Asymptotic behaviour of injection and suction for Hele-Shaw flow in $\mathbb{R}^3$ with surface tension near balls

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Abstract

We discuss long-time behaviour of the Hele-Shaw flow in $\mathbb{R}^3$ with surface tension and injection or suction in the origin, for domains that are small perturbations of balls. After rescaling, radially symmetric solutions become stationary. We study the stability of these solutions. In particular, we show that all liquid can be removed by suction if the suction point and the geometric centre coincide and the ratio of suction speed and surface tension is small enough. Any smaller amount of liquid can be removed if the suction point is near the geometric centre. We use the principle of linearised stability and the abstract theory of quasilinear parabolic equations.

1 Introduction

In the problem of Hele-Shaw flow with surface tension and injection or suction at a single point one seeks both a family of domains $t \mapsto \Omega(t) \subseteq \mathbb{R}^N$, $0 \in \Omega(t)$, parameterised by time t and two functions $\mathbf{v}(\cdot, t) : \Omega(t) \to \mathbb{R}^N$ and $p(\cdot, t) : \Omega(t) \to \mathbb{R}$ such that

$$\operatorname{div} \mathbf{v} = \mu\delta \qquad \text{in } \Omega(t), \tag{1}$$

$$\mathbf{v} = -\nabla p \quad \text{in } \Omega(t), \tag{2}$$

$$p = -\gamma\kappa \quad \text{on } \Gamma(t) := \partial\Omega(t). \tag{3}$$

Here, $\kappa(\cdot, t) : \Gamma(t) \to \mathbb{R}$ stands for the mean curvature of the moving boundary $t \mapsto \Gamma(t)$ of the domain (taken negative if $\Omega(t)$ is convex), μ stands for the injection speed if $\mu > 0$ or the suction speed if $\mu < 0$, γ is a positive constant and δ is the delta distribution. The normal velocity v_n of the moving boundary $\Gamma(t)$ is given by
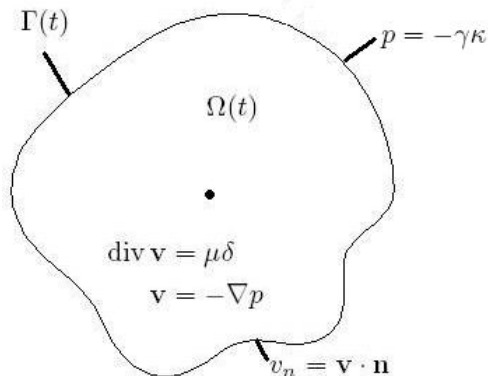
$$v_n = \mathbf{v} \cdot \mathbf{n}. \tag{4}$$

Figure 1: Hele-Shaw flow with surface tension where injection or suction takes place at a single point

From (1), (2) and (3) we get

$$\Delta p = -\mu\delta \quad \text{in } \Omega(t), \tag{5}$$
$$p = -\gamma\kappa \quad \text{on } \Gamma(t) = \partial\Omega(t). \tag{6}$$

Hence, we have a Dirichlet problem for any time t. On $\Gamma(t)$ we have

$$v_n = -\frac{\partial p}{\partial \mathbf{n}}.$$

Besides liquid flow in a Hele-Shaw cell, see Elliott and Ockendon [EO82], the model and variations of it describe the growth of tumors and porous media flow.

For a similar problem, Escher and Simonett [ES97a] proved existence of short-time solutions $t \mapsto \Omega(t)$. Global existence in time and stability for the problem without surface tension and injection for small perturbations of balls has been proved in [Von06]. For the suction problem, Tian [Tia95] proved that if the geometric centre and the suction point do not coincide, then the solution breaks down before all the fluid is sucked out or the domain becomes unbounded with zero area.

Let σ_N be the area of the unit sphere $\mathbb{S}^{N-1}$ in $\mathbb{R}^N$. We will assume that the initial domain $\Omega(0)$ has a volume equal to the volume of the unit ball $\mathbb{B}^N$ in $\mathbb{R}^N$, which is equal to $\frac{\sigma_N}{N}$. The volume $\mathfrak{V}(t)$ of the domain satisfies

$$\mathfrak{V}(t) = \frac{\sigma_N}{N} + \mu t, \tag{7}$$

because of

$$\frac{d\mathfrak{V}(t)}{dt} = \int_{\Gamma(t)} v_n d\sigma = -\int_{\Omega(t)} \Delta p \, dx = \mu.$$

Note that for negative μ, our problem only makes sense if

$$t \le T_\mu := -\frac{\sigma_N}{\mu N}.$$

By radial symmetry, if $\Omega(0) = \mathbb{B}^N$ then $\Omega(t) = s_{N,\mu}(t)\mathbb{B}^N$, where

$$s_{N,\mu}(t) = \sqrt[N]{\frac{\mu N t}{\sigma_N} + 1}.$$

In order to prove stability of these solutions, we rescale by a factor $s_{N,\mu}(t)$ such that $\mathbb{B}^N$ becomes a stationary solution. Small star-shaped perturbations of this stationary solution are described by means of a function $r(\cdot, t) : \mathbb{S}^{N-1} \to \mathbb{R}$. In Section 2 we derive and linearise a nonlinear non-local evolution equation for r, describing the motion of the domain $t \mapsto \Omega(t)$. For $N = 3$ the evolution operator can be treated as autonomous after introducing a new time variable. From Section 3 we restrict our attention to this case. We use the principle of linearised stability to show existence of a unique global solution that decays in little Hölder spaces. In the case of injection, this means that the solution exists for all $t > 0$. In the case of suction, we find that all liquid can be removed under the conditions that suction takes place in the geometric centre and the ratio $\frac{|\mu|}{\gamma}$ is small enough. This gives a partial answer to an open problem posed in 1993 [Hoh94]. In Section 4 we prove that any smaller amount can be removed if the geometric centre is close enough to the suction point. Here we use the fact that the evolution induces a semiflow.

2 The evolution equation for the domain

In this section we derive a nonlinear non-local evolution equation describing the motion of the domain, in a similar way as we did in [Von06] for $\gamma = 0$. Again we determine the linearisation of the evolution operator in terms of the Dirichlet-to-Neumann mapping. Like in [Von06] we describe a domain $\Omega(t)$ by a continuous function $R(\cdot, t) : \mathbb{S}^{N-1} \to (-1, \infty)$ satisfying

$$\Omega(t) = \Omega_{R(\cdot,t)} = \left\{ \mathbf{x} \in \mathbb{R}^N \setminus \{\mathbf{0}\} : |\mathbf{x}| < 1 + R\left(\frac{\mathbf{x}}{|\mathbf{x}|}\right) \right\} \cup \{\mathbf{0}\}.$$

Introduce $r(\cdot, t)$ such that

$$\Omega_{r(\cdot,t)} = s_{N,\mu}(t)^{-1}\Omega_{R(\cdot,t)},$$

which is equivalent to

$$1 + r(t) = \frac{1 + R(t)}{s_{N,\mu}(t)}.$$ (8)

We will often write $r(t)$ instead of $r(\cdot, t)$. Define $\Gamma_{r(\cdot,t)} = \partial\Omega_{r(\cdot,t)}$. Introduce $\tilde{\mathbf{z}}(r, \cdot) : \mathbb{S}^{N-1} \to \Gamma_{r(\cdot,t)}$ as

$$\tilde{\mathbf{z}}(r, \boldsymbol{\xi}) = (1 + r(\boldsymbol{\xi}, t))\, \boldsymbol{\xi}$$

and introduce $\mathbf{n}(r, \cdot)$ as the function that maps an element $\boldsymbol{\xi} \in \mathbb{S}^{N-1}$ to the exterior unit normal vector on $\Gamma_{r(\cdot,t)}$ at the point $\tilde{\mathbf{z}}(r, \boldsymbol{\xi})$. We will often write $\tilde{\mathbf{z}}(r)$ and $\mathbf{n}(r)$ instead of $\tilde{\mathbf{z}}(r, \cdot)$ and $\mathbf{n}(r, \cdot)$. For R we have the evolution equation

$$\frac{\partial R}{\partial t}(\boldsymbol{\xi}) = -\frac{\nabla p(\tilde{\mathbf{z}}(R, \boldsymbol{\xi})) \cdot \mathbf{n}(R, \boldsymbol{\xi})}{\mathbf{n}(R, \boldsymbol{\xi}) \cdot \boldsymbol{\xi}} \quad , \boldsymbol{\xi} \in \mathbb{S}^{N-1}.$$

For this, see [Pro97] or [Von06]. Let $\Psi : \mathbb{R}^N \to \mathbb{R}$ be defined by

$$\Psi(\mathbf{x}) = \begin{cases} -\dfrac{1}{2\pi} \ln |\mathbf{x}| & N = 2, \\ \dfrac{1}{(N-2)\sigma_N |\mathbf{x}|^{N-2}} & N \geq 3. \end{cases}$$ (9)

Define $U : \Omega_R \to \mathbb{R}$ by

$$U = p - \mu\Psi.$$

We get

$$\Delta U = 0 \qquad \text{in } \Omega(t),$$
$$U = -\gamma\kappa_R - \mu\Psi \quad \text{on } \Gamma(t).$$

Here $\kappa_R : \Gamma_R \to \mathbb{R}$ stands for the mean curvature of Γ_R. Analogously to [Von06] we can derive

$$\frac{\partial R}{\partial t}(\boldsymbol{\xi}) = -\frac{\nabla U(\tilde{\mathbf{z}}(R, \boldsymbol{\xi})) \cdot \mathbf{n}(r, \boldsymbol{\xi})}{\mathbf{n}(r, \boldsymbol{\xi}) \cdot \boldsymbol{\xi}} + \frac{\mu}{\sigma_N s_{N,\mu}^{N-1}(1 + r(\boldsymbol{\xi}))^{N-1}}.$$

Define $u : \Omega_r \to \mathbb{R}$ by

$$u(\mathbf{x}) = U(s_N \mathbf{x}).$$

Then $\Delta u = 0$ and on Γ_r

$$u(\mathbf{x}) = -\gamma\kappa_R(s_{N,\mu}\mathbf{x}) - \mu\Psi(s_{N,\mu}\mathbf{x}) = -\gamma s_{N,\mu}^{-1}\kappa_r(\mathbf{x}) - \mu\Psi(s_{N,\mu}\mathbf{x}) =$$
$$= \begin{cases} -\gamma s_{2,\mu}^{-1}\kappa_r(\mathbf{x}) - \mu\Psi(\mathbf{x}) + \dfrac{\mu}{2\pi} \ln s_{2,\mu} & N = 2, \\ -\gamma s_{N,\mu}^{-1}\kappa_r(\mathbf{x}) - \mu s_{N,\mu}^{2-N}\Psi(\mathbf{x}) & N \geq 3. \end{cases}$$

Let $\Lambda_r : \Omega_r \to \mathbb{R}$ be the harmonic function that satisfies

$$\Lambda_r = -\Psi \qquad \text{on } \Gamma_r$$

and define $G_r : \Omega_r \to \mathbb{R}$ as the harmonic function that satisfies

$$G_r = \kappa_r \qquad \text{on } \Gamma_r.$$

We get

$$u = \begin{cases} -\gamma s_{2,\mu}^{-1} G_r + \mu\Lambda_r + \dfrac{\mu}{2\pi} \ln s_{2,\mu} & N = 2, \\ -\gamma s_{N,\mu}^{-1} G_r + \mu s_{N,\mu}^{2-N}\Lambda_r & N \geq 3. \end{cases}$$

For the derivative we have

$$\nabla U(\tilde{\mathbf{z}}(R)) = s_{N,\mu}^{-1}\nabla u(\tilde{\mathbf{z}}(r)) = -\gamma s_{N,\mu}^{-2}\nabla G_r(\tilde{\mathbf{z}}(r)) + \mu s_{N,\mu}^{1-N}\nabla\Lambda_r(\tilde{\mathbf{z}}(r)).$$

By a calculation similar to [Von06] we get

$$\frac{\partial r}{\partial t}(\boldsymbol{\xi}) = \frac{\gamma}{s_{N,\mu}(t)^3}\frac{\nabla G_r(\tilde{\mathbf{z}}(r,\boldsymbol{\xi}))\cdot\mathbf{n}(r,\boldsymbol{\xi})}{\mathbf{n}(r,\boldsymbol{\xi})\cdot\boldsymbol{\xi}} + \frac{\mu}{s_{N,\mu}(t)^N}\left(-\frac{\nabla\Lambda_r(\tilde{\mathbf{z}}(r,\boldsymbol{\xi}))\cdot\mathbf{n}(r,\boldsymbol{\xi})}{\mathbf{n}(r,\boldsymbol{\xi})\cdot\boldsymbol{\xi}} + \frac{1}{\sigma_N(1+r(\boldsymbol{\xi}))^{N-1}} - \frac{1+r(\boldsymbol{\xi})}{\sigma_N}\right).$$
(10)

For any r define $\kappa(r,\cdot)$ as the function that maps an element $\boldsymbol{\xi}$ of the unit sphere to the mean curvature of Γ_r at $\tilde{\mathbf{z}}(r,\boldsymbol{\xi})$. We will often use the notation $\kappa(r)$ instead of $\kappa(r,\cdot)$.

From now on we will assume that r is in the little Hölder space $h^{4,\alpha}(\mathbb{S}^{N-1})$. The little Hölder spaces $h^{k,\alpha}(K)$ on a compact manifold K are defined as the closure of $\mathcal{C}^\infty(K)$ with respect to the norm of $\mathcal{C}^{k,\alpha}(K)$. Let $(\mathcal{T}_i,\Xi_i)_{i=1}^M$ be an atlas for the unit sphere, and define $\tilde{U}_i := \Xi_i(\mathcal{T}_i)$. For any $r \in h^{4,\alpha}(\mathbb{S}^{N-1})$ let Δ_r denote the Laplace-Beltrami operator on the manifold Γ_r. If a part of Γ_r is parameterised by $\tilde{\mathbf{z}}(r)\circ\Xi_l^{-1}$ then we have

$$\Delta_r = \sum_{i,j}\frac{1}{\sqrt{g_r}}\frac{\partial}{\partial\omega_i}\left(\sqrt{g_r}g_r^{ij}\frac{\partial}{\partial\omega_j}\right) \qquad, \omega_i \in \tilde{U}_l.$$
(11)

Here g_r^{ij} are the elements of the inverse $\mathcal{G}_r^{-1}$ of the matrix $\mathcal{G}_r$ given by

$$\mathcal{G}_r = \left(\frac{\partial\left(\tilde{\mathbf{z}}(r)\circ\Xi_l^{-1}\right)}{\partial\boldsymbol{\omega}}\right)^T\frac{\partial\left(\tilde{\mathbf{z}}(r)\circ\Xi_l^{-1}\right)}{\partial\boldsymbol{\omega}}$$

and

$$g_r = \det\mathcal{G}_r$$

Lemma 2.1. *The Laplace-Beltrami operator Δ_r is symmetric on $\mathbb{L}_2(\Gamma_r)$.*

Proof. This is a straightforward calculation. $\qquad\square$

By [Pro97] Chapter 3 Lemma 8 we have

$$\kappa(r) = (\Delta_r\tilde{\mathbf{z}}(r))\cdot\mathbf{n}(r),$$
(12)

where Δ_r acts on every component of $\tilde{\mathbf{z}}(r)$ separately.

Lemma 2.2. *There exists a neighborhood $\mathcal{U}$ of zero in $\mathcal{C}^{4,\alpha}(\mathbb{S}^{N-1})$ such that κ is analytic from $\mathcal{U}$ to $\mathcal{C}^{2,\alpha}(\mathbb{S}^{N-1})$.*

Proof. We use the same procedure as in [Pro97] was applied for Sobolev spaces. Choose a smooth partition of unity $(\chi_i)_{i=1}^M$ subordinate to the covering $(\mathcal{T}_i)_{i=1}^M$. Define $\kappa^{[k]}(r) = \kappa^{[k]}(r,\cdot) = \kappa(r)\circ\Xi_k^{-1}$ and $\mathbf{n}^{[k]}(r) = \mathbf{n}^{[k]}(r,\cdot) = \mathbf{n}(r)\circ\Xi_k^{-1}$ then we have

$$\kappa(r) = \sum_{i=1}^M\chi_i\cdot\left(\kappa^{[i]}(r)\circ\Xi_i\right).$$

We take $\chi_i(\boldsymbol{\xi})\kappa^{[i]}(r,\Xi_i(\boldsymbol{\xi})) = 0$ if $\boldsymbol{\xi}$ is not in $\mathcal{T}_i$. It is sufficient to prove that $\kappa^{[k]}:\mathcal{C}^{4,\alpha}(\mathbb{S}^{N-1})\to\mathcal{C}^{2,\alpha}(\tilde{U}_k)$ is analytic for all k. From equation (12) we get

$$\kappa^{[k]}(r) = \frac{1}{\sqrt{g_r}}\sum_{i,j}\frac{\partial}{\partial\omega_i}\left(\sqrt{g_r}g_r^{ij}\frac{\partial(\tilde{\mathbf{z}}(r)\circ\Xi_k^{-1})}{\partial\omega_j}\right)\cdot\mathbf{n}^{[k]}(r)$$

The mapping $f \mapsto f\circ\Xi_k^{-1}$ is linear and bounded, hence analytic. Now we will use the fact that pointwise multiplication and composition in a Banach algebra preserve analyticity. Because $\tilde{\mathbf{z}}$ is analytic from $\mathcal{C}^{j,\alpha}(\mathbb{S}^{N-1})$ to $\left(\mathcal{C}^{j,\alpha}(\mathbb{S}^{N-1})\right)^N$ and $\mathcal{C}^{j,\alpha}(\mathbb{S}^{N-1})$ is a Banach algebra for all $j\in\mathbb{N}_0$, $r\mapsto\mathcal{G}_r$ is analytic from $\mathcal{C}^{j,\alpha}(\mathbb{S}^{N-1})$ to $\left(\mathcal{C}^{j-1,\alpha}(\mathbb{S}^{N-1})\right)^{N\times N}$ for all $j\in\mathbb{N}_0$. The mappings $r\mapsto\sqrt{g_r}$ and $r\mapsto\mathcal{G}_r^{-1}$ are analytic around zero, because of Cramer's rule and [Pro97] Chapter 3 Lemma 7. The mapping $\mathbf{n}^{[k]}:\mathcal{C}^{2,\alpha}(\mathbb{S}^{N-1})\to\left(\mathcal{C}^{1,\alpha}(\tilde{U}_k)\right)^N$ is analytic around zero, see [Von06] Lemma 2.5, and $\mathcal{C}^{4,\alpha}(\mathbb{S}^{N-1})\hookrightarrow\mathcal{C}^{2,\alpha}(\mathbb{S}^{N-1})$, therefore $\mathbf{n}^{[k]}:\mathcal{C}^{4,\alpha}(\mathbb{S}^{N-1})\to\left(\mathcal{C}^{1,\alpha}(\mathbb{S}^{N-1})\right)^N$ is analytic around zero and we get the desired result. $\qquad\square$

By [Lun95] Theorem 0.3.2 there exists an extension operator $E \in \mathcal{L}(\mathcal{C}^{k,\alpha}(\mathbb{S}^{N-1}), \mathcal{C}^{k,\alpha}(\overline{\mathbb{B}^N}))$ for $k \in \{0, 1, 2\}$ and $\alpha \in [0; 1)$, such that

$$E(r)|_{\mathbb{S}^{N-1}} = r. \tag{13}$$

Define $\mathbf{z} : \mathcal{C}^{2,\alpha}(\mathbb{S}^{N-1}) \to \left(\mathcal{C}^{2,\alpha}(\overline{\mathbb{B}^N})\right)^N$ by

$$\mathbf{z}(r, \mathbf{x}) = (1 + E(r, \mathbf{x})) \, \mathbf{x},$$

where $\mathbf{z}(r, \cdot) = \mathbf{z}(r)$ and $E(r, \cdot) = E(r)$. In [Von06] we saw that there exists a neighborhood $\mathcal{U}$ of 0 in $\mathcal{C}^{2,\alpha}(\mathbb{S}^{N-1})$ and two mappings $\mathcal{A} : \mathcal{U} \to \mathcal{L}(\mathcal{C}^{2,\alpha}(\overline{\mathbb{B}^N}), \mathcal{C}^{0,\alpha}(\overline{\mathbb{B}^N}))$ and $\mathcal{Q} : \mathcal{U} \to \mathcal{L}\left(\mathcal{C}^{2,\alpha}(\overline{\mathbb{B}^N}), \left(\mathcal{C}^{1,\alpha}(\overline{\mathbb{B}^N})\right)^N\right)$ such that

$$\mathcal{A}(r)u = \left(\Delta \left(u \circ \mathbf{z}(r)^{-1}\right)\right) \circ \mathbf{z}(r)$$

and

$$\mathcal{Q}(r)u = \left(\nabla \left(u \circ \mathbf{z}(r)^{-1}\right)\right) \circ \mathbf{z}(r).$$

Let $\mathcal{P} : \mathcal{U} \to \mathcal{L}(\mathcal{C}^{2,\alpha}(\overline{\mathbb{B}^N}), \mathcal{C}^{0,\alpha}(\overline{\mathbb{B}^N}) \times \mathcal{C}^{2,\alpha}(\mathbb{S}^{N-1}))$ be defined by

$$\mathcal{P}(r)u = \begin{pmatrix} \mathcal{A}(r)u \\ \text{Tr}u \end{pmatrix}. \tag{14}$$

Let $\phi : \mathcal{U} \to \mathcal{C}^{2,\alpha}(\mathbb{S}^{N-1})$ be

$$\phi(r, \mathbf{x}) = \Phi((1 + r(\mathbf{x}))\mathbf{x}), \tag{15}$$

where $\mathcal{U}$ is a subset of the unit ball in $\mathcal{C}^{2,\alpha}(\mathbb{S}^{N-1})$, again $\phi(r, \cdot) = \phi(r)$ and $\Phi : \mathbb{R}^N \to \mathbb{R}$ is the fundamental solution

$$\Phi(\mathbf{x}) := \begin{cases} -\dfrac{1}{2\pi} \ln |\mathbf{x}| & N = 2, \\[2mm] \dfrac{1}{(N-2)\sigma_N |\mathbf{x}|^{N-2}} - \dfrac{1}{(N-2)\sigma_N} & N \geq 3. \end{cases} \tag{16}$$

Note that Ψ from (9) differs from Φ only by a constant, so their derivatives are equal. Now we can write

$$\frac{\partial r}{\partial t} = \frac{\gamma}{s_{N,\mu}(t)^3} \mathcal{F}_1(r) + \frac{\mu}{s_{N,\mu}(t)^N} \mathcal{F}_2(r)$$

with

$$\mathcal{F}_1(r)(\boldsymbol{\xi}) = \frac{\text{Tr}\left(\mathcal{Q}(r)\left[\mathcal{P}(r)^{-1}\begin{bmatrix} 0 \\ \kappa(r) \end{bmatrix}\right]\right)(\boldsymbol{\xi}) \cdot \mathbf{n}(r, \boldsymbol{\xi})}{\mathbf{n}(r, \boldsymbol{\xi}) \cdot \boldsymbol{\xi}}$$

and

$$\mathcal{F}_2(r)(\boldsymbol{\xi}) = \frac{\text{Tr}\left(\mathcal{Q}(r)\left[\mathcal{P}(r)^{-1}\begin{bmatrix} 0 \\ \phi(r) \end{bmatrix}\right]\right)(\boldsymbol{\xi}) \cdot \mathbf{n}(r, \boldsymbol{\xi})}{\mathbf{n}(r, \boldsymbol{\xi}) \cdot \boldsymbol{\xi}} + \frac{1}{\sigma_N(1 + r(\boldsymbol{\xi}))^{N-1}} - \frac{1 + r(\boldsymbol{\xi})}{\sigma_N}$$

Lemma 2.3. *The operators* $\mathcal{F}_1 : h^{4,\alpha}(\mathbb{S}^{N-1}) \to h^{1,\alpha}(\mathbb{S}^{N-1})$ *and* $\mathcal{F}_2 : h^{4,\alpha}(\mathbb{S}^{N-1}) \to h^{1,\alpha}(\mathbb{S}^{N-1})$ *are analytic in a neighborhood* $\mathcal{U}$ *of zero in* $h^{4,\alpha}(\mathbb{S}^{N-1})$.

Proof. Analyticity of $\mathcal{F}_2$ follows from [Von06] Lemma 2.10 in combination with the fact that

$$h^{4,\alpha}(\mathbb{S}^{N-1}) \hookrightarrow h^{2,\alpha}(\mathbb{S}^{N-1}).$$

Using Lemma 2.2, analyticity of $\mathcal{F}_1$ can be proven in a similar way as was done for $\mathcal{F}_2$ in [Von06]. $\qquad \square$

In [Von06] we saw that

$$\mathcal{F}_2'(0)[h] = -\frac{1}{\sigma_N} \mathcal{N}h - \frac{N}{\sigma_N} h, \tag{17}$$

where $\mathcal{N}$ is the Dirichlet-to-Neumann operator on $\mathbb{S}^{N-1}$. We want to derive an expression for $\mathcal{F}_1'(0)$ as well.

Lemma 2.4. *The linearisation around zero of the curvature operator κ is given by*

$$\kappa'(0)[h] = \Delta_0 h + (N-1)h$$

where Δ_0 denotes the Laplace-Beltrami operator on the unit sphere.

Proof. See Chapter 6 of [Pro97]. $\qquad\square$

From [Mül66] we have the following expression for the Laplace-Beltrami operator on the unit sphere:

$$\Delta_0 r = -\mathcal{N}^2 r - (N-2)\mathcal{N}r. \tag{18}$$

Lemma 2.5. *We have*

$$\mathcal{F}_1'(0)[h] = \mathcal{N}(\kappa'(0)[h]) = \mathcal{N}\left(\Delta_0 r + (N-1)r\right) = \mathcal{N}\left(-\mathcal{N}^2 r - (N-2)\mathcal{N}r + (N-1)r\right). \tag{19}$$

Proof. Introduce

$$K(r) = \mathcal{P}(r)^{-1}\begin{pmatrix} 0 \\ \kappa(r) \end{pmatrix}.$$

Because $K(0)$ is constant, $\mathcal{Q}(0)K(0)$, $(\mathcal{Q}'(0)[h])K(0)$ and $(\mathcal{A}'(0)[h])K(0)$ vanish. From $\mathcal{A}(r)K(r) = 0$ we get

$$0 = (\mathcal{A}'(0)[h])K(0) + \mathcal{A}(0)K'(0)[h] = \Delta K'(0)[h].$$

Because $\mathrm{Tr}K'(0)[h] = \kappa'(0)[h]$ we have

$$K'(0)[h] = \mathcal{P}(0)^{-1}\begin{pmatrix} 0 \\ \kappa'(0)[h] \end{pmatrix}.$$

The linearisation of $\mathcal{F}_1$ follows:

$$\mathcal{F}_1'(0)[h] = \frac{\mathrm{Tr}\mathcal{Q}(0)K'(0)[h]\cdot\mathbf{n}(0)}{\mathbf{n}(0)\cdot\mathcal{I}} = \mathcal{N}(\kappa'(0)[h]) = \mathcal{N}(\Delta_0 h + (N-1)h) = \mathcal{N}(-\mathcal{N}^2 h - (N-2)\mathcal{N}h + (N-1)h),$$

where $\mathcal{I}$ stands for the identity. $\qquad\square$

From now on we consider the case $N = 3$. We get

$$\frac{\partial r}{\partial t} = \frac{1}{s_{3,\mu}(t)^3}\left(\gamma\mathcal{F}_1(r) + \mu\mathcal{F}_2(r)\right). \tag{20}$$

Introduce a new time variable $\tau = \tau(t)$ such that $\tau(0) = 0$ and

$$\frac{d\tau}{dt} = \frac{1}{s_{3,\mu}(t)^3}. \tag{21}$$

This means that $\tau(t) = \frac{4\pi}{3\mu}\ln(\frac{3\mu t}{4\pi} + 1)$. We get the autonomous equation

$$\frac{\partial \bar{r}}{\partial \tau} = \mathcal{F}_{\gamma,\mu}(\bar{r}), \tag{22}$$

where $\bar{r}(\tau) = r(t)$ and

$$\mathcal{F}_{\gamma,\mu}(\bar{r}) := \gamma\mathcal{F}_1(\bar{r}) + \mu\mathcal{F}_2(\bar{r}).$$

Note that for the suction problem, $t = T_\mu$ is equivalent with $\tau = \infty$. From now on we write r instead of $\bar{r}$. From (17) and (19) we have

$$\mathcal{F}_{\gamma,\mu}'(0)[h] = \gamma\mathcal{N}\left(-\mathcal{N}^2 h - \mathcal{N}h + 2h\right) - \frac{\mu}{4\pi}(\mathcal{N}h + 3h). \tag{23}$$

3 The spectrum of the linearisation and stability for $N = 3$

In this section we apply the principle of linearised stability, see [Lun95], to the evolution equation (22) for the three-dimensional problem in order to derive a stability result for the injection case. For the suction case we will find stability if the suction point is in the geometric centre of the initial domain and the quotient of suction speed and γ is small enough. We need to study the spectral properties of the operator $\mathcal{F}'_{\gamma,\mu}(0) : h^{4,\alpha}(\mathbb{S}^2) \to h^{1,\alpha}(\mathbb{S}^2)$ given in (23).

Let $\mathfrak{S}^3_k$ be the vector space of spherical harmonics that are restrictions to the unit sphere of harmonic homogeneous polynomials of three variables of degree k. From [Mül66] it is known that the point spectrum of $\mathcal{N}$ is given by the natural numbers including zero and the eigenspace corresponding to $k \in \mathbb{N}_0$ is $\mathfrak{S}^3_k$. Together with (23) this implies that the eigenvalues of $\mathcal{F}'_{\gamma,\mu}(0)$ are given by

$$g_k = \gamma k(-k^2 - k + 2) - \frac{\mu}{4\pi}(k + 3)$$

and the corresponding eigenspace is given by $\mathfrak{S}^3_k$. In the case $\mu > 0$ all g_k are negative. Take now $\mu < 0$. In this case, g_0 and g_1 are positive. If

$$\frac{|\mu|}{\gamma} = -\frac{\mu}{\gamma} < \frac{32\pi}{5}, \tag{24}$$

then all other eigenvalues are negative. This follows from the fact that for $k \geq 2$ the sequence (g_k) decreases and $g_2 < 0$. We summarize these results in the following lemma.

Lemma 3.1. *The point spectrum of $\mathcal{F}'_{\gamma,\mu}(0) : h^{4,\alpha}(\mathbb{S}^2) \to h^{1,\alpha}(\mathbb{S}^2)$ is*

$$\pi(\mathcal{F}'_{\gamma,\mu}(0)) = \{g_0, g_1, g_2, \dots\}.$$

The eigenspace for eigenvalue g_k is $\mathfrak{S}^3_k$. If $\mu > 0$ then all eigenvalues of $\mathcal{F}'_{\gamma,\mu}(0) : h^{4,\alpha}(\mathbb{S}^2) \to h^{1,\alpha}(\mathbb{S}^2)$ are negative. If $\mu < 0$ then the eigenvalues g_0 and g_1 are positive. All other eigenvalues are negative if (24) holds.

For two Banach spaces X and Y such that $X \hookrightarrow Y$ we define $\mathcal{H}(X, Y)$ as the collection of operators $A \in \mathcal{L}(X, Y)$ for which $-A$ is the infinitesimal generator of a strongly continuous analytic semigroup. We shall prove that $-\mathcal{F}'_{\gamma,\mu}(0) \in \mathcal{H}\left(h^{4,\alpha}(\mathbb{S}^2), h^{1,\alpha}(\mathbb{S}^2)\right)$. For this we need the two following lemmas.

Lemma 3.2. *Let X and Y be Banach spaces, such that $X \hookrightarrow Y$ and X dense in Y. Suppose that $F : X \to Y$ and $K : X \to Y$ are bounded linear operators such that $F \in \mathcal{H}(X, Y)$ and suppose that K is compact. Then $F + K \in \mathcal{H}(X, Y)$.*

Proof. See [CHA$^+$87] Theorem 5.6. $\qquad\square$

Lemma 3.3. *The mapping $\mathcal{N} : h^{k+1}(\mathbb{S}^2) \to h^k(\mathbb{S}^2)$ is continuous for all $k \in \mathbb{N}$.*

Proof. Define the Banach space X by

$$X = \{\psi \in \mathcal{C}^{k+1,\alpha}(\overline{\mathbb{B}^N}) : \Delta\psi = 0\}.$$

Let it inherit the norm of $\mathcal{C}^{k+1,\alpha}(\overline{\mathbb{B}^N})$. Because of the maximum principle, the mapping $\mathrm{Tr} : X \to \mathcal{C}^{k+1,\alpha}(\mathbb{S}^{N-1})$ is injective. We shall prove that it is surjective as well. Let g be an element of $\mathcal{C}^{k+1,\alpha}(\mathbb{S}^{N-1})$. Let $f : [0, 1] \to [0, 1]$ be a smooth function such that $f(x) = 0$ for all $x \in [0, \frac{1}{3}]$ and $f(x) = 1$ for all $x \in [\frac{2}{3}, 1]$. Define $\tilde{g}(\mathbf{x}) = f(|\mathbf{x}|)g(\frac{\mathbf{x}}{|\mathbf{x}|})$. Then $\tilde{g} \in \mathcal{C}^{k+1,\alpha}(\overline{\mathbb{B}^N})$. By [GT77] Corollary 4.14 we have a $u \in \mathcal{C}^{2,\alpha}(\overline{\mathbb{B}^N})$, such that u is harmonic and equal to $\tilde{g}$ on the boundary. By [GT77] Theorem 6.19 we have $u \in \mathcal{C}^{k+1,\alpha}(\overline{\mathbb{B}^N})$. Therefore the bounded operator $\mathrm{Tr} : X \to \mathcal{C}^{k+1,\alpha}(\mathbb{S}^{N-1})$ is surjective and by the Open Mapping Theorem it has a bounded inverse. From

$$\mathcal{N} = \frac{\partial}{\partial \mathbf{n}} \circ \mathrm{Tr}^{-1}$$

and the boundedness of $\frac{\partial}{\partial \mathbf{n}}$ we get the desired result. $\qquad\square$

Lemma 3.4. *We have $-\mathcal{F}'_{\gamma,\mu}(0) \in \mathcal{H}\left(h^{4,\alpha}(\mathbb{S}^2), h^{1,\alpha}(\mathbb{S}^2)\right)$.*

Proof. The structure of this proof is as follows. We relate $\mathcal{N}^3$ to a Fourier multiplier operator $\hat{\mathcal{N}}_0^3$ on $\mathbb{R}^2$. The operator $-\hat{\mathcal{N}}_0^3$ generates an analytic semigroup. Using techniques from [ES97b], [ES97a] and [EP03] together with additional perturbation arguments we see that $\mathcal{N}^3 \in \mathcal{H}\left(h^{4,\alpha}(\mathbb{S}^2), h^{1,\alpha}(\mathbb{S}^2)\right)$. Since $-\mathcal{F}'_{\gamma,\mu}(0)$ is in highest order equal to $\mathcal{N}^3$ the lemma follows from Lemma 3.2.

1. Let $(\mathcal{T}_i, \Xi_i)_{i=1}^M$ be an atlas of $\mathbb{S}^2$, with $\Xi_i(\mathcal{T}_i) = \tilde{U}_i$ and $\mathbf{0} \in \tilde{U}_i$. Define

$$U_i = \tilde{U}_i \times (0, \varrho),$$

for some $\varrho < \frac{1}{2}$,

$$\mathcal{W}_i = \{\mathbf{x} \in \overline{\mathbb{B}^3} : 1 - \varrho < |\mathbf{x}| < 1, \frac{\mathbf{x}}{|\mathbf{x}|} \in \mathcal{T}_i\}$$

and $\mathcal{X}_i : \mathcal{W}_i \to U_i$ by

$$\mathcal{X}_i(\mathbf{x}) = \left(\Xi_i\left(\frac{\mathbf{x}}{|\mathbf{x}|}\right), 1 - |\mathbf{x}|\right).$$

Let $\hat{\mathcal{A}}_i : h^{2,\alpha}(U_i) \to h^{0,\alpha}(U_i)$ and $\hat{\mathcal{Q}}_i : h^{2,\alpha}(U_i) \to h^{1,\alpha}(\tilde{U}_i)$ be

$$\hat{\mathcal{A}}_i p = \Delta\left(p \circ \mathcal{X}_i\right) \circ \mathcal{X}_i^{-1}$$

$$\hat{\mathcal{Q}}_i p = \frac{\partial}{\partial \mathbf{n}}\left(p \circ \mathcal{X}_i\right) \circ \mathcal{X}_i^{-1} = -\frac{\partial}{\partial x_3},$$

where $\mathbf{n}$ is the normal on $\mathbb{S}^2$ and

$$h^{k,\alpha}(U_i) := \overline{\mathcal{C}_c^\infty(U_i)}^{\mathcal{C}^{k,\alpha}(U_i)}.$$

From now on we restrict our attention to one chart and omit the index i in $\tilde{U}_i$, U_i, $\hat{\mathcal{A}}_i$ and $\hat{\mathcal{Q}}_i$. There exist functions $\hat{a}_{jk}$, $\hat{a}_j \in \mathcal{C}^\infty(U_i)$ such that

$$\hat{\mathcal{A}} = \sum_{j,k=1}^3 \hat{a}_{jk} \frac{\partial^2}{\partial x_j \partial x_k} - \sum_{j=1}^3 \hat{a}_j \frac{\partial}{\partial x_j}.$$

Define

$$\hat{\mathcal{A}}_0 = -1 + \sum_{j,k=1}^3 \hat{a}_{jk}(\mathbf{0}) \frac{\partial^2}{\partial x_j \partial x_k}$$

Note that $\hat{a}_{33}(\mathbf{0}) = 1$ and $\hat{a}_{13}(\mathbf{0}) = \hat{a}_{23}(\mathbf{0}) = \hat{a}_{31}(\mathbf{0}) = \hat{a}_{32}(\mathbf{0}) = 0$. Let Tr denote the trace operator for functions on the halfspace $\mathbb{R}_+^3 = \{\mathbf{x} \in \mathbb{R}^3 : x_3 \geq 0)\}$. Define $\hat{\mathcal{R}}_0 : h^{1,\alpha}(\mathbb{R}^2 \times \{0\}) \to h^{1,\alpha}(\mathbb{R}_+^3)$ as the solution operator $\hat{\mathcal{R}}_0 g = u$, of the problem

$$\begin{cases} -\hat{\mathcal{A}}_0 u &= 0 \qquad \text{in } \mathbb{R}_+^3 \\ \text{Tr} u &= g \qquad \text{in } \mathbb{R}^2 \times \{0\}. \end{cases}$$

Define the operator $\hat{\mathcal{N}}_0$ by

$$\hat{\mathcal{N}}_0 = \hat{\mathcal{Q}}\hat{\mathcal{R}}_0.$$

From [ES97a] we get

$$\mathbb{F}\hat{\mathcal{N}}_0\mathbb{F}^{-1} = \mathcal{M}_{f(\cdot,1)}, \tag{25}$$

where $\mathbb{F}$ denotes Fourier transform, $f : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ is defined by

$$f(\mathbf{x}, y) = \sqrt{y^2 + \sum_{j,k=1}^2 \hat{a}_{jk}(\mathbf{0}) x_j x_k}.$$

and $\mathcal{M}_{f(\cdot,1)}$ stands for multiplication with the function $f(\cdot, 1)$. Because $(f(\mathbf{x}, y))^3$ is positively homogeneous and its derivatives are bounded on $|\mathbf{x}|^2 + y^2 = 1$, $\hat{\mathcal{N}}_0^3 \in \mathcal{H}\left(h^{4,\alpha}(\mathbb{R}^2 \times \{0\}), h^{1,\alpha}(\mathbb{R}^2 \times \{0\})\right)$, see [ES95] Theorem A.2. In [ES97a] Corollary 5.2, the same strategy is used for a different operator.

2. The next step is relating $\hat{\mathcal{N}}_0^3$ to $\mathcal{N}^3$ if the chart domains are small. The following statement holds true. For any $\varepsilon > 0$, $\zeta \in (0,\alpha)$ there is a $\varrho > 0$, an atlas $(\mathcal{T}_i, \Xi_i)_{i=1}^M$, a partition of unity $(\psi_i)_{i=1}^M$ subordinate to $(\mathcal{W}_i)_{i=1}^M$, and a $C > 0$ such that for $l \in \{1,2,3\}$ and $\hat{\mathcal{N}}_0$ constructed from the atlas as described above we have for all $p \in h^{l+1,\alpha}(\mathbb{S}^2)$

$$\|\mathcal{X}_*(\psi\mathcal{N}p) - \hat{\mathcal{N}}_0\mathcal{X}_*(\psi p)\|_{\mathcal{C}^{l,\alpha}(\mathbb{R}^2)} \leq \varepsilon\|\mathcal{X}_*(\psi p)\|_{\mathcal{C}^{l+1,\alpha}(\mathbb{R}^2)} + C\|p\|_{\mathcal{C}^{l+1,\zeta}(\mathbb{S}^2)}. \tag{26}$$

To see this, we argue as in the proof of Theorem B.4 in [EP03] and choose ϱ sufficiently small, depending on ε. Here and in the sequel we identify $\mathcal{C}^{l,\alpha}(\mathbb{R}^2)$ and $\mathcal{C}^{l,\alpha}(\mathbb{R}^2 \times \{0\})$. Functions $\mathcal{X}_*(\psi p)$ can be extended to the entire $\mathbb{R}^2$ because of the smoothness of the partition of unity. Recall that

$$\mathcal{X}_* f := f \circ \mathcal{X}.$$

We want to show that for fixed $\zeta \in (0,\alpha)$ and $\varepsilon > 0$, we can derive from (26) that there is a $C > 0$ such that for all $p \in h^{4,\alpha}(\mathbb{S}^2)$

$$\|\mathcal{X}_*(\psi\mathcal{N}^3 p) - \hat{\mathcal{N}}_0^3\mathcal{X}_*(\psi p)\|_{\mathcal{C}^{1,\alpha}(\mathbb{R}^2)} \leq \varepsilon\|\mathcal{X}_*(\psi p)\|_{\mathcal{C}^{4,\alpha}(\mathbb{R}^2)} + C\|p\|_{\mathcal{C}^{4,\zeta}(\mathbb{S}^2)}. \tag{27}$$

In the sequel, we will often use the fact that for each $k \in \mathbb{N}$

$$\hat{\mathcal{N}}_0 \in \mathcal{L}(h^{k+1,\alpha}(\mathbb{S}^2), h^{k,\alpha}(\mathbb{S}^2)).$$

First we show that there exists a constant C' independent of p such that

$$\|\mathcal{X}_*(\psi\mathcal{N}p)\|_{\mathcal{C}^{3,\alpha}(\mathbb{R}^2)} \leq C'\left(\|\mathcal{X}_*(\psi p)\|_{\mathcal{C}^{4,\alpha}(\mathbb{R}^2)} + \|p\|_{\mathcal{C}^{4,\zeta}(\mathbb{S}^2)}\right) \tag{28}$$

and

$$\|\mathcal{X}_*(\psi\mathcal{N}^2 p)\|_{\mathcal{C}^{2,\alpha}(\mathbb{R}^2)} \leq C'\left(\|\mathcal{X}_*(\psi p)\|_{\mathcal{C}^{4,\alpha}(\mathbb{R}^2)} + \|p\|_{\mathcal{C}^{4,\zeta}(\mathbb{S}^2)}\right). \tag{29}$$

Let us start with the first estimate. Apply (26) with $\varepsilon = 1$ and $l = 3$. We get

$$\begin{aligned}
\|\mathcal{X}_*(\psi\mathcal{N}p)\|_{\mathcal{C}^{3,\alpha}(\mathbb{R}^2)} &\leq \|\mathcal{X}_*(\psi\mathcal{N}p) - \hat{\mathcal{N}}_0\mathcal{X}_*(\psi p)\|_{\mathcal{C}^{3,\alpha}(\mathbb{R}^2)} + \|\hat{\mathcal{N}}_0\mathcal{X}_*(\psi p)\|_{\mathcal{C}^{3,\alpha}(\mathbb{R}^2)} \\
&\leq \|\mathcal{X}_*(\psi p)\|_{\mathcal{C}^{4,\alpha}(\mathbb{R}^2)} + C\|p\|_{\mathcal{C}^{4,\zeta}(\mathbb{S}^2)} + C\|\mathcal{X}_*(\psi p)\|_{\mathcal{C}^{4,\alpha}(\mathbb{R}^2)}.
\end{aligned}$$

Estimate (28) follows. Replace p by $\mathcal{N}p$ in (26) and take $\varepsilon = 1$ and $l = 2$. Using (28) we get

$$\begin{aligned}
\|\mathcal{X}_*(\psi\mathcal{N}^2 p)\|_{\mathcal{C}^{2,\alpha}(\mathbb{R}^2)} &\leq \|\mathcal{X}_*(\psi\mathcal{N}^2 p) - \hat{\mathcal{N}}_0\mathcal{X}_*(\psi\mathcal{N}p)\|_{\mathcal{C}^{2,\alpha}(\mathbb{R}^2)} + \|\hat{\mathcal{N}}_0\mathcal{X}_*(\psi\mathcal{N}p)\|_{\mathcal{C}^{2,\alpha}(\mathbb{R}^2)} \\
&\leq \|\mathcal{X}_*(\psi\mathcal{N}p)\|_{\mathcal{C}^{3,\alpha}(\mathbb{R}^2)} + C\|\mathcal{N}p\|_{\mathcal{C}^{3,\zeta}(\mathbb{S}^2)} + C\|\mathcal{X}_*(\psi\mathcal{N}p)\|_{\mathcal{C}^{3,\alpha}(\mathbb{R}^2)} \\
&\leq C\|\mathcal{X}_*(\psi p)\|_{\mathcal{C}^{4,\alpha}(\mathbb{R}^2)} + C\|p\|_{\mathcal{C}^{4,\zeta}(\mathbb{S}^2)}.
\end{aligned}$$

Now we can prove estimate (27). Let $\varepsilon > 0$. Let $\eta > 0$ be a small number to be chosen later. We have

$$\begin{aligned}
\|\mathcal{X}_*(\psi\mathcal{N}^3 p) - \hat{\mathcal{N}}_0^3\mathcal{X}_*(\psi p)\|_{\mathcal{C}^{1,\alpha}(\mathbb{R}^2)} \leq{}& \|\mathcal{X}_*(\psi\mathcal{N}^3 p) - \hat{\mathcal{N}}_0\mathcal{X}_*(\psi\mathcal{N}^2 p)\|_{\mathcal{C}^{1,\alpha}(\mathbb{R}^2)} + \\
&+ \|\hat{\mathcal{N}}_0\mathcal{X}_*(\psi\mathcal{N}^2 p) - \hat{\mathcal{N}}_0^2\mathcal{X}_*(\psi\mathcal{N}p)\|_{\mathcal{C}^{1,\alpha}(\mathbb{R}^2)} + \\
&+ \|\hat{\mathcal{N}}_0^2\mathcal{X}_*(\psi\mathcal{N}p) - \hat{\mathcal{N}}_0^3\mathcal{X}_*(\psi p)\|_{\mathcal{C}^{1,\alpha}(\mathbb{R}^2)}.
\end{aligned}$$

We will estimate the three terms on the right separately, denoting by C_η constants depending on η while C denotes constants independent of η. Applying (26) to $\mathcal{N}^2 p$ with $l = 1$ and (29) we get

$$\begin{aligned}
\|\mathcal{X}_*(\psi\mathcal{N}^3 p) - \hat{\mathcal{N}}_0\mathcal{X}_*(\psi\mathcal{N}^2 p)\|_{\mathcal{C}^{1,\alpha}(\mathbb{R}^2)} &\leq \eta\|\mathcal{X}_*(\psi\mathcal{N}^2 p)\|_{\mathcal{C}^{2,\alpha}(\mathbb{R}^2)} + C_\eta\|\mathcal{N}^2 p\|_{\mathcal{C}^{2,\zeta}(\mathbb{S}^2)} \\
&\leq \eta C\left(\|\mathcal{X}_*(\psi p)\|_{\mathcal{C}^{4,\alpha}(\mathbb{R}^2)} + \|p\|_{\mathcal{C}^{4,\zeta}(\mathbb{S}^2)}\right) + C_\eta\|p\|_{\mathcal{C}^{4,\zeta}(\mathbb{S}^2)} \\
&\leq \eta C\|\mathcal{X}_*(\psi p)\|_{\mathcal{C}^{4,\alpha}(\mathbb{R}^2)} + C_\eta\|p\|_{\mathcal{C}^{4,\zeta}(\mathbb{S}^2)}.
\end{aligned}$$

Applying (26) with $l = 2$, replacing p by $\mathcal{N}p$ we get

$$\begin{aligned}
\|\hat{\mathcal{N}}_0\mathcal{X}_*(\psi\mathcal{N}^2 p) - \hat{\mathcal{N}}_0^2\mathcal{X}_*(\psi\mathcal{N}p)\|_{\mathcal{C}^{1,\alpha}(\mathbb{R}^2)} &\leq C\|\mathcal{X}_*(\psi\mathcal{N}^2 p) - \hat{\mathcal{N}}_0\mathcal{X}_*(\psi\mathcal{N}p)\|_{\mathcal{C}^{2,\alpha}(\mathbb{R}^2)} \\
&\leq \eta C\|\mathcal{X}_*(\psi\mathcal{N}p)\|_{\mathcal{C}^{3,\alpha}(\mathbb{R}^2)} + C_\eta\|\mathcal{N}p\|_{\mathcal{C}^{3,\zeta}(\mathbb{S}^2)} \\
&\leq \eta C\|\mathcal{X}_*(\psi\mathcal{N}p)\|_{\mathcal{C}^{3,\alpha}(\mathbb{R}^2)} + C_\eta\|p\|_{\mathcal{C}^{4,\zeta}(\mathbb{S}^2)}.
\end{aligned}$$

From (28) we get

$$\|\hat{\mathcal{N}}_0 \mathcal{X}_*(\psi \mathcal{N}^2 p) - \hat{\mathcal{N}}_0^2 \mathcal{X}_*(\psi \mathcal{N} p)\|_{\mathcal{C}^{1,\alpha}(\mathbb{R}^2)} \leq \eta C \|\mathcal{X}_*(\psi p)\|_{\mathcal{C}^{4,\alpha}(\mathbb{R}^2)} + C_\eta \|p\|_{\mathcal{C}^{4,\varsigma}(\mathbb{S}^2)}.$$

Analogously,

$$\|\hat{\mathcal{N}}_0^2 \mathcal{X}_*(\psi \mathcal{N} p) - \hat{\mathcal{N}}_0^3 \mathcal{X}_*(\psi p)\|_{\mathcal{C}^{1,\alpha}(\mathbb{R}^2)} \leq C \|\mathcal{X}_*(\psi \mathcal{N} p) - \hat{\mathcal{N}}_0 \mathcal{X}_*(\psi p)\|_{\mathcal{C}^{3,\alpha}(\mathbb{R}^2)}$$
$$\leq \eta C \|\mathcal{X}_*(\psi p)\|_{\mathcal{C}^{4,\alpha}(\mathbb{R}^2)} + C_\eta \|p\|_{\mathcal{C}^{4,\varsigma}(\mathbb{S}^2)}.$$

Finally,
$$\|\mathcal{X}_*(\psi \mathcal{N}^3 p) - \hat{\mathcal{N}}_0^3 \mathcal{X}_*(\psi p)\|_{\mathcal{C}^{1,\alpha}(\mathbb{R}^2)} \leq \eta C \|\mathcal{X}_*(\psi p)\|_{\mathcal{C}^{4,\alpha}(\mathbb{R}^2)} + C_\eta \|p\|_{\mathcal{C}^{4,\varsigma}(\mathbb{S}^2)}.$$

We take $\eta = \frac{\varepsilon}{C}$ and get the desired result (27).

3. The next step is proving that for all $\lambda > 0$,

$$\lambda \mathcal{I} + \mathcal{N}^3 : h^{4,\alpha}(\mathbb{S}^2) \to h^{1,\alpha}(\mathbb{S}^2)$$

is an isomorphism. Note that

$$\lambda \mathcal{I} + \mathcal{N}^3 = (\sqrt[3]{\lambda} \mathcal{I} + \mathcal{N})(\sqrt[3]{\lambda} e^{\frac{2\pi}{3} i} \mathcal{I} + \mathcal{N})(\sqrt[3]{\lambda} e^{-\frac{2\pi}{3} i} \mathcal{I} + \mathcal{N}). \tag{30}$$

Parallel to the proof of Lemma 3.13 in [Von06] we can derive surjectivity of $\mu \mathcal{I} + \mathcal{N} : h^{k+1,\alpha}(\mathbb{S}^2) \to h^{k,\alpha}(\mathbb{S}^2)$, for $\mu \in \mathbb{C} \setminus -\mathbb{N}_0$ and for all $k \in \mathbb{N}$. Surjectivity of $\lambda \mathcal{I} + \mathcal{N}^3 : h^{4,\alpha}(\mathbb{S}^2) \to h^{1,\alpha}(\mathbb{S}^2)$ follows if we apply this result for $k = 1, 2, 3$ and $\mu = \sqrt[3]{\lambda}, \sqrt[3]{\lambda} e^{\frac{2\pi}{3} i}, \sqrt[3]{\lambda} e^{-\frac{2\pi}{3} i}$.

4. There exist $C > 0$ and $\lambda_* > 0$ such that for all $r \in h^{4,\alpha}(\mathbb{S}^2)$ and $\lambda \in \mathbb{C}$ with $\operatorname{Re} \lambda \geq \lambda_*$ we have

$$|\lambda| \|r\|_{h^{1,\alpha}(\mathbb{S}^2)} + \|r\|_{h^{4,\alpha}(\mathbb{S}^2)} \leq C \|(\lambda \mathcal{I} + \mathcal{N}^3) r\|_{h^{1,\alpha}(\mathbb{S}^2)}. \tag{31}$$

This can be obtained from (27) via exactly the same procedure that is used in [EP03] in the proof of Theorem B.4. The estimate (31) and the fact that

$$\lambda_* \mathcal{I} + \mathcal{N}^3 : h^{4,\alpha}(\mathbb{S}^2) \to h^{1,\alpha}(\mathbb{S}^2)$$

is an isomorphism imply that $\mathcal{N}^3 \in \mathcal{H}\left(h^{4,\alpha}(\mathbb{S}^2), h^{1,\alpha}(\mathbb{S}^2)\right)$, see [Ama95] Remark I.1.2.1.(a).

$\square$

Lemma 3.5. *The spectrum of $\mathcal{F}'_{\gamma,\mu}(0) : h^{4,\alpha}(\mathbb{S}^2) \to h^{1,\alpha}(\mathbb{S}^2)$ consists entirely of eigenvalues and*

$$\sigma(\mathcal{F}'_{\gamma,\mu}(0)) = \{g_0, g_1, g_2, \dots\}.$$

The resolvent $\left(\lambda \mathcal{I} - \mathcal{F}'_{\gamma,\mu}(0)\right)^{-1} : h^{1,\alpha}(\mathbb{S}^2) \to h^{1,\alpha}(\mathbb{S}^2)$ is compact for all $\lambda \notin \sigma(\mathcal{F}'_{\gamma,\mu}(0))$.

Proof. For every $\lambda \in \mathbb{R}$ define the polynomial

$$p_\lambda(X) = \gamma(X^3 + X^2 - 2X) + \frac{\mu}{4\pi}(X + 3) + \lambda.$$

Note that $p_\lambda(\mathcal{N}) = \lambda \mathcal{I} - \mathcal{F}'_{\gamma,\mu}(0)$. Take λ^* large, such that p_{λ^*} has one negative zero ζ_1 and two zeros ζ_2 and $\zeta_3 = \overline{\zeta_2}$ in $\mathbb{C} \setminus \mathbb{R}$. Then

$$\left(\lambda^* \mathcal{I} - \mathcal{F}'_{\gamma,\mu}(0)\right)^{-1} = -\frac{1}{\gamma} (\zeta_1 \mathcal{I} - \mathcal{N})^{-1} (\zeta_2 \mathcal{I} - \mathcal{N})^{-1} (\zeta_3 \mathcal{I} - \mathcal{N})^{-1}.$$

Because $\zeta_i \notin \mathbb{N}_0$ for $i \in \{1, 2, 3\}$ we see from [Von06], Lemma 3.13 that $(\zeta_i \mathcal{I} - \mathcal{N})^{-1} : h^{1,\alpha}(\mathbb{S}^2) \to h^{1,\alpha}(\mathbb{S}^2)$ is compact. Therefore $\left(\lambda^* \mathcal{I} - \mathcal{F}'_{\gamma,\mu}(0)\right)^{-1} : h^{1,\alpha}(\mathbb{S}^2) \to h^{1,\alpha}(\mathbb{S}^2)$ is a compact mapping as well. From the Hille-Yosida Theorem and Lemma 3.4 we see that $\mathcal{F}'_{\gamma,\mu}(0)$ is a closed operator on $h^{1,\alpha}(\mathbb{S}^2)$ with domain $h^{4,\alpha}(\mathbb{S}^2)$. Applying [Kat95] Theorem III.6.29 we see that the spectrum of $\mathcal{F}'_{\gamma,\mu}(0) : h^{2,\alpha}(\mathbb{S}^2) \to h^{1,\alpha}(\mathbb{S}^2)$ consists entirely of eigenvalues and the resolvent $\left(\lambda \mathcal{I} - \mathcal{F}'_{\gamma,\mu}(0)\right)^{-1} : h^{1,\alpha}(\mathbb{S}^2) \to h^{1,\alpha}(\mathbb{S}^2)$ is compact for all $\lambda \notin \sigma(\mathcal{F}'_{\gamma,\mu}(0))$. We determined the eigenvalues in Lemma 3.1. $\square$

Theorem 3.6. *Let $\mu > 0$ and $0 < \lambda_0 < \frac{3\mu}{4\pi}$. There exists a $\delta > 0$ and a $M > 0$ such that the problem*

$$\frac{\partial r}{\partial \tau} = \mathcal{F}_{\gamma,\mu}(r)$$

with $r(0) = r_0 \in h^{4,\alpha}(\mathbb{S}^2)$ and $\|r_0\|_{\mathcal{C}^{4,\alpha}(\mathbb{S}^2)} < \delta$, has a solution $r \in \mathcal{C}\left([0,\infty), h^{4,\alpha}(\mathbb{S}^2)\right) \cap \mathcal{C}^1\left([0,\infty), h^{1,\alpha}(\mathbb{S}^2)\right)$ satisfying

$$\|r(\tau)\|_{\mathcal{C}^{4,\alpha}(\mathbb{S}^2)} \leq Me^{-\lambda_0\tau}\|r_0\|_{\mathcal{C}^{4,\alpha}(\mathbb{S}^2)}.$$

Proof. Combining Lemma 3.4 with [RR93] Theorem 11.31 we see that $\mathcal{F}'_{\gamma,\mu}(0)$ is sectorial. Note that $-\frac{3\mu}{4\pi}$ is the largest eigenvalue of $\mathcal{F}'_{\gamma,\mu}(0)$. The theorem follows from Lemma 2.3, Lemma 3.4, Lemma 3.5 and [Lun95] Theorem 9.1.2. $\square$

If we combine this estimate with (21) we get for the non-autonomous problem (20),

$$\|r(t)\|_{\mathcal{C}^{4,\alpha}(\mathbb{S}^2)} \leq M \left(\frac{3\mu t}{4\pi} + 1\right)^{-\zeta} \|r_0\|_{\mathcal{C}^{4,\alpha}(\mathbb{S}^2)},$$

for $\zeta = \frac{4\pi}{3\mu}\lambda_0$.

The case $\mu < 0$ is more complicated. We need some extra conditions for certain Richardson moments of the initial domain in order to get results similar to Theorem 3.6. Define

$$\mathfrak{M}^3_1 = \left\{ r \in h^{1,\alpha}(\mathbb{S}^2) : \int_{\Omega_r} d\mathbf{x} = \frac{4\pi}{3}, \int_{\Omega_r} x_j d\mathbf{x} = 0, \ j \in \{1,2,3\} \right\},$$

where x_j denotes the j-th component of $\mathbf{x}$. Note that $r \in \mathfrak{M}^3_1$ if and only if the corresponding domain Ω_r has the volume of the unit ball and its geometric centre is at the origin.

Lemma 3.7. *Suppose that r satisfies (22). If $r_0 \in \mathfrak{M}^3_1$ then $r(t) \in \mathfrak{M}^3_1$, for all $t > 0$.*

Proof. It is easy to check that if $\Omega_{r(0)}$ has the volume of the unit ball, then $\Omega_{r(t)}$ has the volume of the unit ball for all t. By Green's second identity, (2), (3), (5) and Lemma 2.1 we have

$$\begin{aligned}
\frac{d}{dt} \int_{\Omega_{R(t)}} x_j d\mathbf{x} &= \int_{\Gamma_{R(t)}} x_j(\mathbf{v}, \mathbf{n}) d\mathbf{x} = \int_{\Gamma_{R(t)}} -x_j \frac{\partial p}{\partial \mathbf{n}} d\mathbf{x} \\
&= \int_{\Omega_{R(t)}} p\Delta x_j d\mathbf{x} - \int_{\Omega_{R(t)}} x_j \Delta p d\mathbf{x} - \int_{\Gamma_{R(t)}} p\frac{\partial x_j}{\partial \mathbf{n}} d\mathbf{x} \\
&= -\int_{\Gamma_{R(t)}} p\frac{\partial x_j}{\partial \mathbf{n}} d\mathbf{x} = \gamma \int_{\Gamma_{R(t)}} \kappa n_j d\mathbf{x} = \gamma \int_{\Gamma_{R(t)}} \Delta_{R(t)} x_j d\mathbf{x} = 0.
\end{aligned}$$

The lemma follows from this. $\square$

Now we prove a theorem about global existence for the suction case for domains for which the zeroth and first moments vanish.

Theorem 3.8. *Let $\mu < 0$ be such that (24) holds and let $0 < \lambda_0 < \frac{5}{4\pi}\mu + 8\gamma$. There exists a $\delta > 0$ and a $M > 0$ such that the problem*

$$\frac{\partial r}{\partial \tau} = \mathcal{F}_{\gamma,\mu}(r) \tag{32}$$

with $r(0) = r_0 \in h^{4,\alpha}(\mathbb{S}^2) \cap \mathfrak{M}^3_1$ and $\|r_0\|_{\mathcal{C}^{4,\alpha}(\mathbb{S}^2)} < \delta$, has a solution $r \in \mathcal{C}\left([0,\infty), h^{4,\alpha}(\mathbb{S}^2)\right) \cap \mathcal{C}^1\left([0,\infty), h^{1,\alpha}(\mathbb{S}^2)\right)$ satisfying

$$\|r(\tau)\|_{\mathcal{C}^{4,\alpha}(\mathbb{S}^2)} \leq Me^{-\lambda_0\tau}\|r_0\|_{\mathcal{C}^{4,\alpha}(\mathbb{S}^2)}.$$

Proof. Define for all $L \in \mathbb{N}_0$ the subspaces $h^{L,\alpha}_1(\mathbb{S}^2)$ by

$$h^{L,\alpha}_1(\mathbb{S}^2) = \left\{ r \in h^{L,\alpha}(\mathbb{S}^2) : (r,s)_{\mathbb{L}_2(\mathbb{S}^2)} = 0, \forall s \in \mathfrak{S}^3_0 \oplus \mathfrak{S}^3_1 \right\},$$

Introduce

$$\mathcal{G}_{1,\gamma,\mu} = \mathcal{F}_{\gamma,\mu}\big|_{h^{4,\alpha}_1(\mathbb{S}^2)}.$$

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We get

$$\mathcal{G}'_{1,\gamma,\mu}(0) = \mathcal{F}'_{\gamma,\mu}(0)\big|_{h_1^{4,\alpha}(\mathbb{S}^2)}.$$

Because $\mathcal{F}'_{\gamma,\mu}(0)$ is invariant with respect to the decomposition $h^{k,\alpha}(\mathbb{S}^2) = h_1^{k,\alpha}(\mathbb{S}^2) \oplus \mathfrak{S}_0^3 \oplus \mathfrak{S}_1^3$ we have

$$\mathcal{G}'_{1,\gamma,\mu}(0)\left[h_1^{4,\alpha}(\mathbb{S}^2)\right] \subseteq h_1^{1,\alpha}(\mathbb{S}^2).$$

Combining Lemma 3.4 and Lemma 3.5 with [Von06] Lemma 4.2 we get the following results:

- The operator $\mathcal{G}'_{1,\gamma,\mu}(0) : h_1^{1,\alpha}(\mathbb{S}^2) \to h_1^{1,\alpha}(\mathbb{S}^2)$ with $\mathcal{D}(\mathcal{G}'_{1,\gamma,\mu}(0)) = h_1^{4,\alpha}(\mathbb{S}^2)$ is closed.

- The spectrum

$$\sigma(\mathcal{G}'_{1,\gamma,\mu}(0)) = \{g_2, g_3, g_4, \dots\}$$

  consists of negative real numbers. The largest element is $-\frac{5}{4\pi}\mu - 8\gamma$.

- The operator $\mathcal{G}'_{1,\gamma,\mu}(0)$ is sectorial because $\mathcal{F}'_{\gamma,\mu}(0)$ is sectorial. To see this, combine Lemma 3.4 and [RR93] Theorem 11.31.

Define $f_1 : h^{4,\alpha}(\mathbb{S}^2) \to \mathbb{R} \times \mathbb{R}^3$ by

$$f_1(r) = \left(\begin{array}{c} \int_{\Omega_r} d\mathbf{x} - \frac{4\pi}{3} \\ \int_{\Omega_r} \mathbf{x}\,d\mathbf{x} \end{array} \right).$$

Let $\mathcal{P}_1 : h^{4,\alpha}(\mathbb{S}^2) \to h_1^{4,\alpha}(\mathbb{S}^2)$ be the orthogonal projection on $h_1^{4,\alpha}(\mathbb{S}^2)$ with respect to the $\mathbb{L}_2(\mathbb{S}^2)$-inner product. Let $\phi_1 : h^{4,\alpha}(\mathbb{S}^2) \to \mathbb{R} \times \mathbb{R}^3 \times h_1^{4,\alpha}(\mathbb{S}^2)$ be defined by

$$\phi_1(r) = \left(\begin{array}{c} f_1(r) \\ \mathcal{P}_1 r \end{array} \right).$$

Like in [Von06] we can prove that ϕ_1 is invertible in a neighborhood $\mathcal{U}$ of zero. There exists a $\mathcal{V} \subseteq h_1^{4,\alpha}(\mathbb{S}^2)$ such that $\{0\} \times \{\mathbf{0}\} \times \mathcal{V} \subseteq \phi_1(\mathcal{U})$. Define the analytic mapping $\psi_1 : \mathcal{V} \to \mathfrak{M}_1^3$ by

$$\psi_1(\tilde{r}) = \phi^{-1}(0, \mathbf{0}, \tilde{r}).$$

It has been proved that for $h \in h_1^{4,\alpha}(\mathbb{S}^2)$ we have

$$\psi_1'(0)[h] = h. \tag{33}$$

Assume for the moment that r is a solution to (32) and $r(t) \in \mathfrak{M}_1^3 \cap \mathcal{U}$. Then $\mathcal{P}_1 r$ satisfies

$$\frac{\partial(\mathcal{P}_1 r)}{\partial \tau} = (\mathcal{P}_1 \circ \mathcal{F}_{\gamma,\mu} \circ \psi_1)(\mathcal{P}_1 r). \tag{34}$$

We will discuss the solvability of (34) first. Because of (33) and the chain rule of differentiation, the linearisation around zero of the evolution operator on the right-hand side is

$$(\mathcal{P}_1 \circ \mathcal{F}_{\gamma,\mu} \circ \psi_1)'(0) = \mathcal{G}'_{1,\gamma,\mu}(0).$$

Applying [Lun95] Theorem 9.1.2, we get a $\delta > 0$ such that if $\tilde{r}_0 = \mathcal{P}_1 r_0 \in h_1^{4,\alpha}(\mathbb{S}^2)$ with $\|\tilde{r}_0\|_{\mathcal{C}^{4,\alpha}(\mathbb{S}^2)} < \delta$, then the problem

$$\frac{\partial \tilde{r}}{\partial \tau} = (\mathcal{P}_1 \circ \mathcal{F}_{\gamma,\mu} \circ \psi_1)\tilde{r},$$

with $\tilde{r}(0) = \tilde{r}_0$ has a unique solution $\tilde{r} \in \mathcal{C}\left([0,\infty), h_1^{4,\alpha}(\mathbb{S}^2)\right) \cap \mathcal{C}^1\left([0,\infty), h_1^{1,\alpha}(\mathbb{S}^2)\right)$. Furthermore there exists a $M' > 0$ independent of $\tilde{r}_0$ such that

$$\|\tilde{r}(\tau)\|_{\mathcal{C}^{4,\alpha}(\mathbb{S}^2)} \leq M' e^{-\lambda_0 \tau} \|\tilde{r}_0\|_{\mathcal{C}^{4,\alpha}(\mathbb{S}^2)}.$$

Take

$$r = \psi_1(\tilde{r}).$$

We get

$$\frac{\partial r}{\partial \tau} = \psi_1'(\tilde{r}) \left[\frac{\partial \tilde{r}}{\partial \tau}\right] = \psi_1'(\tilde{r}) \left[\mathcal{P}_1 \mathcal{F}_{\gamma,\mu}(r)\right] = \psi_1'(\mathcal{P}_1 r)[\mathcal{P}_1 \mathcal{F}_{\gamma,\mu}(r)].$$

Because $\psi_1(\mathcal{P}_1 r) = r$ for all $r \in \mathfrak{M}_1^3 \cap \mathcal{U}$, we have

$$\psi_1'(\mathcal{P}_1 r)[\mathcal{P}_1 h] = h,$$

for all $h \in T_r \mathfrak{M}_1^3$. Because of Lemma 3.7 we have $\mathcal{F}_{\gamma,\mu}(r) \in T_r \mathfrak{M}_1^3$ and therefore

$$\frac{\partial r}{\partial \tau} = \mathcal{F}_{\gamma,\mu}(r).$$

There exists a $\delta > 0$ such that for $r_0 \in \mathfrak{M}_1^3$ with $\|r_0\|_{\mathcal{C}^{4,\alpha}(\mathbb{S}^2)} < \delta$ we have

$$\|r(\tau)\|_{\mathcal{C}^{4,\alpha}(\mathbb{S}^2)} = \|(\psi_1 \circ \mathcal{P}_1)r(\tau)\|_{\mathcal{C}^{4,\alpha}(\mathbb{S}^2)} \leq C\|\mathcal{P}_1 r(\tau)\|_{\mathcal{C}^{4,\alpha}(\mathbb{S}^2)} \leq$$

$$\leq C e^{-\lambda_0 \tau} \|\mathcal{P}_1 r_0\|_{\mathcal{C}^{4,\alpha}(\mathbb{S}^2)} \leq C e^{-\lambda_0 \tau} \|r_0\|_{\mathcal{C}^{4,\alpha}(\mathbb{S}^2)}.$$

This proves the theorem. $\qquad\square$

If we combine this estimate with (21) we get for the non-autonomous problem (20),

$$\|r(t)\|_{\mathcal{C}^{4,\alpha}(\mathbb{S}^2)} \leq M \left(\frac{3\mu t}{4\pi} + 1\right)^{\zeta} \|r_0\|_{\mathcal{C}^{4,\alpha}(\mathbb{S}^2)},$$

for $\zeta = -\frac{4\pi}{3\mu}\lambda_0$ and $t \in [0, T_\mu)$.

4 Stability for perturbations of the suction point

If the suction point is not in the geometric centre of the initial domain we can not derive a result like Theorem 3.8. The solution either becomes unbounded or breaks down before all liquid is sucked out. However, in this section we show that an arbitrarily large portion of the liquid can be removed. More precisely, for every $\varepsilon > 0$ there exists a neighborhood of the suction point, such that if the geometric centre is in this neighborhood and the conditions of Theorem 3.8 are satisfied except for $r_0 \in \mathfrak{M}_1^3$, then a solution on $(0, T_\mu - \varepsilon)$ exists.

Let X be a metric space and let $T^+ : X \to (0, \infty) \cup \{\infty\}$ be some mapping. Define

$$V := \left\{(x, \tau) \in X \times [0, \infty) : \tau < T^+(x)\right\}.$$

A mapping $f : V \to X$ is called a semiflow on X if

1. V is open in $X \times (0, \infty)$;

2. $f \in \mathcal{C}(V, X)$;

3. $f(\cdot, 0) = \mathcal{I}$;

4. if $x \in X$ and $\tau \in [0, T^+(x))$, and if $\tau^* \in [0, T^+(f(x, \tau)))$ then $\tau + \tau^* < T^+(x)$ and $f(x, \tau + \tau^*) = f(f(x, \tau), \tau^*)$.

Define $\mathcal{E} : \mathcal{U} \to \mathcal{L}(\mathcal{C}^{2,\alpha}(\mathbb{S}^2), \mathcal{C}^{1,\alpha}(\mathbb{S}^2))$ and $l : \mathcal{U} \to \mathcal{C}^{2,\alpha}(\mathbb{S}^2)$ by

$$(\mathcal{E}(r)\psi)(\boldsymbol{\xi}) = -\frac{\mathrm{Tr}\left(\mathcal{Q}(r)\left[\mathcal{P}(r)^{-1}\begin{bmatrix} 0 \\ \psi \end{bmatrix}\right]\right)(\boldsymbol{\xi}) \cdot \mathbf{n}(r, \boldsymbol{\xi})}{\mathbf{n}(r, \boldsymbol{\xi}) \cdot \boldsymbol{\xi}}$$

and

$$l(r) = \frac{1}{\sigma_N(1+r)^{N-1}} - \frac{1+r}{\sigma_N}.$$

where $\mathcal{U}$ is a suitable neighborhood of zero in $\mathcal{C}^{2,\alpha}(\mathbb{S}^2)$. We have

$$\frac{\partial r}{\partial \tau} = -\gamma \mathcal{E}(r)\kappa(r) - \mu \mathcal{E}(r)\phi(r) + \mu l(r).$$

Lemma 4.1. *The mapping $\mathcal{E}$ is analytic around zero from $\mathcal{U}$ to $\mathcal{L}(\mathcal{C}^{2,\alpha}(\mathbb{S}^2),\mathcal{C}^{1,\alpha}(\mathbb{S}^2))$*

Proof. The analyticity of $\mathcal{P}$, $\mathcal{Q}$ and $\mathbf{n}$ around zero is proved in [Von06]. The mapping $r \mapsto \mathcal{P}(r)^{-1}$ is analytic around zero as well because inversion is an analytic mapping, see [Pro97] Chapter 3 Lemma 7, and compositions of analytic mapping are analytic. Analyticity of $\mathcal{E}$ around zero follows from this. $\qquad\square$

Lemma 4.2. *Let $\mu < 0$, $\alpha_1 \in (0,\alpha)$, $\beta \in (\alpha,1)$ and assume that (24) holds. There exists a neighborhood $\mathcal{U}$ of 0 in $h^{3,\beta}(\mathbb{S}^2)$ such that the problem*

$$\frac{\partial r}{\partial \tau} = \mathcal{F}_{\gamma,\mu}(r)$$

has for each $r(0) = r_0 \in \mathcal{U} \cap h^{4,\alpha_1}(\mathbb{S}^2)$ a unique maximal solution

$$r \in \mathcal{C}([0,T^+(r_0)),h^{4,\alpha_1}(\mathbb{S}^2)) \cap \mathcal{C}^{0,\eta}([0,T^+(r_0)),h^{1,\alpha}(\mathbb{S}^2)),$$

where $\eta = 1 - \frac{\alpha-\alpha_1}{3}$. The mapping $(r_0,\tau) \mapsto r(\tau)$ is a semiflow on $\mathcal{U} \cap h^{4,\alpha_1}(\mathbb{S}^2)$.

Proof. Let $\beta \in (\alpha,1)$. According to [ES97a] Lemma 3.1, there exists a neighborhood $\hat{\mathcal{U}}$ of 0 in $h^{2,\beta}(\mathbb{S}^2)$,

$$\kappa_1 \in \mathcal{C}^{\omega}(\hat{\mathcal{U}},\mathcal{L}(h^{3,\alpha}(\mathbb{S}^2),h^{1,\alpha}(\mathbb{S}^2)))$$

and

$$\kappa_2 \in \mathcal{C}^{\omega}(\hat{\mathcal{U}},h^{1,\beta}(\mathbb{S}^2))$$

such that

$$\kappa(r) = \kappa_1(r)r + \kappa_2(r).$$

From (12), we see that κ_1 is a quasilinear differential operator of second order and κ_2 is of first order. Therefore there exists a small neighborhood of zero $\mathcal{U} \subset \hat{\mathcal{U}}$ in $h^{3,\beta}(\mathbb{S}^2)$ such that

$$\kappa_1 \in \mathcal{C}^{\omega}(\mathcal{U},\mathcal{L}(h^{4,\alpha}(\mathbb{S}^2),h^{2,\alpha}(\mathbb{S}^2)))$$

and

$$\kappa_2 \in \mathcal{C}^{\omega}(\mathcal{U},h^{2,\beta}(\mathbb{S}^2)).$$

Combining this with Lemma 4.1 we can choose $\mathcal{U}$ such that

$$r \mapsto \mathcal{E}(r)\kappa_1(r) \in \mathcal{C}^{\omega}(\mathcal{U},\mathcal{L}(h^{4,\alpha}(\mathbb{S}^2),h^{1,\alpha}(\mathbb{S}^2))).$$

Because of [ES97a] Remark 3.3 we have

$$\mathcal{E}(0)\kappa_1(0) = \mathcal{N}^3 + p(\mathcal{N}),$$

where p is a polynomial of degree 2 and therefore we get from Lemma 3.2 and Lemma 3.4

$$\mathcal{E}(0)\kappa_1(0) \in \mathcal{H}(h^{4,\alpha}(\mathbb{S}^2),h^{1,\alpha}(\mathbb{S}^2)).$$

By [Ama95] Theorem I.1.3.1, $\mathcal{H}(h^{4,\alpha}(\mathbb{S}^2),h^{1,\alpha}(\mathbb{S}^2))$ is open in $\mathcal{L}(h^{4,\alpha}(\mathbb{S}^2),h^{1,\alpha}(\mathbb{S}^2))$. This implies that we can choose $\mathcal{U}$ such that

$$r \mapsto \gamma\mathcal{E}(r)\kappa_1(r) \in \mathcal{C}^{\omega}(\mathcal{U},\mathcal{H}(h^{4,\alpha}(\mathbb{S}^2),h^{1,\alpha}(\mathbb{S}^2))). \tag{35}$$

From Lemma 4.1, [Von06] Lemma 2.4 and the proof of Lemma 2.9, we see that $\mathcal{E}$, ϕ and l are analytic around zero. We can choose $\mathcal{U}$ such that

$$r \mapsto \gamma\mathcal{E}(r)\kappa_2(r) + \mu\mathcal{E}(r)\phi(r) + \mu l(r) \in \mathcal{C}^{\omega}(\mathcal{U},h^{1,\beta}(\mathbb{S}^2)). \tag{36}$$

The little Hölder spaces satisfy

$$\left(h^{4,\alpha}(\mathbb{S}^2),h^{1,\alpha}(\mathbb{S}^2)\right)^0_{1-\frac{\alpha-\alpha_1}{3},\infty} = h^{4,\alpha_1}(\mathbb{S}^2).$$

For more information about continuous interpolation of Hölder spaces, see [Lun95] Chapter 1. The result follows from (35), (36) and [Ama93] Theorem 12.1. $\qquad\square$

Theorem 4.3. *Let $T > 0$ and let $\eta \in (0,1)$. Let $\mu < 0$, such that (24) holds. Define*

$$\alpha_1 = \alpha + 3(\eta - 1).$$

There exists a $\delta > 0$ such that the problem

$$\frac{\partial r}{\partial \tau} = \mathcal{F}_{\gamma,\mu}(r)$$

with $r(0) = r_0 \in h^{4,\alpha_1}(\mathbb{S}^2)$ and $\|r_0\|_{\mathcal{C}^{4,\alpha_1}(\mathbb{S}^2)} < \delta$, has a solution $r \in \mathcal{C}\left([0,T), h^{4,\alpha_1}(\mathbb{S}^2)\right) \cap \mathcal{C}^{0,\eta}\left([0,T), h^{1,\alpha}(\mathbb{S}^2)\right)$.

Proof. From the semiflow property proved in Lemma 4.2 we see that the set

$$V = \left\{ (r_0, \tau) \in \mathcal{U} \times (0, \infty) : \tau < T^+(x) \right\}$$

is open in $h^{4,\alpha_1}(\mathbb{S}^2) \times (0, \infty)$. Because of

$$T^+(0) = \infty,$$

the point $(0, T)$ is an interior point of V. Therefore there exists a neighborhood $\tilde{\mathcal{U}}$ of zero in $h^{4,\alpha_1}(\mathbb{S}^2)$ such that for all $r_0 \in \tilde{\mathcal{U}}$ we have

$$T^+(r_0) \geq T.$$

$\square$

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