadrangle which is embedded in a projective space is classical. Characterizations using certain (transitivity) properties of the automorphism group have been given by Tits [14] and Walker [19]. Thas and Payne (see e.g. [13]) have given a number of characterizations based on geometic and combinatorial assumptions.

4. SUMMARY OF THE FIVE PAPERS

The first paper [A] is on nearaffine planes. Nearaffine planes (and more generally nearaffine spaces) were introduced by J. André (see e.g. [1]) to describe geometrically vector spaces over nearfields. By definition a *nearfield* $(F,+,\cdot)$ is a quasifield (as defined in section 1) with the additional property that $(F\setminus\{0\},\cdot)$ is a group. Let $(F,+,\cdot)$ be a nearfield and set $V=F^2$. With addition and scalar multiplication on the left (by elements of F) defined componentwise on V, V is called a *vector space of dimension* 2 *over* F. For $x,y\in V$, $x\neq y$, define the line $x\sqcup y$ from x to y by

$$x\sqcup y := F\cdot(y-x)+x.$$

If F happens to be a field, then V is just the standard 2-dimensional vector space over F and the lines $x\sqcup y$ coincide with the ordinary lines in the Desarguesian affine plane. If F is a proper nearfield, then in general $u,v\in x\sqcup y$ does not imply $x\sqcup y = u\sqcup v$ and a rather complicated set of axioms is necessary to describe this geometry. The axioms for a near-affine plane are chosen in such a way that we get the ordinary affine planes back if the additional property $x\sqcup y = y\sqcup x$ holds for all $x,y\in V$.

What we do in this paper is to set up a theory for nearaffine planes which generalizes the theory of *translation planes*, i.e. affine planes which can be coordinatized by a quasifield in the sense as described in section 1. This leads us to what we have called *nearaffine translation planes*. As for ordinary translation planes, it is possible to give equivalent algebraic, geometric and group theoretic descriptions of nearaffine translation planes. For us nearaffine planes are especially important due to certain connections with Minkowski planes, the subject of papers [B] and [C] which we shall now discuss.

Consider the hyperboloid in projective 3-space over $\mathbb{F}_q$, i.e. the set of totally singular points of the quadratic form $Q(x) = \xi_1\xi_2 + \xi_3\xi_4$ on $\mathbb{F}_q^4$. The picture to keep in mind here is that of the hyperboloid $x^2 - y^2 + z^2 = 1$ (use the transformation $\xi_1 = x - y$, $\xi_2 = x + y$, $\xi_3 = z - t$, $\xi_4 = z + t$ and take $t = 1$). There are two families $\mathcal{L}^+$ and $\mathcal{L}^-$ of totally singular lines on the hyperboloid. Explicitly these lines are (in ξ-coordinates)

$$\ell^+_{a,b} := \langle (a,0,b,0),(0,b,0,-a)\rangle \text{ and}$$

$$\ell^-_{a,b} := \langle (a,0,0,b),(0,b,-a,0)\rangle$$

where $a,b \in \mathbb{F}_q$ and at least one of a and b is not equal to zero. We have already pointed out that the totally singular lines form the rather trivial structure of a $(q+1)\times(q+1)$ grid (see Fig.2). To obtain an interesting geometry we proceed as in the case of the Möbius planes and add the conic intersections of the planes with the set of totally singular points as objects to our geometry. These plane sections are called *circles*. Any three distinct points on the hyperboloid with the property that no two are on a totally singular line determine a unique plane and therefore a unique circle. In this way we arrive at an icidence structure with a set M of *points*, two collections $\mathcal{L}^+$ and $\mathcal{L}^-$ of subsets of M called *lines*, and a collection $\mathcal{C}$ of subsets of M called *circles* satisfying the following axioms.

(M1) $\mathcal{L}^+$ and $\mathcal{L}^-$ are partitions of M,

(M2) $|\ell^+ \cap \ell^-| = 1$ for all $\ell^+ \in \mathcal{L}^+$, $\ell^- \in \mathcal{L}^-$,

(M3) any three points, no two on a line, determine a unique circle $c \in \mathcal{C}$, (M4) $|\ell \cap c| = 1$ for all $\ell \in \mathcal{L}^+ \cup \mathcal{L}^-$, $c \in \mathcal{C}$,

(M5) there exist three points, no two of which are on a line.

Such an incidence structure is called a *Minkowski plane*. Let us prove some elementary properties of Minkowski planes. From (M1) and (M2) it follows that $|\ell^+| = |\mathcal{L}^-|$ for all $\ell^+ \in \mathcal{L}^+$ and $|\ell^-| = |\mathcal{L}^+|$ for all $\ell^- \in \mathcal{L}^-$. By (M1) and (M4) we have $|\mathcal{L}^+| = |c|$ and $|\mathcal{L}^-| = |c|$ for all $c \in \mathcal{C}$. Since $\mathcal{C} \neq \emptyset$ by (M3) and (M5), we have proved that $|\mathcal{L}^+| = |\mathcal{L}^-| = |\ell| = |c|$ for all $\ell \in \mathcal{L}^+ \cup \mathcal{L}^-$, $c \in \mathcal{C}$. The number $n := |c| - 1$ is called the *order* of the Minkowski plane. It is often convenient to think of the points and lines of a Minkowski plane as being arranged in an $(n+1) \times (n+1)$ square grid.

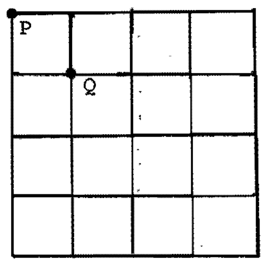


Figure 3.

Every circle then corresponds to a transversal of this grid intersecting each horizontal and vertical line exactly once. An important property (which for infinite Minkowski planes is an additional axiom) is

(M6) given a circle c, a point $P \in c$ and a point $Q \notin c$, P and Q not on a line, there is a unique circle d such that $P, Q \in d$ and $c \cap d = \{P\}$.

To prove this, note that the two noncollinear points P and Q are on $n-1$ circles (Figure 3 shows that there are $(n-1)^2$ points not collinear with P or Q; each circle through P and Q contains $n-1$ of these). Since there are $n-2$ points on c not equal to P and noncollinear with Q, there must be exactly $(n-1)-(n-2) = 1$ circle through P and Q which does not intersect c in a point distinct from P. With the help of (M6) it is not very hard to see that with every point Z of a Minkowski plane we can associate an affine plane (the *derived* plane at Z) as follows. The points of the affine plane are the points which are not collinear with Z. The lines of the affine plane are the lines of the Minkowski plane missing Z and the circles containing Z. Axiom (A2) for affine planes now corresponds to (M6). In the hyperboloid model this affine plane is clearly visible if we use stereographic projection from Z onto a plane.

It is possible to construct Minkowski planes which are not isomorphic to a Minkowski plane associated with a quadratic form on $\mathbb{F}_q^4$. In [B] we show that the known Minkowski planes are characterized by the fact that a certain geometrical condition (called (D) in [B]) holds. The idea behind the proof of this lies in the observation that with any point Z of one of the known Minkowski planes we can also associate a nearaffine plane. The points of the nearaffine plane are again the points which are not collinear with Z. The lines of the nearaffine plane correspond to the lines and circles missing Z. Viewed in this way, condition (D) is nothing but a special case of Desargues' theorem in the nearaffine plane. One can show that (D) implies that all nearaffine planes are nearaffine translation planes. The automorphisms of the nearaffine planes extend to automorphisms of the Minkowski plane. These in turn enable one to reconstruct the algebraic representation of the known Minkowski planes.

In [C] we have generalized the theorem of Ostrom & Wagner for projective planes (Theorem 2) and Hering's result for Möbius planes (see [9]) to Minkowski planes: if the automorphism group of a Minkowski plane is transitive on pairs of noncollinear points, then the plane is one of the known Minkowski planes. The technique used here is very much the same as in the proof of the Ostrom & Wagner theorem. Again the basic tool is to study involutions in the automorphism group. Here some rather deep group theory is necessary to reduce to the case where there is an involution which has a subplane as a set of fixed points. Once this is achieved, induction is possible to finish the proof.

In [D] we have characterized the unitary geometry on $\mathbb{F}_{q^2}^3$ which we shall now describe in some detail. Let q be a prime power and $V = \mathbb{F}_{q^2}^3$. Define a nondegenerate hermitian form (,) on V by

$$(x,y) = \bar{\xi}_1\eta_1 + \bar{\xi}_2\eta_2 + \bar{\xi}_3\eta_3 \ ,$$

for $x = (\xi_1,\xi_2,\xi_3)$, $y = (\eta_1,\eta_2,\eta_3) \in V$. Here $\bar{\lambda} = \lambda^q$ for all $\lambda \in \mathbb{F}_{q^2}$.

Let U be the set of totally isotropic points, i.e.

$$U = \{ \langle x \rangle \mid (x,x) = 0,\ x \in V \setminus \{0\} \} .$$

Let $\langle x \rangle \in U$ and let $\langle y \rangle$ be any any other point. A point $\langle \lambda x + y \rangle$ on the line $\langle x,y \rangle$ joining $\langle x \rangle$ and $\langle y \rangle$ is in U if $0 = (\lambda x + y, \lambda x + y) = \mathrm{Tr}(\lambda(x,y)) + (y,y)$, where $\mathrm{Tr}\colon \mathbb{F}_{q^2} \to \mathbb{F}_q$ is the trace map given by $\mathrm{Tr}(\alpha) = \alpha + \overline{\alpha}$, $\alpha \in \mathbb{F}_{q^2}$. We claim that it is impossible that all points $\langle \lambda x + y \rangle$ are in U, i.e., that $(x,y) = 0$ and $(y,y) = 0$. Suppose on the contrary that $(x,y) = (y,y) = 0$. Take any point $\langle z \rangle$ not on the line $\langle x \rangle^{\perp} := \{ x' \mid (x,x') = 0 \}$. Then

$$u = - \frac{(y,z)}{(x,z)} + y$$

satisfies $(u,x) = (u,y) = (u,z) = 0$, so $(u,v) = 0$ for all $v \in V$, a contradiction. This shows that $\langle x \rangle$ is the only point of U on the line $\langle x \rangle^{\perp}$ and that every other line through $\langle x \rangle$ contains q points $\neq \langle x \rangle$ of U (for Tr is an $\mathbb{F}_q$-linear map with a kernel of dimension 1, so $\mathrm{Tr}(\lambda(x,y)) = -(y,y)$ has q solutions λ if $(x,y) \neq 0$). Since there are $q^2 + 1$ lines through $\langle x \rangle$, one of which is $\langle x \rangle^{\perp}$, it follows that $|U| = 1 + q \cdot q^2 = 1 + q^3$. Also, every two distinct points of U are on a unique line of $q + 1$ U-points, i.e., we have constructed a 2-$(q^3 + 1, q + 1, 1)$ design. A 2-$(n^3 + 1, n + 1, 1)$ design is called a *unital* ($n \in \mathbb{N}$). For $q = 2$ the 2-(9,3,1) design is the unique affine plane of order 3. But already for $q = 3$ numerous 2-(28,4,1) designs are known (see Brouwer [4]) and so we are left with the question what properties are characteristic for the unitals associated with a unitary geometry. It is conjectured that the following "anti-Pasch" axiom will do:

No four distinct points intersect in six distinct lines.

It is easy to show that this property holds for the classical unitals. Suppose $\langle x \rangle, \langle y \rangle, \langle a \rangle, \langle b \rangle, \langle c \rangle, \langle d \rangle$ are six distinct points of U such that they form the configuration of Figure 4.

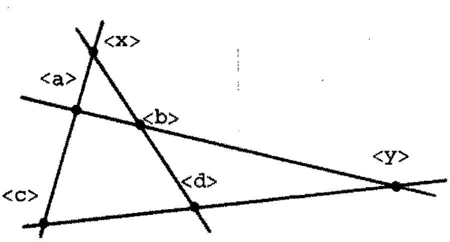


Figure 4.

Since a,b,c and d are linearly dependent, we may assume that

$$a + b + c + d = 0$$

and therefore also that $x = a + c$, $y = a + b$. From $(x,x) = 0$ it follows that $(a,c) + (c,a) = 0$. Similarly, $(a,b) + (b,a) = 0$ (from $(y,y) = 0$) and $(b,c) + (c,b) = 0$ (from $(d,d) = 0$ and the other relations). Since a,b and c are linearly independent the Gram matrix

$$\begin{pmatrix} 0 & (a,b) & (a,c) \\ (b,a) & 0 & (b,c) \\ (c,a) & (c,b) & 0 \end{pmatrix}$$

is nonsingular. Hence $0 \neq (a,b)(b,c)(c,a) + (a,c)(b,a)(c,b)$. This contradicts the other relations.

In [D] we have characterized the classical unitals under additional geometric assumptions. The basic steps in the proof are as follows. Using nontrivial group theory it is easy to prove that once the automorphism group of the unital is large enough, we can only have a classical unital. The geometrical conditions we impose ensure the existence of such an automorphism group. More precisely, for the classical unital we have for $\langle x\rangle \in U$ that the linear transformation

$$v \mapsto v + \alpha(x,v)x, \quad v \in V$$

respects the hermitian form (,) if $Tr(\alpha) = 0$ and so acts as an automorphism of the unital fixing all lines through $\langle x\rangle$. These transformations are called the *unitary transvections*. The geometrical conditions imply the existence of all possible unitary transvections and these generate a 2-transitive group of automorphisms.

We conclude with a discussion of the last paper [E] on semi-partial geometries. The concept of generalized quadrangle has been generalized in a number of ways by replacing the key axiom iv) as formulated in Theorem 5, by a similar axiom. Most of these axioms can be formulated as:

For every point x and every line L with $x \notin L$,

$$|\{ y \in L \mid x \text{ and } y \text{ collinear} \}| \in S,$$

where S is some finite subset of $\mathbb{N} \cup \{0\}$. By taking $S = \{0,\alpha\}$ one gets the essential axiom for a semi-partial geometry (for a complete definition see [E]). In this paper we show that certain semi-partial geometries are already determined by some numerical data. There are two cases to consider, namely $\mu = \alpha^2$ and $\mu = \alpha(\alpha + 1)$ in the notation of [E]. The line of proof in both cases is essentially identical and roughly reads as follows. By results of Debroey [7] it suffices to show that the points and lines of such a semi-partial geometry satisfy the dual of the axiom of Pasch (for obvious reasons called the *diagonal axiom*). For both the conditions $\mu = \alpha^2$ and $\mu = \alpha(\alpha + 1)$ there is a straightforward geometric interpretation. The hard part of the proof consists in using this over and over again to show that any two intersecting lines generate a well-behaved "subspace". Once this has been achieved it is no longer hard to show that the diagonal axiom holds provided the semi-partial geometry properly contains such a subspace.

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NEARAFFINE PLANES

ABSTRACT. In this paper we develop a theory for nearaffine planes analogous to the theory of ordinary affine translation planes. In a subsequent paper we shall use this theory to give a characterization of a certain class of Minkowski planes.

1. INTRODUCTION

Nearaffine spaces were introduced by J. André as a generalization of affine spaces (see e.g., [1], [2], [3]). We shall restrict our attention to nearaffine spaces of dimension 2, the nearaffine planes. Our set of axioms, defining nearaffine planes is weaker than the one used by André. If, however, the so-called Veblen-condition is assumed to hold (see Section 3), our definition coincides with the one given by André in [2]. Our main goal will be to generalize the theory of translation planes to the case of nearaffine planes. In a second paper, we shall show the relationship between certain nearaffine planes and Minkowski planes.

In Section 2 we give the definition of a near affine plane and some basic results. Section 3 is devoted to the so-called Veblen-axiom. In Section 4 we consider automorphisms of nearaffine planes, in particular translations and dilatations. In Section 5 we show that translations exist whenever a certain Desarguers configuration holds. In Section 6 we give an algebraic representation for nearaffine translation planes. Section 7 contains some information on the relationship with Latin squares. Finally, in Section 8, we give a construction of a class of nearaffine planes. More detailed information, especially on the construction of nearaffine planes, can be found in [12].

2. DEFINITION AND BASIC RESULTS

Let X be a nonempty set of elements called *points*, L a set of subsets of X called *lines*. Let $\sqcup$ be an operation called *join* mapping the ordered pairs $(x, y), x, y \in X, x \neq y$, onto L (the join from x to y is denoted by $x \sqcup y$), and $\|$ an equivalence relation called *parallelism* on L (l parallel to m is denoted by $l \| m$).

We say that $(X, L, \sqcup, \|)$ is a *nearaffine plane* if the following three groups of axioms are satisfied.

Axioms on Lines:

(L1) $\quad x, y \in x \sqcup y \quad$ for all $x, y \in X, x \neq y$.

(L2) $\quad z \in x \sqcup y \setminus \{x\} \Leftrightarrow x \sqcup y = x \sqcup z \quad$ for all $x, y, z \in X, x \neq y$.

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(L3) $x \sqcup y = y \sqcup x = x \sqcup z \Rightarrow x \sqcup z = z \sqcup x$ for all $x, y, z \in X$, $y \neq x \neq z$.

The point x is called a *basepoint* of the line $x \sqcup y$. It is not difficult to show the following proposition (see [2]).

PROPOSITION 2.1. *The following are equivalent.*

(i) $x \sqcup y$ *has a basepoint* $\neq x$,

(ii) *each point of* $x \sqcup y$ *is a base point of* $x \sqcup y$,

(iii) $x \sqcup y = y \sqcup x$.

Therefore we may define: a line $x \sqcup y$ is called *straight* iff $x \sqcup y = y \sqcup x$. The set of all straight lines is denoted by G. The lines in $L \backslash G$ are called *proper* lines.

Axioms of parallelism:

(P1) for all $l \in L$, $x \in X$ there exists exactly one line with base point x parallel to l.
We denote this line by $(x \parallel l)$.

(P2) $x \sqcup y \parallel y \sqcup x$ for all $x, y \in X, x \neq y$.

(P3) $(g \parallel l) \Rightarrow l \in G$ for all $g \in G, l \in L$.

Axioms on richness:

(R1) There exists at least two non-parallel straight lines.

(R2) Every line l meets every straight line g with $g \nparallel l$ in exactly one point.

We state some basic results which follow immediately from our axioms (see e.g. [2], [11]).

PROPOSITION 2.2. *Two distinct lines with the same base point have no other point in common.*

PROPOSITION 2.3. *Two distinct straight lines intersect in one point unless they are parallel in which case they are disjoint.*

THEOREM 2.4. *A nearaffine plane with commutative join is an affine plane.*

We shall only consider finite nearaffine planes, i.e., nearaffine planes with a finite number of points. The following result is easy to prove (see, e.g., [2], [11]).

PROPOSITION 2.5. *All lines of a nearaffine plane have the same number of points.*

The number of points on a line, which equals the number of parallel straight lines in one equivalence class, is denoted by n and called the *order* of the nearaffine plane.

PROPOSITION 2.6. $|X| = n^2$.

PROPOSITION 2.7. *There are exactly $n+1$ lines with a given base point.*

We denote by $s+1$ the number of equivalence classes containing straight lines. By (R1) we have $s \geqslant 1$.

PROPOSITION 2.8. *Every point is on $s+1$ straight lines,* $|G| = n(s+1)$, $|L \setminus G| = n^2(n-s)$.

3. The Veblen-condition

Many interesting examples of nearaffine planes (e.g., the nearaffine planes associated with Minkowski planes) satisfy the following version of the Veblen-condition (named (V') in [2]).

(V') Let g be a straight line, P, Q, R distinct points on g, $l \neq g$ a line with base point P and $S \in l \setminus \{P\}$. Then $(R \| Q \sqcup S) \cap l \neq \varnothing$ (see Figure 1).

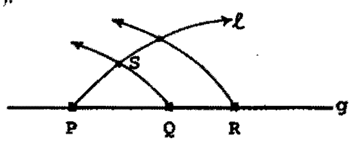

Fig 1

Before we prove the main result on nearaffine planes which satisfy (V'), we prove a proposition valid in any nearaffine plane. Notice that until now we have not used axiom (P2) and that the proof of this proposition only requires the following weakened version of (P2) (this will be important in our paper on Minkowski planes).

(P2′) Let g and h be two distinct parallel straight lines, $x, x' \in g$ and $y, y' \in h$. Then $x \sqcup y \| x' \sqcup y' \Leftrightarrow y \sqcup x \| y' \sqcup x'$.

PROPOSITION 3.1. *Two parallel lines which have their base point on one straight line are disjoint or identical.*

Proof. Let l and l' be two parallel lines with base points x and x' respectively on the straight line g. If $y \in l \cap l'$, $y \neq x, x'$ then $x \sqcup y = l \| l' = x' \sqcup y$, hence $y \sqcup x \| y \sqcup x'$ by (P2′) and so $y \sqcup x = y \sqcup x'$ by (P1). Therefore $x = x'$ by (R2) and so $l = l'$ by (P1).

THEOREM 3.2. (André [2]). *Let $\mathfrak{A} = (X, L, \sqcup, \|)$ be a nearaffine plane satisfying (V') and g a straight line of $\mathfrak{A}$. Then the point set X and the line set $L_g := \{l \in L \mid l$ has base point on $g\} \cup \{h \in G \mid h \| g\}$ constitute an affine plane $\mathfrak{A}_g = (X, L_g)$.*

Proof. Let $l, m \in L_g$, $l \neq m$. If $l \| m$ then $|l \cap m| = 0$ by 2.3 and 3.1. If $l \nparallel m$

then $|l \cap m| = 1$. This follows from (R2) if $l \| g$ or $m \| g$. Suppose, therefore, that l and m have base points on g. The n line in L_g parallel to m partition X by 3.1. Hence, at least one of these lines contains a point of l. Therefore, by (V') and 2.5, each of these lines, so in particular m, contains exactly one point of l. Since $|L_g| = n(n+1)$ and $|l| = n$ for every $l \in L_g$ it follows from [5, result 3.2.4c, p. 139] that $\mathscr{N}_g$ is an affine plane. □

Remark. Notice that two lines of $\mathscr{N}_g$ are parallel in $\mathscr{N}_g$ (i.e., disjoint) iff they are parallel in $\mathscr{N}$.

4. Automorphisms

In this section we generalize such notions as automorphism, dilatation etc. to the case of nearaffine planes. Proofs which do not differ essentially from the corresponding proofs for affine planes (see e.g., [4]) will be omitted.

DEFINITION 4.1. Let $\mathscr{N} = (X, L, \sqcup, \|)$ and $\mathscr{N}' = (X', L', \sqcup', \|')$ be two nearaffine planes. A bijection $\alpha \colon X \to X'$ is called an *isomorphism* of $\mathscr{N}$ and $\mathscr{N}'$ if

(i) $(P \sqcup Q)^\alpha = P^\alpha \sqcup' Q^\alpha$ for all $P, Q \in X, P \neq Q$,
and
(ii) $l \| m \Leftrightarrow l^\alpha \|' m^\alpha$ for all $l, m \in L$.

If $\mathscr{N} = \mathscr{N}'$, then α is called an *automorphism of* $\mathscr{N}$. A permutation α of the points of $\mathscr{N}$ is called a dilatation if $P \sqcup Q \| P^\alpha \sqcup Q^\alpha$ for all $P \neq Q$.

The automorphisms of a nearaffine plane form a group $\mathscr{A}$, the dilatations form a group $\mathscr{D}$.

THEOREM 4.2. $\mathscr{D} \trianglelefteq \mathscr{A}$.

LEMMA 4.3. *Suppose $\delta \in \mathscr{D}$ fixes $P \in X$. Then $Q^\delta \in P \sqcup Q$ for all $Q \in X$, $X \neq P$.*

THEOREM 4.4. *Suppose $\delta \in \mathscr{D}$ fixes two distinct points P and Q. Then $\delta = 1$.*

Proof. Take $R \in X$. If $R = P$ or $R = Q$, then $R^\delta = R$. if $R \neq P, Q$ we have by 4.3: $R^\delta \in P \sqcup R$ and $R^\delta \in Q \sqcup R$. By (R1) there is at least one straight line $g \neq P \sqcup Q$ through P, so for $R \in g$ we have $R^\delta \in (P \sqcup R) \cap (Q \sqcup R) = \{R\}$, i.e., $R^\delta = R$. For an arbitrary $R \notin g$ we replace P by a point P' in such a way that $P' \sqcup R$ is straight and Q by some point $Q \in g \backslash \{P'\}$. It follows that $R^\delta \in (P' \sqcup R) \cap (Q' \sqcup R) = \{R\}$. □

COROLLARY 4.5. *Let $\delta_1, \delta_2 \in \mathscr{D}$ and suppose $P^{\delta_1} = P^{\delta_2}, Q^{\delta_1} = Q^{\delta_2}$ for distinct points P and Q. Then $\delta_1 = \delta_2$.*

DEFINITION 4.6. A dilatation τ is called a *translation* if $\tau = 1$ or if $P \sqcup P^\tau \| Q \sqcup Q^\tau$ for all $P, Q \in X$. The parallel class containing $P \sqcup P^\tau$ is called the *direction* of $\tau \neq 1$. The translation τ is *straight* if $P \sqcup P^\tau$ is straight. We denote by $\mathscr{T}$ the set of all translations.

A translation $\tau \neq 1$ has no fixd point. Suppose $P^\tau = P$; then for any point $Q \neq P$ we have $Q^\tau \neq Q$ by 4.4 and $Q^\tau \in P \sqcup Q$ by 4.3. Hence, if $P \sqcup Q$ is straight, $Q \sqcup Q^\tau = P \sqcup Q$.

This is a contradiction since there are at least two nonparallel straight lines through P.

LEMMA 4.7. *If $\alpha \in \mathscr{A}$ and $\tau \in \mathscr{T}$, then $\alpha\tau\alpha^{-1} \in \mathscr{T}$. If in addition $\alpha \in \mathscr{D}$ and $\tau \neq 1$, then τ and $\alpha\tau\alpha^{-1}$ have the same direction.*

THEOREM 4.8. *Let C be a parallel class consisting of straight lines and $\mathscr{T}(C) := \{\tau \in \mathscr{T} \mid \tau$ has direction $C\} \cup \{1\}$. Then $\mathscr{T}(C) \trianglelefteq D$.*

LEMMA 4.9. *Let C and D be two distinct parallel classes consisting of straight lines. Then $\sigma\tau = \tau\sigma$ for all $\sigma \in \mathscr{T}(C)$, $\tau \in \mathscr{T}(D)$.*

LEMMA 4.10. *Let C and D be two parallel classes containing straight lines, $\sigma \in \mathscr{T}(C)$ and $\tau \in \mathscr{T}(D)$. If $\sigma\tau \neq 1$, then $\sigma\tau$ has no fixed points.*

Proof. If $C = D$ or if σ or $\tau = 1$, this is a consequence of 4.8. If $C \neq D$ and $\sigma, \tau \neq 1$, then $P^{\sigma\tau} = P$ for some $P \in X$ implies $P \sqcup P^\sigma \in C$, $P \sqcup P^{\tau^{-1}} \in D$, $P^\sigma = P^{\tau^{-1}}$, a contradiction.

For nearaffine planes the product of two translations need not be a translation. For straight translations the following theorem holds.

THEOREM 4.11. *Let C, D and E be three distinct parallel classes consisting of straight lines. Suppose $\rho \in \mathscr{T}(C)$, $\sigma \in \mathscr{T}(D)$, $\tau \in \mathscr{T}(E)$ and $P \in X$ satisfy $P^{\rho\sigma} = P^\tau$. Then $\rho\sigma = \tau$.*

Proof. If $\tau = 1$, then $P^{\rho\sigma} = P$, hence $\rho\sigma = 1$ by 4.10. If $\tau \neq 1$, then $P^\tau \neq P$. From 4.9 it follows that $(P^\tau)^\tau = (P^{\rho\sigma})^\tau = (P^\tau)^{\rho\sigma}$. Hence, $\tau = \rho\sigma$ by 4.5. □

THEOREM 4.12. *Let C and D be two distinct parallel classes consisting of straight lines with $|\mathscr{T}(C)| = |\mathscr{T}(D)| = n$. Then*

$$\mathscr{T} \subseteq \langle \mathscr{T}(C), \mathscr{T}(D) \rangle = \mathscr{T}(C)\mathscr{T}(D).$$

If in addition $\mathscr{T}(C)$ and $\mathscr{T}(D)$ are Abelian, then $\mathscr{T} = \mathscr{T}(C)\mathscr{T}(D)$.

Proof. By 4.9, $\langle \mathscr{T}(C), \mathscr{T}(D) \rangle = \mathscr{T}(C)\mathscr{T}(D)$ and $|\mathscr{T}(C)\mathscr{T}(D)| = n^2$. By 4.10, $\mathscr{T}(C)\mathscr{T}(D)$ is the Frobenius kernel of $\mathscr{D}$, hence it contains all fixed-points free dilatations. Therefore $\mathscr{T} \subseteq \mathscr{T}(C)\mathscr{T}(D)$. Suppose $\mathscr{T}(C)$ and $\mathscr{T}(D)$ are Abelian. Take $\rho \in \mathscr{T}(C)$, $\sigma \in \mathscr{T}(D)$ and $P, Q \in X$. There exist $\rho_1 \in \mathscr{T}(C)$, $\sigma_1 \in \mathscr{T}(D)$ such that $P^{\rho_1\sigma_1} = Q$. Hence,

$$P \sqcup P^{\rho\sigma} \parallel (P \sqcup P^{\rho\sigma})^{\rho_1\sigma_1} = P^{\rho_1\sigma_1} \sqcup P^{\rho_1\sigma_1\rho\sigma} = Q \sqcup Q^{\rho\sigma},$$

i.e., $\rho\sigma \in \mathscr{T}$. □

A nearaffine plane having two distinct parallel classes C and D consisting of straight lines such that $|\mathscr{T}(C)| = |\mathscr{T}(D)| = n$ is called a *nearaffine translation plane*. Notice that this definition is consistent with the definition of translation plane.

THEOREM 4.13. *Let C, D and E be three distinct parallel classes consisting of straight lines. If* $|\mathscr{T}(C)| = |\mathscr{T}(D)| = n$, *then*
(a) $\mathscr{T}(E)$ *is Abelian,*
(b) $\mathscr{T}(C) \simeq \mathscr{T}(D)$.

Proof. (a) Let $\tau_1, \tau_2 \in \mathscr{T}(E)$. By 4.12 there exist $\rho_1 \in \mathscr{T}(C)$, $\sigma_1 \in \mathscr{T}(D)$ such that $\tau_1 = \rho_1\sigma_1$. By 4.9,

$$\tau_1\tau_2 = \rho_1\sigma_1\tau_2 = \tau_2\rho_1\sigma_1 = \tau_2\tau_1.$$

(b) Define the automorphism $\phi: \mathscr{T}(C) \to \mathscr{T}(D)$ as follows: Fix a line $g \in E$. For each $\rho \in \mathscr{T}(C)$ let $\phi(\rho) \in \mathscr{T}(D)$ be determined by $g^{\rho\phi(\rho)} = g$. □

COROLLARY 4.14. *If in addition to the hypothesis of* 4.13, $|T(E)| = n$, *then* $\mathscr{T}(C) \simeq \mathscr{T}(D) \simeq \mathscr{T}(E)$ *and these groups are Abelian.*

So far we have not used (P2) in this section. Using (P2) it is possible to prove the following theorem.

THEOREM 4.15. *The order n of a nearaffine translation plane is odd or a power of* 2.

Proof. Suppose n is even and let C and D be two distinct parallel classes consisting of straight lines such that $|\mathscr{T}(C)| = |\mathscr{T}(D)| = n$. There exists $\rho \in \mathscr{T}(C)$ such that $\rho^2 = 1$, $\rho \neq 1$. Take $\sigma \in \mathscr{T}(D)$ and $P \in X$. Then,

$$P \sqcup P^{\rho\sigma} \parallel P^{\rho\sigma^{-1}} \sqcup (P^{\rho\sigma})^{\rho\sigma^{-1}} = P^{\rho\sigma^{-1}} \sqcup P \parallel P \sqcup P^{\rho\sigma^{-1}}.$$

Therefore $P^{\rho\sigma}$, $P^{\rho\sigma^{-1}} \in P \sqcup P^{\rho\sigma} \notin D$. Since $P^{\rho\sigma^{-1}}$ and $P^{\rho\sigma} = (P^{\rho\sigma^{-1}})^{\sigma^2}$ are on the same straight line of D it follows that $P^{\rho\sigma^{-1}} = P^{\rho\sigma}$, i.e., $\sigma^2 = 1$. Hence. $\mathscr{T}(D)$ is an (elementary Abelian) 2-group. □

5. A DESARGUES CONFIGURATION

Let $\mathscr{N} = (X, L, \sqcup, \parallel)$ be a nearaffine plane and C a parallel class consisting of straight lines. Consider the following condition (cf. [2], [3]).

(D1) Little Desargues configuration. If $P, P', Q, Q', R, R' \in X$ are distinct points such that $P \sqcup P'$, $Q \sqcup Q'$, $R \sqcup R'$ are distinct lines of C, then $P \sqcup Q \parallel P' \sqcup Q'$ and $P \sqcup R \parallel P' \sqcup R'$ imply $Q \sqcup R \parallel Q' \sqcup R'$ (see Figure 2).

Analogous to the situation for affine planes, the validity of (D1) is seen to be equivalent to the existence of all possible translations with direction C.

THEOREM 5.1. *C satisfies* (D1)$\Leftrightarrow |\mathscr{T}(C)| = n$.

The following theorem will be useful in our paper on Minkowski planes. Again notice that we only make use of (P2′).

THEOREM 5.2. *Let $\mathscr{N} = (X, L, \sqcup, \parallel)$ be a nearaffine plane in which the Veblen-condition holds, and let C be a parallel class of straight lines. Then*

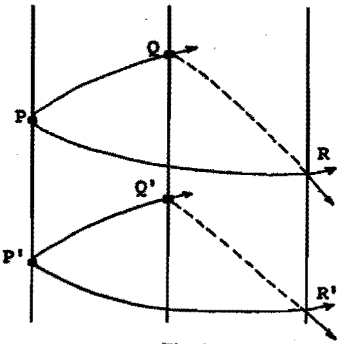

Fig. 2.

(*using the notation of* 3.2), *C satisfies* (D1) *in* $\mathcal{N} \Leftrightarrow C$ *satisfies* (D1) *in* $\mathcal{N}_g$ *for all* $g \in C$.

Proof. $\Rightarrow$: Every translation of $\mathcal{N}$ with direction C is easily seen to induce a translation of $\mathcal{N}_g$ with direction C for every $g \in G$.

$\Leftarrow$: Let P, P', Q, Q', R, R' be distinct points such that $P \sqcup P'$, $Q \sqcup Q'$, and $R \sqcup R'$ are distinct straight lines of C and such that $P \sqcup Q \parallel P' \sqcup Q'$, $P \sqcup R \parallel P' \sqcup R'$. Let S (resp. S') be the base point of the line in $\mathcal{N}_{P \sqcup P'}$ passing through Q and R (resp. Q' and R'), (see Figure 3). Application of (D1) in $\mathcal{N}_{P \sqcup P'}$ yields $S \sqcup Q \parallel S' \sqcup Q'$.

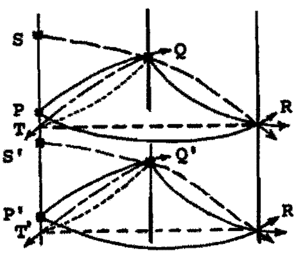

Fig. 3.

Let D be a parallel class of straight lines different from C, and let T (resp. T') be the point of intersection of $P \sqcup P'$ and the straight line of D passing through R (resp. R'). Application of (D1) in $\mathcal{N}_{P \sqcup P'}$ to the triangles TQR and $T'Q'R'$ yields $T \sqcup Q \parallel T' \sqcup Q'$, hence $Q \sqcup T \parallel Q' \sqcup T'$. Finally apply (D1) in $\mathcal{N}_{Q \sqcup Q'}$ to the triangle TQR and $T'Q'R'$ to obtain $Q \sqcup R \parallel Q' \sqcup R'$. □

6. Algebraic representation

In this section an algebraic representation is given of the nearaffine translation planes. The tedious but straightforward proofs are omitted. For details see [12].

Let G and G' be two groups of order n written additively. We do not assume that G or G' is Abelian or that $G \simeq G'$ (although the same symbol + is used for addition in both groups). Let $\mathscr{F}$ be a set of $(n-1)$ mappings $f_i : G \to G'$, $i = 1,\ldots,n-1$, such that the following conditions are satisfied.

(i) f_i is a bijection for all $i = 1,\ldots,n-1$.
(ii) $f_i(0) = 0$ for all $i = 1,\ldots,n-1$.
(iii) $f_i(\alpha) = -f_i(-\alpha)$ for all $i = 1,\ldots,n-1, \alpha \in G$.
(iv) $f_i(\alpha) \neq f_j(\alpha)$ for $1 \leqslant i < j \leqslant n-1, \alpha \in G\backslash\{0\}$.
(v) For all $i = 1,\ldots,n-1$ either,

$$\forall_{\alpha \in G\backslash\{0\}} \exists_{\beta \in G} [f_i(\alpha + \beta) \neq f_i(\alpha) + f_i(\beta)]$$

or

$$\forall_{\alpha,\beta \in G} [f_i(\alpha + \beta) = f_i(\alpha) + f_i(\beta)]$$

and $f_i - f_j$ is a bijection for $j = 1,\ldots,n-1, j \neq i$.

Given such a set of mappings $\mathscr{F}$ it is possible to construct a nearaffine translation plane in the following way. Put $X := G \times G'$. For $x, y \in X$, $x = (\xi, \xi'), g = (\eta, \eta'), x \neq y$, define:

$$x \sqcup y := \begin{cases} \{(\xi, \alpha') \mid \alpha' \in G'\} & \text{if } \xi = \eta, \\ \{(\alpha, \xi') \mid \alpha \in G\} & \text{if } \xi' = \eta', \\ \{(\xi + \alpha, \xi' + f_i(\alpha) \mid \alpha \in G\} & \text{if } \xi \neq \eta, \xi' \neq \eta' \text{ and} \\ & f_i(-\xi + \eta) = -\xi' + \eta'. \end{cases}$$

The line set L is just the set of all $x \sqcup y$, $x \neq y$. For any line $l = x \sqcup y$ we let $\mathrm{d}(l) \in \{0, 1,\ldots,n-1, \infty\}$ be determined by

$$\mathrm{d}(l) := \begin{cases} \infty & \text{if } \xi = \eta, \\ 0 & \text{if } \xi' = \eta', \\ \mathrm{i} & \text{if } \xi' \neq \eta', \xi' \neq \eta' \text{ and} f_i(-\xi + \eta) = -\xi' + \eta'. \end{cases}$$

Notice that $\mathrm{d}(l)$ only depends on l and not on the special choice of x and y. Define parallelism by

$$l \parallel m :\Leftrightarrow \mathrm{d}(l) = \mathrm{d}(m),$$

then $\mathscr{N} = (X, L, \sqcup, \parallel)$ is a nearaffine translation plane. Conversely, every nearaffine translation plane can be described in this way. The parallel classes $C_0 := \{l \in L \mid \mathrm{d}(l) = 0\}$, $C_\infty := \{l \in L \mid \mathrm{d}(l) = \infty\}$ consist of straight lines. For each $\alpha \in G$, the mapping $(\xi, \xi') \to (\alpha + \xi, \xi')$ is a translation with direction C_0. For each $\alpha' \in G'$, the mapping $(\xi, \xi') \to (\xi, \alpha' + \xi')$ is a translation with direction C_∞. For $i = 1,\ldots,n-1$, $C_i := \{l \in L \mid \mathrm{d}(l) = i\}$ consists of straight lines iff i satisfies the second alternative of (v).

The Veblen-condition (V') is satisfied if for $1 \leqslant i < j \leqslant n-1$,

(a) $f_i - f_j : G \to G'$ is a bijection,
(b) $\bar{f}_i - \bar{f}_j : G' \to G$ is a bijection,
(c) for all $k \in \{1,\ldots,n-1\}$ which satisfy the second alternative of (v) and for all $\gamma \in G$ there is a unique solution α of $f_k(\gamma) = f_j(\gamma + \alpha) - f_i(\alpha)$.

7. Nearaffine planes and Latin squares

It is well known that the existence of an affine plane of order n is equivalent to the existence of $n-1$ mutually orthogonal Latin squares (MOLS) of order n (see [5]). For nearaffine plane the following result holds.

THEOREM 7.1. *If $\mathcal{N}$ is a nearaffine plane of order n with $s+1$ parallel classes containing straight lines ($s<n$), then there exist s MOLS of order n.*

Proof. The $n(s+1)$ lines in the $s+1$ parallel classes consisting of straight lines together with n lines from a parallel class consisting of proper lines, all having their base points on a fixed straight line, constitute an $(s+2)$ – net of order n. This is equivalent to the existence of s MOLS of order n (see, e.g., [5]). □

Let N be an integer, $N \geqslant 2$, and suppose N has prime decomposition $N = = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$. Define

$$s(N) := \min_{1 \leqslant i \leqslant k} p_i^{\alpha_i} - 1.$$

It is well known (see, e.g., [5]) that there exist at least $s(N)$ MOLS of order N (the so-called MacNeish bound). The following theorem shows therefore that, as far as nearaffine translation planes are concerned, we cannot hope for interesting applications of 7.1

THEOREM 7.2. *Let $\mathcal{N}$ be a nearaffine translation plane of order n with $s+1$ parallel classes containing straight lines. Then $s \leqslant s(n)$.*

Proof. Notice that $s-1$ of the f_i's associated with $\mathcal{N}$, say $f_1, f_2, \dots, f_{s-1}$, satisfy the second alternative of (v) of Section 6. Put $\phi_i := \bar{f}_1 \circ f_i$, $i = 1, 2, \dots, s$. Then $\phi_i - \phi_j : G \to G$ is a permutation of the elements of G, $1 \leqslant i < j \leqslant s$. Hence, the Latin squares $A^{(i)} = [a^{(i)}_{x,y}]$ defined by

$$a^{(i)}_{x,y} := \phi_i(x) + y, \quad i = 1, \dots, s, \quad x, y \in G,$$

are mutually orthogonal. Since $\phi_1, \phi_2, \dots, \phi_{s-1}$ are automorphisms of G it follows by a theorem of H. B. Mann (see [6] or [9]) that $s-1 \leqslant s(n)$. Suppose $s-1 = s(n) = p^\alpha - 1$, p a prime, $\alpha \in \mathbb{N}$. It follows from the proof of Mann's theorem that the elements $\neq 0$ of a Sylow p-subgroup P of G are all in different conjugacy classes. Thus $-y+x+y \in P \Rightarrow -y+x+y = x$ for all $x \in P$, $y \in G$. In particular, if $y \in N_G(P)$ then $x+y = y+x$ for all $x \in P$, i.e., $P \leqslant Z(N_G(P))$. By a theorem of Burnside (see [7] or [8]), G contains a normal p-complement N. Since $|G \backslash N|$ and $|N|$ are coprime, N is a characteristic subgroup of G. Thus, the rows and columns of $A^{(1)}, \dots, A^{(s-1)}$ which correspond to the elements of N, yield mutually orthogonal Latin subsquares of order n/p^α. By a theorem of Parker (see [10]) such a set of $s-1$ MOLS cannot be extended to a set of s MOLS, a contradiction. Hence, $s-1 < s(n)$, i.e., $s \leqslant s(n)$. □

8. Construction of Nearaffine Planes

Using the representation of nearaffine translation planes of Section 6, we treat a special case of the more general construction described in [12]. The nearaffine planes thus obtained turn out to be associated with certain Minkowski planes. Let p be a prime, h a positive integer and $n = p^h$. For the groups G and G' of Section 6 we take the additive group of $GF(n)$. Fix an automorphism ϕ of $GF(n)$, and for each $a \in GF(n)^*$ define $f_a : GF(n) \to GF(n)$ by $f_a(0) := 0$ and

$$f_a(x) := ax^{-1}, \quad x \in GF(n)^*, \text{ if } a \text{ is a square,}$$

$$f_a(x) := a(x^{-1})^{\phi}, \quad x \in GF(n)^*, \text{ if } a \text{ is a nonsquare.}$$

The set $\mathscr{F} := \{f_a \mid a \in GF(n)^*\}$ is easily seen to satisfy the properties (i),...,(v) of Section 6. The corresponding nearaffine plane is of order n, and $s = 1$. It is also not hard to show that the Veblen-condition holds in these nearaffine planes.

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FINITE MINKOWSKI PLANES

ABSTRACT. In this paper we give second characterizations of a certain class of finite Minkowski planes.

1. INTRODUCTION

It is well known, see e.g. [5], that with each point of a Minkowski plane there is associated an affine plane, its so-called derived plane. It is the purpose of this paper to show that, under certain additional hypotheses, with each point of a Minkowski plane there is also associated a nearaffine plane, its *residual* plane. In addition we show that the 'known' Minkowski plane are characterized by the fact that these nearaffine planes are nearaffine translation planes (see [9]). Using this result a configurational condition is obtained in a completely natural way which characterizes the known Minkowski planes.

2. BASIC CONCEPTS

Let M be a set of *points* and $\mathscr{L}^+, \mathscr{L}^-, \mathscr{C}$ three collections of subsets of M. The elements of $\mathscr{L} := \mathscr{L}^+ \cup \mathscr{L}^-$ are called *lines* or *generators*, the elements of $\mathscr{C}$ are called *circles*. We say that $\mathscr{M} = (M, \mathscr{L}^+, \mathscr{L}^-, \mathscr{C})$ is a *Minkowski plane* if the following axioms are satisfied (cf. [5]):

(M1): $\mathscr{L}^+$ and $\mathscr{L}^-$ are partitions of M.

(M2): $|l^+ \cap l^-| = 1$ for all $l^+ \in \mathscr{L}^+, l^- \in \mathscr{L}^-$.

(M3): Given any three points no two on a line, there is a unique circle passing through these three points.

(M4): $|l \cap c| = 1$ for all $l \in \mathscr{L}$, $c \in \mathscr{C}$.

(M5): There exist three points no two of which are on one line.

(M6): Given a circle c, a point $P \in c$ and *a* point $Q \notin c$, P and Q not on one line, there is a unique circle d such that $P, Q \in d$ and $c \cap d = \{P\}$.

Two points P and Q are called *plus-parallel* (notation $P \parallel_+ Q$) if P and Q are on a line of $\mathscr{L}^+$, *minus-parallel* ($P \parallel_- Q$) if P and Q are on a line of $\mathscr{L}^-$. *Parallel* ($P \parallel Q$) means either $P \parallel_+ Q$ or $P \parallel_- Q$. For $P \in M$, $\epsilon = +, -$ we denote by $[P]_\epsilon$ the unique line in $\mathscr{L}^\epsilon$ incident with P. If P, Q and R are (distinct) nonparallel points, then we denote by (P, Q, R) the unique circle containing P, Q and R. Two circles c and d *touch* in a point P if $c \cap d = \{P\}$.

Fix a point Z and put

$$M_Z := M \backslash ([Z]_+ \cup [Z]_-),$$
$$L_Z := \{c^* \mid c \in \mathscr{C}, Z \in c\} \cup \{l^* \mid l \in \mathscr{L} \backslash \{[Z]_+, [Z]_-\}\},$$

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where the * indicates that we have removed the point that the circle or line has in common with $[Z]_+ \cup [Z]_-$. Then $\mathcal{M}_Z := (M_Z, L_Z)$ is an affine plane with pointset M_Z and lineset L_Z (see, e.g., [5]). We call $\mathcal{M}_Z$ the *derived plane* with respect to the point Z. We shall only consider finite Minkowski planes, i.e., Minkowski planes with a finite number of points. For finite Minkowski planes (M6) is a consequence of the other axioms (see [5]). It is easily seen that $|\mathcal{L}^+| = |\mathcal{L}^-| = |l| = |c| = :n+1$ for all $l \in \mathcal{L}, c \in \mathcal{C}$. The integer n is called the *order* of the Minkowski plane. Notice that n is also the order of the derived planes $\mathcal{M}_Z$.

Following Benz [1] we sketch the close relationship between (finite) Minkowski planes and sharply 3-transitive sets of permutations. Let Ω be a finite set, $|\Omega| = n+1 \geqslant 3$, and G a subset of S^Ω, the symmetric group on Ω, acting sharply triply transitively on Ω.

Define

$$\begin{aligned} M &:= \Omega \times \Omega, \\ \mathcal{L}^+ &:= \{\{(\alpha,\beta) \mid \alpha \in \Omega\} \mid \beta \in \Omega\}, \\ \mathcal{L}^- &:= \{\{(\alpha,\beta) \mid \beta \in \Omega\} \mid \alpha \in \Omega\}, \\ \mathcal{C} &:= \{\{(a,\alpha^g) \mid \alpha \in \Omega\} \mid g \in G\}. \end{aligned}$$

Then $\mathcal{M} := (\Omega, G) := (M, \mathcal{L}^+, \mathcal{L}^-, \mathcal{C})$ is a Minkowski plane of order n. Conversely, every Minkowski plane can be obtained in this way.

Two Minkowski planes $\mathcal{M} = (\Omega, G) = (M, \mathcal{L}^+, \mathcal{L}^-, \mathcal{C})$ and $\mathcal{M}' = (\Omega', G') = (M', \mathcal{L}^{+\prime}, \mathcal{L}^{-\prime}, \mathcal{C}')$ are said to be *isomorphic* if there is a bijection $s : M \to M'$ such that

$$\mathcal{L}^s = \mathcal{L}' \quad \text{and} \quad \mathcal{C}^s = \mathcal{C}'.$$

Since s maps the disjoint lines of $\mathcal{L}^+$ onto disjoint lines there are only two possibilities, either $(\mathcal{L}^\varepsilon)^s = \mathcal{L}^\varepsilon$ or $(\mathcal{L}^\varepsilon)^s = \mathcal{L}^{-\varepsilon}$, $\varepsilon = +, -$. In the first case s is called a *positive isomorphism* in the second case a *negative isomorphism*. If s is a positive isomorphism then there exist bijections $a, b : \Omega \to \Omega'$ such that $(\alpha,\beta)^s = (\alpha^a, \beta^b)$ for all $\alpha, \beta \in \Omega$, and $G' = a^{-1}Gb$. If s is a negative isomorphism then there exist bijections $a, b : \Omega \to \Omega'$ such that $(\alpha,\beta)^s = (\beta^b, \alpha^a)$, and $G' = b^{-1}G^{-1}a$. It follows that we may assume w.l.o.g that $\mathrm{id} \in G$.

A (*positive, negative*) automorphism of a Minkowski plane $\mathcal{M}$ is a (positive, negative) isomorphism of $\mathcal{M}$ onto itself. The automorphism group $\mathrm{Aut}(\Omega, G) \leqslant S^{\Omega \times \Omega}$ of the Minkowski plane (Ω, G) is given by

$$\mathrm{Aut}(\Omega, G) = \{(a,b) \mid a^{-1}Gb = G\} \cup \{(a,b) \mid a^{-1}Gb = G^{-1}\}\tau$$

where τ is the permutation which sends (α,β) to (β,α).

3. The residual plane

Let $\mathcal{M} = (M, \mathcal{L}^+, \mathcal{L}^-, \mathcal{C})$ be a Minkowski plane. Fix a point $Z \in M$ and define $M_z = M \backslash ([Z]_+ \cup [Z]_-)$. We have already remarked that the lines $\neq$

$[Z]_+, [Z]_-$ together with the circles which are incident with Z are the lines of an affine plane with pointset M_Z. We shall show that the lines $\neq [Z]_+$, $[Z]_-$ together with the circles not incident with Z are the lines of a nearaffine plane with the same pointset if suitable conditions are assumed to hold in $\mathscr{M}$.

For each point $P \in M_Z$ we let the points P^+ and P^- be defined by $P^+ := [Z]_+ \cap [P]_-$, $P^- := [Z]_- \cap [P]_+$. The restriction of a line l or circle c to M_Z is denoted by $l^* := l \cap M_Z$ resp. $c^* := c \cap M_Z$. For any two distinct points $P, Q \in M_Z$ we define

$$P \sqcup Q := \begin{cases} l^* & \text{iff } P, Q \in l \in \mathscr{L}, \\ \{P\} \cup (P^+, P^-, Q)^* & \text{iff } P \text{ and } Q \text{ are nonparallel.} \end{cases}$$

Since two circles can have at most two points in common it follows that $P \sqcup Q = Q \sqcup P$ if and only if $P \sqcup Q = l^*$ for some $l \in \mathscr{L}$, provided the order n of $\mathscr{M}$ is at least 5. The verification of the axioms (L1), (L2) and (L3) (see [9]) is now straightforward. In order to define parallelism we have to require that the following condition holds in $\mathscr{M}$ for every point Z.

(A) Let $P_1, Q_1, P_2, Q_2 \in M_Z$ and suppose that P_1 and Q_1, P_2 and Q_2, P_1 and P_2 are nonparallel. If there exists a circle c touching (P_1^+, P_1^-, Q_1) in P_1^- and touching (P_2^+, P_2^-, Q_2) in P_2^+, then there also exists a circle d touching (P_1^+, P_1^-, Q_1) in P_1^+ and touching (P_2^+, P_2^-, Q_2) in P_2^- (see Figure 1).

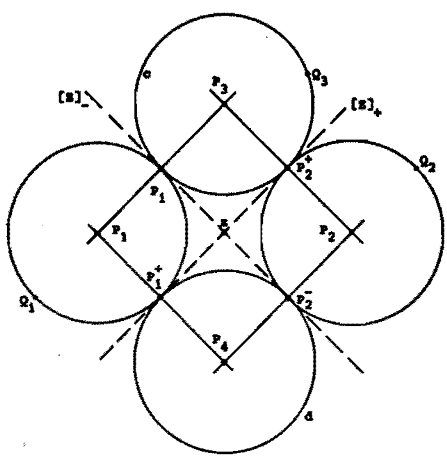

Fig. 1.

In the definition of $P_1 \sqcup Q_1 \parallel P_2 \sqcup Q_2$ we have to distinguish several cases.

Case 1. P_1 and Q_1 parallel, say $P_1 \sqcup Q_1 = l_1^*$ for some $l_1 \in \mathscr{L}^\varepsilon$.

$$P_1 \sqcup Q_1 \parallel P_2 \sqcup Q_2 :\Leftrightarrow P_2 \sqcup Q_2 = l_2^* \quad \text{for some } l_2 \in \mathscr{L}^\varepsilon.$$

Case 2. P_1 and Q_1 nonparallel, P_1, P_2 parallel, say $P_1, P_2 \in l\mathscr{L}^\varepsilon$. From [9], proposition 3.1, it is clear that we have to define

$$P_1 \sqcup Q_1 \parallel P_2 \sqcup Q_2 :\Leftrightarrow P_1 \sqcup Q_1 = P_2 \sqcup Q_2$$

or

$$(P_1 \sqcup Q_1) \cap (P_2 \sqcup Q_2) = \emptyset.$$

Case 3. P_1 and Q_1 nonparallel and P_1, P_2 nonparallel. Put $P_3 = [P_1]_+ \cap \cap [P_2]_-$ and $P_4 := [P_1]_- \cap [P_2]_+$ (see Figure 1). $P_1 \sqcup Q_1 \parallel P_2 \sqcup Q_2 :\Leftrightarrow \Leftrightarrow$ There exists $P_3 \sqcup Q_3$ such that

$$(P_3 \sqcup Q_3) \cap (P_1 \cap Q_1) = \emptyset = (P_3 \cap Q_3) \cap (P_2 \sqcup Q_2).$$

Notice that condition (A) is equivalent to: $P_1 \sqcup Q_1 \parallel P_2 \sqcup Q_2$ implies $P_2 \sqcup Q_2 \parallel P_1 \sqcup Q_1$, i.e., parallelism is a symmetric relation. We prove that parallelism is a transitive relation. Suppose $P_1 \sqcup Q_1 \parallel P_2 \sqcup Q_2$ and $P_2 \sqcup Q_2 \parallel P_3 \sqcup Q_3$ (with distinct P_1, P_2, P_3). We prove that $P_1 \sqcup Q_1 \parallel P_3 \sqcup Q_3$.

Case (a). $P_1 \parallel Q_1$. Trivial

Case (b). $P_1 \nparallel Q_1, P_2, P_3 \in l$ for some $l \in \mathscr{L}$. The transitivity follows at once from the following observation. If $c, d, e, \in \mathscr{C}$ and c and d touch in a point P, d and e touch in the same point P, then c and e touch in P. To show this suppose $Q \in c \cap e, Q \neq P$, then there are two circles through Q, namely c and e, touching d in P. This contradicts (M6).

Case (c). $P_1 \nparallel Q_1, P_1 \in [P_2]_\varepsilon, P_3 \in [P_2]_{-\varepsilon}$ for some $\varepsilon = +, -$. By definition $P_1 \sqcup Q_1 \parallel P_3 \sqcup Q_3$.

Case (d). $P_1 \nparallel Q_1, P_1 \parallel_\varepsilon P_2$ for some $\varepsilon = +, -, P_3 \nparallel P_1, P_3 \nparallel P_2$. Put $P_4 := [P_2]_\varepsilon \cap [P_3]_{-\varepsilon}$. Since $P_2 \sqcup Q_2 \parallel P_3 \sqcup Q_3$ there exists Q_4 such that $P_2 \sqcup Q_2 \parallel P_4 \sqcup Q_4 \parallel P_3 \sqcup Q_3$. Apply case (b) to find $P_1 \sqcup Q_1 \parallel P_4 \sqcup Q_4$ and case (c) to find $P_1 \sqcup Q_1 \parallel P_3 \sqcup Q_3$.

Case (e). $P_1 \nparallel Q_1, P_1 \parallel_\varepsilon P_3$ for some $\varepsilon = +, -, P_2 \nparallel P_1, P_2 \nparallel P_3$. Put $P_4 := = [P_1]_\varepsilon \cap [P_2]_{-\varepsilon}$. There exists Q_4 such that $P_1 \sqcup Q_1 \parallel P_4 \sqcup Q_4$ and $P_4 \sqcup Q_4 \parallel P_3 \sqcup Q_3$. Apply case (b).

Case (f). $P_1 \nparallel Q_1, P_1, P_2, P_3$ mutually nonparallel. Put $P_4 := [P_1]_+ \cap [P_2]_-$. There exists Q_4 and that $P_1 \sqcup Q_1 \parallel P_4 \sqcup Q_4$ and $P_4 \sqcup Q_4 \parallel P_2 \sqcup Q_2$. Apply case (d) to find $P_4 \sqcup Q_4 \parallel P_3 \sqcup Q_3$ and so $P_1 \sqcup Q_1 \parallel P_3 \sqcup Q_3$.

Let L^Z be the set of all $P \sqcup Q, P, Q \in M_Z$, $P \neq Q$. It is not hard to show that $\mathscr{M}^Z := (M_z, L^Z, \sqcup, \parallel)$ satisfies all the axioms of a nearaffine plane except possibly (P2) or (P2'). For (P2) to hold we have to require:

(B) Let P_1, Q_1, P_2, Q_2 be points as in (A). If $P_1 \in (P_2^+, P_2^-, Q_2)$ and $P_2 \in (P_1^+, P_1, Q_1)$. Then circles c and d as described in (A) exist.

If we content ourself with the weaker (P2') we have to require:

(C) Let ε be $+$ or $-$, A and B to distinct points on $[Z]_\varepsilon$, $A \neq Z \neq B$

and c_1 and c_2 two circles touching in A. Put (see Figure 2)

$$C_i := [Z]_{-\varepsilon} \cap c_i, \quad i = 1, 2,$$
$$P_i := [A]_{-\varepsilon} \cap [C_i]_\varepsilon, \quad i = 1, 2,$$
$$Q_i := [B]_\varepsilon \cap c_i, \quad i = 1, 2,$$
$$D_i := [Q_i]_\varepsilon \cap [Z]_{-\varepsilon}, \quad i = 1, 2,$$
$$d_i := (P_i, D_i, B), \quad i = 1, 2.$$

Then d_1 and d_2 touch in B.

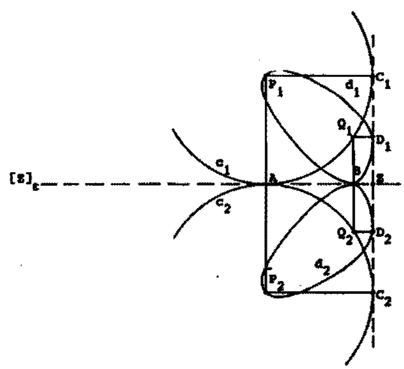

Fig. 2.

If $\mathcal{M}$ is a Minkowski plane satisfying the conditions (A) and (B) or (A) and (C) and Z a point of $\mathcal{M}$, then the nearaffine plane $\mathcal{M}^Z$ is called the *residual plane* with respect to Z.

For the remainder of the section let $\mathcal{M} = (M, \mathcal{L}^+, \mathcal{L}^-, \mathcal{C})$ be a Minkowski plane satisfying the conditions (A) and (C). Since $\sqcup$ and $\|$ are defined strictly in terms of the incidence in $\mathcal{M}$ it follows at once that an automorphism of $\mathcal{M}$ fixing a point Z, induces an automorphism of $\mathcal{M}^Z$, i.e., Aut $(\mathcal{M})_Z \lesssim$ Aut $(\mathcal{M}^Z)$. In fact, Aut $(\mathcal{M})_Z \simeq$ Aut $(\mathcal{M}^Z)$ as we shall see in a moment. The crucial observation is the following lemma.

3.1. LEMMA. *Let Z be a point of $\mathcal{M}$. For any two nonparallel points A and B of M_Z let $[A, B]$ be the set of points consisting of A, B, Z and the points $C \in M_Z$, nonparallel to A and B, for which there is no set $P \sqcup Q \setminus \{P\}$ containing A, B and C. Then*

$$[A, B] = (A, B, Z).$$

Proof. Clearly both $[A, B]$ and (A, B, Z) contain $A, B,$ and Z. Let $C \in (A, B, Z)$, $C \neq A, B, Z$ then $(A, B, C) = (A, B, Z)$. Suppose for some $P, Q \in M_Z$ we have $A, B, C \in P \sqcup Q \setminus \{P\}$. Then $A, B, C \in (P^+, P^-, Q) \setminus \{P^+, P^-\}$, so $(A, B, C) = (P^+, P^-, C)$ a circle not passing through Z, a contradiction. Conversely, let $C \in [A, B]$, $C \neq A, B, Z$ and suppose $C \in (A, B, Z)$. Then

$Z \notin (A, B, C)$ and so (A,B,C) intersects $[Z]_+$ and $[Z]_-$ in points P^+ and P^- respectively, different from Z. So, with P defined by $P = [P^+]_- \cap [P^-]_+$, A, B, C are on $P \sqcup Q \setminus \{P\}$, a contradiction. □

The lemma just proved shows that the residual plane $\mathcal{M}^Z$ completely determines the Minkowski plane $\mathcal{M}$. The lines of $\mathcal{M}$ can be recovered from the straight lines of $\mathcal{M}^Z$, the circles not containing Z from the proper lines of M^Z, and the circles containing Z from the sets $[A, B]$. This proves the following theorem.

3.2. THEOREM. *Let Y and Z be the points of* $\mathcal{M}$. *Then*

(a) $\mathcal{M}^Y \simeq \mathcal{M}^Z$ *iff there exists* $\phi \in \mathrm{Aut}(\mathcal{M})$ *such that* $Y^\phi = Z$.
(b) *Any automorphism of* $\mathcal{M}^Z$ *can be extended to an automorphism of* $\mathcal{M}$ *fixing z.*
(c) $\mathrm{Aut}(\mathcal{M})_Z \simeq \mathrm{Aut}(\mathcal{M}^z)$.

It is not hard to show that for any point Z of $\mathcal{M}$ the residual plane $\mathcal{M}^Z$ satisfies the Veblen-condition (V'). In fact we can prove somewhat more.

3.3. THEOREM. *Let* $Z \in M, l \in \mathcal{L}, l \neq [Z]_+, [Z]_-$ *and let Y be defined by* $Y = l \cap ([Z]_+ \cup [Z]_-)$. *Then* $\mathcal{M}^Z_{l^*} \simeq \mathcal{M}_Y$, *where* l^* *is the straight line* $l \setminus \{Y\}$ of $\mathcal{M}^Z$ (notation as in [9]).

Proof. Define an isomorphism $\phi: M_Z \to M_Y$ of $\mathcal{M}^Z_{l^*}$ onto $\mathcal{M}_Y$ as follows. For $P \in M_Z, P \notin l^*$ we define $P^\phi := P$, and for $P \in M_Z, P \in l^*, P^\phi := [P]_{-\varepsilon} \cap [Z]_\varepsilon$, where ε is determined by $l \in \mathcal{L}^\varepsilon$. □

As a direct consequence of this theorem we have the following result.

3.4. THEOREM. *If the derived* $\mathcal{M}_Z$ *is a translation plane for every* $Z \in M$, *then the residual plane* $\mathcal{M}^Z$ *is a nearaffine translation plane for every* $Z \in M$.

Proof. Apply 3.3 and 5.2 of [9]. □

As a converse to this theorem we mention the following theorem.

3.5. THEOREM. *Let Z be a point of* $\mathcal{M}$. *if* $\mathcal{M}^Z$ *is a nearaffine translation plane, then* $\mathcal{M}_Z$ *is a translation plane and* $\mathcal{M}^Z$ *and* $\mathcal{M}_Z$ *have the same translation group.*

Proof. By 3.2 every automorphism of $\mathcal{M}^Z$ is also an automorphism of $\mathcal{M}_Z$, and it is not hard to show that a straight translation of $\mathcal{M}^Z$ with a direction corresponding to $\mathcal{L}^\varepsilon$ is also a translation of $\mathcal{M}$. Let $\mathcal{T}_+$ and $\mathcal{T}_-$ be the translation groups of $\mathcal{M}^Z$ with directions $\mathcal{L}^+$ and $\mathcal{L}^-$ respectively. Since $\mathcal{T}_+$ and $\mathcal{T}_-$ are also translation groups of $\mathcal{M}_Z$ it follows that $\mathcal{T}_+$ and $\mathcal{T}_-$ are elementary abelian. Hence, by 4.12 of of [9], the set $\mathcal{T}$ of all translation of $\mathcal{M}^Z$ is a group and $\mathcal{T} = \mathcal{T}_+ \mathcal{T}_-$ = the full translation group of M_Z. □

4. Characterizations of the Known Finite Models

Using the correspondence with sharply triply transitive sets of permutations all known (finite) Minkowski planes can be described as follows. Let P be a prime, h a positive integer, $q := p^h$ and ϕ an automorphism of $GF(q)$. Let $G(\phi)$ be the set of permutations acting on the projective line $\Omega := PG(1,q) = GF(q) \cup \cup \{\infty\}$ given by

$$x \to \frac{ax+b}{cx+d}, \quad a,b,c,d \in GF(q), ad-bc = \text{(nonzero) square in}$$

$GF(q)$,

$$x \to \frac{ax^{\phi}+b}{cx^{\phi}+d}, \quad a,b,c,d \in GF(q), ad-bc = \text{nonsquare in } GF(q),$$

i.e., $G(\phi) = G_1 \cup \phi G_2$, where $G_1 := PSL(2,q)$ and $G_2 := PG(2,q) \backslash PSL(2,q)$. Then $G(\phi)$ is sharply triply transitive on Ω (cf. [7], [8], [10]). The residual planes of $(\Omega, G(\phi))$ are easily seen to be the nearaffine translation planes described in [9], Section 8. We shall show that a Minkowski plane whose residual planes are nearaffine translation planes, is isomorphic to an $(\Omega, G(\phi))$.

Let c be a circle of a Minkowski plane $\mathcal{M}$ of order n and Z a point of $\mathcal{M}$, $Z \notin c$. If $\mathcal{M}_Z$ is augmented to a projective plane, then the points of $c^* = c \backslash ([Z]_+ \cup [Z]_-)$ together with the two ideal points corresponding to $\mathcal{L}^+$ and $\mathcal{L}^-$ constitute an oval in this projective plane. In n is even, there exists a point (the *nucleus* of the oval) in the projective plane such that the $n+1$ lines incident with this point are the $n+1$ tangents of the oval. If n is odd, each point of the projective plane is incident with 0 or 2 tangents (see [3]). From this observation we deduce the following lemma.

4.1. LEMMA. *Let $\mathcal{M}$ be a Minkowski plane of order n. If n is even, there cannot exist 3 distinct circles c_1, c_2, d such that c_1 and c_2 touch in a point Z and c_i touches d in $P_i \neq Z$, $i = 1, 2$. In any case there cannot exist 4 distinct circles c_1, c_2, c_3 and d such that c_1, c_2, c_3 touch in a point Z and such that c_i touches d in a point $p_i \neq Z$, $i = 1, 2, 3$.*

Proof. Case n is even. Suppose circles c_1, c_2 and d as described exist. The lines $[[Z]_+ \cap d]_-$ and $[[Z]_- \cap d]_+$ are tangents to the oval corresponding with d in the projective plane associated with $\mathcal{M}_Z$. They intersect in a point of M_Z. Also c_1 and c_2 are tangents to the oval. They intersect in an ideal point of the projective plane, a contradiction.

Case n is odd. Now c_1, c_2 and c_3 correspond to tangents of the oval d in the projective plane associated with $\mathcal{M}_Z$. They intersect in one (ideal) point, a contradiction. □

4.2. THEOREM. *Let $\mathcal{M} = (\Omega, G) = (M, \mathcal{L}^+, \mathcal{L}^-, \mathcal{C})$ be a Minkowski of order $n \geq 5$. Suppose conditions* (A) *and* (C) *hold in $\mathcal{M}$ and that $\mathcal{M}^Z$ is a nearaffine translation plane for every point Z. Then $\mathcal{M} \simeq (\Omega, G(\phi))$.*

Proof. Fix $\alpha_1 \in \Omega$. For each point $(\alpha_1, \beta) \in M$ there is an elementary Abelian group $\mathscr{T}_-(\alpha_1, \beta)$ of translations of $\mathscr{M}^{(\alpha_1,\beta)}$ and $\mathscr{M}_{(\alpha_1,\beta)}$, and $\mathscr{T}_-(\alpha_1, \beta) \lesssim \text{Aut}(\mathscr{M})$ (3.2, 3.4, 3.5). Each $\mathscr{T}_-(\alpha_1, \beta)$ fixes all lines of $\mathscr{L}^-$ and one lines of $\mathscr{L}^+$ (namely the line $\{(\alpha, \beta) | \alpha \in \Omega\}$). Using the notation of Section 2, each $\mathscr{T}_-(\alpha_1, \beta)$ consists of positive automorphisms of the form (1, b), where $b \in S^\Omega$ fixes β and $Gb = G$, i.e., for each $\beta \in \Omega$ there is an elementary Abelian group $B(\beta)$ which fixes β, acts regularly on $\Omega \backslash \{\beta\}$, and for which $GB(\beta) = G$. Define $B := \langle B(\beta) | \beta \in \Omega \rangle$, then B is doubly transitive on Ω and $GB = G$. Therefore, G is a union of cosets of B and in particular $B \subseteq G$. Hence, no nontrivial permutation in B leaves 3 letters fixed. By a theorem of Feit ([4]), B contains a normal subgroup of order $n+1$ or there exists an exactly triply transitive permutation group B_0 containing B such that $[B_0 : B] \leqslant 2$. Suppose B contains a normal subgroup of order $n+1$, then B also contains a sharply doubly transitive subgroup B^*. The circles $\{(\alpha, \alpha^g) | \alpha \in \Omega\}, g \in B^*$ together with the lines $l \in \mathscr{L}$ now constitute an affine plane of order $n+1$ and hence configuration as described in 4.1 exist, a contradiction. Therefore $B \leqslant B_0$, where B_0 is sharply 3-transitive, and $[B_0 : B] \leqslant 2$. All sharply triply transitive groups are known (see [6]). If n is even, then $B_0 \simeq PSL(2, n)$ and so $B = G = PSL(2, n)$, i.e. $\mathscr{M}$ is the classical Minkowski plane of order $n = 2^h$. If n is odd, there are at most two sharply 3-transitive groups of degree $n+1$ and such a group certainly contains $PSL(2, n)$. The Sylow p-subgroups $B(\beta)$ of B are the Sylow p-subgroups of $PSL(2, n)$. Therefore $B \leqslant PSL(2, n)$ and since $|B| \geqslant \frac{1}{2}(n+1)(n)(n-1)$ it follows that $B \simeq PSL(2, n)$. Thus, with $G_1 := PSL(2, n)$ and $G_2 := PGL(2, n) \backslash PSL(2, n)$,

$$G = G_1 \cup \phi G_2$$

for some $\phi \in S^\Omega$. It remains to show that ϕ is an automorphism of $GF(n)$. If x, y and z are three distinct points of Ω, then there is a $g \in G_1$ such that $x^\phi = x^g$, $y^\phi = y^g$, $z^\phi = z^g$ for otherwise there exists $h \in G_2$ such that $x^\phi = x^{\phi h}$, $y^\phi = y^{\phi h}$, $z^\phi = z^{\phi h}$, i.e., $h = 1$, contradicting $h \in G_2$. It follows that we may assume w.l.o.g. that ϕ fixes 0, 1 and ∞. If we do so it also follows that

$$\frac{x^\phi - y^\phi}{x - y} = \text{square in } GF(n) \text{ for all } x, y \in GF(n), \quad x \neq y,$$

for $g \in G_1$ determined by $x^\phi = x^g$, $y^\phi = y^g$, $\infty = \infty^\phi = \infty^g$ has determinant $(x^\phi - y^\phi)/(x - y)$. By a theorem of Bruen and Levinger (see [2]) it follows that ϕ is an automorphism of $GF(n)$. □

Using the previous theorems it is possible to give a geometric characterization of the Minkowski planes $(\Omega, G(\phi))$. Consider the following configurational condition:

(D) Let ε be $+$ or $-$, $l \in \mathscr{L}^\varepsilon$ and V, W to distinct points on l. Suppose c and c' are to distinct circles touching in V. Let Y and Q be two distinct points on c, $Y \nparallel W$, $Q \nparallel W$. Define

$$Y' := c' \cap [Y]_{-\varepsilon'}$$
$$Q' := c' \cap [Q]_{-\varepsilon'}$$
$$d := (Y, Q, W),$$
$$d' := (Y', Q', W).$$

Then d and d' touch in W (see Figure 3).

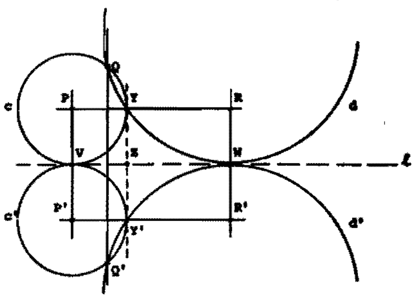

Fig. 3.

Notice that (D) is nothing but a special case of the Desarques configuration (D1) in $\mathcal{M}^Z$ on the points P, Q, R, P', Q', R'.

4.3. THEOREM. *Let $\mathcal{M}$ be a Minkowski plane of order $n \geqslant 5$, and suppose* (D) *holds in $\mathcal{M}$. Then $\mathcal{M}$ is isomorphic to one of the planes* $(\Omega, G(\phi))$.

Of course the proof of 4.3 is based on 4.2 and it is clear that (D) implies (A). Also (C) is a consequence of (D).

4.4. LEMMA. *Let $\mathcal{M}$ be a Minkowski plane of order n*

(a) *If n is even then* (A) *implies* (B) (*hence* (C)).

(b) *In any case* (D) *implies* (C).

Proof. (a) The following statement is easily seen to be equivalent to (B): If the circles c and d as described in (A) exist, then $P_1 \in (P_2^+, P_2^-, Q_2) \Leftrightarrow$ $\Leftrightarrow P_2 \in (P_1^+, P_1^-, Q_1)$. To prove this last statement, consider the configuration of condition (A) and suppose c and d exist, $P_2 \in (P_1^+, P_1^-, Q)$ but $P_1 \notin (P_2^+, P_2^-, Q_2)$. Let e be the circle through P_1 touching (P_2^+, P_2^-, Q_2) and c in P_2^+, f the circle through P_1 touching (P_1^+, P_1^-, Q_1) in P_2. By (A) e and f touch in P_1. Similarly it follows that the circle g through P_1 touching (P_2^+, P_2^-, Q_2) in P_2 touches f in P_1. Therefore, g and e touch in P_1 and so the circles g, e, (P_2^+, P_2^-, Q_2) touch each other in P_2^+, P_2^-, P_1. This contradicts 4.1 since n is even.

(b) Consider the configuration of condition (C). We claim that (P_1, Q_1, Z) and (P_2, Q_2, Z) touch in Z. If (P_i, Q_i, Z) touches c_i in Q_i for $i = 1, 2$, this follows from (A). Suppose, therefore, that (P_1, Q_1, Z) does not touch c_1 in Q_1, i.e., suppose that (P_1, Q_1, Z) has another point $E_1 \neq Q_1$ in common with

c_1. Put $E_2 := [E_1]_{-\varepsilon} \cap c_2$. By (D) the circles (E_2, Q_2, Z) and $(E_1, Q_1, Z) = (P_1, Q_1, Z)$ touch in Z. Suppose (E_2, Q, Z) intersects $[A]_{-\varepsilon}$ in a point $P'_2 \neq P_2$. Let Y be the point of intersection of $[Z]_\varepsilon$ and (E_2, P'_2, C_2). If we apply (D) twice it follows that (E_1, P_1, Y) and (E_1, C_1, Y) both touch (E_2, P'_2, C_2) in Y. Hence $(E_1, P_1, Y) = (E_1, C_1, Y)$ and impossibility because $P_1 \| C_1$. We have proved $P_2 \in (E_2, Q_2, Z)$, i.e., (P_1, Q_1, Z) and (P_2, Q_2, Z) touch in Z. So: c_1 and c_2 touch in A implies (P_1, Q_1, Z) and (P_2, Q_2, Z) touch in Z. It is easily seen that the converse also holds. If we replace c_i by d_i, $i = 1, 2$, it follows that d_1 and d_2 touch in B. □

To finish the proof of 4.3 we have to show that all residual planes $\mathscr{M}^Z$ are nearaffine translation planes. By 3.4 it suffices to show that all derived planed $\mathscr{M}_Z$ are translation planes.

4.5. LEMMA. *Let $\mathscr{M}$ be a Minkowski plane satisfying* (D), *then $\mathscr{M}_Z$ is a translation plane for every point Z.*

Proof. Let $Z \in M$ and $P, Q, R, P', Q', R' \in M_Z$ such that $P \|_- P'$, $Q \|_- Q'$, $R \|_- R'$, the line PQ (in $\mathscr{M}_Z$) is parallel to $P'Q'$ and PR is parallel to $P'R'$. We have to show that QR is parallel to $Q'R'$, i.e., we have to show that the circles (Z, Q, R) and (Z, Q', R') touch in Z. We assume here that P, Q, R (and also P', Q', R') are mutually nonparallel. The other cases follow from the cases we do consider. Put $Y = (P, Q, R) \cap [Z]_+$. If we apply (D) to (P, Q, Z), (P', Q', Z), $(P, Q, Y) = (P, Q, R)$ and (P', Q', Y), it follows that (P, Q, R) and (P', Q', Y) touch in Y. Application of (D) to (P, R, Z), (P', R', Z), $(P, R, Y) = (P, Q, R)$ and (P', R', Y) yields (P, Q, R) and (P', R', Y) touch in Y. Hence $(P', Q', Y) = (P', R', Y) = (P', Q', R')$. Finally we apply (D) to $(Q, R, Y)(Q', R', Y)$, (Q, R, Z) and (Q', R', Z) and obtain the desired result. □

Notice that it is possible to give a proof of 4.3 without using the theory of nearaffine planes. Show directly, using (D), that any translation of a desired planes $\mathscr{M}_Z$ extends to an automorphism of $\mathscr{M}$. Then argue as we did in 4.2.

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TWO-TRANSITIVE MINKOWSKI PLANES

ABSTRACT. In this paper we determine all finite Minkowski planes with an automorphism group which satisfies the following transitivity property: any ordered pair of nonparallel points can be mapped onto any other ordered pair of nonparallel points.

1. INTRODUCTION

All known finite inversive planes have a two-transitive group of automorphisms. Conversely, every inversive plane admitting an automorphism group which is two-transitive on the points, is of a known type (cf. [9]).

For Minkowski planes the situation is quite similar. All known finite Minkowski planes have an automorphism group acting two-transitively on non-parallel points. In this note we shall show that this property is characteristic for the known Minkowski planes. More precisely, we shall prove the following theorem.

THEOREM. *Let $\mathscr{M}$ be a finite Minkowski plane of odd order n, and suppose that $\mathscr{M}$ admits an automorphism group Γ acting two-transitively on non-parallel points. Then n is a prime power, $\mathscr{M} \simeq \mathscr{M}(n, \phi)$ for some field automorphism ϕ of $GF(n)$, and Γ contains $PSL(2, n) \times PSL(2, n)$.*

For a definition of $\mathscr{M}(n, \phi)$ see Section 2. As Minkowski planes of even order n only exist for n a power of 2 and are unique for given order $n = 2^a$, this result completes the classification of the Minkowski planes with an automorphism group acting two-transitively on nonparallel points.

2. DEFINITIONS, NOTATION AND BASIC RESULTS

Let M be a set of *points* and $\mathscr{L}^+, \mathscr{L}^-, \mathscr{C}$ three collections of subsets of M. The elements of $\mathscr{L} := \mathscr{L}^+ \cup \mathscr{L}^-$ are called *lines or generators*, the elements of $\mathscr{C}$ are called *circles*. We say that $\mathscr{M} = (M, \mathscr{L}^+, \mathscr{L}^-, \mathscr{C})$ is a *Minkowski plane* if the following axioms are satisfied (cf. [8]).

(M1): $\mathscr{L}^+$ and $\mathscr{L}^-$ are partitions of M.
(M2): $|l^+ \cap l^-| = 1$ for all $l^+ \in \mathscr{L}, l \in \mathscr{L}^-$.
(M3): Given any three points no two on a line, there is a unique circle passing through these three points.
(M4): $|l \cap c| = 1$ for all $l \in \mathscr{L}, c \in \mathscr{C}$.
(M5): There exist three points no two of which are on one line.

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(M6): Given a circle c, a point $P \in c$ and a point $Q \notin c$, P and Q not on one line, there is a unique circle d such that $P, Q \in d$ and $c \cap d = \{P\}$.

Two points P and Q are called *plus-parallel* (notation $P \parallel_+ Q$) if P and Q are on a line of $\mathscr{L}^+$, *minus-parallel* (notation $P \parallel_- Q$) if P and Q are on a line of $\mathscr{L}^-$. *Parallel* (notation $P \parallel Q$) means either $P \parallel_+ Q$ or $P \parallel_- Q$. For $P \in M$ we denote by $[P]_+$ (resp. $[P]_-$) the unique line in $\mathscr{L}^+$ (resp. $\mathscr{L}^-$) incident with P. If P, Q and R are (distinct) nonparallel points, then we denote by (P, Q, R) the unique circle containing P, Q and R. Two circles c and d *touch* in a point P if $c \cap d = \{P\}$.

We shall only consider finite Minkowski planes, i.e., Minkowski planes with a finite number of points. For finite Minkowski planes (M6) is a consequence of the other axiom (see [8]). It is easily seen that $|\mathscr{L}^+| = |\mathscr{L}^-| = |l| = |c| = : n + 1$ for all $l \in \mathscr{L}$, $c \in \mathscr{C}$. The integer n is called the order of the Minkowski plane. Fix a point P and put

$$M_P := M \backslash ([P]_+ \cup [P]_-),$$
$$L_P := \{c^* \mid c \in \mathscr{C}, P \in c\} \cup \{l^* \mid l \in \mathscr{L} \backslash \{[P]_+, [P]_-\}\},$$

where the * indicates that we have removed the point that the circle or line has in common with $[P]_+ \cup [P]_-$. Then $\mathscr{M}_P := (M_P, L_P)$ is an affine plane with point set M_P and line set L_P (see, e.g., [8]). The projective plane associated with $\mathscr{M}_P$ will be denoted by $\tilde{\mathscr{M}}_P$. We call $\mathscr{M}_P$ the *derived plane* with respect to the point P.

Following Benz [2], we sketch the close relationship between finite Minkowski planes and sharply triply transitive sets of permutations. Let Ω be a finite set, $|\Omega| = n + 1$, and let G be a subset of Sym (Ω), the symmetric group on Ω, acting sharply triply transitively on Ω. Define

$$M := \Omega \times \Omega,$$
$$\mathscr{L}^+ := \{\{(\alpha, \beta) \mid \alpha \in \Omega\} \mid \beta \in \Omega\},$$
$$\mathscr{L}^- := \{\{(\alpha, \beta) \mid \beta \in \Omega\} \mid \alpha \in \Omega\},$$
$$\mathscr{C} := \{\{(\alpha, \alpha^g) \mid \alpha \in \Omega\} \mid g \in G\}.$$

Then $\mathscr{M} := (\Omega, G) := (M, \mathscr{L}^+, \mathscr{L}^-, \mathscr{C})$ is a Minkowski plane of order n. Conversely, every Minkowski plane can be obtained in this way.

Two Minkowski planes $\mathscr{M} = (\Omega, G) = (M, \mathscr{L}^+, \mathscr{L}^-, \mathscr{C})$ and $\mathscr{M}' = (\Omega', G') = (M', \mathscr{L}^{+\prime}, \mathscr{L}^{-\prime}, \mathscr{C}')$ are said to be *isomorphic* if there is a bijection $s: M \rightarrow M'$ such that

$$\mathscr{L}^s = \mathscr{L}' \quad \text{and} \quad \mathscr{C}^s = \mathscr{C}'.$$

Either $(\mathscr{L}^+)^s = \mathscr{L}^{+\prime}$ and $(\mathscr{L}^-)^s = \mathscr{L}^{-\prime}$ or $(\mathscr{L}^+)^s = \mathscr{L}^{-\prime}$ and $(\mathscr{L}^-)^s = \mathscr{L}^{+\prime}$. In the first case, s is called a *positive isomorphism*, in the second case, a *negative isomorphism*. If s is a positive isomorphism then there exist bijections a, b: $\Omega \to \Omega'$ such that $(\alpha, \beta)^s = (\alpha^a, \beta^b)$ for all $\alpha, \beta \in \Omega$, and $a^{-1}Gb = G'$. If s is a negative isomorphism, then there exist bijections $a, b: \Omega \to \Omega'$ such that $(\alpha, \beta)^s = (\beta^b, \alpha^a)$ and $b^{-1}G^{-1}a = G'$. It follows that we may assume w.l.o.g. that G contains the identity permutation on Ω.

A (*positive, negative*) *automorphism* of a Minkowski plane $\mathscr{M}$ is a (positive, negative) isomorphism of $\mathscr{M}$ onto itself. The automorphism group

$\mathrm{Aut}(\Omega, G) \leqslant \mathrm{Sym}(\Omega \times \Omega)$ of the Minkowski plane (Ω, G) is given by

$$\mathrm{Aut}(\Omega, G) = \{(a, b) \in \mathrm{Sym}(\Omega) \times \mathrm{Sym}(\Omega) \mid a^{-1}Gb = G\}$$
$$\cup \{(a, b) \in \mathrm{Sym}(\Omega) \times \mathrm{Sym}(\Omega) \mid a^{-1}Gb = G^{-1}\}\tau$$

where $\tau \in \mathrm{Sym}(\Omega \times \Omega)$ is defined by $(\alpha, \beta)^\tau = (\beta, \alpha)$ for all $(\alpha, \beta) \in \Omega \times \Omega$.

We shall now describe all known finite Minkowski planes (cf. [14]).

Let q be a prime power and let ϕ be a field automorphism of GF(q). We shall denote by $\mathscr{M}(q, \phi)$ the Minkowski plane (Ω, G) with $\Omega = \mathrm{PG}(1, q)$, the projection line of order q, and G the subset of Sym (Ω) consisting of the permutations

$$x \mapsto \frac{ax+b}{cx+d}, \quad ad - bc = \text{a non-zero square of GF}(q),$$

and

$$x \mapsto \frac{ax^\phi+b}{cx^\phi+d}, \quad ad - bc = \text{a nonsquare of GF}(q).$$

Of course, if q is even, we always get $G = \mathrm{PSL}(2, q)$, and it can be shown that these are the only Minkowski planes of even order (see [7]). For q odd, G is a group if and only if $\phi^2 = 1$ (see [10]), and nonisomorphic Minkowski planes of the same order q can exist. Notice that $\mathscr{M}(q, \phi)$ has an automorphism group containing $\mathrm{PSL}(2, q) \times \mathrm{PSL}(2, q)$ which is two-transitive on nonparallel points, i.e., if P, Q, P', Q' are points such that $P \nparallel Q$ and $P' \nparallel Q'$, then there is an automorphism g satisfying $P^g = P'$ and $Q^g = Q'$.

We conclude this section by listing some theorems on permutation groups which will be fundamental in our investigations. For the more standard results on (permutation) groups, the reader is referred to [11] or [17].

Result 1 (Gleason's lemma). Let Γ be a permutation group of a finite set M such that, for some prime p, every element of M is fixed by a permutation in Γ which has order p and fixes no other element. Then Γ is transitive on M (see [5], 4.3.15, p.191).

A transitive permutation group which has the property that only the identity fixes more than one letter, but the subgroup fixing a letter is nontrivial, is called a *Frobenius group.*

Result 2. In a Frobenius group the elements which fix no letter together with the identity form a regular normal subgroup (see [11], p. 495).

The regular normal subgroup in Result 2 is called the *Frobenius kernel.*

Result 3. Let Γ be a 2-transitive permutation group on a finite set M with an even number of letters such that only the identity fixes more than two letters. Then either Γ contains a sharply 2-transitive normal subgroup and $|M|$ is a power of 2, or Γ contains PSL(2, q) as a normal subgroup of index $\leqslant 2$ (see [6] and [12]).

Result 4. Let Γ be a 2-transitive permutation group on a finite set M. If every element of Γ which fixes an element of M has odd order, then either Γ is solvable (in which case Γ is isomorphic to a subgroup of the group of semilinear transformations of a Galois field of characteristic 2) or Γ contains as normal subgroup isomorphic to PSL(2, q) (see [1]).

3. PROOF OF THEOREM

For the proof of our theorem we require a number of lemmas. The first lemma shows that we can assume without loss of generality that an automorphism group which is two-transitive on nonparallel points, contains positive automorphisms only.

LEMMA 1. *Let* $\mathcal{M} = (M, \mathcal{L}^+, \mathcal{L}^-, \mathcal{C})$ *be a Minkowski plane of odd order* n *and let* Γ^* *be a group of automorphisms of* $\mathcal{M}$ *two-transitive on nonparallel points. Then* $\Gamma := \Gamma_{\mathcal{L}^+} = \Gamma_{\mathcal{L}^-}$ *is also two-transitive on nonparallel points* ($\Gamma_{\mathcal{L}^+}$ *is the set-wise stabilizer of* $\mathcal{L}^+$ *in* Γ^*).

Proof. Let P and Q be two points, $P \nparallel Q$. Then

$$[\Gamma_P : \Gamma_{PQ}] = [\Gamma_P^* : \Gamma_{PQ}^*][\Gamma_{PQ}^* : \Gamma_{PQ}][\Gamma_P^* : \Gamma_P]^{-1},$$

Now $[\Gamma_P^* : \Gamma_{PO}^*] = |M_P| = n^2$ (as before $M_P = M \backslash ([P]_+ \cup [P]_-) = \{R \mid R \nparallel P\}$), and $[\Gamma_{PQ}^* : \Gamma_{PQ}]$, $[\Gamma_P^* : \Gamma_P] \in \{1, 2\}$ since $[\Gamma^* : \Gamma] \in \{1, 2\}$. Since n is odd it follows that $[\Gamma_P : \Gamma_{PQ}] = n^2$, i.e., Γ_P is transitive on M_P. Hence, Γ is two-transitive on nonparallel points. □

From now on $\mathcal{M} = (M, \mathcal{L}^+, \mathcal{L}^-, \mathcal{C}) = (\Omega, G)$ is a Minkowski plane of odd order $n \geqslant 5$ with a group Γ of positive automorphisms acting two-transitively on nonparallel points. (For $n = 3$ the theorem follows readily from [4].) We denote by $\Gamma(\mathcal{L}^\varepsilon)$ the subgroup of Γ fixing all lines of $\mathcal{L}^\varepsilon$, $\varepsilon = +, -$. Notice that $\Gamma(\mathcal{L}^{-\varepsilon})$ has a faithful representation on the $(n+1)$ lines of $\mathcal{L}^\varepsilon$, $\varepsilon = +, -$.

LEMMA 2. *If* $\Gamma(\mathscr{L}^{\varepsilon})$ *contains* PSL(2, *n*) *for* $\varepsilon = +$ *or* $-$, *then* $\mathscr{M} \simeq \mathscr{M}(n, \phi)$ *for some* $\phi \in AUT(\mathrm{GF}(n))$ *and* Γ *contains* $\mathrm{PSL}(2, n) \times \mathrm{PSL}(2, n)$.

Proof. For convenience we take $\varepsilon = -1$. As a permutation group on $M = \Omega \times \Omega$, Γ consists of permutations $(a_\gamma, b_\gamma) \in \mathrm{Sym}(\Omega) \times \mathrm{Sym}(\Omega)$ satisfying $a_\gamma^{-1} G b_\gamma = G$, $\gamma \in \Gamma$. Clearly, $\Sigma \leqslant \Gamma(\mathscr{L}^-)$ is equivalent to $a_\sigma = 1$ for all $\sigma \in \Sigma$. Hence $B := \{b_\sigma \mid \sigma \in \Sigma\}$ is a subgroup of $\mathrm{Sym}(\Omega)$ satisfying $GB = G$. Therefore G consists of a number of cosets of B, in particular $B \subseteq G$ since we are assuming that $1 \in G$. If $\Sigma \simeq B = G_1 := \mathrm{PSL}(2, n)$ then $G = G_1 \cup \phi G_2$ for some $\phi \in \mathrm{Sym}(\Omega)$ where $G_2 := \mathrm{PGL}(2, n) \backslash G_1 (|G_1| = \frac{1}{2}(n+1)n(n-1)$ and $|G| = |\mathscr{C}| = (n+1)n(n-1))$. Viewing Ω as the projective line $\mathrm{GF}(n) \cup \{\infty\}$ in the appropriate way, we claim that we may take $\phi \in \mathrm{Aut}(\mathrm{GF}(n))$. Let x, y and z be three distinct points of Ω. Since G is sharply triply transitive on Ω, there exists a $g \in G$ such that $x^\phi = x^g$, $y^\phi = y^g$ and $z^\phi = z^g$. Suppose $g \in \phi G_2$, i.e., $g = \phi g_2$ for some $g_2 \in G_2$, then $x^\phi = (x^\phi)^{g_2}$, $y^\phi = (y^\phi)^{g_2}$, $z^\phi = (z^\phi)^{g_2}$, and we get the contradiction $1 = g_2 \in G_2$.

We have shown: for any three distinct $x, y, z \in \Omega$ there is a $g_1 \in G_1$ such that $x^\phi = x^{g_1}$, $y^\phi = y^{g_1}$ and $z^\phi = z^{g_1}$. It follows that we may assume without loss of generality that ϕ fixes 0, 1 and ∞. If we do so it also follows that

$$\frac{x^\phi - y^\phi}{x - y} \text{ is a square in } \mathrm{GF}(n) \text{ for all } x, y \in \mathrm{GF}(n), x \neq y,$$

for $g_1 \in G_1$ determined by $x^\phi = x^g$, $y^\phi = y^g$, $\infty^\phi = \infty = \infty^g$ is the permutation $(\xi \mapsto ((x^\phi - y^\phi)/(x - y))(\xi - y) + y^\phi) \in G_1$. By a theorem of Bruen and Levinger (see [3]) it follows that $\phi \in \mathrm{Aut}(\mathrm{GF}(n))$. It remains to show that $\Gamma(\mathscr{L}^+)$ also contains PSL(2, *n*). Let $\gamma = (a_\gamma, b_\gamma) \in \Gamma$, then $a_\gamma^{-1} b_\gamma \in a_r^{-1} G b_\gamma = G \subseteq \mathrm{P\Gamma L}(2, n)$. Hence,

$$G_1^{a_\gamma} \subseteq G^{a_\gamma} = a_\gamma^{-1} G a_\gamma = a_\gamma^{-1}(a_\gamma G b_\gamma^{-1}) a_\gamma = G(a_\gamma^{-1} b_\gamma)^{-1} \subseteq \mathrm{P\Gamma L}(2, n).$$

Since $G_1^{a_\gamma}$ is a two-transitive subgroup of $\mathrm{P\Gamma L}(2, n)$, $G_1^{a_\gamma}$ contains G_1 so $G_1^{a_\gamma} = G_1$. Therefore $a_\gamma \in \mathrm{P\Gamma L}(2, n)$. Now $\{a_\gamma \mid \gamma \in \Gamma\}$ is a two-transitive subgroup of $\mathrm{P\Gamma L}(2, n)$, hence contains G_1. Since $a_\gamma^{-1} b_\gamma \in G = G_1 \cup \phi G_2$ and $a_\gamma^{-1} G_1 b_\gamma = G_1^{a_\gamma}(a_\gamma^{-1} b_\gamma) = G_1(a_\gamma^{-1} b_\gamma)$ either $a_\gamma^{-1} G_1 b_\gamma = G_1$ or $a_\gamma^{-1} G_1 b_\gamma = \phi G_2$. Since G_1 does not contain a subgroup of index 2, $\{a_\gamma \mid \gamma \in \Gamma, a_\gamma^- G_1 b_\gamma = G_1\}$ contains G_1. Let $a \in G_1$, then there is a $\gamma \in \Gamma$ such that $\gamma = (a, b)$, $a^{-1} G_1 b = G_1$. Since $a \in G_1$ also $b \in G_1$. Hence $(1, b^{-1}) \in \Gamma$ and so $(a, 1) = (a, b)(1, b^{-1}) \in \Gamma(\mathscr{L}^+)$. □

LEMMA 3. *Let* ε *be* $+$ *or* $-$. *If* $\Sigma \leqslant \Gamma(\mathscr{L}^\varepsilon)$ *is transitive on* $\mathscr{L}^{-\varepsilon}$ *and* $\Sigma_{l,m} = 1$ *for all* $l, m \in \mathscr{L}^{-\varepsilon}$, $l \neq m$, *then* $|\Sigma_l| \leqslant 3$ *for all* $l \in \mathscr{L}^{-\varepsilon}$. *If* $\Gamma(\mathscr{L}^\varepsilon)_{l,m} = 1$ *for all* $l, m \in \mathscr{L}^{-\varepsilon}$, *then* $|\Gamma(\mathscr{L}^\varepsilon)_l| \leqslant 3$ *for all* $l \in \mathscr{L}^{-\varepsilon}$.

Proof. Let $\Sigma \leqslant \Gamma(\mathscr{L}^\varepsilon)$ be transitive on $\mathscr{L}^{-\varepsilon}$. Then G contains a subgroup $H \simeq \Sigma$ (as permutation groups, see proof of Lemma 2). If $\Sigma_{l,m} = 1$ for distinct l, m in $\mathscr{L}^{-\varepsilon}$, then $H_{\alpha,\beta} = 1$ for distinct $\alpha, \beta \in \Omega$. It follows that the circles corresponding to the elements of H cannot intersect each other in more than one point. Moreover, by Result 2, these circles fall into $|H_\alpha|$ classes of $n+1$ disjoint circles (each class corresponding to a coset of the Frobenius kernel of H). Thus each point is on $|H_\alpha|$ of these circles, one from each class, and circles in distinct classes intersect in exactly one point. Now, if $|\Sigma_l| = |H_\alpha| > 3$ we can find four circles c_1, c_2, c_3 and d such that the c_i touch each other in a point P not on d and such that the c_i touch d in three distinct points. However, this means that in the projective plane $\tilde{\mathscr{M}}_P$, the oval corresponding to d has three tangents through a common point. As n, the other of $\tilde{\mathscr{M}}_P$, is odd, this is a contradiction.

Suppose $\Gamma(\mathscr{L}^\varepsilon)_{l,m} = 1$ for distinct $l, m \in \mathscr{L}^{-\varepsilon}$. If $\Gamma(\mathscr{L}^\varepsilon) = 1$ for some (hence all) $l \in \mathscr{L}^{-\varepsilon}$ there is nothing to prove. If $|\Gamma(\mathscr{L}^\varepsilon)| > 1$, then $\Gamma(\mathscr{L}^\varepsilon)$ is transitive on $\mathscr{L}^{-\varepsilon}$ by Result 1 and we can take $\Sigma = \Gamma(\mathscr{L}^\varepsilon)$.

LEMMA 4. *Let ε be $+$ or $-$. If $\Gamma(\mathscr{L}^\varepsilon)$ is two-transitive on $\mathscr{L}^{-\varepsilon}$, then n is a prime power, $\mathscr{M} \simeq \mathscr{M}(n, \phi)$ for some $\phi \in \mathrm{Aut}(\mathrm{GF}(n))$ and Γ contains* $\mathrm{PSL}(2, n \times \mathrm{PSL}(2, n)$.

Proof. As G is sharply triply transitive on Ω, $\Gamma(\mathscr{L}^\varepsilon)_{l,m,n} = 1$ for distinct lines $l, m, n \in \mathscr{L}^{-\varepsilon}$. By Result 3, either $\Gamma(\mathscr{L}^\varepsilon)$ contains a sharply two-transitive subgroup, or $\Gamma(\mathscr{L}^\varepsilon)$ contains $\mathrm{PSL}(2, n)$ as a normal subgroup of index $\leqslant 2$. The first alternative is impossible by Lemma 3. Lemma 2 now completes the proof. □

LEMMA 5. *If $\Gamma(\mathscr{L}^\varepsilon)$ contains a nontrivial element fixing two lines of $\mathscr{L}^{-\varepsilon}$ ($\varepsilon = +$ or $-$), then n is a prime power, $\mathscr{M} \simeq \mathscr{M}(n, \phi)$ for some $\phi \in \mathrm{Aut}(\mathrm{GF}(n))$ and Γ contains* $\mathrm{PSL}(2, n) \times \mathrm{PSL}(2, n)$.

Proof. Suppose $1 \neq \gamma \in \Gamma(\mathscr{L}^\varepsilon)$ fixes $l, m \in \mathscr{L}^{-\varepsilon}$, $l \neq m$. We may assume that γ has prime order. As remarked in the proof of Lemma 4, γ fixes no other lines of $\mathscr{L}^{-\varepsilon}$ besides l and m. Since $\Gamma(\mathscr{L}^\varepsilon)$ is a normal subgroup of Γ, $\langle \gamma^\alpha | \alpha \in \Gamma_l \rangle \leqslant \Gamma(\mathscr{L}^\varepsilon)$. By Result 1, it follows that $\langle \gamma^\alpha | \alpha \in \Gamma_l \rangle$ is transitive on $\mathscr{L}^{-\varepsilon} \backslash \{l\}$. Hence $\langle \gamma^\alpha \alpha \in \Gamma \rangle$ is two-transitive on $\mathscr{L}^{-\varepsilon}$. Now apply Lemma 4. □

From the foregoing lemmas it is clear that our main objective will be to show that $\Gamma(\mathscr{L}^\varepsilon)$ is nontrivial. For this it is necessary first to investigate how Γ acts on $\mathscr{C}$ and how Γ_P acts on $\mathscr{M}_P$, $P \in M$. Define a *pencil* to be any maximal set of mutually tangent circles through a common point P, called the *carrier* of the pencil. Thus the pencils with given carrier P are essentially identical with parallel classes of lines in the affine plane $\mathscr{M}_P$. Every pencil contains n circles. Every point is carrier of $n-1$ pencils.

LEMMA 6. *For every point P and pencil $\mathscr{P}$ with carrier P, $\Gamma_{P,\mathscr{P}}$ is transitive on the n circles of $\mathscr{P}$.*

Proof. Since Γ is two-transitive on nonparallel points, Γ_P is transitive on the points of $\mathscr{M}_P$. By Theorem 3 of [16] we are done. □

Thus, if circles c and d touch, then there exists $\gamma \in \Gamma$ such that $c^\gamma = d$. This shows that every Γ-orbit on $\mathscr{C}$ consists of a number of components of the *touch-graph* defined on $\mathscr{C}$ by: $c, d \in \mathscr{C}$ are *adjacent* iff c and d touch.

LEMMA 7. *The touch-graph has* 1 *or* 2 *components. If it has* 2 *components, then each component contains* $\frac{1}{2}(n+1)n(n-1)$ *circles and every point is incident with* $\frac{1}{2}n(n-1)$ *circles of each component.*

Proof. Let c_1, c_2 and c_3 be three distinct circles and P a point, $P \notin c_1, c_2, c_3$. The ideal line of the affine plane $\mathscr{M}_P$ consists of the ideal points (i.e., parallel classes of $\mathscr{M}_P$) $\mathscr{L}^+ \backslash \{[P]_+\}$, $\mathscr{L}^- \backslash \{[P]_-\}$ and the $(n-1)$ pencils with carrier P. The circles c_1, c_2 and c_3 correspond to ovals intersecting the ideal line in $\mathscr{L}^+ \backslash \{[P]_+\}$ and $\mathscr{L}^- \backslash \{[P]_-\}$. Thus, since n is odd, for each c_i there are $\frac{1}{2}(n-1)$ ideal points which are exterior with respect to c_i (i.e., are the point of intersection of two tangents of c_i) and $\frac{1}{2}(n-1)$ ideal points which are interior with respect to c_i. This shows that at least two of c_1, c_2 and c_3 have an exterior point on the ideal line in common, hence are in the same component of the touch-graph. Therefore, the number of components is at most 2. If there are 2 components and c_1 and c_2, say, are in distinct components, then the ideal points corresponding to the pencils fall into two classes: $\frac{1}{2}(n-1)$ are exterior with respect to c_1 and the other $\frac{1}{2}(n-1)$ are exterior with respect to c_2. Hence P is incident with $\frac{1}{2}n(n-1)$ circles of each component and an easy counting argument shows that each component contains $\frac{1}{2}(n+1)n(n-1)$ circles. □

Remark. The touch-graph of $\mathscr{M}(q, \phi)$, q odd, actually has two components.

By Lemmas 6 and 7, if t is the number of Γ-orbits on $\mathscr{C}$, $t \in \{1, 2\}$ and $[\Gamma : \Gamma_c] = t^{-1}(n+1)n(n-1)$ for all $c \in \mathscr{C}$. Using this result we can show the transitivity properties stated in the next lemma.

LEMMA 8.

- (i) *If c is a circle, then Γ_c is two-transitive on c.*
- (ii) *If P is a point, then Γ_P has t orbits of length $t^{-1}(n-1)$ on the pencils with carrier P.*
- (iii) *If P and Q are distinct points, $P \not\parallel Q$, then $\Gamma_{P,Q}$ has t orbits of length $t^{-1}(n-1)$ on the circles containing P and Q.*
- (iv) *If P and Q are distinct points of the circle c, then* $|\Gamma| = (n+1)^2 n^2 (n-1) t^{-1} |\Gamma_{P,Q,c}|$.

Proof. Let P and Q be distinct points of the circle c, and let $\mathscr{P}$ be the pencil with carrier P containing c. Denote by s the number of pencils in the Γ_P-orbit containing $\mathscr{P}$. Then $[\Gamma_P : \Gamma_{P,\mathscr{P}}] = s$ and $[\Gamma_P : \Gamma_{P,c}] = ns$ by Lemma 6. Hence,

$$(n+1) \geqslant [\Gamma_c : \Gamma_{c,P}] = \frac{|\Gamma_c|}{|\Gamma|} \cdot \frac{|\Gamma|}{|\Gamma_P|} \cdot \frac{|\Gamma_P|}{|\Gamma_{P,c}|}$$
$$= \frac{1}{t^{-1}(n+1)n(n-1)} \cdot (n+1)^2 \cdot ns$$
$$= \frac{st(n+1)}{n-1} = st + \frac{2st}{n-1}.$$

Thus, $st = \frac{1}{2}(n-1)u$ with $u \in \mathbb{N}$, and so $(n+1) \geqslant [\Gamma_c : \Gamma_{c,P}] = \frac{1}{2}(n+1)u$, i.e., $u \in \{1, 2\}$. As $s = \frac{1}{2}t^{-1}(n-1)u$ with $u, t \in \{1, 2\}$ and n is odd, $(n, s) = 1$. Therefore it follows from

$$n \geqslant \frac{|\Gamma_{c,P}|}{|\Gamma_{c,P,Q}|} = \frac{|\Gamma_{P,c}|}{|\Gamma_P|} \cdot \frac{|\Gamma_P|}{|\Gamma_{P,Q}|} \cdot \frac{|\Gamma_{P,Q}|}{|\Gamma_{P,Q,c}|}$$
$$= \frac{1}{ns} \cdot n^2 \cdot [\Gamma_{P,Q} : \Gamma_{P,Q,c}] = \frac{n}{s}[\Gamma_{P,Q} : \Gamma_{P,Q,c}]$$

that $[\Gamma_{c,P} : \Gamma_{c,P,Q}] = n$ and $[\Gamma_{P,Q} : \Gamma_{P,Q,c}] = s$. Now from $[\Gamma_{c,P} : \Gamma_{c,P,Q}] = n$ it follows that $\Gamma_{c,P}$ is transitive on $c \backslash \{P\}$, hence, since P was an arbitrary point of c, Γ_c is two-transitive on c. Therefore $(n+1) = [\Gamma_c : \Gamma_{c,P}] = \frac{1}{2}(n+1)u$, so $u = 2$ and $s = t^{-1}(n-1)$. Finally,

$$|\Gamma| = \frac{|\Gamma|}{|\Gamma_P|} \cdot \frac{|\Gamma_P|}{|\Gamma_{P,Q}|} \cdot \frac{|\Gamma_{P,Q}|}{|\Gamma_{P,Q,c}|} \cdot |\Gamma_{P,Q,c}| = (n+1)^2 n^2 (n-1) t^{-1} \cdot |\Gamma_{P,Q,c}|$$

which proves (iv). □

LEMMA 9. *Let P be a point. If Γ_P has odd order, then n is a power of a prime, $\mathscr{M} \simeq \mathscr{M}(n, \phi)$ for some $\phi \in \mathrm{Aut}(\mathrm{GF}(n))$ and Γ contains* $\mathrm{PSL}(2, n) \times \mathrm{PSL}(2, n)$.

Proof. Fix a line $l \in \mathscr{L}^+$ and let $\Delta \simeq \Gamma_l / (\Gamma(\mathscr{L}^-) \cap \Gamma_l)$ be the permutation group on l induced by Γ_l. As Γ is two-transitive on the nonparallel points of $\mathscr{M}$, Δ is two-transitive on l. As Γ_P has odd order, Δ_P has odd order for all $P \in l$. By Result 4, either Δ is solvable or Δ contains $\mathrm{PSL}(2, n)$ as a normal subgroup. If Δ is solvable, then Δ is isomorphic to a subgroup of the group of semilinear transformations of a Galois field of characteristic 2, i.e., $n + 1 = 2^a$ for some $a \in \mathbb{N}$ and $|\Delta| \mid (n+1)na$. If Δ contains $\mathrm{PSL}(2, n)$ as a normal subgroup, then $n = p^b$ for some prime p and $b \in \mathbb{N}$ and Δ is a subgroup of $\mathrm{P\Gamma L}(2, n)$, i.e., $|\Delta| \mid (n+1)n(n-1)b$. By Lemma 8(iv), the order of Γ_l is $(n+1)n^2(n-1)t^{-1} \cdot |\Gamma_{P,Q,c}|$. In both cases it follows from $n \geqslant 5$ that $|\Gamma(\mathscr{L}^-) \cap \Gamma_l| = |\Gamma(\mathscr{L}^-)_l| > 3$.

By Lemma 3 there exists a nontrivial element of $\Gamma(\mathscr{L}^-)$ fixing two distinct lines of $\mathscr{L}^+$. Lemma 5 now completes the proof. □

By the previous lemma we may assume from now on that Γ_P has even order. More in particular, Γ_P contains involutions. Since n is odd, every involution $\tau \in \Gamma_P$ either induces a homology of the projective plane $\tilde{\mathscr{M}}_P$ associated with the affine plane $\mathscr{M}_P$, or the τ-fixed points and lines of $\tilde{\mathscr{M}}_P$ constitute a Baer subplane of $\tilde{\mathscr{M}}_P$(cf. [5], p. 172). Our next lemma deals with the case where Γ_P contains a homology.

LEMMA 10. *Let $P \in M$ and suppose that $\tau \in \Gamma_P$ is an involution which, considered as a collineation of $\tilde{\mathscr{M}}_P$, is a homology. Then n is a prime power, $\mathscr{M} \simeq \mathscr{M}(n, \phi)$ for some $\phi \in \mathrm{Aut}(\mathrm{GF}(n))$ and Γ contains* $\mathrm{PSL}(2, n) \times \mathrm{PSL}(2, n)$. *If Γ_P has even order and*

(i) *n is not a square, or*
(ii) *$t = 1$ (i.e., Γ is transitive on $\mathscr{C}$), then Γ_P contains homologies.*

Proof. We distinguish two cases:

Case (*a*). The axis of τ is the ideal line of $\mathscr{M}_P$. Now, since Γ_P is transitive on M_P, $\mathscr{M}_P$ is a translation plane and Γ_P contains the full translation group of $\mathscr{M}_P$(see [5], p. 187, result 4.3.1). Let $\Sigma^{(P)}$ be the subgroup of Γ_P consisting of those translations of $\mathscr{M}_P$ which fix all lines of $\mathscr{L}^-$. Then $\Sigma^{(P)}$ is transitive on $\mathscr{L}^+ \backslash \{[P]_+\}$, hence $\Sigma := \langle \Sigma^{(P)} | P \in M \rangle$ is two-transitive on $\mathscr{L}^+$. Since $\Sigma \leqslant \Gamma(\mathscr{L}^-)$ we are done by Lemma 4.

Case (*b*). The axis of τ is an affine line of $\mathscr{M}_P$. Clearly, the axis of τ corresponds to a line $l \neq [P]_+, [P]_-$ of $\mathscr{M}$, say $l \in \mathscr{L}^+ \backslash \{[P]_+\}$. Now $1 \neq \tau \in \Gamma(\mathscr{L}^-)$ and τ fixes the two distinct lines $[P]_+$ and l of $\mathscr{L}^+$. By Lemma 5 we have completed the proof of our first claim.

The order of a Baer subplane of $\mathscr{M}_P$ is $\sqrt{n}$. Hence, if n is not a square, every involution in Γ_P acts as a homology of $\tilde{\mathscr{M}}_P$. Suppose $t = 1$. Let Λ be a Sylow 2-subgroup of Γ_P and let τ be an involution in the center of Λ. Suppose the τ-fixed points and lines of $\tilde{\mathscr{M}}_P$ constitute a Baer subplane. The two ideal points of $\mathscr{M}_P$ corresponding to $\mathscr{L}^+$ and $\mathscr{L}^-$ are fixed by Γ_P, and by Lemma 8 (ii) Γ_P is transitive on the remaining $n - 1$ ideal points. Let $2^a \| (n - 1)$. By [17], Theorem 3.4, every Λ-orbit on these $n - 1$ ideal points has length divisible by 2^a. The ideal line of $\mathscr{M}_P$ is fixed by τ and contains therefore, apart from the ideal points corresponding to $\mathscr{L}^+$ and $\mathscr{L}^-$, $\sqrt{n} - 1$ fixed points. Since $\tau \in Z(\Lambda)$, Λ permutes these $\sqrt{n} - 1$ points. However, $2^b \| (\sqrt{n} - 1)$ with $b < a$, contradicting the fact that each of these $\sqrt{n} - 1$ points is in a Λ-orbit with length divisible by 2^a. □

For the proof of our main result we need one more definition and lemma.

DEFINITION. Suppose $M_1 \subseteq M$; $\mathscr{L}_1^\varepsilon \subseteq \mathscr{L}^\varepsilon$, $\varepsilon = +, -$; $\mathscr{C}_1 \subseteq \mathscr{C}$. Put $\mathscr{L}_1^{\varepsilon *} := \{ l \cap M_1 | l \in \mathscr{L}_1^\varepsilon \}$, $\varepsilon = +, -$; $\mathscr{C}_1^* := \{ c \cap M_1 | c \in \mathscr{C}_1 \}$. If $\mathscr{M}_1 := (M_1, \mathscr{L}_1^{+*}, \mathscr{L}_1^{-*}, \mathscr{C}_1^*)$ is a Minkowski plane with the property that any two circles which touch in $\mathscr{M}_1$, touch in $\mathscr{M}$, then $\mathscr{M}_1$ is called a *subplane* of $\mathscr{M}$ (compare [5], p. 258).

LEMMA 11. *Let Δ be a group of positive automorphisms of $\mathscr{M}$. Let M_1 be the set of points left fixed by Δ; $\mathscr{L}_1^+$ (resp. $\mathscr{L}^-$) the set of lines of $\mathscr{L}^+$ (resp. $\mathscr{L}^-$) left fixed by Δ; and $\mathscr{C}_1$ the set of circles left fixed by Δ. Then $\mathscr{M}_1 := (M_1, \mathscr{L}_1^{+*}, \mathscr{L}_1^{-*}, \mathscr{C}_1^*)$ is a subplane of $\mathscr{M}$ if and only if M_1 contains, at least three mutually nonparallel points.*

Proof. Straightforward verification. □

We are now ready to prove our main result.

THEOREM. *Let $\mathscr{M} = (M, \mathscr{L}^+, \mathscr{L}^-, \mathscr{C})$ be a finite Minkowski plane of odd order n, and suppose that $\mathscr{M}$ admits an automorphism group Γ two-transitive on nonparallel points. Then n is a prime power, $\mathscr{M} \simeq \mathscr{M}(n, \phi)$ for some $\phi \in$* Aut(GF(n)) *and Γ contains* PSL(2, n) × PSL(2, n).

Proof. Suppose $\mathscr{M}$ is a counter example to the theorem of minimal order. By Lemma 1 we may assume that Γ contains positive automorphisms only. By Lemma 9, Γ_P has even order for all $P \in M$. By Lemma 10 every involution in Γ_P has $(\sqrt{n} + 1)^2$ fixed points. Hence, if Λ is a 2-subgroup of Γ maximal with respect to fixing at least three mutually nonparallel points, $\Lambda \neq 1$. Let $\mathscr{M}_1 = (M_1, \mathscr{L}_1^{+*}, \mathscr{L}_1^{-*}, \mathscr{C}_1^*)$ be the subplane of $\mathscr{M}$ consisting of the Λ-fixed points, lines and circles of $\mathscr{M}$ of order n_1, say. Clearly n_1 is odd, and since $\Lambda \neq 1$ we have $n_1 < n$. We claim that $N_\Gamma(\Lambda)$, considered as an automorphism group of $\mathscr{M}_1$, acts two-transitively on the nonparallel points of $\mathscr{M}_1$. To see this, let $c \in \mathscr{C}_1$. Then $\Lambda \leqslant \Gamma_c$ and Λ, considered as a permutation group on c, is a 2-subgroup of Γ_c maximal with respect to fixing at least three points of c. By Lemma 8(i), Γ_c is two-transitive on c, hence $N_{\Gamma_c}(\Lambda)$ is two-transitive on $c^* := c \cap M_1$ (see [1], Lemma 3.3). Now let A_1, A_2 and B_1, B_2 be two pairs of nonparallel points of $\mathscr{M}_1$. If $A_i \nparallel B_j$, $i, j = 1, 2$, and c_1 is the unique circle containing A_2, B_1, B_2, and c_2 is the unique circle containing A_2, B_1, B_2, then there is a $\gamma_1 \in N_{\Gamma_{c_1}}(\Lambda)$ and a $\gamma_2 \in N_{\Gamma_{c_2}}(\Lambda)$ such that $A_1^{\gamma_1} = A_2$, $A_2^{\gamma_1} = B_1$, $A_1^{\gamma_2} = B_1$, $B_1^{\gamma_2} = B_2$. Hence $\gamma = \gamma_1\gamma_2 \in N_\Gamma(\Lambda)$ satisfies $A_1^\gamma = B_1$ and $A_2^\gamma = B_2$. Repeated application of this result in case $A_i \parallel B_j$ for some i and j, proves our claim. Since $\mathscr{M}$ was supposed to be a minimal counter example, n_1 is a prime power, say $n_1 = p^a$ with p prime and $a \in \mathbb{N}$. If $P \in M_1$, then the projective plane $(\tilde{\mathscr{M}}_1)_P$ associated with $(\mathscr{M}_1)_P$ is a subplane of the projective plane $\tilde{\mathscr{M}}_P$ associated with $\mathscr{M}_P$ (this is why we required in the definition of a subplane of a Minkowski plane, that circles tangent in $\mathscr{M}_1$ are also tangent in $\mathscr{M}$). In fact $(\tilde{\mathscr{M}}_1)_P$ is a 2-subplane of $\tilde{\mathscr{M}}_P$ is the sense of Ostrom and Wagner [15]. By their Theorem 6, $n = n_1^{2^g}$ for some integer g. Hence, also n is a prime power, $n = p^b$ with $b = a2^g$. Let Π be a Sylow p-subgroup of Γ_P, $P \in M$. Let π be an element in the centre of Π. Since π fixed the two ideal points corresponding to $\mathscr{L}^+$ and $\mathscr{L}^-$ of $\mathscr{M}_P$, π also fixes an affine line L of $\mathcal{M}_P$. Suppose L intersects the ideal line of $\mathcal{M}_P$ in a point A. Then Π fixes A for if $A^\sigma \neq A$ for some $\sigma \in \Pi$, then L^σ and L intersect in an affine point Q of $\mathcal{M}_P$. Since Π permutes the fixed objects of π, L^σ hence Q is fixed by π. Since Γ_P, hence Π, is transitive on the n^2 affine points of $\mathcal{M}_P$, every affine point of $\mathcal{M}_P$ is fixed by π, i.e., $\pi = 1$ a contradiction. By Theorem 3 of [16] $\Gamma_{P,A}$, hence Π is transitive on the n affine lines through A. Therefore, π fixes all lines through A, i.e., π is an elation of $\bar{\mathcal{M}}_P$ with centre A and axis the ideal line of $\mathcal{M}_P$. Suppose A is the ideal point corresponding to $\mathcal{L}^{-\varepsilon}$ for $\varepsilon = +$ or $-$, then $\pi \in \Gamma(\mathcal{L}^{-\varepsilon})_{[P]_\varepsilon}$. By Lemma 5, $\Gamma(\mathcal{L}^{-\varepsilon})_{l,m} = 1$ for distinct lines $l, m \in \mathcal{L}^\varepsilon$, so by Lemma 3, $p \leqslant$ order of $\pi \leqslant |\Gamma(\mathcal{L}^{-\varepsilon})_{[P]_\varepsilon}| \leqslant 3$, i.e., $p = 3$. Also $\Gamma(\mathcal{L}^{-\varepsilon})$ is a Frobenuis group on the $(n+1)$ lines of $\mathcal{L}^\varepsilon$, $\Gamma(\mathcal{L}^{-\varepsilon}) \trianglelefteq \Gamma$ and Γ acts two-transitively on $\mathcal{L}^\varepsilon$, hence the Frobenius kernel of $\Gamma(\mathcal{L}^{-\varepsilon})$ is an elementary abelian 2-group and in particular $n + 1 = 2^c$ for some $c \in \mathbb{N}$. However, $n + 1 = p^b + 1 = 3^{a2^g} + 1 \equiv 2(4)$ and so we have shown that A is an ideal point corresponding to a pencil with carrier P. Let T be the group of translations of $\mathcal{M}_P$ contained in Γ_P and for each pencil $\mathcal{P}$ with carrier P let $T(\mathcal{P})$ be the group of translations of T fixing all circles of $\mathcal{P}$. By Lemma 10 and Lemma 8(ii), Γ_P has two orbits of length $\frac{1}{2}(n-1)$ on the pencils with carrier P. Put $x = |T(\mathcal{P})|$ for $\mathcal{P}$ in the first, and $y = |T(\mathcal{P})|$ for $\mathcal{P}$ in the second orbit. It follows that

$$(1) \qquad |T| = 1 + (x-1)\cdot\tfrac{1}{2}(n-1) + (y-1)\cdot\tfrac{1}{2}(n-1) = = 1 + \tfrac{1}{2}(x+y-2)(n-1),$$

and one of x and $y \geqslant p$, so $x + y \geqslant p + 1$. Also, if s is the number of T-orbits on M_P,

$$(2) \qquad s|T| = n^2.$$

Since $x + y \geqslant p + 1 \geqslant 4$, it follows that $|T| \geqslant n$, hence $s \leqslant n$. From (1) and (2) it also follows that $s \equiv 1(\mathrm{mod}\,\frac{1}{2}(n-1))$. Since T is not transitive on M_P, $s > 1$. Therefore $s = n$, $|T| = n$ and $p = 3$. We list some properties of T.

(i) As a translation group containing translations in different directions, T is elementary abelian,
(ii) $T \triangleleft \Gamma_P$,
(iii) T acts regularly on the lines of $\mathcal{L}^\varepsilon \setminus \{[P]_\varepsilon\}$, $\varepsilon = +, -$,
(iv) the subgroups $\langle \tau \rangle$, $\tau \in T$ are in $1-1$ correspondence with the $\frac{1}{2}(n-1)$ pencils with carrier P in a Γ_P-orbit: $\tau \leftrightarrow$ pencil $\mathcal{P}$ iff centre of $\tau = \mathcal{P}$; Γ_P acts on this orbit as Γ_P acts on $\{\langle \tau \rangle | \tau \in T\}$ by conjugation.

Take $Q \in M_P$. By Lemma 8(iii), $\Gamma_{P,Q}$ is still transitive on the pencils with carrier P in a Γ_P-orbit, so $\Gamma_{P,Q}$ acts by conjugation transitively on the subgroups $\langle \tau \rangle$, $\tau \in T$. By (ii) and (iii), T is a regular normal subgroup of Γ_P considered as a permutation group on $\mathcal{L}^+ \setminus \{[P]_+\}$. Since $\Gamma_{P,Q} \leqslant \Gamma_{P,[Q]_\varepsilon}$,

$\Gamma_{P,Q}$ acts on $\mathscr{L}^+\backslash\{[P]_+, [Q]_+\}$ as it does on $T\backslash\{1\}$ by conjugation. It follows that either $\Gamma_{P,Q}$ is transitive or has two orbits of length $\frac{1}{2}(n-1)$ on $\mathscr{L}^+\backslash\{[P]_+, [Q]_+\}$. The former alternative is impossible: an involution in the center of a Sylow 2-subgroup of Γ_P is a homology (see the last part of the proof of Lemma 10). Therefore $\Gamma_{P,Q}$ has 2 orbits of length $\frac{1}{2}(n-1)$ on $\mathscr{L}^+\backslash\{[P]_+, [Q]_+\}$ and it acts on both orbits as it acts on the subgroups $\langle\tau\rangle, \tau\in T$ by conjugation. Let c be a circle through P and Q in the pencil $\mathscr{P}$, where $\mathscr{P}$ is the centre of $\langle\tau\rangle$, say. Then $\Gamma_{P,Q,c}$ fixes $\mathscr{P}$, hence $\Gamma_{P,Q,c}$ fixes $\langle\tau\rangle$ by conjugation and therefore also two distinct lines $l, m\in\mathscr{L}^+\backslash\{[P]_+, [Q]_+\}$. Therefore also $l\cap c$ and $m\cap c$ are fixed by $\Gamma_{P,Q,c}$. By Lemma 11, $\Gamma_{P,Q,c}$ has a subplane $\mathscr{M}_2$ as a set of fixed points. Let n_2 be the order of $\mathscr{M}_2$ and let c^* be the set of points left fixed by $\Gamma_{P,Q,c}$. With $\mathscr{B} = \{c^{*\gamma}|\gamma\in\Gamma_c\}$ we get a $2-(n+1, n_2+1, 1)$ design on c (see [13]). The number of blocks through a point is $n/n_2 = 3^b/n_2$. Hence $n_2 = 3^d$ for some $d\in\mathbb{N}$. The total number of blocks equals $(n+1)n/(n_2+1)n = (3^b+1/3^d+1)\cdot 3^{b-d}$. Hence $b/d\in 2\mathbb{N}+1$. Since b is even, d is even so $10\leqslant n_2+1 = 3^d+1 \equiv 2 \pmod 4$. However, $\Gamma_{c^*} = N_{\Gamma_c}(\Gamma_{P,Q,c})$ is sharply 2-transitive on the n_2+1 points of c^*, and so n_2+1 is a power of 2. This was our final contradiction. □

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A characterization of the classical unitals

by

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ABSTRACT

A characterization of the classical unitals is given in terms of certain geometrical properties.

KEY WORDS & PHRASES: *Unital, inversive plane, generalized quadrangle, Minkowski plane*

1. INTRODUCTION

A *unital* or *unitary block design* is a $2-(q^3+1,q+1,1)$ design, i.e. an incidence structure of q^3+1 points, $q^2(q^2-q+1)$ lines, such that each line contains $q+1$ points and any two distinct points are on a unique line. If q is a prime power, the absolute points and non-absolute lines of a unitary polarity of $PG(2,q^2)$ form a unital (see [2]). These unitals are called *classical*.

In [6], O'NAN showed that a classical unital satisfies the following condition.

(I) No four distinct lines intersect in six distinct points (see Figure 1).

No:

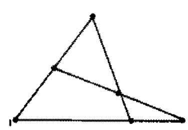

Fig. 1

In [5], PIPER conjectured that this property characterizes the classical unitals. Here we shall give a characterization for even q under the assumption that also the following condition holds.

(II) If L is a line, x a point not on L, M a line through x meeting L and and $y \neq x$ a point on M, then there exists a line $L' \neq M$ through y intersecting all lines through x which meet L.

To achieve this result we shall give another characterization for all q under the additional assumption that a third condition holds. To formulate this condition we need some notation. If x and y are distinct points, then we denote by xy the line through x and y. Given a point x, two lines L and L' missing x are called *x-parallel* (notation $L \parallel_x L'$) if and only if they intersect the same lines through x. Clearly, $\parallel_x$ is an equivalence relation on the set of lines missing x, and by (I) and (II), each equivalence class consists of q disjoint lines. Our third condition now reads as follows.

(III) Given a point x, three distinct lines M_1, M_2, M_3 through x and points y_i, z_i on M_i $(i=1,2,3)$ such that $(y_1y_2)\parallel_x(z_1z_2)$ and $(y_1y_3)\parallel_x(z_1z_3)$, then also $(y_2y_3)\parallel_x(z_2z_3)$.

Clearly, the presence of unitary transvections in PΓU(3,q) implies that the classical unitals satisfy conditions (II) and (III). In Section 2 we shall study unitals satisfying (I) and (II). Section 3 is devoted to the proof that unitals satisfying all three conditions are classical. Finally, in Section 4, we shall show that for even q, (III) is a consequence of (I) and (II).

2. UNITALS SATISFYING (I) AND (II)

Throughout this section $\mathcal{U}$ is a unital on q^3+1 points with point set X and line set $\mathcal{L}$ satisfying (I) and (II) above. If $x \in X$, then we denote by $\mathcal{L}^x$ the set of lines incident with x, and $\mathcal{L}_x$ will be the set of lines missing x. Furthermore, $\mathcal{C}_x$ will stand for the set of $\parallel_x$-equivalence classes on $\mathcal{L}_x$. From [1] it is clear that we want to show that the incidence structure which has $\mathcal{L}^x$ as the set of points, $\mathcal{C}_x$ as the set of blocks and LIC $(L \in \mathcal{L}^x, C \in \mathcal{C}_x)$ iff L meets one (hence all) lines of C, is the residual of an inversive plane of order q. We denote this incidence structure by $\mathcal{I}^*(x) = (\mathcal{L}^x, \mathcal{C}_x)$. Clearly, $\mathcal{I}^*(x)$ is a $2-(q^2,q+1,q)$ design.

<u>LEMMA 1</u>. *If* $x \in X$ *and* $L, L' \in \mathcal{L}_x$ *such that* L *and* L' *both meet three distinct lines* $M_1, M_2, M_3 \in \mathcal{L}^x$, *then* $L \parallel_x L'$, *i.e. three distinct points of* $\mathcal{I}^*(x)$ *are in at most one block of* $\mathcal{I}^*(x)$.

<u>PROOF</u>. Let $y \in M_1 \cap L'$ and let L" be the line through y which is x-parallel to L, then $L' \neq L''$ contradicts (I). □

If M and M' are two distinct lines through a point x, then an easy counting argument shows that there are q −2 lines $N_1,\ldots,N_{q-2}$ through x such that no line of $\mathcal{L}_x$ meets M, M' and N_i, $i = 1,\ldots,q-2$. Put $C^*(M,M') := \{M,M'\} \cup \{N_1,N_2,\ldots,N_{q-2}\}$. We have to show that the $C^*(M,M')$ correspond to circles which will make $\mathcal{I}^*(x)$ into an inversive plane. We have to show that $N,N' \in C^*(M,M') \Rightarrow C^*(M,M') = C^*(N,N')$. Clearly, $C^*(M,M') = C^*(M',M)$ and so it suffices to show that $M'' \in C^*(M,M') \Rightarrow C^*(M,M') = C^*(M,M'')$. This is the contents of the next lemma.

LEMMA 2. *Fix a line* $M \in L$ *and two distinct points* x *and* y *in* M. *For* $M', M'' \in L^x \setminus \{M\}$ *write* $M' \sim M''$ *iff no line of* $L^y \setminus \{M\}$, *intersects both* M' *and* M'' *or* $M' = M''$. *Then* $\sim$ *is an equivalence relation on* $L^x \setminus \{M\}$.

PROOF. For $u,v \in M$ let $A^*(u,v)$ be the incidence structure with $L^u \setminus \{M\}$ as points, $L^v \setminus \{M\}$ as lines, and incidence defined by PIB iff P and B meet ($P \in L^u \setminus \{M\}$, $B \in L^v \setminus \{M\}$). If u, v, w are distinct points of M, then clearly the mapping $\tilde{\tau}^u_{v,w}: A^*(u,v) \to A^*(u,w)$ defined by

$$\tilde{\tau}^u_{v,w}(P) := P, \qquad P \in L^u \setminus \{M\},$$

$$\tilde{\tau}^u_{v,w}(B) := \text{u-parallel of B through w}, \qquad B \in L^v \setminus \{M\},$$

is an isomorphism of $A^*(u,v)$ onto $A^*(u,w)$. Now fix $x,y \in X$, $x \neq y$. If $q > 2$ and u, v are distinct points in M, $u,v \neq x,y$, then

$$\tilde{\delta}^{x,y}_{u,v} := \tilde{\tau}^y_{v,x}\, \tilde{\tau}^v_{u,y}\, \tilde{\tau}^u_{x,v}\, \tilde{\tau}^x_{y,u},$$

is an automorphism of $A^*(x,y)$.

Now we claim that

(i) For all $u,v \in M \setminus \{x,y\}$, $u \neq v$ and for all $P \in L^x \setminus \{M\}$, $\tilde{\delta}^{x,y}_{u,v}(P) \neq P$ and $\tilde{\delta}^{x,y}_{u,v}(P) \sim P$.

(ii) For all $u,v,v' \in M \setminus \{x,y\}$, $u \neq v \neq v' \neq u$ and for all $P \in L^x \setminus \{M\}$, $\tilde{\delta}^{x,y}_{u,v}(P) \neq \tilde{\delta}^{x,y}_{u,v'}(P)$ and $\tilde{\delta}^{x,y}_{u,v}(P) \sim \tilde{\delta}^{x,y}_{u,v'}(P)$.

To prove these claims, write $u P v$ for the u-parallel of P incident with v. Then $\tilde{\delta}^{x,y}_{u,v}(P) = y(uPv)x$. Suppose $y(uPv)x \not\sim P$ or $y(uPv)x = P$. Then there is a line L incident with x intersecting P and $y(uPv)x$. Then L intersects uPv in a point a, say. Since au intersects P, we now have an O'Nan configuration on the lines M, P, L and au, contradicting (I).

Suppose $y(uPv)x \not\sim y(uPv')x$ or $y(uPv)x = y(uPv')x$. Let L be the line through y intersecting $y(uPv)x$ and $y(uPv')x$. Then L intersects uPv and uPv' in points a and a', say. Since au intersects uPv', we have an O'Nan configuration on M, L, uPv' and au, again in contradiction with (I).

For a given $P \in \mathcal{L}^x\backslash\{M\}$, there are $q-2$ $Q \in \mathcal{L}^x\backslash\{M\}$, $Q \neq P$ such that $Q \sim P$. Fixing u we can make $q-2$ choices for $v \in M\backslash\{x,y,u\}$. Thus, each $Q \in \mathcal{L}^x\backslash\{M\}$, $Q \neq P$ can be written as $Q = \tilde{\delta}^{x,y}_{u,v}(P)$. If $Q = \tilde{\delta}^{x,y}_{u,v}(P) \sim P$ and $Q' = \tilde{\delta}^{x,y}_{u,v'}(P) \sim P$, then $Q \sim Q'$ by (ii). □

Lemma 2 and its proof have a number of important corollaries.

COROLLARY 3. *Let $x \in X$ and let ∞_x be a new symbol. Put*

$$\mathcal{C}^x := \{C^*(M,M') \cup \{\infty_x\} \mid M,M' \in \mathcal{L}^x,\ M \neq M'\}.$$

Then $\mathcal{I}(x) := (\mathcal{L}^x \cup \{\infty_x\}, \mathcal{C}^x \cup \mathcal{C}_x)$ is an inversive plane of order q with point set $\mathcal{L}^x \cup \{\infty_x\}$ and block set $\mathcal{C}^x \cup \mathcal{C}_x$ and incidence defined in the obvious way.

PROOF. See the discussion preceding Lemma 2. □

COROLLARY 4. *For $x,y \in X$, $x \neq y$, the incidence structure $\mathcal{A}^*(x,y)$ of Lemma 2 is isomorphic to the derived design $\mathcal{I}(x)^{xy}$ with ∞_x and the lines through ∞_x removed. The affine plane $\mathcal{I}(x)^{xy}$ admits a dilatation group of order $q-1$ with centre ∞_x.*

PROOF. The automorphisms $\tilde{\delta}^{x,y}_{u,v}$ of $\mathcal{A}^*(x,y)$ induce $q-2$ distinct nonidentity dilatations with centre ∞_x on $\mathcal{I}(x)^{xy}$. Since $\mathcal{I}(x)^{xy}$ has order q, these are the non-identity elements of the dilatation group with centre ∞_x of order $q-1$. □

COROLLARY 5. *Let $L \in \mathcal{L}$ and let $x_1,x_2,\ldots,x_{q+1}$ be the points on L. It is possible to partition the set of lines which meet L into classes A_{ij}, $1 \leq i,j \leq q+1$, such that for all i and j*

(i) $|A_{ij}| = q-1$

(ii) $M \in A_{ij} \Rightarrow x_i \in M$,

(iii) every point $x \in X\backslash L$ is on exactly one line of $\bigcup_k A_{kj}$,

(iv) no line which meets L in a point $\neq x_i$, meets two lines of A_{ij},

(v) for all k, i', $1 \leq k,i' \leq q+1$, $k \neq i,i'$ and for all $M \in A_{ij}$, the x_k-parallel of M through $x_{i'}$ is in $A_{i'j}$,

(vi) if $1 \leq i' \leq q+1$, $i' \neq i$ and $M \in A_{ij}$, $M' \in A_{i',j}$ then there exists a unique $k \in \{1,\ldots,q+1\}$ such that M and M' are x_k-parallel.

PROOF. Consider $I(x_1)$. Number the circles of $I(x_1)$ through the two points ∞_{x_1} and L of $I(x_1)$ from 1 upto q+1. Apart from ∞_{x_1} and L each such circle contains (q-1) lines through x_1. These will be the sets $A_{1,j}$, $j = 1,\dots,q+1$. For $i > 1$ and $1 \le j \le q+1$, let $A_{i,j}$ consist of the (q-1) lines through x_i which in $I(x_1)$ correspond to the (q-1) circles (not through ∞_{x_1}) in the pencil with carrier L and which contains circle j through ∞_{x_1} and L. Now (i) and (ii) are trivially satisfied. For (iii), note that the q+1 lines xx_i, $i = 1,\dots,q+1$ are in A_{ij}'s with distinct j since the circles in a pencil with carrier L partition the set of points $\neq$ L of $I(x_1)$. To prove the other cases, observe that our subdivision of the set of lines meeting L into the classes A_{ij} would have remained the same if we had started by considering $I(x_i)$, $i > 1$ instead of $I(x_1)$. Thus, to prove (iv), it suffices to show that no line $M \in L^{x_1}\backslash\{L\}$ can intersect two distinct lines $N_1,N_2 \in A_{ij}$ with $i > 1$. This follows at once, since N_1 and N_2 correspond to tangent circles in $I(x_1)$. Also (v) is clear if we take $k = 1$ for then M and the x_k-parallel of M through x_i, represent the same circle in $I(x_1)$. Finally (vi) follows from (i), (v) and the easily shown fact that two lines which meet L cannot be x_k- and x_ℓ-parallel for distinct k and ℓ. □

Following PIPER [5], we are now able to associate with each line L of U an incidence structure $GQ(L)$ as follows. The points of $GQ(L)$ are the points $x \in X\backslash L$ and the sets A_{ij}, $1 \le i,j \le q+1$. The lines of $GQ(L)$ are the lines M of U meeting L, and 2(q+1) new lines, $A_1,A_2,\dots,A_{q+1}$, $B_1,B_2,\dots,B_{q+1}$. Incidence in $GQ(L)$ is defined as displayed in the following table.

	line of type M	line of type A_k or B_ℓ
point of type $x \in X\backslash L$	$x \in M$	never
point of type A_{ij}	$M \in A_{ij}$	$i=k$ or $j=\ell$

Incidence in $GQ(L)$

THEOREM 6. *Let $\mathcal{U} = (X, \mathcal{L})$ be a unital with* $q+1$ *points on a line satisfying* (I) *and* (II). *Then for each line* $L \in \mathcal{L}$, $GQ(L)$ *is a generalized quadrangle with* $q+1$ *points on a line and* $q+1$ *lines through a point. Moreover, any two nonintersecting lines* m_1 *and* m_2 *of* $GQ(L)$ *form a regular pair (in the sense of* [7]*) provided* m_1 *and* m_2 *do not correspond to lines* M_1 *and* M_2 *of* $\mathcal{U}$ *such that* $M_1 \in A_{ij}$ *and* $M_2 \in A_{k\ell}$ *with* $i \neq k$ *and* $j \neq \ell$. *In particular, the lines* $A_1, \ldots, A_{q+1}, B_1, \ldots, B_{q+1}$ *of* $GQ(L)$ *are regular.*

PROOF. Straightforward verification. □

We shall see in Section 4 that if all lines of $GQ(L)$ are regular, then $\mathcal{U}$ is classical.

3. UNITALS SATISFYING (I), (II) AND (III)

Let $\mathcal{U} = (X, \mathcal{L})$ be a unital satisfying (I), (II) and (III). Using (III) it is easy to see that for any three distinct points x, y, z on a line L there is a unique automorphism $\tau^x_{y,z}$ of $\mathcal{U}$ fixing x and all lines through x and mapping y onto z: if $u \notin L$ then $\tau^x_{y,z}(u)$ is defined to be the point of intersection of xu and the x-parallel of yu through z, if $v \in L \setminus \{x\}$, fix a point $u \notin L$ and define $\tau^x_{y,z}(v)$ to be the point of intersection of L and the x-parallel of uv through $\tau^x_{y,z}(u)$.

THEOREM 7. *Let $\mathcal{U} = (X, \mathcal{L})$ be a unital with* $q+1$ *points on a line satisfying* (I), (II) *and* (III), *and let* G *be the automorphism group of* $\mathcal{U}$ *generated by the* $\tau^x_{y,z}$. *Then* $\mathcal{U}$ *is classical,* G *is isomorphic to* $PSU(3,q^2)$ *and acts on* $\mathcal{U}$ *in the usual way.*

PROOF. Clearly G is transitive on X. We claim that G acts 2-transitively on X if $q > 2$ (the case $q = 2$ is left to the reader). To prove this, note that the mappings $\tilde{\tau}^x_{y,z}$ of Lemma 2 are induced by the automorphisms $\tau^x_{y,z}$ of $\mathcal{U}$. Hence, also the mappings $\tilde{\delta}^{x,y}_{u,v}$ of Lemma 2 are induced by automorphisms $\delta^{x,y}_{u,v} \in G$ of $\mathcal{U}$. Since incidence in the inversive plane $\mathcal{I}(x)$ is determined by incidence in $\mathcal{U}$, $\delta^{x,y}_{u,v}$ induces an automorphism of $\mathcal{I}(x)$. By Corollary 4, this is a dilatation of $\mathcal{I}(x)^{xy}$ with centre ∞_x. Therefore, it can also be viewed as a dilatation of $\mathcal{I}(x)^{\infty_x}$ with centre xy. Thus in the affine plane $\mathcal{I}(x)^{\infty_x}$, each point is the centre of a dilatation. Hence $I(x)^{\infty x}$ is a translation plane and the group generated by the dilatations contains the full translation group of $I(x)^{\infty x}$ ([2, p.187]). Let T(x) be the normal subgroup of G_x consisting of elements which induce (possibly identity) translations of $I(x)^{\infty x}$. Then T(x) acts regularly on the points of $I(x)^{\infty x}$, i.e. on L^x, and for each line $L \in L^x$, $T(x)_L$ acts regularly on $L\setminus\{x\}$. Thus T(x) is a normal subgroup of G_x acting regularly on $X\setminus\{x\}$, and G is 2-transitive. Applying [4] we get that G has a normal subgroup M such that $M \le G \le \mathrm{Aut}\, M$ and M acts on X as one of the following groups in its usual 2-transitive representation: a sharply 2-transitive group, $PSL(2,q^3)$, $Sz(q^{3/2})$, $PSU(3,q^2)$, or a group of Ree type. Since $q^3 + 1 = (q+1)(q^2-q+1)$ is not a prime power for $q > 2$, the first alternative will not occur. If $H \le G$ and x, y, z are three distinct points of X, then the H_{xy}-orbit of z is contained in xy, so has length $\le q-1$. This excludes $M = PSL(2,q^3)$ and $M = Sz(q^{3/2})$. Moreover, this argument shows that if $M = PSU(3,q^2)$ then U is classical, for M_{xy} has a unique orbit of length q-1 on $X\setminus\{x,y\}$, all other orbits have length $(q^2-1)/(q+1,3)$ ([6, p. 499]). Now the $\tau^x_{y,z}$ can be identified with the unitary transvections and it follows that $G \simeq PSU(3,q^2)$. Thus we are left with the case that M is a group of Ree type. Since $q = 3^{2a+1}$, G contains an involution δ fixing at least two points $x,y \in X$ (Corollary 4). By [4], Lemma 3.3(v) and (ix), $\delta \in M$ and δ fixes q+1 points. Since δ is a dilatation on $I(x)^{\infty x}$ these must be the q+1 points of xy and so U is nothing but the Ree unital associated with M. Now, for $L \in L$, $\langle\delta\rangle \times PSL(2,q) \simeq M_L \trianglelefteq G_L$ and so $\langle\tau^x_{y,z} \mid x,y,z \in L\rangle \le \mathrm{Aut}(PSL(2,q)) = P\Gamma L(2,q)$, which shows that at least one, and hence all, $\tau^x_{y,z} \in M$, i.e. $G = M$ of order $(q^3+1)q^3(q-1)$. Now for a 3-Sylow group T(x) of G, $T(x)/T(x)_L$ (x on L) is the elementary abelian translation group of $I(x)^{\infty x}$. Hence, for the derived group $T(x)^{(1)}$ of T(x) we find $|T(x)^{(1)}| \le |T(x)_L| = q$, contradicting Lemma 3.3(iii) of [4]. □

4. MORE CHARACTERIZATIONS

Let $U = (X,L)$ be a unital satisfying (I) and (II). Consider the following two conditions.

(III') Given a point x and three distinct lines M_1, M_2, M_3 through x and points y_i, z_i on M_i (i = 1,2,3) such that $(y_1y_2)\parallel_x(z_1z_2)$, $(y_1y_3)\parallel_x(z_1z_3)$

and one of the lines (y_iy_j) or (z_iz_j) meets all three of M_1, M_2 and M_3, then $(y_2y_3)\|_x(z_2z_3)$.

(IV) Given a point x and two distinct lines M_1 and M_2 through x and points y_1, y_3, z_1, z_3 on M_1, y_2, y_4, z_2, z_4 on M_2 such that $(y_1y_2)\|_x(z_1z_2)$, $(y_1y_4)\|_x(z_1z_4)$ and $(y_2y_3)\|_x(z_2z_3)$, then also $(y_3y_4)\|_x(z_3z_4)$.

Clearly, (III) implies (III') and (IV). The converse is also true.

<u>LEMMA 8</u>. *Let* $\mathcal{U} = (X,\mathcal{L})$ *be a unital satisfying* (I), (II), (III') and (IV), *then also* (III) *holds.*

<u>PROOF</u>. Let x, M_i, y_i, z_i, i = 1,2,3 be as in (III). Suppose that M_1, M_2 and M_3 determine a circle in $\mathcal{I}(x)$ not containing ∞_x, i.e. suppose there is a line through y_1 intersecting M_2 in u_2 and M_3 in u_3, say. Let v_2 (v_3) be the point of intersection of the x-parallel of y_1u_2 through z_1 and $M_2(M_3)$. Using (III') we find that $(u_2y_3)\|_x(v_2z_3)$ and $(u_3y_2)\|_x(v_3z_2)$. Hence by (IV), $(y_2y_3)\|_x(z_2z_3)$ and (III) is shown to hold in this case. The remaining case is where M_1, M_2 and M_3 are on a circle of $\mathcal{I}(x)$ containing ∞_x, i.e. no line of $\mathcal{L}_x$ meets all three of M_1, M_2 and M_3. Since the two circles of $\mathcal{I}(x)$ corresponding to y_1y_2 and y_1y_3 cannot be tangent (for otherwise $y_1y_2 = y_1y_3$ and there is a line intersecting M_1, M_2 and M_3), there is a line M_4 through x which meets y_1y_2 in y_4 and z_1z_2 in z_4, say, and which also meets y_1y_3 and z_1z_3. Now looking at M_1, M_3 and M_4 are applying (III') we see that $(y_3y_4)\|_x(z_3z_4)$. Since M_2, M_3 and M_4 are not on a circle of $\mathcal{I}(x)$ (for otherwise this would be the circle determined by M_1, M_2 and M_3), we can apply the previous case and conclude that $(y_2y_3)\|_x(z_2z_3)$. □

The reason for considering (III') and (IV) is that in both cases there is a line M_i which is intersected by all lines mentioned in the condition. Thus, both (III') and (IV) have a (no doubt awkward) equivalent formulation $(\overline{\text{III}}')$ respectively $(\overline{\text{IV}})$ into terms of $\mathcal{GQ}(M_i)$. Since the classical unital satisfies (III') and (IV), the classical generalized quadrangle Q(4,q) on the points and lines of a hyperquadric in PG(4,q) must satisfy $(\overline{\text{III}}')$ and $(\overline{\text{IV}})$. So, conversely, if a unital U satisfying (I) and (II) has the property that $\mathcal{GQ}(L)$ is isomorphic to Q(4,q) for each line L of $\mathcal{U}$, then $\mathcal{U}$ is classical.

THEOREM 9. *Let $\mathcal{U}$ be a unital with* q+1 *points on a line satisfying* (I) *and* (II). *If for each line* L *of* $\mathcal{U}$, $\mathcal{GQ}(L) \simeq Q(4,q)$, *i.e. if every line of* $\mathcal{GQ}(L)$ *is regular, then* $\mathcal{U}$ *is classical.*

We are now in a position to prove that for even q, (I) and (II) suffice to characterize $\mathcal{U}$.

THEOREM 10. *Let* $\mathcal{U} = (X,\mathcal{L})$ *be a unital with* q+1 *points on a line satisfying* (I) *and* (II). *If* q *is even, then* $\mathcal{U}$ *is classical.*

PROOF. Let L be a line of $\mathcal{U}$ and let A_{ij}, $1 \le i,j \le q+1$ and A_i, B_i, $i = 1,\ldots$ $\ldots,q+1$ be defined as before. For each $x \in X\backslash L$ put

$$C(x) := \{A_{ij} \mid \exists \text{ line } M \in A_{ij} \text{ incident with } x\}.$$

By Corollary 5, C(x) has exactly one point on each of the lines A_i and B_i, $i = 1,\ldots,q+1$. We claim that if $x,y \in X\backslash L$, $x \neq y$, then $|C(x) \cap C(y)| \le 2$. First suppose xy is a line meeting L, $xy \in A_{ij}$, say. then by Corollary 5(iv), $C(x) \cap C(y) = \{A_{ij}\}$. Now consider the case where xy is a line of $\mathcal{U}$ not meeting L. Suppose x_1, x_2, x_3 are distinct points of L such that $xx_1, yx_1 \in A_{1,1}$, $xx_2, yx_2 \in A_{2,2}$ and $xx_3, yx_3 \in A_{3,3}$. In $\mathcal{I}(x)$, L, yx, yx_2, yx_3 correspond to circles with the following properties: yx_1, yx_2, yx_3 all go through the point xy of $\mathcal{I}(x)$ and are tangent to L in respectively xx_1, xx_2 and xx_3. Since q is even, there is a point $\neq$ xy of $\mathcal{I}(x)$ which is also on the circles yx_1, yx_2, yx_3, i.e. there is a line $M \neq xy$ through x intersecting yx_i, $i = 1,2,3$. By Lemma 1, $L \Vert_y M$ and so xy intersects L, a contradiction. We have shown that each triple A_{i_1,j_1}, A_{i_2,j_2}, A_{i_3,j_3} with $|\{i_1,i_2,i_3\}| = |\{j_1,j_2,j_3\}| = 3$ is covered at most once by a C(x). Since there are $q^3 - q$ C(x), each such triple is covered exactly once. Thus, with the A_{ij} as points, the A_i and B_i as lines and the C(x) as circles, we have obtained a Minkowski plane $\mathcal{M}(L)$ of even order q. By [3], $\mathcal{M}(L)$ is isomorphic to the geometry of points, lines and plane sections of a quadric of index two in PG(3,q). Since $\mathcal{GQ}(L)$ is determined by $\mathcal{M}(L)$ (the points of $\mathcal{GQ}(L)$ correspond to the points and circles of $\mathcal{M}(L)$, the lines of $\mathcal{GQ}(L)$ correspond to the lines and pencils of $\mathcal{M}(L)$, etc.) $\mathcal{GQ}(L)$ is isomorphic to Q(4,q) and so $\mathcal{U}$ is classical. □

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A characterization of two classes of semi partial geometries by their parameters

by

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ABSTRACT

We show that, under mild restrictions on the parameters, semi-partial geometries with $\mu = \alpha^2$ or $\mu = \alpha(\alpha+1)$ are determined by their parameters.

KEY WORDS & PHRASES: *Semi-partial geometry, partial geometry, strongly regular graph*

1. INTRODUCTION

Let X be a (finite) nonempty set and L a set of subsets of X. Elements of X are called *points*, elements of L are called *lines*. The pair (X,L) is called a *partial linear space* if any two distinct points are on at most one line.

Two distinct points x and y are called *collinear* if there exists $L \in L$ such that $x,y \in L$, *noncollinear* otherwise. Two distinct lines L and M are called *concurrent* if $|L \cap M| = 1$.

We write $x \sim y$ ($x \not\sim y$) to denote that x and y are collinear (noncollinear). Similarly $L \sim M$ ($L \not\sim M$) means $|L \cap M| = 1$ ($|L \cap M| = 0$).

If $x \sim y$ ($L \sim M$) we denote by xy (LM) the line (point) incident with x and y (L and M).

For a nonincident point-line pair (x,L) we define:

$$[L,x] := \{y \in X | y \in L,\ y \sim x\},$$

$$[x,L] := \{M \in L | x \in M,\ L \sim M\}.$$

Given positive integers s,t,α,μ, the partial linear space (X,L) is called a *semi-partial geometry* (s.p.g) with parameters s,t,α,μ if:

(i) every line contains s+1 points,

(ii) every point is on t+1 lines,

(iii) for all $x \in X$, $L \in L$, $x \notin L$ we have $|[x,L]| \in \{0,\alpha\}$,

(iv) for all $x,y \in X$ with $x \not\sim y$ the number of points z such that $x \sim z \sim y$ equals μ.

A semi-partial geometry which satisfies $|[x,L]| = \alpha$ for all $x \in X$, $L \in L$ with $x \notin L$, or equivalently which satisfies $\mu = \alpha(t+1)$, is also called a *partial geometry* (p.g).

The *point-graph* of the partial linear space (X,L) is the graph with vertex set X, two distinct vertices x and y being adjacent iff $x \sim y$. The point-graph of a semi-partial geometry is easily seen to be strongly regular. Let (X,L) be a semi-partial geometry.

For $x,y \in X$, $x \not\sim y$ we define

$$[x,y] := \{L \in \mathcal{L} \mid x \in L,\ |[L,y]| = \alpha\}.$$

It is easy to see that $\alpha = s+1$ iff any two distinct points are collinear iff $(X,\mathcal{L})$ is a Steiner system $S(2,s+1,|X|)$. We shall always assume $s \geq \alpha$, hence noncollinear points exist.

Let $x,y \in X$, $x \not\sim y$. Then $\mu = |[x,y]|\alpha$ and $|[x,y]| \geq |[x,L]| = \alpha$ if $L \in [y,x]$. Hence, $\mu \geq \alpha^2$ and

(*) $\mu = \alpha^2 \Longleftrightarrow \forall K \in [x,y],\ L \in [y,x]:\ K \sim L,$

(* *) $\mu = \alpha(\alpha+1) \Longleftrightarrow$ every line $K \in [x,y]$ intersect every line $L \in [y,x]$ but one.

This is the basic observation we use in showing that, under mild restrictions on the parameters, semi partial geometries with $\mu = \alpha^2$ or $\mu = \alpha(\alpha+1)$ satisfy the Diagonal Axiom (D).

(D) : Let x_1,x_2,x_3,x_4 be four distinct points no three on a line, such that $x_1 \sim x_2 \sim x_3 \sim x_4 \sim x_1 \sim x_3$.
Then also $x_2 \sim x_4$.

From DEBROEY [1], it then follows that such a semi-partial geometry is known.

2. SEMI-PARTIAL GEOMETRIES WITH $\mu = \alpha^2$.

Our first theorem deals with the case $\alpha = 1$, $\mu = 1$.

THEOREM 1. *Every strongly regular graph with parameters* $(n,k,\lambda,\mu = 1)$ *is the point-graph of a s.p.g. with* $s = \lambda+1$, $t = \frac{k}{\lambda+1} - 1$, $\alpha=1$, $\mu=1$.

PROOF. Let (X,E) be a strongly regular graph with $\mu = 1$, and let $x \in X$. Since two nonadjacent points in $\Gamma(x)$ cannot have a common neighbour in $\Gamma(x)$, the induced subgraph on $\Gamma(x)$ in the union of cliques. This induced subgraph has valency λ, so it is the union of $\frac{k}{\lambda+1}$ cliques of size $\lambda+1$. □

Next we deal with the case $\alpha = 2$, $\mu = 4$.

THEOREM 2. *Let* (X,L) *be a s.p.g. with parameters* s,t, $\alpha = 2$, $\mu = 4$. *Then* (X,L) *satisfies* (D).

PROOF. Let x_1,x_2,x_3,x_4 be four distinct points no three on a line, such that $x_1 \sim x_2 \sim x_3 \sim x_4 \sim x_1 \sim x_3$. If $x_2 \not\sim x_4$, then we can apply (*) to the points x_2 and x_4. Since $x_1x_4 \in [x_4,x_2]$ and $x_2x_3 \in [x_2,x_4]$, x_1x_4 and x_2x_3 intersect in a point $\neq x_2,x_3$. Now $3 \leq |[x_1,x_2x_3]| \leq \alpha=2$, a contradiction. □

Let U be a set containing t+3 elements. Then we denote by $U_{2,3}$ the s.p.g. which has as points the 2-subsets of U, as lines the 3-subsets of U together with the natural incidence.

The parameters are $s=2$, t, $\alpha=2$, $\mu=4$.

DEBROEY [1] showed that a s.p.g. with $t>1$, $\alpha=2$, $\mu=4$ satisfying (D) is isomorphic to a $U_{2,3}$. Hence we have the following theorem.

THEOREM 3. *A s.p.g. with* $t>1$, $\alpha=2$, $\mu=4$ *is isomorphic to a* $U_{2,3}$. □

REMARK. A s.p.g. with $t=1$, $\alpha=2$, $\mu=4$ is isomorphic to the geometry of edges and vertices of the complete graph K_{s+2}

We now consider the case $\alpha>2$. For the remainder of this section let (X,L) be a s.p.g with $\alpha>2$ and $\mu = \alpha^2$.

LEMMA 1. *Let* $x \in X$, $L \in L$, $x \notin L$ *such that* $[L,x] = \{z_1,\ldots,z_\alpha\}$. *Let* M *be a line through* z_1 *intersecting* xz_2 *in a point* $u \neq x,z_2$. *Suppose there exists* $y \in L$, $y \neq z_1,\ldots,z_\alpha$ *with* $u \not\sim y$. *Then* M *intersects* xz_i *for all* $i = 1,\ldots,\alpha$ (*see figure* 1).

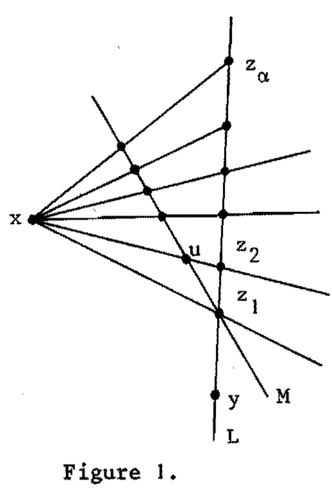

Figure 1.

PROOF. By (*) applied to x and y, the α lines $L = L_1, L_2, \dots, L_\alpha$ of $[y,x]$ intersect the α lines $xz_1, \dots, xz_\alpha$ of $[x,y]$. In particular $L_1, \dots, L_\alpha$ intersect xz_2. Hence $[y,u] = [y,x] = \{L_1, \dots, L_\alpha\}$.

Since $M \in [u,y]$, M intersects $L_1, \dots, L_\alpha$ in points $v_1 = z_1, v_2, \dots, v_\alpha$ respectively. If $x \sim v_i$ for all i, then the $\alpha+1$ points $u, v_1 v_2, \dots, v_\alpha$ on M are all collinear with x, a contradiction. Hence $x \not\sim v_i$ for some i. Since L_i intersects $xz_1, \dots, xz_\alpha$ it follows that $[x,v_i] = [x,y] = \{xz_1, \dots, xz_\alpha\}$. Since $M \in [v_i, x]$, M intersects all lines in $[x,v_i]$. □

LEMMA 2. *Let* $x \in X$, $L \in \mathcal{L}$, $x \notin L$ *such that* $[L,x] = \{z_1, \dots, z_\alpha\}$. *Let* M *be a line through* z_1 *intersecting* xz_2 *in a point* $u \neq x, z_2$. *If* $s > \alpha$, *then* M *intersects* xz_i *for all* $i = 1, \dots, \alpha$.

PROOF. Assume that M intersects xz_i, $i = 1, \dots, \beta$ $(2 \leq \beta < \alpha)$ in points $u_1 = z_1$, $u_2 = u, \dots, u_\beta$ respectively and does not intersect $xz_{\beta+1}, \dots, xz_\alpha$. Take $y \in L$, $y \neq z_1, \dots, z_\alpha$. By lemma 1 $y \sim u_i$, $i = 1, \dots, \beta$.

Since $|[M,x]| = \alpha$, there is a $v \in M$ such that $v \sim x$, $v \neq u_1, \dots, u_\beta$. Also $v \sim z$ for all $z \in \bigcup_{i=1}^{\beta} [yu_i, x]$, for if $v \not\sim z$ for some $z \in [yu_i, x]$, then $vx \in [v,z]$ and $yu_i \in [z,v]$. Hence $vx \sim yu_i$ and so yu_i intersects the $\alpha+1$ lines $xv, xz_1, \dots, xz_\alpha$ through x, a contradiction. The points of $\bigcup_{i=1}^{\beta} [yu_i, x]$ are therefore on the α lines $M = vz_1, vz_2, \dots, vz_\alpha$ of $[v,y]$.

Since $s > \alpha$ we can take $y' \in L$ such that $y' \neq y, z_1, \dots, z_\alpha$.

Now if $z \in [yu_2, x]$, then $z \sim y'$. Indeed, as shown z is on some vz_i and since vz_i intersects at most $\alpha-1$ of the lines $xz_1, \dots, xz_\alpha$, it follows from Lemma 1 that every point of intersection of vz_i and a line xz_j, so in particular z, is collinear with y'.

But now we have $|[yu_2, y']| \geq |[yu_2, x] \cup \{y\}| = \alpha+1$, a contradiction. □

LEMMA 3. *Let* $x \in X$, $L \in \mathcal{L}$, $x \notin L$ *such that* $[L,x] = \{z_1, \dots, z_\alpha\}$. *If* $s > \alpha$, *then every line* M *not through* x *which intersects two lines of* $[x,L] = \{xz_1, \dots, xz_\alpha\}$ *also intersects* L *and all lines of* $[x,L]$.

PROOF. The number of pairs $(u,v) \neq (z_1, z_2)$ such that $u \in xz_1$, $v \in xz_2$, $u,v \neq x$, $u \sim v$ equals $s(\alpha-1)-1$. Every line $M \neq xz_1, \dots, xz_\alpha$ which intersects L and $xz_1, \dots, xz_\alpha$ gives rise to such a pair (u,v). By (*) and lemma 2 the number of these lines equals $(s+1-\alpha)(\alpha-1) + \alpha(\alpha-2) = s(\alpha-1)-1$. □

Let $L_1, L_2 \in \mathcal{L}$ intersect in a point x. If L is any line intersecting L_1 and L_2 not in x, we let $L_3, L_4, \ldots, L_\alpha$ be the other lines in $[x,L]$. By lemma 3, $L_3, L_4, \ldots, L_\alpha$ are independent of the choice of L. Put

$$\mathcal{L}(L_1,L_2) := \{L_1,L_2,\ldots,L_\alpha\} \cup \{L \in \mathcal{L} \mid L \sim L_1,L_2, LL_1 \neq x \neq LL_2\},$$

$$X(L_1,L_2) := \bigcup_{L \in \mathcal{L}(L_1,L_2)} L$$

LEMMA 4. *Let* $L_1, L_2 \in \mathcal{L}$, $L_1 \sim L_2$. *If* $s > \alpha$, *then* $\langle L_1,L_2\rangle := (X(L_1,L_2), \mathcal{L}(L_1,L_2))$ *is a partial geometry (in fact a dual design) with parameters* $\tilde{s} = s$, $\tilde{t} = \alpha-1$, $\tilde{\alpha} = \alpha$.

PROOF. Clearly two points are on at most one line and each line contains s+1 points. Using (*) and Lemma 3 it follows immediately that every point $x \in X(L_1,L_2)$ is on α lines of $\mathcal{L}(L_1,L_2)$ so $\tilde{t}+1 = \alpha$. It also follows immediately that any two lines of $\mathcal{L}(L_1,L_2)$ intersect, hence $\tilde{\alpha} = \tilde{t}+1 = \alpha$. □

Notice that for $M_1,M_2 \in \mathcal{L}(L_1,L_2)$, $M_1 \neq M_2$, $M_1 \sim M_2$ we have $\langle M_1,M_2\rangle = \langle L_1,L_2\rangle$. Notice also that for any two noncollinear points x and y of $\langle L_1,L_2\rangle$ there are $\tilde{\mu} = \tilde{\alpha}(\tilde{t}+1) = \alpha^2 = \mu$ points $z \in X(L_1,L_2)$ collinear with both x and y, i.e. the common neighbours of x and y in $(X,\mathcal{L})$ are the common neighbours of x and y in $\langle L_1,L_2\rangle$.

THEOREM 4. *Let* $(X,\mathcal{L})$ *be a s.p.g. with parameters* $s,t,\alpha(>2)$, $\mu = \alpha^2$. *If* $s > \alpha$ *and* $t \geq \alpha$, *then* $(X,\mathcal{L})$ *satisfies* (D).

PROOF. Let x_1,x_2,x_3,x_4 be four distinct points no three on a line, such that $x_1 \sim x_2 \sim x_3 \sim x_4 \sim x_1 \sim x_3$.

Suppose $x_2 \not\sim x_4$. Since $x_2 \sim x_1 \sim x_4$ it follows that

$$x_1 \in \langle x_4x_3, x_2x_3\rangle \qquad (\dagger)$$

In $(X,\mathcal{L})$ there are $\lambda = s-1 + (\alpha-1)t$ points collinear with both x_1 and x_3. In $\langle x_4x_3,x_2x_3\rangle$ there are $\tilde{\lambda} = \tilde{s}-1 + (\tilde{\alpha}-1)\,\tilde{t} = (s-1) + (\alpha-1)^2$ points collinear with both x_1 and x_3. Since $t \geq \alpha = \tilde{t} + 1$ it follows that $\tilde{\lambda} < \lambda$ and so there exists $x_5 \in X \setminus X(x_4x_3,x_2x_3)$ such that $x_1 \sim x_5 \sim x_3$. Now application of

(†)

to x_1, x_5, x_3, x_4 yields $x_5 \sim x_4$,
to x_1, x_2, x_3, x_5 yields $x_5 \sim x_2$,
to x_4, x_1, x_2, x_5 yields $x_2 \sim x_4$. □

DEBROEY [1] showed that a s.p.g. with parameters $s, t, \alpha(>2)$, $\mu = \alpha^2$ satisfying (D) is of the following type: the "points" are the lines of $PG(d,q)$, the "lines" are the planes in $PG(d,q)$ for some prime power q and $d \in \mathbb{N}$, $d \geq 4$. In this case $s = q(q+1)$, $t = (q-1)^{-1}(q^{d-1}-1)-1$, $\alpha = q+1$, $\mu = (q+1)^2$.

<u>THEOREM 5</u>. *Let* (X,L) *be a s.p.g. with parameters* $s, t, \alpha(>2)$, $\mu = \alpha^2$. *If* $s > \alpha$ *and* $t \geq \alpha$, *then* (X,L) *is isomorphic to the s.p.g. consisting of the lines and planes in* $PG(d,q)$. *In particular* $s = q(q+1)$, $t = (q-1)^{-1}(q^{d-1}-1)-1$, $\alpha = q+1$, $\mu = (q+1)^2$.

The only interesting case remaining is $s = \alpha$. Now if (X,E) is a Moore graph of valency r, i.e. a strongly regular graph with $\lambda = 0$, $\mu = 1$, then $(X, \{\Gamma(x) \mid x \in X\})$ is easily seen to be a s.p.g. with parameters $s = t = \alpha = r-1$, $\mu = (r-1)^2$ (here $\Gamma(x) = \{y \in X \mid (x,y) \in E\}$). The point graph of this s.p.g. is the complement of (X,E). Such a s.p.g. does not satisfy (D) for $r > 2$. From the following theorem follows immediately that a s.p.g. with $\mu = \alpha^2$, $s = \alpha$ is necessarily of this type.

<u>THEOREM 6</u>. *Let* (X,L) *be a s.p.g. with* $t \geq \alpha$, $\mu = \alpha^2$ *and* $s = \alpha$. *Then* $t = \alpha$.

<u>PROOF</u>. Let $x, y \in X$, $x \not\sim y$. Let $[x,y] = \{L_1, \ldots, L_\alpha\}$, $[y,u] = \{M_1, \ldots, M_\alpha]$ and put $z_{ij} = L_i M_j$, $i,j = 1, \ldots, \alpha$ (see figure 2).

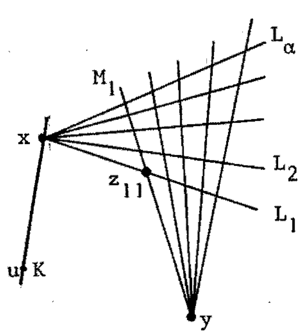

Figure 2.

The number of $(z_{ij},z_{k\ell})$ with $i \neq k$, $j \neq \ell$, $z_{ij} \sim z_{k\ell}$ equals $\alpha^2 \cdot (\alpha-1)(\alpha-2)$. Now let K be a line through x, $K \neq L_1,\ldots,L_\alpha$, and let u be a point on K, $u \neq x$.

Then u is collinear with $(\alpha-1)$ of the α points $z_{i,1},\ldots,z_{i,\alpha}$, for $i = 1,\ldots,\alpha$. Since $u \not\sim y$, u is collinear with all of $z_{1,j},\ldots,z_{\alpha,j}$ or with none, for $j = 1,\ldots,\alpha$.

It follows that there are α lines through u intersecting $(\alpha-1)$ of the α lines $M_1,\ldots,M_\alpha$. Hence each point $u \neq x$ on K gives rise to $\alpha(\alpha-1)(\alpha-2)$ pairs $(z_{ij},z_{k\ell})$ as described, so K gives rise to all $\alpha^2(\alpha-1)(\alpha-2)$ pairs $(z_{ij},z_{k\ell})$.

Suppose $t > \alpha$, then we can find two such lines K and K'. It follows that for $u \in K$, the α lines through u intersecting $(\alpha-1)$ of the α lines $M_1,\ldots,M_\alpha$ also intersect K'. But now $|[u,K']| = \alpha+1$, a contradiction. □

3. SEMI-PARTIAL GEOMETRIES WITH $\mu = \alpha(\alpha+1)$.

In this section (X,L) is a semi-partial geometry with parameters s,t,α and $\mu = \alpha(\alpha+1)$.

If $x,y \in X$, $x \not\sim y$ we shall always denote the $\alpha+1$ lines in $[x,y]$ by $K_1,\ldots,K_{\alpha+1}$, and the $(\alpha+1)$ lines in $[y,x]$ by $L_1,\ldots,L_{\alpha+1}$. By (**) we can number these lines in such a way that $K_i \cap L_i = \emptyset$, $i = 1,\ldots,\alpha+1$ and $K_i \cap L_j \neq \emptyset$, $i,j = 1,\ldots,\alpha+1$, $i \neq j$ (see figure 3).

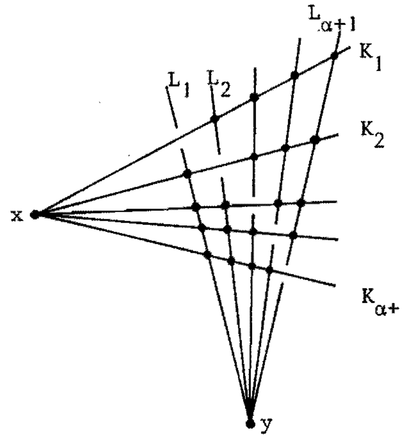

Figure 3.

Again our aim will be to show that the diagonal axiom (D) holds. We first deal with the case $\alpha = 2$.

LEMMA 5. *If* $\alpha = 2$ *and* $t > s$, *then a set of* 3 *collinear points not on one line can be extended to a set of* 4 *collinear points no 3 on a line.*

PROOF. Let x, a and b be three distinct collinear points not on one line. There are t-1 lines $\neq$ xa,ab through a and on each of those lines there is a point $y_i \sim b$, $y_i \neq a$, $i = 1,\ldots,t-1$. Suppose $y_i \not\sim x$ for all $i = 1,\ldots,t-1$. Now for each $i = 1,\ldots,t-1$, $ay_i \not\sim xb$ (for otherwise $|[a,xb]| \geq 3$) and $by_i \not\sim xa$. Also $xa,xb \in [x,y_i]$ and $ay_i,by_i \in [y_i,x]$. Hence, by (**) there is a third line through y_i intersecting xa and xb in points u_i and v_i respectively. Clearly $u_i \neq u_j$ if $i \neq j$, for $u_i = u_j$ implies $x,v_i,v_j \in [u_i,xb]$. Thus xa contains $t+1 > s+1$ points (namely $x,a,u_1,\ldots,u_{t-1}$), a contradiction. □

LEMMA 6. *Suppose* $\alpha = 2$. *If* x_1,x_2,x_3,x_4 *are four distinct collinear points, no three on a line, then no point can be collinear with exactly three of these four points.*

PROOF. Suppose x_5 is collinear with x_2,x_3,x_4 and $x_1 \not\sim x_5$. Clearly $x_5 \notin x_2x_3$, x_2x_4,x_3x_4. Hence $\{x_1x_2,x_1x_3,x_1x_4\} = [x_1,x_5]$ and $\{x_5x_2,x_5x_3,x_5x_4\} = [x_5,x_1]$ so x_5x_2 has to intersect x_1x_3 or x_1x_4 by (**). But then $|[x_2,x_1x_3]|$ or $|[x_2,x_1x_4]| > 2$, a contradiction. □

LEMMA 7. *Same hypothesis as in lemma 6. Then the only points collinear with exactly two points of* $\{x_1,x_2,x_3,x_4\}$ *are the points on the lines* x_ix_j, $i \neq j$.

PROOF. Suppose $x_5 \sim x_1,x_4$ and $x_5 \not\sim x_2,x_3$, $x_5 \notin x_1x_4$ (see figure 4).

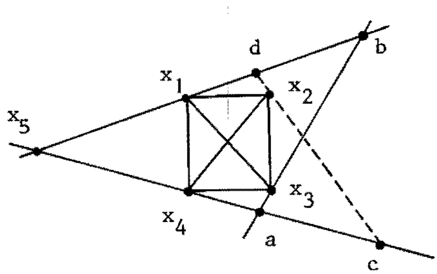

Figure 4.

Apply (**) to x_3 and x_5 to get a line ab through x_3 with $a \in x_5x_4$, $b \in x_5x_1$. Similarly (**) applied to x_5 and x_2 gives us a line cd through x_2 with $c \in x_5x_4$, $d \in x_5x_1$. Clearly $b \not\sim c$ so we can apply (**) to b and c. It follows that $ab \cap cd = \emptyset$. Also $x_2 \not\sim a$ and (**) applied to x_2 and a yields: $ab \cap cd \neq \emptyset$ or $ab \cap x_2x_4 \neq \emptyset$. Hence $ab \cap x_2x_4 \neq \emptyset$, a contradiction since $\{x_2,x_4\} = [x_2x_4,x_3]$. □

<u>THEOREM 7</u>. *If* (X,L) *is a s.p.g with parameters* $s,t,\alpha = 2$, $\mu = 6$ and $t>s$, *then* (X,L) *satisfies* (D).

<u>PROOF</u>. Let x_1,x_2,x_3 and x_4 be four distinct points no three on a line such that $x_4 \sim x_1 \sim x_2 \sim x_3 \sim x_4 \sim x_2$. By Lemma 5 there exists $x_5 \sim x_2,x_3,x_4$.

By Lemmas 6 and 7 $x_1 \sim x_3,x_5$. □

<u>REMARK</u>. If (X,L) is a s.p.g but not a partial geometry, then $t \geq s$ (see DEBROEY & THAS [2]). Using the integrality conditions for the multiplicities of the eigenvalues of a strongly regular graph it follows that a s.p.g with $s=t$, $\alpha=2$ and $\mu=6$ satisfies $(8s^2-24s+25) \mid \{8(s+1)(2s^3-9s^2+19s-30)\}^2$. From this one easily deduces an upper bound for s. The remaining cases were checked by computer and only $s=t=28$ survived. Thus, every s.p.g which is not a partial geometry satisfies (D) or has $s=t=28$ (and 103125 points).

We now turn to the case $\alpha \geq 3$. We shall make two additional assumptions in this case. The first assumption is $\alpha \neq 3$, the second assumption is $s \geq f(\alpha)$ where f is defined in Lemma 9. Notice that this bound on s is used only in the proof of Lemma 9.

<u>LEMMA 8</u>. *Let* $x,y \in X$, $x \not\sim y$ *and suppose* $[x,y] = [K_1,\ldots,K_{\alpha+1}]$, $[y,x] = [L_1,\ldots,L_{\alpha+1}]$ *such that* $K_i \cap L_i = \emptyset$, $i = 1,\ldots,\alpha+1$. *If* M *is a line intersecting* $\sigma \geq 1$ *lines of* $[x,y]$, $\tau \geq 1$ *lines of* $[y,x]$ *and* $\sigma < \tau$, *then* $\sigma = \alpha-1$ *and* $\tau = \alpha$.

<u>PROOF</u>. Since $\sigma < \tau$, there exists a point of intersection u of M with a line $L_i \in [y,x]$ such that u is not on one of the lines of $[x,y]$. Then $u \not\sim x$ and so, applying (**) to u and x, it follows that $M \in [u,x]$ intersects $\alpha-1$ of the α lines $K_1,K_2,\ldots,K_{i-1},K_{i+1},\ldots,K_{\alpha+1} \in [x,u]$. Thus $\alpha-1 \leq \sigma < \tau \leq \alpha$, which proves our claim. □

LEMMA 9. *Let* $x \in X$ *and* $L \in \mathcal{L}$ *such that* $x \notin L$ *and* x *is collinear with* α *points* $z_2, z_3, \ldots, z_{\alpha+1}$ *on* L. *Let* M *be a line through* $z_{\alpha+1}$ *meeting* xz_α *in a point* $u \neq x, z_\alpha$. *Suppose* $s \geq f(\alpha)$ *where* $f(4) = 12$, $f(5) = 16$, $f(6) = f(7) = 17$, $f(8) = 18$, $f(9) = 19$, $f(10) = 21$, $f(11) = 23$, $f(\alpha) = 2\alpha$ $(\alpha \geq 12)$. *Then* M *intersects at least* $\alpha-1$ *lines of* $[x,L]$.

PROOF. Suppose M does not meet at least two lines of $[x,L]$, xz_2 and xz_3, say. Since $s \geq 2\alpha$ we can find $y \in L$ such that $x \not\sim y \not\sim u$. Let $[x,y] = \{K_1, K_2 = xz_2, \ldots, K_{\alpha+1} = xZ_{\alpha+1}\}$ and $[y,x] = \{L_1 = L, L_2, L_3, \ldots, L_{\alpha+1}\}$ with $K_i \cap L_i = \emptyset$.

Looking at u and y we find that M intersects $\alpha-1$ of the α lines L_i, $i \neq \alpha$. Every point L_iM which is collinear with x is on a line K_j, $j \neq \alpha$. If $L_iM \sim x$ for these $\alpha-1$ i's, we find that M meets at least α of the lines $K_1, \ldots, K_{\alpha+1}$, hence at least $\alpha-1$ of the lines $K_2, \ldots, K_{\alpha+1}$, a contradiction. Let $t = L_iM$ be a point not collinear with x. Considering $x \not\sim t$ we see that M intersects $\alpha-1$ of the α lines in $[x,y] \backslash \{K_i\}$. This shows that $i = 2$ or 3, so there are at most two such points t, and that M meets $K_1, K_4, K_5, \ldots, K_{\alpha+1}$. Let $V = \{K_4M, K_5M, \ldots, K_\alpha M\}$ and count pairs (y,v), $y \in L$, $y \not\sim x$, $v \in V$, $v \sim y$. The number of such pairs is at least $(s-\alpha+1)(\alpha-5)$ (first choose y, $s-\alpha+1$ possibilities, then given y we can find $\alpha-3$ points $L_iM \sim x$ as above, possibly one on $K_1(y)$, and one is $z_{\alpha+1}$), and at most $(\alpha-3)(\alpha-2)$ (first choose v, then y). It follows that for $\alpha > 5$, $s \leq 2\alpha - 1 + \lfloor \frac{6}{\alpha-5} \rfloor$. Let $W = V \cup \{q, q'\} = \{w \in M \mid w \sim x\}$ and count pairs (y,w), $y \in L$, $y \not\sim x$, $w \in W$, $w \sim y$. This yields $(s-\alpha+1)(\alpha-4) \leq (\alpha-3)(\alpha-2) + 2(\alpha-1)$, hence $s \leq 2\alpha + \lfloor \frac{8}{\alpha-4} \rfloor$ if $\alpha > 4$. Above we saw that for any $y \in L$ with $x \not\sim y \not\sim u$, $K_1 = K_1(y)$ meets M. But if $s+1 > \alpha + (\alpha-2) + 2(\alpha-1) = 4\alpha-4$, we can find $y \in L$ such that $y \not\sim x$, u, q and q', a contradiction. Therefore we have $s < 4\alpha-4$. We now have obtained a contradiction for all $\alpha \geq 4$ and the lemma is proved. □

LEMMA 10. *Some hypotheses as in Lemma 9. Then* M *intersects exactly* $\alpha-1$ *lines of* $[x,L]$.

PROOF. Take $y \in L$, $y \not\sim x$ and let K_i and L_i be defined as before. Put $K := K_{\alpha+1}$ and let $A(x,L)$ be the set of lines $\neq K, L$ through $z_{\alpha+1}$ intersecting at least $\alpha-1$ lines of $[x,L]$, $A(y,K)$ the set of lines $\neq K, L$ through $z_{\alpha+1}$ intersecting at least $\alpha-1$ lines of $[y,K]$. Suppose a lines of $A(x,L)$ intersects $\alpha-1$ lines of $[x,L]$ and b lines of $A(x,L)$ intersect α lines of $[x,L]$. Counting the points $u \sim z_{\alpha+1}$ on $K_2, K_3, \ldots, K_\alpha$, such that $u \neq x, z_2, \ldots, z_\alpha$ yields $a(\alpha-2) + b(\alpha-1) = (\alpha-1)(\alpha-2)$. Hence $a = 0$ and $b = \alpha-2$ or $a = \alpha-1$ and $b = 0$. Thus $|A(x,L)| = \alpha-2$ or $\alpha-1$ according as every line in $A(x,L)$ intersects all lines or all but one line in $[x,L]$. A similar result holds for $A(y,K)$. Now $A(x,L) = A(y,K)$, for suppose $N \in A(x,L)$ then by Lemma 8, N intersects at least $\alpha-1$ lines of $[y,x]$, so at least $\alpha-2 \geq 2$ lines of $[y,K]$. Hence $N \in A(y,K)$ by Lemma 9. Similarly, $N \in A(y,k)$ implies $N \in A(x,L)$. Suppose $|A(x,L)| = \alpha-2$, i.e. there are $\alpha-2$ lines through $z_{\alpha+1}$ intersecting all lines of $[x,L] \cup [y,K]$. It follows that $K_2L_{\alpha+1} \not\sim z_{\alpha+1}$ so we can apply (**) to $K_2L_{\alpha+1}$ and $z_{\alpha+1}$. This shows that $L_{\alpha+1} \in [K_2L_{\alpha+1}, z_{\alpha+1}]$ intersects all $N \in A(y,K) \subseteq [z_{\alpha+1}, K_2L_{\alpha+1}]$, a contradiction, for $L_{\alpha+1} \sim N$ implies $|[y,N]| \geq \alpha+1$. □

<u>LEMMA 11</u>. *Let* $x \in X$, $L \in \mathcal{L}$ *such that* x *is collinear with* α *points* $z_2, \ldots, z_{\alpha+1}$ *on* L. *Let* M *be a line through* $z_{\alpha+1}$ *intersecting* $\alpha-1$ *lines of* $[x,L]$ *and let* $y \in L$, $y \not\sim x$. *Then, if* $[x,y] = \{K_1(y), K_2 = xz_2, \ldots, K_{\alpha+1} = xz_{\alpha+1}\}$, M *intersects* $K_1(y)$.

<u>PROOF</u>. Suppose M does not intersect K_2, say. As shown in Lemma 10, M also intersects $\alpha-1$ lines of $[y, K_{\alpha+1}] = \{L_1 = L, L_2, \ldots, L_\alpha\}$. So M intersects at least one of $L_{\alpha-1}$ and L_α and since $\alpha \geq 4$, $L_2 \neq L_{\alpha-1}, L_\alpha$. Suppose M intersects $L_{\alpha-1}$ (L_α) in a point v. If $v \not\sim x$ then apply (**) to v and x. It follows that $M \in [v,x]$ intersects $K_1(y) \in [x,v]$ for M misses $K_2 \in [x,v]$. If $v \sim x$ then $v = L_{\alpha-1}K_i$ ($v = L_\alpha K_i$) for some i. By Lemma 10 applied to x and $L_{\alpha-1}$ (L_α) it follows that M intersects $K_1(y) \in [x, L_{\alpha-1}]$ ($K_1(y) \in [x, L_\alpha]$), for M does not intersect $K_2 \in [x, L_{\alpha-1}]$ ($K_2 \in [x, L_\alpha]$). □

<u>COROLLARY</u>. *The line* $K_1(y)$ *is the same for all* $y \in L$, $y \not\sim x$.

<u>LEMMA 12</u>. *Let* $x \in X$, $L \in \mathcal{L}$ *such that* x *is collinear with* α *points* $z_2, z_3, \ldots, z_{\alpha+1}$ *on* L. *Put* $K_i = xz_i$, $i = 2, \ldots, \alpha+1$ *and let* K_1 *be defined by* $\{K_1, K_2, \ldots, K_{\alpha+1}\} = [x,y]$ *for any* $y \in L$, $y \not\sim x$. *Then every line which intersects* K_1 *and a* $K_i (i \neq 1)$ *not in* x, *intersects* L *and therefore exactly* α *lines of* $\{K_1, \ldots, K_{\alpha+1}\}$.

<u>PROOF</u>. Fix $i \in \{2, \ldots, \alpha+1\}$. The number of pairs (u,v) such that

$u \in K_1\setminus\{x\}$, $v \in K_i\setminus\{x\}$, $u\sim v$ equals $s(\alpha-1)$. If $y \in L$, $y \not\sim x$ and $[y,x] = \{L_1=L,L_2,\ldots,L_{\alpha+1}\}$, then each of the $\alpha-1$ lines $L_2,L_3,\ldots,L_{i-1},L_{i+1},\ldots,L_{\alpha+1}$ gives rise to such a pair (u,v). Each point z_j, $j = 2,3,\ldots,i-1,i+1,\ldots,\alpha+1$ is on $\alpha-1$ lines $\neq K_j,L$ which intersect α lines of $\{K_1,\ldots,K_{\alpha+1}\}$. They all intersect K_1 by Lemma 11 and no two miss the same K_k since otherwise some K_ℓ would be hit $\alpha+1$ times. Thus each point z_j, $j=2,3,\ldots,i-1,i+1,\ldots,\alpha+1$ gives rise to $(\alpha-2)$ pairs (u,v). Finally there are $(\alpha-1)$ pairs (u,v) with $v = z_i$. In all, the lines intersecting L contain $(s+1-\alpha)(\alpha-1) + (\alpha-1)(\alpha-2) + (\alpha-1) = s(\alpha-1)$, i.e. all, pairs (u,v). □

If in Lemma 12 we replace $L = L_1$ by a line L_j missing K_j, then it follows that every line intersecting two lines of $\{K_1,\ldots,K_{\alpha+1}\}$ not in x, intersects exactly α lines of $\{K_1,\ldots,K_{\alpha+1}\}$. Using this result and the foregoing lemmas we can now proceed as in the case $\mu = \alpha^2$. For any two intersecting lines L_1,L_2 we can define in an obvious way a partial geometry $\langle L_1,L_2\rangle = (X(L_1,L_2),L(L_1,L_2))$, now with parameters $\tilde{s} = s$, $\tilde{t} = \alpha$, $\tilde{\alpha} = \alpha$ (so $\langle L_1,L_2\rangle$ is an $(\alpha+1)$-net of order $s+1$). Again $\tilde{\mu} = \tilde{\alpha}(\tilde{t}+1) = \alpha(\alpha+1) = \mu$, so with the same proof as the proof of Theorem 4 we have the following theorem.

<u>THEOREM 8</u>. *Let* (X,L) *be a s.p.g. with parameters* $s,t,\alpha,\mu = \alpha(\alpha+1)$. *If* $\alpha \geq 4$, $s \geq f(\alpha)$ (f as in Lemma 9) *and* $t \geq \alpha+1$ (i.e. *if* (X,L) *is not a p.g.*), *then* (X,L) *satisfies* (D).

Fix a $(d-2)$-dimensional subspace S of PG(d,q), q a prime power, $d \in \mathbb{N}$. Then with the lines of PG(d,q) which have no point with S in common as "points" and with the planes of PG(d,q) intersecting S in exactly one point as "lines" and with the natural incidence relation, one obtains a s.p.g. with parameters $s = q^2-1$, $t = (q-1)^{-1}(q^{d-1}-1)-1$, $\alpha = q$, $\mu = q(q+1)$.

DEBROEY [1] showed that a s.p.g. with parameters $s,t,\alpha \geq 2$, $\mu = \alpha(\alpha+1)$ and satisfying (D) is of this type. Combining this result with Theorems 7 and 8 we arrive at the following theorem.

<u>THEOREM 9</u>. *Let* (X,L) *be a s.p.g. with parameters* $s,t,\alpha,\mu = \alpha(\alpha+1)$ *which is not a p.g.. If* $\alpha = 2$ *and not* $s = t = 28$ *or if* $\alpha \geq 4$ *and* $s \geq f(\alpha)$, *then* (X,L) *is isomorphic to a s.p.g. consisting of the lines in* PG(d,q) *missing a given* $(d-2)$-*dimensional subspace of* PG(d,q) *and the planes inter-*

secting this subspace in one point. In particular $s = q^2-1$, $t = (q-1)^{-1}(q^{d-1}-1)-1$, $\alpha = q$, $\mu = q(q+1)$ *for some prime power* q *and* $d \in \mathbb{N}$ *and any s.p.g. with these parameters with* $q \neq 3$ *and* $d \geq 4$ *is of this type.*

REFERENCES

[1] DEBROEY, I., *Semi partial geometries satisfying the diagonal axiom*, J. Geometry, vol 13/2 (1979) 171-190.

[2] DEBROEY, I. & J.A. THAS, *On semi partial geometries*, J. Comb. Th. (A) 25 (1978) 242-250.

SAMENVATTING

Dit proefschrift bestaat uit vijf artikelen en een inleidend hoofdstuk. In elk van de vijf artikelen wordt een karakterisering gegeven van een object uit de eindige meetkunde. Het inleidende hoofdstuk bestaat uit een overzicht van vergelijkbare resultaten uit de literatuur (de stelling van Veblen & Young over projectieve ruimtes, de stelling van Ostrom & Wagner over projectieve vlakken met een 2-transitieve automorfismengroep, de stelling van Buekenhout & Shult over polaire ruimtes, etc.), en inleidingen in elk van de vijf artikelen.

Het eerste artikel gaat over bijna affiene vlakken. Evenals bij gewone affiene vlakken is het ook hier mogelijk het begrip translatie te definiëren. Aangetoond wordt dat het bestaan van translaties equivalent is met de geldigheid van een "Stelling van Desargues", en dat bijna affiene vlakken met een transitieve groep van translaties op een bepaalde algebraïsche manier kunnen worden beschreven.

In het tweede artikel wordt aangetoond dat er een verband bestaat tussen bijna affiene vlakken en Minkowski vlakken. Dit gegeven wordt gebruikt om een meetkundige karakterisering te geven van alle, tot nu toe bekende, Minkowski vlakken. In essentie komt deze karakterisering neer op de eis dat alle bijna affiene vlakken die met een Minkowski vlak zijn geassocieerd, moeten voldoen aan de Stelling van Desargues.

In het derde artikel wordt een tweede karakterisering gegeven van de op dit moment bekende Minkowski vlakken. Het blijken precies die Minkowski vlakken te zijn waarvan de automorfismengroep transitief is op paren niet-collineaire punten.

Het vierde artikel geeft een meetkundige karakterisering van de klassieke unital (dit is het $2\text{-}(q^3+1, q+1, 1)$ design van de absolute punten en niet absolute lijnen van een unitaire polariteit van $PG(2,q^2)$). De gekozen meetkundige condities zijn zodanig dat een op de punten 2-transitieve groep van automorfismen geconstrueerd kan worden, die vervolgens geïdentificeerd wordt als $PSU(3,q^2)$.

Het vijfde en laatste artikel geeft een karakterisering van twee klassen van semi-partiële meetkundes die geconstrueerd kunnen worden uit projectieve ruimtes. Bij deze karakterisering wordt alleen uitgegaan van de speciale vorm van de parameters. Het doel wordt hier bereikt door aan te tonen dat in deze semi-partiële meetkundes de duale versie van het axioma van Pasch geldt.

STELLINGEN

behorende bij het proefschrift

CHARACTERIZATION THEOREMS IN FINITE GEOMETRY

van

H.A. WILBRINK

1. De bewijstechniek van [1] is mogelijkerwijs ook te gebruiken om het niet bestaan van een sterk reguliere graaf op 99 punten van graad 14 aan te tonen.

   [1] H.A. Wilbrink & A.E. Brouwer, *A (57,14,1) strongly regular graph does not exist*, Proc. KNAW A 86 (1), 1983.

2. Er bestaat tenminste een (symmetrisch) 2-(49,16,5) design.

   A.E. Brouwer & H.A. Wilbrink, *A symmetric design with parameters* 2-(49,16,5), to appear.

3. De punten en lijnen die geheel buiten een niet ontaarde hyperkwadriek in PG(2n-1,2) liggen, vormen een semi-partiële meetkunde.

4. Veronderstel dat een rang 3 Zara graaf de volgende eigenschap heeft. Voor ieder tweetal disjuncte vlakken bestaan er partities in lijnen van die vlakken zò dat iedere lijn van elke partitie in een vlak is met een lijn uit de andere partitie. Dan vormen de vlakken en lijnen met de natuurlijke incidentie de punten en lijnen van een bijna-zeshoek.

   A. Blokhuis, *Few-distance sets*, Proefschrift T.H.E., 1983.

   E. Shult & A. Yanushka, *Near n-gons & line systems*, Geom. Dedicata 9 (1980), 1-72.

5. Vermoedelijk geldt de volgende stelling. Als n de orde is van een projectief vlak met een reguliere abelse automorfismengroep en p is een priemdeler van n, dan is $n = p$ of p^2 deelt n. Voor p=2 en p=3 is dit bewezen.

   H.A. Wilbrink, *A note on planar difference sets*, to appear.

6. De grafen op de inwendige en uitwendige punten van een niet ontaarde hyperkwadriek in PG(2n,5), met als kanten de paren onderling loodrechte punten, zijn sterk regulier.

7. Veel bewijzen in de combinatoriek kunnen met 50% worden ingekort door gebruik te maken van matrices. Vergelijk: E. Artin, *Geometric Algebra*, Interscience, New York, 1957, p 14.

8. Het adagium "een plaatje is geen bewijs" dient zeker niet te worden geïnterpreteerd als een aanbeveling tot het niet gebruiken van plaatjes in de wiskunde.

9. Het gebruik van computers binnen de wiskunde kan een remmende invloed hebben op de ontwikkeling van de wiskunde, en dient daarom met de nodige terughoudendheid te gebeuren.