A FORMALISATION OF DESIGN METHODS

A λ-calculus Approach to System Design with an Application to Text Editing

L.M.G. FEIJS

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PROEFSCHRIFT

ter verkrijging van de graad van doctor aan de Technische Universiteit Eindhoven, op gezag van de Rector Magnificus, prof. ir. M. Tels, voor een commissie aangewezen door het College van Dekanen in het openbaar te verdedigen op vrijdag 23 februari 1990 te 16.00 uur

door

LAURENTIUS MICHIEL GERARDUS FEIJS

geboren te Sittard

Dit proefschrift is goedgekeurd
door de promotoren

prof. dr. F.E.J. Kruseman Aretz
en
prof. dr. J.A. Bergstra

The work described in this thesis has been carried out at the Philips Research Laboratories, Eindhoven as part of the Philips Research programme.

This work has been done in the context of ESPRIT project 432 (Meteor).

Aan mijn ouders

Contents

Chapter 1

Overview and introduction

1.1 Structure of this monograph

This monograph is concerned with models of software development. The foundations for the models studied are based on a λ-calculus approach to design structures. It also contains an application of the concepts studied to text editing. This first chapter is an overview chapter.

The main body of the monograph is formed by the Chapters 2, 3, 4 and 5. This body is divided into two parts. The first part consists of the Chapters 2 and 3 which are about research of a fundamental nature, dealing with models of software development based on a λ-calculus approach to design structures. The second part consists of Chapter 4 and 5, which are about work of an engineering nature, viz. the application of the concepts studied in the first part to text editing. Each chapter is structured as a unit with its own introduction, bibliography and appendices.

Chapter 1 is self-contained and requires no preliminaries. Within Chapter 1 there are, amongst others, sections dedicated to the Chapters 2, 3, 4 and 5 – one section for each chapter. Chapter 1 serves to place the other chapters in a broader context and to make the relation between them explicit.

Chapter 2 is concerned with a formalisation of design structures. It is self-contained and its preliminaries are an elementary knowledge of mathematics and computer science as well as a certain familiarity with lambda calculus.

Chapter 3 is concerned with correctness-preserving transformations of designs. This chapter is the heart of the monograph in the sense that it is devoted entirely to the dynamic aspects of the process of software development. Chapter 3 is a natural continuation of Chapter 2. Results from Chapter 2 are used in Chapter 3, but no additional preliminaries are required.

Chapter 4 is concerned with the formal specification of a text editor. The text editor is introduced by way of illustration of the concepts from Chapter 2 and 3, but at the same time Chapter 4 can be viewed as a case study in the use of formal specification techniques. Chapter 4 is self-contained and its preliminaries are an elementary knowledge of mathematics and computer science as well as a certain familiarity with the COLD-K language.

Chapter 5 is concerned with the systematic design of a text editor and is a natural continuation of Chapter 4. The definitions from Chapter 4 are used in Chapter 5, but no additional preliminaries are required.

The remainder of this first chapter is organised as follows. Section 1.2 is entitled "the nature of software" and it serves to introduce and demarcate the field of software engineering, which is the branch of applied science in which the work presented in this monograph fits. Section 1.3 is about informal models of software development and in Section 1.4 we sketch a line of thought which gradually moves away from the informal models, leading to the formalisation of design structures and the correctness-preserving transformations of designs presented in the subsequent sections.

Section 1.5 and 1.6 refer to Chapter 2 and Chapter 3 respectively. These sections serve to present the main achievements of the corresponding chapters and to relate them to other work.

The results of Chapters 2 and 3 are generic in the sense that they can be instantiated with a particular formalism. In order to apply the results it is necessary to do this and therefore Section 1.7 is about formal specification techniques in general and Section 1.8 is about the particular formalism we have chosen to be used in Chapters 4 and 5.

Sections 1.9 and 1.10 refer to Chapters 4 and 5 respectively. These sections serve to present the main achievements of the corresponding chapters and to relate them to other work. Finally, Section 1.11 discusses a number of options for future work.

1.2 The nature of software

We approach the nature of software from the viewpoint that software construction is an *engineering* activity (see Section 1.2.1 below) with certain peculiar characteristics, viz. formality, the role of languages, the role of tools, and application domain evolution (Sections 1.2.2 to 1.2.5 below). The most important of these characteristics is *formality* which acts as a recurring theme in the discussion of the other characteristics.

1.2.1 Software engineering

The history of software engineering started in 1944 when Eckert, Mauchly and von Neumann adopted the idea of storing the instructions for a computer in electronic form internally in the memory of the computer [1]. The size of computer memories has grown exponentially since then and as a consequence there are hardly any physical limitations upon the size of the computer programs themselves. Also from a physical viewpoint, computer programs are extremely flexible in the sense that it is easy to copy and to modify them. Probably this situation has led to the adoption of the term 'software' as a synonym for 'computer program'. The first computer programs were just sequences of instructions, but it did not take long until program structuring mechanisms were introduced. As Turing [1] put it:

> When we wish to start on a subsidiary operation we need only make a note of where we left off the major operation and then apply the first instruction of the subsidiary. When the subsidiary is over we look up the note and continue with the major operation. ... We have only to think how this is to be done once, and then forget how it is done.

Also it was soon realised that computers and computer programs were going to serve as tools for the programmer. As Turing [1] wrote:

> This process of constructing instruction tables should be very fascinating. There need be no real danger of it ever becoming a drudge, for any processes that are quite mechanical may be turned over to the machine itself.

But despite these very promising aspects of software construction, the by now well-known problem of program correctness was present from the very beginning. Again we quote Turing [1]:

> Delay there must be, due to the virtually inevitable snags, for up to a point it is better to let the snags be there than to spend such time in design that there are none (how many decades would this course take).

Although the above-mentioned characteristics of software guarantee its construction to be a fascinating activity, it takes two more characteristics to turn it into a real *engineering* activity.

The first of these characteristics is the possibility to construct mathematical models of computers and of the calculations performed by them. The second characteristic is the existence of a wide range of practical applications.

We shall briefly discuss the latter two characteristics now. Certain mathematical models of computers and of the calculations performed by them were already available *before* the actual construction of computers took place. Church's λ-calculus model and the Turing-machine model already existed in 1936 and they provided very abstract mathematical models of computation. Although these models were tuned to the analysis of a certain theoretical notion of *computability*, they mark the beginning of fruitful research which has produced an enormous amount of results and concepts. Let us highlight just a few areas. Of course the following list is a gross simplification of a huge field of research.

- The invention and analysis of numerous efficient algorithms for searching, sorting and numerical calculations (see e.g. [2]).
- The theory of program correctness started by Floyd, Hoare, Dijkstra and others (see [3,4,5]).
- Specification techniques and denotational semantics, which have led to the conception of special specification languages [6,7,8] and to a rich collection of techniques for describing semantics [9,10,4].
- Chomsky's theory of formal languages [11] and parsing theory [12].
- Type theory [13,14,15,16], providing theoretical foundations for the concept of type-checking.
- Models of communicating processes starting with Petri nets and Milner's CCS [17,18].
- Programming language concepts, embodied in a never-ending stream of new programming languages: parameter mechanisms, programming variables, scoping, etc. Let us mention just a few important languages: FORTRAN, ALGOL, COBOL, LISP, Pascal, Modula, C, SETL, Ada, FP, ML, OCCAM, POOL and Prolog.

Programmed computers turn out to have many practical applications. Initially the applications were restricted to numerical computations, but soon they included also bookkeeping, administration, language processing, data manipulation and process control. Programmed computers have become an integral part of almost all technical systems. Completely new applications have come into existence in areas such as telecommunications, entertainment, manufacturing, safety, medicine and education, but regretfully also in warfare. By now, programmed computers have deeply penetrated all systems of infrastructure upon which our society is built.

As shown above, the term 'software engineering' is quite justified. Let us mention two other terms as well: 'computer science' and 'computing science'. The former term is chosen somewhat unfortunately whereas the latter is not generally accepted (yet). Both 'computer science' and 'computing science' are often used in order to stress the role of the mathematical discipline. The term 'software engineering' is also used in connection with the project management and the organisational aspects of software construction.

It is easy to think that software engineering is related to electronic engineering, but in fact this is not really the case at all. It is true that computers are built up from electronic components; yet programmers need not know about the actual construction of the computer. The situation becomes even clearer when programming languages are employed which abstract more and more from the underlying machine.

Software engineering has a significant overlap with mathematics, although of course there are many branches of mathematics which are just not applicable. The following branches of mathematics are relevant [19]:

- formal logic,
- set theory,
- theory of relations, maps and functions,
- λ-calculus,
- automata theory,
- theory of formal languages,
- discrete mathematics.

Also queueing theory, category theory, graph theory and the theory of partial orders and lattices should be mentioned. The following definition of software engineering is given by Boehm [20]. It is quite consistent with the discussion presented above:

> the practical application of scientific knowledge in the design and construction of computer programs and the associated documentation required to develop, operate and maintain them.

1.2.2 Formality

There is one characteristic which distinguishes software engineering from most other engineering areas: formality. Every task to be performed by a computer must be formulated as a program. The computer executes such a program precisely *as it is*. In particular, the computer will not complain if the program is wrong, or if the program solves another problem than the one meant by the programmer. Computer programs must be constructed with extreme care. One mistake may cause a program to produce nonsensical results. With respect to the results, there is no difference between small mistakes and large mistakes: forgetting a semicolon can be as disastrous as a collection of huge design errors.

The relevance of this characteristic is amplified by the size and the complexity of the programs. For simple programs there is a possibility that testing or even trial and error methods will eventually lead to acceptable results. For large software products such an approach is fatal. When the complexity of software products increases, program correctness becomes a very serious issue. Remember the fact that *one* mistake may cause the program to produce nonsensical results. Furthermore, this complexity may be such that it is impossible for one person to understand and remember all details of a program. It is a consequence of this complexity that communication, documentation, modularisation, specification and abstraction play an increasingly important role in the production of large software systems.

Many software projects exhibit phenomena which can easily be interpreted as the symptoms of inadequately approaching the formality and the complexity characteristics sketched above. One such phenomenon is a strong emphasis on project management, risk analysis, planning, baseline documents and milestones, whereas on the other hand it turns out to be very hard to put the planning into practice. Some authors even introduced the term 'software crisis'. We quote Sommerville and Morrison [21].

> ... in the late 1960s after the so-called software crisis was identified. This was the name given to the failure of the software developers to build large systems which were required to run on the then-new third-generation computer hardware.

Very much related to this, there is an increasing interest for modelling the software development process. We shall come back to these models later and in fact we shall construct a special kind of such models ourselves (this will be discussed in Section 1.6). We view the 'software crisis' simply as a symptom of poorly understood deeper causes such as the lack of formality and the complexity of software.

The formality issues mentioned above can be explained better by referring to the notion of a formal system [19]. A formal system consists of: (1) a syntax, by which we mean an alphabet and rules which define a set of formulae, and (2) a number of rules which can be used for the derivation of new formulae from some given formula. The rules of a formal system are given without any reference to their intuition or meaning. The formal system just defines a game. A computer program can be viewed as a formal system and the computer executing it can be viewed as the player of the game. The number of steps performed in a calculation may become so large that one person cannot imagine and understand the calculation as a sequence of steps.

Instead of that, reasoning about program execution must be based on more powerful (mathematical) approaches, using some implicit or explicit induction principle. If we want to consider the formal system as a black box and try to verify its correctness just by testing, then in general we can never be sure that the formal system is correct. Program testing can be used to show the presence of errors, but not their absence.

There is a second point where formal systems enter into software engineering, viz. as systems of reasoning. For this purpose many relatively simple formal systems have been proposed. The following list is of course not complete.

- Equational logic for specification of data types [22]. These specifications are often called 'algebraic specifications'. They work well for data types such as natural numbers, Boolean values, stacks and queues. In the typical 'stacks' example, one uses equations like $pop(push(s,x)) = s$ and $top(push(s,x)) = x$. Efforts have been made to push the approach further by describing more complex systems, such as simple programming languages [23], file systems [24] and data base systems [25].
- First-order predicate logic for describing the various states involved in a program execution. Imperative programs can be viewed as state transformers and hence there is a need to describe properties of states. A typical example of such a property is that in some state an array $a[0..n-1]$ is sorted. The state itself remains implicit and to express this property we would have a first-order predicate logic formula $\forall i, j : \mathbb{N}(i < n \wedge j < n \wedge i \leq j \Rightarrow a[i] \leq a[j])$. This logic is already classic [26], but in the 1970s it has received much attention in connection with Hoare's logic and Dijkstra's *wp* calculus (see below).
- Hoare's logic for reasoning about the correctness of sequential programs. This is a formal system for reasoning about triples $\langle P, s, Q\rangle$ where P is a precondition, s is a program and Q is a postcondition. This logic can be interpreted as dealing with partial correctness of programs. It was one of the first formal systems in software engineering which was really presented and widely recognised as a formal system. By now it has been incorporated into the field of mathematical logic, where its generalisations are known as *dynamic logic* [27].
- Dijkstra's *wp* calculus [5] is a formal system which can be interpreted as dealing with total correctness. One writes $P \Rightarrow wp(s, Q)$ where P is a precondition, s is a program and Q is a postcondition.

There are several related reasons for the introduction of these formal systems of logic into software engineering. The first is the program correctness problem: formal systems of logic provide mental tools for reasoning about programs and data types. We could express this as follows: ***one way of solving the problems caused by the introduction of a formal system, is to introduce another formal system, viz. a logic for reasoning.***

Another closely related reason is that formal systems of logic make it possible to construct tools that assist in reasoning about programs – at least in principle. The current state of the art is that the reasoning power of theorem provers is too limited to take full advantage of this possibility; this situation might change in the future. Even if there are no tools for reasoning yet, syntax checking and type checking the formulae of some system of logic already give useful results.

The size and complexity of programs make it necessary to split programs into modules which have a specification associated with them. Such specifications should contain precise properties of data types and of programs; the formulae of some system of logic are very suited for expressing these properties.

A question which has until now not been solved entirely satisfactorily is the integration of these formal systems of logic. One promising approach to this integration leads to the so-called wide-spectrum languages. These are languages combining an expressive logic with programming language constructs. Examples of these are VDM [6], Z [7] and COLD-K [8]. A further integration would have to incorporate models of communicating processes. A somewhat different approach is taken in the Genesis project [28], where tools are constructed which are parameterised over systems of logic.

An interesting option which comes with the introduction of formal systems of logic is the possibility to provide the proofs and have them checked automatically. The proofs must be coded as expressions of a special version of λ-typed λ-calculus and then proof-checking becomes a matter of type-checking – which turns out to be decidable. The checkers are called verification systems or justification systems; the Automath checker [16] is one of the first examples.

An approach which might become important in the long term is to have a formalisation of typical software development processes. This formalisation then might be turned into an automation of (part of) the software development process [29].

1.2.3 The role of languages

There are many different languages in use in software engineering and much progress is language-driven. Each language L is a set of sentences and when e.g. L is a programming language, these are the well-formed programs that can be compiled and executed. To characterise a language one usually employs a grammar G – which is a formal system – and we have $L = L(G)$. In order to structure our discussion, we adopt the following classification scheme:

- programming languages,
- specification languages,
- special-purpose languages.

We begin with a discussion of programming languages. These have always played an important role throughout the history of software engineering, which started with the machine language of the early computers. A machine-language program contains precisely the sequences of 0s and 1s that can be interpreted directly by the computer. For example, an instruction to load the accumulator register with a binary number 111, for example, could look as follows:

```
10000110 00000111.
```

Clearly this was a user-unfriendly approach and the wish to abstract from these bit sequences soon gave rise to a following generation of languages. These were called *assembly languages*; an assembly-language program has precisely the same structure as the corresponding machine-language program, but the sequences of 0s and 1s have been replaced by symbols. For example one could write

```
LDA A #7
```

rather than 10000110 00000111. In the 1950s FORTRAN [30] was introduced, in which one could write down formulae such as

```
X1 = (-B + SQRT(B**2 - 4.0*A*C))/(2.0*A)
```

which is relatively close to the usual mathematical notation. This was a great step forward, although control-flow constructs and parameter passing were not very well-chosen from today's point of view.

Let us interrupt our historical survey now to make a few observations; these apply to the historical development sketched above, but in fact they apply to the later developments as well. First, there has been a tendency to make the languages more abstract in the sense that the programs have more in common with the mathematical descriptions of the underlying models and less with the corresponding machine-language programs. The second observation is that many software developers care a lot about the language they use. The language in use serves as a medium for thinking about the problem at hand and the program to solve it.

The last observation is that the introduction of new concepts into the practice of software engineering can be done by embodying each concept as a language construct in a new language. But somehow the converse of this statement seems to be the case as well: it is hard to introduce methodological concepts unless these are very concretely available as constructs in a programming language. To give one example, the concept of parameterisation was from a mathematical point of view already well-described and modelled since the formulation of the λ-calculus; still it required effort and discipline to have well-identified parameters when writing subroutines in an assembly language. FORTRAN provided for functions and subroutines with parameters which made it much easier to use the concept of parameterisation – although of course the FORTRAN and the λ-calculus parameter mechanism are quite different.

In the 1960s and 1970s the historical development has been more complex. Several new concepts were introduced, but not in such a way that there is precisely one sequence of languages which is monotonically increasing with respect to abstractness and expressiveness. Instead of trying to provide a complete historical survey, we shall just mention a number of important concepts:

- modularisation,
- parameterisation,
- concurrency,
- higher-order functions,
- types,
- abstract data types,
- logic programming,
- term rewriting,
- inheritance.

For a systematic comparison of a number of languages, see [31] and [32].

A problem with the introduction of new concepts via a programming language is that usually several concepts are presented in a somewhat intertwined fashion. Typically the intertwining is between concepts of a methodological nature and concepts dealing with efficient program execution mechanisms. For example, the modules in [33] are primarily intended as a rather syntactic means of ordering a sequence of procedures, data types and variables, together with static ways of prohibiting access to constructs which should be considered local. But at the same time modules can have initialisation statements; moreover there are special kinds of modules, e.g. the so-called interface modules which offer solutions to the mutual exclusion problems of communicating processes.

Much progress of a methodological nature has been achieved by introducing new concepts such as modularisation, parameterisation, types, concurrency, and higher-order functions into programming languages. However, the problem with programming languages is that all constructs must be executable and as a consequence a need arose for another kind of languages which are optimal with respect to expressivity and abstractness. The resulting languages are called *specification languages*. This term may be somewhat misleading because these are not only useful for *specifying* programs to be constructed, but they can also be used for describing programs and systems which are already available. This need for another kind of languages has been expressed elegantly by Hoare [34]:

> So in the specification y, you should take advantage of the full range of concepts and notations of mathematics, including concepts that cannot be represented on a computer and operations that could not be implemented in a programming language.

Many of the issues occurring in relation with modularisation, parameterisation, typing, concurrency, higher-order functions, etc. apply also to specification languages. For most of these concepts it seems a good idea to make them available in specification languages as well – although their integration still is a major technical problem. In [35] a specification language is defined as a language satisfying the following requirements:

- the language concepts enable the description of digital systems,
- the language constructions are derived from mathematical logic,
- the language has a precise syntax,
- the language has a precise semantics.

If furthermore the language allows for descriptions at several levels of abstraction, it is called a *wide-spectrum* specification language. Typically the wide-spectrum languages have a subset which can be considered as executable. However it is a misconception to postulate that as an ideal it should be tried to make *all* constructs in the language executable. By doing so, one just gets another programming language. It is the very purpose of a specification language to avoid the conflicts between executability on the one hand and expressivity and abstractness on the other hand; in a specification language the choice should be in favour of expressivity and abstractness. In [35] a classification of specification languages is presented. We summarise it here:

- thematic languages, tuned to feature a theoretical approach,
- combination languages,
- mid-spectrum languages,
- wide-spectrum languages.

In this monograph, the main focus will be upon the mid-spectrum and wide-spectrum languages since these are currently reaching a level of maturity at which they can be applied in an industrial context. Furthermore in section 1.5 we shall propose a class of syntactic structures called 'designs' and hence the set D of all designs can be viewed as a language. Yet, D does not fit in the classification programming/specification languages; instead of that, it provides a kind of 'missing link' between these two language-classes because it serves for putting programs and their specifications together in an orderly fashion.

There is a great diversity of special-purpose languages. Because these can be tuned to very specific needs, they are less subject to conflicting requirements than the general-purpose programming languages and the general-purpose specification languages. As examples we have data-base query languages, job-control languages, input languages for parser-generators etc.

1.2.4 The role of tools

In this section we shall briefly discuss the role of *tools*, i.e. computer-programs which are meant to be of help in software engineering. The role of tools is to relieve the software designer from certain dull and error-prone subtasks – as already indicated by the second citation of Turing in Section 1.2.1. Of course only those subtasks can be automated which are completely understood and which are formalised. We list a number of tools.

- Syntax checkers which serve to determine whether an input text belongs to the language for which the checker has been built.
- Type checkers which serve to determine whether an input text is correct according to the type system for which the checker has been built.
- Code generators which serve to perform meaning-preserving translations from a source language to a target language.

- Compilers for programming languages. Typically each compiler contains a syntax checker, a type checker and a code generator.
- Text editors, structure editors and graphics editors which serve to enter and modify programs, specifications and other documents which play a role in software engineering.
- Data-base systems, file systems, and configuration-management systems which provide systematic ways of storing programs, specifications and other documents.
- Mail systems, calendars, news systems etc. which serve to structure the communication between software engineers.
- Text-processing tools and type-setters which serve to produce nice-looking and well-structured documents.
- Interactive program transformation systems which assist the software engineer in transforming specifications into programs using correctness-preserving transformations.
- Theorem provers, proof assistants and proof checkers which assist the software engineer in constructing and checking proofs in a formal system of logic.

The full potential of possibilities has by no means been fully exploited so far. A potential bottleneck is the lack of integration of all these tools [36]. This integration is complicated by the existence of many programming/specification languages and systems of logic. Also the diversity of available computers and operating systems is a complicating factor in practice.

Each tool must be based on a formal system and/or an underlying theory; for example, a syntax checker is based on a particular grammar and the general theory of languages and parsing whereas a type checker is based on a particular version of type theory. In general, the construction of tools is a natural follow-up of language definition or of methodological and theoretical advancement.

1.2.5 Application domain evolution

The characteristics of the process of software development are not the same for all application domains. It is not possible to give an overview of all application domains, but some useful remarks can be made by distinguishing application domains with respect to their maturity. In particular, the degree of achieved formality turns out to be a dominant factor. This section serves to provide a slightly relativising context for the discussions of the preceding sections and it also plays a role in positioning our own work.

We summarise the idea as worked out in [37], where Sikkel and Van Vliet discuss domain-oriented reuse of software. They state that the possibilities for reusing software depend on the maturity of the application domain.

Application domains are subject to a certain development and there are four distinct phases:

- no reuse,
- ad-hoc reuse,
- structured reuse,
- automation of the domain.

It is argued that the introduction and standardisation of useful ideas, concepts and structures starts in the second phase. This introduction is an iterative process which takes quite a lot of time. Only after that is it possible to make the transition to the third phase. Formalisation is viewed as a tool which can help in establishing this transition.

Sikkel and Van Vliet focus attention on this transition from the second phase to the third phase. Indeed, for the first and second phases it is hardly possible to provide useful models of software development, whereas in the third phase the software development itself can proceed in an orderly fashion. Therefore it seems worthwhile to speed-up this transition whenever possible. Just as Sikkel and Van Vliet, we believe in formal specification techniques as a tool for doing this. In the third phase, the software development itself is to a certain extent already amenable to mathematical analysis.

In Section 1.6 we shall discuss certain models of the software development process, which can indeed be viewed as examples of such a mathematical analysis. In the fourth phase the formalisation of the software development process leads to replacing an entire design activity by a simple push-button operation. Furthermore, when this push-button operation has become sufficiently efficient, the application domain will have become quite unproblematic from a software engineering viewpoint.

1.3 Informal models of software development

1.3.1 General

Below we shall summarise some informal models of software development. Most of these informal models of software development are not based on a deep analysis of formal systems, but rather they are based on common sense, social observations and empirical data. Each of these models conveys some useful insight about software development. There is a large variety of such models and quite arbitrarily we selected a few representative models for a brief discussion. No attempt has been made to give a complete overview of this type of models. For an overview we refer to [38] or [39].

1.3.2 Waterfall model

The waterfall model [20], also called the life-cycle model, states that, roughly speaking, the software development process goes through the following phases:

- system feasibility,
- software plans and requirements,
- product design,
- detailed design,
- code,
- integration,
- implementation,
- operations and maintenance.

At the end of each phase a kind of verification or validation takes place and if its outcome is unsatisfactory, a return to a previous phase may take place. This model and variants of it have been used in many projects and these models have a certain value by providing guidance with respect to the order in which a project should carry out its major tasks.

1.3.3 Boehm's spiral model

Boehm [40] states that many software life-cycle models of software development are inappropriate in many situations. He proposes a so-called spiral model of software development and enhancement. This model is an elaboration of the waterfall model of Section 1.3.2. The spiral model adopts a similar sequence of phases as the waterfall model, but the sequence begins with having a prototype. Furthermore, this model has the additional feature that this sequence should be done several times as an iterative process. Every time before the sequence is entered, a risk-analysis must take place. The model can be viewed as a kind of repetitive statement and our own presentation of it below shows it as a **while** ... **do** ... loop.

Boehm has a pictorial notation for it where the total execution sequence is shown as a spiral.

```
while risk < max do
    • prototype,
    • software plans and requirements,
    • ...
    • operations and maintenance.
end
```

Every iteration cycle yields some product which enters the next cycle as a prototype. The idea of risk-analysis gets a lot of attention in this model and it is extensively discussed in [40].

1.3.4 A layered behavioural model

Curtis, Krasner and Iscoe [41] report about a field study of the software design process for large systems. It is argued that human and organisational factors seriously influence the execution of software development tasks. A layered behavioural model of software development is given. It has five layers, each of which has a certain influence on the software development process. The model focuses on the behaviour of the people creating the software, rather than on the software itself. The five layers are as follows:

- individual,
- team,
- project,
- company,
- business milieu.

By means of a field study based on interview techniques the three most salient problems of software development were identified. Each problem is extensively discussed for each of the layers of the model:

- the thin spread of application domain knowledge,
- fluctuating and conflicting requirements,
- communication and coordination problems.

These problems have survived for several decades despite serious efforts to improve software productivity and quality. Software development tools and practices are said to have disappointingly small effects in earlier studies, because they did not improve the most troublesome processes in software development.

The proposition that human and organisational factors affect the execution of software development tasks is convincingly shown in [41]. However there are opportunities to make progress by studying simplified models of software development at a technical level. Also the reported communication problems suggest that it is useful to improve upon the specification techniques employed for the products.

Let us say a few words about the relation of the model and the work on models of software development of this monograph. We do not contribute to the ecological research suggested in [41] nor to the studies of processes such as learning, requirements negotiation, etc. Instead we focus on the software product and its descriptions and on the evolutionary behaviour of these through certain development stages.

1.3.5 Koomen's iteration, learning and detailing model

In [42] the software development model is viewed as a step from a specification S to an implementation I. In general, the implementation is more complex than the specification; this is because of decomposition and the introduction of additional information. Koomen explicitly mentions the knowledge K which is needed by the developer to perform the step $S \rightarrow I$. This is denoted as follows:

$$S \xrightarrow[K]{} I$$

It is argued that the developer must acquire the knowledge K by learning, which in most cases takes place by means of iterations and making errors. Furthermore the transformation process is not just one step, but at another level this step consists of iterations and recursions.

Also in [42] a distinction is made which in our opinion is quite useful, viz. the distinction between an *idealised design process* (IDP) and an *actual design process* (ADP). It seems worthwhile to study IDPs and even to formalise them, but because of the errors which occur during the learning proces, it is inevitable that there is a difference between ADP and IDP.

Before setting-up our own line of development aiming at certain formal models of software development, let us explicitly mention some useful ideas that can be taken from the above informal models and that also act as ingredients of the formal models. From the waterfall and spiral models, we take the use of sequential composition and repetition as structuring mechanisms at process level. From Curtis, Krasner and Iscoe's report we recall that there are many aspects of software development that seem not amenable to formalisation (yet). But we also mention the reported communication problems which we shall translate into the need for having 'redundant' specifications (see Section 1.6). From Koomen's model we adopt the idea that software development is a process of detailing and adding information; this will come back in our top-down and bottom-up models (see Section 1.5) which are strategies of growing (= adding information to) structures called ***designs***.

1.4 Towards formal models of software development

1.4.1 Modularisation

Although the informal models of software development of Sections 1.3.2, 1.3.3, and 1.3.4 convey certain information about software and its construction, they are still somewhat unsatisfactory. We view it as a weakness of these and similar models that they impose a management-oriented and project-oriented structure upon a process of software development which is in fact still poorly understood. Indeed, the situation is even worse in practice, for often the software itself is poorly understood.

We shall study models of software development in this monograph, but we want to start from a position where we can understand the structure of the software. Then we take this structure as a starting point for making meaningful statements about models of software development. Our main mental tool will be that of formalisation. We begin with having a look at modularisation, since this seems to be the right way ahead to impose meaningful structure upon the software itself.

To get started, it is sufficient to have an informal understanding of the question *what is a module?*. A module is a piece of software which more or less constitutes a coherent unit. One may think of it as a bunch of procedures which together provide a certain functionality; one may also think of it as an abstract data type.

Let us assume that modules can be fitted together to form larger modules by composition mechanisms such as **import** and that it is possible to associate names to modules. Then it becomes possible to represent the module structure of a software product as a directed acyclic graph (DAG) where the nodes are modules and where the edges correspond with the 'is part of' relation.

For example, consider a module structure which is as follows:

- **let** m1 := **module** ... **end**;
- **let** m2 := **module** ... **end**;
- **let** m3 := **import** m1 **into module** ... **end**;
- **let** m4 := **import** m1, m2 **into module** ... **end**;
- **let** m5 := **import** m2, m3, m4 **into module** ... **end**;

This is represented by the figure below, where an arrow indicates that one module is part of another module.

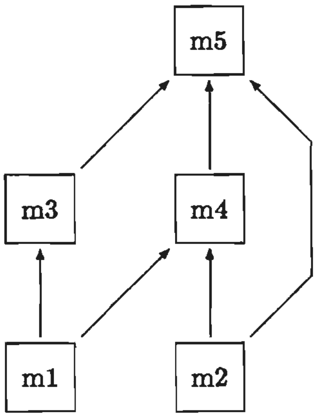

Fig 1.1. Example of DAG module structure.

To have a well-chosen module structure in a software product is the key to avoiding many problems in later stages of the software development process. For a discussion of the criteria for choosing a module structure, we refer to [43]. Although DAG structures as sketched above are attractive because of their simplicity, they are in practice hardly adequate for describing the structure of real software systems. This is because they can not cope with parameterisation (generic modules) and specification ('redundant' modules for specification purposes only).

1.4.2 Parameterisation

Parameterisation of modules is a technique for improving the reusability of modules. Software reuse is currently a hot topic in software engineering literature [44] and it is important indeed, for at least the following reasons.

- It is hoped that more frequent reuse of software will lead to higher productivity of software developers and to significant cost reductions.
- Reuse of modules within one and the same design may lead to significant reductions with respect to the complexity of the design.
- Frequently used software packages could be better certificated for proper construction and operation. For these packages, it is worthwhile to put more effort in obtaining optimal results.

Parameterisation comes from the insight that often modules could be reused, provided they have suitable parameters. E.g. the Ada language provides a mechanism to indicate that a package (= module) is *generic*, which means that the package has one or more parameters. Both types and functions can occur as parameters. We quote Sommerville and Morrison [21].

> One classic example of how generics are useful is in sorting procedures. In most languages two different procedures are required to sort an array of integers and an array of real numbers. Indeed, the situation is much worse than that since we require a different sort procedure for every different type.

When a generic package is used, the actual parameters must be chosen and the substitution of the actual parameters for the formal parameters is supposed to be done at compile time.

Reusability of modules can be improved by providing them with parameters but one can conceive of a large number of options for such a parameter mechanism. Let us have a look at two approaches, viz. the parameterisation of the algebraic specification languages CLEAR and ASL.

In CLEAR, the parameterisation is uniform in terms of modules: one can parameterise a module with respect to a parameter (x say) ranging over modules. In the design of CLEAR [50] it has been recognised that the formal parameters of modules must satisfy certain requirements before an instantiation can take place. There is no full λ-calculus, but there is a special construct called *theory procedure*. For example in CLEAR one writes

$$\textbf{procedure}\ \mathit{Sorting}(x : R) = M$$

to introduce a parameterised module named *Sorting*. *M* is the actual specification of sorting which of course must be parameterised with respect to the sort of things to be sorted and the relevant ordering relation. The latter sort and relation are described in the parameter restriction *R*. The colon is somewhat misleading since it does not refer to a classical (irreflexive) typing relation but to the implementation relation.

In ASL [52] there is an explicit lambda notation for parameterisation, but the connection with the formal implementation relation is not made. In ASL one writes

$$\lambda\ \mathbf{spec}\ x,\ \mathbf{sort}\ s,\ \mathbf{opn}\ p : R.M$$

to parameterise the module *M* with respect to a module (**spec**) x which contains a sort s and an operation p. Again R is a parameter restriction, but in ASL it can be either a module or a Boolean expression.

It is not obvious how to reflect parameterisation and instantiation of modules in the DAG structures discussed before, which means that actually the DAG approach is somewhat too naive in practice. In Section 1.5 improved design structures and a parameterisation mechanism are proposed.

1.4.3 Simple top-down and bottom-up models

When the DAG model for the modular software products applies, this gives an opportunity to impose structure upon the development process. In this way one easily arrives at simple versions of top-down and bottom-up models and these are nothing but ways of letting the DAG grow.

We depict a few states from a typical bottom-up development process.

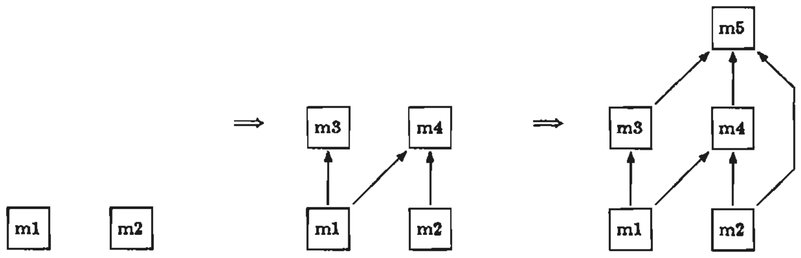

Fig 1.2. The simple bottom-up model.

The three distinguishing properties of this simple bottom-up development model are the following.

- Initially there are one or more unrelated modules. Let us call these the *machine* modules.
- Modules are only added, and never removed.
- All modules are always constructed in terms of the machine modules – again either directly or indirectly.

We depict a few states from a typical top-down development process.

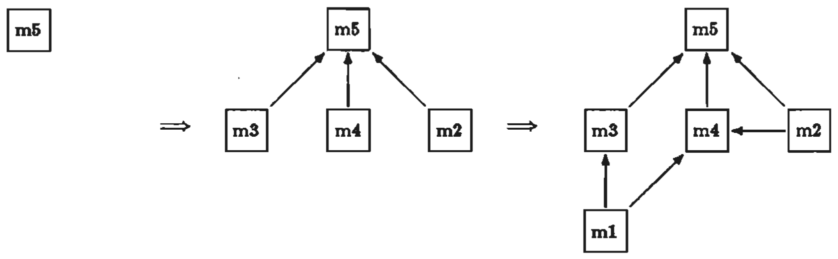

Fig 1.3. The simple top-down model.

The three distinguishing properties of this simple top-down development model are the following.

- Initially there are one or more unrelated modules. Let us call these the *system* modules.
- Modules are only added, and never removed.
- All modules are always either directly or indirectly 'part of' at least one system module.

It is not hard to formalise the DAG structure and the simple top-down and bottom-up models discussed above[1]. Similar discussions are presented in [46] and [47]. The approach can also be pushed further by including middle-out and outside-in models.

However, the approach is not entirely satisfactory, mainly because the need for specifications is not taken into account. Especially the simple top-down

[1]It is interesting to compare this with M. Feather's [45] view of growing specifications as a process of applying elaborations e_step$_1$, e_step$_2$ etc. upon an initial specification m_0. Feather explores the combination of parallel elaborations e_step$_1$(m_0) and e_step$_2$(m_0) say, which e.g. for independent steps could mean to take e_step$_2$(e_step$_1$(m_0)).

approach can hardly work in practice, because it is problematic to use a non-existing module unless a sufficiently precise specification of it is available. This defect becomes even more clear when we aim at *reuse* of modules. Parameterisation is a mechanism for improving module reusability, but at the same time it leads to an increasing need for specifications. Even when the functionality of a reusable module is sufficiently known, there is still a need to have clear (formal) specifications of the parameters and of their effect upon module behaviour.

The simple top-down and bottom-up models are too poor to describe this, but there is a clear route for improving this situation. The main extension is to associate specifications with modules. In order to make this precise, a formalisation of the resulting design structures (= structure of modules + specifications) will be needed. This is the subject of Chapter 2, summarised in Section 1.5. After that we shall be in a better position to have a second look at the dynamic aspects of the process of software development and this is the subject of Chapter 3, summarised in Section 1.6.

1.5 A formalisation of design structures

This section is just a very informal and incomplete introduction to the contents of Chapter 2. The current section is meant to fit logically in the main lines of thought presented in the current chapter, but the detailed technical work involved is done in Chapter 2.

Chapter 2 begins with a formalisation of the module structure of software products. A mathematical model is introduced, which covers from an abstract point of view several distinct languages. According to the model, modules are nothing but *terms* and the ways of fitting modules together are *algebraic operators* in the language of an algebraic system.

This algebraic approach to module composition is investigated by several authors. For example, in Module Algebra [48] modules are terms consisting of module constants/variables and the operators Σ (the visible signature of a module), . (renaming of a module), T (conversion of a signature to a module without axioms), + (combination/union of modules), and $\Box$ (restriction of the visible signature of a module). The ACT ONE MOD language [49] provides algebraic operators for basic module specification, extension, renaming, union, composition and actualisation. The CLEAR language [50] provides algebraic operators for basic module specification (either with loose semantics or initial-algebra semantics), enrichment and derivation (= hiding + renaming). Closely related to Module Algebra is the Class Algebra underlying the COLD-K design language [8], the main difference from Module

Algebra being related to the fact that in COLD-K one can describe systems having a *state.*

In Chapter 2, there is no fundamental difference between a specification module and an implementation module. The only difference is a matter of *role* and this is formalised by assuming a binary relation $\sqsubseteq$ on modules where

$$M_1 \sqsubseteq M_2$$

means that M_1 is an implementation of M_2. This relation is required to be a preorder (= reflexive + transitive). Starting from this simple model, a λ-calculus approach to parameterisation is developed in Chapter 2. However, unlike classical (untyped) λ-calculus, there are restrictions to the actual parameters which may be provided for a parameterised term. The resulting calculus is called $\lambda\pi$-calculus. In classical λ-calculus [51] one would write $\lambda x.M$ to indicate that x is a formal parameter, which may occur in M. Instead of that, in $\lambda\pi$-calculus we get

$$\lambda x \sqsubseteq R.M$$

where R is a so-called parameter restriction. The calculus is formalised by means of a collection of rules, where the most important rule is called (π), because it is a kind of *partial* version of the well-known rule of classical λ-calculus, which is called (β). Somewhat simplified, the rule (π) is as follows:

$$(\lambda x \sqsubseteq R.M)A \to M[x := A], \qquad \text{provided } A \sqsubseteq R.$$

This calculus can be put on top of an arbitrary algebraic system with preorder, which means that it works for any formalism with an algebraic approach to module composition.

It is interesting to compare this $\lambda\pi$-calculus with the parameterisation of CLEAR and ASL as mentioned in Section 1.4.2. CLEAR does not provide a set of rules as in $\lambda\pi$-calculus but instead of that the parameterised specifications get a semantics directly in terms of mappings from algebras to algebras. Just as CLEAR, ASL does not provide a set of rules and the parameterised specifications get a semantics in terms of mappings from algebras to algebras. Furthermore the ASL mechanism is not uniformly in terms of modules: one can parameterise a module with respect to one or more sorts and/or operations.

The $\lambda\pi$-calculus can be provided with a simple system of types and then it is shown to have the Church-Rosser property (also called confluence) and the strong normalisation property.

Using $\lambda\pi$-calculus, Chapter 2 describes a formalisation of design structures by introducing the concept of a ***design***. Intuitively, a design is a (possibly large) hierarchically structured and component-wise specified software system. Precisely the fact that a design is component-wise specified implies that issues of information-hiding arise, which cannot adequately be dealt with by just the algebraic approach to module composition alone; as it turns out, they can however be dealt with by the $\lambda\pi$-calculus.

The designs of Chapter 2 are related to the so-called books in Automath [54] and we shall sketch the relation here using De Bruijn's terminology – which is quite recent. Every lambda term can be viewed as a tree, having nodes of two kinds.

- Application (a) nodes, where a parameterised term gets an argument,
- Typing nodes (t), which are just λ-abstractions. Let us adopt this term, though formally the parameter restrictions of $\lambda\pi$-calculus are not types.

Now one can organise lambda terms so that they become a special kind of slanted trees, built up from typing-with-immediate-application pairs (*at* pairs) and single typing nodes (t nodes). Hence the trees are named *at&t* systems. This is precisely the way Automath books are organised and it can be used for coding and type-checking mathematics. De Bruijn noted this 'books-as-lambda-terms' analogy.

Now it turns out that *at&t* systems in $\lambda\pi$-calculus can be used for giving a meaning to a formal notion of design, corresponding to the intuition sketched above. In this way there is a 'designs-as-lambda-terms' analogy.

A design can be represented by a structure which looks like this:

$$\begin{array}{llll}
x_1 & := & P_1 & \sqsubseteq & Q_1 \\
x_2 & := & P_2 & \sqsubseteq & Q_2 \\
 & \ldots & & & \\
x_n & := & P_n & \sqsubseteq & Q_n \\
\multicolumn{2}{l}{\textbf{system}\ S} & & &
\end{array}$$

where each line is called a ***component***, each P_i is called a ***glass-box description*** (= implementation module) and each Q_i is called a ***black-box description*** (= specification module).

It is optional to omit the glass-box descriptions for certain components and in that case the corresponding P_is are just **prim**, which is a dummy place-holder. Such components are called ***primitive***. Primitive components act as the parameters of the design in which they occur; typically something will be filled-in for the **prim**s in a later stage of the development.

In Chapter 2 two notions of correctness for designs are defined, called *glass-box correctness* and *black-box correctness*. Both forms of correctness are very important from a methodological point of view. Glass-box correctness is based on the principle that all details of the implementation modules may be known when using these modules. No information-hiding is required.

Black-box correctness is based on the principle that no details of the implementation modules may be known when using these modules. Instead of that, the user of a module must rely exclusively on specifications – called black-box descriptions in our terminology. The principle of black-box correctness is probably already old and has for example been expressed in [55].

> We feel it is essential that the user of an interface should not have to know anything about the details of the implementation. In particular, the fact that this interface may be formed by a single body or a set of several bodies should not make any difference.

However, to the best of our knowledge, it has never been formalised adequately and analysed with the techniques of mathematical logic, as in Chapter 2.

The definitions of these correctness notions provide a starting point for a systematic study of *transformational development*. In particular, a design is built up from components and the correctness of a complete design follows from the correctness of its components. In Chapter 2 we investigate the precise conditions so that when modifying *one* component, the correctness of the resulting design follows from the correctness of the modified component. Black-box correctness gives rise to a certain locality principle with respect to modifications of implementations. Such modifications of implementations occur very frequently in practice, typically because of requirements to achieve certain efficiency improvements. For large designs, a complete re-verification of the (black-box) correctness is out of the question in practice, and as a consequence it is of key importance to have locality principles. In Chapter 2 a mathematical basis is provided for this and a number of locality principles and a number of pitfalls are formulated and demonstrated in a very precise way.

In Chapter 2 it is shown that several interesting properties of designs have simple analogons in the $\lambda\pi$-calculus. The most striking analogy is that black-box correctness holds in a design if and only if the corresponding lambda term in $\lambda\pi$-calculus can be reduced so that no candidate redex remains uncontracted and the contractions are performed from right to left. The new and innovative aspect of these analogies is that reduction strategies of lambda terms are used to characterise methodological notions.

It is important to note that in this monograph, both the design structures and a number of methodological notions are defined formally and are analysed using techniques from mathematical logic. This is a great step forward with respect to the usual informal and intuitive approach to the same methodological notions (as e.g. in the above quotation from [55]).

1.6 Correctness-preserving transformations of designs

This section is an overview of the contents of Chapter 3. In view of the achievements mentioned in Section 1.5 we are now in a better position to continue the line of thought presented in Section 1.4. We aim at a precise description of the top-down and bottom-up models of software development, but now we consider them as ways of growing ***designs*** rather than just DAGs.

In order to have a systematic approach, Chapter 3 begins with an examination of algebraic operations at the level of designs. So given two designs d_1 and d_2, we investigate the mechanisms to combine these to another design, d_3 say. There are two operations of this type:

- a binary operation $*$ to be used for combining unrelated designs,
- a binary operation $\circ$ which can be used for combining designs where the system provided by one design is plugged into the list of assumed primitives of another design.

Of course the question arises under which circumstances these operations preserve black-box correctness. This leads to a formal notion called *validation.* As it turns out, the question of preservation of black-box correctness can be answered satisfactorily. Again this question involves non-trivial issues of information hiding, requiring an analysis in the tradition of mathematical logic, using results from Chapter 2.

After that, a formalisation of the top-down model and the bottom-up model is undertaken, employing techniques which are taken from the product level: first the pre- and postconditions are made explicit and after that an iteration construct is introduced based on an invariant assertion. In particular, the top-down model is based on an invariant assertion TD_INV which asserts that the current design is black-box correct and that all components play a role in the system of the design. In Chapter 3 we describe the models as ***design-programs*** and by way of example we show a part of the top-down design-program, which is called td. The input-parameters of td are denoted as d_b and d_t which – roughly speaking – are the external interfaces of the design to be constructed. Below, *d* corresponds with a variable design (the 'current design') which is modified in a step-wise manner,

```
td := technique $d_b, d_t$
def  $d := d_t$;
     while not bot($d$) = $d_b$ do $d$ := td_step($d$); od;
     $d$.
```

This td_step is such that it preserves TD_INV. The design-development language used to express the design-programs has a well-defined syntax and semantics which is given in an appendix of Chapter 3. For the formal details we refer to Chapter 3.

It is remarkable that the top-down and bottom-up models of the development process arise in a systematic manner by applying (at the design-program level) an approach which comes from the field of classical sequential programming. The possibilities for deriving invariants and design-programs describing design creation by this approach have by no means been exhausted in Chapter 3. It certainly is interesting to investigate other possibilities.

The fact that it is possible to split designs and reassemble them again makes it possible to discuss models of the development process where two (or more) developers each operate on a part of a design so that when each of them has finished his part, their results are fitted together to yield a new design again, which furthermore is black-box correct.

The formal notion of validation makes it possible to construct also several simple models of design evolution. Summarising, Chapter 3 provides three types of models of the software development process:

- models of design creation,
- models of design partition,
- models of design evolution.

In each case the models are very simple – at least from the viewpoint of practical industrial applications. But they are formal and they are based on a non-trivial approach to modularisation and information hiding and in that sense they constitute a step forward with respect to the DAG-growing models of Section 1.4.3. Also, it is very important that these models are characterised formally and that now we know very precisely the invariant assertions on which the top-down and bottom-up design programs are based. The fact that our models are formal can be viewed as an advantage over informal and semi-formal process models, as in Section 1.3.

The important question of course is whether these models can be made useful for the practice of developing non-trivial software products. Actually, a very similar question applies to $\lambda\pi$-calculus and the formal notion of ***design*** of Chapter 2. Turning this theory into practice is a large undertaking of an engineering nature. This has been undertaken indeed, viz. in the context of a 'project' around the COLD-K language. This project aimed (and aims) at achieving a significant development in formal specification techniques and at industrialising the results. Clearly this is a task involving many people, the author being just one of them. At the present time not much tool-support related to the results of Chapters 2 and 3 is available, but as argued in Section 1.2.4, the construction of these tools may be a natural follow-up of the methodological and theoretical results of Chapter 2 and 3. Presently only the well-formedness condition 'wf' on designs can be checked, but one can imagine more sophisticated checks. Also a tool to list all proof-obligations related to the black-box correctness of a given design could be useful.

Next we would like to relate and compare the achievements from Chapter 3 with another, somewhat similar approach, viz. L. Williams' approach [56]. In [56] a software process model (SPM) describes software development as a sequence of *activities*. An activity is defined as a 4-tuple consisting of a set of *preconditions*, an *action*, a set of *postconditions*, and a set of *messages*. Williams describes by means of a kind of grammar which sequence of activities are possible for a certain approach. In particular, his design programs (to use our own terminology) take the shape of regular expressions with sequential composition, $\vee$ (choice), Δ (interleaved parallelism) and $*$ (repetition). Using this as a description technique, he gives design programs for several models, such as Boehm's spiral model, Lehman's contractual model and the JSD method.

Although at the process level a certain formal machinery is used (regular expressions), at a lower level the activities (forming the alphabet for the regular expressions) are described informally and do not refer to product artifacts with a formal definition. When we compare this with the process models (design-programs) of Chapter 3, we see that the formality of [56] is lacking a basis, which *is* present for our approach – viz. in Chapter 2.

It is also interesting to compare our design programs with the DEVA approach of [29]. DEVA can be used for (a.o.) describing functions, statements, programs, contexts, modules, etc. and also the transformations and tactics operating on these. The design-development language of Chapter 3 is of a much less general-purpose nature and the design programs of Chapter 3 describe strategies for growing and combining the design structures of Chapter 2. The kernel of DEVA is an applicative language whereas we employ an imperative design-development language. Probably the most important difference is that DEVA design programs can (at least in principle) be executed automatically, whereas the semantics of our design programs is relational to reflect the phenomenon of human 'creative freedom' and there is hardly any sense in letting a computer evaluate the top-down design program, for example.

The work presented in Section 1.5 (which is a summary of Chapter 2) and Section 1.6 (which is a summary of Chapter 3) has served as a contribution to the design of COLD-K. As a result, the parameterisation of COLD-K modules is based on $\lambda\pi$-calculus. Furthermore, COLD-K provides constructs for components and designs, based on the concepts developed in Chapter 2 and Chapter 3.

The next two sections (1.7 and 1.8) are about formal specification techniques and they discuss several specification languages, with a special emphasis on COLD-K. Section 1.8 should be viewed as a kind of **import** at the level of this monograph: we import COLD-K because it is used in Section 1.9 (which is a summary of Chapter 4) and Section 1.10 (which is a summary of Chapter 5). Chapters 4 and 5 illustrate the applicability and usefulness of the theory developed in Chapter 2 and Chapter 3. They contain an example of a large formal specification and a subsequent development process based on the formal models of software development.

1.7 Formal specification techniques

1.7.1 General

In Section 1.2.3 we have already discussed the role of specification languages. In this section we will have a closer look at a few approaches to formal specifications, each with its own specification language.

The discussion in Section 1.2.3 suggests that in many respects the practical progress in software engineering is *language-driven*: it is relatively hard to introduce new methodological concepts unless these are very concretely available as constructs in the language in use. Probably this is a major motivation behind the need for formal specification languages. In practice it is not enough to have a number of good concepts dealing with the methodology of writing formal specifications: in order to make these concepts transferable to a large audience of potential users, one needs a language as a vehicle. The design of such a language involves many choices and clearly the outcome of these choices will determine the restrictions to be imposed upon the users – who may or may not like that. But this seems by far outweighed by the benefits of having a very concrete, more or less standardised language.

We do not attempt to provide a complete overview of all relevant languages; for that we refer the reader to [57] or [35]. The main purpose of this section is just to mention some specification languages which are relatively close to COLD-K with respect to their aims and their technical contents. In particular this means that we shall restrict ourselves to so-called mid-spectrum and wide-spectrum languages. Quite arbitrarily we have chosen to discuss VDM and Z, but there are other important approaches as well, notably CIP-L and Larch. For a comparison with these we refer to [58].

1.7.2 VDM

VDM is a formal-specification approach which has led to the design of several related specification languages, most of which are often just called VDM. The roots of VDM lie in the formal specification of programming languages (PL/I) and in the early days one would refer to the specification language as the *meta-language*, just to distinguish it from the programming language being specified.

The VDM version Meta-IV [6] (1978) has attracted considerable attention and it has been used to specify many complex systems. Its strong points are higher-order functions, a rich collection of built-in data types and the existence of a large body of pragmatics and examples. The built-in data types include sets, pairs, maps, tuples, trees. Each of these data types comes with a rich collection of operators, most of which are denoted in a mathematical style. E.g. for sets one can use symbols such as $\cup, \cap, \subseteq$ and set comprehension $\{x \in S \mid p(x)\}$. For functions there is even a λ-calculus notation, including a fixed-point operator $\mathbf{Y}$.

In [59] an improvement of the VDM language is described. It reflects some influence of both the achievements in the theory of program correctness and the work on algebraic specifications. There is an explicit treatment of the problem of partial functions. This topic arises naturally in many practical situations and it is closely related to the idea of associating *preconditions* with functions. Jones uses a pre- and postcondition style for specifying functions. The following toy-example of a partial function has been taken from [59] (p. 74).

$$\begin{array}{l} \mathit{subp}\,(i : \mathrm{N}, j : \mathrm{N})\; r : \mathrm{N} \\ \text{pre } j \leq i \\ \text{post } r + j = i \end{array}$$

The problem of partial functions is solved by the introduction of the logic of partial functions (LPF). In this way it becomes possible to treat the set of truth-values as any set and to view a predicate P as a Boolean-valued function. This leads to the situation that $P(x)$ fails to denote a Boolean value when x does not satisfy the precondition of P. In that case neither $P(x) = \text{true}$ nor $P(x) = \text{false}$ holds.

VDM is based on set-theory which is very flexible; one can freely construct subsets of given sets and each of these subsets can occur again as the domain or range of a function. This implies that there is no concept of ***typing*** – except of course that the number of arguments of a function can be viewed as a simple type.

VDM has special constructs for so-called ***operations*** which are similar to functions, except for the fact that operations can have side-effects. The following example of an operation has been taken from [59] (p. 86).

$$\begin{array}{l} LOAD\ (i : \mathrm{N}) \\ \text{ext wr } reg : \mathrm{N} \\ \text{post } reg = \dot{i} \end{array}$$

LOAD is introduced as an operation which has write-read access (wr) with respect to a so-called external variable (ext) called *reg*. This variable can contain natural numbers. The availability of these operations and algorithmic definitions for them in addition to the specification techniques for mathematical functions makes VDM into a true mid-spectrum or wide-spectrum language (see [35]). Currently there is still a lot of development going on around VDM and the language has reached a degree of maturity and acceptation exceeding that of most other specification languages. An interesting development is Middelburg's VVSL language [60] where the COLD-K modularisation constructs and $\lambda\pi$-based parameterisation constructs have been put on top of an (enriched) version of VDM.

1.7.3 Z

The specification language Z has emerged from a certain style of specification developed at the Oxford Programming Research Group. Z was first proposed by Abrial and it has been evolving for several years, but the recent book of Spivey has set a rigorous standard now [61]. Z is essentially based on Zermelo-Fraenkel set-theory described in first-order predicate logic. It offers a rich collection of notations for operations on sets such as $\cup, \cap, \subseteq$ and $\{x \in S \mid p(x)\}$. Relations, maps and functions are considered as special kinds of sets and they all have their own special notations. It is possible to introduce new primitive sets and if one does so, this means that the remainder of the specification is parameterised with respect to these primitive sets.

We show an example in Z below. It introduces a so-called schema *subp* which can be interpreted as an operation from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$ satisfying the specification of our earlier VDM example.

subp

$i?, j? : \mathbb{N}$
$r! : \mathbb{N}$

$j? \leq i?$
$r! + j? = i?$

Although formally $i?$, $j?$ and $r!$ are just logical variables, it is a convention that variables ending in ? are inputs and those ending in ! are outputs. The entire double-box construction is called a schema. A schema consists of a signature part, declaring one or more variables and an optional axiom part which serves as a predicate relating these variables.

It is also possible to describe operations having side-effects. This is done by formally introducing the two states as logical variables in the signature part of the schema of the operation.

Let us try the second VDM example as well. The only interesting state component in this example is a register which we formally introduce by a schema having a signature part only.

REG

$reg : \mathbb{N}$

Now we can mention both *REG* and *REG'* in the signature part of the schema of *LOAD* in order to indicate that *LOAD* has write-read access with respect to the register.

By way of convention, the dashed version refers to the *new* state.

```
┌── LOAD ───────────────────────┐
│ REG
│ REG'
│ i? : ℕ
├──────────────
│ reg' = i?
└───────────────────────────────┘
```

Z provides a collection of operators on schemas, some of which are best understood in a rather syntactic way. By way of example we mention two such operators.

- Inclusion: if S is a schema, it is allowed to mention S in the signature part of another schema T. This is semantically equivalent to combining the signature parts of S and T and taking the conjunction of their axiom parts.

- Hiding: in order to hide a variable v from a schema S we can write $S \setminus (v)$. It is semantically equivalent to removing the variable from the signature part and prefixing the axiom part with a quantification $\exists v : \mathcal{T} \cdot$ where $\mathcal{T}$ is the type of v.

Spivey [61] gives a formal definition of Z, using Z as a metalanguage. As it turns out, the meaning of a schema with signature part Σ and axiom part φ is a so-called variety. For (Σ, φ) this variety is the collection of structures (= models or heterogeneous algebras) having the signature Σ and which satisfy φ. So for a Σ-structure $\mathcal{A}$ we have $\mathcal{A} \in \text{meaning}(\Sigma, \varphi) \Leftrightarrow \mathcal{A} \models \varphi$. There is also a concept of typing in Z, but this is set-oriented rather than function-oriented and as a result there is only a limited kind of type checking possible. We shall come back to that later.

1.8 COLD-K

COLD-K is a wide-spectrum specification language developed at the Philips Research Laboratories in Eindhoven within the framework of the Meteor project. The language has been designed mainly by H.B.M. Jonkers, with technical contributions from C.P.J. Koymans, G.R. Renardel de Lavalette, the author, and to a lesser degree also from J.H. Obbink and P.H. Rodenburg. COLD-K is meant to be used throughout several stages of the software development process, including the specification and implementation stages. Actually there is not just one language COLD-K but there is a sequence of subsequent versions, such as COLD-S, COLD-flat, COLD-K, COLD-K-RTL and COLD-1. Amongst these, the COLD-K language plays a special role, since it is a *kernel* language which is meant to serve as a fixed point in the development of the language. There is a mathematically defined syntax and semantics for COLD-K [53]. Other language versions (notably COLD-1) can be defined in terms of COLD-K, just by adding constructs of a purely syntactic nature. In the remainder of this chapter we shall restrict ourselves to COLD-K.

In order to give an impression of COLD-K, let us do the VDM and Z examples in COLD-K. We assume that we have the sort of natural numbers denoted by `Nat` and operations on `Nat` such as `leq` and `add`.

```
FUNC subp: Nat # Nat -> Nat

AXIOM FORALL i:Nat, j:Nat
      ( leq(j,i) => add(subp(i,j),j) = i )
```

The second example involves a register which is introduced as a so-called *variable* function (keyword `VAR`).

```
FUNC reg: -> Nat VAR
```

The LOAD operation becomes a procedure, because of its side-effect. This procedure has modification rights (keyword `MOD`) with respect to the register reg. In COLD-K there is no built-in construct for indicating a postcondition, but the assertion sub-language includes Harel's dynamic logic [27]. This logic is employed in the axiom given below. This axiom should be read as follows: *in all states and for all i, whenever we reach a new state by executing* `LOAD(i)`, *the assertion* `reg = i` *holds in this new state.*

```
PROC LOAD: Nat ->
MOD  reg

AXIOM FORALL i:Nat
      ( [ LOAD(i) ] reg = i )
```

In fact the examples above only show definitions which could occur within a COLD-K scheme (= module) which in turn could be part of an entire design. It is outside the scope of this monograph to give a more or less complete introduction to COLD-K; for that we refer to [8], [53] and [62].

Here we would like to point out a few differences with respect to the versions of VDM and Z as discussed in the sections 1.7.2 and 1.7.3.

First of all the COLD-K solution to the problem of functions being partial differs from the solutions of VDM and Z. The assertion language of COLD-K is based on a special version of typed predicate calculus called MPL_ω [53] which at its turn is derived from Scott's E-logic. It is possible to state that an expression e is defined and this is done by means of the postfix operator ! which can be used to write $e!$. There are rules which relate this notion of definedness to the built-in equality and to the quantifiers **FORALL** and **EXISTS**. The problem of the undefined truth-values vanishes because one of these rules says that a predicate applied to an undefined expression just yields **FALSE**. The point is that predicates are normal mathematical predicates and not Boolean-valued functions.

It is also interesting to note how programming variables and operations with side-effects are described in COLD-K. Programming variables are modelled in COLD-K by having variable functions and variable predicates. The operations which modify them are called ***procedures*** and there are mechanisms that regulate the modification rights of procedures with respect to variables. There are several ways of defining procedures: they can be given algorithmically or they can be described axiomatically because dynamic logic is built in to the assertion sub-language. This is a difference with respect to VDM, where there are special constructs with keywords pre and post, rather than a more powerful logic. In Z there is nothing special about states and side-effects at all: most of this is dealt with by having a certain pragmatic style relying on conventions about identifiers such as $i?$, $r!$ and reg'.

The expressions in COLD-K are strongly typed, which has the important advantage that mechanical type-checking is possible. A syntax- and type-checker exists. This is a difference with VDM and Z, since in the latter languages one can just define any set, e.g. $E \triangleq \{i : \mathbb{N} \mid \exists j : \mathbb{N} \cdot i = 2 \times j\}$ introduces $E \subset \mathbb{N}$ as the set of even numbers. Now one could introduce a function $f : E \to E$ and then to find out that $f(77)$ is wrong requires reasoning about the assertion $\exists j : \mathbb{N} \cdot i = 2 \times j$ which was used to define E. In Z there is a rather coarse concept of type checking which cannot detect the problem with $f(77)$ but which could at least find out that $f(-3)$ is nonsense.

Probably the most important difference between COLD-K on the one hand and VDM and Z on the other hand is the presence of powerful constructs for modularisation, parameterisation and designs in COLD-K. The modularisation of COLD-K is based on an algebraic approach to modularisation. The basic module construct is called a ***class scheme*** and it consists of a list of definitions of sorts, functions, predicate, procedures and axioms. These can be combined into larger schemes (= modules) by means of import, export, renaming and by using names for schemes. Furthermore COLD-K provides for parameterisation of schemes over schemes. The work presented in Chapters 2 and 3 has served as a contribution to the design of COLD-K in the sense that the parameterisation of COLD-K schemes is based on $\lambda\pi$-calculus and that COLD-K provides constructs for components and designs. In COLD-K one writes `LAMBDA x : ITEM OF BODY`, corresponding with a $\lambda\pi$-calculus term $\lambda x \sqsubseteq$ `ITEM.BODY`, and one writes `APPLY PARAM TO ARG` corresponding with a $\lambda\pi$-calculus term `(PARAM ARG)`. The top-level construct which can be denoted in COLD-K is a *design*. In Jones' VDM there is nothing comparable to the COLD-K modularisation, parameterisation and designs. In Z there are schemas comparable to the COLD-K schemes. The parameterisation of Z is just over sets rather than over entire schemes, whereas there are no such things as components and designs in Z.

Another difference between COLD-K and VDM/Z is the absence of fixed built-in notions like Booleans, natural number, tuples, sequences, sets and maps in COLD-K. Instead of that, the language provides facilities to *define* them. In practice such data types are taken from standard library – written *in* the language, cf. Appendix B of Chapter 4.

COLD-K can be used for expressing programs and specifications, but it is not meant for denoting proofs. Of course, languages to denote proofs exist, ranging from plain English for convincing arguments to λ-typed λ-calculus for automated checking – as in the Automath approach. Yet COLD-K gives rise to proof obligations; more precisely, the principle of black-box correctness requires for each component `COMP X : X_SPEC := X_IMPL` that $\Gamma \vdash$ `X_IMPL` $\sqsubseteq$ `X_SPEC` for a certain well-defined Γ. In the editor case study of Chapter 5, some proof obligations are treated informally, whereas many others are tacitly dealt with.

After the intermezzo about formal specification techniques of Sections 1.7 and 1.8, we can proceed with the main line of this monograph, viz. the formalisation of design structures (= structure of modules + specifications) and the models of software development based on that. We are now in a position where we can indeed attach 'redundant' formal specifications to modules, viz. by using VDM, Z or COLD-K. Using one of these, we shall present a case study concerning a text editor, which has been developed as an application and illustration of the results of Chapter 2 and Chapter 3.

1.9 Formal specification of a text editor

This section is an overview of Chapter 4 which concerns the formal specification of a multi-buffer and display-oriented text editor. This specification illustrates the use of COLD-K as a specification technique. In particular, the editor specification is modularised and certain parts of it are generic and as such it illustrates the COLD-K algebraic approach to module composition and the $\lambda\pi$-calculus. There is no real development process yet in Chapter 4, except for the bottom-up construction of the formal specification, where we mean bottom-up in the DAG sense of Section 1.4.3.

The specification of the editor consists of four parts:

- a library,
- an application domain formalisation: texts and operations on texts,
- models of the available primitives, such as a file system and a video display unit,
- specification of the actual editor including features such as buffer management and keybinding.

When the construction of this specification began, there was already a small library of data type specifications, containing Booleans, natural numbers, sets, sequences, bags etc. Most of the library had been constructed earlier, mainly by H.B.M. Jonkers and the author. The library is given in an appendix of Chapter 4.

The need to formalise the application domain is very typical: almost every specific application domain has its own concepts, notations, conventions and jargon. Before any system specification can be written down, these concepts must be introduced formally. As argued already in Section 1.2.4, each tool must be based on a formal system and/or an underlying theory and for the text editor – considered as a tool – the underlying theory is a formalisation of 'text' and operations on texts. For example, in Chapter 4 there is a sort of texts, denoted as `Text` where each text consist of a sequence of lines. Consider the following text:

```
first line of text
second line
```

Fig 1.4. Example of text.

which is viewed as a sequence of lines where the first line has 18 characters viz. "`first line of text`" and where the second line has 11 characters, viz.

`"second line"`. An equally valid, but somewhat more abstract approach is to model texts by focusing on their 'contour' only. In this approach the above text is modelled by just the sequence $\langle 18, 11 \rangle$ alone. Of course there is a kind of forgetful mapping from the first model to the second model in the sense that the information conveyed by the actual characters in the text is lost in the second model. Such approaches at distinct levels of abstraction play a role in the formalisation of the notion of text.

The sort of lines is called `Line` and each line consists of a sequence of characters (sort `Char`). The relevant sorts can be denoted in COLD-K as

```
SORT Char
SORT Line
SORT Text
```

and there are operations to select a line from a text and a character from a line.

```
FUNC sel: Text # Nat -> Line
FUNC sel: Line # Nat -> Char
```

Starting from this very simple model, a rich collection of operations on texts is defined, including a variety of cut and paste operations. For example, there is a `paste` operation such that `paste`(t, u, k, l) means to take a text t and to insert another text u into it immediately before the position with given coordinates (k, l). As a kind of inverse of paste there is an operation cut such that `cut`(t, k, l, i, j) means taking a text t and cutting out the piece of text beginning at position (k, l) and ending at position (i, j). It yields a pair (t_1, t_2) where t_1 = 'remaining text' and t_2 = 'deleted text'.

```
FUNC paste: Text # Text # Nat # Nat -> Text
FUNC cut: Text # Nat # Nat # Nat # Nat -> Text # Text
```

The following picture may give an idea of both `cut` and `paste`.

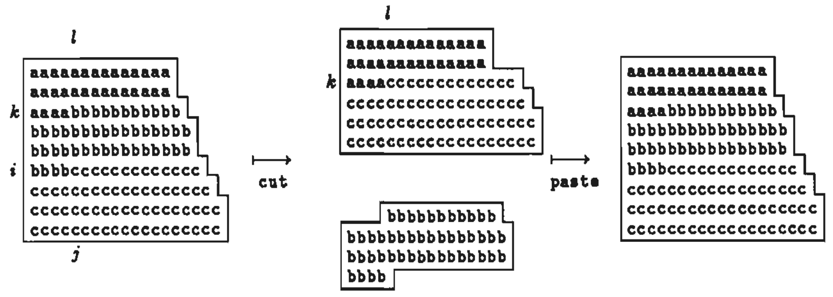

Fig 1.5. Cut and paste operations on text.

Algebraic laws for the operations are investigated and an example is the equation $\texttt{paste}(\texttt{cut}(t,k,l,i,j),k,l) = t$, which holds conditionally. It is shown that this helps in establishing a set of well-understood operations with useful notations. In this way an elegant and useful formalisation of the application domain is obtained.

To interface the editor with its environment it is necessary to model the available primitives such as a file system and a video display unit. By way of example we shall have a closer look at the video display unit. Its state space is spanned by two variable functions

```
FUNC screen: -> Text VAR
FUNC cursor: -> Nat # Nat VAR
```

where `Nat` refers to the natural numbers employed to represent the vertical and horizontal co-ordinates of the cursor. All display operations are described by their effect on either `screen` or `cursor` or both. For example `PROC nl: -> MOD screen, cursor` serves for sending a new-line command to the display, thereby possibly modifying the screen and the cursor.

To describe the operation of the actual editor, the notion of 'marked text' is introduced first. A marked text is a composite object that consists of a text and two co-ordinate pairs called mark and dot. The *dot* serves a kind of 'current' location in the text whereas as the *mark* is a marker that can be put on any position in the text. The specification of the editor is based on a variable map from buffer names to marked texts (sometimes also called 'buffers') and on a notion of 'current' marked text.

On several occasions in Chapter 4, the technique of invariants is used to describe essential aspects of the editor. There is a so-called 'text-invariant' that says that for each marked text in the editor, both the dot and mark are positions that exist in the text of that marked text. Another interesting example is the so-called 'window-invariant', which is introduced as a predicate `PRED WI`. It describes the relation between the current marked text on the one hand and the screen and the cursor of the display on the other hand. What `WI` states in formal terms boils down (in informal terminology) to the statement that the window should correspond with a 'look' to the text, if necessary filled with blanks, such that the dot is visible as the cursor. The following picture sketches part of the situation.

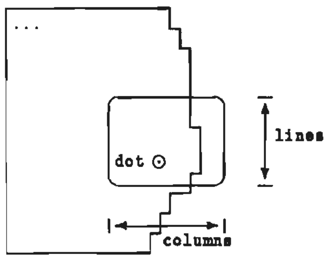

Fig 1.6. The relation between window and text.

The editor supports a complete set of editing commands like insert_character, set_mark, beginning_of_line, backward_character, delete_next_character, search_forward, write_named_file, delete_to_killbuffer etc. The editor specification is completed by a simple keybinding to associate command-invocations with key-strokes.

In fact the specification of the editor itself is only a fragment of the total specification. The relative fractions of the number of lines in each of the four parts mentioned above are about 35%, 20%, 20% and 25% respectively (from these the first 35% are general purpose and usable in any specification case study; the next 20% are reusable within the same application domain, i.e. when 'text' plays a role, whereas the remaining 45% are dedicated to this application).

Chapter 4 covers a number of important aspects of a text editor, although the editor described is relatively poor in its bells and whistles. However it is far from trivial and its functionality makes it a usable editor. It covers several interesting features also present in other text editors and in particular in EMACS. For a more detailed discussion and evaluation we refer to the relevant sections at the end of Chapter 4.

Chapter 4 shows how $\lambda\pi$-calculus and COLD-K can be used as an instrument for the description of a relatively large and complex software system. At the same time it illustrates a number of general-purpose specification techniques and as such, it can be viewed as a contribution to the advancement of formal specification techniques in general. In the context of the main line of thought of this monograph, the role of Chapter 4 is twofold: firstly, it shows a number of examples of parameterised modules, using the $\lambda\pi$ mechanism of Chapter 2. Secondly, the resulting specification serves as a starting point for a systematic development process (using the results of Chapter 3) which is described in Chapter 5.

1.10 Systematic design of a text editor

This section is a short summary of Chapter 5 which is a continuation of the editor case study of Chapter 4. The formal specification is taken as a *specification* and a systematic development process aiming at the actual construction of the editor is undertaken. The top-down model of Chapter 3 is employed. This model turns out to be workable and because the editor design is quite complex, it is shown how the top-down model helps in mastering this complexity. Realistic efficiency considerations and data-reification issues are taken into account. In particular the design is based on the assumption that it is important to economise with respect to memory usage and with respect to communication with the video display unit.

In Chapter 5 there is not just one editor design, but two:

- a design d_{editor} which is mainly concerned with topics such as buffer management, file handling, window management and key binding in terms of a.o. sequences and maps,
- a design d_{basic} which is concerned with efficient implementations of sequences and maps.

These designs can be fitted together with the $\circ$ operation on designs as mentioned in Section 1.6. In this way Chapter 5 illustrates also one of the models of design partition. Chapter 5 focuses on the top-down development of d_{editor}, whereas d_{basic} can be found in an appendix of Chapter 5.

In the development of d_{editor} many classical efficiency considerations concerning memory usage, execution time and communication overhead play a guiding role. The important issue of data reification is addressed on several occasions – e.g. when choosing a representation of the marked texts. Each text buffer is represented as an array with a gap and a group of pointers and co-ordinate pairs. To describe this, the well-known machinery of abstraction functions and representation invariants is employed. The top-down approach is exploited to achieve a separation of concerns on several occasions – e.g. to separate the buffer-management from the window-management. Let us have a closer look at the latter example. When programming the editor operations insert_character, beginning_of_line, backward_character etc. no details of the window-management play a role. This is made possible by a postulated component named `WI_PACKAGE` which is specified to offer a procedure `mod_text_restore` which re-establishes the window-invariant `WI` after an arbitrary modification of the current text buffer. In a later stage of the development process this postulated component is implemented.

The resulting editor, viewed as a product plus its documentation is in a number of aspects quite satisfactory. In particular, the editor design is with respect to most operations, reasonably efficient. It should be noted that the editor design is quite complex, especially because it is a *display-oriented multi-buffer* editor, which means that it is more than just a small toy example.

It can be expected that due to its completeness and its component structure, the editor design will be 'robust' for various forms of design evolution and efficiency improvements. The theory of correctness-preserving transformations on designs of Chapter 2 and Chapter 3 is applicable here: in particular, since the principle of black-box correctness has been adopted, many efficiency improvements can take the shape of black-box correctness preserving glass-box modifications – meaning that there is a 'locality principle' which can yield a significant reduction of the verification task.

The algorithmic COLD-K texts are translated manually to C to get a working editor. The details of this translation process are given in an appendix of Chapter 5.

The editor case study is quite large in view of the fact that it was meant as an example. This is justified by the need to have a non-toy example illustrating how the theoretical concepts of $\lambda\pi$-calculus, components, designs, algebraic operations on designs and formal development models can be turned into real *engineering* concepts. The fact that the example grew large is a direct consequence of the level at which our concepts apply, viz. programming-in-the-large, rather than programming-in-the-small. After reaching the milestone in the evolution of COLD of finishing COLD-K, it was important to have this kind of an exercise before the design of yet another language version started. The example has served already for educational purposes in the COLD-K workshop of the Philips Centre for Software Technology (CST) in November 1988.

In Chapters 4 and 5 we employed the COLD-K language (which certainly was not the only option) and we showed that this is useful as an instrument when constructing software following a systematic approach and employing formal specifications.

For a more detailed discussion and evaluation we refer to the relevant sections at the end of Chapter 5.

In the context of the main line of thought of this monograph, the role of Chapter 5 is twofold: firstly, it shows a number of examples of designs, thereby illustrating the notions of component and design from Chapter 2. Secondly, Chapter 5 is an application of the results from Chapter 3 and it demonstrates the applicability and usefulness of these results.

1.11 Options for future work

In this section we shall mention a number of topics which could be the subject of further research or of further development.

The first topic concerns the definition of a number of useful language features which possibly could be added to wide-spectrum languages such as COLD-K and which could make the task of constructing large designs easier. Amongst these we have higher-order logic, concurrency and – at a more syntactical level – flexible mechanisms for dealing with user-defined operations and binders.

The second topic concerns the precise definition of the implementation relation $\sqsubseteq$ for wide-spectrum languages – and for COLD-K in particular. For the editor case study of Chapter 4 and 5 we adopted a relation based on signature inclusion and theory inclusion. This is certainly not the only possibility and it would be interesting to have a look at other notions of implementation. Furthermore, the precise conditions that guarantee that the module composition operations of the so-called 'class-algebra' of COLD-K are monotonic with respect to $\sqsubseteq$ have not been formulated yet. For an analysis of the problem we refer to [63].

The third topic is the construction of tools supporting the development process based on the instantiation of the approach from Chapters 2 and 3 with a given formalism. We have already given a list of tools in Section 1.2.4 and most of these are relevant when using a wide-spectrum language. We can not give a minimal list of tools which are absolutely necessary, but clearly the attractiveness of a certain formalism is increased by providing a syntax checker, type checker, module library, code generator and a tool to create/manipulate graphical representations.

Finally we mention the topic of practical applications. In view of the fact that formal techniques are meant to be a mental tool, it is not enough just to develop the mathematical foundations of a language and the theoretical models of software development based on them. It is also necessary to build up experience in using formal methods and although it is inevitable that the first case studies are small toy examples, the next step is to scale them up and seriously attack complex systems. Performing large case studies is a time-consuming and resource-consuming activity but there is more than one yield. Case studies yield feedback on the language, the method and the theory and insight in the nature of the problems which are relevant in practice.

By way of a concluding remark we state that the theory of Chapter 2 and Chapter 3 in combination with a particular formalism (such as the COLD-K language) provides a potential starting point for improving the process of software development. It is also clear that this involves much work of an engineering nature. Chapter 4 and Chapter 5 are a contribution to that, but much remains to be done.

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Chapter 2

A Formalisation of Design Structures

2.1 Introduction

This chapter presents a formal theory dealing with the component-wise construction and specification of complex systems, addressing issues of modularisation, parameterisation, abstraction and information hiding. The theory consists of two parts where the first part mainly serves to introduce an extensive formal machinery which is employed in the second part.

The first part of this theory introduces the notion of an algebraic system with associated preorder relation. Furthermore a special version of lambda-calculus is developed which is called $\lambda\pi$-calculus. More precisely, there is an instance of this calculus for every algebraic system with preorder. The most important characteristic of the calculus is that associated with every abstraction (λx say) there is a so-called parameter-restriction.

The motivation for the introduction of these notions is as follows. An algebraic system can be used to model the modular structuring of systems. In particular, we view module-composition mechanisms as operators of an algebraic system – following [1]. The associated preorder (denoted by $\sqsubseteq$) is used to model the so-called implementation-relation – which is an important relationship between modules and their specifications. We do not partition the modules into two kinds (implementation/specification) but we adopt a single-sorted approach where $\sqsubseteq$ is used to model the implementation and specification roles of modules. This allows for a smooth transition from specification to implementation and in particular we can have the situation that $m_1 \sqsubseteq m_2$ and $m_2 \sqsubseteq m_3$. The reflexivity and transitivity properties of this relation are adopted for a-priori and intuitive reasons [2]. The purpose of the $\lambda\pi$-calculus is to describe the introduction of names and the abstraction-step taking place when introducing specifications for yet unknown modules. At the same time this $\lambda\pi$-calculus serves as a parameterisation mechanism for modules (this calculus was used for the *parameterisation* of the design language COLD-K [3], [4]). In this calculus we have abstractions of the form $\lambda x \sqsubseteq R.M(x)$ where R is meant as a restriction upon the possible actual parameters A to which $\lambda x \sqsubseteq R.M(x)$ can be applied. To make the restriction R effective, we adopt a certain mechanism regulating the equalities that can be derived about application-expressions. In particular, there is a *partial* version of the classical rule (β) in the sense that $(\lambda x \sqsubseteq R.M(x))A = M(A)$ can only be concluded when it has been shown that A is an implementation of R, i.e. $A \sqsubseteq R$. This $\lambda\pi$-calculus is a theory where equations play an important role and just as for classical lambda calculus [5], it is useful to study reduction. In particular the reduction relations $\rightarrow$ and $\twoheadrightarrow$ allow for a detailed study of the (modified) rule (β).

The second part of this theory introduces the notion 'design of a system'. The intuition for this is that a design is a hierarchically-structured and component-wise specified software system. We shall define a *black-box description* as the specification part of a component and a *glass-box description* as the implementation part of a component. Although it may be redundant, a black-box description can be useful when it is less complex or easier to read than the glass-box description. Even when the black-box description is not less complex, there may be advantages for having an alternative representation. There is a correctness criterion for components because the two descriptions must be related by the preorder $\sqsubseteq$. We define a *design* as a collection of components, where some components serve as building blocks for other components. A design need not be finished, but it may correspond to an intermediate stage in a development. In particular, a design may contain components which have no glass-box description yet.

There are non-trivial issues of information hiding that arise in connection with designs. This leads to definitions of *black-box correctness* (based on the exclusive use of specifications) and *glass-box correctness* (using implementation knowledge). As it turns out, $\lambda\pi$-calculus can be used to give an interpretation of designs. The reduction relations can be used to give alternative characterisations of black-box correctness and glass-box correctness.

The second part of this theory provides a starting point for a systematic study of transformational development. In particular, a design is built-up from components and the correctness of a complete design follows from the correctness of its components. We shall investigate the precise conditions such that when modifying *one* component, the correctness of the resulting design follows from the correctness of the modified component. As it turns out, this depends both on the chosen notion of correctness and on the question whether an implementation module or a specification module is modified.

In view of the above introduction, we adopt the following structure for this chapter. In Section 2.2 we shall introduce *algebraic systems with preorder.* In Section 2.3 we introduce the $\lambda\pi$-calculus and we study some of its properties. This covers the first part of the theory. For a given algebraic system with preorder, we can define the notion of a *component.* Furthermore we shall define what it means that a component is *correct* – in a given context. This will be done in Section 2.4. Using the notions of component and correctness, we can define the notion of a *design* and its correctness. In Section 2.5 we introduce designs and we study their properties. Also in Section 2.5 we study a certain class of correctness-preserving transformations of designs. This covers the second part of the theory.

In Chapter 3 the study of correctness-preserving transformations of designs will be continued in a somewhat more general setting and with a focus on the dynamic aspects of the software development process.

2.2 Algebraic systems with preorder

2.2.1 Motivation

We adopt the following minimal set of notions to begin the development of our theory.

- a set of *constructs* which are the objects to be created and/or used by software developers. For each construct there are two possibilities. A construct can be given as *primitive*, by which we mean that the developer can not or should not investigate how it was constructed. So a construct is primitive if it can be fitted together from zero components. Alternatively, a construct can be built by fitting together a finite nonzero number of constructs.
- *composition mechanisms* (or 'combinators'), by which we mean ways of combining existing constructs into new constructs.
- an *implementation relation*, corresponding to the possibility that one construct is viewed as an implementation of another construct.

As an example of these notions we consider (software) modules. A primitive module can contain several sort definitions, function definitions, procedure definitions and axioms. At this level we have module composition mechanisms such as *import*. To illustrate the implementation relation, we consider two modules m_1 and m_2 where m_1 is an algorithmic description which is characterised axiomatically by m_2; then the pair (m_1, m_2) is in the implementation relation. The motivation for the choice of these notions is as follows. First of all it is clear that without constructs or without composition mechanisms no constructive activity is possible at all. We adopt an implementation relation because it seems a very general starting point for a discussion of abstraction and information hiding. It is clear that these notions are essential for applying the paradigm of hierarchical decomposition. The latter paradigm appears frequently in literature dealing with the manageability of the software development process [6] and it certainly is of great value.

The notions of constructs, composition mechanisms and an implementation relation can be formalised by introducing the mathematical concept of an *algebraic system*. Constructs correspond to terms and composition mechanisms to function symbols. This approach is also chosen in [1], where a so-called 'module-algebra' is studied. We shall choose one kind of algebraic system which exhibits precisely the kind of implementation relation in which we are interested. It would be nice if we could give one single formal definition of some kind of mathematical structure (which we might call a *construction system*) in which we formalise these notions once and for all. However, one can imagine several kinds of implementation relations and they differ strongly, both in their mathematical properties and in their role. Therefore it seems better to focus on one kind of implementation relation first.

At this point we must be more precise about the nature of the implementation relation. Therefore we must be specific about the kind of constructs we have in mind. As an important case, we think of constructs at the level of (software) modules. A module can be used as a specification but certain modules can also be used as executable implementations. If m_1 and m_2 are modules such that – viewing m_1 as an implementation and viewing m_2 as a specification – m_1 implements m_2, then we say that the pair (m_1, m_2) is in the implementation relation. We can make a list of properties which from an intuitive point of view should hold for this implementation relation [2].

- Each module implements itself, i.e. the implementation relation should be reflexive.
- If m_1 implements m_2 and m_2 implements m_3, then m_1 should implement m_3, i.e. the implementation relation should be transitive.
- Suppose that m_1 implements m_2. Now if m_2 occurs as a subconstruct in a larger construct, $m(m_2)$ say, then $m(m_1)$ – i.e. the construct m containing m_1 instead of m_2 – should implement $m(m_2)$. It follows that we

shall prefer composition mechanisms which are monotonic with respect to the implementation relation. Alternative terms for 'monotonic' are: 'compatible' and 'compositional'.

There exist algebraic systems for which a precise definition of the implementation relation has been given. In [7] it has been defined what it means that m_1 implements m_2 for the case where m_1 is a sequential program and m_2 is a pair (φ, ψ) consisting of a precondition and a postcondition (the usual notation for this is $\{\varphi\}m_1\{\psi\}$). In [8] this has been done for the case where m_1 is an implementation of a data type and m_2 is a specification of a data type.

2.2.2 Formalisation

We have gathered enough properties of our minimal set of notions in order to formalise them. After we shall have done the formalisation we shall often not use the above terminology, but rather adopt a terminology which fits best to the results of the formalisation. In particular, instead of constructs we shall have terms and instead of combinators we shall have constants (which act as the primitive constructs) and function symbols. We shall often give examples which are extremely small, e.g. by taking symbols denoting natural numbers rather than realistic software modules. Nevertheless, such examples are related to problems which occur in connection with realistic software modules also. We shall describe an algebraic system as a *structure* or *model* and we shall adopt several notations and conventions from logic [9].

Definition 2.2.1 An *algebraic system with preorder* is a quadruple

$$\mathfrak{R} = (A, R, \{F_j \mid j \in J\}, \{C_i \mid i \in I\})$$

$(J, I$ index sets) where A is a set (called the *domain* of $\mathfrak{R}$), R is a preorder, each F_j is a function and each C_i is an element (a *constant*) of A. Recall that a *preorder* is a relation which is reflexive and transitive. We assume that the arity of each F_j is given as a natural number a_j. □

We also allow for many-sorted algebraic systems with preorder. However we require that there is one *domain of interest* (A), on which the preorder R is defined. In such a case we shall say that there are *secondary domains*. In order to keep things simple, we shall focus our discussion on the single-sorted case. When necessary, we can always deal with secondary domains somewhat informally. Often we shall refer to R as the 'implementation relation'. Clearly we shall prefer algebraic systems in which each F_j is a monotonic function, but in general we shall not require this. The advantage of a monotonic function is that knowledge about the constituents of a composite construct, allows one to infer properties of the composite construct itself. Monotonicity is a relevant issue in this context – but we have no reason here to consider some notion of continuity of functions.

We shall distinguish between the elements of the domain of an algebraic system with preorder, and the terms (consisting of symbols) used to denote these elements.

Definition 2.2.2 The *alphabet* to be used for constructing the set of terms for a $\mathfrak{R}$ as above, consists of the following *symbols*:

1. *function symbols*: f_j (one for each F_j),
2. *constant symbols*: c_i (one for each C_i),
3. *variables*: x_i (one for each $i \in \mathbb{N}$)

where $\mathbb{N}$ denotes the set of natural numbers. For this alphabet, the collection of symbols $\{\sqsubseteq\} \cup \{f_j \mid j \in J\} \cup \{c_i \mid i \in I\}$ is called the *signature* of $\mathfrak{R}$. We denote it by Sig($\mathfrak{R}$). □

The function symbols and constant symbols from Sig($\mathfrak{R}$) can be combined with variables to build terms in the usual way. Recall that the arity of each function F_j with function symbol f_j is given as a_j.

Definition 2.2.3 The set of *terms* for $\mathfrak{R}$, denoted as $T_\mathfrak{R}$ is inductively defined by:

1. $c_i \in T_\mathfrak{R}$ for all $i \in I$,
2. $x_i \in T_\mathfrak{R}$ for all $i \in \mathbb{N}$,
3. if $t_1, \ldots, t_{a_j} \in T_\mathfrak{R}$ then $f_j(t_1, \ldots, t_{a_j}) \in T_\mathfrak{R}$ for all $j \in J$. □

The terms at their turn can be used to build formulae.

Definition 2.2.4 ($T_\mathfrak{R}$-formulae) The set of *atomic $T_\mathfrak{R}$-formulae* is defined as the set of formulae $P = Q$ and $P \sqsubseteq Q$ for $P, Q \in T_\mathfrak{R}$.

The set of *$T_\mathfrak{R}$-formulae* is the smallest set containing $\top, \bot$, all atomic $T_\mathfrak{R}$-formulae and which is closed under the usual connectives $\neg, \wedge, \vee, \rightarrow$ and the quantifier $\forall$. More precisely, the set of $T_\mathfrak{R}$-formulae is inductively defined by:

1. $\top$ and $\bot$ are $T_\mathfrak{R}$-formulae,
2. if φ is an atomic $T_\mathfrak{R}$-formula, then φ is a $T_\mathfrak{R}$-formula,

3. if φ is a $T_{\mathfrak{R}}$-formulae, then so is $\neg\varphi$,
4. if φ and ψ are $T_{\mathfrak{R}}$-formulae, then so are $\varphi \wedge \psi$, $\varphi \vee \psi$ and $\varphi \rightarrow \psi$,
5. if x is a variable and φ is a $T_{\mathfrak{R}}$-formula, then $\forall x\,(\varphi)$ is a $T_{\mathfrak{R}}$-formula. $\square$

An (atomic) $T_{\mathfrak{R}}$-formula is said to be *closed* if no free variables occur in it. Note that (atomic $T_{\mathfrak{R}}$-formulae) $\subset$ ($T_{\mathfrak{R}}$-formulae). Sometimes we shall write just 'formula' instead of '$T_{\mathfrak{R}}$-formula'. Now we have the syntactic definition of (atomic) $T_{\mathfrak{R}}$-formulae, the next step is to define what it means that such a formula is true.

Definition 2.2.5 *Truth* in $\mathfrak{R}$ is defined as usual where it is understood that $\sqsubseteq$ corresponds to R. When the $T_{\mathfrak{R}}$-formula φ holds in $\mathfrak{R}$, we denote this by $\mathfrak{R} \models \varphi$. See [9] for the details. In particular, the symbol $=$ is interpreted by (mathematical) equality on A. Furthermore $\top, \bot, \neg, \wedge, \vee, \rightarrow$ and $\forall$ correspond with truth, absurdity, negation, conjunction, disjunction, implication and universal quantification respectively. $T_{\mathfrak{R}}$-formulae which are not closed, are interpreted as implicitly universally quantified. $\square$

In the following table we put together some of the notions introduced so far. We simplify the presentation in the table, in the sense that the entry 'symbols' does not show the variables x_i. The entries for 'terms' and 'atomic formulae' speak for themselves. The entry 'statements' shows the notation for the statement that a $T_{\mathfrak{R}}$-formula φ holds in $\mathfrak{R}$. The entry 'model' is trivial, but we introduce it here already because later we shall add a second column to this table where the entry 'model' will be less trivial.

symbols	Sig($\mathfrak{R}$)
terms	$T_{\mathfrak{R}}$
atomic formulae	$\{P = Q, P \sqsubseteq Q \mid P, Q \in T_{\mathfrak{R}}\}$
statements	$\mathfrak{R} \models \varphi$
model	$\mathfrak{R}$

Recall that we distinguish between the elements of the domain of an algebraic system with preorder, and the terms, (consisting of symbols) used to denote these elements. We adopt the point of view that a typical software developer manipulates *symbols*. In fact the developer might not be able to find out for all φ whether $\mathfrak{R} \models \varphi$ or $\mathfrak{R} \not\models \varphi$. Therefore it might seem strange that we nevertheless introduce the actual algebraic system itself. The reason is that for an understanding of the notions of component, black-box description and design we need not to know the precise rules by which the developer reasons for finding out if $\mathfrak{R} \models \varphi$. We simply introduce the algebraic system as a *source of true facts*. This is somewhat similar to the way in which one simply imports all valid assertions about the basic data types into a proof system for imperative programs [10] (p.61).

2.2.3 Examples

In this section we give two extremely simple examples and a realistic, relevant example of an algebraic system with preorder.

As an extremely simple example, consider $\mathfrak{R}_1 := (\mathbb{N}, \leq, \{+\}, \{0, 1, 2, \ldots\})$ where $\mathbb{N}$ is the set of natural numbers, $\leq$ is the usual "less than or equal" ordering on $\mathbb{N}$, and $+$ is binary addition. Trivially $\leq$ is a preorder and $+$ is monotonic in both arguments. We assume $\mathrm{Sig}(\mathfrak{R}_1) = \{\sqsubseteq, +, 0, 1, 2, \ldots\}$ and use infix notation for $+$. We can consider a statement like $\mathfrak{R}_1 \models x_0{+}1 \sqsubseteq x_0{+}2$ (which holds).

As a better example, consider some set S and take its powerset as a domain. Assume constants for the subsets of S. Let $\mathfrak{R}_2 := (\mathcal{P}(S), \subseteq, \{\cap\}, \{c_i \mid c_i \subseteq S\})$. Notice that $\subseteq$ is a preorder and $\cap$ is monotonic in both arguments. This example can have many faces, depending on the choice for the symbols. If we adopt (infix) $\wedge$ for $\cap$, $\top$ for S and $\bot$ for $\emptyset$ then the example is about proposition logic and $\sqsubseteq$ becomes *logical implication*. But we can also adopt (mixfix) **import** .. **into** .. for $\cap$, **module end** for S and **module axiom false end** for $\emptyset$, making the example look as a simplified kind of module algebra.

Next we shall discuss an example of an algebraic system which is interesting for practical applications, viz. Jonkers' so-called class algebra CA underlying COLD-K [4] which is related to Bergstra's *Module Algebra* [1]. This CA only serves as an example here; in particular, no knowledge about CA or COLD-K is required to read or apply the theory presented in this chapter. The reader who is not interested in COLD-K may very well decide to skip the current section and proceed with Section 2.3. The fact that the theory presented in this chapter is independent of COLD-K has several advantages. We mention two advantages. First, by abstracting from COLD-K, it is easier to study the mathematical properties of the notions in which we are interested. Secondly, the notions component, black-box description and design are applicable to a wider class of design languages.

In COLD-K [4] one has a set of (schemes denoting) so-called *class descriptions*, which we simply view as constant symbols (the c_i). There are a number of function symbols (among which one for import) by which one can combine class descriptions to build larger class descriptions. For class descriptions P

and Q one can define what it means that P ***implements*** Q and we view such a fact as a pair in the implementation relation. So CA $\models P \sqsubseteq Q$ means that P is an implementation of Q. Formally we view CA as an algebraic system with preorder

$$(\mathsf{CDescription}, \sqsubseteq, \{\Sigma, \mathrm{T}, \Box, \bullet, +\}, \{C_i \mid i \in \mathsf{CDescription}\})$$

with secondary domains CSignature and CRenaming containing class signatures and class renamings respectively. We sketch CA briefly. CDescription is the set of class descriptions modulo semantic equivalence. Each C_i is the equivalence class of i. Σ is a unary function which takes an element of CDescription and yields its class signature. T is an embedding of class signatures in CDescription. $\Box$ (for export) is a binary function which takes a class signature and an element of CDescription and yields another (restricted) element of CDescription. $\bullet$ (for renaming) is a binary function which takes a class renaming and an element of CDescription and yields an element of CDescription. $\sqsubseteq$ is the implementation relation. Of course the above presentation of CA is somewhat simplified.

2.3 Lambda calculus

2.3.1 Introduction

By way of preparation for the introduction of components and designs, we shall extend an arbitrary algebraic system with preorder such that we can deal with abstraction and with the introduction of names in a formal way. We do this by putting a version of lambda calculus 'on top of' the algebraic system.

At first sight it is not obvious that one benefits from introducing lambda calculus in order to describe components and designs. However, we can already at this point indicate that certain connections can be made. First of all lambda calculus offers the possibility to introduce names which may occur within terms and which become bound to some other term (by application). Also for the formalisation of the notion of a design we shall need names, because we want to be able to *refer* to components. The act of providing the specification of a term instead of the term itself can be viewed as a kind of abstraction. In classical lambda calculus [5], abstraction is done by putting $\lambda x.$ in front of a term. It is possible to combine both kinds of abstraction in an elegant way, viz. by having lambda abstraction where one puts $\lambda x \sqsubseteq P$. in front of a term, where P is a specification of the term to which x is going to be bound by application.

It will turn out that one such version of lambda calculus can be used for several purposes, viz. describing parameterisation (e.g. of software modules) and for describing the notions of component and design. Furthermore we shall see in Section 2.5 that certain methodologically relevant properties of designs have very simple counterparts in this lambda calculus. The calculus is named $\lambda\pi$-calculus, where the π refers to a rule (π) which resembles the rule (β) of classical lambda calculus, but which is *partial.*

Before we embark on the detailed definition of the $\lambda\pi$-calculus, let us give a preview of it by means of a table. We have the same entries as before, but in addition to the column for $\mathfrak{R}$, there is a second column for $\lambda\pi$-calculus.

symbols	Sig($\mathfrak{R}$)	Sig($\mathfrak{R}$) $\cup \{\lambda, .\}$
terms	$T_{\mathfrak{R}}$	$\Lambda_{\mathfrak{R}}$
atomic formulae	$\{P = Q, P \sqsubseteq Q \mid P, Q \in T_{\mathfrak{R}}\}$	$\{P = Q, P \sqsubseteq Q \mid P, Q \in \Lambda_{\mathfrak{R}}\}$
statements	$\mathfrak{R} \models \varphi$	$\vdash \varphi$
model	$\mathfrak{R}$	$\mathfrak{R}^+$

For the entries 'symbols', 'terms' and 'atomic formulae' we have a simple inclusion. This reflects the fact that we shall add new symbols which are used to construct new terms and new atomic formulae. In particular, $T_{\mathfrak{R}} \subset \Lambda_{\mathfrak{R}}$, where $\Lambda_{\mathfrak{R}}$ is a set of so-called lambda terms. The entry 'statements' requires more explanation. The $\lambda\pi$-calculus introduces a collection of rules which have an axiomatic status, and that are adopted for a-priory and intuitive reasons. The simplest statements in this calculus are of the form $\vdash \varphi$. The calculus is such that true facts φ from $\mathfrak{R}$ automatically yield a statement $\vdash \varphi$ in $\lambda\pi$. This will be achieved, roughly speaking, by adopting the principle $(\mathfrak{R} \models \varphi) \Rightarrow (\vdash \varphi)$ as one of the rules. In order to avoid confusion, we must clearly state that $\lambda\pi$-calculus is *not* about some proof system for $\mathfrak{R}$ in the tradition of propositional logic or first-order predicate-logic, say – as the notation $\vdash \varphi$ might suggest. Of course many such proof systems exist already and various soundness and completeness results are known [9]. Instead of that, the theme of $\lambda\pi$ is a certain non-conventional approach to reasoning about lambda abstraction. Because the rules of $\lambda\pi$ have an axiomatic status, the question whether they have a model arises naturally. This question is not completely trivial. It will be addressed in Appendix A where we shall construct a model which will be denoted by $\mathfrak{R}^+$. This explains the entry 'model' in the table.

In subsection 2.3.2 we shall define the set of terms and the rules of the $\lambda\pi$-calculus. In subsection 2.3.3 we shall give some derived rules. In subsection

2.3.4 we shall briefly discuss monotonicity and in subsection 2.3.5 we shall define reduction for the $\lambda\pi$-calculus. Finally subsections 2.3.6 and 2.3.7 are about normalisation and confluence for this reduction. In Appendix A we shall construct a model for the $\lambda\pi$-calculus.

2.3.2 Definition of the Calculus

We assume an algebraic system with preorder

$$\Re = (A, R, \{F_j \mid j \in J\}, \{C_i \mid i \in I\})$$

with its associated signature Sig($\Re$) etc. as before. We shall put a version of lambda calculus 'on top of' $\Re$ to get the $\lambda\pi$-calculus and in this way the algebraic system $\Re$ acts as a parameter of the calculus! The detailed definition of the calculus begins with the definition of the type symbols and the lambda terms.

Definition 2.3.1 The set of *type symbols* is inductively defined by

1. 0 is a type symbol,
2. if σ, τ are type symbols, then so is $(\sigma \rightarrow \tau)$. □

These type symbols are usually known as 'simple types'. Roughly speaking, there are two classical approaches for associating such type symbols with terms. The first approach is Curry's approach where the types are not textually part of the terms. Instead of that there is a derivation system for associating types with terms. E.g. $\lambda x.x$ has type $(0 \rightarrow 0)$, but also e.g. $((0 \rightarrow 0) \rightarrow (0 \rightarrow 0))$, whereas $(\lambda x.(xx))(\lambda x.(xx))$ has no type at all. The second approach is Church's approach where type-information is an essential part of the terms. One way of achieving this is to associate explicitly one type with each variable. E.g. x^0 and $x^{(0\rightarrow 0)}$ could be distinct variables of types 0 and $(0 \rightarrow 0)$ respectively. In this approach $\lambda x^0.x^0$ has type $(0 \rightarrow 0)$ only. Each term has one type and the problematic $(\lambda x.(xx))(\lambda x.(xx))$ is excluded from the set of terms. We use the type symbols along the lines of Church's approach. In order to avoid confusion, we must state explicitly that these simple types are not the main issue of $\lambda\pi$; the types are used in a classical way and their only purpose is to exclude certain problematic terms.

Definition 2.3.2 We assume infinitely many variables x_i^τ of each type τ $(i \in \mathbb{N})$. The alphabet to be used for constructing the set of lambda terms consists of the following *symbols*:

1. *function symbols*: f_j (one for each F_j),
2. *constant symbols*: c_i (one for each C_i),
3. *variables*: x_i^τ (one for each (τ, i) with $\tau \in$ type symbols, $i \in \mathbb{N}$),
4. *auxiliary symbols*: . , (,), $\lambda, \sqsubseteq$. □

The type superscript x_i^τ is used with variables only when necessary to avoid ambiguity. Recall that the arity of each function F_j with function symbol f_j is given as a_j.

Definition 2.3.3 The set of *lambda terms* for $\mathfrak{R}$, denoted as $\Lambda_{\mathfrak{R}}$, and the *type* of each lambda term are inductively defined by

1. $x_i^\tau \in \Lambda_{\mathfrak{R}}$ $(i \in \mathbb{N})$ with type τ,
2. $c_i \in \Lambda_{\mathfrak{R}}$ $(i \in I)$ with type 0 ,
3. if $P_1, \ldots, P_{a_j} \in \Lambda_{\mathfrak{R}}$ with types 0, then $f_j(P_1, \ldots, P_{a_j}) \in \Lambda_{\mathfrak{R}}$ with type 0,
4. if $P, Q \in \Lambda_{\mathfrak{R}}$ and P is of type $(\sigma \to \tau)$ and Q is of type σ, then $(PQ) \in \Lambda_{\mathfrak{R}}$ with type τ,
5. if $P, Q \in \Lambda_{\mathfrak{R}}$ where P is of type σ, Q is of type τ and x_i^σ does not occur in P,
   then $(\lambda x_i^\sigma \sqsubseteq P.Q) \in \Lambda_{\mathfrak{R}}$ with type $(\sigma \to \tau)$.

Note that clauses 1., 2. and 3. yield the terms in $T_{\mathfrak{R}}$ if we restrict ourselves to variables of type 0, identifying x_i and x_i^0. Also note that in the last clause, $\sqsubseteq$ occurs as a *symbol* in lambda terms.

We use $\equiv$ to denote syntactical equality. For $P \in \Lambda_{\mathfrak{R}}$, $FV(P)$ denotes the set of *free variables* of P which is defined as usual.

We shall identify α-congruent terms, as is usually done in classical lambda calculus; see e.g. [5] 2.1.12. We adopt the usual variable convention [5] 2.1.13 which is as follows. If $M_1, \ldots, M_n$ occur in a certain mathematical context (e.g. definition, proof), then in these terms all bound variables are chosen to be different from the free variables. To work according to the variable convention means that sometimes a systematic renaming of bound variables (a so-called α-conversion) has to take place. □

There is an option to simplify the calculus by strengthening the last clause to exclude *all* variables from the parameter restriction P in $(\lambda x_i^\sigma \sqsubseteq P.Q)$. We have chosen not to adopt this simplification and we shall explicitly study some of the consequences of the fact that such P may contain variables. When it comes to applications of the calculus where these consequences are felt as an unnecessary complication, it is straightforward to adapt the calculus correspondingly. Unless stated otherwise, we shall assume that parameter restrictions may contain variables.

As typical elements of $\Lambda_{\Re}$ we shall use A, B, C etc. and their indexed versions. We shall sometimes be somewhat sloppy in the distinction between actual variables ($x_0, x_1, \ldots$) and the typical elements of the set of variables (x, y, z and their indexed versions).

Example 2.3.4 Consider $\Re_1 = (\mathbb{N}, \leq, \{+\}, \{0, 1, 2, \ldots\})$ where $\mathbb{N}$ is the set of natural numbers, $\leq$ is the usual "less than or equal" ordering on $\mathbb{N}$, and $+$ is binary addition. Assume an alphabet containing the symbols $+$, $0, 1, \ldots$ with the obvious interpretation. The function symbol $+$ is used in infix notation. We use the same notation for the symbols and for the elements of the domain of the algebraic system with preorder, but this cannot cause confusion, since lambda terms can contain *symbols* only.

(i) $1 + (x_0 + x_1)$ is a lambda term with type 0, provided x_0, x_1 of type 0.

(ii) $\lambda x_1 \sqsubseteq 2.x_1$ is a lambda term of type $(0 \to 0)$, provided x_1 of type 0.

(iii) $\lambda x_0 \sqsubseteq (\lambda x_1 \sqsubseteq 2.x_1).(x_0 1)$ is a lambda term of type $((0 \to 0) \to 0)$, provided x_0 of type $(0 \to 0)$, x_1 of type 0. □

We must make a remark with respect to the above example where we used $\Re_1$ as an example of an algebraic system with preorder. We shall use this $\Re_1$ very often to illustrate some formal and technical detail of our theory and for that purpose this trivial $\Re_1$ is usable. But we must warn the reader that from a software-engineering point of view it is a misleading example and that it even may suggest an intuition for $\Re$ which is quite wrong. The examples $\Re_2$ and CA from Section 2.2.3 are much better in this respect.

Recall that the algebraic system $\Re$ has the implementation relation R and that we write $\Re \models P \sqsubseteq Q$ if the elements denoted by P and Q are in this relation. Now we want to extend this relation, or more precisely, we want to compare lambda terms as well. We refer to $\sqsubseteq$ as the 'implementation relation'. From the terms one can again build formulae $P = Q$ and $P \sqsubseteq Q$. The following should be compared with definition 2.2.4.

Definition 2.3.5 ($\Lambda_{\Re}$-formulae) The set of *atomic $\Lambda_{\Re}$-formulae* is defined as the set of formulae $P = Q$ and $P \sqsubseteq Q$ for $P, Q \in \Lambda_{\Re}$, where we require that P and Q have the same type.

The set of *$\Lambda_{\Re}$-formulae* is the smallest set containing $\top, \bot$, all atomic $\Lambda_{\Re}$-formulae and which is closed under the usual logical connectives $\neg, \wedge, \vee, \to$ and the quantifier $\forall$. □

An (atomic) $\Lambda_{\Re}$-formula is said to be *closed* if no free variables occur in it. Sometimes we write just 'formula' instead of '$\Lambda_{\Re}$-formula'. Let us say a few words about the interpretation of formulae. Because $T_{\Re} \subset \Lambda_{\Re}$, also ($T_{\Re}$-formulae) $\subset$ ($\Lambda_{\Re}$-formulae). Clearly $T_{\Re}$-formulae φ can be interpreted in $\Re$. These are precisely the formulae where all (sub)terms have type 0 and do not contain the symbol λ. For these formulae the statement $\Re \models \varphi$ makes sense and has been introduced already in definition 2.2.5. For other formulae, no interpretation in $\Re$ is defined.

Definition 2.3.6 (Context)

(i) An *assumption* is an atomic $\Lambda_{\Re}$-formula, put in square brackets. In particular, if $P, Q \in \Lambda_{\Re}$ where P and Q have the same type then $[P \sqsubseteq Q]$ is an assumption (viz. an *inequality*), and $[P = Q]$ is an assumption (viz. an *equality*).

(ii) A *context* is a finite set set of assumptions, e.g. $\{[P_1 \sqsubseteq Q_1], \ldots, [P_n \sqsubseteq Q_n]\}$. We shall use Γ, Γ' etc. to denote contexts.

(iii) If $\Gamma = \{[\varphi_1], \ldots, [\varphi_n]\}$ with $\Gamma \neq \emptyset$, then $\bigwedge\Gamma$ abbreviates $\varphi_1 \wedge \ldots \wedge \varphi_n$. If $\Gamma = \emptyset$, then $\bigwedge\Gamma$ abbreviates $\top$.

(iv) We shall write $\Gamma, [\varphi]$ for $\Gamma \cup \{[\varphi]\}$ and we shall write $[\varphi]$ for $\{[\varphi]\}$.

(v) As a convenient notation we shall write $x \in \Gamma$ when x occurs freely in Γ. More precisely $x \in \Gamma :\Leftrightarrow \exists P, Q \in \Lambda_{\Re} \cdot (([P = Q]) \in \Gamma \vee ([P \sqsubseteq Q]) \in \Gamma) \wedge x \in FV(P) \cup FV(Q)$. □

In our preview above, we indicated already that the simplest statements in $\lambda\pi$ are of the form $\vdash \varphi$. The general form is $\Gamma \vdash \varphi$ which intuitively corresponds with "φ follows from the assumptions in Γ". The technical term for such a statement is a 'sequent'. We shall adopt a rather syntactic point of view and we shall define a notion of *derivability* for these sequents, based on a set of rules, which will be given in a Gentzen-style formulation. We shall simultaneously define derivability for sequents of the form $\Gamma \vdash P = Q$ and of the form $\Gamma \vdash P \sqsubseteq Q$.

Definition 2.3.7 A *sequent* is a pair (Γ, φ), written as $\Gamma \vdash \varphi$, where Γ is a context and where φ is an atomic $\Lambda_{\Re}$-formula. So φ is either $P \sqsubseteq Q$ or $P = Q$, ($P, Q \in \Lambda_{\Re}$ and of equal type). We shall write $\vdash \varphi$ for $\emptyset \vdash \varphi$. □

The first group of rules serves for importing facts about $\Re$ into the calculus. There is a rule denoted by ($\models_1$), expressing that for each monotonic function F_j of $\Re$, this monotonicity can be used in the calculus. To state this somewhat more clearly: the algebraic system with preorder has an indexed collection of functions $\{F_j \mid j \in J\}$. Some of these functions may be monotonic, others are not. As usual, a function F_j is said to be monotonic if $\forall x, y, \vec{m}, \vec{n}\,(x \sqsubseteq y \Rightarrow F_j(\vec{m}, x, \vec{n}) \sqsubseteq F_j(\vec{m}, y, \vec{n}))$. We do not require that all functions F_j are monotonic, but we assume that somehow we can tell the monotonic functions from the non-monotonic ones. Then the rule applies to the monotonic functions only. To simplify the notation, we write $f_j(\ldots, P, \ldots) \sqsubseteq f_j(\ldots, Q, \ldots)$ instead of $f_j(\vec{M}, P, \vec{N}) \sqsubseteq f_j(\vec{M}, Q, \vec{N})$.

The second rule can be read as follows: whenever a closed formula φ is true in the algebraic system $\mathfrak{R}$, then φ can be considered derivable in the calculus. The statement $\mathfrak{R} \models \varphi$ is in the upper part of this rule, which has the effect that it acts as a premiss for the applicability of the rule. The sequent $\Gamma \vdash \varphi$ is in the lower part of the rule, which has the effect that it acts as the conclusion of the rule.

Definition 2.3.8 (Rules $\models_1$, $\models_2$)

$$\frac{\begin{array}{l}\mathfrak{R} \models F_j \text{ monotonic}\\ \Gamma \vdash P \sqsubseteq Q\end{array}}{\Gamma \vdash f_j(\ldots, P, \ldots) \sqsubseteq f_j(\ldots, Q, \ldots)}\,(\models_1)$$

$$\frac{\begin{array}{l}\varphi \text{ closed}\\ \mathfrak{R} \models \varphi\end{array}}{\Gamma \vdash \varphi}\,(\models_2)$$

□

Note that the well-formedness of the sequents involved in rule $(\models_1)$ requires that both P and Q have type 0. By these rules $(\models_i)$ one can import certain relevant facts about $\mathfrak{R}$ into the calculus. In this way we need not include a rule which expresses monotonicity of all functions. In subsection 2.3.4 we shall see that a rule expressing monotonicity for *all* lambda terms would not even be acceptable. The next rule is just necessary for manipulating contexts in the obvious way.

Definition 2.3.9 (Rule context)

$$\frac{}{\Gamma, [\varphi] \vdash \varphi}\,(\text{context})$$

□

Now we proceed with some more rules. Reflexivity and transitivity have been added as rules; for those closed formulae which can be interpreted in $\mathfrak{R}$, this reflexivity and transitivity are already given by rule ($\models_2$). But for the other formulae, these rules are essentially new. E.g. for $\mathfrak{R}_1$ as in example 2.3.4, $\vdash 1 \sqsubseteq 1$ follows from $\mathfrak{R}_1 \models 1 \sqsubseteq 1$, but $\vdash x_0^{(0\to 0)} \sqsubseteq x_0^{(0\to 0)}$ does not. In general $\mathfrak{R}$ does not tell how to compare all lambda terms. But reflexivity and transitivity are properties which are associated with a implementation relation on a-priory grounds and this is formalised by the rules (refl.) and (trans.).

For the 'λ-introduction' rules (λI_1), (λI_2) and the 'application rule' (ap.) we add some remarks. Abstraction is covariant with respect to $\sqsubseteq$ in the second argument and it is contravariant with respect to $\sqsubseteq$ in its first argument (!). Application is covariant in its first argument. There is no general covariance of application in its second argument which would be monotonicity (we shall discuss this later). These covariance and contravariance issues are important characteristics of the calculus.

Definition 2.3.10 (Rules refl. trans. λI_1, λI_2, ap.)

$$\frac{}{\Gamma \vdash P \sqsubseteq P}\ \text{(refl.)}$$

$$\frac{\Gamma \vdash P_1 \sqsubseteq P_2, P_2 \sqsubseteq P_3}{\Gamma \vdash P_1 \sqsubseteq P_3}\ \text{(trans.)}$$

$$\frac{\Gamma, [x \sqsubseteq P] \vdash Q_1 \sqsubseteq Q_2 \qquad (x \notin \Gamma)}{\Gamma \vdash (\lambda x \sqsubseteq P.Q_1) \sqsubseteq (\lambda x \sqsubseteq P.Q_2)}\ (\lambda I_1)$$

$$\frac{\Gamma \vdash P_1 \sqsubseteq P_2}{\Gamma \vdash (\lambda x \sqsubseteq P_2.Q) \sqsubseteq (\lambda x \sqsubseteq P_1.Q)}\ (\lambda I_2)$$

$$\frac{\Gamma \vdash P_1 \sqsubseteq P_2}{\Gamma \vdash (P_1 Q) \sqsubseteq (P_2 Q)}\ \text{(ap.)}$$

□

Note that the well-formedness of the sequents involved in rule (λI_1) requires that $x \notin FV(P)$. Similarly for (λI_2) we must have $x \notin FV(P_i)$ for $i = 1, 2$. The definition of the next rule requires the notion of substitution.

Definition 2.3.11 (Substitution) Let x_i be a variable and P a term such that x_i and P have the same type. Then we define the *substitution* operator $[x_i := P]$ inductively by

1. $x_i[x_i := P] \equiv P$,
   $x_i[x_j := P] \equiv x_i \ (i \neq j)$,
2. $c_i[x_j := P] \equiv c_i$,
3. $f_j(P_1, \ldots, P_{a_j})[x_i := P] \equiv f_j(P_1[x_i := P], \ldots, P_{a_j}[x_i := P])$,
4. $(Q_1 Q_2)[x_i := P] \equiv ((Q_1[x_i := P])(Q_2[x_i := P]))$,
5. $(\lambda x_i \sqsubseteq Q_1.Q_2)[x_j := P] \equiv (\lambda x_i \sqsubseteq Q_1[x_j := P].Q_2[x_j := P])$. □

Note that for clause 5. above $x_i \not\equiv x_j$ by the variable convention and that $x_i \notin FV(P)$ – also by the variable convention. In this way we avoid clashes of free and bound variables and violation of the restriction that a variable bound by a λ does not occur in the parameter restriction of that same λ.

The next rule is called (π) and it resembles the well-known rule (β) of classical lambda calculus, but is *partial*, by which we mean that contraction is conditional. In the applicability condition of the rule (π) an actual parameter is compared with a parameter restriction. Here is the very essential difference between $\lambda\pi$ and classical typed λ-calculi: the parameter restrictions are *not* used to restrict the set of well-formed terms, but instead of that they are used to regulate the contractions.

Definition 2.3.12 (Rule π)

$$\frac{\Gamma \vdash P_2 \sqsubseteq P_1}{\Gamma \vdash (\lambda x \sqsubseteq P_1.Q)P_2 = Q[x := P_2]} \ (\pi)$$

□

Note that the property that a variable bound by a λ does not occur in the parameter restriction of that same λ, is preserved by the conversion from $(\lambda x \sqsubseteq P_1.Q)P_2$ to $Q[x := P_2]$. Finally we have a rule that makes $\sqsubseteq$ into a partial order and a general substitution rule. We write $\varphi(P)$ to denote $\varphi[x := P]$ and $\varphi(Q)$ to denote $\varphi[x := Q]$. We add a remark about the substitution rule below. In the transition from $\varphi(P)$ to $\varphi(Q)$, it is not required that *all* occurrences of P are replaced by Q.

Definition 2.3.13 (Rules $=I$, subst.)

$$\frac{\begin{array}{l}\Gamma \vdash P_1 \sqsubseteq P_2 \\ \Gamma \vdash P_2 \sqsubseteq P_1\end{array}}{\Gamma \vdash P_1 = P_2}\,(=I)$$

$$\frac{\begin{array}{l}\Gamma \vdash \varphi(P) \\ \Gamma \vdash P = Q\end{array}}{\Gamma \vdash \varphi(Q)}\,(\text{subst.})$$

□

Now we present the intuition behind the rules (λI_1), (λI_2) and (π). Each lambda term $(\lambda x \sqsubseteq P.Q)$ can be viewed as a function having a restriction concerning its argument. Therefore it is reasonable that the evaluation of an application term cannot take place unless the argument provably meets this restriction (rule π). The rules (λI_1) and (λI_2) describe the conditions under which one function can be viewed as the implementation of another function. First of all, two functions with the same argument restriction are in the implementation relation if for every acceptable argument their results are in the implementation relation (rule λI_1). Secondly, if two functions have equal function bodies but the restriction of one function is weaker than the restriction of the other function, then the function with the *weakest* restriction implements the other function (rule λI_2). The fact that in the rule (λI_1) an assumption $[x \sqsubseteq P]$ is discharged is motivated as follows: by the condition in the rule (π) we know that whatever is going to be substituted for x will meet the restriction $x \sqsubseteq P$ and therefore it is reasonable that the assumption $[x \sqsubseteq P]$ can be used when comparing the function bodies Q_1 and Q_2.

Definition 2.3.14 $(\lambda\pi)$ *Derivability* for sequents is defined inductively by:

1. If φ is closed and $\Re \models \varphi$, then $\Gamma \vdash \varphi$ is derivable (*by rule* $(\models_2)$). If $\Gamma \vdash \varphi$ is the conclusion of a rule from (context, refl.), then $\Gamma \vdash \varphi$ is derivable.

2. If $\Gamma \vdash \varphi$ is the conclusion of a rule from ($\models_1$, trans., λI_1, λI_2, ap., π, $=I$, subst.) and all premises of this rule are derivable, then $\Gamma \vdash \varphi$ is derivable, provided that the monotonicity condition and the variable condition are satisfied for the cases of $(\models_1)$ and (λI_1) respectively.

We write $\Gamma \vdash \varphi$ if $\Gamma \vdash \varphi$ is derivable. □

The *typing* of terms should not be confused with the *restrictions* associated with each λ. Actually the correctness with respect to the typing has been dealt with by the definition of the set of lambda terms, whereas the restrictions associated with each λ play a role in the calculus. E.g. for $\Re_1$ as in example 2.3.4, $(\lambda x_1 \sqsubseteq 2.x_1)(\lambda x_1 \sqsubseteq 1.x_1 + x_1)$ is not a lambda term but $(\lambda x_1 \sqsubseteq 2.x_1)3$ is a lambda term. In the latter term the effect of the restriction $(\sqsubseteq 2)$ is that we cannot apply the rule (π) to this term.

Let us point out explicitly that $\lambda\pi$-calculus and λ-typed λ-calculus are not just almost the same. There are fundamental differences between the nature of the implementation relation '$\sqsubseteq$' of $\lambda\pi$-calculus and the typing relation ':' of λ-typed λ-calculus. One might be tempted to think that it is just a matter of notation and that instead of writing $\lambda x \sqsubseteq P.Q$ we could as well replace the symbol '$\sqsubseteq$' by the symbol ':' to get $\lambda x : P.Q$ which looks the same as a term of λ-typed λ-calculus. The difference is that '$\sqsubseteq$' as a relation between terms is reflexive and transitive, whereas in λ-typed λ-calculus the typing relation ':' is neither reflexive nor transitive.

Example 2.3.15 Consider two terms P_1, P_2 of equal type. Let Γ be a context. Then we have

$$\Gamma \vdash P_1 = P_2 \;\Rightarrow\; \Gamma \vdash P_1 \sqsubseteq P_2,$$

which we show as follows. First, by rule (refl.), $\Gamma \vdash P_1 \sqsubseteq P_1$ and this can be combined with $\Gamma \vdash P_1 = P_2$ by rule (subst.) when we replace the second occurrence of P_1 in $P_1 \sqsubseteq P_1$ by P_2. This yields $\Gamma \vdash P_1 \sqsubseteq P_2$. The result of this example can be viewed as a derived rule of the calculus. Similarly

$$\Gamma \vdash P_1 = P_2 \;\Rightarrow\; \Gamma \vdash P_2 \sqsubseteq P_1,$$

which we show again with (refl.) and (subst.). The latter result can be viewed as a second derived rule of the calculus. □

Example 2.3.16 Consider $\Re_1 = (\mathbb{N}, \leq, \{+\}, \{0, 1, 2, \ldots\})$. Define the term P by

$$P \equiv \lambda x_0 \sqsubseteq 1.((\lambda x_1 \sqsubseteq x_0 + 1.x_1)(x_0 + x_0)).$$

By rule (context) we have $[x_0 \sqsubseteq 1] \vdash x_0 \sqsubseteq 1$. Because in $\Re_1$ the function $+$ is monotonic, we have from $(\models_1)$ that $[x_0 \sqsubseteq 1] \vdash x_0 + x_0 \sqsubseteq x_0 + 1$ and therefore we can apply rule (π) to get $[x_0 \sqsubseteq 1] \vdash (\lambda x_1 \sqsubseteq x_0 + 1.x_1)(x_0 + x_0) = x_0 + x_0$. By the first derived rule of the previous example this yields $[x_0 \sqsubseteq 1] \vdash (\lambda x_1 \sqsubseteq x_0 + 1.x_1)(x_0 + x_0) \sqsubseteq x_0 + x_0$ and similarly by the second derived rule of the previous example $[x_0 \sqsubseteq 1] \vdash x_0 + x_0 \sqsubseteq (\lambda x_1 \sqsubseteq x_0 + 1.x_1)(x_0 + x_0)$.

Now we can apply rule (λI_1) twice and finally use rule $(=I)$ to get $\vdash P = \lambda x_0 \sqsubseteq 1.x_0 + x_0$. □

In Appendix A we study a model for the calculus of this section. To summarise the results of that appendix: assuming certain restrictions upon $\Re$, there is a model $\Re^+$ such that for atomic formula φ in the language of $\Re$ we have $\Re \models \varphi \Leftrightarrow \Re^+ \models \varphi$ and furthermore for arbitrary φ

$$\Gamma \vdash \varphi \;\Rightarrow\; \Re^+ \models \bigwedge \Gamma \to \varphi.$$

2.3.3 Derived rules

We shall list a number of lemmas that can be viewed as derived rules. The first lemma is very simple and it shows that we can perform a kind of α-conversion at the level of sequents.

Lemma 2.3.17 If $\Gamma \vdash \varphi$ then for fresh y, $\Gamma[x := y] \vdash \varphi[x := y]$.

Proof. The proof is by induction over the length of the derivation of $\Gamma \vdash \varphi$.

- If $\Gamma \vdash \varphi$ is a direct consequence of rule $(\models_1)$ with $\varphi \equiv f(A) \sqsubseteq f(B)$ and $\Gamma \vdash A \sqsubseteq B$, then by (i.h.) $\Gamma[x := y] \vdash A[x := y] \sqsubseteq B[x := y]$. Apply rule $(\models_1)$ again.
- If $\Gamma \vdash \varphi$ is a direct consequence of $\Re \models \varphi$ by rule $(\models_2)$, then φ is closed and hence $\varphi \equiv \varphi[x := y]$. Apply rule $(\models_2)$.
- If $\Gamma \vdash \varphi$ is $\Gamma \vdash (\lambda z \sqsubseteq R.S_1) \sqsubseteq (\lambda z \sqsubseteq R.S_2)$ as a direct consequence of rule (λI_1) with $\Gamma, [z \sqsubseteq R] \vdash S_1 \sqsubseteq S_2$ and $z \notin \Gamma$, then we distinguish two cases. If $x \equiv z$, then x does not occur in $\Gamma \vdash \varphi$ and we are done immediately. Otherwise use (i.h.) and apply rule (λI_1) again.
- Other rules: analogously. □

Lemma 2.3.18 (Weakening) If $\Gamma \vdash \varphi$, then $\Gamma, [\psi] \vdash \varphi$.

Proof. The proof is by induction over the length of the derivation of $\Gamma \vdash \varphi$.

- If $\Gamma \vdash \varphi$ is a direct consequence of rule $(\models_1)$ with $\varphi \equiv f(A) \sqsubseteq f(B)$ and $\Gamma \vdash A \sqsubseteq B$, then by (i.h.) $\Gamma, [\psi] \vdash A \sqsubseteq B$. Apply rule $(\models_1)$ again.
- If $\Gamma \vdash \varphi$ is a direct consequence of $\Re \models \varphi$ by rule $(\models_2)$, then so is $\Gamma, [\psi] \vdash \varphi$.
- If $\varphi \in \Gamma$, then $\Gamma, [\psi] \vdash \varphi$ by rule (context) again.

- If $\varphi \equiv (\lambda x \sqsubseteq R.S_1) \sqsubseteq (\lambda x \sqsubseteq R.S_2)$ and $\Gamma \vdash \varphi$ by rule (λI_1), then we know $\Gamma, [x \sqsubseteq R] \vdash S_1 \sqsubseteq S_2$ and $x \notin \Gamma$.

  If $x \in FV(\psi)$ then we substitute a fresh z for x (using lemma 2.3.17), yielding $\Gamma, [z \sqsubseteq R] \vdash S_1[x := z] \sqsubseteq S_2[x := z]$. Apply (i.h.) and rule (λI_1).

  Otherwise we have $x \notin FV(\psi)$ and by (i.h.) $\Gamma, [\psi], [x \sqsubseteq R] \vdash S_1 \sqsubseteq S_2$ so by rule (λI_1) again we have $\Gamma, [\psi] \vdash (\lambda x \sqsubseteq R.S_1) \sqsubseteq (\lambda x \sqsubseteq R.S_2)$.

- Other rules: analogously. □

Lemma 2.3.19 If $\Gamma \vdash \varphi$, then $\Gamma[x := P] \vdash \varphi[x := P]$.

Proof. The proof is by induction over the length of the derivation of $\Gamma \vdash \varphi$.

- If $\Gamma \vdash \varphi$ is $\Gamma \vdash f(A) \sqsubseteq f(B)$ because $\Gamma \vdash A \sqsubseteq B$ for f corresponding with a monotonic function, then by (i.h.) $\Gamma[x := P] \vdash A[x := P] \sqsubseteq B[x := P]$. Use rule $(\models_1)$ again.
- If $\Gamma \vdash \varphi$ is a direct consequence of $\mathfrak{R} \models \varphi$, then note that (since φ closed) $\varphi \equiv \varphi[x := P]$. Use rule $(\models_2)$ again. [1]
- If $\varphi \in \Gamma$, then $\varphi[\;\;] \in \Gamma[\;\;]$, so apply rule (context) again.
- If $\varphi \equiv (R \sqsubseteq R)$, apply rule (refl.) again.
- If $\varphi \equiv (P_1 \sqsubseteq P_3)$ because $\Gamma \vdash P_1 \sqsubseteq P_2$ and $\Gamma \vdash P_2 \sqsubseteq P_3$ by rule (trans.), then by (i.h.) $\Gamma[\;\;] \vdash P_1[\;\;] \sqsubseteq P_2[\;\;]$ and $\Gamma[\;\;] \vdash P_2[\;\;] \sqsubseteq P_3[\;\;]$. Apply rule (trans.) again.
- If $\Gamma \vdash \varphi$ is $\Gamma \vdash (\lambda y \sqsubseteq R.Q_1) \sqsubseteq (\lambda y \sqsubseteq R.Q_2)$ as a direct consequence of rule (λI_1) with $\Gamma, [y \sqsubseteq R] \vdash Q_1 \sqsubseteq Q_2$ and $y \notin \Gamma$, then we distinguish two cases.

  If $x \equiv y$ then $x \notin \Gamma$ and x does not occur freely in $\Gamma \vdash \varphi$. Therefore $\Gamma[x := P] \vdash \varphi[x := P]$ is just the same as $\Gamma \vdash \varphi$.

  If $x \not\equiv y$ then two possibilities arise. The simplest possibility is that $y \notin P$. Then we can apply (i.h.) to get $\Gamma[x := P], [y \sqsubseteq R[x := P]] \vdash Q_1[x := P] \sqsubseteq Q_2[x := P]$. Now apply rule (λI_1) again.

  If $y \in P$ then we first substitute some fresh z for y to get $\Gamma, [z \sqsubseteq R] \vdash Q_1[y := z] \sqsubseteq Q_2[y := z]$. Now use (i.h.) and apply rule (λI_1) again.

- Other rules: analogously. □

Theorem 2.3.20 (Cut-rule) If $\Gamma \vdash \varphi$, and $\Gamma, [\varphi] \vdash \psi$, then $\Gamma \vdash \psi$.

[1] here is a reason why we require φ closed in $(\models_2)$

Proof. Induction over the length of the derivation of $\Gamma, [\varphi] \vdash \psi$.

- If $\Gamma, [\varphi] \vdash \psi$ is $\Gamma, [\varphi] \vdash f(A) \sqsubseteq f(B)$ because $\Gamma, [\varphi] \vdash A \sqsubseteq B$ for f corresponding with a monotonic function, then by (i.h.) $\Gamma \vdash A \sqsubseteq B$. Use rule ($\models_1$) again.
- If $\Gamma, [\varphi] \vdash \psi$ is a direct consequence of $\Re \models \psi$, then use ($\models_2$) again.
- If $\psi \in \Gamma$, then $\Gamma \vdash \psi$. If $\psi \equiv \varphi$ then $\Gamma \vdash \varphi$ is $\Gamma \vdash \psi$.
- If $\psi \equiv (\lambda x \sqsubseteq R.S_1) \sqsubseteq (\lambda x \sqsubseteq R.S_2)$ and $\Gamma, [\varphi] \vdash \psi$ by rule (λI_1), then we know $\Gamma, [\varphi], [x \sqsubseteq R] \vdash S_1 \sqsubseteq S_2$ and $x \notin (\Gamma, \varphi)$ so certainly $x \notin \Gamma$. By (i.h.) $\Gamma, [x \sqsubseteq R] \vdash S_1 \sqsubseteq S_2$ so by rule (λI_1) we have $\Gamma \vdash (\lambda x \sqsubseteq R.S_1) \sqsubseteq (\lambda x \sqsubseteq R.S_2)$.
- If $\psi \equiv \psi(Q)$ and $\Gamma, [\varphi] \vdash \psi(Q)$ by rule (subst.), then we know $\Gamma, [\varphi] \vdash \psi(P)$ and $\Gamma, [\varphi] \vdash P = Q$ for some P. By (i.h.) $\Gamma \vdash \psi(P)$ and $\Gamma \vdash P = Q$ so by rule (subst.) again we have $\Gamma \vdash \psi(Q)$.
- Other rules: analogously. □

Remark 2.3.21 In earlier versions of the calculus we employed a much more powerful rule ($\models$), saying $\Re \models \bigwedge\Gamma \rightarrow \varphi \;\Rightarrow\; \Gamma \cup \Gamma' \vdash \varphi$ but this caused problems when proving the lemmas of this section. In particular, for the proof of the cut-rule we would have to use the model construction $\Re^+$. □

Lemma 2.3.22 If $\Gamma \vdash P = Q$, then $\Gamma \vdash R[x := P] = R[x := Q]$.

Proof. We want to use the rule (subst.) and we take $\varphi(P) \equiv (R[x := P] = R[x := P])$ which we denote as $\varphi(P) \equiv (R(P) = R(P))$. Clearly $\Gamma \vdash \varphi(P)$. Now consider the second occurrence of P and apply rule (subst.). We obtain $\Gamma \vdash \varphi(Q)$, which is $\Gamma \vdash R(P) = R(Q)$. □

Lemma 2.3.23 (Generalised cut-rule). If $\Gamma \vdash \varphi_1, \ldots, \Gamma \vdash \varphi_n$ and $\Gamma, [\varphi_1], \ldots, [\varphi_n] \vdash \psi$ then $\Gamma \vdash \psi$.

Proof. By iterated application of the cut-rule. □

We listed a number of lemmas about the $\lambda\pi$-calculus above. For our discussion of components and designs in later sections of this chapter, one can simply accept the propositions of these lemmas as additional rules of the $\lambda\pi$-calculus.

2.3.4 Monotonicity

We now give a small derivation that shows that even if the functions F_j are monotonic, there is a kind of consistency problem if we would adopt a rule expressing monotonicity for all lambda terms. The rule (λI_2) plays a key role in this matter. By monotonicity we mean the following rule.

Definition 2.3.24 (Rule mon.)

$$\frac{\Gamma \vdash Q_1 \sqsubseteq Q_2}{\Gamma \vdash (PQ_1) \sqsubseteq (PQ_2)}\ (\text{mon.}) \qquad (\notin \lambda\pi!)$$

□

Example 2.3.25 Consider the lambda term $(\lambda x_0 \sqsubseteq P.(\lambda x_1 \sqsubseteq x_0.x_1))$, then we would have the following derivation – when adopting (mon.). Assume $Q_1, Q_2 \in \Lambda_{\Re}$ such that $\vdash Q_1 \sqsubseteq Q_2 \sqsubseteq P$. By assuming monotonicity we have

$$\vdash (\lambda x_0 \sqsubseteq P.(\lambda x_1 \sqsubseteq x_0.x_1))Q_1 \sqsubseteq (\lambda x_0 \sqsubseteq P.(\lambda x_1 \sqsubseteq x_0.x_1))Q_2$$

and from this we obtain by the rule (π)

$$\vdash (\lambda x_1 \sqsubseteq Q_1.x_1) \sqsubseteq (\lambda x_1 \sqsubseteq Q_2.x_1).$$

But by the rule (λI_2) we have

$$\vdash (\lambda x_1 \sqsubseteq Q_1.x_1) \sqsupseteq (\lambda x_1 \sqsubseteq Q_2.x_1)$$

and hence

$$\vdash (\lambda x_1 \sqsubseteq Q_1.x_1) = (\lambda x_1 \sqsubseteq Q_2.x_1).$$

This shows that if we assume monotonicity for all terms, we would have that Q_1 plays no role any more in $\lambda x_1 \sqsubseteq Q_1.x_1$. □

Under certain conditions and with certain modifications it is possible to adopt the rule (mon.) however. We sketch one workable approach, which yields a slightly different calculus, $\lambda\pi_{\text{mon}}$, say. It can be done only with an obvious additional requirement on $\Re$.

Definition 2.3.26 $(\lambda\pi_{\text{mon}})$ Assume an algebraic system as before $\Re$ where furthermore *all* functions F_j are monotonic with respect the preorder R. Define a restricted set of lambda terms $\Lambda_{\Re(\text{mon})}$ which is like $\Lambda_{\Re}$ except for the additional restriction that parameter restrictions contain *no* variables at all. The calculus $\lambda\pi_{\mathrm{mon}}$ is about formulae and sequents based on $\Lambda_{\Re(\mathrm{mon})}$. It has the same rules as $\lambda\pi$ and in addition to these also the rule (mon.). □

A model $\Re^{+}_{\mathrm{mon}}$ can be constructed along the same lines as $\Re^{+}$, except that the function domains may contain monotonic functions only. For most of the theory about components and designs later in this chapter, there is hardly any difference between $\lambda\pi$ an $\lambda\pi_{\mathrm{mon}}$.

2.3.5 Reduction

In classical lambda calculus every term $(\lambda x.Q)R$ can be contracted and therefore such a term is called a *redex*. In the $\lambda\pi$-calculus it is *not* the case that every term $(\lambda x \sqsubseteq P.Q)R$ can be contracted. It follows that for the $\lambda\pi$-calculus some care is needed in using the word *redex*. We reserve the word *redex* for those terms which can be contracted (in a given context). We also introduce the term *candidate-redex*. The following should be contrasted with [5] 3.1.8.

Definition 2.3.27 A *candidate-redex* is a term $(\lambda x \sqsubseteq P.Q)R$. A candidate-redex $M \equiv (\lambda x \sqsubseteq P.Q)R$ can be *contracted* (is a *redex*) in context Γ if $\Gamma \vdash R \sqsubseteq P$. In this case $Q[x := R]$ is called a *contractum* of M. □

The following definition can be viewed as a reformulation of [5] 3.1.5 and 3.1.17 for the $\lambda\pi$-calculus.

Definition 2.3.28 The relation $\rightarrow$ is defined inductively by:

1. $\Gamma \vdash R \sqsubseteq A \Rightarrow \Gamma \vdash (\lambda x \sqsubseteq A.B)R \rightarrow B[x := R]$,
2. $\Gamma \vdash M \rightarrow N \Rightarrow \Gamma \vdash f_j(\ldots, M, \ldots) \rightarrow f_j(\ldots, N, \ldots)$,
3. $\Gamma \vdash M \rightarrow N \Rightarrow \Gamma \vdash ZM \rightarrow ZN$,
4. $\Gamma \vdash M \rightarrow N \Rightarrow \Gamma \vdash MZ \rightarrow NZ$,
5. $\Gamma \vdash P \rightarrow Q \Rightarrow \Gamma \vdash (\lambda x \sqsubseteq P.M) \rightarrow (\lambda x \sqsubseteq Q.M)$,
6. $\Gamma, [x \sqsubseteq P] \vdash M \rightarrow N,\ x \notin \Gamma\ \Rightarrow \Gamma \vdash (\lambda x \sqsubseteq P.M) \rightarrow (\lambda x \sqsubseteq P.N)$.

A *reduction (path)* is a sequence $\Gamma \vdash M_0 \rightarrow M_1 \rightarrow M_2 \rightarrow \ldots$ □

The following definition can be viewed as a reformulation of (part of) [5] 3.1.5 for the $\lambda\pi$-calculus.

Definition 2.3.29 The relation $\twoheadrightarrow$ is defined inductively by:

1. $\Gamma \vdash M \to N \Rightarrow \Gamma \vdash M \twoheadrightarrow N$,
2. $\Gamma \vdash M \twoheadrightarrow M$,
3. $\Gamma \vdash M \twoheadrightarrow N,\ \Gamma \vdash N \twoheadrightarrow L \Rightarrow \Gamma \vdash M \twoheadrightarrow L$,

and the relation $=_\pi$ is defined inductively by:

1. $\Gamma \vdash M \twoheadrightarrow N \Rightarrow \Gamma \vdash M =_\pi N$,
2. $\Gamma \vdash M =_\pi N \Rightarrow \Gamma \vdash N =_\pi M$,
3. $\Gamma \vdash M =_\pi N,\ \Gamma \vdash N =_\pi L \Rightarrow \Gamma \vdash M =_\pi L$. □

The following lemma can be viewed as a justification of the definitions of $\to, \twoheadrightarrow$ *and* $=_\pi$.

Lemma 2.3.30

(i) $\Gamma \vdash M \to N \Rightarrow \Gamma \vdash M = N$,

(ii) $\Gamma \vdash M \twoheadrightarrow N \Rightarrow \Gamma \vdash M = N$,

(iii) $\Gamma \vdash M =_\pi N \Rightarrow \Gamma \vdash M = N$.

Proof. (i) The proof is by induction over the definition of $\to$.

1. $\Gamma \vdash (\lambda x \sqsubseteq P.Q)R \to Q[x := R]$. Use rule (π).
2. $\Gamma \vdash f_j(\ldots, M, \ldots) \to f_j(\ldots, N, \ldots)$ because $\Gamma \vdash M \to N$. By (i.h.) $\Gamma \vdash M = N$. Use rule (subst.).
3. $\Gamma \vdash ZM \to ZN$ because $\Gamma \vdash M \to N$. By (i.h.) $\Gamma \vdash M = N$. Use rule (subst.).
4. $\Gamma \vdash MZ \to NZ$ because $\Gamma \vdash M \to N$. As 3.
5. $\Gamma \vdash (\lambda x \sqsubseteq P.M) \to (\lambda x \sqsubseteq Q.M)$ because $\Gamma \vdash P \to Q$. By (i.h.) $\Gamma \vdash P = Q$. Use rule (subst.).
6. $\Gamma \vdash (\lambda x \sqsubseteq P.M) \to (\lambda x \sqsubseteq P.N)$ because $\Gamma, [x \sqsubseteq P] \vdash M \to N$. By (i.h.) and the rules (refl.) and (subst.) $\Gamma, [x \sqsubseteq P] \vdash M \sqsubseteq N$. By rule (λI_1) $\Gamma \vdash (\lambda x \sqsubseteq P.M) \sqsubseteq (\lambda x \sqsubseteq P.N)$. In the same way we get $\Gamma \vdash (\lambda x \sqsubseteq P.N) \sqsubseteq (\lambda x \sqsubseteq P.M)$. Finally use rule $(=I)$.

(ii) The proof is by induction over the definition of $\twoheadrightarrow$.

1. Use (i),
2. Use reflexivity.
3. Use (i.h.) and transitivity.

(iii) The proof is by induction over the definition of $=_\pi$.

1. Use (ii),
2. Use (i.h.) and rule (subst.),
3. Use (i.h.) and transitivity. □

2.3.6 Normalisation

A very desirable property of a notion of reduction is the strong normalisation property, denoted by SN. We write SN(M) if M strongly normalises by which we mean that it does not have an infinite reduction path. We say that SN holds if SN(M) for all terms M. Classical untyped λ-calculus does not have this property, and the famous counter-example is $(\lambda x.(xx))(\lambda x.(xx))$ which reduces to itself. In $\lambda\pi$-calculus there is a system of simple types which excludes such problematic terms and as it turns out, this is sufficient to get SN for our notion of reduction. Note that this is not completely trivial, because of the presence of parameter restrictions: in these parameter restrictions also reduction steps can be done. Although typically most of these parameter restrictions will eventually disappear themselves by reductions, this idea cannot be considered as a proof of the fact that the parameter restrictions are harmless.

The remainder of this section is about proving SN. The reader may want to skip this proof and in that case he can proceed with Section 2.3.7. We write Λ_σ to denote the set of terms from $\Lambda_\mathfrak{R}$ with type σ. We write Λ_σ^0 to denote the closed terms from Λ_σ.

Definition 2.3.31 We define $\lambda\beta$ as the calculus with terms from $\Lambda_\mathfrak{R}$ and with all rules of $\lambda\pi$-calculus but for the rule (π) which has been replaced by the rule (β):

$$(\beta) \qquad \frac{}{\Gamma \vdash (\lambda x \sqsubseteq P_1.Q)P_2 = Q[x := P_2]}$$

We adopt the obvious notion of reduction for $\lambda\beta$. We shall prove strong normalisation SN for $\lambda\beta$, using Tait's so-called 'computability' argument as given in e.g. [5] A2. Then SN for $\lambda\pi$ follows easily from SN for $\lambda\beta$.

Let us explain the basic structure of this computability argument. As a kind of naive approach one could try to prove SN by induction on the structure of the terms. Indeed, SN(x), SN(c_i), SN(P_1)$\wedge\ldots\wedge$SN(P_n) $\Rightarrow$ SN($f_j(P_1,\ldots,P_n)$) and SN(P) $\wedge$ SN(Q) $\Rightarrow$ SN($\lambda x \sqsubseteq P.Q$), but the approach fails when it comes to application. From the fact that SN(P) and SN(Q) hold one can not conclude SN(PQ) – as is easily seen from the counterexample of the (not well-formed) term $P \equiv Q \equiv \lambda x.(xx)$.

Therefore we have to strengthen the property SN to another property with will be denoted by C_τ (one for each τ). This strengthening is done by way of *induction loading*, i.e. $M \in C_\tau$ implies SN(M). The property C_τ is sometimes referred to as 'computable', 'reducible' or 'stable'.

Definition 2.3.32 Define the following classes of terms inductively:

$$C_0 = \{M \in \Lambda_0^0 \mid \mathrm{SN}(M)\},$$
$$C_{\sigma\to\tau} = \{M \in \Lambda_{\sigma\to\tau}^0 \mid \forall N \in C_\sigma(MN \in C_\tau)\}.$$

To prove that $M \in C_\sigma$ implies SN(M), requires an induction on the structure of σ. In fact we have another very simple induction loading at this level, which is the reason for adding the proposition (ii) below. As usual we write $\sigma \to \tau_1 \to \tau_2$ to denote $\sigma \to (\tau_1 \to \tau_2)$, i.e. we adopt the convention that type construction is right-associative. Note that every type σ is of the form $\sigma_1 \to \ldots \to \sigma_n \to 0$ which is $\sigma_1 \to (\ldots \to (\sigma_n \to 0)\ldots)$.

Lemma 2.3.33 (i) $M \in C_\sigma \Rightarrow \mathrm{SN}(M)$,
(ii) $\lambda x^{\sigma_1} \ldots x^{\sigma_n}.c \in C_\sigma$ if $\sigma = \sigma_1 \to \ldots \to \sigma_n \to 0$, where c is some constant.

Proof. The proof is by simultaneous induction on σ.

Basis: obviously $M \in C_0 \Rightarrow \mathrm{SN}(M)$ and $c \in C_0$. For the induction step we consider the type $\sigma \to \tau = \sigma \to \tau_1 \to \ldots \to \tau_m \to 0$ and we assume (i) and (ii) for $\sigma, \tau, \tau_1, \ldots, \tau_m$ (i.h.). We must prove (i) $M \in C_{\sigma\to\tau} \Rightarrow \mathrm{SN}(M)$ and (ii) $\lambda x^\sigma x^{\tau_1} \ldots x^{\tau_m}.c \in C_{\sigma\to\tau}$.

First we treat (i). Let $M \in C_{\sigma\to\tau}$, so $\forall N \in C_\sigma(MN \in C_\tau)$. We have $\sigma = \sigma_1 \to \ldots \to \sigma_n \to 0$ for some $\sigma_1, \ldots, \sigma_n$. Hence $M(\lambda x^{\sigma_1} \ldots x^{\sigma_n}.c) \in C_\tau$ and by i.h. has no infinite reduction. So certainly M has not, i.e. SN(M). In fact we only use $\lambda x^{\sigma_1} \ldots x^{\sigma_n}.c$ to show that C_σ is inhabited. The proof-step works for any term N in C_σ. Next we treat (ii), and we reason as follows:

$\lambda x^\sigma x^{\tau_1} \ldots x^{\tau_m}.c \in C_{\sigma\to\tau}$ if

for $N \in C_\sigma$, $(\lambda x^\sigma x^{\tau_1} \ldots x^{\tau_m}.c)N \in C_\tau$ if

for $P_1 \in C_{\tau_1}, \ldots, P_m \in C_{\tau_m}$, $(\lambda x^\sigma x^{\tau_1} \ldots x^{\tau_m}.c)NP_1 \ldots P_m \in C_0$ which holds since one can do at most $m+1$ reductions + finitely many ones in N and $P_1, \ldots, P_m$. In the last step we used the induction hypothesis which gives us SN(N), and SN(P_1), …, SN(P_m). □

Definition 2.3.34 Define the following classes of terms:

$$\begin{aligned} C &= \bigcup \{C_\sigma \mid \sigma \text{ is a type }\}, \\ C^*_\sigma &= \{M \in \Lambda_\sigma \mid M[\vec{x} := \vec{P}] \in C \text{ for } \vec{P} \subseteq C\}. \end{aligned}$$

where $\vec{P} \subseteq C$ denotes an arbitrary sequence of terms in C and where $\vec{x}$ should always contain all free variables of M.

The intuition behind C^*_σ is the set of terms M in Λ_σ such that every instance of M with elements in C is in C. In particular, if $M^* \equiv M[\vec{x} := \vec{P}]$ then $M \in C^*_\sigma \Rightarrow M^* \in C_\sigma$. By introducing this C^*_σ we pushed the induction loading somewhat further and this will allow us to have a proof by induction on the structure of the terms. In this inductive proof we shall use the following preservation properties.

Lemma 2.3.35 (Preservation properties).

(i) $M \in C \wedge M \twoheadrightarrow M' \Rightarrow M' \in C$,

(ii) $M \in C^*_\sigma \wedge M \twoheadrightarrow M' \Rightarrow M' \in C^*_\sigma$,

(iii) $M \in C^*_\sigma \wedge \vec{P} \subseteq C \Rightarrow M' :\equiv M[\vec{x} := \vec{P}] \in C^*_\sigma$,

(iv) $M \in C^*_\sigma \Rightarrow \mathrm{SN}(M)$.

Proof. (i) The proof is by induction on the type of M. If $M \in C_0$, i.e. $\mathrm{SN}(M)$ then $\mathrm{SN}(M')$ for if not, then $M \twoheadrightarrow M' \rightarrow \dots$, contradiction. If $M \in C_{\sigma\rightarrow\tau}$, then for arbitrary $N \in C_\sigma$ we have $MN \in C_\tau$ so by i.h. $M'N \in C_\tau$. This shows $M' \in C$.

(ii) Assume $M \in C^*_\sigma$ and $M \twoheadrightarrow M'$. It follows that $M[\vec{x} := \vec{P}] \twoheadrightarrow M'[\vec{x} := \vec{P}]$. This can be proved by induction on the derivation of $\twoheadrightarrow$, cf. [5] 3.1.14. Let $\vec{P} \subseteq C$, then we know $M[\vec{x} := \vec{P}] \in C$ and by (i), $M'[\vec{x} := \vec{P}] \in C$. This shows $M' \in C^*_\sigma$.

(iii) Let $\vec{y}$ contain the free variables of M'. We must show that for $\vec{Q} \subseteq C$ we have $M'[\vec{y} := \vec{Q}] \in C$, i.e. $M[\vec{x} := \vec{P}][\vec{y} := \vec{Q}] \in C$, i.e. $M[\vec{x}, \vec{y} := \vec{P}, \vec{Q}] \in C$, which holds because $M \in C^*_\sigma$.

(iv) Assume $M \in C^*_\sigma$ and let $\vec{P} \subseteq C$, then for $\vec{x} \supseteq FV(M)$ we have $M[\vec{x} := \vec{P}] \in C$ and hence by 2.3.33 (i) $\mathrm{SN}(M[\vec{x} := \vec{P}])$. Therefore certainly SN holds for M itself. □

Lemma 2.3.36 $M \in \Lambda_\sigma \Rightarrow M \in C^*_\sigma$.

Proof. The proof is by induction on the structure of M.

The first three cases are easy: Case 1 is $M \equiv x$. Now for all $P \in C$ one has $x[x := P] \in C$. Case 2 is $M \equiv c_i$ which is trivial. Case 3 is $M \equiv f_j(M_1, \ldots, M_{a_j})$. Use i.h. and the fact that each M_i is of type 0.

Case 4. $M \equiv AB$ with $A \in \Lambda_{\sigma\to\tau}$, $B \in \Lambda_\sigma$. Let $\vec{P} \subseteq C$. By i.h. $A \in C^*_{\sigma\to\tau}$ and $B \in C^*_\sigma$, i.e. $A[\vec{x} := \vec{P}] \in C_{\sigma\to\tau}$ and $B[\vec{x} := \vec{P}] \in C_\sigma$. Now $M[\vec{x} := \vec{P}] \equiv A[\vec{x} := \vec{P}]B[\vec{x} := \vec{P}]$ which is in C_τ and hence is in C. Hence $M \in C^*_\tau$.

Case 5. $M \equiv \lambda y \sqsubseteq A.B$ with $A \in \Lambda_\sigma, B \in \Lambda_\tau$. Let $\vec{P} \subseteq C$. By i.h. $A \in C^*_\sigma$, $B \in C^*_\tau$. We may assume $y \notin \vec{x}$. Abbreviate $A^* :\equiv A[\vec{x} := \vec{P}]$, $B' :\equiv B[\vec{x} := \vec{P}]$ so $A^* \in C_\sigma$ and by 2.3.35 (iii) $B' \in C^*_\tau$. Note that y may still occur free in B'. Let $\tau = \tau_1 \to \ldots \to \tau_n \to 0$.

> $M \in C^*_{\sigma\to\tau}$ if
>
> $M^* :\equiv \lambda y \sqsubseteq A^*.B' \in C_{\sigma\to\tau}$ if
>
> for $S \in C_\sigma$, $(\lambda y \sqsubseteq A^*.B')S \in C_\tau$ if
>
> for $T_1 \in C_{\tau_1}, \ldots, T_n \in C_{\tau_n}$, $\mathrm{SN}((\lambda y \sqsubseteq A^*.B')ST_1 \ldots T_n)$ which holds, for if not, then three possibilities arise, each leading to a contradiction.

Either (a) A^* or B' have an infinite reduction. But $B' \in C^*_\tau$ so by lemma 2.3.35 (iv) $\mathrm{SN}(B')$ and by lemma 2.3.33 (i) also $\mathrm{SN}(A^*)$. Contradiction.

or (b) S or $T_1 \ldots T_n$ have an infinite reduction which contradicts $S \in C$, $T_1 \in C, \ldots, T_n \in C$ and lemma 2.3.33(i).

or (c) $(\lambda y \sqsubseteq A^*.B')ST_1 \ldots T_n \twoheadrightarrow (\lambda y \sqsubseteq A^{*\prime}.B'')S'T_1' \ldots T_n') \to B''[y := S']T_1' \ldots T_n' \to \ldots$. But by lemma 2.3.35 (i) $S' \in C$, $T_1' \ldots T_n' \in C$ and by lemma 2.3.35 (ii) $B'' \in C^*_\tau$. Therefore by the definition of C^*_τ we have $B''^* :\equiv B''[y := S'] \in C$. From the definition of C we have $B''^*T_1' \ldots T_n' \in C_0$, i.e. $\mathrm{SN}(B''[\ldots]T_1' \ldots T_n')$. Contradiction. □

Theorem 2.3.37 (SN). In $\lambda\pi$ every term strongly normalises.

Proof. By lemma 2.3.36 and lemma 2.3.35 (iv) we have that in $\lambda\beta$ every term strongly normalises. It follows that in $\lambda\pi$ every term strongly normalises, for suppose that M has an infinite reduction in $\lambda\pi$, then so it has in $\lambda\beta$ which cannot be the case. □

Remark 2.3.38 An alternative proof of SN for $\lambda\pi$-calculus can be given by using a technique due to Plotkin (who actually considered λ-typed λ-calculus), suggested to us by L.S. van Benthem Jutting. Let $\Lambda_{\Re}$ denote the set of terms of $\lambda\pi$-calculus and let Λ denote the set of terms of classical simple-typed λ-calculus. Define a mapping $\overline{} : \Lambda_{\Re} \to \Lambda$ as follows:

$$\begin{aligned}
&\overline{x^\tau} :\equiv x^\tau,\\
&\overline{c_i} :\equiv c_i,\\
&\overline{f_j(P_1,\ldots,P_{a_j})} :\equiv f_j(\overline{P_1},\ldots,\overline{P_{a_j}}),\\
&\overline{(MN)} :\equiv (\overline{M}\,\overline{N}),\\
&\overline{(\lambda x^\sigma \sqsubseteq M.N)} :\equiv ((\lambda y^\sigma.\lambda x^\sigma.\overline{N})\overline{M}) && (y^\sigma \text{ fresh}).
\end{aligned}$$

By induction on the structure of M it can be shown that $\overline{M[x := N]} \equiv \overline{M}[x := \overline{N}]$. Using this, one shows by induction on the derivation of $\Gamma \vdash M \to N$ that

$$\Gamma \vdash M \to N \;\Rightarrow\; \overline{M} \overset{\neq}{\twoheadrightarrow} \overline{N}.$$

Now if a term $M \in \Lambda_{\Re}$ has an infinite reduction path, then so does $\overline{M}$, which cannot be the case by SN for classical simple-typed λ-calculus. □

2.3.7 Confluence

The diamond property (= confluence = Church-Rosser property) holds for $\lambda\pi$. This means that when a term M allows for two different reductions, $\Gamma \vdash M \twoheadrightarrow M_1, M \twoheadrightarrow M_2$ say, then these can always be 'brought together' by further reductions, i.e. there is an M_3 with $\Gamma \vdash M_1 \twoheadrightarrow M_3, M_2 \twoheadrightarrow M_3$. The remainder of this section is about proving this property. The reader may want to skip this proof and in that case he can proceed with Section 2.4. By way of preparation for this proof, we introduce the so-called weak diamond property and we shall show that the latter holds for $\lambda\pi$ first. This weak diamond property is relatively easy to prove. In combination with the SN property it will give us a cheap route towards proving the full diamond property.

Definition 2.3.39 A one-step reduction relation $\to$ satisfies the *weak diamond property* if ($\twoheadrightarrow$ denoting the reflexive transitive closure)

$$\Gamma \vdash M \to M_1, M \to M_2 \;\Rightarrow\; \exists M_3\,(\Gamma \vdash M_1 \twoheadrightarrow M_3, M_2 \twoheadrightarrow M_3). \qquad \square$$

Lemma 2.3.40 (Substitution)

(i) $\Gamma \vdash B \to B' \;\Rightarrow\; \Gamma[x := R] \vdash B[x := R] \to B'[x := R]$,

(ii) $\Gamma \vdash R \to R' \;\Rightarrow\; \Gamma \vdash B[x := R] \twoheadrightarrow B[x := R']$.

Proof. (i) By induction on the derivation of $\Gamma \vdash B \to B'$. (ii) By induction on the structure of B. □

Lemma 2.3.41 The reduction relation $\rightarrow$ satisfies the weak diamond property.

Proof. By induction on the derivation of $\Gamma \vdash M \rightarrow M_1$ it will be shown that for all M_2 such that $\Gamma \vdash M \rightarrow M_2$ there is an M_3 such that $\Gamma \vdash M_1 \twoheadrightarrow M_3$ and $\Gamma \vdash M_2 \twoheadrightarrow M_3$. For the trivial $M_2 \equiv M_1$ we can always take $M_3 \equiv M_1$ so in some cases we shall not mention this trivial M_2. The numbering of the cases below corresponds to the numbering of the cases in 2.3.28.

1. $\Gamma \vdash M \rightarrow M_1$ is $\Gamma \vdash (\lambda x \sqsubseteq A.B)R \rightarrow B[x := R]$ because $\Gamma \vdash R \sqsubseteq A$.

   (i) if $M \rightarrow M_2$ is $M \rightarrow B[x := R]$ we are done immediately: $M_3 \equiv M_1 \equiv M_2$.

   (ii) if $M \rightarrow M_2$ is $M \rightarrow (\lambda x \sqsubseteq A'.B)R$, take $M_3 \equiv M_1$ (noting $\Gamma \vdash R \sqsubseteq A'$).

   (iii) if $M \rightarrow M_2$ is $M \rightarrow (\lambda x \sqsubseteq A.B')R$ then take $M_3 \equiv B'[x := R]$ and apply lemma 2.3.40 (i).

   (iv) if $M \rightarrow M_2$ is $M \rightarrow (\lambda x \sqsubseteq A.B)R'$ then take $M_3 \equiv B[x := R']$ and apply lemma 2.3.40 (ii).

2. $\Gamma \vdash M \rightarrow M_1$ is $\Gamma \vdash f_j(\ldots, X, \ldots) \rightarrow f_j(\ldots, X_1, \ldots)$ because $\Gamma \vdash X \rightarrow X_1$. Two cases arise:

   (i) $M \rightarrow M_2$ is $M \rightarrow f_j(\ldots, X_2, \ldots)$. By i.h. there exists an X_3 such that $X_1 \twoheadrightarrow X_3$ and $X_2 \twoheadrightarrow X_3$. Take $M_3 \equiv f_j(\ldots, X_3, \ldots)$.

   (ii) $M \rightarrow M_2$ is $f_j(\ldots, X, \ldots, Y, \ldots) \rightarrow f_j(\ldots, X, \ldots, Y_1, \ldots)$. Take $M_3 \equiv f_j(\ldots, X_1, \ldots, Y_1, \ldots)$.

3. $\Gamma \vdash M \rightarrow M_1$ is $\Gamma \vdash ZX \rightarrow ZX_1$ because $\Gamma \vdash X \rightarrow X_1$.

   (i) if $M \rightarrow M_2$ is $ZX \rightarrow Z'X$, then take $M_3 \equiv Z'X_1$.

   (ii) if $M \rightarrow M_2$ is $ZX \rightarrow ZX_2$, then take $M_3 \equiv ZX_3$ where X_3 is given by i.h.

   (iii) if $M \rightarrow M_2$ is $(\lambda x \sqsubseteq A.B)X \rightarrow B[x := X]$, then take $M_3 \equiv B[x := X_1]$ and apply lemma 2.3.40 (ii).

4. $\Gamma \vdash M \rightarrow M_1$ is $\Gamma \vdash XZ \rightarrow X_1Z$ because $\Gamma \vdash X \rightarrow X_1$.

   (i) if $M \rightarrow M_2$ is $XZ \rightarrow X_2Z$, then take $M_3 \equiv X_3Z$ where X_3 is given by i.h.

   (ii) if $M \rightarrow M_2$ is $XZ \rightarrow XZ'$, then take $M_3 \equiv X_1Z'$.

(iii) if $M \to M_2$ is $(\lambda x \sqsubseteq A.B)Z \to B[x := Z]$, then either $X \to X_1$ by $A \to A_1$ or $X \to X_1$ by $B \to B_1$. If by $A \to A_1$ then take $M_3 \equiv B[x := Z]$. If by $B \to B_1$ then take $M_3 \equiv B_1[x := Z]$ and apply lemma 2.3.40 (i).

5. $\Gamma \vdash M \to M_1$ is $\Gamma \vdash (\lambda x \sqsubseteq P.X) \to (\lambda x \sqsubseteq P_1.X)$ because $\Gamma \vdash P \to P_1$.

(i) if $M \to M_2$ by $P \to P_2$ then take $M_3 \equiv \lambda x \sqsubseteq P_3.X$ where P_3 is given by i.h.

(ii) if $M \to M_2$ by $X \to X'$ then take $M_3 \equiv \lambda x \sqsubseteq P_1.X'$.

6. $\Gamma \vdash M \to M_1$ is $\Gamma \vdash (\lambda x \sqsubseteq P.X) \to (\lambda x \sqsubseteq P.X_1)$ because $\Gamma, [x \sqsubseteq P] \vdash X \to X_1$ and $x \notin \Gamma$.

(i) if $M \to M_2$ by $P \to P'$ then take $M_3 \equiv \lambda x \sqsubseteq P'.X_1$ and use 2.3.30 (i), rule (subst.) and the cut-rule 2.3.20 to show that $\Gamma, [x \sqsubseteq P'] \vdash X \to X_1$.

(ii) if $M \to M_2$ by $X \to X_2$ then take $M_3 \equiv \lambda x \sqsubseteq P.X_3$ where X_3 is given by i.h. (in context $\Gamma, [x \sqsubseteq P]$). □

The following is known as Newman's lemma and it provides us with an easy way of proving the diamond property.

Lemma 2.3.42 If a one-step reduction relation $\to$ satisfies the weak diamond property and has the strong normalisation property (SN), then its reflexive transitive closure $\twoheadrightarrow$ has the diamond property ($\diamond$).

Proof. Newman's result. See [5] proposition 3.2.25. □

Theorem 2.3.43 ($\diamond$). $\twoheadrightarrow$ has the diamond property, i.e. if $\Gamma \vdash M \twoheadrightarrow M_1, M \twoheadrightarrow M_2$ then there is an M_3 with $\Gamma \vdash M_1 \twoheadrightarrow M_3, M_2 \twoheadrightarrow M_3$.

Proof. By 2.3.42, 2.3.41 and 2.3.37. □

The diamond property for $\lambda\pi$ does not essentially depend on SN. As G.R. Renardel de Lavalette has shown, it is possible to define an untyped version of $\lambda\pi$ which satisfies the diamond property. Tait's proof method (see [5] pp. 59-62) is applicable, both for the untyped and typed versions of the calculus.

2.4 Components

2.4.1 Introduction

Let us assume that an algebraic system $\Re$ is given. Roughly speaking, $\Re$ provides us with a notion of a construct 'being constructed from' a number of primitive constructs and with ways of fitting constructs together. Furthermore the possibility that one construct is a specification of another construct exists. On top of this we have the $\lambda\pi$-calculus and we assume that the developer is allowed to make use of the possibilities of this lambda calculus whenever possible.

$\Lambda_{\Re}$ is a starting point from which a software developer can start his constructive activity. The developer could try to write one (probably large) term such that his customer is satisfied with this term. Let us call such a term an *end-product.* Clearly this is not a feasible approach, so there must be a top-level specification which serves as a kind of contract between developer and customer. But even with a top-level specification it is difficult to have a manageable development process. Therefore we shall choose another representation for the end-product, viz. a representation in which the term consists of sub-terms which can be referred to and such that some redundant terms have been added. The latter terms serve as specifications and we shall call them *black-box descriptions.*

For the customer, the references and the blackbox descriptions are probably not interesting; in any case they do no harm, since it is always possible to look-up the references and to throw away the black-box descriptions. For the developer, the references and the black-box descriptions are extremely helpful *during* the development since they make it easier to manipulate the product.

We adopt the viewpoint that introducing references and adding black-box descriptions should be combined, in the sense that whenever a name (a rerefence) is given to some term P, there should also be a black-box description associated with P. This viewpoint immediately leads us to a formal notion of a component. We formalise components as triples, consisting of a name and two terms, where the second term is the black-box description.

We must add one remark here. Strictly speaking, it is possible to have black-box descriptions without using references. This approach is often used in combination with Hoare's logic. The black-box descriptions correspond to pairs of pre- and postconditions. The black-box descriptions are textually included (as a comment) at the appropriate places in the program text. However, the use of references is more general.

We shall say that a component is *correct* if the two terms of the component are in the implementation relation. In order to benefit from the introduction of components, one should require that all components are correct. This requirement can serve as an invariant of the development process. We shall formalise what it means that a component is correct – in a given context.

We shall also have the possibility to have references to constructs which are outside the scope of the developer and which are given by a name and a black-box description only. We introduce a symbol **prim** which in these cases will serve as a placeholder.

2.4.2 Formal definitions

For given $\mathfrak{R}$ we could, as a first attempt, define the set of components as $\{x_i \mid i \in \mathbb{N}\} \times (\Lambda_{\mathfrak{R}} \cup \{\mathbf{prim}\}) \times \Lambda_{\mathfrak{R}}$. However, we must take care to respect the typing, so we adopt the following definition:

Definition 2.4.1 Let $\mathfrak{R}$ be an algebraic system with preorder.

(i) Define $C'_{\mathfrak{R}} = \{x_i \mid i \in \mathbb{N}\} \times (\Lambda_{\mathfrak{R}} \cup \{\mathbf{prim}\}) \times \Lambda_{\mathfrak{R}}$.

(ii) The set $C_{\mathfrak{R}}$ of *components* is defined by $C_{\mathfrak{R}} = \{(x_i^\sigma, P, Q) \in C'_{\mathfrak{R}} \mid Q$ has type σ and if $P \not\equiv \mathbf{prim}$ then P has type $\sigma\}$.

(iii) For a component $c = (x, P, Q)$ we say that x is the *name* of c, P is the *glass-box description* of c and Q is the *black-box description* of c. We shall often just write C for $C_{\mathfrak{R}}$. □

Recall that we have the implementation relation denoted by $\sqsubseteq$.

Definition 2.4.2 A component (x, P, Q) where $P \not\equiv \mathbf{prim}$ is *correct* in context Γ if

$$\Gamma \vdash P \sqsubseteq Q.$$

A component $(x, \mathbf{prim}, Q)$ is always correct in any context. □

Notation 2.4.3 (Concrete syntax). We shall write

$$x \;:=\; P \;\sqsubseteq\; Q$$

to denote the component (x, P, Q). □

The concrete syntax is intended to be used in very small examples. In the practical applications we have in mind, we have at least two possibilities.

The first possibility is to have a so-called design-engineering database with operations among which an operation that yields the glass-box description of a component and an operation that yields the black-box description of a component. The second possibility is to include the notion of a component in a design language (as done for COLD-K) and we refer to [11] for a concrete syntax used to represent components.

Example 2.4.4 Consider $\mathfrak{R}_1$ as before. Now the component

$$y \quad := \quad x + x \quad \sqsubseteq \quad 3$$

is correct in the context $[x \sqsubseteq 1]$ since in the $\lambda\pi$-calculus we have

$$[x \sqsubseteq 1] \vdash x + x \sqsubseteq 3. \qquad \square$$

Components are somewhat similar to 'lines' in Automath [13], [14]. The similarity exists only on the syntactical level because the implementation relation of Automath (a *typing* relation) is completely different from '$\sqsubseteq$'.

2.5 Designs

2.5.1 Introduction

In the previous section we discussed the notion of a component. Based on this notion we shall define the notion of a *design*[2]. First of all, it is clear that a design should contain a collection of components. In fact we shall define a design such that it contains a sequence of components. In a design it is possible that a component contains references to other components, but we want to exclude the possibility of circular referencing. Because the components are elements of a sequence, this can be achieved simply by requiring that no name is used before it has been introduced. We shall formulate a well-formedness predicate on designs.

Besides this sequence of components, each design contains one additional term, which we shall call the *system* of the design. The system is to be viewed as an indication of the actual product to be delivered to the customer. In most cases this system will be a very simple term which contains the names of one or more components of the design.

We shall define two notions of correctness for designs, which we shall call *glass-box correctness* and *black-box correctness*. Both notions are interesting

[2]We use 'design' in the sense of 'design object' rather than 'design operation'.

from a methodological point of view. The syntax and the correctness of designs will be defined in Section 2.5.2.

It is possible to translate each well-formed design into a lambda term. For an arbitrary design d the lambda term resulting from this translation can be viewed as the *meaning* of d. In this translation, there is an abstraction-application pair corresponding to each non-prim component. This translation will be defined in subsection 2.5.3. Also in subsection 2.5.3, we shall see that several interesting properties of designs have simple counterparts in the $\lambda\pi$-calculus. In subsection 2.5.4 we shall investigate *correctness-preserving modifications* of designs.

2.5.2 Syntax and Correctness

Definition 2.5.1 The set $D_{\Re}$ of *designs* is defined by

$$D_{\Re} = (C_{\Re})^* \times \Lambda_{\Re}.$$

For a design $d = ((c_1, \ldots, c_n), S)$ we say that $\{c_1, \ldots, c_n\}$ is the *set of components* of d and we denote it by $cset(d)$ and we call S the *system* of d. We shall often just write D for $D_{\Re}$. □

Notation 2.5.2 We shall employ the following *concrete syntax*

$$\begin{array}{llll} x_1 & := & P_1 & \sqsubseteq & Q_1 \\ x_2 & := & P_2 & \sqsubseteq & Q_2 \\ \ldots & & & & \\ x_n & := & P_n & \sqsubseteq & Q_n \\ \multicolumn{3}{l}{\textbf{system } S} & & \end{array}$$

to denote the design $(((x_1, P_1, Q_1), (x_2, P_2, Q_2), \ldots, (x_n, P_n, Q_n)), S)$. □

Again the concrete syntax is intended to be used for very small examples. Designs are somewhat similar to Automath 'books' [13], [14] where the components correspond to Automath 'lines'. However, there is no such thing as a 'system' in Automath books.

Definition 2.5.3 Let d be given as $d = ((c_1, \ldots, c_j, \ldots, c_n), S)$, then we call the design $((c_j, \ldots, c_n), S)$ the *j-th sub-design* of d and we denote it as $d_{(j)}$. □

Definition 2.5.4 A design d is *well-formed* (abbreviated as *wf*) if the following conditions hold:

(i) All components of d have distinct names.

(ii) No variable occurring in d is used before it has been introduced as the name of a component. Formally, if d is given as in 2.5.2 (concrete syntax), then for $1 \leq j \leq n$, $FV(P_j) \cup FV(Q_j) \subseteq \{x_1, \ldots, x_{j-1}\}$, taking $FV(\mathbf{prim}) := \emptyset$.

(iii) Let S be the system of d, then each variable in $FV(S)$ must be the name of one of the components of d. □

We defined already what it means that a component is correct in a given context. Now we want to define correctness for wf designs. Roughly speaking, we shall define this in such a way that a design is correct iff each of its components is correct. In order to make this idea precise we must be explicit about the contexts in which the correctness of the components is to be derived. There are two reasonable possibilities for defining these contexts. Therefore we shall have two notions of correctness for designs.

The first notion of correctness corresponds to the possibility that there is no information hiding: if the developer reasons about a name x_k for which there is a component (x_k, P_k, Q_k) in the design, then the developer may use the fact that x_k stands for P_k. If $P_k \equiv \mathbf{prim}$, then the best assumption he can make about x_k is $[x_k \sqsubseteq Q_k]$.

Definition 2.5.5 (gbc) Let the design d be given as

$$\begin{array}{lllll} x_1 & := & P_1 & \sqsubseteq & Q_1 \\ & \ldots & & & \\ x_n & := & P_n & \sqsubseteq & Q_n \\ \multicolumn{5}{l}{\mathbf{system}\ S.} \end{array}$$

Assume that d is wf. We say that d is *glass-box correct* (abbreviated *gbc*) if for each component $(x_j, P_j, Q_j) \in cset(d)$ where $P_j \not\equiv \mathbf{prim}$ we have

$$\Gamma_j \vdash P_j \sqsubseteq Q_j$$

where $\Gamma_j = \varphi_1, \ldots, \varphi_{j-1}$ and where for $1 \leq k \leq j-1$ the φ_k are defined by

(i) $\varphi_k = [x_k = P_k]$ $(P_k \not\equiv \mathbf{prim})$,

(ii) $\varphi_k = [x_k \sqsubseteq Q_k]$ $(P_k \equiv \mathbf{prim})$. □

The second notion of correctness corresponds to the possibility that the glass-box descriptions are hidden: if the developer reasons about a name x_k for which there is a component (x_k, P_k, Q_k) in the design, then he may only use the fact that the term for which x_k stands, is specified by Q_k, i.e. he may use the assumption $[x_k \sqsubseteq Q_k]$.

Definition 2.5.6 (bbc) Let the design d be given as

$$\begin{array}{llll} x_1 & := & P_1 & \sqsubseteq\ Q_1 \\ & \ldots & & \\ x_n & := & P_n & \sqsubseteq\ Q_n \\ \multicolumn{4}{l}{\textbf{system}\ S.} \end{array}$$

Assume that d is wf. We say that d is *black-box correct* (abbreviated *bbc*) if for each component $(x_j, P_j, Q_j) \in cset(d)$ where $P_j \not\equiv$ **prim** we have

$$\Gamma_j \vdash P_j \sqsubseteq Q_j$$

where $\Gamma_j = \varphi_1, \ldots, \varphi_{j-1}$ and where for $1 \leq k \leq j-1$ the φ_k are defined by

$$\varphi_k = [x_k \sqsubseteq Q_k]. \qquad \square$$

Example 2.5.7 Consider $\Re_1$ as before. The following design is gbc but not bbc.

$$\begin{array}{llll} x_1 & := & 5 & \sqsubseteq\ 10 \\ x_2 & := & x_1 & \sqsubseteq\ 7 \\ \multicolumn{4}{l}{\textbf{system}\ x_2.} \end{array}$$

$\square$

We defined two notions of correctness, viz. gbc and bbc. Intuitively it is clear that bbc is stronger than gbc, since the facts $P_j \sqsubseteq Q_j$ to be derived are the same for gbc and bbc, but for bbc these facts should be derived with less knowledge than for gbc. This intuition is made more precise below.

Theorem 2.5.8 Let d be a wf design. Then we have

$$d \text{ is bbc} \quad \Rightarrow \quad d \text{ is gbc}.$$

Proof. We give the details for a design d with two components. After that we give a proof-sketch for the general case. Let d be given as in 2.5.6 (definition bbc). Assume that d is bbc. We distinguish two cases, where each case falls apart into two subcases.

1. $P_1 \equiv$ **prim**. The first component is trivially correct. For the second component we distinguish two cases.

   (i) $P_2 \equiv$ **prim**. The second component is also trivially correct. Therefore d is gbc.

   (ii) $P_2 \not\equiv$ **prim**. For the second component we must show $[x_1 \sqsubseteq Q_1] \vdash P_2 \sqsubseteq Q_2$, which follows from the fact that d is bbc. Therefore d is gbc.

2. $P_1 \not\equiv$ **prim**. The bbc-correctness condition for the first component is the same as its gbc-correctness condition, viz. $\vdash P_1 \sqsubseteq Q_1$. So the first component is correct. For the second component we distinguish two cases.

   (i) $P_2 \equiv$ **prim**. The second component is now trivially correct. Therefore d is gbc.

   (ii) $P_2 \not\equiv$ **prim**. From the fact that d is bbc we have $\vdash P_1 \sqsubseteq Q_1$ and $[x_1 \sqsubseteq Q_1] \vdash P_2 \sqsubseteq Q_2$. From $\vdash P_1 \sqsubseteq Q_1$ by (subst.) we get $[x_1 = P_1] \vdash x_1 \sqsubseteq Q_1$ and now we can use the cut-rule to get $[x_1 = P_1] \vdash P_2 \sqsubseteq Q_2$. This shows that the second component is correct in the gbc sense. Therefore d is gbc.

This concludes the proof for a design d with two components. For the general case, the following assertion is proved by induction on the number of components (n).

$$\bigwedge_{j=1}^{n} \left(P_j \not\equiv \mathbf{prim} \Rightarrow \Gamma_j \vdash P_j \sqsubseteq Q_j \right)$$

where $\Gamma_j = \varphi_1, \ldots, \varphi_{j-1}$ and where for $1 \leq k \leq n-1$ the φ_k are defined by

(i) $\varphi_k = [x_k = P_k] \quad (P_k \not\equiv \mathbf{prim})$,

(ii) $\varphi_k = [x_k \sqsubseteq Q_k] \quad (P_k \equiv \mathbf{prim})$.

Again use rule (subst.) and the (generalised) cut-rule. □

2.5.3 Semantics

It is possible to translate each design d into a lambda term[3]. The lambda term resulting from this translation can be viewed as the *meaning* of d and we shall denote it as $[\![d]\!]$. For each component of d for which the glass-box description is not **prim**, there is an abstraction-application pair in $[\![d]\!]$. For each component of d for which the glass-box description is **prim**, there is an abstraction in $[\![d]\!]$. A similar technique of using abstraction-application pairs and abstractions to describe the role of names has been proposed by N.G. de Bruijn in the context of Automath. Readers who are not familiar with Automath could skip this explanation and proceed with the next paragraph (example 2.5.9). In particular each so-called 'book', which is a structure as for example

nat	:=	prim	:	*type*
zero	:=	prim	:	nat
suc	:=	prim	:	(nat $\rightarrow$ nat)
one	:=	suc(zero)	:	nat

can be viewed as a λ-term, viz. $\lambda\,$nat : *type*.$\lambda\,$zero : nat.$\lambda\,$suc : (nat $\rightarrow$ nat).($\lambda\,$one : nat.*type*)(suc(zero)). De Bruijn noted this 'books-as-lambda-terms' analogy where the correctness of a book follows from the well-typedness of its λ-term. In fact the analogy is somewhat complicated when considering δ-reductions. Of course, de Bruijn's typing relation ':' is a kind of implementation relation which has completely different properties as our '$\sqsubseteq$'. Yet the idea works for designs in the sense that the author noted a 'designs-as-lambda-terms' analogy.

We explain the translation procedure for designs in a step by step manner for a simple case. After that we give a formal definition which deals with the general case.

Example 2.5.9 Consider the design d of example 2.5.7. To translate this d we begin with its first component $(x_1, 5, 10)$ which we translate into the following 'term with hole' $(\lambda x_1 \sqsubseteq 10.\cdots)5$ where the dots $\cdots$ indicate the hole. Later the hole gets filled with the translation of the rest of the design. Next, we proceed with the second component $(x_2, x_1, 7)$ which we translate into $(\lambda x_2 \sqsubseteq 7.\cdots)x_1$. The latter 'term with hole' is put in the first hole and this yields $(\lambda x_1 \sqsubseteq 10.(\lambda x_2 \sqsubseteq 7.\cdots)x_1)5$. Now we are left with one hole again and the last step is to put the system in that hole. So $[\![d]\!] = (\lambda x_1 \sqsubseteq$

[3]The idea of using lambda terms as a semantic domain is not new and has e.g. been used by Landin [12] who mapped ALGOL60 to an enriched lambda calculus; in particular, it is interesting to note that **let** $x = Z$; X is mapped to $(\lambda x.X)Z$.

10.$(\lambda x_2 \sqsubseteq 7.x_2)x_1)5$. □

For components with **prim**, a slightly different treatment is needed, which is not shown yet by this example. By way of preparation for the precise definition of the translation procedure, we need an auxiliary definition.

Definition 2.5.10 For a design $d = ((c_1, \ldots, c_n), S)$ with $n \geq 1$ we define $\text{first}(d) := c_1$ and $\text{rest}(d) := ((c_2, \ldots, c_n), S)$. □

So first(d) yields the first component of d and rest(d) is d with its first component removed. Note that 'rest' does not preserve wf. We give the formal definition, which is defined by recursion.

Definition 2.5.11 (Meaning function). Define $\mathcal{M} : D \to \Lambda_{\Re}$. Write $[\![d]\!]$ for $\mathcal{M}(d)$.

1. $[\![\,\textbf{system}\ S\,]\!] \equiv S$,
2. $[\![d]\!] \equiv (\lambda x \sqsubseteq Q.[\![\text{rest}(d)]\!])P$ if $\text{first}(d) = (x, P, Q)$ and $P \not\equiv \textbf{prim}$,

   $\equiv (\lambda x \sqsubseteq Q.[\![\text{rest}(d)]\!])$ if $\text{first}(d) = (x, \textbf{prim}, Q)$. □

The recursive formulation of $\mathcal{M}$ was originally defined by the author. A slightly different formulation has been employed later by H.B.M. Jonkers. It is a continuation semantics where a translation is given first for each component.

Definition 2.5.12 (Alternative formulation) First define $\mathcal{M} : C \to \Lambda_{\Re} \to \Lambda_{\Re}$, where C is the set of components. Write $[\![c]\!]$ for $\mathcal{M}(c)$.

1. $[\![(x, P, Q)]\!](R) :\equiv (\lambda x \sqsubseteq Q.R)P$ if $P \not\equiv \textbf{prim}$,
2. $[\![(x, \textbf{prim}, Q)]\!](R) :\equiv (\lambda x \sqsubseteq Q.R)$.

Define $\mathcal{M} : D \to \Lambda_{\Re}$. Write $[\![d]\!]$ for $\mathcal{M}(d)$.

1. $[\![((c_1, \ldots, c_n), S)]\!] :\equiv ([\![c_1]\!] \circ \ldots \circ [\![c_n]\!])(S)$

where $\circ$ denotes function composition. □

We can make several observations which support the idea that $[\![d]\!]$ can be viewed as the *meaning* of d. First of all, the components $(x_k, \textbf{prim}, Q_k)$ become lambda abstractions where the black-box description of the component becomes the restriction associated with this abstraction $(\lambda x_k \sqsubseteq Q_k.\ldots)$. This corresponds precisely to the idea that when the design is applied in the real world, something satisfying the black-box description has to be filled in for a

prim. We see that the components with a **prim** glass-box description can be viewed as the formal *parameters* of the design in which they occur. Secondly, we can observe that reducing $[\![d]\!]$ corresponds to replacing the names by the things they stand for (i.e. by the corresponding glass-box descriptions). If a component (x_j, P_j, Q_j) is not correct, then we may consider replacing x_j by P_j as illegal. This is precisely reflected in $[\![d]\!]$, where a candidate-redex $(\lambda x_j \sqsubseteq Q_j.\ldots)P_j$ cannot be contracted unless $P_j \sqsubseteq Q_j$.

Example 2.5.13 Assume variables w, x, y, z and let Q_1, Q_2, Q_3, Q_4, P_2, P_4, $S \in \Lambda_{\Re}$. Let the design d be given by

$$\begin{array}{lllll} w & := & \mathbf{prim} & \sqsubseteq & Q_1 \\ x & := & P_2 & \sqsubseteq & Q_2 \\ y & := & \mathbf{prim} & \sqsubseteq & Q_3 \\ z & := & P_4 & \sqsubseteq & Q_4 \\ \multicolumn{2}{l}{\mathbf{system}} & S. & & \end{array}$$

The meaning of d, denoted as $[\![d]\!]$ is given by

$$[\![d]\!] \equiv \lambda w \sqsubseteq Q_1.(\lambda x \sqsubseteq Q_2.(\lambda y \sqsubseteq Q_3.(\lambda z \sqsubseteq Q_4.S)P_4))P_2. \qquad \square$$

Theorem 2.5.14 d is wf $\Rightarrow$ $[\![d]\!]$ is closed. **Proof.** Obvious from $\mathcal{M}$. $\square$

We want to have a look at reduction strategies for terms $[\![d]\!]$ and in order to formulate these strategies precisely, we introduce a bit of machinery for marking candidate-redexes and keeping track of what happens with these marked candidate-redexes during reduction. This technique is known as *lifting* and we adopt the notations and conventions from [5] 11.1.2, 11.2.1 – adapted to $\lambda\pi$-calculus. We mark a candidate-redex by giving an index to its first lambda. For this purpose we introduce an auxiliary extension of the set of terms $\Lambda_{\Re}$. The following should be compared with definition 2.3.3.

Definition 2.5.15 (Terms with indices)

$\Lambda'_{\Re}$ is a set of terms constructed over the alphabet of $\Lambda_{\Re}$ extended with symbols $\lambda_0, \lambda_1, \lambda_2, \ldots$. This set $\Lambda'_{\Re}$ is inductively defined by the clauses below, where it is understood that furthermore the same rules and restrictions with respect to the types as for $\Lambda_{\Re}$, apply in the obvious way.

1. $x_i \in \Lambda'_{\Re}$ $(i \in \mathbb{N})$,
2. $c_i \in \Lambda'_{\Re}$ $(i \in I)$,
3. if $P_1, \ldots, P_{a_j} \in \Lambda'_{\Re}$ then $f_j(P_1, \ldots, P_{a_j}) \in \Lambda'_{\Re}$,
4. if $P, Q \in \Lambda'_{\Re}$ then $(PQ) \in \Lambda'_{\Re}$,

5. if $P, Q \in \Lambda'_{\Re}$ and $x_i \notin FV(P)$, then $(\lambda x_i \sqsubseteq P.Q) \in \Lambda'_{\Re}$,
6. if $P, Q, R \in \Lambda'_{\Re}$, $j \in \mathbb{N}$ and $x_i \notin FV(P)$, then $((\lambda_j x_i \sqsubseteq P.Q)R) \in \Lambda'_{\Re}$. In this case we say that the candidate-redex $((\lambda_j x_i \sqsubseteq P.Q)R)$ *has index* j.

If $P \in \Lambda'_{\Re}$, then $|P| \in \Lambda_{\Re}$ is P with all indices removed. □

This provides us with an extended set of terms and the obvious next step is to say how the indices are treated during reduction.

Definition 2.5.16 (Reduction with indices)

(i) Substitution on $\Lambda'_{\Re}$ is defined in the usual way. See [5] 11.1.3.

(ii) In addition to normal π-reduction $\to$, we define π_j-reduction (also denoted by $\to$) which means to contract a candidate-redex having index j (one notion of reduction for each index $j \in \mathbb{N}$). In particular, $\Gamma \vdash |R| \sqsubseteq |A| \Rightarrow \Gamma \vdash (\lambda_j x \sqsubseteq A.B)R \to B[x := R]$ and we refer to the latter $\to$ step as a π_j-reduction step.

(iii) The notion of reduction π' is $\pi \cup \bigcup_{j \in \mathbb{N}} \pi_j$. □

So now we have the features that candidate-redexes can be indexed optionally and that we can keep track of these candidate-redexes during reduction. The following should be compared with [5] 11.1.6 (i).

Lemma 2.5.17 (Lifting) Let $P, Q \in \Lambda_{\Re}$ such that $\Gamma \vdash P \twoheadrightarrow Q$ where $\twoheadrightarrow$ is based on π-reduction, then for all $P' \in \Lambda'_{\Re}$ with $|P'| \equiv P$ there is a Q' with $|Q'| \equiv Q$ such that $\Gamma \vdash P' \twoheadrightarrow Q'$, where the latter $\twoheadrightarrow$ is based on π'-reduction.

Proof. If $\Gamma \vdash P \twoheadrightarrow Q$ is a one-step π-reduction, where a candidate-redex is contracted, Δ say, then it is possible to identify a corresponding candidate-redex Δ', say in P'. Take the Q' obtained by contracting this Δ' in P'. If $Q \equiv P$ then take $Q' \equiv P'$. If $P \twoheadrightarrow Q$ involves more than one step, then use transitivity. □

The following should be compared with [5] 11.1.6 (ii).

Lemma 2.5.18 (Projecting) Let $P', Q' \in \Lambda'_{\Re}$ such that $\Gamma \vdash P' \twoheadrightarrow Q'$ where $\twoheadrightarrow$ is based on π'-reduction, then $\Gamma \vdash |P'| \twoheadrightarrow |Q'|$ where the latter $\twoheadrightarrow$ is based on π-reduction.

Proof. Leave out all indices from a reduction path from P' to Q'. □

When translating designs to $\lambda\pi$-terms, each non-**prim** component gives rise to a candidate-redex and we shall associate indices with these candidate-redexes in a systematic way: index 1 for a candidate-redex derived from the first component, index 2 for a candidate-redex derived from the second component etc. In order to formulate this precisely we give a re-formulation of the meaning function $\mathcal{M}$ introduced before. Therefore we introduce an extended meaning function $\mathcal{M}'$ using an auxiliary which takes two arguments and which is also denoted by $\mathcal{M}'$.

Definition 2.5.19 (Meaning function with indices). First define $\mathcal{M}' : D \times \mathbb{N} \to \Lambda'_{\Re}$. Write $[\![d]\!]'_j$ for $\mathcal{M}'(d, j)$. Intuitively $[\![d]\!]'_j$ is the meaning of d where j is the next number to be used for indexing.

1. $[\![\,\mathbf{system}\ S\,]\!]'_j \equiv S$,
2. $[\![d]\!]'_j \equiv (\lambda_j x \sqsubseteq Q.[\![\mathrm{rest}(d)\,]\!]'_{j+1})P \quad$ if $\mathrm{first}(d) = (x, P, Q)$, $P \not\equiv \mathbf{prim}$,

$$\equiv (\lambda x \sqsubseteq Q.[\![\mathrm{rest}(d)\,]\!]'_{j+1}) \qquad \text{if } \mathrm{first}(d) = (x, \mathbf{prim}, Q).$$

Using this we define $\mathcal{M}' : D \to \Lambda'_{\Re}$, writing $[\![d]\!]'$ for $\mathcal{M}'(d)$, as follows: $[\![d]\!]' :\equiv [\![d]\!]'_1$. □

This has the effect that we can explicitly refer to a candidate-redex derived from the j-th component by means of the index j. Note that each j refers to at most one candidate-redex occurrence in $[\![d]\!]'$ and that π'-reduction preserves this property – which is because of the special structure of the terms resulting from the translation of designs by $\mathcal{M}'$. The following lemma relates both formulations of the translation from designs to lambda terms, $\mathcal{M}$ and $\mathcal{M}'$.

Lemma 2.5.20 Let d be a design, then $|[\![d]\!]'| \equiv [\![d]\!]$.

Proof. The following stronger statement is proved by induction on the number of components of the design: $\forall j \in \mathbb{N}\,(\,|[\![d]\!]'_j| \equiv [\![d]\!]\,)$. □

We illustrate $\mathcal{M}'$ with a small example.

Example 2.5.21 Assume variables x, y, z and let $Q_1, Q_2, Q_3, P_2, P_3, S \in \Lambda_{\Re}$. Let the design d be given by

$$\begin{array}{lll} x & := & \mathbf{prim} \sqsubseteq Q_1 \\ y & := & P_2 \quad\ \sqsubseteq Q_2 \\ z & := & P_3 \quad\ \sqsubseteq Q_3 \\ \multicolumn{3}{l}{\mathbf{system}\ \ S.} \end{array}$$

The meaning with indices of d is given by

$$[\![d]\!]' \equiv \lambda x \sqsubseteq Q_1.(\lambda_2 y \sqsubseteq Q_2.(\lambda_3 z \sqsubseteq Q_3.S)P_3)P_2.$$

In this example the first lambda can not have an index because this lambda does not belong to a candidate-redex. The second component (y, P_2, Q_2) gives rise to a candidate-redex having index 2 and the third component gives rise to a candidate-redex having index 3. Note that also after a reduction step, the residuals of a certain candidate-redex can still be identified by their index. E.g. suppose that for $[\![d]\!]'$ as above, $[\![d]\!]'$ reduces to $\lambda x \sqsubseteq Q_1.(\lambda_3 z \sqsubseteq Q_3[y := P_2].S[y := P_2])P_3[y := P_2]$, then the original candidate-redex indexed 3 need not be present any more, but λ_3 still marks the beginning of a candidate-redex which is reminiscent of the the third component. □

There is a very remarkable relation between our notions of correctness and certain reduction strategies for lambda terms. If d is wf and gbc, then $[\![d]\!]$ is outermost reducible, by which we mean that the candidate-redexes introduced by the translation of d by $\mathcal{M}$, can be contracted in an outside-in (i.e. left-to-right) order. If d is wf and bbc, then $[\![d]\!]$ is innermost reducible, by which we mean that the candidate-redexes introduced by the translation of d by $\mathcal{M}$, can be contracted in a inside-out (i.e. right-to-left) order. The formulation of these notions of outermost reducible and innermost reducible will be relative to a given design d because we want to restrict our considerations to those candidate-redexes which are introduced by the translation of d by $\mathcal{M}$. In order to formulate this precisely we use the technique of lifting introduced above. We observe that in a term $[\![d]\!]'$ the indices occur in increasing order, i.e. when this term contains indexed lambdas λ_j, λ_k with $j < k$ then the λ_j occurs to the left of the λ_k. This observation justifies the definitions below where 'outermost' will be defined as 'in increasing order of indices' and where 'innermost' will be defined as 'in *de*creasing order of indices'.

Definition 2.5.22 (omr) For wf design d, we say that $[\![d]\!]'$ is *outermost reducible* if there exist terms $R_1, \ldots, R_m$ such that there is a reduction path ($\rightarrow$ denoting π'-reduction)

$$\vdash \quad [\![d]\!]' \rightarrow R_1 \rightarrow \ldots \rightarrow R_m$$

where indexed candidate-redexes are contracted in increasing order of indices and such that the last term R_m contains no more indices. We say that $[\![d]\!]$ is *outermost reducible with respect to the candidate-redexes introduced by the translation of d by $\mathcal{M}$* if $[\![d]\!]'$ is outermost reducible. When d is clear from the context, we shall just say that $[\![d]\!]$ is *outermost reducible* (abbreviated *omr*) and then it is to be understood that we mean that it is outermost reducible with respect to the candidate-redexes which are introduced by the translation of d by $\mathcal{M}$. □

Very much in the same way we define 'innermost reducible' below.

Definition 2.5.23 (imr) For wf design d, we say that $[\![d]\!]'$ is *innermost reducible* if there is a reduction path starting with $[\![d]\!]'$ where indexed candidate-redexes are contracted in *de*creasing order of indices and such that the last term contains no more indices. We adopt an obvious terminology analogous to that of 2.5.22. □

Before we prove the relation between the notions of correctness and the reduction strategies, we first give an example. The following example shows three very simple designs and we verify for these designs that gbc corresponds to omr and that bbc corresponds to imr.

Example 2.5.24 Consider $\mathfrak{R}_1$ as before.

(i) The following design d is neither gbc nor bbc.

$$\begin{array}{lllll} x_1 & := & 5 & \sqsubseteq & 4 \\ x_2 & := & x_1 & \sqsubseteq & 7 \\ \textbf{system} & x_2. & & & \end{array}$$

$[\![d]\!]' \equiv (\lambda_1 x_1 \sqsubseteq 4.(\lambda_2 x_2 \sqsubseteq 7.x_2)x_1)5$ which is not omr because $\not\vdash 5 \sqsubseteq 4$. It is not imr because although $[\![d]\!]' \rightarrow (\lambda_1 x_1 \sqsubseteq 4.x_1)5$, we cannot contract the candidate-redex $(\lambda_1 x_1 \sqsubseteq 4.x_1)5$.

(ii) The following design d' is gbc but not bbc.

$$\begin{array}{lllll} x_1 & := & 5 & \sqsubseteq & 10 \\ x_2 & := & x_1 & \sqsubseteq & 7 \\ \textbf{system} & x_2. & & & \end{array}$$

$[\![d']\!]' \equiv (\lambda_1 x_1 \sqsubseteq 10.(\lambda_2 x_2 \sqsubseteq 7.x_2)x_1)5$ which is omr because $[\![d']\!]' \rightarrow (\lambda_2 x_2 \sqsubseteq 7.x_2)x_1[x_1 := 5] \equiv (\lambda_2 x_2 \sqsubseteq 7.x_2)5 \rightarrow x_2[x_2 := 5] \equiv 5$. It is not imr because we cannot make the reduction step $\vdash (\lambda_1 x_1 \sqsubseteq 10.(\lambda_2 x_2 \sqsubseteq 7.x_2)x_1)5 \rightarrow (\lambda_1 x_1 \sqsubseteq 10.x_2[x_2 := x_1])5$. To make the latter reduction step requires (cf. 2.3.28 clauses 4, 6 & 1) that $[x_1 \sqsubseteq 10] \vdash x_1 \sqsubseteq 7$ which is not the case.

(iii) The following design d'' is both gbc and bbc.

$$\begin{array}{lllll} x_1 & := & 5 & \sqsubseteq & 5 \\ x_2 & := & x_1 + 1 & \sqsubseteq & x_1 + 5 \\ \textbf{system} & x_1 + x_2. & & & \end{array}$$

$[\![d'']\!]' \equiv (\lambda_1 x_1 \sqsubseteq 5.(\lambda_2 x_2 \sqsubseteq x_1 + 5.x_1 + x_2)(x_1 + 1))5$ which is omr because $[\![d'']\!]' \rightarrow (\lambda_2 x_2 \sqsubseteq 5 + 5.5 + x_2)(5 + 1) \rightarrow 5 + (5 + 1)$. It is imr because $[\![d'']\!]' \rightarrow (\lambda_1 x_1 \sqsubseteq 5.x_1 + (x_1 + 1))5 \rightarrow 5 + (5 + 1)$. □

It is easy to see why omr corresponds with gbc. When performing an outermost π-reduction, the resulting substitutions will affect the parameter restrictions occurring deeper inside the term. This shows that the glass-box descriptions P_i are relevant in the sense that substitutions $[x_i := P_i]$ take place, influencing preconditions of subsequent steps. Indeed, for gbc the P_i are relevant for the correctness conditions of subsequent components. On the other hand, when performing an innermost reduction, the early reduction steps do not influence the preconditions of the subsequent steps. Similarly for bbc, the P_i are not relevant for the contexts of the correctness conditions.

By way of preparation for the theorem "d is gbc $\Leftrightarrow$ $[\![d]\!]$ is omr" we need a definition.

Definition 2.5.25 Let the wf design d be given as in 2.5.2 (concrete syntax). The *unfolding* of d, denoted as *unf*(d) is defined as the design

$$\begin{array}{lll} x_1 & := & P_1S_1 \sqsubseteq Q_1S_1 \\ & \dots & \\ x_n & := & P_nS_n \sqsubseteq Q_nS_n \\ \textbf{system} & S & \end{array}$$

where each S_j $(1 \leq j \leq n)$ is a sequence of substitutions $S_j \equiv \sigma_1, \dots, \sigma_{j-1}$ where for $(1 \leq k \leq n)$ the σ_k are defined as follows:

$$\begin{array}{ll} \sigma_k \equiv \quad \text{(empty)} & (P_k \equiv \textbf{prim}), \\ \sigma_k \equiv [x_k := P_kS_k] & (P_k \not\equiv \textbf{prim}), \end{array}$$

where for $S_k \equiv \sigma_1, \dots, \sigma_{k-1}$ we write P_kS_k to denote $P_k\sigma_1, \dots, \sigma_{k-1}$. □

The intuition behind *unf*(d) is the design obtained from d by replacing all names by the things they stand for (if possible) while preserving the structure of d.

Theorem 2.5.26 Let d be a wf design. Now we have

$$d \text{ is gbc} \quad \Leftrightarrow \quad [\![d]\!] \text{ is omr.}$$

Proof. We give the details for a design d with two components. After that we give a proof-sketch for the general case. Let d be given as in 2.5.6 (definition bbc). We distinguish four cases.

1. $P_1 \equiv \textbf{prim} \wedge P_2 \equiv \textbf{prim}$. In this case $[\![d]\!]' \equiv \lambda x_1 \sqsubseteq Q_1.\lambda x_2 \sqsubseteq Q_2.S$. Now d is trivially gbc and also $[\![d]\!]$ is trivially omr.

2. $P_1 \equiv \textbf{prim} \wedge P_2 \not\equiv \textbf{prim}$. In this case $[\![d]\!]' \equiv \lambda x_1 \sqsubseteq Q_1.(\lambda_2 x_2 \sqsubseteq Q_2.S)P_2$. Now d is gbc iff $[x_1 \sqsubseteq Q_1] \vdash P_2 \sqsubseteq Q_2$ which is precisely the case iff $\vdash \lambda x_1 \sqsubseteq Q_1.(\lambda_2 x_2 \sqsubseteq Q_2.S)P_2 \to \lambda x_1 \sqsubseteq Q_1.S[x_2 := P_2]$ which means that $[\![d]\!]$ is omr.

3. $P_1 \not\equiv \textbf{prim} \wedge P_2 \equiv \textbf{prim}$. In this case $[\![d]\!]' \equiv (\lambda_1 x_1 \sqsubseteq Q_1.\lambda x_2 \sqsubseteq Q_2.S)P_1$. Now d is gbc iff $\vdash P_1 \sqsubseteq Q_1$ which is precisely the case iff $\vdash (\lambda_1 x_1 \sqsubseteq Q_1.\lambda x_2 \sqsubseteq Q_2.S)P_1 \to (\lambda x_2 \sqsubseteq Q_2.S)[x_1 := P_1]$ which means that $[\![d]\!]$ is omr.

4. $P_1 \not\equiv \textbf{prim} \wedge P_2 \not\equiv \textbf{prim}$. In this case $[\![d]\!]' \equiv (\lambda_1 x_1 \sqsubseteq Q_1.(\lambda_2 x_2 \sqsubseteq Q_2.S)P_2)P_1$. Now d is gbc iff $\vdash P_1 \sqsubseteq Q_1$ and $[x_1 = P_1] \vdash P_2 \sqsubseteq Q_2$. First note that $\vdash P_1 \sqsubseteq Q_1$ is precisely the case iff the candidate redex of λ_1 can be reduced. The result of this first reduction step is $(\lambda_2 x_2 \sqsubseteq Q_2[x_1 := P_1].S[x_1 := P_1])P_2[x_1 := P_1]$.

   Secondly, we shall show that $[x_1 = P_1] \vdash P_2 \sqsubseteq Q_2$ is precisely the case iff in the result of the first reduction we can perform the λ_2 reduction step. Therefore we show $[x_1 = P_1] \vdash P_2 \sqsubseteq Q_2 \Leftrightarrow \vdash P_2[x_1 := P_1] \sqsubseteq Q_2[x_1 := P_1]$.

   To show ($\Rightarrow$), assume $[x_1 = P_1] \vdash P_2 \sqsubseteq Q_2$. By (subst.) $[x_1 = P_1] \vdash P_2[x_1 := P_1] \sqsubseteq Q_2[x_1 := P_1]$. Now x_1 does not occur any more at the right-hand side of the '$\vdash$' and therefore the assumption $[x_1 = P_1]$ is non-essential. To see this, apply lemma 2.3.19, performing the substitution $[x_1 := P_1]$; this yields $[P_1 = P_1] \vdash P_2[x_1 := P_1] \sqsubseteq Q_2[x_1 := P_1]$ whence by (refl.) and the cut-rule $\vdash P_2[x_1 := P_1] \sqsubseteq Q_2[x_1 := P_1]$.

   To show ($\Leftarrow$) assume $\vdash P_2[x_1 := P_1] \sqsubseteq Q_2[x_1 := P_1]$. By weakening and rule (subst.) we get $[x_1 = P_1] \vdash P_2 \sqsubseteq Q_2$. This concludes ($\Leftarrow$). So d is gbc iff $[\![d]\!]$ is omr.

For the general case, first prove

(i) d is gbc $\Leftrightarrow$ *unf*(d) is gbc,

(ii) *unf*(d) is gbc $\Leftrightarrow$ $[\![d]\!]$ is omr.

The theorem is a direct consequence of (i) $\wedge$ (ii). □

Theorem 2.5.27 Let d be a wf design. Now we have

$$d \text{ is bbc} \quad \Leftrightarrow \quad [\![d]\!] \text{ is imr.}$$

Proof. Again we give the details for a design d with two components and after that a proof-sketch for the general case. Let d be given as in 2.5.6 (definition bbc). We distinguish four cases.

1. $P_1 \equiv \textbf{prim} \wedge P_2 \equiv \textbf{prim}$. In this case $[\![d]\!]' \equiv \lambda x_1 \sqsubseteq Q_1.\lambda x_2 \sqsubseteq Q_2.S$. Now d is trivially bbc and also $[\![d]\!]$ is trivially imr.
2. $P_1 \equiv \textbf{prim} \wedge P_2 \not\equiv \textbf{prim}$. In this case $[\![d]\!]' \equiv \lambda x_1 \sqsubseteq Q_1.(\lambda_2 x_2 \sqsubseteq Q_2.S)P_2$. Now d is bbc iff $[x_1 \sqsubseteq Q_1] \vdash P_2 \sqsubseteq Q_2$ which is precisely the case iff $\vdash \lambda x_1 \sqsubseteq Q_1.(\lambda_2 x_2 \sqsubseteq Q_2.S)P_2 \rightarrow \lambda x_1 \sqsubseteq Q_1.S[x_2 := P_2]$ which means that $[\![d]\!]$ is imr.
3. $P_1 \not\equiv \textbf{prim} \wedge P_2 \equiv \textbf{prim}$. In this case $[\![d]\!]' \equiv (\lambda_1 x_1 \sqsubseteq Q_1.\lambda x_2 \sqsubseteq Q_2.S)P_1$. Now d is bbc iff $\vdash P_1 \sqsubseteq Q_1$ which is precisely the case iff $\vdash (\lambda_1 x_1 \sqsubseteq Q_1.\lambda x_2 \sqsubseteq Q_2.S)P_1 \rightarrow (\lambda x_2 \sqsubseteq Q_2.S)[x_1 := P_1]$ which means that $[\![d]\!]$ is imr.
4. $P_1 \not\equiv \textbf{prim} \wedge P_2 \not\equiv \textbf{prim}$. In this case $[\![d]\!]' \equiv (\lambda_1 x_1 \sqsubseteq Q_1.(\lambda_2 x_2 \sqsubseteq Q_2.S)P_2)P_1$. Now d is bbc iff $\vdash P_1 \sqsubseteq Q_1$ and $[x_1 \sqsubseteq Q_1] \vdash P_2 \sqsubseteq Q_2$.

   First note that $[x_1 \sqsubseteq Q_1] \vdash P_2 \sqsubseteq Q_2$ is precisely the case iff the candidate-redex of λ_2 can be reduced. The result of this first reduction step is $(\lambda_1 x_1 \sqsubseteq Q_1.S[x_2 := P_2])P_1$.

   Secondly we see that $\vdash P_1 \sqsubseteq Q_1$ is precisely the case iff in the result of the first reduction, we can perform the λ_1 reduction step. So d is bbc iff $[\![d]\!]$ is imr.

The general case goes in essentially the same way, noting that for performing a contraction internally within a term $(\lambda_j x \sqsubseteq Q. \;\ldots\;)$ or within a term $(\lambda x \sqsubseteq Q. \;\ldots\;)$ we may work in a context with $[x \sqsubseteq Q]$ (cf. 2.3.28 clauses 4, 6 & 1). Therefore the fact that $\Gamma_j \vdash P_j \sqsubseteq Q_j$ is precisely the condition that is needed in order to perform the contraction (Γ_j as in 2.5.6, definition bbc). □

2.5.4 Correctness-preserving Modifications

One of the ideas behind the notion of a design is that there is a locality-principle which, roughly speaking, can be stated as follows: "it should be possible to implement each component in a design without worrying about the implementation of the other components in the design". In order to make this idea more precise we shall define several kinds of so-called *correctness-preserving modifications* of designs and we shall investigate their properties. There are two (binary) criteria to classify the modifications and therefore we shall have four kinds of modifications. The first criterion deals with "what is modified?". We consider two cases: either some glass-box description is modified, or some black-box description is modified. The second criterion deals with "which notion of correctness is adopted?". Again we consider two cases: glass-box correctness and black-box correctness. The modifications are defined to establish the adopted notion of correctness – at least locally for the modified component. The scheme is that we have X-preserving Y-modifications for $X \in \{\text{gbc}, \text{bbc}\}$ and $Y \in \{\text{glass-box}, \text{black-box}\}$. We study what happens if we take a correct design (in the X sense) and modify a Y-description. The interesting question is "is the result correct again in the X sense?", or alternatively "has X been preserved?".

Definition 2.5.28 Let the wf design d be given as

$$\begin{array}{lllll} x_1 & := & P_1 & \sqsubseteq & Q_1 \\ & \ldots & & & \\ x_n & := & P_n & \sqsubseteq & Q_n \\ \multicolumn{5}{l}{\textbf{system } S.} \end{array}$$

We say that d' is a *gbc-preserving glass-box modification* of d, (abbreviated as d' is *gbc-gb-mod* of d, if d' is obtained from d via the replacement:

$$(x_j, P_j, Q_j) := (x_j, P'_j, Q_j)$$

for some $j \in \{1, \ldots, n\}$ such that $P_j \not\equiv$ **prim**, provided d' is wf and for Γ_j as in definition 2.5.5 (definition gbc)

$$\Gamma_j \vdash P'_j \sqsubseteq Q_j.$$

In the same way we define what it means that d' is a *bbc-preserving glass-box modification* of d, by taking Γ_j as in 2.5.6. □

The intuition behind a gbc-preserving glass-box modification is that a modification takes place within one component while preserving the local correctness (in the gbc sense) of that component. The question if preserving local correctness implies preserving correctness of the whole design, is investigated below.

Remark 2.5.29 The proposition that for d' is gbc-gb-mod of d we have

$$d \text{ is gbc} \quad \Rightarrow \quad d' \text{ is gbc}.$$

just does not hold in general.

The counter-example is as follows. Consider $\mathfrak{R}_1$ as before and let the design d given as

$$\begin{array}{llll} x_1 & := & 5 & \sqsubseteq 10 \\ x_2 & := & x_1 & \sqsubseteq 5 \\ \textbf{system} & x_2 & & \end{array}$$

and consider a replacement as follows:

$$(x_1, 5, 10) := (x_1, 6, 10).$$

Clearly d is gbc but d' is not, because $[x_1 = 6] \not\vdash x_1 \sqsubseteq 5$. □

Theorem 2.5.30 If d' is bbc-gb-mod of d, we have

$$d \text{ is bbc} \quad \Rightarrow \quad d' \text{ is bbc}.$$

Proof. Assume d is as in 2.5.28 (definition bbc-gb-mod) and consider the replacement $(x_j, P_j, Q_j) := (x_j, P_j', Q_j)$ for some $j \in \{1, \ldots, n\}$ such that $P_j \not\equiv \textbf{prim}$, provided d' is wf and for Γ_j as in definition 2.5.6 (definition bbc) $\Gamma_j \vdash P_j' \sqsubseteq Q_j$. For all $i \in \{1, \ldots, n\}$ where $P_i \not\equiv \textbf{prim}$ we must show

$$\Gamma_i \vdash P_i \sqsubseteq Q_i \quad (P_i' \text{ if } i = j)$$

where $\Gamma_i = [x_1 \sqsubseteq Q_1], \ldots, [x_{i-1} \sqsubseteq Q_{i-1}]$. For $i \neq j$ we are done since d is bbc. For $i = j$ we are also done since for Γ_j as in 2.5.6 (definition bbc) we have by the definition of bbc-gb-mod $\Gamma_j \vdash P_j' \sqsubseteq Q_j$. □

The intuition behind the remark and the theorem given above is that gbc designs do not offer *implementation freedom* but bbc designs *do*.

Instead of considering correctness-preserving glass-box modifications one can also consider correctness-preserving black-box modifications. The intuition behind the latter kind of modification is a change of specification for a component which has already been implemented. It does not come as a surprise that when one adopts glass-box correctness, these modifications preserve correctness for the whole design – since the black-box descriptions are in fact not used. It is also easy to see that when one adopts black-box correctness, these modifications may disturb the correctness of the whole design.

Definition 2.5.31 Let the wf design d be given as

$$\begin{array}{llll} x_1 & := & P_1 & \sqsubseteq Q_1 \\ & \ldots & & \\ x_n & := & P_n & \sqsubseteq Q_n \\ \textbf{system} & S. & & \end{array}$$

We say that d' is a *gbc-preserving black-box modification* of d, if d' is obtained from d via the replacement:

$$(x_j, P_j, Q_j) := (x_j, P_j, Q'_j)$$

for some $j \in \{1, \ldots, n\}$ such that $P_j \not\equiv$ **prim**, provided d' is wf and for Γ_j as in 2.5.5 (definition gbc)

$$\Gamma_j \vdash P_j \sqsubseteq Q'_j$$

In the same way we define what it means that d' is a *bbc-preserving black-box modification* of d, by taking Γ_j as in 2.5.6. □

Remark 2.5.32 The proposition that for d' is bbc-bb-mod of d we have

$$d \text{ is bbc} \quad \Rightarrow \quad d' \text{ is bbc.}$$

just does not hold in general.

The counter-example is as follows. Consider $\mathfrak{R}_1$ as before and let the design d be given as

$$\begin{array}{lllll} x_1 & := & 1 & \sqsubseteq & 2 \\ x_2 & := & x_1 & \sqsubseteq & 2 \\ \multicolumn{5}{l}{\textbf{system } x_2} \end{array}$$

and consider a replacement as follows:

$$(x_1, 1, 2) := (x_1, 1, 3).$$

Now d is bbc but d' is not. □

Theorem 2.5.33 If d' is gbc-bb-mod of d, we have

$$d \text{ is gbc} \quad \Rightarrow \quad d' \text{ is gbc.}$$

Proof. Assume d is as in 2.5.31 (definition gbc-bb-mod) and consider the replacement $(x_j, P_j, Q_j) := (x_j, P_j, Q'_j)$ for some $j \in \{1, \ldots, n\}$ such that $P_j \not\equiv$ **prim**, provided d' is wf and for Γ_j as in definition 2.5.5 (definition gbc) $\Gamma_j \vdash P_j \sqsubseteq Q'_j$. For all $i \in \{1, \ldots, n\}$ where $P_i \not\equiv$ **prim** we must show

$$\Gamma_i \vdash P_i \sqsubseteq Q_i \quad (Q'_i \text{ if } i = j)$$

where $\Gamma_i = \varphi_1, \ldots, \varphi_{i-1}$ and where $\varphi_k = [x_k = P_k]$ ($P_k \not\equiv$ **prim**), $\varphi_k = [x_k \sqsubseteq Q_k]$ ($P_k \equiv$ **prim**).

Now notice that neither Q_j nor Q'_j occurs in Γ_i so we are done for $i \neq j$ because d is gbc. For $i = j$ we know $\Gamma_j \vdash P_j \sqsubseteq Q'_j$ by the definition of gbc-bb-mod. □

The following table summarises the last two theorems and remarks. For the modifications which lead to a correct design, the corresponding entry in the table contains a '+'. For the modifications which may lead to an incorrect design, the corresponding entry in the table contains a '−'.

preserving	*modifying*	
	glass-box description	black-box description
black-box correctness	+	−
glass-box correctness	−	+

In the above table we see two entries where the modification may lead to an incorrect design, although locally (i.e. for the replaced component) correctness is preserved. One is tempted to think that by restricting the modifications to those modifications where a term is replaced by a term which is less than or equal to that term, the resulting designs are still correct. This turns out to be the case indeed for bbc-preserving black-box modifications, but it does not hold for gbc-preserving glass-box modifications.

Theorem 2.5.34 Let the wf and bbc design d be given as

$$\begin{array}{lllll} x_1 & := & P_1 & \sqsubseteq & Q_1 \\ & \dots & & & \\ x_n & := & P_n & \sqsubseteq & Q_n \\ \multicolumn{5}{l}{\textbf{system } S} \end{array}$$

and let d' be obtained from d via the replacement

$$(x_j, P_j, Q_j) := (x_j, P_j, Q'_j)$$

for some $j \in \{1, \dots, n\}$ such that $P_j \not\equiv$ **prim**, provided d' is wf and

$$[x_1 \sqsubseteq Q_1], \dots, [x_{j-1} \sqsubseteq Q_{j-1}] \vdash P_j \sqsubseteq Q'_j \sqsubseteq Q_j.$$

Then d' is bbc.

Proof. Clearly the first $j-1$ components are still correct. The condition $[x_1 \sqsubseteq Q_1], \dots, [x_{j-1} \sqsubseteq Q_{j-1}] \vdash P_j \sqsubseteq Q'_j$ tells us that the j−th component is correct. Now consider the i–th component for $j < i \leq n$. Define $\Gamma \equiv [x_1 \sqsubseteq Q_1], \ldots, [x_j \sqsubseteq Q_j], \ldots, [x_{i-1} \sqsubseteq Q_{i-1}]$. Define $\Gamma' \equiv [x_1 \sqsubseteq Q_1], \ldots, [x_j \sqsubseteq Q'_j], \ldots, [x_{i-1} \sqsubseteq Q_{i-1}]$. Since d is bbc we have

$$\Gamma \vdash P_i \sqsubseteq Q_i.$$

Now $[x_1 \sqsubseteq Q_1], \ldots, [x_{j-1} \sqsubseteq Q_{j-1}] \subset \Gamma'$ so $\Gamma' \vdash Q'_j \sqsubseteq Q_j$, and since $[x_j \sqsubseteq Q'_j] \in \Gamma'$ we can use the rule (trans.) and we obtain $\Gamma' \vdash x_j \sqsubseteq Q_j$. By the generalised cut-rule we get

$$\Gamma' \vdash P_i \sqsubseteq Q_i. \qquad \square$$

Remark 2.5.35 The proposition that for the wf and gbc design d given as

$$\begin{array}{lcl} x_1 := & P_1 & \sqsubseteq \; Q_1 \\ \quad \ldots & & \\ x_n := & P_n & \sqsubseteq \; Q_n \\ \textbf{system } S & & \end{array}$$

and for d' obtained from d via the replacement

$$(x_j, P_j, Q_j) := (x_j, P'_j, Q_j)$$

for some $j \in \{1, \ldots, n\}$ such that $P_j \not\equiv$ **prim**, provided d' is wf and for Γ_j as in 2.5.5 (definition gbc)

$$\Gamma_j \vdash P'_j \sqsubseteq P_j \sqsubseteq Q_j$$

we have d' is gbc, just does not hold in general.

The counter-example is as follows. Consider $\mathfrak{R}_1$ as before and let the design d be given as

$$\begin{array}{lcl} x_1 := & 5 & \sqsubseteq \; 10 \\ x_2 := & 5 & \sqsubseteq \; x_1 \\ \textbf{system } x_2 & & \end{array}$$

and consider a replacement as follows:

$$(x_1, 5, 10) := (x_1, 4, 10).$$

Clearly d is gbc but d' is not, because $[x_1 = 4] \not\vdash 5 \sqsubseteq x_1$. $\square$

2.6 Looking back

Recall the aim of this chapter, as explained in the introduction, viz. to obtain a theory about the component-wise construction and specification of complex systems, with a focus on issues of modularisation, parameterisation, abstraction and information hiding. Our main tool when constructing such a theory is *formalisation.* In order to obtain the necessary formalisations, in Section 2.2 we abstract from the details of the underlying design language, by viewing a design language (without parameterisation) as the set of terms $T_{\Re}$ of an algebraic system with preorder $\Re$. After that a version of lambda calculus, called $\lambda\pi$ is introduced, which serves two purposes. First of all it adds *parameterisation* to the algebraic system with preorder $\Re$ and secondly it allows us to give a meaning to the notion of a design by means of a 'designs-as-lambda-terms' analogy. The set of terms in this calculus is denoted as $\Lambda_{\Re}$. In $\lambda\pi$-calculus one can derive facts of the form $\Gamma \vdash P \sqsubseteq Q$ with intuition '*P implements Q in context* Γ'. The most significant differences between $\lambda\pi$-calculus and classical λ-calculus are that there is a restriction $x \sqsubseteq P$ associated with each abstraction and that a term $(\lambda x \sqsubseteq P.Q)A$ can only be contracted if the argument A meets its restriction in the sense that $A \sqsubseteq P$. This calculus is shown to have reasonable properties such as cut-elimination, strong normalisation and confluence. This $\lambda\pi$ is not the same as typed λ-calculus with subtypes (e.g. [15]). In $\lambda\pi$, the ordering relation $\sqsubseteq$ is on the λ-terms whereas in [15] there is a relation $\leq$ on their types. Intuitively, in $\lambda x \sqsubseteq P$ the x is 'comparable with' P rather than 'in' P. A connection with λ-typed λ-calculus can be made by viewing a term $\lambda x \sqsubseteq P.Q$ as the image of $\lambda x : *.\lambda y : (x \sqsubseteq P).Q$ under a forgetful mapping, since in $\lambda\pi$ the proof (y) of $(x \sqsubseteq P)$ remains implicit. This connection is not worked out further here.

The formal definition of the set C of *components* and the set D of *designs* is straightforward: a *component* is a triple (x, P, Q) (where x is a new name and where P is either a term or prim) and it is *correct* in context Γ if $\Gamma \vdash P \sqsubseteq Q$. We cal P the *glass-box description* and Q the *black-box description.* A *design* consists of a list of components and one additional term. Designs can be represented as objects of the form:

$$\begin{array}{llll}
x_1 & := & P_1 & \sqsubseteq & Q_1 \\
x_2 & := & P_2 & \sqsubseteq & Q_2 \\
x_3 & := & P_3 & \sqsubseteq & Q_3 \\
\multicolumn{2}{l}{\text{system}} & S & &
\end{array}$$

where P_1, P_2, P_3 can be prim's or terms and where Q_1, Q_2, Q_3 are terms.

The definition of *correctness* for designs is non-trivial because of the contexts in which components must be shown correct. Two possibilities arise, leading us to the definitions of *glass-box correctness* and *black-box correctness.* A translation from designs into lambda terms is defined, thereby obtaining a 'meaning' for our notion of a design. The meaning of a design d is a term $[\![d]\!] \in \Lambda_{\aleph}$. For each component of d for which the glass-box description is not prim there is an abstraction-application pair in $[\![d]\!]$. For each component of d for which the glass-box description is prim there is an abstraction in $[\![d]\!]$. This translation provides us with a better insight in the nature of components with a prim glass-box description: these components are the formal *parameters* of the design in which they occur. Furthermore there is a very remarkable relation between our notions of correctness and certain reduction strategies for lambda terms: glass-box correctness corresponds to the possibility to perform a *outermost* reduction and black-box correctness corresponds to the possibility to perform a *innermost* reduction.

There exist various kinds of *modifications* of designs, with the intuition that a modification takes place within one component while preserving the local correctness of that component. The question if preserving local correctness implies preserving correctness of the whole design, is investigated for various kinds of modifications. There are a number of positive and negative results. For example, that glass-box correct designs do not offer *implementation freedom* but black-box correct designs *do.* These results are summarised by the following table:

preserving	*modifying*	
	glass-box description	black-box description
black-box correctness	+	−
glass-box correctness	−	+

It is interesting to compare the results achieved in this chapter with the approach sketched by Girard et.al. in [16] (p. 17). The principle of black-box correctness is implicitly already present in this citation.

> At a more general level, abstracting away from any peculiar syntactic choice, one should see a type as an instruction for *plugging* things together. Let us imagine that we program with *modules*, i.e. closed units, which we can plug together. A module is absolutely closed, we have no right to open it. We just have the ability to use it or not, and to choose the manner of use (plugging). The type of a module is of course completely determined by all the possible *pluggings* it allows without crashing. In particular, one can always substitute a module with another of the same type, in the event of a

breakdown, or for the purpose of optimisation. This idea of ***arbitrary pluggings*** seems ***mathematisable***, but to attempt this would lead us too far astray.

In a certain sense, our research can be viewed a 'mathematisation' (= formalisation) of these ideas. A precise understanding of the notions of component, black-box description and design is of great methodological importance and our formalisation and results should be of help in the systematic development of designs in realistic applications.

2.7 Acknowledgements

The formalisation of algebraic systems with preorder and the definition of the $\lambda\pi$-calculus and its properties have been done in cooperation with H.B.M. Jonkers, C.P.J. Koymans and G.R. Renardel de Lavalette.

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Appendix A

Model Construction

There are several reasons why one wants to have a model for the $\lambda\pi$-calculus. First of all, if we have a non-trivial model, then we know that the calculus is consistent in the sense that not every equality (or every inequality) is derivable. Secondly, by constructing a model we can make our intuition that a lambda term $\lambda x \sqsubseteq P.Q$ denotes a function more precise. In this section we shall construct such a model, denoted by $\mathfrak{R}^+$, provided $\mathfrak{R}$ satisfies certain restrictions. The calculus is not complete with respect to this $\mathfrak{R}^+$, i.e. $(\mathfrak{R}^+ \models \varphi) \Rightarrow (\vdash \varphi)$ does not hold. We do not consider this as a disadvantage of our approach, because the calculus has a certain value in its own right, which does not depend on completeness. The calculus is sound with respect to $\mathfrak{R}^+$, i.e. $(\vdash \varphi) \Rightarrow (\mathfrak{R}^+ \models \varphi)$ – otherwise we should not call it a model. Under certain assumptions, the model $\mathfrak{R}^+$ is obtained as an extension of the underlying algebraic system with preorder $\mathfrak{R}$. Lambda terms are interpreted as normal set-theoretic functions. The model contains elements $*_\tau$ which correspond to the 'undefined' terms, by which we mean the terms which cannot be contracted because of the condition in the rule (π).

In order to keep things simple, we shall adopt two additional assumptions about the underlying model $\mathfrak{R}$ and we shall discuss the various possibilities arising if these additional assumptions do not hold. The first assumption is that in $\mathfrak{R}$ the preorder R satisfies the additional property that $\forall x, y \; (x \sqsubseteq y \wedge y \sqsubseteq x \Rightarrow x = y)$ which means that R is a partial order. The second assumption is that there is a maximum element, i.e. $\exists x \forall y \; (y \sqsubseteq x)$. Note that the example CA of Section 2.2.3 satisfies this property with '**class none end**' as maximum element.

If R is not a partial order, then several possibilities arise. The first possibility is that $\mathfrak{R}$ has at least a substitution property for the relation $=_R$ defined as $R \cap R^{-1}$, i.e. $x =_R y :\Leftrightarrow x \sqsubseteq y \wedge y \sqsubseteq x$. The required substitution property is that $x =_R y \Rightarrow f_j(\ldots,x,\ldots) =_R f_j(\ldots,y,\ldots)$ for all $j \in J$, which means that $=_R$ is a congruence. For this first possibility we have the option that $=$ is interpreted by $=_R$. With this option, the model construction can still be done, but things become a bit more complicated and also the fact that $=$ can not be interpreted by real equality is not elegant.

When R is not a partial order, the second possibility is that $\mathfrak{R}$ does not even have the substitution property for $=_R$ mentioned above. In that case the rules $(=I)$ and (subst.) are essentially non-conservative. The calculus with these rules is still usable and it deals just with implementation (denoted by $\sqsubseteq$) and bi-implementation (denoted by $=$) where the rules $(=I)$ and (subst.) have an axiomatic status. Another option is to remove the rule $(=I)$ and to adapt the calculus correspondingly – as has been done in [4].

If there is no maximum element, then the model construction must be done somewhat differently. We shall discuss this below. So from now on we restrict ourselves to an algebraic system $\mathfrak{R}$ where R is a partial order and we require the existence of the maximum element.

Notation A.1.1 If z is a function $z : X \to A$ then $z[x \to a]$ denotes the function which everywhere equals z but for the argument x, for which it results a; formally $z[x \to a](y) = z(y)$ for $y \neq x$ and $z(x) = a$. □

First we define the function domains and we define the ordering of functions.

Definition A.1.2 $(A_\tau, \sqsubseteq_\tau)$. Consider $\mathfrak{R}$ as before. We define A_τ and $\sqsubseteq_\tau$ by induction on the structure of the type symbol τ. Let $*_0$ denote the maximum element, which is already present.

1. $A_0 := A$,
   $a \sqsubseteq_0 b :\Leftrightarrow aRb$.
2. $A_{\sigma\to\tau} := A_\sigma \to A_\tau$,
   $f \sqsubseteq_{\sigma\to\tau} g :\Leftrightarrow \forall x \in A_\sigma(f(x) \sqsubseteq_\tau g(x))$.

When there is no maximum element, we add $*_0$ as a fresh 'junk object' and we define $A_0 := A \cup \{*_0\}$ and $a \sqsubseteq_0 b :\Leftrightarrow aRb \vee b = *_0$. The fact that this works depends on the restriction to closed φ in rule $(\models_2)$; to see this consider a 'one-object' $\mathfrak{R}$, by which we mean a $\mathfrak{R}$ such that $\mathfrak{R} \models x = y$, where adding an object makes the formula $\forall x \forall y((x = y))$ false.

We shall write $*_{\sigma\to\tau}$ for $\lambda\!\!\lambda\ a \in A_\sigma . *_\tau$, where $\lambda\!\!\lambda$ denotes functional abstraction. If no confusion arises we omit the subscripts for $*$ and $\sqsubseteq$. Note that $*_{\sigma\to\tau}$ is the maximum element of $A_{\sigma\to\tau}$. Note also that $\sqsubseteq_\tau$ is a partial order again for all type symbols τ, as is easily shown by induction on the structure of τ. □

The model $\mathfrak{R}^+$ is a structure somewhat similar to $\mathfrak{R}$, where the most important difference is that $\mathfrak{R}^+$ has a collection of domains rather than just one. There is one domain A_τ for each type τ and consequently there is also a collection of relations $\sqsubseteq_\tau$, rather than just one. Again we have functions F_j and constants C_i, but these are related to A_0 only.

Definition A.1.3 ($\mathfrak{R}^+$). Consider $\mathfrak{R}$ as before and let Typ be the set of type symbols.

(i) The model $\mathfrak{R}^+$ is defined as $(\{A_\tau \mid \tau \in Typ\}, \{\sqsubseteq_\tau \mid \tau \in Typ\}, \{F_j \mid j \in J\}, \{C_i \mid i \in I\})$.

(ii) An *assignment* z is a map variables $\rightarrow \bigcup\{A_\tau \mid \tau \in Typ\}$ such that $z(x^\tau) \in A_\tau$, i.e. such that z respects the typing. □

Definition A.1.4 The *interpretation* of terms in $\mathfrak{R}^+$ under an assignment z is defined by induction over the structure of terms. We write $P\langle z\rangle$ to denote the interpretation of term P under assignment z.

1. $x_i\langle z\rangle = z(x_i)$,
2. $c_i\langle z\rangle = C_i$,
3. $f_j(P_1, \dots, P_n)\langle z\rangle = F_j(P_1\langle z\rangle, \dots, P_n\langle z\rangle)$,
4. $(PQ)\langle z\rangle = P\langle z\rangle(Q\langle z\rangle)$
5. $(\lambda x^\sigma \sqsubseteq P.Q)\langle z\rangle = \lambda\!\!\lambda\, a \in A_\sigma.$ if $a \sqsubseteq P\langle z\rangle$ then $Q\langle z[x^\sigma \rightarrow a]\rangle$ else $*$. □

Note that for a term M the relevant part of an assignment is the restriction of the assignment to the free variables $\vec{x} = FV(M)$; in particular, if $\forall y \notin \vec{x}(z_1(y) = z_2(y))$ then $M\langle z_1\rangle = M\langle z_2\rangle$.

Definition A.1.5 We define $\mathfrak{R}^+ \models \varphi$ for $\Lambda_{\mathfrak{R}}$-formulae φ built from atoms, conjunction and implication as follows.

(i) $\mathfrak{R}^+ \models (P = Q)\langle z\rangle :\Leftrightarrow \quad P\langle z\rangle = Q\langle z\rangle$.

(ii) $\mathfrak{R}^+ \models (P \sqsubseteq Q)\langle z\rangle :\Leftrightarrow \quad P\langle z\rangle \sqsubseteq Q\langle z\rangle$.

(iii) $\mathfrak{R}^+ \models (\varphi \wedge \psi)\langle z\rangle \;:\Leftrightarrow\; \mathfrak{R}^+ \models \varphi\langle z\rangle$ and $\mathfrak{R}^+ \models \psi\langle z\rangle$.

(iv) $\mathfrak{R}^+ \models (\varphi \rightarrow \psi)\langle z\rangle \;:\Leftrightarrow\; \mathfrak{R}^+ \models \varphi\langle z\rangle \Rightarrow \mathfrak{R}^+ \models \psi\langle z\rangle$.

(v) $\mathfrak{R}^+ \models \varphi \;:\Leftrightarrow$ for all z $\mathfrak{R}^+ \models \varphi\langle z\rangle$. □

Lemma A.1.6 For atomic $T_{\mathfrak{R}}$-formulae we have

$$\mathfrak{R} \models \varphi \Leftrightarrow \mathfrak{R}^+ \models \varphi.$$

Proof. Immediate from the definitions of $\mathfrak{R} \models \varphi$ and $\mathfrak{R}^+ \models \varphi$. $\square$

As a next step we want to show the soundness of $\mathfrak{R}^+$ and for this purpose we introduce several lemmas about assignments and substutitions.

Lemma A.1.7 $R[x := P]\langle z\rangle \ = \ R\langle z[x \to P\langle z\rangle]\rangle$.

Proof. By induction over the structure of R.

- Let $R \equiv x$. Then $R[x := P]\langle z\rangle = P\langle z\rangle$ and $R\langle z[x \to P\langle z\rangle]\rangle = x\langle z[x \to P\langle z\rangle]\rangle = P\langle z\rangle$ Otherwise $R \equiv y \not\equiv x$. Then $R[x := P]\langle z\rangle = z(y)$ and $R\langle z[x \to P\langle z\rangle]\rangle = y\langle z[x \to \ldots]\rangle = z(y)$.
- $R \equiv c_i$: trivial.
- $R \equiv f_j(P_1, \ldots, P_n)$. Then $R[x := P]\langle z\rangle = f_j(P_1[x := P], \ldots, P_n[x := P])\langle z\rangle = F_j(P_1[x := P]\langle z\rangle, \ldots, P_n[x := P]\langle z\rangle) =$ (using i.h.) $F_j(P_1\langle z[x \to P\langle z\rangle]\rangle, \ldots, P_n\langle z[x \to P\langle z\rangle]\rangle) = f_j(P_1, \ldots P_n)\langle z[x \to P\langle z\rangle]\rangle$ $= R\langle z[x \to P\langle z\rangle]\rangle$.
- $R \equiv (P_1P_2)$. Then $R[x := P]\langle z\rangle = P_1[x := P]\langle z\rangle(P_2[x := P]\langle z\rangle)$ which we write as $f(a)$. Similarly we write $(P_1P_2)\langle z[x \to P\langle z\rangle]\rangle$ as $g(b)$. By i.h. we have $f = g$ and $a = b$. Therefore $f(a) = g(b)$.
- $R \equiv \lambda y \sqsubseteq Q_1.Q_2$. By the variable convention $x \not\equiv y$. Define $f := R[x := P]\langle z\rangle$ which equals $\lambda\!\!\lambda\ a.$ if $a \sqsubseteq Q_1[x := P]\langle z\rangle$ then $Q_2[x := P]\langle z[y \to a]\rangle$ else $*$, and define $g := R\langle z[x \to P\langle z\rangle]\rangle =$ which equals $\lambda\!\!\lambda\ a.$ if $a \sqsubseteq Q_1\langle z[x \to P\langle z\rangle]\rangle$ then $Q_2\langle z[x \to P\langle z\rangle][y \to a]\rangle$ else $*$. Now consider an arbitrary argument, c say and prove (using i.h.) that $f(c) = g(c)$. In particular, the induction hypothesis yields $Q_2[x := P]\langle z[y \to a]\rangle = Q_2\langle z[y \to a][x \to P\langle z[y \to a]\rangle]\rangle$. We must use that – by the variable convention – P does not contain y whence $P\langle z[y \to a]\rangle = P\langle z\rangle$. $\square$

Lemma A.1.8 (Substitution) Let z be an assignment. Assume $P\langle z\rangle = Q\langle z\rangle$.

(i) $R[x := P]\langle z\rangle = R[x := Q]\langle z\rangle$.

(ii) If $R_1[x := P]\langle z\rangle = R_2[x := P]\langle z\rangle$,
then $R_1[x := Q]\langle z\rangle = R_2[x := Q]\langle z\rangle$.

(iii) If $R_1[x := P]\langle z\rangle \sqsubseteq R_2[x := P]\langle z\rangle$,
then $R_1[x := Q]\langle z\rangle \sqsubseteq R_2[x := Q]\langle z\rangle$.

Proof. (i) We have $R\langle z[x \to P\langle z\rangle]\rangle = R\langle z[x \to Q\langle z\rangle]\rangle$ and hence by

lemma A.1.7 we obtain $R[x := P]\langle z\rangle = R[x := Q]\langle z\rangle$.
(ii) Write $R[x := P]$ as $R(P)$ etc. By (i) we have $R_1(Q)\langle z\rangle = R_1(P)\langle z\rangle$ and by assumption $R_1(P)\langle z\rangle = R_2(P)\langle z\rangle$ and by (i) again $R_2(P)\langle z\rangle = R_2(Q)\langle z\rangle$. Use transitivity of $=$.
(iii) By (i) we have $R_1(Q)\langle z\rangle = R_1(P)\langle z\rangle$, and by assumption $R_1(P)\langle z\rangle \sqsubseteq R_2(P)\langle z\rangle$ and by (i) again $R_2(P)\langle z\rangle = R_2(Q)\langle z\rangle$. Use reflexivity and transitivity of $\sqsubseteq$. □

Theorem A.1.9 (Soundness).

$$\Gamma \vdash \varphi \;\Rightarrow\; \mathfrak{R}^+ \models \bigwedge\Gamma \rightarrow \varphi.$$

Proof. The proof is by induction on the length of the derivation of $\Gamma \vdash \varphi$.

- ($\models_1$) If $\Gamma \vdash \varphi$ is $\Gamma \vdash f_j(A) \sqsubseteq f_j(B)$ because $\Gamma \vdash A \sqsubseteq B$ for f_j corresponding with a monotonic function. Just use i.h. and the monotonicity of the corresponding F_j.
- ($\models_2$) If $\Gamma \vdash \varphi$ is a direct consequence of $\mathfrak{R} \models \varphi$. This case is obvious.
- (context). $\Gamma, \varphi \vdash \varphi$. We must show that $\mathfrak{R}^+ \models (\bigwedge\Gamma \wedge \varphi)\langle z\rangle$ implies $\mathfrak{R}^+ \models \varphi\langle z\rangle$ which is obvious.
- (refl.). $\Gamma \vdash P \sqsubseteq P$. We must show that $\mathfrak{R}^+ \models \bigwedge\Gamma$ implies $\mathfrak{R}^+ \models P \sqsubseteq P$ which follows from the fact that the relations $\sqsubseteq_\tau$ of $\mathfrak{R}^+$ are partial orders.
- (trans.). As refl.
- (λI_1). $\Gamma \vdash (\lambda x \sqsubseteq P.Q_1) \sqsubseteq (\lambda x \sqsubseteq P.Q_2)$. We have $\mathfrak{R}^+ \models \bigwedge\Gamma \wedge x \sqsubseteq P \rightarrow Q_1 \sqsubseteq Q_2$ (i.h.). Take an arbitrary z and assume $\mathfrak{R}^+ \models \bigwedge\Gamma\langle z\rangle$.
  Define $f := (\lambda x \sqsubseteq P.Q_1)\langle z\rangle$ and $g := (\lambda x \sqsubseteq P.Q_2)\langle z\rangle$. For $a \sqsubseteq P\langle z\rangle$ we shall show $f(a) \sqsubseteq g(a)$. Since $x \notin \Gamma$ we have $\mathfrak{R}^+ \models \bigwedge\Gamma\langle z[x \rightarrow a]\rangle$. From i.h. $Q_1\langle z[x \rightarrow a]\rangle \sqsubseteq Q_2\langle z[x \rightarrow a]\rangle$, i.e. $f(a) \sqsubseteq g(a)$. This shows $\mathfrak{R}^+ \models \bigwedge\Gamma\langle z\rangle \;\Rightarrow\; f \sqsubseteq g$.
- (λI_2). $\Gamma \vdash (\lambda x \sqsubseteq P_2.Q) \sqsubseteq (\lambda x \sqsubseteq P_1.Q)$. We have $\mathfrak{R}^+ \models \bigwedge\Gamma \rightarrow P_1 \sqsubseteq P_2$ (i.h.). Take an arbitrary z and assume $\mathfrak{R}^+ \models \bigwedge\Gamma\langle z\rangle$.
  Define $f := (\lambda x \sqsubseteq P_2.Q)\langle z\rangle$ and $g := (\lambda x \sqsubseteq P_1.Q)\langle z\rangle$. We show for arbitrary a that $f(a) \sqsubseteq g(a)$.
  If $a \not\sqsubseteq P_1\langle z\rangle$ and $a \not\sqsubseteq P_2\langle z\rangle$ then $f(a) = g(a) = *$.
  If $a \sqsubseteq P_2\langle z\rangle$ but $a \not\sqsubseteq P_1\langle z\rangle$ then $f(a) \sqsubseteq g(a) = *$.
  If $a \sqsubseteq P_1\langle z\rangle \sqsubseteq P_2\langle z\rangle$ then $f(a) = g(a)$.
  This shows $\mathfrak{R}^+ \models \bigwedge\Gamma\langle z\rangle \;\Rightarrow\; f \sqsubseteq g$ so $\mathfrak{R}^+ \models \bigwedge\Gamma \rightarrow (\lambda x \sqsubseteq P_2.Q) \sqsubseteq (\lambda x \sqsubseteq P_1.Q)$.
- (ap.). $\Gamma \vdash (P_1 Q) \sqsubseteq (P_2 Q)$. Take an arbitrary z. Assume $\mathfrak{R}^+ \models \bigwedge\Gamma\langle z\rangle$.

We have $\mathfrak{R}^+ \models \bigwedge\Gamma \to P_1 \sqsubseteq P_2$ (i.h.). Define $a := Q\langle z\rangle$, $f := P_1\langle z\rangle$ and $g := P_2\langle z\rangle$. We have $f \sqsubseteq g$ (by i.h.). We must show $f(a) \sqsubseteq g(a)$. This follows directly from the definition of $\sqsubseteq$.

- (π). $\Gamma \vdash (\lambda x \sqsubseteq P_1.Q)P_2 = Q[x := P_2]$. We have $\mathfrak{R}^+ \models \bigwedge\Gamma \to P_2 \sqsubseteq P_1$ (i.h.). Take an arbitrary z. Assume $\mathfrak{R}^+ \models \bigwedge\Gamma\langle z\rangle$.
$(\lambda x \sqsubseteq P_1.Q)P_2\langle z\rangle = (\lambda\!\!\lambda\, a.$ if $a \sqsubseteq P_1\langle z\rangle$ then $Q\langle z[x \to a]\rangle$ else $*)(P_2\langle z\rangle)$ $=$ (i.h.) $Q\langle z[x \to P_2\langle z\rangle]\rangle = Q[x := P_2]\langle z\rangle$ where we used lemma A.1.7.
- $(=I)\Gamma \vdash P_1 = P_2$ because $\Gamma \vdash P_1 \sqsubseteq P_2$ and $\Gamma \vdash P_2 \sqsubseteq P_1$. We have $\mathfrak{R}^+ \models \bigwedge\Gamma \to P_1 \sqsubseteq P_2$ and $\mathfrak{R}^+ \models \bigwedge\Gamma \to P_2 \sqsubseteq P_1$ (i.h.). Use the fact that $\sqsubseteq$ is a partial order.
- (subst.). $\Gamma \vdash R_1(Q) = R_2(Q)$ where we assume that φ is an equality. We have $\mathfrak{R}^+ \models \bigwedge\Gamma \to R_1(P) = R_2(P)$ and $\mathfrak{R}^+ \models \bigwedge\Gamma \to P = Q$ (i.h.). Take an arbitrary z and assume $\mathfrak{R}^+ \models \bigwedge\Gamma\langle z\rangle$. From (i.h.) $R_1(P)\langle z\rangle = R_2(P)\langle z\rangle$ and $P\langle z\rangle = Q\langle z\rangle$ so by the substitution lemma (ii) we get $R_1(Q)\langle z\rangle = R_2(Q)\langle z\rangle$, i.e. $\mathfrak{R}^+ \models (R_1(Q) = R_2(Q))\langle z\rangle$. If φ contains the symbol $\sqsubseteq$ instead of $=$, we proceed in a similar way, using the substitution lemma (iii). □

Appendix B

List of symbols

In this appendix we give a list of the symbols used. For each symbol the list contains a very short informal description. The list has been subdivided into a number of sub-lists. The first sub-list contains general mathematical symbols. The second sub-list contains the symbols which are introduced and/or used first in Section 2. In a similar way the third sub-list contains the symbols which are introduced and/or used first in Section 3, and so on. The list does not include symbols which denote in some sense the negation of the meaning of an another symbol e.g. $\notin$ negates $\in$ and therefore $\notin$ is not in the list. For some symbols the list contains a relevant page number – usually the defining occurrence of the symbol.

General mathematical symbols

$\Rightarrow, \Leftarrow$	Logical implication
$\Leftrightarrow$	Logical equivalence
$\wedge, \vee$	Conjunction, disjunction
$\forall$	Universal quantification
$\bigwedge$	Generalised conjunction
$\equiv$	Syntactical equality
$=$	Equality
$\{\ \}$	Set construction
$\{\ \mid\ \}$	Set comprehension
$\in$	Set membership
$\emptyset$	Empty set
$\cup$	Set union
$\cap$	Set intersection
$\setminus$	Set difference
$\subseteq$	Set inclusion
$\bigcup$	Generalised union

$\| \|$	Cardinality
$\mathcal{P}$	Powerset
$\rightarrow$	Function space
$(C)^*$	The set of sequences with elements from C
$\times$	Cartesian product
$(\, , \,)$	Pair
$(\, , \ldots , \,)$	Tuple (= sequence)
$\mathbb{N}$	The set of natural numbers
$\leq$	Less than or equal to
$+$	Addition
$0, 1, 2, \ldots$	Natural numbers
$f : A \rightarrow B$	f is a function from A to B
$:=$	Substitution
R^{-1}	Inverse of relation R
$i.h.$	Induction hypothesis
$\lambda\!\!\lambda$	Functional abstraction
$\circ$	Function composition

Symbols concerning constructs

$m_1, m_2, \ldots$	Typical modules	54
$\mathfrak{R}$	Typical algebraic system with preorder	55
A	Typical domain	55
R	Typical preorder	55
F_j	Typical function	55
J	Typical index set of functions	55
a_j	Arity of function F_j	55
C_i	Typical constant	55
I	Typical index set of constants	55
f_j	Typical function symbol	56
c_i	Typical constant symbol	56
x_0, x_1, x_2	Variables	56
$T_{\mathfrak{R}}$	Set of terms for $\mathfrak{R}$	56
Sig	Signature of ...	56
$=$	Equality symbol	56
$\sqsubseteq$	Preorder (symbol)	56
$\models$	Truth (in a model)	57
φ, ψ	Typical formulae	56
$\mathfrak{R}_1$	$\mathbb{N}$ as algebraic system with preorder	58
S	Arbitrary set	58
$\mathfrak{R}_2$	$\mathcal{P}(S)$ as algebraic system with preorder	58
CA	The Class Algebra of COLD	58

CDescription	The set of class descriptions	58
CSignature	The set of class signatures	58
CRenaming	The set of class renamings	58
$\sqsubseteq$	Implementation relation of CA	58
Σ	Signature operation	58
T	Embedding of class signatures	58
$\square$	Export	58
$\bullet$	Application of renaming	58
$+$	Import	58

Symbols concerning lambda calculus

(β)	Classical Contraction rule	60
$\lambda\pi$	Lambda calculus based on rule (π)	60
λ	Abstractor (symbol)	61
$\Lambda_{\mathfrak{K}}$	Set of terms for $\lambda\pi$	62
0	Basic type symbol	61
σ, τ	Typical type symbols	61
$\rightarrow$	Constructor for type symbols	61
$x_0, x_1, \ldots$	Variables	62
x_i^τ	Typical variable of type τ	62
$\sqsubseteq$	Partial order (symbol)	62
A, B, P, Q, R, S, Z	Typical elements of $\Lambda_{\mathfrak{K}}$	63
x, y, z	Typical variables	63
$FV(\ )$	Set of free variables	62
$[\ :=\ ]$	Substitution in a lambda term	67
$\top$	Constant true	63
$\wedge$	Conjunction	63
$\rightarrow$	Implication	63
$\bigwedge$	Conjuction (etc.)	64
$[\varphi]$	Formula φ as an assumption	64
Γ	Typical context	64
$\vdash$	Derivation symbol	64
$(\models_1)$	Rule of $\lambda\pi$ (algebraic system oracle)	65
$(\models_2)$	Rule of $\lambda\pi$ (algebraic system oracle)	65
(context.)	Rule of $\lambda\pi$ (context rule)	65
(refl.)	Rule of $\lambda\pi$ (reflexivity)	66
(trans.)	Rule of $\lambda\pi$ (transitivity)	66
(λI_1)	Rule of $\lambda\pi$ (lambda introduction)	66
(λI_2)	Rule of $\lambda\pi$ (lambda introduction)	66
$(ap.)$	Rule of $\lambda\pi$ (application)	66
(π)	Rule of $\lambda\pi$ (partial contraction rule)	67

$(=I)$	Rule of $\lambda\pi$ (= introduction)	68
(subst.)	Rule of $\lambda\pi$ (substitution)	68
$\mathfrak{R}^+$	Model for $\lambda\pi$	70
(mon.)	Rule for monotonicity	73
$\lambda\pi_{\text{mon}}$	Version of $\lambda\pi$ with built-in monotonicity	73
$\mathfrak{R}^+_{\text{mon}}$	Model for $\lambda\pi_{\text{mon}}$	74
$\rightarrow$	One step π Reduction	74
$\twoheadrightarrow$	π Reduction	74
$=_\pi$	π Equality	75
SN	Strong normalisation property	76
$\lambda\beta$	Calculus with β instead of π	76
C, C_τ	'Computable' terms	77
C^*_τ	'Substitutive-computable' terms	78
$\diamond$	Diamond property (confluence, Church-Rosser)	80
Λ	Terms of classical simple-typed λ-calculus	79
$\overline{}$	Coding: $\Lambda_{\mathfrak{R}} \rightarrow \Lambda$	79
$\overset{\neq}{\twoheadrightarrow}$	Reduction with at least one step	80

Symbols concerning components

prim	Symbol serving as a placeholder for constructs	84
$C'_{\mathfrak{R}}$	The set of triples (x, P, Q)	84
$C_{\mathfrak{R}}, C$	The set of components	84
P	Typical element of $\Lambda_{\mathfrak{R}} \cup \{\mathbf{prim}\}$	84
c	Typical component	84

Symbols concerning designs

$D_{\mathfrak{R}}, D$	The set of designs	86
d, d', d'', r	Typical designs	86
$cset(\ )$	Set of components of a design	86
$d_{(j)}$	The j-th sub-design of d	86
$:=$	Symbol used in concrete syntax of designs	86
$\sqsubseteq$	Symbol used in concrete syntax of designs	86
system	Symbol used in concrete syntax of designs	86
wf	Well-formed	87
i, j, k, l, n, m	Typical natural numbers	87
gbc	Glass-box correct	87
bbc	Black-box correct	88
$\mathcal{M}$, $[\![\ ]\!]$	Meaning function for designs	91
λ_i	Lambda with index	92
$\Lambda'_{\mathfrak{R}}$	Set of terms with indices	92
π_j	reduction of index j	93

π'	reduction with indices	93
$\mid\ \mid$	mapping forgetting indices	93
$\mathcal{M}'$, $[\![\ ]\!]'$	Meaning function with indices	94
omr	Outermost reducible	95
imr	Innermost reducible	96
$unf(d)$	Unfolding of d	97
σ_k, ρ_k	Typical substitutions	97
S	Typical sequence of substitutions	97
gbc-gb-mod	gbc-preserving glass-box modification	100
bbc-gb-mod	bbc-preserving glass-box modification	100
gbc-bb-mod	gbc-preserving black-box modification	101
bbc-bb-mod	bbc-preserving black-box modification	102

Symbols concerning Model Construction

$\varphi\langle z\rangle$	Interpretation onder assignment z	112
$*_\tau$	Interpretation of undefined terms	110
A_0	Basic domain	111
$A_{\sigma\to\tau}$	Function domain	111
$z[x \to a]$	Modification of an assignment	112
Typ	Set of type symbols	112
$*_\tau$	Maximal element	111

Chapter 3

Correctness-Preserving Transformations of Designs

3.1 Introduction

In this chapter we want to study correctness preserving transformations of designs. We refer to Chapter 2 for the definition of the syntax and the semantics of designs. In Chapter 2 we investigated already a number of simple transformations such as the black-box correctness preserving glass-box modifications (bbc-gb-mod). These modifications take one design as an input and yield one design as a result. Now we shall also consider *binary operations* $(*, \circ)$ on designs. This implies that we consider the construction of a design by fitting together two designs. Conversely, these operations can be used for describing the process of splitting a given design into smaller designs. The interesting point, of course, is that under certain conditions the result of applying a binary operation to two bbc designs is a bbc design again. We must investigate these conditions and the algebraic laws that hold for the binary operations (both at the level of designs and of their semantics in terms of $\lambda\pi$, via $[\![\ ]\!]$). It turns out that in order to define these operations on designs it is convenient to use sequences within $\lambda\pi$. Therefore we need an extension of the $\lambda\pi$-calculus. Such an extension is given in appendix A.

In Section 2 we give a new definition of the set of designs. This definition is somewhat more general than the definition given in Chapter 2. Furthermore we discuss two conditions on designs which enable us to restrict ourselves to a convenient kind of designs.

In Section 3 we shall investigate several operations on designs. We shall have binary operations $*$ and $\circ$. We briefly investigate a binary operation $\sharp$ which however turns out to be somewhat disappointing. We also have two unary operations called bot and top. Furthermore there are two binary predicates called gbv and bbv. In this way we obtain an ***algebra of designs***. At the end of Section 3 we shall include a summary of this algebra of designs.

In Section 4 we shall investigate models of the development process which describe design *creation*. Therefore we shall start with a discussion of the *validation* of a given design with respect to a given machine&user-context. We shall describe models of the development process as highly non-deterministic design-programs. These design-programs are of an imperative nature, by which we mean that they may contain assignment statements. We need a simple language which serves as a design-development language in which we express these design-programs. Such a language, inspired by COLD, but precisely tailored to our needs is given in appendix B. In Section 3.4 we include a brief sketch of the main ingredients of this design-development language – see subsection 3.4.3. The binary operation $\circ$ will play a role in our discussion of validation. The binary operation $*$ will be used in our description of design creation.

In Section 5 we shall discuss the *evolution* of a given design with respect to its context in connection with issues of design *validation*. We shall present some simple models of the development process which deal with design evolution. The binary operation $\circ$ will be used in our description of design evolution.

The fact that we can partition designs and reassemble them again makes it possible to discuss models of the development process where two (or more) developers each operate on a part of a design such that when each of them has finished his part, their results are fitted together to yield a new design which is both bbc and valid. We shall refer to this type of development process as *parallel development*. Both binary operations $*$ and $\circ$ will be used in our description of parallel development. Section 6 is devoted to design partition and to parallel development.

In this chapter we shall focus on those aspects of the software development process which are connected with the idea of structuring the software product as a design. In view of the fact that the notion of a design is a formalisation of the approach of having 'black-box descriptions', this focusing leads us to an investigation of concepts which deal with the manageability of the software development process: top-down development, layered designs, validation, design evolution. It will be the 'leitmotiv' of this chapter that whenever possible we shall cast the relevant notions in the form of designs and algebraic operations on designs.

The models of the development process described in this chapter are most certainly not meant to be exclusive: any technique for obtaining valid and correct designs is a good technique as such. Rather than being prescriptive, we aim at the formulation of some general principles which may be of help for realistic development processes.

The proofs of the theorems and lemmas in this chapter are given with a level of detail which is meant for convincing the reader rather than for enabling mechanical proof-checking.

The Sections 1, 2, 3, 4, 5 and 6 constitute the main line of development of this chapter whereas appendix A and appendix B are rather technical digressions. Definitions from appendix A are used in Sections 2 and 3. Definitions from appendix B are used in Sections 4, 5 and 6. It has been tried to make both appendix A and appendix B independent of the Sections 1-6. We suggest the following order to the reader: 1-2-3-4-5-6 (possibly reading parts of A and B on a need-to-know basis), followed by appendices A and B. Appendix C contains a list of symbols.

3.2 Designs

3.2.1 Designs where the system is a sequence

In this section we give a new definition of the set of designs. This definition is somewhat more general than the definition given in Chapter 2. Furthermore we introduce some notation.

In Chapter 2 the set $D_{\mathfrak{R}}$ of designs has been defined for a given algebraic system with preorder $\mathfrak{R}$. This notion of 'design' is not symmetric in the sense that a design can have several **prim** components whereas its system consists of just one term. This fact seems to be an obstacle when we want to define binary operations on designs. We view it as an obstacle because we want to define an operation $\circ$ on designs where we replace the **prims** of one design by elements from the system of another design. Therefore we extend our definition of 'design' and we shall use the $\lambda\pi$-calculus with sequences as given in appendix A. We consider an algebraic system with preorder $\mathfrak{R}$. The set of terms of $\lambda\pi$-calculus is denoted as $\Lambda_{\mathfrak{R}}$. Among the terms we have so-called sequences which are of the form $[P_1, \ldots, P_m]$ and which have a type of the form $(\tau_1, \ldots, \tau_m)$; the introduction of sequences does not disturb the nice properties of $\lambda\pi$ such as confluence and strong normalisation. The set $C_{\mathfrak{R}}$ of *components* is defined as in Chapter 2, except for the fact that we use the $\lambda\pi$-calculus with sequences.

Definition 3.2.1 Consider the $\lambda\pi$-calculus with sequences and whose set of terms is $\Lambda_{\mathfrak{R}}$. Consider also $C_{\mathfrak{R}}$ which is the set of components constructed over this $\Lambda_{\mathfrak{R}}$. Each component is either a triple (x, P, Q) or $(x, \mathbf{prim}, Q)$ for

$P, Q \in \Lambda_{\Re}$. Then the set $D'_{\Re}$ of *designs* is defined as the subset of $(C_{\Re})^* \times \Lambda_{\Re}$ given by

$$\{((c_1, \ldots, c_k), S) \mid S \text{ has a type } (\tau_1, \ldots, \tau_m)\}$$

for $k \geq 0, m \geq 0$, (k and m not fixed). □

So in fact we have very much the same kind of designs as in Chapter 2. The only differences are that now we adopted a richer set of lambda terms and that now the system of each design must be a sequence. We adopt an obvious notation for components and designs with symbols $:=$, $\sqsubseteq$ and **system**. Note that we can embed $D_{\Re}$ in $D'_{\Re}$ by mapping '...**system** S' to '...**system** $[S]$'. We adopt the same definition of wellformedness (wf), glass-box correctness (gbc), black-box correctness (bbc) and semantics ($[\![\ ]\!]$) etc. for designs as in Chapter 2. For the remainder of this chapter we restrict ourselves to designs from $D'_{\Re}$.

Example 3.2.2 Consider $\Re_1 = (\mathbb{N}, \leq, \{+\}, \{0, 1, 2, \ldots\})$. The following design is an element of $D'_{\Re}$ because its system is a sequence.

$$\begin{array}{lll} x & := & \textbf{prim} \sqsubseteq 7 \\ y & := & x + 2 \sqsubseteq 9 \\ \multicolumn{3}{l}{\textbf{system}\ [x, y, x + y + 1].} \end{array}$$

□

Remark 3.2.3 It is important to realise that the mechanism of introducing components, '$x := P \sqsubseteq Q$' say, is much more complex than an abbreviation mechanism as in '$x := P$'. The essential difference is that with components non-trivial issues of information-hiding arise. Yet we think that it is a good idea to employ an abbreviation mechanism as well: it should be possible to write '$x := P$' and from then on use x, knowing that x can be replaced by P. Abbreviations which have a scope consisting of one design are called *local*. It is possible to view a local abbreviation '$x := P$' as a special kind of component. Abbreviations which have a scope consisting of a collection of designs are called *global*. We think that both a facility for local abbreviations and a facility for global abbreviations are useful. In this chapter we shall not investigate such abbreviation facilities any further. □

We shall need some notation.

Definition 3.2.4 Let d be a wf design.

(i) cset(d) := the set of component names of d,

(ii) sys(d) := the set of comp. names that occur freely in the system of d.

Note that $\text{sys}(d) \subseteq \text{cset}(d)$. □

Definition 3.2.5 (arity). (i) For a design d, the *arity* of d, notation $\text{arity}(d)$ is the pair $((\sigma_1, \ldots, \sigma_n), (\tau_1, \ldots, \tau_m))$ where the $\sigma_1, \ldots, \sigma_n$ are the types of the names of the **prim** components of d and where $(\tau_1, \ldots, \tau_m)$ is the type of the system of d.

(ii) For a design d, the *reduced arity* of d, also denoted as $\text{arity}(d)$ is the pair (n, m) where n is the number of **prim** components of d and where m is the length of the system of d. □

Throughout this chapter we shall restrict ourselves to designs where all glass-box descriptions have type 0 and where the system has type $(0, ..., 0)$, i.e. all system elements have type 0. This restriction gives us the advantage that we need not worry about the types of terms and the arities of designs. Under this restriction it will be sufficient to employ the reduced arity only. The restriction is by no means essential and the extension of our definitions and results towards arbitrary designs is straightforward. In view of this restriction we shall simply use the term 'arity' for 'reduced arity'.

We must be precise about *equality* of designs.

Definition 3.2.6 (=). Let d_1, d_2 be wf designs. $d_1 = d_2$ means that d_2 can be obtained from d_1 by means of a systematic renaming of bound variables. In this definition it is understood that both the variables bound by lambdas and the names of components are considered as bound variables. It is also understood that clashes of free and bound variables must be avoided. Sometimes we shall state explicitly that two designs d_1 and d_2 must have disjoint sets of component names ($\text{cset}(d_1) \cap \text{cset}(d_2) = \emptyset$) in order that some operation is defined. □

Recall that in Chapter 2 we defined a mapping $[\![\ ]\!]$ from designs to lambda terms, viewing $[\![d]\!]$ as the semantics of d. (Of course it is possible to push the process of semantical interpretation one step further by interpreting $[\![d]\!]$ in some model of $\lambda\pi$, but throughout this chapter we shall not do that.) We shall say that two wf designs d_1 and d_2 are *semantically equivalent* if $[\![d_1]\!]$ and $[\![d_2]\!]$ are equal in the sense of $\lambda\pi$-calculus, i.e. if $\vdash [\![d_1]\!] = [\![d_2]\!]$ is derivable. Sometimes we write $d_1 =_{[\![\ ]\!]} d_2$ if $\vdash [\![d_1]\!] = [\![d_2]\!]$. Note that $d_1 = d_2$ implies $d_1 =_{[\![\ ]\!]} d_2$ by rule (refl.).

Even without referring to properties of the algebraic system with preorder $\Re$, each design can be shown to have a lot of semantical equivalents. Under certain conditions, this fact will give us some freedom in choosing a convenient kind of representatives from the classes of semantically equivalent designs. We shall have two such conditions, viz. a condition called *directly specified* (abbreviated ds) to be introduced in Section 3.2.3 and glass-box correctness (gbc).

3.2.2 Permuting Components

This section contains preparations which we shall need in Section 3.2.3 when we shall introduce a condition called '**prims**-first' (abbreviated pf). These preparations include a lemma which we shall call the 'component swap lemma'.

From an intuitive point of view, it is clear that the order of the non-**prim** components in a design to a certain extent is irrelevant. First note that the order of the **prim** components is relevant however, as is illustrated by the following example.

Example 3.2.7 Consider the algebraic system with preorder $\mathfrak{R}_1$ as before. Let d and d' respectively be given by

$$\begin{array}{lllll} y & := & \textbf{prim} & \sqsubseteq & 1 \\ x & := & \textbf{prim} & \sqsubseteq & 2 \\ \multicolumn{5}{l}{\textbf{system}\ [\,x\,],} \end{array} \qquad \begin{array}{lllll} x & := & \textbf{prim} & \sqsubseteq & 2 \\ y & := & \textbf{prim} & \sqsubseteq & 1 \\ \multicolumn{5}{l}{\textbf{system}\ [\,x\,].} \end{array}$$

Now $[\![d]\!] = \lambda y \sqsubseteq 1.\lambda x \sqsubseteq 2.[x]$ whereas $[\![d']\!] = \lambda x \sqsubseteq 2.\lambda y \sqsubseteq 1.[x]$ whence $\not\vdash [\![d]\!] = [\![d']\!]$. □

The next example shows a case where two components can be swapped.

Example 3.2.8 Consider the algebraic system with preorder $\mathfrak{R}_1$ as before. Let d and d' respectively be given by

$$\begin{array}{lllll} y & := & 1 & \sqsubseteq & 2 \\ x & := & \textbf{prim} & \sqsubseteq & 2 \\ z & := & x+y & \sqsubseteq & 4 \\ \multicolumn{5}{l}{\textbf{system}\ [\,z\,],} \end{array} \qquad \begin{array}{lllll} x & := & \textbf{prim} & \sqsubseteq & 2 \\ y & := & 1 & \sqsubseteq & 2 \\ z & := & x+y & \sqsubseteq & 4 \\ \multicolumn{5}{l}{\textbf{system}\ [\,z\,].} \end{array}$$

Now $[\![d]\!] = (\lambda y \sqsubseteq 2.\lambda x \sqsubseteq 2.(\lambda z \sqsubseteq 4.[z])(x+y))1$ and $\vdash [\![d]\!] = \lambda x \sqsubseteq 2.[x+1]$. Also $[\![d']\!] = \lambda x \sqsubseteq 2.(\lambda y \sqsubseteq 2.(\lambda z \sqsubseteq 4.[z])(x+y))1$ and $\vdash [\![d']\!] = \lambda x \sqsubseteq 2.[x+1]$. □

We formalise the idea behind the latter example below.

Lemma 3.2.9 (Component swap lemma). Let d and d' be wf designs. Let d be gbc. Let d' be obtained from d by swapping two adjacent components, where at least one of these components is non-**prim**. Then we have:

(i) d' is gbc,

(ii) d is bbc $\Leftrightarrow$ d' is bbc,

(iii) $\vdash [\![d]\!] = [\![d']\!]$.

Proof. Let d and d' respectively be given by

$$\begin{array}{lllll} x_1 & := & P_1 & \sqsubseteq & Q_1 \\ & \dots & & & \\ x_i & := & P_i & \sqsubseteq & Q_i \\ x_{i+1} & := & P_{i+1} & \sqsubseteq & Q_{i+1} \\ & \dots & & & \\ x_n & := & P_n & \sqsubseteq & Q_n \\ \multicolumn{5}{l}{\mathbf{system}\ [\vec{M}],} \end{array} \qquad \begin{array}{lllll} x_1 & := & P_1 & \sqsubseteq & Q_1 \\ & \dots & & & \\ x_{i+1} & := & P_{i+1} & \sqsubseteq & Q_{i+1} \\ x_i & := & P_i & \sqsubseteq & Q_i \\ & \dots & & & \\ x_n & := & P_n & \sqsubseteq & Q_n \\ \multicolumn{5}{l}{\mathbf{system}\ [\vec{M}].} \end{array}$$

First of all we note that since d is wf, x_{i+1} does not occur in P_i, Q_i, P_{i+1} and Q_{i+1}. Similarly, since d' is wf, x_i does not occur in P_i, Q_i, P_{i+1} and Q_{i+1}.

(i) We know that d is gbc and we must prove that d' is gbc. Throughout this part (i) we shall employ several of the derived rules of $\lambda\pi$ as worked out in Chapter 2 (weakening, cut-rule etc.). Clearly, for all components but for those named x_i and x_{i+1}, the correctness condition is not affected by the swapping. The correctness of the component named x_i in d' follows from its correctness in d by weakening. For the component named x_{i+1} in d' we reason as follows. If $P_{i+1} \equiv \mathbf{prim}$ then the correctness condition is trivially true, so we assume that $P_{i+1} \not\equiv \mathbf{prim}$. Let us consider the case where also $P_i \not\equiv \mathbf{prim}$. We use the fact that d is gbc and hence $\Gamma, [x_i = P_i] \vdash P_{i+1} \sqsubseteq Q_{i+1}$ where Γ is the context with $i-1$ assumptions as given by the definition of gbc. Now note that x_i occurs neither in Γ nor in P_i, P_{i+1}, Q_{i+1}. By performing the substitution $[x_i := P_i]$) (both left and right of the '$\vdash$') we get $\Gamma, [P_i = P_i] \vdash P_{i+1} \sqsubseteq Q_{i+1}$. Now we can use the reflexivity of $=$ in $\lambda\pi$ and the cut-rule to get $\Gamma \vdash P_{i+1} \sqsubseteq Q_{i+1}$. We might summarise this by saying that the assumption $[x_i = P_i]$ does not play an essential role in the derivation of $P_{i+1} \sqsubseteq Q_{i+1}$. We have shown that the component named x_{i+1} is correct in d'. In the case where $P_i \equiv \mathbf{prim}$ we can have a similar reasoning. This shows that d' is gbc.

(ii) We can show ($\Rightarrow$) in a similar way as (i), where instead of an assumption $[x_i = P_i]$ we have an assumption $[x_i \sqsubseteq Q_i]$. For the converse ($\Leftarrow$) we note that if d' is obtained from d by swapping, then also d can be obtained from d' by swapping.

(iii) Since d is gbc, we can contract all abstraction-application pairs of $[\![d]\!]$ which correspond to a non-prim component and we get $\vdash [\![d]\!] = (\mathcal{L}[\vec{M}])S$ where

- $\mathcal{L}$ is the sequence of abstractions $\lambda x_j \sqsubseteq Q_j$. for $P_j \equiv$ **prim** with increasing j.
- $\mathcal{S}$ is the sequence of substitutions $[x_j := P_j]$ for $P_j \not\equiv$ **prim** with decreasing j.

Similarly let $\vdash [\![d']\!] = (\mathcal{L}[\vec{M}])\mathcal{S}'$. If $P_i \equiv$ **prim** or $P_{i+1} \equiv$ **prim**, then x_i is bound in $\mathcal{L}$ whereas the component named x_{i+1} plays a role in $\mathcal{S}$ (or conversely). In both cases the order of these components in the design does not matter, by which we mean that $(\mathcal{L}[\vec{M}])\mathcal{S} \equiv (\mathcal{L}[\vec{M}])\mathcal{S}'$. The remaining case is $P_i \not\equiv$ **prim** and $P_{i+1} \not\equiv$ **prim**. Let $\mathcal{S}$ be given as $\mathcal{S}_1[x_{i+1} := P_{i+1}][x_i := P_i]\mathcal{S}_2$, then $\mathcal{S}'$ must be $\mathcal{S}_1[x_i := P_i][x_{i+1} := P_{i+1}]\mathcal{S}_2$. Now note that $\mathcal{S}$ and $\mathcal{S}'$ are effectively the same substitutions because $x_i \notin FV(P_{i+1})$ and $x_{i+1} \notin FV(P_i)$. This shows $\vdash [\![d]\!] = [\![d']\!]$. □

By repeated application of the component swap lemma it is also possible to swap components that are not adjacent. Some care is needed in order to preserve the applicability conditions of the component swap lemma during the intermediate swap steps. Several cases arise and we show one of them as a lemma below.

Lemma 3.2.10 Let d and d' be wf designs. Let d be gbc. Let d' be obtained from d by swapping two non-**prim** components. Then the same conclusions (i), (ii) and (iii) as in lemma 3.2.9 hold.

Proof. Let d and d' respectively be given by

$$\begin{array}{llll}
x_1 & := & P_1 & \sqsubseteq Q_1 \\
 & \cdots & & \\
x_i & := & P_i & \sqsubseteq Q_i \\
 & \cdots & & \\
x_{i+k} & := & P_{i+k} & \sqsubseteq Q_{i+k} \\
 & \cdots & & \\
x_n & := & P_n & \sqsubseteq Q_n \\
\multicolumn{4}{l}{\mathtt{system}\ [\vec{M}],}
\end{array}
\qquad
\begin{array}{llll}
x_1 & := & P_1 & \sqsubseteq Q_1 \\
 & \cdots & & \\
x_{i+k} & := & P_{i+k} & \sqsubseteq Q_{i+k} \\
 & \cdots & & \\
x_i & := & P_i & \sqsubseteq Q_i \\
 & \cdots & & \\
x_n & := & P_n & \sqsubseteq Q_n \\
\multicolumn{4}{l}{\mathtt{system}\ [\vec{M}].}
\end{array}$$

which means that in d the two components to be swapped are separated by $k-1$ components in between. Our strategy is to move component i downwards past component $i+k$ by k swaps first. The essential observation is that since d' is wf, x_i does not occur in $P_i, Q_i, P_{i+1}, Q_{i+1}, \ldots, P_{i+k}, Q_{i+k}$. Therefore the intermediate designs resulting after $1, 2, \ldots, k$ steps are all wf. So for each swap the conclusions of the component swap lemma carry over to the resulting intermediate design. In particular, the intermediates are gbc.

As the second part of our strategy, we make component $i+k$ (by which we mean $x_{i+k} := P_{i+k} \sqsubseteq Q_{i+k}$) bubble upwards by $k-1$ swaps. We can apply the component swap lemma $k-1$ times. Note also that the $k-1$ 'in-between' components are allowed to contain **prim**'s because the components i and $i+k$ are both non-**prim**. Hence in each swap, at least one of the components involved is non-**prim**. □

Very much in the same way we have that the conclusions of the component swap lemma also hold if the components are not adjacent, provided the components to be swapped have no **prim** component between them.

3.2.3 The conditions 'pf' and 'ds'

We think that it is convenient to restrict ourselves to designs organised as indicated by the following condition:

Definition 3.2.11 (pf) Let d be a design. We say that d is **prim**s-*first* (abbreviated pf) if its **prim** components occur before its non-**prim** components. □

We give an example.

Example 3.2.12 Suppose that in the design d given below $P_1, \ldots, P_m$ are not **prim**. Then d is pf.

$$\begin{array}{lllll}
x_1 & := & \mathbf{prim} & \sqsubseteq & M_1 \\
 & \ldots & & & \\
x_n & := & \mathbf{prim} & \sqsubseteq & M_n \\
y_1 & := & P_1 & \sqsubseteq & Q_1 \\
 & \ldots & & & \\
y_m & := & P_m & \sqsubseteq & Q_m \\
\multicolumn{5}{l}{\mathbf{system}\ [S_1, \ldots, S_l].}
\end{array}$$

□

Under a certain condition (called 'directly specified') to be worked out below, a design which is wf and gbc can be transformed into a semantically equivalent pf design.

Definition 3.2.13 (ds). We say that a design d is *directly specified* (abbreviated ds) if it is wf and no black-box description contains the name of a component. □

Let us have a look at an example of a design which is *not* ds. Consider the algebraic system with preorder $\Re_1 = (\mathbb{N}, \leq, \{+\}, \{0, 1, 2, \ldots\})$. Now consider the following design:

Example 3.2.14

$$\begin{array}{lll} x & := & \textbf{prim} \sqsubseteq 7 \\ y & := & P \quad\ \sqsubseteq x \\ z & := & \textbf{prim} \sqsubseteq y \\ \multicolumn{3}{l}{\textbf{system } [S_1, \ldots, S_l].} \end{array}$$ □

In order to see that there is something strange with this design, assume that we are interested in choosing P such that we can show the correctness of the second component, while adopting black-box correctness. We must prove $P \sqsubseteq x$ under the assumption $x \sqsubseteq 7$. Since we can think of $\sqsubseteq$ as the ordering $\leq$ on natural numbers, we are very limited in finding some $P \leq x$ if all we know about x is $x \leq 7$ (of course we could take x itself). Furthermore, as another problem, it should be be noted that that if we swap the second and the third component, we get a design which is not even wf. The condition ds serves to avoid this and similar situations.

Lemma 3.2.15 Let d be a ds design. Then there is a wf and pf design d' such that

(i) d is gbc $\Leftrightarrow$ d' is gbc,

(ii) d is bbc $\Leftrightarrow$ d' is bbc,

(iii) d is gbc $\Rightarrow$ $\vdash [\![d]\!] = [\![d']\!]$.

Proof. The fact that d is ds enables us to put the **prim** components before the non-**prim** components without violating the condition wf. The fact that this transformation preserves glass-box correctness, black-box correctness and semantics follows by repeated application of the component swap lemma 3.2.9. The possibility to apply lemma 3.2.9 repeatedly depends on the fact that after each swap we get a wf and ds design again; this is the case since the design is ds and one of the components involved in the swap is always a **prim** component moving upwards. □

We are often interested in designs which are ds and gbc. In fact we shall prefer designs which are even bbc but this condition seems not needed here yet. If all designs are ds and gbc then by lemma 3.2.15 we can transform each design into an equivalent pf one. Therefore we shall from now on focus on designs which are pf and also ds. It follows that for each algebraic operation which we shall define, we shall have to verify that the result of applying the operation to pf and ds arguments yields a pf and ds design again.

We end this section with a lemma which sometimes is useful for simplifying designs.

Lemma 3.2.16 (Component cancellation lemma). Let d and d' be pf $\wedge$ ds designs. Assume that in d there are components $(x_i := P_i \sqsubseteq Q_i)$ and $(x_j := x_i \sqsubseteq Q_i)$ where $i \neq j$. Let d' be obtained from d by removing the component $(x_j := x_i \sqsubseteq Q_i)$ and replacing all occurrences of x_j by x_i. Then the following hold:

(i) d is bbc $\Rightarrow$ d' is bbc,

(ii) $\vdash [\![d]\!] = [\![d']\!]$.

Proof. Let d and d' respectively be given by

$$\begin{array}{llll}
x_1 & := & P_1 & \sqsubseteq \; Q_1 \\
 & \dots & & \\
x_i & := & P_i & \sqsubseteq \; Q_i \\
 & \dots & & \\
x_{j-1} & := & P_{j-1} & \sqsubseteq \; Q_{j-1} \\
x_j & := & x_i & \sqsubseteq \; Q_i \\
x_{j+1} & := & P_{j+1} & \sqsubseteq \; Q_{j+1} \\
 & \dots & & \\
x_n & := & P_n & \sqsubseteq \; Q_n \\
\multicolumn{4}{l}{\textbf{system } S,}
\end{array}
\qquad
\begin{array}{llll}
x_1 & := & P_1 & \sqsubseteq \; Q_1 \\
 & \dots & & \\
x_i & := & P_i & \sqsubseteq \; Q_i \\
 & \dots & & \\
x_{j-1} & := & P_{j-1} & \sqsubseteq \; Q_{j-1} \\
x_{j+1} & := & P_{j+1}[x_j := x_i] & \sqsubseteq \; Q_{j+1} \\
 & \dots & & \\
x_n & := & P_n[x_j := x_i] & \sqsubseteq \; Q_n \\
\multicolumn{4}{l}{\textbf{system } S[x_j := x_i].}
\end{array}$$

(i) Components named $x_1, \dots, x_{j-1}$ are correct in d iff they are correct in d'. For a component $(x_k := P_k \sqsubseteq Q_k)$ with $j < k \leq n$ the correctness condition in d is

$\Gamma, [x_j \sqsubseteq Q_i], [x_{j+1} \sqsubseteq Q_{j+1}], \dots, [x_{k-1} \sqsubseteq Q_{k-1}] \vdash P_k \sqsubseteq Q_k$ ...(*)

and in d' it is

$\Gamma, [x_{j+1} \sqsubseteq Q_{j+1}], \dots, [x_{k-1} \sqsubseteq Q_{k-1}] \vdash P_k[x_j := x_i] \sqsubseteq Q_k$ ...(**)

where Γ abbreviates $[x_1 \sqsubseteq Q_1], \dots, [x_{j-1} \sqsubseteq Q_{j-1}]$. Assume (*). By performing the substitution $[x_j := x_i]$) (both to the left and right of the '$\vdash$') we get $\Gamma, [x_i \sqsubseteq Q_i], [x_{j+1} \sqsubseteq Q_{j+1}], \dots, [x_{k-1} \sqsubseteq Q_{k-1}] \vdash P_k[x_j := x_i] \sqsubseteq Q_k$. Here we used the fact that x_j does not occur in Γ and hence $\Gamma[x_j := x_i] = \Gamma$. Now note that the assumption $[x_i \sqsubseteq Q_i]$ is already in Γ. This shows (**).

(ii) Consider the term $[\![d]\!]$. This term contains the abstraction-application pair $(\lambda x_j \sqsubseteq Q_i.\cdots)x_i$ which corresponds to the component $(x_j := x_i \sqsubseteq Q_i)$. This abstraction-application pair occurs in the scope of an abstraction $\lambda x_i \sqsubseteq Q_i.$ and therefore it can be contracted via π-reduction. This π-reduction yields precisely $[\![d']\!]$. □

Remark 3.2.17 (Strengthening of the component cancellation lemma). The proposition that in the component cancellation lemma 3.2.16 we have d' is bbc $\Rightarrow$ d is bbc, (i.e. the converse of (i)) is *false*. This can be shown by the following counterexample. Let the pf $\wedge$ ds designs d and d' respectively be given as

$$\begin{array}{lll} x_1 & := & \textbf{prim} \sqsubseteq Q_1 \\ x_2 & := & x_1 \quad \sqsubseteq Q_1 \\ x_3 & := & (\lambda z \sqsubseteq x_1.Q_2)x_2 \sqsubseteq Q_2 \\ \multicolumn{3}{l}{\textbf{system } S,} \end{array} \qquad \begin{array}{lll} x_1 & := & \textbf{prim} \sqsubseteq Q_1 \\ x_3 & := & (\lambda z \sqsubseteq x_1.Q_2)x_1 \sqsubseteq Q_2 \\ \multicolumn{3}{l}{\textbf{system } S.} \end{array}$$

Now d' is bbc because $\vdash (\lambda z \sqsubseteq x_1.Q_2)x_1 =$ (rule π, refl.) $Q_2[z := x_1] \equiv Q_2 \sqsubseteq$ (example 2.3.15) Q_2 whereas in d we can not prove the black-box correctness of the third component.
If for d and d' as in the component cancellation lemma 3.2.16 we have additionally that there is no glass-box description P_k $(j + 1 \leq k \leq n)$ in which *both* x_i and x_j occur then the converse of (i) holds: d' is bbc $\Rightarrow$ d is bbc. □

Sometimes it will be convenient to represent a context (i.e. a set of assumptions) by a design and this is formalised in definition 3.2.18 below.

Definition 3.2.18 Let the pf $\wedge$ ds design d be given as

$$\begin{array}{lllll} x_1 & := & \textbf{prim} & \sqsubseteq & M_1 \\ & \dots & & & \\ x_n & := & \textbf{prim} & \sqsubseteq & M_n \\ y_1 & := & P_1 & \sqsubseteq & Q_1 \\ & \dots & & & \\ y_m & := & P_m & \sqsubseteq & Q_m \\ \multicolumn{5}{l}{\textbf{system } [S_1, \dots, S_l].} \end{array}$$

where $P_1, \dots, P_m$ are not equal to **prim**. Then we define the *black-box context* of d and the *glass-box context* of d respectively as follows.

(i) $\Gamma_{\text{bb}}(d)$ is $[x_1 \sqsubseteq M_1], \dots, [x_n \sqsubseteq M_n], [y_1 \sqsubseteq Q_1], \dots, [y_m \sqsubseteq Q_m]$,

(ii) $\Gamma_{\text{gb}}(d)$ is $[x_1 \sqsubseteq M_1], \dots, [x_n \sqsubseteq M_n], [y_1 = P_1], \dots, [y_m = P_m]$. □

3.3 Algebraic Operations on Designs

3.3.1 The operation $*$

The first operation on designs is called *concatenation* and is denoted by $*$. The concatenation of d_1 and d_2 yields a design containing the components of d_1 and the components of d_2 and having as its system simply the concatenation of the systems of d_1 and d_2.

We have the following intuition. The designs d_1 and d_2 are considered as 'disjoint' designs and by constructing $d_1 * d_2$, we simply take a kind of union of their component sets. It is important to realise that designs d_1 and d_2 have no such thing as "free parameters" and that each design constitutes very much a closed unit. Conversely, if we can split a design d into d_1 and d_2, such that $d = d_1 * d_2$ then this means that d consisted in fact already of two unrelated parts.

This binary operation $*$ will be used in our description of design creation (Section 3.4); there $*$ turns out to be useful for describing the addition of one ore more **prim** components to a given design. Furthermore, since $*$ can be used for describing the splitting of a design into two smaller designs, it will be used in our description of design partition and parallel development (Section 3.6).

Definition 3.3.1 ($*$). Assume pf $\wedge$ ds designs d_1 and d_2 where $\text{cset}(d_1) \cap \text{cset}(d_2) = \emptyset$. Let d_1 and d_2 respectively be given by

$$\begin{array}{lllll@{\qquad\qquad}lllll}
x_1 & := & \mathbf{prim} & \sqsubseteq & M_1 & u_1 & := & \mathbf{prim} & \sqsubseteq & N_1 \\
 & \ldots & & & & & \ldots & & & \\
x_{n_1} & := & \mathbf{prim} & \sqsubseteq & M_{n_1} & u_{n_2} & := & \mathbf{prim} & \sqsubseteq & N_{n_2} \\
y_1 & := & P_1 & \sqsubseteq & Q_1 & v_1 & := & A_1 & \sqsubseteq & B_1 \\
 & \ldots & & & & & \ldots & & & \\
y_{l_1} & := & P_{l_1} & \sqsubseteq & Q_{l_1} & v_{l_2} & := & A_{l_2} & \sqsubseteq & B_{l_2} \\
\multicolumn{5}{l}{\mathbf{system}\ [S_1, \ldots, S_{m_1}],} & \multicolumn{5}{l}{\mathbf{system}\ [T_1, \ldots, T_{m_2}]}
\end{array}$$

where $P_1, \ldots, P_{l_1}$ are not equal to **prim** and where $A_1, \ldots, A_{l_2}$ are not equal to **prim**. Then we define $d_1 * d_2$ as

$$\begin{array}{lllll}
x_1 & := & \mathbf{prim} & \sqsubseteq & M_1 \\
 & \ldots & & & \\
x_{n_1} & := & \mathbf{prim} & \sqsubseteq & M_{n_1} \\
u_1 & := & \mathbf{prim} & \sqsubseteq & N_1 \\
 & \ldots & & & \\
u_{n_2} & := & \mathbf{prim} & \sqsubseteq & N_{n_2} \\
y_1 & := & P_1 & \sqsubseteq & Q_1 \\
 & \ldots & & & \\
y_{l_1} & := & P_{l_1} & \sqsubseteq & Q_{l_1} \\
v_1 & := & A_1 & \sqsubseteq & B_1 \\
 & \ldots & & & \\
v_{l_2} & := & A_{l_2} & \sqsubseteq & B_{l_2} \\
\multicolumn{5}{l}{\mathbf{system}\ [S_1, \ldots, S_{m_1},\ T_1, \ldots, T_{m_2}].}
\end{array}$$

Note that $d_1 * d_2$ is a pf $\wedge$ ds design again. Note also that if $\text{arity}(d_1) =$

(n_1, m_1) and arity$(d_2) = (n_2, m_2)$, then arity$(d_1 * d_2) = (n_1 + n_2, m_1 + m_2)$. Finally note that cset$(d_1 * d_2) = $ cset$(d_1) \cup$ cset(d_2). □

In Section 3.1 we already announced an *algebra of designs*. We shall now show two simple algebraic laws which state that the set of pf $\wedge$ ds designs together with the operation $*$ constitutes a monoid. Strictly speaking, it is a partial monoid, but when we take into account that we always can perform a systematic renaming, we can also view it as a non-partial monoid.

Lemma 3.3.2 (Algebraic properties of $*$). Let d, d_1, d_2 and d_3 be pf $\wedge$ ds designs and let $e = \mathbf{system}\,[]$. Then $d * e$ and $e * d$ are defined and we have

(i) $d * e = e * d = d$,

(ii) $(d_1 * d_2) * d_3 = d_1 * (d_2 * d_3)$

provided in (ii) everything is defined, i.e. cset$(d_i) \cap$ cset$(d_j) = \emptyset$ for $i \neq j$.

Proof. Directly from definition 3.3.1. □

Of course we are also interested in the behaviour of $*$ with respect to glass-box correctness (gbc) and black-box correctness (bbc). Since the intuition behind $*$ is that of taking the union of the component sets of two 'disjoint' designs the following should not come as a surprise.

Lemma 3.3.3 (Correctness-preserving properties of $*$). Let d_1 and d_2 be pf $\wedge$ ds designs with disjoint sets of component names.

(i) d_1 and d_2 are gbc $\Leftrightarrow$ $d_1 * d_2$ is gbc,

(ii) d_1 and d_2 are bbc $\Leftrightarrow$ $d_1 * d_2$ is bbc.

Proof. (i) Let d_1 and d_2 be as in definition 3.3.1. Throughout this part (i) we shall employ several of the derived rules of $\lambda\pi$ as worked out in Chapter 2 again (weakening, cut-rule etc.). We show ($\Rightarrow$) first. Assume that both d_1 and d_2 are gbc. We must show that $d_1 * d_2$ is gbc. If in showing the correctness of some component in d_1 we have the condition $\Gamma \vdash P_j \sqsubseteq Q_j$, where Γ is the context as given by the definition of gbc, then the same component appears also in $d_1 * d_2$ with correctness condition $\Gamma' \vdash P_j \sqsubseteq Q_j$ for some $\Gamma' \supseteq \Gamma$. By weakening it can be shown that $\Gamma \vdash P_j \sqsubseteq Q_j$ implies $\Gamma' \vdash P_j \sqsubseteq Q_j$. Therefore this component is also correct as a component in $d_1 * d_2$. In a similar way can see that each component from d_2 is also correct in $d_1 * d_2$.

Next we shall show ($\Leftarrow$). Assume $d_1 * d_2$ is gbc. First we must show that d_1 is gbc. Consider a non-prim component in $d_1 * d_2$ which comes from d_1. Suppose that we have the correctness condition $\Gamma' \vdash P_j \sqsubseteq Q_j$ for this

component in $d_1 * d_2$. We must prove $\Gamma \vdash P_j \sqsubseteq Q_j$ for the same component in d_1, and we have $\Gamma' = \Gamma \cup \Delta$ for some set of assumptions Δ.
Δ contains assumptions of the form $[u_i \sqsubseteq N_i]$ for $1 \leq i \leq n_2$ where each u_i occurs at most once in the left hand side of an assumption. Since Δ stems from d_2, we have $u_i \notin \mathrm{cset}(d_1)$ and hence the u_i do not occur freely in Γ or in P_j, Q_j. Therefore the assumptions in Δ do not play an essential role in the derivation of $P_j \sqsubseteq Q_j$ (show this by again using the technique of substituting both left and right of the '$\vdash$'). Hence $\Gamma \vdash P_j \sqsubseteq Q_j$. This shows that d_1 is gbc. The fact that d_2 is gbc can be shown in a similar way where we deal with non-essential assumptions $[x_i \sqsubseteq M_i]$ for $1 \leq i \leq n_1$ and $[y_i = P_i]$ for $1 \leq i \leq l_1$.
(ii) Similarly as (i). □

Note that neither $d_1 * d_2 = d_2 * d_1$ (obvious) nor $\vdash [\![d_1 * d_2]\!] = [\![d_2 * d_1]\!]$ – essentially for the same reasons that forbid us to swap **prim** components.

The following lemma states that the operation $*$ on designs can be interpreted (via $[\![\ ]\!]$) as the operation $*$ of the $\lambda\pi$-calculus with sequences (see appendix A).

Lemma 3.3.4 (Semantics of $*$). Let d_1 and d_2 be pf $\wedge$ ds and gbc designs with disjoint sets of component names, then we have

$$\vdash [\![d_1 * d_2]\!] = [\![d_1]\!] * [\![d_2]\!].$$

Proof. Let d_1, d_2 be given as in definition 3.3.1. Since $d_1 * d_2$ is gbc, we can contract all abstraction-application pairs of $[\![d_1 * d_2]\!]$ which correspond to non-**prim** components and we get $\vdash [\![d_1 * d_2]\!] = \mathcal{L}_1\mathcal{L}_2([\vec{S}\vec{T}]S_2S_1)$ where

- $\mathcal{L}_1$ is the sequence of abstractions $\lambda x_j \sqsubseteq M_j.$ for $1 \leq j \leq n_1$ with increasing j,
- $\mathcal{L}_2$ is the sequence of abstractions $\lambda u_j \sqsubseteq N_j.$ for $1 \leq j \leq n_2$ with increasing j,
- S_2 is the sequence of substitutions $[v_j := A_j]$ for $1 \leq j \leq l_2$ with decreasing j,
- S_1 is the sequence of substitutions $[y_j := P_j]$ for $1 \leq j \leq l_1$ with decreasing j.

Now we have the following calculation in $\lambda\pi$-calculus:

$$\begin{aligned} \vdash [\![d_1 * d_2]\!] &= \mathcal{L}_1\mathcal{L}_2([\vec{S}\vec{T}]S_2S_1) \\ &\equiv \mathcal{L}_1\mathcal{L}_2[\vec{S}S_1\vec{T}S_2] \end{aligned}$$

$$\begin{aligned}
&\overset{*1}{=} \mathcal{L}_1\mathcal{L}_2([\vec{S}S_1] * [\vec{T}S_2])\\
&\overset{*2}{=} \mathcal{L}_1([\vec{S}]S_1) * (\mathcal{L}_2([\vec{T}]S_2))\\
&= [\![d_1]\!] * [\![d_2]\!].
\end{aligned}$$

□

3.3.2 The operation ∘

The second (partial) operation is called *composition* and is denoted by $\circ$. It is related to functional composition, usually also denoted by $\circ$. Roughly speaking $d_1 \circ d_2$ is the design which is obtained by appending d_1 to d_2 while replacing the prims of d_1 by the elements of the system of d_2. Furthermore there is a notion of *validation*, by which we mean that we shall define when the composition of d_1 with d_2 is *valid*. In Section 3.4.3 we shall argue that this use of the terms 'valid' and 'validation' is consistent with the usual terminology (as used in e.g. [1]).

This binary operation $\circ$ will play a key role in the discussion of validation (Section 3.4.3) and in the description of design evolution (Section 3.5). Furthermore $\circ$ will be used in the description of parallel development (Section 3.6).

Definition 3.3.5 ($\circ$).
Assume pf $\wedge$ ds designs d_1 and d_2 where $\mathrm{cset}(d_1) \cap \mathrm{cset}(d_2) = \emptyset$. Let d_1 and d_2 respectively be given by

$$\begin{array}{llll}
x_1 & := & \mathbf{prim} & \sqsubseteq \; M_1\\
 & \ldots & & \\
x_n & := & \mathbf{prim} & \sqsubseteq \; M_n\\
y_1 & := & P_1 & \sqsubseteq \; Q_1\\
 & \ldots & & \\
y_m & := & P_m & \sqsubseteq \; Q_m\\
\multicolumn{4}{l}{\mathbf{system}\ L,}
\end{array}
\qquad
\begin{array}{llll}
z_1 & := & A_1 & \sqsubseteq \; B_1\\
 & \ldots & & \\
z_l & := & A_l & \sqsubseteq \; B_l\\
\multicolumn{4}{l}{\mathbf{system}\ [S_1,\ldots,S_n].}
\end{array}$$

We assume that $P_1,\ldots,P_m$ are not prim, whereas some of the A_i may be prim $(1 \le i \le l)$. We define $d_1 \circ d_2$ as the design given by

$$\begin{array}{llll}
z_1 & := & A_1 & \sqsubseteq \; B_1\\
 & \ldots & & \\
z_l & := & A_l & \sqsubseteq \; B_l\\
x_1 & := & S_1 & \sqsubseteq \; M_1\\
 & \ldots & &
\end{array}$$

$$\begin{array}{llll}
x_n & := & S_n & \sqsubseteq \; M_n \\
y_1 & := & P_1 & \sqsubseteq \; Q_1 \\
 & \dots & & \\
y_m & := & P_m & \sqsubseteq \; Q_m \\
\multicolumn{2}{l}{\mathbf{system}} & L. &
\end{array}$$

Note that $d_1 \circ d_2$ is a pf $\wedge$ ds design again. Note also that if arity$(d_1) = (n, k)$ and arity$(d_2) = (h, n)$, then arity$(d_1 \circ d_2) = (h, k)$. □

The design $d_1 \circ d_2$ can be viewed as a 'layered' design with layers d_1 and d_2.

Remark 3.3.6 We get a highly intuitive view of the construction of $d_1 \circ d_2$ if we omit the keyword **system** in d_2 and the keywords **prim** in d_1 and write the system of d_2 as a column vector. We show this below. Write d_2 as

$$\begin{array}{llll}
z_1 & := & A_1 & \sqsubseteq \; B_1 \\
 & \dots & & \\
z_l & := & A_l & \sqsubseteq \; B_l \\
 & & \begin{bmatrix} S_1 \\ \vdots \\ S_n \end{bmatrix} &
\end{array}$$

and write d_1 as

$$\begin{array}{llll}
x_1 & := & & \sqsubseteq \; M_1 \\
 & \dots & & \\
x_n & := & & \sqsubseteq \; M_n \\
y_1 & := & P_1 & \sqsubseteq \; Q_1 \\
 & \dots & & \\
y_m & := & P_m & \sqsubseteq \; Q_m \\
\multicolumn{2}{l}{\mathbf{system}} & L. &
\end{array}$$

Now one can view the construction of $d_1 \circ d_2$ as a matter of plugging the system of d_2 into the hole corresponding to the **prims** of d_1. □

We have one simple algebraic law for $\circ$.

Lemma 3.3.7 (Algebraic properties of $\circ$). Let d_1, d_2 and d_3 be pf $\wedge$ ds designs.

$$(d_1 \circ d_2) \circ d_3 = d_1 \circ (d_2 \circ d_3).$$

provided everything is defined.

Proof. Directly from definition 3.3.5. □

Remark 3.3.8 (i) There is no neutral element e_o such that for all designs d we have $d \circ e_o = e_o \circ d = d$. Note that $e =$ **system** $[]$ certainly does not do the job, since in general the arity of e need not match the arity of an arbitrary d.

(ii) If we would adopt a 'top' element $\top$ in $\lambda\pi$-calculus, then for any n, the design e_n given by

$$\begin{array}{lll} x_1 & := & \textbf{prim} \sqsubseteq \top \\ & \ldots & \\ x_n & := & \textbf{prim} \sqsubseteq \top \\ \multicolumn{3}{l}{\textbf{system } [x_1, \ldots, x_n]} \end{array}$$

would have the property that for every design d with arity (m, n) the following holds:
$\vdash [\![e_n \circ d]\!] = [\![d]\!]$. Therefore it would be possible to view such an e_n as a left neutral element. There is no such right neutral element. This is because when $\vdash [\![d \circ e]\!] = [\![d]\!]$, then $[\![d \circ e]\!]$ must have the same parameter restrictions for its un-applied lambdas as $[\![d]\!]$. But the parameter restrictions $[\![d \circ e]\!]$ come from the black-box descriptions of its **prim**-components, which can not be chosen to fit all d.

(iii) We could have defined $d_1 \circ d_2$ differently, e.g. by having no restrictions on the arities of d_1 and d_2. In that case the elements of the system of d_2 for which there is no **prim** in d_1 could appear again in the system of the resulting $d_1 \circ d_2$. In a similar way we could deal with the possibility that there are **prim**s for which there is no system element. If we would have defined $d_1 \circ d_2$ in such a way, we would have $e =$ **system** $[]$ as a neutral element. □

The following example shows one of the applications for the operation $\circ$.

Example 3.3.9 Let us assume that we have a library consisting of two implemented components and that we have a design d which must use this library. Let the library be given as a design d_{lib}. Let d_{lib} and d respectively be given as

$$\begin{array}{lllll} x_1 & := & P_1 & \sqsubseteq & Q_1 \\ x_2 & := & P_2 & \sqsubseteq & Q_2 \\ \multicolumn{5}{l}{\textbf{system } [x_1, x_2],} \end{array} \qquad \begin{array}{lll} x'_1 & := & \textbf{prim} \sqsubseteq Q_1 \\ x'_2 & := & \textbf{prim} \sqsubseteq Q_2 \\ x_3 & := & P_3(x'_1, x'_2) \sqsubseteq Q_3 \\ x_4 & := & P_4(x'_1, x'_2, x_3) \sqsubseteq Q_4 \\ \multicolumn{3}{l}{\textbf{system } S(x'_1, x'_2, x_3, x_4).} \end{array}$$

The instantiation of d with d_{lib} can be described by the composition $d \circ d_{lib}$.

We can simplify $d \circ d_{lib}$ to a design d'. The designs $d \circ d_{lib}$ and d' respectively are given as

$$\begin{array}{lllll} x_1 & := & P_1 & \sqsubseteq & Q_1 \\ x_2 & := & P_2 & \sqsubseteq & Q_2 \\ x_1' & := & x_1 & \sqsubseteq & Q_1 \\ x_2' & := & x_2 & \sqsubseteq & Q_2 \\ x_3 & := & \multicolumn{3}{l}{P_3(x_1', x_2') \sqsubseteq Q_3} \\ x_4 & := & \multicolumn{3}{l}{P_4(x_1', x_2', x_3) \sqsubseteq Q_4} \\ \multicolumn{2}{l}{\mathbf{system}} & \multicolumn{3}{l}{S(x_1', x_2', x_3, x_4),} \end{array} \qquad \begin{array}{lllll} x_1 & := & P_1 & \sqsubseteq & Q_1 \\ x_2 & := & P_2 & \sqsubseteq & Q_2 \\ x_3 & := & \multicolumn{3}{l}{P_3(x_1, x_2) \sqsubseteq Q_3} \\ x_4 & := & \multicolumn{3}{l}{P_4(x_1, x_2, x_3) \sqsubseteq Q_4} \\ \multicolumn{2}{l}{\mathbf{system}} & \multicolumn{3}{l}{S(x_1, x_2, x_3, x_4).} \end{array}$$

The fact that $\vdash [\![d \circ d_{lib}]\!] = [\![d']\!]$ follows from the component cancellation lemma 3.2.16. □

In the above example we see that the facts that the library is modeled as a design and that the components using it are modeled as a design introduce some overhead. In the example this overhead consists of the components $(x_1' := x_1 \sqsubseteq Q_1)$ and $(x_2' := x_2 \sqsubseteq Q_2)$. However the additional components can easily be removed, as shown by the example. The feasibility of this approach in practice may depend on the existence of an abbreviation facility which should provide for 'global' abbreviations (as mentioned in remark 3.2.3). If there is no such abbreviation facility, then the overhead due to the duplication of black-box descriptions (the Q_1 and Q_2 in the example) may become too large.

We have one more algebraic law, in which both $*$ and $\circ$ occur. By way of introduction to this law we first give an example.

Example 3.3.10 Let the pf $\wedge$ ds designs d_1 and d_2 respectively be given by

$$\begin{array}{lllll} x_1 & := & \mathbf{prim} & \sqsubseteq & A_1 \\ x_2 & := & A_2 & \sqsubseteq & A_3 \\ \multicolumn{2}{l}{\mathbf{system}} & \multicolumn{3}{l}{[A_4],} \end{array} \qquad \begin{array}{lllll} y_1 & := & \mathbf{prim} & \sqsubseteq & B_1 \\ y_2 & := & B_2 & \sqsubseteq & B_3 \\ \multicolumn{2}{l}{\mathbf{system}} & \multicolumn{3}{l}{[B_4]} \end{array}$$

and let the pf $\wedge$ ds designs d_3 and d_4 respectively be given by

$$\begin{array}{lllll} z_1 & := & \mathbf{prim} & \sqsubseteq & C_1 \\ z_2 & := & C_2 & \sqsubseteq & C_3 \\ \multicolumn{2}{l}{\mathbf{system}} & \multicolumn{3}{l}{[C_4],} \end{array} \qquad \begin{array}{lllll} u_1 & := & \mathbf{prim} & \sqsubseteq & D_1 \\ u_2 & := & D_2 & \sqsubseteq & D_3 \\ \multicolumn{2}{l}{\mathbf{system}} & \multicolumn{3}{l}{[D_4],} \end{array}$$

then the designs $(d_1 * d_2) \circ (d_3 * d_4)$ and $(d_1 \circ d_3) * (d_2 \circ d_4)$ respectively are given by

$$\begin{array}{lllll} z_1 & := & \mathbf{prim} & \sqsubseteq & C_1 \\ u_1 & := & \mathbf{prim} & \sqsubseteq & D_1 \end{array} \qquad \begin{array}{lllll} z_1 & := & \mathbf{prim} & \sqsubseteq & C_1 \\ u_1 & := & \mathbf{prim} & \sqsubseteq & D_1 \end{array}$$

$$\begin{array}{llllllllllll}
z_2 & := & C_2 & \sqsubseteq & C_3 & \qquad & z_2 & := & C_2 & \sqsubseteq & C_3 \\
u_2 & := & D_2 & \sqsubseteq & D_3 & & x_1 & := & C_4 & \sqsubseteq & A_1 \\
x_1 & := & C_4 & \sqsubseteq & A_1 & & x_2 & := & A_2 & \sqsubseteq & A_3 \\
y_1 & := & D_4 & \sqsubseteq & B_1 & & u_2 & := & D_2 & \sqsubseteq & D_3 \\
x_2 & := & A_2 & \sqsubseteq & A_3 & & y_1 & := & D_4 & \sqsubseteq & B_1 \\
y_2 & := & B_2 & \sqsubseteq & B_3 & & y_2 & := & B_2 & \sqsubseteq & B_3 \\
\multicolumn{5}{l}{\mathbf{system}\ [A_4, B_4],} & & \multicolumn{5}{l}{\mathbf{system}\ [A_4, B_4].}
\end{array}$$

The latter two designs are semantically equivalent by the component swap lemma 3.2.9 provided they are gbc. □

Lemma 3.3.11 (Interchange law for $*$ and $\circ$). Consider pf $\wedge$ ds designs d_1, d_2, d_3 and d_4. Assume that $(d_1 * d_2) \circ (d_3 * d_4)$ is gbc, then

$$\vdash [\![(d_1 * d_2) \circ (d_3 * d_4)]\!] = [\![(d_1 \circ d_3) * (d_2 \circ d_4)]\!].$$

provided $(d_1 \circ d_3) * (d_2 \circ d_4)$ is defined.

Proof. First note that the left-hand side of the equation must be defined – as is easily seen by analysing the arities of $d_1 * d_2$ and $d_3 * d_4$. The design $(d_1 * d_2) \circ (d_3 * d_4)$ has precisely the same components as the design $(d_1 \circ d_3) * (d_2 \circ d_4)$. Furthermore these designs have the same sequence of **prim** components: first the **prim** components of d_3, followed by the **prim** components of d_4. These designs also have the same system: the concatenation of the systems of d_1 and d_2. Finally the fact that these designs are semantically equivalent follows by repeated application of the component swap lemma 3.2.9. □

Remark 3.3.12 The term 'interchange law' is taken from [2] (page 44). In order to have an intuitive understanding of the interchange law we shall consider a very simple algebraic system in which this law also holds. Consider an algebraic system with *walls* as objects and with two operations: putting one wall next to another and putting one wall on top of another. Of course we are given a number of *bricks*. We can put one wall beside another wall:

$$\boxed{\text{A}} * \boxed{\text{B}} = \begin{array}{|c|c|}\hline \text{A} & \text{B} \\ \hline \end{array}.$$

We can put one wall on top of another wall:

$$\boxed{\text{A}} \circ \boxed{\text{B}} = \begin{array}{|c|}\hline \text{A} \\ \hline \text{B} \\ \hline \end{array}.$$

Now we have

$$(\boxed{\text{A}} * \boxed{\text{B}}) \circ (\boxed{\text{C}} * \boxed{\text{D}}) = \begin{array}{|c|c|}\hline \text{A} & \text{B} \\ \hline \end{array} \circ \begin{array}{|c|c|}\hline \text{C} & \text{D} \\ \hline \end{array} = \begin{array}{|c|c|}\hline \text{A} & \text{B} \\ \hline \text{C} & \text{D} \\ \hline \end{array}.$$

But we also have

$$(\boxed{\mathrm{A}}\circ\boxed{\mathrm{C}})*(\boxed{\mathrm{B}}\circ\boxed{\mathrm{D}}) = \begin{array}{|c|}\hline \mathrm{A}\\\hline \mathrm{C}\\\hline\end{array} * \begin{array}{|c|}\hline \mathrm{B}\\\hline \mathrm{D}\\\hline\end{array} = \begin{array}{|c|c|}\hline \mathrm{A} & \mathrm{B}\\\hline \mathrm{C} & \mathrm{D}\\\hline\end{array}.$$

□

Very much in the same way as we have two notions of correctness for designs, we shall have two different definitions of 'valid', viz. glass-box valid and black-box valid (written as $\mathrm{gbv}(d_1, d_2)$ and $\mathrm{bbv}(d_1, d_2)$ respectively).

Definition 3.3.13 Assume pf $\wedge$ ds designs d_1 and d_2 such that $d_1 \circ d_2$ is defined and let d_1 and d_2 be given as in definition 3.3.5.
(i) the pair (d_1, d_2) is *glass-box valid* (notation $\mathrm{gbv}(d_1, d_2)$) if

$$\forall i\ (1 \leq i \leq n) \cdot \Gamma \vdash S_i \sqsubseteq M_i$$

where $\Gamma := [z_1 \sqsubseteq B_1], \ldots, [z_h \sqsubseteq B_h], [z_{h+1} = A_{h+1}], \ldots, [z_l = A_l]$, assuming that h is the number of prim components of d_2.
(ii) the pair (d_1, d_2) is *black-box valid* (notation $\mathrm{bbv}(d_1, d_2)$) if

$$\forall i\ (1 \leq i \leq n) \cdot \Delta \vdash S_i \sqsubseteq M_i$$

where $\Delta := [z_1 \sqsubseteq B_1], \ldots, [z_l \sqsubseteq B_l]$. □

Before actually employing these notions gbv and bbv for formulating the correctness-preserving properties of $\circ$, we have a look at a few simple properties.

For the left neutral element e_n we have for all d that both $\mathrm{gbv}(e_n, d)$ and $\mathrm{bbv}(e_n, d)$, provided the arities match. But $\mathrm{gbv}(d, e_n)$, $\mathrm{bbv}(d, e_n)$ need not hold in general. The proposition that for all d_1, d_2 we have $\mathrm{bbv}(d_1, d_2) \Rightarrow$ (d_1, d_2 are bbc) just does not hold in general. The counter-example is very simple. Use $\Re_1$ as before and take the d_2 with one component $x := 2 \sqsubseteq 1$ and with system [ ]. This is because bbv is about the "plug-ability" of d_2 with respect to d_1 rather than about the internals of d_1 and d_2. Similarly glass-box validation does not imply glass-box correctness – as is easily seen by considering the same counter-example.

As a non-trivial property we have $\mathrm{bbv}(d_1, d_2) \Rightarrow \mathrm{gbv}(d_1, d_2)$, provided d_2 is glass-box correct. This can be shown by similar techniques as employed to prove '$\mathrm{bbc}(d) \Rightarrow \mathrm{gbc}(d)$' in Chapter 2.

Next we investigate the behaviour of $\circ$ with respect to gbc and bbc. Intuitively it is clear that the validation conditions gbv and bbv will play a role here.

Lemma 3.3.14 (Correctness-preserving properties of $\circ$). Let d_1 and d_2 be pf $\wedge$ ds designs.

(i) $(d_1,\ d_2$ are gbc $\wedge$ gbv$(d_1, d_2))\ \Rightarrow\ d_1 \circ d_2$ is gbc,

(ii) $(d_1,\ d_2$ are bbc $\wedge$ bbv$(d_1, d_2))\ \Leftrightarrow\ d_1 \circ d_2$ is bbc.

Proof . Let d_1 and d_2 be given as in definition 3.3.5. Let Γ and Δ be given by definition 3.3.13.
(i) Assume that d_1 and d_2 are gbc and that gbv(d_1, d_2) holds. We must show that $d_1 \circ d_2$ is gbc. We investigate three kinds of components in $d_1 \circ d_2$.

- A component from d_2 has the same correctness condition in $d_1 \circ d_2$ as in d_2.
- For a component $(x_i := S_i \sqsubseteq M_i)$ where $1 \leq i \leq n$ the correctness condition is $\Gamma, [x_1 = S_1], \ldots, [x_{i-1} = S_{i-1}] \vdash S_i \sqsubseteq M_i$. This correctness condition follows by weakening from the assumption that gbv(d_1, d_2) holds.
- For a component $(y_j := P_j \sqsubseteq Q_j)$ where $1 \leq j \leq m$ the correctness condition is $\Gamma, [x_1 = S_1], \ldots, [x_n = S_n], [y_1 = P_1], \ldots, [y_{j-1} = P_{j-1}] \vdash P_j \sqsubseteq Q_j$ (*).
We have $\Gamma \vdash [S_1 \sqsubseteq M_1], \ldots, [S_n \sqsubseteq M_n]$ and hence $\Gamma, [x_1 = S_1], \ldots, [x_n = S_n] \vdash [x_1 \sqsubseteq M_1], \ldots, [x_n \sqsubseteq M_n]$ which at its turn together with the assumption that d_1 is gbc shows the condition (*).

(ii) We show ($\Rightarrow$) first. Assume that d_1 and d_2 are bbc and that bbv(d_1, d_2) holds. We must show that $d_1 \circ d_2$ is bbc. We investigate three kinds of components in $d_1 \circ d_2$.

- A component from d_2 has the same correctness condition in $d_1 \circ d_2$ as in d_2.
- For a component $(x_i := S_i \sqsubseteq M_i)$ where $1 \leq i \leq n$ the correctness condition is $\Delta, [x_1 \sqsubseteq M_1], \ldots, [x_{i-1} \sqsubseteq M_{i-1}] \vdash S_i \sqsubseteq M_i$. This correctness condition follows by weakening from the assumption that bbv(d_1, d_2) holds.
- For a component $(y_j := P_j \sqsubseteq Q_j)$ where $1 \leq j \leq m$ the correctness condition is $\Delta, [x_1 \sqsubseteq M_1], \ldots, [x_n \sqsubseteq M_n], [y_1 \sqsubseteq Q_1], \ldots, [y_{j-1} \sqsubseteq Q_{j-1}] \vdash P_j \sqsubseteq Q_j$, which follows by weakening from the assumption that d_1 is bbc.

Next we shall show ($\Leftarrow$). Assume that $d_1 \circ d_2$ is bbc. First we must show

that d_1 is bbc. For a non-**prim** component in d_1, $(y_j := P_j \sqsubseteq Q_j)$ say, the correctness condition is $[x_1 \sqsubseteq M_1], \ldots, [x_n \sqsubseteq M_n], [y_1 \sqsubseteq Q_1], \ldots, [y_{j-1} \sqsubseteq Q_{j-1}] \vdash P_j \sqsubseteq Q_j$. Now we note that the assumptions in Δ do not play an essential role in the derivation of $P_j \sqsubseteq Q_j$. This shows that d_1 is gbc.
For a component in d_2 the correctness condition is the same as in $d_1 \circ d_2$. This shows that d_2 is bbc. Finally we must show that $\mathrm{bbv}(d_1, d_2)$ holds, i.e. for arbitrary i such that $1 \leq i \leq n$ we must show $\Delta \vdash S_i \sqsubseteq M_i$. Now we note that the assumptions $[x_1 \sqsubseteq M_1], \ldots, [x_{i-1} \sqsubseteq M_{i-1}]$ do not play an essential role in the derivation of the correctness condition for the component $(x_i := S_i \sqsubseteq M_i)$ in $d_1 \circ d_2$. □

Remark 3.3.15 The proposition that we have

$$d_1 \circ d_2 \text{ is gbc} \;\Rightarrow\; d_1,\, d_2 \text{ are gbc}$$

does not hold in general.

The counter-example is as follows. Consider $\Re_1$ as before. Let d_1 and d_2 respectively be given by

$$\begin{array}{llll} y & := & \mathbf{prim} & \sqsubseteq\ 3 \\ z & := & y & \sqsubseteq\ 2 \\ \multicolumn{4}{l}{\mathbf{system}\ [y,z],} \end{array} \qquad\qquad \begin{array}{llll} x & := & 1 & \sqsubseteq\ 2 \\ \multicolumn{4}{l}{\mathbf{system}\ [x].} \end{array}$$

We see that d_1 is not gbc. The design $d_1 \circ d_2$ is gbc. We give $d_1 \circ d_2$ below:

$$\begin{array}{llll} x & := & 1 & \sqsubseteq\ 2 \\ y & := & x & \sqsubseteq\ 3 \\ z & := & y & \sqsubseteq\ 2 \\ \multicolumn{4}{l}{\mathbf{system}\ [y,z].} \end{array}$$ □

We have a law for bbv which is somewhat similar to the interchange law of lemma 3.3.11.

Lemma 3.3.16 (Interchange law for bbv). Consider pf $\wedge$ ds designs d_1, d_2, d_3 and d_4 such that the compositions $d_1 \circ d_3$ and $d_2 \circ d_4$ are defined.

$$\mathrm{bbv}(d_1 * d_2, d_3 * d_4) \;\Leftrightarrow\; (\mathrm{bbv}(d_1, d_3) \wedge \mathrm{bbv}(d_2, d_4)).$$

Proof. The condition $\mathrm{bbv}(d_1, d_3)$ requires that one can prove that the system elements of d_3 satisfy the restrictions of the **prim** components of d_1 in a context given by d_3. The same facts must also be proved for $\mathrm{bbv}(d_1 * d_2, d_3 * d_4)$, but now in the combined context of d_3 and d_4. We note that the assumptions from d_4 do not play an essential role.
The condition $\mathrm{bbv}(d_2, d_4)$ requires that one can prove that the system ele-

ments of d_4 satisfy the restrictions of the **prim** components of d_2 in a context given by d_4. The same facts must also be proved for $\text{bbv}(d_1 * d_2, d_3 * d_4)$, but now in the combined context of d_3 and d_4. We note that the assumptions from d_3 do not play an essential role. □

The operation $\circ$ on designs is interpreted (via $[\![\ ]\!]$) as the operation $\circ$ of the $\lambda\pi$-calculus with sequences (see appendix A).

Lemma 3.3.17 (Semantics of $\circ$). Let d_1 and d_2 be pf $\wedge$ ds designs and let d_2 be gbc.

$$\vdash [\![d_1 \circ d_2]\!] = [\![d_1]\!] \circ [\![d_2]\!].$$

Proof. As 3.3.4 (Semantics of $*$). □

3.3.3 The operations bot and top

We would like to isolate the parts of d_1 which for given d_2 play a role in $\text{bbv}(d_1, d_2)$ and the parts of d_2 which for given d_1 play a role in $\text{bbv}(d_1, d_2)$. We cast these parts in the form of designs.

Definition 3.3.18 (bot, top). Consider a pf $\wedge$ ds design d which is given as

$$\begin{array}{lllll} x_1 & := & \textbf{prim} & \sqsubseteq & M_1 \\ & \cdots & & & \\ x_n & := & \textbf{prim} & \sqsubseteq & M_n \\ y_1 & := & P_1 & \sqsubseteq & Q_1 \\ & \cdots & & & \\ y_m & := & P_m & \sqsubseteq & Q_m \\ \multicolumn{5}{l}{\textbf{system}\ [S_1, \ldots, S_l]} \end{array}$$

where $P_1, \ldots, P_m$ are not equal to **prim**. We define $\text{bot}(d)$ and $\text{top}(d)$ respectively as the designs

$$\begin{array}{lllll} x_1 & := & \textbf{prim} & \sqsubseteq & M_1 \\ & \cdots & & & \\ x_n & := & \textbf{prim} & \sqsubseteq & M_n \\ \multicolumn{5}{l}{\textbf{system}\ [],} \end{array} \qquad \begin{array}{lllll} x_1 & := & \textbf{prim} & \sqsubseteq & M_1 \\ & \cdots & & & \\ x_n & := & \textbf{prim} & \sqsubseteq & M_n \\ y_1 & := & \textbf{prim} & \sqsubseteq & Q_1 \\ & \cdots & & & \\ y_m & := & \textbf{prim} & \sqsubseteq & Q_m \\ \multicolumn{5}{l}{\textbf{system}\ [S_1, \ldots, S_l]} \end{array}$$

where it is understood that in top(d) only those components are retained whose name (x_i for some i with $1 \leq i \leq n$ or y_j for some j with $1 \leq j \leq m$) occurs in the system $[S_1, \ldots, S_l]$. Note that bot(d) and top(d) are pf $\wedge$ ds designs again. Note also that if arity(d) $= (n, l)$ then arity(bot(d)) $= (n, 0)$ and arity(top(d)) $= (n', l)$ where $n' \leq n + m$. We call bot(d) the ***bottom*** of d and we call top(d) the ***top*** of d. □

Remark 3.3.19 bot(d) and top(d) are *not* related to the idea of having terms $\perp$ and $\top$ which are minimal and maximal with respect to $\sqsubseteq$ in $\lambda\pi$-calculus. A motivation of the terms ***bottom*** and ***top*** will be given in remark 3.3.22 below. □

We have a number algebraic properties.

Lemma 3.3.20 (Algebraic properties of bot, top). Consider a pf $\wedge$ ds design d and let $e =$ **system** [].

(i) bot(bot(d)) $=$ bot(d),

(ii) top(top(d)) $=$ top(d),

(iii) top(bot(d)) $= e$.

Proof. Directly from the definition 3.3.18. □

The following lemma gives some algebraic properties which relate the unary operations bot and top with the binary operations $*$ and $\circ$.

Lemma 3.3.21 Assume pf $\wedge$ ds designs d_1 and d_2 and let $d_1 \circ d_2$ be defined.

(i) bot($d_1 * d_2$) $=$ bot(d_1) $*$ bot(d_2),

(ii) top($d_1 * d_2$) $=$ top(d_1) $*$ top(d_2),

(iii) bot($d_1 \circ d_2$) $=$ bot(d_2),

(iv) top($d_1 \circ d_2$) $=$ top(d_1).

Proof. (i) and (ii) follow directly from the definitions 3.3.18 and 3.3.1. whereas (iii) and (iv) follow directly from the definitions 3.3.18 and 3.3.5. □

A pf $\wedge$ ds design d for which bot(d) $= d$ is called a ***bottom*** design and a design d for which top(d) $= d$ is called a ***top*** design.

Remark 3.3.22 We have chosen the terms ***bottom*** and ***top*** because in our view these notions are related to the so-called top-down and bottom-up models of the software development process. For example in a top-down development process one starts with a given top design and then adds components and implements components, until the remaining **prim** components correspond to the primitives which actually are available. In a bottom-up development process one starts with a given bottom design and then adds components and adds system elements until the resulting design meets the actual requirements of the user of the design. We shall investigate the top-down and bottom-up models of the software development process in Sections 3.4.4 and 3.4.5 respectively. □

The following lemma confirms our intuition that bot(d) is precisely the part of d which is relevant for bbv(d, d_1) and that top(d) is precisely the part of d which is relevant for bbv(d_2, d).

Lemma 3.3.23 Consider pf ∧ ds designs d, d_1 and d_2.

(i) bbv(d, d_1) $\Leftrightarrow$ bbv(bot$(d), d_1)$,

(ii) bbv(d_2, d) $\Leftrightarrow$ bbv$(d_2$, top$(d))$.

Proof. Let d be given as in definition 3.3.18. Let d_1 and d_2 respectively be given by

$$\begin{array}{llll} z_1 & := & A_1 & \sqsubseteq B_1 \\ & \dots & & \\ z_k & := & A_k & \sqsubseteq B_k \\ \textbf{system} & [C_1, \dots, C_n], & & \end{array} \qquad \begin{array}{llll} v_1 & := & \textbf{prim} & \sqsubseteq R_1 \\ & \dots & & \\ v_l & := & \textbf{prim} & \sqsubseteq R_l \\ w_1 & := & X_1 & \sqsubseteq Y_1 \\ & \dots & & \\ w_h & := & X_h & \sqsubseteq Y_h \\ \textbf{system} & L. & & \end{array}$$

(i) bbv(d, d_1) holds iff $\forall i\ (1 \le i \le n) \cdot [z_1 \sqsubseteq B_1], \dots, [z_k \sqsubseteq B_k] \vdash C_i \sqsubseteq M_i$, which is precisely the condition bbv(bot$(d), d_1)$.

(ii) bbv(d_2, d) holds iff $\forall i\ (1 \le i \le l) \cdot [x_1 \sqsubseteq M_1], \dots, [x_n \sqsubseteq M_n], [y_1 \sqsubseteq Q_1], \dots, [y_m \sqsubseteq Q_m] \vdash S_i \sqsubseteq R_i$. The condition bbv$(d_2$, top$(d))$ requires that the same formulae $S_i \sqsubseteq R_i$ can be derived, but with fewer assumptions, since the following assumptions have been removed: those assumptions $[x_i \sqsubseteq M_i]$ and those assumptions $[y_j \sqsubseteq Q_j]$ $(1 \le i \le n,\ 1 \le j \le m)$ for which the variable (x_i or y_j) does not occur in the system $[S_1, \dots, S_l]$. Now $(\Leftarrow)$ follows by weakening and $(\Rightarrow)$ follows from the observation that the removed assumptions do not play an essential role in derivation of the formulae $S_i \sqsubseteq R_i$. □

The following fallacy is essentially due to the fact that glass-box correct designs do not offer implementation freedom.

Remark 3.3.24 The proposition that for pf $\wedge$ ds designs d_1 and d_2 we have

$$\mathsf{gbv}(d_1, d_2) \Rightarrow \mathsf{gbv}(d_1, \mathsf{top}(d_2)).$$

just does not hold in general.

The counter-example is as follows

$$\mathsf{gbv}\begin{pmatrix} x := \mathbf{prim} \sqsubseteq 1 & y := \ \ 1 \ \ \sqsubseteq 2 \\ \mathbf{system}\ [x] \quad , & \mathbf{system}\ [y] \end{pmatrix} \text{ holds because } [y = 1] \vdash y \sqsubseteq 1,$$

$$\mathsf{gbv}\begin{pmatrix} x := \mathbf{prim} \sqsubseteq 1 & y := \mathbf{prim} \sqsubseteq 2 \\ \mathbf{system}\ [x] \quad , & \mathbf{system}\ [y] \end{pmatrix} \text{ does not, for } [y \sqsubseteq 2] \nvdash y \sqsubseteq 1. \qquad \Box$$

Recall that we took the decision to cast the top of a design into the form of a design. This can be viewed as the choice of a very specific representation. For a given design d_2, we have $\mathsf{top}(d_2)$ as the representation of the parts of d_2 which for given d_1 play a role in $\mathsf{bbv}(d_1, d_2)$. First we note that the order of the **prim** components in $\mathsf{top}(d_2)$ is not relevant (from the current point of view). This idea is formalised by introducing a binary predicate $=_{\mathrm{pp}}$.

Definition 3.3.25 Let d and d' be pf $\wedge$ ds designs. We say that d' is a **prim** permutation of d, notation $d =_{\mathrm{pp}} d'$, if d' can be obtained from d by permuting the order of the **prim** components. $\Box$

Example 3.3.26 Let d and d' respectively be given by

$$\begin{array}{llll} x_1 & := & \mathbf{prim} \sqsubseteq & M_1 \\ x_2 & := & \mathbf{prim} \sqsubseteq & M_2 \\ x_3 & := & P \quad\ \ \sqsubseteq & Q \\ \multicolumn{4}{l}{\mathbf{system}\ [x_1, x_2, x_3],} \end{array} \qquad\qquad \begin{array}{llll} x_2 & := & \mathbf{prim} \sqsubseteq & M_2 \\ x_1 & := & \mathbf{prim} \sqsubseteq & M_1 \\ x_3 & := & P \quad\ \ \sqsubseteq & Q \\ \multicolumn{4}{l}{\mathbf{system}\ [x_1, x_2, x_3].} \end{array}$$

We have $d =_{\mathrm{pp}} d'$. Of course we also have $d' =_{\mathrm{pp}} d$. $\Box$

Lemma 3.3.27 Consider pf $\wedge$ ds designs d_1, d_2 and d_2' such that $d_2 =_{\mathrm{pp}} d_2'$ and such that $d_1 \circ d_2$ is defined, then we have

$$\mathsf{bbv}(d_1, d_2) \Leftrightarrow \mathsf{bbv}(d_1, d_2').$$

Proof. Just note that $\mathsf{bbv}(d_1, d_2)$ is defined (in definition 3.3.13) as $\forall i\ (1 \leq i \leq n) \cdot \Delta \vdash S_i \sqsubseteq M_i$ for suitable n, Δ, S_i and M_i where the elements of Δ are derived from the components of d_2, where the S_i stem from the system of d_2 and the M_i stem from d_1. Finally recall that the context Δ is a set (of assumptions) and that therefore Δ is independent of the order of the components in d_2. □

If we view a top design as a representation of the parts of a design d_2 which are relevant for $\text{bbv}(d_1, d_2)$, then we might identify even more top designs than indicated by $=_{\text{pp}}$.

Definition 3.3.28 Let d_2 and d_2' be top designs with arities (h, n) and (h', n) respectively. Then we define

$$\begin{aligned} d_2 =_{\text{top}} d_2' \ :\Leftrightarrow\ & \text{forall } d_1 \text{ with } d_1 \circ d_2 \text{ defined:} \\ & (\text{bbv}(d_1, d_2) \Leftrightarrow \text{bbv}(d_1, d_2')). \end{aligned}$$ □

If for two top designs d_2 and d_2' we have $d_2 =_{\text{top}} d_2'$ we say that d_2 and d_2' are *top-equivalent*.

Example 3.3.29 Consider an algebraic system $\Re$ which provides a binary function with symbol $+$ which is written in infix notation. Let the top designs d_2 and d_2' respectively be given as

$$\begin{array}{lll} x_1 & := & \textbf{prim} \sqsubseteq M_1 \\ x_2 & := & \textbf{prim} \sqsubseteq M_2 \\ x_3 & := & \textbf{prim} \sqsubseteq M_3 \\ \multicolumn{3}{l}{\textbf{system}\ [x_1 + x_2, x_1 + x_3],} \end{array} \qquad \begin{array}{lll} x_1 & := & \textbf{prim} \sqsubseteq M_1 \\ x_2 & := & \textbf{prim} \sqsubseteq M_2 \\ x_1' & := & \textbf{prim} \sqsubseteq M_1 \\ x_3 & := & \textbf{prim} \sqsubseteq M_3 \\ \multicolumn{3}{l}{\textbf{system}\ [x_1 + x_2, x_1' + x_3].} \end{array}$$

Then $d_2 =_{\text{top}} d_2'$ because if we consider the composition of d_2 (or of d_2') with some design d_1 of the form

$$\begin{array}{lll} y_1 & := & \textbf{prim} \sqsubseteq R_1 \\ y_2 & := & \textbf{prim} \sqsubseteq R_2 \\ & \cdots & \end{array}$$

(constructing $d_1 \circ d_2$ or $d_1 \circ d_2'$), then the validity conditions $\text{bbv}(d_1, d_2)$ and $\text{bbv}(d_1, d_2')$ are equivalent:

$$\text{bbv}(d_1, d_2) \Leftrightarrow [x_1 \sqsubseteq M_1], [x_2 \sqsubseteq M_2], [x_3 \sqsubseteq M_3] \vdash x_1 + x_2 \sqsubseteq R_1,\ x_1 + x_3 \sqsubseteq R_2$$

$$\Leftrightarrow [x_1 \sqsubseteq M_1], [x_2 \sqsubseteq M_2], [x_1' \sqsubseteq M_1], [x_3 \sqsubseteq M_3] \vdash x_1 + x_2 \sqsubseteq R_1,\ x_1' + x_3 \sqsubseteq R_2$$

$$\Leftrightarrow \text{bbv}(d, d_2').$$ □

In the above example we have transformed the top design d_2 into d'_2 where d'_2 can be viewed as the concatenation of two simpler designs. This idea is formulated in a general form in lemma 3.3.30 given below.

Lemma 3.3.30 Let d_2 be a top design with arity (h, n). Then for all $n', n'' \in \mathbb{N}$ with $n' + n'' = n$ we can find d'_2, d''_2, k' and k'' such that d'_2 and d''_2 have arities (k', n') and (k'', n'') and furthermore

$$d'_2 * d''_2 =_{\text{top}} d_2.$$

Proof. As in example 3.3.29 introduce additional **prim** components and rewrite system elements in terms of the names of these new components. In this way it is possible to make the set of component names in the first n' system elements disjoint with respect to the set of component names in the remaining n'' system elements. Then permute the order of the components such that first we have the components whose name is in the first n' system elements, followed by the remaining components. Finally take d'_2 and d''_2 in the obvious way such that $d'_2 * d''_2 =_{\text{top}} d_2$. □

Example 3.3.31 Consider an algebraic system $\Re$ which provides a binary function with symbol $+$ which is written in infix notation. Let the top design d_2 be given as in example 3.3.29. Then for $n' = n'' = 1$ lemma 3.3.30 says that there exist d'_2, d''_2 with $d'_2 * d''_2 =_{\text{top}} d_2$. Indeed, we can take d'_2 and d''_2 as follows

$$\begin{array}{llll} x_1 & := & \textbf{prim} \sqsubseteq & M_1 \\ x_2 & := & \textbf{prim} \sqsubseteq & M_2 \\ \multicolumn{4}{l}{\textbf{system } [x_1 + x_2],} \end{array} \qquad \begin{array}{llll} x'_1 & := & \textbf{prim} \sqsubseteq & M_1 \\ x_3 & := & \textbf{prim} \sqsubseteq & M_3 \\ \multicolumn{4}{l}{\textbf{system } [x'_1 + x_3].} \end{array}$$ □

Remark 3.3.32 For bottom designs the situation is much easier. The most interesting notion of equality is $=_{[\![\ ]\!]}$. Furthermore each bottom design d of arity $(n, 0)$ can be split for every n' and n'' with $n' + n'' = n$ into d' and d'' with arities $(n', 0)$ and $(n'', 0)$ such that $d' * d'' = d$. □

We see that $=_{\text{pp}}$ is nothing but a special case of $=_{\text{top}}$ and lemma 3.3.27 just says that $d =_{\text{pp}} d' \Rightarrow d =_{\text{top}} d'$. The advantage of considering $=_{\text{pp}}$ is the fact that $=_{\text{pp}}$ is a simple syntactical notion.

3.3.4 The operation $\sharp$

We shall briefly discuss a binary operation denoted by $\sharp$. The operation is suggested by the transitivity of the implementation relation $\sqsubseteq$. This operation will turn out to be somewhat disappointing but nevertheless we believe that it is illustrative to see what goes wrong. Yet the idea behind the operation seems attractive and we shall sketch one possible line of formalisation to exploit it. The results of this section will not be used in the remainder of this chapter (Section 3.6.5 being the only exception).

Definition 3.3.33 ($\sharp$) Assume pf $\wedge$ ds designs d_1 and d_2. Let d_1 and d_2 respectively be given by

$$\begin{array}{lllll} x_1 & := & \mathbf{prim} & \sqsubseteq & M_1 \\ & \dots & & & \\ x_n & := & \mathbf{prim} & \sqsubseteq & M_n \\ y_1 & := & P_1 & \sqsubseteq & Q_1 \\ & \dots & & & \\ y_m & := & P_m & \sqsubseteq & Q_m \\ \multicolumn{3}{l}{\mathbf{system}\ L,} \end{array} \qquad \begin{array}{lllll} x_1 & := & \mathbf{prim} & \sqsubseteq & M_1 \\ & \dots & & & \\ x_n & := & \mathbf{prim} & \sqsubseteq & M_n \\ y_1 & := & Q_1 & \sqsubseteq & R_1 \\ & \dots & & & \\ y_m & := & Q_m & \sqsubseteq & R_m \\ \multicolumn{3}{l}{\mathbf{system}\ L.} \end{array}$$

So d_1, d_2 have the same component names, the same black-box descriptions for the **prim** components, the same system and furthermore they have one 'column' in common. Then we define $d_1 \sharp d_2$ as

$$\begin{array}{lllll} x_1 & := & \mathbf{prim} & \sqsubseteq & M_1 \\ & \dots & & & \\ x_n & := & \mathbf{prim} & \sqsubseteq & M_n \\ y_1 & := & P_1 & \sqsubseteq & R_1 \\ & \dots & & & \\ y_m & := & P_m & \sqsubseteq & R_m \\ \multicolumn{3}{l}{\mathbf{system}\ L.} \end{array}$$

Note that $d_1 \sharp d_2$ is a pf $\wedge$ ds design again. Note also that if arity$(d_1) = (n,k)$ and arity$(d_2) = (n,k)$, then arity$(d_1 \sharp d_2) = (n,k)$. □

Lemma 3.3.34 (Algebraic properties of $\sharp$).

$$(d_1 \sharp d_2) \sharp d_3 = d_1 \sharp (d_2 \sharp d_3).$$

Proof. Directly from the definition 3.3.33 □

The following remark shows that $\sharp$ lacks a desirable property.

Remark 3.3.35 The proposition that we have

$$d_1, d_2 \text{ are bbc } \Rightarrow \ d_1 \sharp d_2 \text{ is bbc}$$

just does not hold in general.

The counter-example is as follows. Consider $\mathfrak{R}_1$ as before. Let d_1 and d_2 respectively be given by

$$\begin{array}{llll} x & := & 1 & \sqsubseteq 2 \\ y & := & x & \sqsubseteq 2 \\ \multicolumn{4}{l}{\textbf{system } [y],} \end{array} \qquad \begin{array}{llll} x & := & 2 & \sqsubseteq 3 \\ y & := & 2 & \sqsubseteq 2 \\ \multicolumn{4}{l}{\textbf{system } [y].} \end{array}$$

So both d_1 and d_2 are both bbc but $d_1 \sharp d_2$ is not bbc. We give $d_1 \sharp d_2$ below.

$$\begin{array}{llll} x & := & 1 & \sqsubseteq 3 \\ y & := & x & \sqsubseteq 2 \\ \multicolumn{4}{l}{\textbf{system } [y].} \end{array}$$

□

The above is a bit disappointing, but when adopting glass-box correctness, there is no problem.

Lemma 3.3.36 Assume pf $\wedge$ ds designs d_1, d_2. Then we have

$$d_1, d_2 \text{ are gbc } \Rightarrow \ d_1 \sharp d_2 \text{ is gbc}$$

provided $d_1 \sharp d_2$ is defined.

Proof. For each component use rule (trans.) and the fact that neither the black-box descriptions nor the glass-box descriptions of d_2 can contain component names. □

As a direct consequence of this lemma we also have d_1, d_2 are bbc $\Rightarrow d_1 \sharp d_2$ is gbc. Just use 'd_1 is bbc $\Rightarrow d_1$ is gbc' and 'd_2 is bbc $\Rightarrow d_2$ is gbc'.

It is not hard to see why $\sharp$ fails to preserve bbc. The obvious idea – which is to suppose that rule (trans.) applies – fails because the facts $P_i \sqsubseteq Q_i$ and $Q_i \sqsubseteq R_i$ are not given for the same set of assumptions. In particular the assumptions $y_j \sqsubseteq Q_j$ $(j < i)$ are stronger than the $y_j \sqsubseteq R_j$ which determine the black-box correctness of $d_1 \sharp d_2$.

As a possible solution, we could generalise the notions of correctness so as not to have two notions gbc and bbc, but just one notion, 'correct' say, which is parameterised over contexts – writing 'Γ-correct' for context Γ.

Definition 3.3.37 (Γ-correct) Let the pf $\wedge$ ds design d be given as

$$\begin{array}{llll} x_1 & := & \text{prim} \sqsubseteq & M_1 \\ & \cdots & & \\ x_n & := & \text{prim} \sqsubseteq & M_n \\ y_1 & := & P_1 \quad \sqsubseteq & Q_1 \\ & \cdots & & \end{array}$$

y_m := P_m $\sqsubseteq$ Q_m
system L.

For a given context Γ we say that d is Γ-correct, if for every component $y_i := P_i \sqsubseteq Q_i$ we have

$$\Gamma \vdash P_i \sqsubseteq Q_i. \qquad \square$$

Recall that $\Gamma_{\text{gb}}(d)$ denotes the glass-box context of d and that $\Gamma_{\text{bb}}(d)$ denotes the black-box context of d (see definition 3.2.18). Now we get the specialised notions gbc and bbc back in the sense that d is gbc $\Leftrightarrow$ d is $\Gamma_{\text{gb}}(d)$-correct and similarly that d is bbc $\Leftrightarrow$ d is $\Gamma_{\text{bb}}(d)$-correct. (To see these equivalences is not completely trivial: for $\Rightarrow$ we must use weakening and for $\Leftarrow$ we must use that d is wf and that for each $P_i \sqsubseteq Q_i$ the assumptions $y_j = Q_j$ or $y_j \sqsubseteq Q_j$ for $j \geq i$ are non-essential).

With the above machinery of generalised correctness it is straightforward to see precisely in which way $\sharp$ preserves black-box correctness.

Lemma 3.3.38 Assume pf $\wedge$ ds designs d_1, d_2. Then we have

$$d_1, d_2 \text{ are } \Gamma_{\text{bb}}(d_2)\text{-correct} \Rightarrow d_1 \sharp d_2 \text{ is bbc}$$

provided $d_1 \sharp d_2$ is defined.

Proof. Just use transitivity for each component. $\square$

Using transitivity of $\sqsubseteq$ means to work by means of a certain type of stepwise refinement. For example, when constructing $d_1 \sharp d_2$ we could consider a $y_i := P_i \sqsubseteq Q_i$ from d_1 and a $y_i := Q_i \sqsubseteq R_i$ from d_2. Now this situation has the interpretation that R_i is a 'high-level' specification which is refined by the 'intermediate-level' specification Q_i, which at its turn is refined by the 'low-level' description P_i.

A typical development process could be to construct the R_i first, followed by the Q_i and after that the P_i. Another option is to work the other way around, i.e. first the P_i, then the Q_i and the R_i. Of course the length of these sequences can also be extended to more than three.

The analysis of $\sharp$ also suggests a somewhat different line of development where we do not generalise the notions of correctness, but rather generalise the notion of design. This suggestion arises naturally from the observation that when $d_1 \sharp d_2$ is defined, d_1 and d_2 must have already much in common. Therefore we could view them as parts of one generalised design. We do not give formal definitions here, but we rather give a sketch. We could adopt *generalised designs* which are of the following form:

$$\begin{array}{llllll}
x_1 & := & \text{prim} \sqsubseteq & M_1 & & \\
& \ldots & & & & \\
x_n & := & \text{prim} \sqsubseteq & M_n & & \\
y_1 & := & P_1 \quad \sqsubseteq & Q_1 & \sqsubseteq & R_1 \\
& \ldots & & & & \\
y_m & := & P_m \quad \sqsubseteq & Q_m & \sqsubseteq & R_m \\
\textbf{system} & L. & & & &
\end{array}$$

We conclude this section with mentioning the advantages and disadvantages of these generalised designs over our earlier designs.

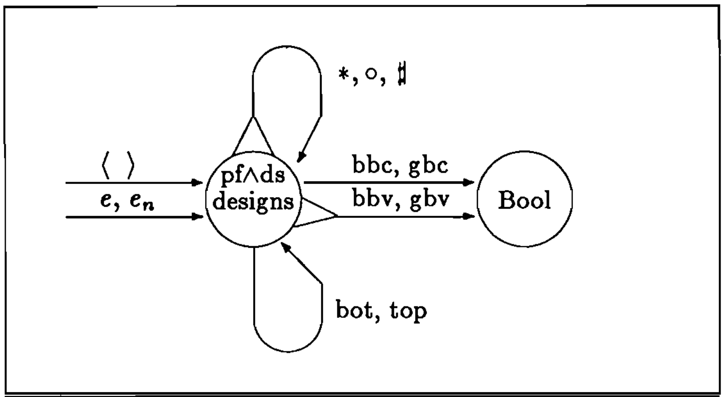
The main advantage is that generalised designs carry more information for each component. The intermediate-level specifications (the Q_i) can play the role of 'part of a correctness proof' for each component. The second advantage is that certain types of step-wise refinement – just as sketched above – become correctness-preserving transformations of designs in a natural way.

The main disadvantage is that many different generalisations are conceivable: what about n instead of three descriptions for each component (the P_i, Q_i, R_i)? what about a different n_i for each component? how should we generalise ds and pf? do we besides gbc and bbc also allow for something like 'intermediate-box' correctness?. We expect that these technicalities tend to complicate the theory significantly.

A second disadvantage is that it is not entirely clear how to adapt the mapping $[\![\ ]\!]$ from designs to lambda terms to make it work for generalised designs.

3.3.5 Summary

We end this section with a summary of the algebra of designs. The signature of the algebra of designs is summarised by the picture given below. We did not include all predicates. E.g. the predicate $=_{\text{pp}}$ is not shown. The collection of all pf $\wedge$ ds designs is shown as a circle. For the sake of the picture we view predicates as functions to Boolean values. The set of Boolean values is shown as a circle named Bool. The operations on designs are shown as arrows. The arrow which has been labeled $\langle\ \ \rangle$ corresponds with the possibility to construct a design directly, without using algebraic operations on designs. The constants e and e_n are given in lemma 3.3.2 and remark 3.3.8 (ii). We also give a number of equations and equivalences which correspond with some of our lemmas. We write $d_1 =_{[\![\,]\!]} d_2$ if $\vdash [\![d_1]\!] = [\![d_2]\!]$.



$$
\begin{aligned}
& d * e = e * d = d \\
& (d_1 * d_2) * d_3 = d_1 * (d_2 * d_3) \\
& e_n \circ d =_{[\![\,]\!]} d \\
& (d_1 \circ d_2) \circ d_3 = d_1 \circ (d_2 \circ d_3) \\
& (d_1 * d_2) \circ (d_3 * d_4) =_{[\![\,]\!]} (d_1 \circ d_3) * (d_2 \circ d_4) \\
& \mathrm{bbc}(d_1) \wedge \mathrm{bbc}(d_2) \;\Leftrightarrow\; \mathrm{bbc}(d_1 * d_2) \\
& \mathrm{bbc}(d_1) \wedge \mathrm{bbc}(d_2) \wedge \mathrm{bbv}(d_1, d_2) \;\Leftrightarrow\; \mathrm{bbc}(d_1 \circ d_2) \\
& \mathrm{bot}(\mathrm{bot}(d)) = \mathrm{bot}(d) \\
& \mathrm{top}(\mathrm{top}(d)) = \mathrm{top}(d) \\
& \mathrm{top}(\mathrm{bot}(d)) = e \\
& \mathrm{bot}(d_1 \circ d_2) = \mathrm{bot}(d_2) \\
& \mathrm{top}(d_1 \circ d_2) = \mathrm{top}(d_1) \\
& \mathrm{bot}(d_1 * d_2) = \mathrm{bot}(d_1) * \mathrm{bot}(d_2) \\
& \mathrm{top}(d_1 * d_2) = \mathrm{top}(d_1) * \mathrm{top}(d_2) \\
& \mathrm{bbv}(d_1, d_2) \;\Leftrightarrow\; \mathrm{bbv}(\mathrm{bot}(d_1), d_2) \\
& \mathrm{bbv}(d_1, d_2) \;\Leftrightarrow\; \mathrm{bbv}(d_1, \mathrm{top}(d_2)) \\
& \mathrm{bbv}(d_1 * d_2, d_3 * d_4) \;\Leftrightarrow\; (\mathrm{bbv}(d_1, d_3) \wedge \mathrm{bbv}(d_2, d_4))
\end{aligned}
$$

Fig 3.1. The algebra of designs.

3.4 Design Creation

3.4.1 General

In this section we want to derive several design-programs which describe the creation of a design. We shall use the design-development language of appendix B for expressing these design-programs. A design-program can be executed by a developer (or a team of developers). Since in general our design-programs will be highly non-deterministic, the developer(s) must make choices during the execution of a design-program. The fact that these choices exist, corresponds with the necessity of 'creative freedom' for the developer(s). Sometimes we shall use the term 'model of the development process' as a synonym for 'design-program'.

The derivation of such design-programs is quite non-trivial. The control structure of a design-program is not essentially complexer than that of a classical simple 'while program', but the point is that the design-programs describe the manipulation of complex data types: designs, components etc.

Therefore we shall spend quite some effort to the presentation of an elaborate example about a top-down development process first and only after that treat the formalisation of the top-down design-program in a more general setting. This has the advantage that during the formal treatment of the top-down design-program we can refer to the example. Once we have done the top-down design-program, the bottom-up design program is relatively easy.

The next section (Section 3.4.2) is devoted to the example and after that we proceed with the derivation of two design-programs, which we begin in Section 3.4.3. Section 3.4.3 presents the common setting of our design-programs which applies to the next two sections (3.4.4 and 3.4.5). Section 3.4.4 is about top-down development. Section 3.4.5 is about bottom-up development.

3.4.2 Top-down example

Before embarking on a formal treatment of top-down and bottom-up developments, we first give an elaborate example of a top-down development which is taken from the area of electronic digital hardware [3]. We have no claim whatsoever that the hardware circuit of the example is efficient or fast. In fact our main interest is not in hardware design at all, and the purpose of the example is to illustrate the notion of design and some of the transformation steps operating on designs. Of course we could take examples using the class-algebra CA of COLD-K, but for illustrating the design concept it is beneficial to have a kind of 'stand-alone' example as well.

The example is about so-called *logical circuits* and the composition mechanisms for these are interconnection *wirings*. We have two kinds of descriptions for logical circuits, viz. equations and interconnection diagrams. The equations are based on two-valued Boolean logic with constants $0, 1$, multiplication (= and) such that $0.0 = 0.1 = 1.0 = 0, 1.1 = 1$, addition (= or) such that $0+0 = 0, 1+0 = 0+1 = 1+1 = 1$ and inverting (= not) such that $\overline{0} = 1$ and $\overline{1} = 0$. We write $x \oplus y$ for $x.\overline{y} + \overline{x}.y$. Logical circuits have so-called ports, which act as a kind of variables (a, b, c, etc) and which are grouped into two categories, viz. input ports and output ports. An equation is always written with the output ports occurring at the left-hand side and the input ports at the right-hand side of the equation. E.g. $z = \overline{(a.b)}$ is an equation which specifies a 'nand' logical circuit with input ports a, b and output port z. The interconnection diagrams represent algebraic terms corresponding to some algebraic approach to logical-circuit composition. We do not provide a formalisation of these representation issues – although of course this could be done. The interconnection diagrams may contain component names and we assume that this is also the case for the terms represented by the interconnection diagrams. Intuitively, the interconnection diagrams will speak for themselves.

We shall use numbers such as 7400, 7404, ... to act as component names. The use of these names is consistent with the terminology of the well-known transistor transistor logic (TTL) family of integrated logical circuits [3].

Let us assume – for sake of the example – that it is our task to develop a so-called four-bit adder. We need some notation to specify the four-bit adder. For a bit-sequence $\vec{b}$ we write $\mathrm{int}(\vec{b})$ to denote the integer which is binary represented by $\vec{b}$. In particular, $\mathrm{int}(0,0,0,0) = 0$, $\mathrm{int}(0,0,0,1) = 1$, $\mathrm{int}(0,0,1,0) = 2$ and $\mathrm{int}(0,0,1,1) = 3$. A four-bit adder is a logical circuit with input ports $a_3, a_2, a_1, a_0, b_3, b_2, b_1, b_0$, output ports s_4, s_3, s_2, s_1, s_0 and which is specified by the equation $\mathrm{int}(s_4, s_3, s_2, s_1, s_0) = \mathrm{int}(a_3, a_2, a_1, a_0) + \mathrm{int}(b_3, b_2, b_1, b_0)$ which we abbreviate as $\mathrm{int}(\vec{s}) = \mathrm{int}(\vec{a}) + \mathrm{int}(\vec{b})$. The component name of the four-bit adder is 74283. Therefore the top of the design d to be developed is given by the following design.

$$\begin{array}{lll} 74283 := \text{prim} & \sqsubseteq & (\mathrm{int}(\vec{s}) = \mathrm{int}(\vec{a}) + \mathrm{int}(\vec{b})) \\ \textbf{system}\ [\ 74283\ ] & & \end{array}$$

This is the initial design of the top-down development. With respect to the available primitive building blocks, we adopt a minimalistic approach by restricting ourselves to two simple logical circuits called nand-gate and ground-connection. It is a well-known fact that these are sufficient to construct all other logical circuits. A nand-gate has two input ports a, b and one output port z. It is specified by $z = \overline{(a.b)}$. A ground-connection has no input ports and one output port g, specified by $g = 0$. Therefore the bottom of the design d to be developed is given by the following design.

$$\begin{array}{lll} 7400 := \text{prim} & \sqsubseteq & (z = \overline{(a.b)}) \\ \text{GND} := \text{prim} & \sqsubseteq & (g = 0) \\ \textbf{system } [\,] & & \end{array}$$

Now the top-down development can really begin. In order to decompose the 74283 four-bit adder we employ a component providing for a so-called single-bit full adder. A single-bit full adder is a logical circuit with three input ports a, b, c_i and two output ports s and c_o. The ports c_i and c_o are usually known as 'carry-in' and 'carry-out' respectively. It is specified by the equations $s = a \oplus b \oplus c_i$ and $c_o = a.b + c_i.(a \oplus b)$. So $s = 1$ iff the number of inputs that equal 1 is odd. Also $c_o = 1$ iff there are two or more inputs that equal 1. We shall introduce a new component named 74183 to provide the functionality of a single-bit full adder.

Using four instances of the 74183 and one ground-connection, the four-bit adder can be implemented. This is known as a ripple-carry configuration [3] (p. 87). We refer to the following interconnection diagram as 74283IMPL.

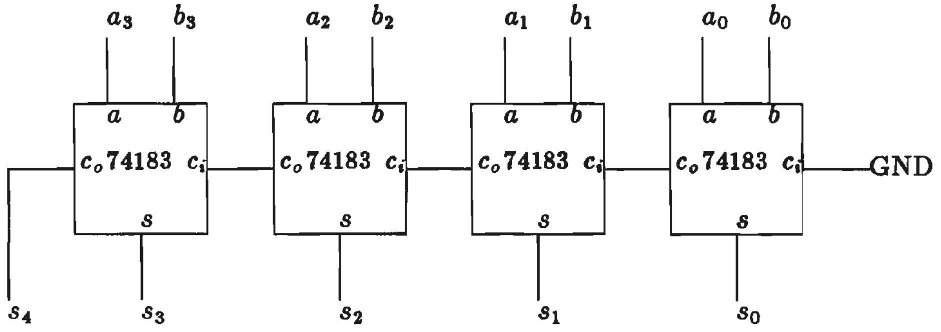


Fig 3.2. Interconnection diagram 74283IMPL.

The design-step to be taken involves two modifications. First of all, the new components 74183 and GND must be added to the design. These new components can be viewed as a two-components 'mini-design', d' say.

$$\begin{array}{lll} \text{GND} := \text{prim} & \sqsubseteq & (g = 0) \\ 74183 := \text{prim} & \sqsubseteq & (s = a \oplus b \oplus c_i,\ c_o = a.b + c_i.(a \oplus b)) \\ \textbf{system } [\,] & & \end{array}$$

We must concatenate d' with our initial design d, i.e. construct $d' * d$. Secondly, the old 74283 must be updated by means of a kind of overwriting insert operation, inserting 74283 := 74283IMPL $\sqsubseteq$ $(\mathrm{int}(\vec{s}) = \mathrm{int}(\vec{a}) + \mathrm{int}(\vec{b}))$. After this design step, our design d is as follows:

$$\begin{array}{llll}
\text{GND} & := & \textbf{prim} & \sqsubseteq \; (g = 0) \\
74183 & := & \textbf{prim} & \sqsubseteq \; (s = a \oplus b \oplus c_i,\ c_o = a.b + c_i.(a \oplus b)) \\
74283 & := & \text{74283IMPL} & \sqsubseteq \; (\mathrm{int}(\vec{s}) = \mathrm{int}(\vec{a}) + \mathrm{int}(\vec{b})) \\
\multicolumn{4}{l}{\textbf{system}\ [\ 74283\ ]}
\end{array}$$

In order to decompose the 74183 single-bit full adder, we employ and-gates, or-gates and xor-gates which are provided by new components 7408, 7432 and 7486 respectively. These are specified by $z = x.y$, $z = x + y$ and $z = x \oplus y$ respectively.

Using and-gates, or-gates and xor-gates the single-bit full adder can be implemented. We refer to the following interconnection diagram as 74183IMPL. We can easily verify that this diagram satisfies the equations of 74183. In particular, it is helpful to note that the output of the leftmost 7408 equals $a.b$ and that the output of the leftmost 7486 equals $a \oplus b$. The output of the rightmost 7408 equals $c_i.(a \oplus b)$. Having noticed this, it follows that for the output $s = a \oplus b \oplus c_i$ and $c_o = a.b + c_i.(a \oplus b)$.

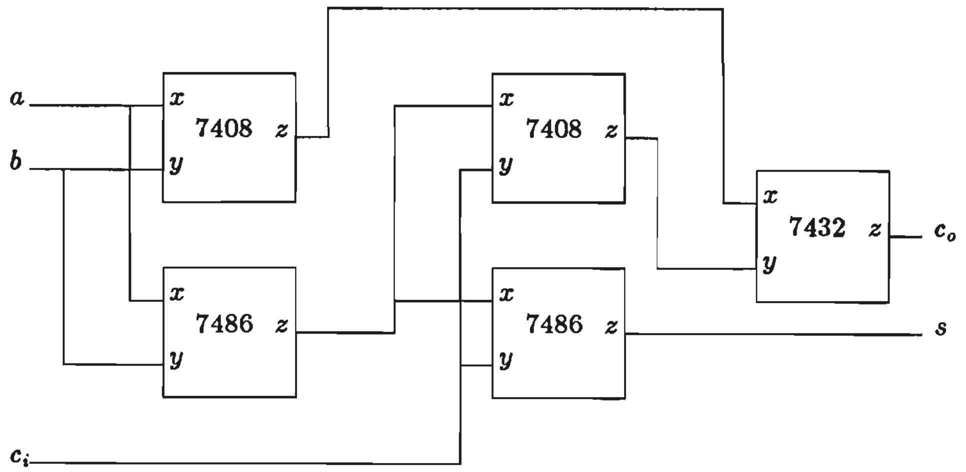


Fig 3.3. Interconnection diagram 74183IMPL.

As a matter of fact, verifying that this diagram satisfies the equations of its specification essentially means to verify a pair in the implementation relation $\sqsubseteq$. The black-box correctness of d requires that each pair (glass-box description, black-box description) is in this relation – for a suitable context. Here and in the general formulation of the top-down model in Section 3.4.4, black-box correctness of the current design serves as an invariant of the development process. The current section is one of the few places in this monograph where we give detailed proofs for the proof obligations arising in connection with $\sqsubseteq$. In Chapter 5, most proofs remain more or less implicit.

Again the design-step to be taken involves two modifications. First, the new components must be added to the design. These new components can be viewed as a three-components mini-design, d', say.

$$\begin{array}{llll}
7408 & := & \text{prim} & \sqsubseteq \ (z = x.y) \\
7432 & := & \text{prim} & \sqsubseteq \ (z = x + y) \\
7486 & := & \text{prim} & \sqsubseteq \ (z = x \oplus y) \\
\textbf{system} \ [\,] & & &
\end{array}$$

We must concatenate d' with our current design d. Secondly, we must insert a new version of 74183 which has 74183IMPL instead of **prim**. After this design step, our design d is as follows:

$$\begin{array}{llll}
7408 & := & \text{prim} & \sqsubseteq \ (z = x.y) \\
7432 & := & \text{prim} & \sqsubseteq \ (z = x + y) \\
7486 & := & \text{prim} & \sqsubseteq \ (z = x \oplus y) \\
\text{GND} & := & \text{prim} & \sqsubseteq \ (g = 0) \\
74183 & := & \text{74183IMPL} & \sqsubseteq \ (s = a \oplus b \oplus c_i,\ c_o = a.b + c_i.(a \oplus b)) \\
74283 & := & \text{74283IMPL} & \sqsubseteq \ (\text{int}(\vec{s}) = \text{int}(\vec{a}) + \text{int}(\vec{b})) \\
\textbf{system} \ [\ 74283\ ] & & &
\end{array}$$

We could select the last **prim**-component as the next candidate to be implemented – which typically is a kind of default in top-down development. However, we have no intention to decompose GND, so we permute some **prim** components, putting GND first (i.e. constructing a design that is pp-equivalent, cf. definition 3.3.25).

In order to decompose the 7486 xor-gate, we employ inverters, which are provided by a new component 7404. An inverter is a logical circuit with one input port p and one output port q. It is specified by $q = \overline{p}$. Using and-gates, or-gates and inverters it is easy to implement a xor-gate. We refer to the following interconnection diagram as 7486IMPL. For its verification it is useful to note that the output of the upper 7408 is $\overline{x}.y$ and that the output of the lower 7408 is $x.\overline{y}$. Therefore the final output z is $\overline{x}.y + x.\overline{y}$ which equals $x \oplus y$.

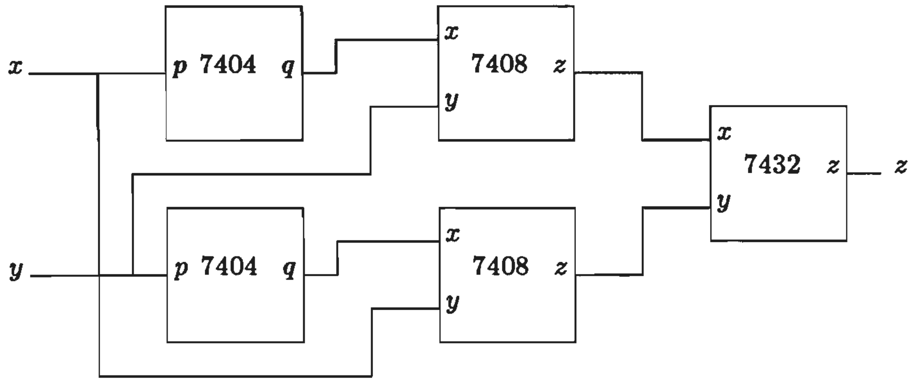


Fig 3.4. Interconnection diagram 7486IMPL.

Now the design-step is the addition of the 7404 and the insertion of the implemented 7486 with 7486IMPL instead of **prim**. Once more we put GND first. We do not show the resulting d and we proceed immediately with the next design-step.

In order to decompose the 7432 or-gate, we shall employ the 7400 nand-gate – which is an available primitive. Furthermore we employ two 7404 inverters. We refer to the following interconnection diagram as 7432IMPL. It is useful to note that the inputs of its 7400 equal $\overline{x}$ and $\overline{y}$. Therefore the final output z is $\overline{\overline{x}.\overline{y}}$ which equals $x + y$.

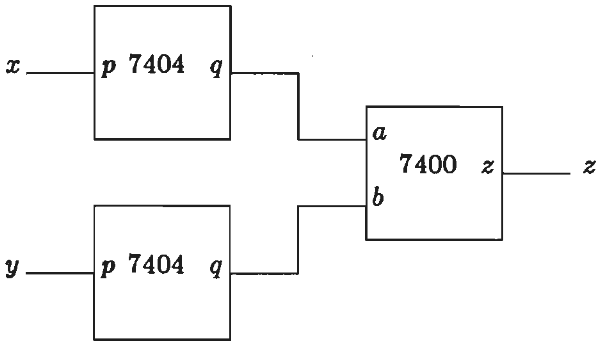


Fig 3.5. Interconnection diagram 7432IMPL.

The design-step is to add the 7400 component and to insert a modified 7432 component, with 7432IMPL instead of **prim**. To decompose the 7408 and-gate we need no new components. It can be done easily with one 7400 nand-gate and one 7404 inverter. We refer to the following interconnection diagram as 7408IMPL. It is useful to note that the output of its 7400 equals $\overline{x.y}$. Therefore the final output z is $x.y$.

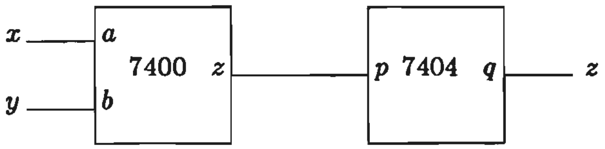


Fig 3.6. Interconnection diagram 7408IMPL.

Finally we implement the 7404 inverter. We employ one 7400 whose two inputs are connected. We refer to the following interconnection diagram as 7404IMPL.

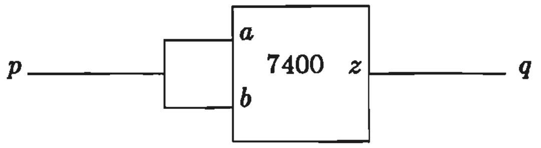


Fig 3.7. Interconnection diagram 7404IMPL.

The resulting design is finished because its bottom equals the agreed bottom design with 7400 and GND. We show the resulting design d below.

7400	:=	**prim**	$\sqsubseteq$	$(z = \overline{(a.b)})$
GND	:=	**prim**	$\sqsubseteq$	$(g = 0)$
7404	:=	7404IMPL	$\sqsubseteq$	$(q = \overline{p})$
7408	:=	7408IMPL	$\sqsubseteq$	$(z = x.y)$
7432	:=	7432IMPL	$\sqsubseteq$	$(z = x + y)$
7486	:=	7486IMPL	$\sqsubseteq$	$(z = x \oplus y)$
74183	:=	74183IMPL	$\sqsubseteq$	$(s = a \oplus b \oplus c_i,\ c_o = a.b + c_i.(a \oplus b))$
74283	:=	74283IMPL	$\sqsubseteq$	$(\text{int}(\vec{s}) = \text{int}(\vec{a}) + \text{int}(\vec{b}))$

system [74283]

This concludes the example. A systematic discussion of top-down development in general will be undertaken in Section 3.4.4.

3.4.3 Design-programs

In this section we turn our attention to design-programs for design creation in a very general setting. We shall use the design-development language of appendix B. The design-programs will be highly non-deterministic. The data that are manipulated by the developer(s) during the execution of a design-program may include designs, components, terms of $\lambda\pi$ and names. In this section we view these data as belonging to given data types which bring with them certain predicates and operations. In particular, we have predicates bbc, bbv, = etc. and operations $*$, $\circ$, bot, top etc.

We shall briefly sketch the main ingredients of the design-development language. For the details we refer to appendix B. In the design-development language we have assignment statements (an example of an assignment statement is $d', d'' :=$ bot(d), top(d);). Statements can be composed with sequential composition, a non-deterministic choice construct (symbol $[\,]$) and a repetition construct (keywords **while**, **do** and **od**). We have expressions and expression lists. Expressions may contain operation symbols and variables (an example of an expression is top(d)). Expression lists may contain operation symbols, variables and procedure calls. Assertions may contain expressions, predicate symbols, logical connectives and quantifiers (an example of an assertion is **forall** d (bbc(d))). Procedures have a list of input parameters and sometimes a list of result parameters. A procedure is either given axiomatically (keywords **pre** and **post**) or it is defined explicitly (keyword **def**). If a procedure is intended to be executed by a developer, it is called a *technique* (keyword **technique**). If a procedure is meant as a description of an event which is not performed by a developer it is called an *event* (keyword **event**).

We assume that in the design-development language we have variables of distinct sorts: $d, \ldots$ for pf $\wedge$ ds designs, $c, \ldots$ for components, $v, w \ldots$ for names and P, $Q, \ldots$ for terms.

In order to have a systematic approach for deriving design-programs we shall use methods which come from the field of classical sequential programming [4], [7]. In particular, if we want to derive a design-program with a repetition construct then we shall first look for a suitable invariant.

We are interested in design-programs which describe the creation of a design which is valid in a given context. Therefore we must start with a formalisation of what it means that a design is valid in a given context.

Definition 3.4.1 (machine&user-context, d valid in w.)

(i) A *machine&user-context* w is a pair (d_m, d_s) of pf $\wedge$ ds designs where

d_m is called the ***machine*** of w and d_s is called the ***system user*** of w.

(ii) Let $w = (d_m, d_s)$, then we say that d is ***valid*** in w if $\text{bbv}(d, d_m) \wedge \text{bbv}(d_s, d)$. □

In a certain way, a machine&user-context w constitutes a (simplified) view of the external world – at least from the viewpoint of a developer who has to create a design which is valid in w. The definition can be motivated as follows. The prim components of d can be viewed as a specification of all building blocks that are needed by d. When the product described by the design d becomes somehow operational, then the actual building blocks are provided. We view such a collection of actual building blocks as an underlying 'machine'. The design d itself can be viewed as a description of a product to be delivered to the user of d. In general this user has certain requirements to the product described by the design. In our view both providing a machine for a design and providing a product to its user can be described by the binary operation $\circ$. In particular, the components provided by the the machine can (via $\circ$) be plugged into the prims of the design. Similarly we can imagine that the system user is assuming a number of primitives (also prims) which become available by means of the system of the design.

Of course, in many practical situations it is not the case that the system user and the machine are formalised as designs. Often the system user is not formalised at all and validation becomes a matter of informal reasoning and negotiating. Nevertheless we believe that our abstraction might provide some insight for such situations also.

If we call the activity of showing that a design d is (black-box) correct *verification* and if we call the activity of showing that a design d is valid in a machine&user-context w *validation*, then our terminology is consistent with the usual terminology [1]: verification = 'are we building the product right?' and validation = 'are we building the right product?'.

Let us illustrate this with the example of the four-bit adder of Section 3.4.2. In this example the machine&user-context consists of two "designs". The first is the design of the 7400 and GND; typically this is a design where transistors and resistors occur as components. This 7400 + GND design is the underlying machine. The second is the design of a higher layer, where the 74283 four-bit adder is just a primitive building block; typically this could be the design of a digital computer. This digital-computer design is the system user.

We see that the validity of a design d in a given machine&user-context w depends only on bot(d), top(d) and on w. This observation can from an intuitive point of view be explained as follows. First of all bot(d) contains precisely all prim components of d. The prim components of d can be viewed as a summary of all building blocks that are needed by d. Secondly top(d) has the same system as d and top(d) contains a number of **prim** components which can be viewed as specifications of the names occurring in the system of d. The system of d (with these names eliminated) can be viewed as the top-level product to be delivered to the user of d.

In general, it is the task of the developer to (re-)establish an invariant, INV say, which depends both on the design d and on the machine&user-context (d_m, d_s). This invariant should consist of two parts where the first part deals with validation and the second part deals with verification. So INV must express that d is valid in its machine&user-context (d_m, d_s) and that d is black-box correct and therefore we define it as follows:

$$\text{INV} \;:\equiv\; \text{bbv}(d, d_m) \textbf{ and } \text{bbv}(d_s, d) \textbf{ and } \text{bbc}(d).$$

Now we turn our attention to the problem of design creation, which in its most general form is to create a design d such that INV is established. In order to keep things simple, we shall study the problem of design creation in a restricted setting where we focus on the verification aspect. Therefore we assume that before the actual execution of a design-program starts the boundaries of the design to be created are already fixed, by which we mean that somehow the bottom and the top of the design to be created are determined. Since the bottom and the top of a design are designs themselves we can model this situation by assuming that there are two given designs d_b and d_t. In view of lemma 3.3.27 we consider the top as given up to permutation of **prim** components. Therefore we have the following postcondition of the of design creation.

$$\text{POST} \;:\equiv\; \text{bot}(d) = d_b \textbf{ and } \text{top}(d) =_{\text{pp}} d_t \textbf{ and } \text{bbc}(d).$$

It is possible to give a criterion for the selection of d_b and d_t. We see that if the machine&user-context $w = (d_m, d_s)$ then d_b and d_t should be chosen such that bbv(d_b, d_m) and bbv(d_s, d_t) hold. Using lemma 3.3.23 and lemma 3.3.27 one can verify that bbv(d_b, d_m) and bbv(d_s, d_t) and POST implies INV. Of course it is always possible to derive a d_b and d_t from (d_m, d_s) mechanically, but our approach is somewhat more general.

Let us illustrate this with the example of the four-bit adder of Section 3.4.2. Indeed, in this example we have chosen a certain d_t, viz. the design with one **prim**-component 74283 and with **system** [74283]. Also we have chosen a certain d_b, viz. the design with two **prim**-components 7400 and GND and with **system** [ ].

It is not reasonable to assume that the developer can create a large design in one step; instead he adds components and modifies existing components, one at a time. Therefore we assume that there is one variable d which always contains the 'current design' and we shall focus on design-programs that modify this d in a repetitive manner.

There are many (loop-) invariants which could be derived from POST. We shall investigate two possibilities. The first possibility will be investigated in Section 3.4.4 and it leads to the derivation of a design-program which corresponds to the top-down approach. The second possibility will be investigated in Section 3.4.5 and it leads to the derivation of a design-program which corresponds to the bottom-up approach. These design-programs are related to the models of the development process given in [5].

We shall consider the partial correctness of these design-programs; it may very well be the case that a (partially) correct design-program cannot succesfully terminate for a given input. In this case we say that the execution of the design-program *fails*. From this it should be clear that a correctness formula $\{A_1\}s\{A_2\}$ does not say anything about failure or successful termination of s. We shall not use an axiomatisation in the style of Hoare's logic [6] for reasoning about these formulae $\{A_1\}s\{A_2\}$. Of course this could be done, but we think it would push the level of formalisation too far.

3.4.4 Top-down development

In order to obtain an invariant, we take the postcondition POST as a starting point. POST consists of three conjuncts and a candidate invariant is obtained by simply omitting the first conjunct (as suggested in [7] Section 16.2). This yields $\text{top}(d) =_{\text{pp}} d_t$ **and** $\text{bbc}(d)$, i.e. during the development process the top of the design remains constant (up to permutation of prim components) and furthermore black-box correctness is adopted as a methodological principle. We strengthen this assertion by requiring that all components (except possibly those in d_b) play a role in the system of the design d. In order to formulate this precisely we need an auxiliary definition:

Definition 3.4.2 Let d be a pf $\wedge$ ds design.

(i) The binary relation $<^d_1$ on the set of component names of d is defined by $x_1 <^d_1 x_2 \;:\Leftrightarrow\; x_1$ occurs freely in the glass-box description of the component named x_2.
(ii) The binary relation $<^d$ is defined as the transitive closure of $<^d_1$.
(iii) Let γ be some subset of cset(d) then we define
$x_1 \leq^d \gamma \;:\Leftrightarrow\; x_1 \in \gamma \vee \exists x_2 \in \gamma \cdot x_1 <^d x_2.$ □

Roughly speaking, we can view a design d as a set of components, identified by names, where some components are part of other components. Therefore we shall sometimes refer to $<^d$ as the 'part of' relation.

Now we can express the requirement that all components (except possibly those in d_b) play a role in the system of the design d as a condition **forall** v $(v \in \mathrm{cset}(d) \rightarrow (v \in \mathrm{cset}(d_b)$ **or** $v \leq^d \mathrm{sys}(d)))$. This condition guarantees that no implementation effort is spent on components which will not be used. This yields the following invariant (where we use the notation sys(d) from definition 3.2.4):

$$\begin{array}{lll} \mathrm{TD\_INV} & :\equiv & \mathrm{top}(d) =_{\mathrm{pp}} d_t \textbf{ and} \\ & & \mathrm{bbc}(d) \textbf{ and} \\ & & \textbf{forall } v\ (v \in \mathrm{cset}(d) \rightarrow (v \in \mathrm{cset}(d_b) \textbf{ or } v \leq^d \mathrm{sys}(d))). \end{array}$$

Now we can develop a design-program based on this invariant. The technique td given below has two input parameters (d_b and d_t). It is given by an explicit definition and it uses one variable (d). After execution of an initialisation statement and a repetition construct it yields the value of d as its result.

```
td := technique $d_b, d_t$
def  $d := d_t$;
     while not bot($d$) = $d_b$ do $d$ := td_step($d$); od;
     $d$
```

where td_step satisfies the partial-correctness assumption $\{\mathrm{TD\_INV} \wedge \mathrm{bot}(d) \neq d_b\}$ $d := \mathrm{td\_step}(d);$ $\{\mathrm{TD\_INV}\}$.

Let us illustrate this with the example of the four-bit adder of Section 3.4.2. Recall that in the example the assignment $d := d_t$ was executed indeed, viz. at the point where we said "This is the initial design of the top-down development." Recall also that in the example we had several points where we said "In order to decompose the ..., we shall employ the ... " etc.; each such point marks the beginning of the execution of $d := \mathrm{td\_step}(d)$; . Finally recall also that in the example we reached a point where we could say: "The resulting design is finished because its bottom equals the agreed bottom design with 7400 and GND." This corresponds with a positive outcome of the stop-criterion $\mathrm{bot}(d) = d_b$.

Remark 3.4.3 Under the given assumption for td_step we have

$$\{\mathrm{bot}(d_b) = d_b \textbf{ and } \mathrm{top}(d_t) = d_t\}\ \ d := \mathrm{td}(d_b, d_t);\ \ \{\mathrm{POST}\}.$$

(where the precondition simply expresses that d_b is a bottom design and that d_t is a top design). This can be proved as follows. First we verify that

after execution of $d := d_t$; the invariant TD_INV holds. We have $d = d_t$, so top(d) $=_{\text{pp}}$ d_t. Since d_t is a top design it contains only **prim** components and therefore it is trivially bbc. Take some $v \in$ cset(d), then $v \leq^d$ sys(d) because d is a top design. As a next step we must show that execution of d := td_step(d); does not violate TD_INV. This follows from the assumption for td_step. Finally we note that bot(d) $= d_b$ and TD_INV implies POST.
□

We now turn our attention to td_step. It will turn out that there exist several techniques which satisfy the assumption for td_step. Therefore we shall investigate several techniques which we shall call td_step$_0$, td_step$_1$ etc.

The techniques td_step$_0$ and td_step$_1$ describe the transformation of a **prim** component into a non-**prim** component. The latter component should be provably correct in a suitable context. Therefore we must describe how the developer should select a **prim** component and its context (represented as a design) from a design d. Actually the developer has no choice in this selection: he must select the last **prim** component, otherwise the result of re-inserting it (in its original relative position) as an implemented component could violate the condition pf.

Definition 3.4.4 Let the pf $\wedge$ ds design d be given as follows:

$$\begin{array}{lll} x_1 := & \textbf{prim} \sqsubseteq & M_1 \\ \quad \ldots & & \\ x_n := & \textbf{prim} \sqsubseteq & M_n \\ y_1 := & P_1 \qquad \sqsubseteq & Q_1 \\ \quad \ldots & & \\ y_m := & P_m \qquad \sqsubseteq & Q_m \\ \textbf{system} & [S_1, \ldots, S_l]. & \end{array}$$

where $P_1, \ldots, P_m$ are not equal to **prim**.

(i) last_prim(d) is defined as the component (x_n := **prim** $\sqsubseteq M_n$),

(ii) last_prim_context(d) is defined as the bottom design given by

$$\begin{array}{lll} x_1 := & \textbf{prim} \sqsubseteq & M_1 \\ \quad \ldots & & \\ x_{n-1} := & \textbf{prim} \sqsubseteq & M_{n-1} \\ \textbf{system} & []. & \end{array}$$

□

We assume that there is an operation 'insert' which serves for inserting a component into a design by overwriting an existing component (which has the same name as the component to be inserted). We do not give a formal definition of 'insert'. For a name v and a term P we write $v \in P$ if v occurs freely in P.

Of course the developer cannot select a **prim** component from d if d does not contain at least one **prim** component. The fact that d contains at least one **prim** component can be expressed as **not** bot(d) = **system** [].

The technique td_step$_0$ is a kind of naive approach. It uses an auxiliary technique called td_impl which takes a design d and which describes the selection of the last **prim** component and its transformation into a non-**prim** component (c'). We use the notation Γ_{bb} from definition 3.2.18.

```
td_impl := technique $d \to c'$
pre not bot($d$) = system []
post exists $v$ (exists $P$ (exists $Q$
        (last_prim($d$) = ($v$ := prim $\sqsubseteq Q$) and
        $c'$ = ($v := P \sqsubseteq Q$) and
        forall $w$ ($w \in P \to w \in$ cset(last_prim_context($d$))) and
        $\Gamma_{\mathrm{bb}}$(last_prim_context($d$)) $\vdash P \sqsubseteq Q$)))

td_step$_0$ := technique $d$
def insert($d$,td_impl($d$))
```

Let us illustrate this with the example of the four-bit adder of Section 3.4.2. Indeed, at a certain point we said: "To decompose the 7408 and-gate we need no new components. It can be done easily with one 7400 nand-gate and one 7404 inverter. We refer to the following interconnection diagram as 7408IMPL." etc. Formally this corresponds with a last_prim(d) which is (7408 := **prim** $\sqsubseteq$ ($z = x.y$)). The last_prim_context(d) here contains three **prim**-components, viz. GND, 7400 and 7404.

Lemma 3.4.5 (Partial correctness of td_step$_0$).

{TD_INV} d := td_step$_0$(d); {TD_INV}.

Proof. Assume that initially d is the design given in definition 3.4.4. We assume that this design satisfies TD_INV. If this design has no **prim** components then td_step$_0$ fails because the precondition of td_impl does not hold. Recall that we deal with *partial* correctness, so td_step$_0$ is allowed to fail. Therefore we proceed with the assumption that there is at least one **prim** component and hence last_prim(d) = (x_n := **prim** $\sqsubseteq M_n$). Then it follows that td_impl returns a component ($x_n := P \sqsubseteq M_n$) for some P. This P contains only names from $\{x_1, \ldots, x_{n-1}\}$ and it satisfies $[x_1 \sqsubseteq M_1], \ldots, [x_{n-1} \sqsubseteq M_{n-1}] \vdash P \sqsubseteq M_n$. Execution of 'insert' yields the design

$$\begin{array}{lll}
x_1 & := & \textbf{prim} \sqsubseteq M_1 \\
& \ldots & \\
x_{n-1} & := & \textbf{prim} \sqsubseteq M_{n-1} \\
x_n & := & P \sqsubseteq M_n \\
y_1 & := & P_1 \sqsubseteq Q_1 \\
& \ldots & \\
y_m & := & P_m \sqsubseteq Q_m \\
\textbf{system} & [S_1, \ldots, S_l] &
\end{array}$$

which is pf $\wedge$ ds and bbc. We note that the top of the latter design equals precisely the top of the initial design. Finally we show that all names from $\{x_1, \ldots, x_n, y_1, \ldots, y_m\}$ which are not in cset(d_b) are part of the system. This follows from the fact that TD_INV holds for the initial design. □

The definition of td_step$_0$ is a kind of naive approach because td_step$_0$ does not allow for the creation and insertion of new **prim** components. We shall now describe an improvement with respect to td_step$_0$. We could define a technique which describes the creation and insertion of *one* new **prim** component; however, we shall not do this because it is essential for the top-down approach as expressed by TD_INV that no new **prim** component is introduced unless it is used immediately. We prefer a technique which describes both the creation of new **prim** components and the transformation of an existing **prim** component into a non-**prim** component whose glass-box description uses all new **prim** components. This leads us to td_step$_1$ which is an improved version of td_step$_0$. It uses an auxiliary technique called td_spec_impl which describes the selection of the last **prim** component, the creation of a set of new **prim** components and the transformation of the selected **prim** component into a non-**prim** component. The set of new **prim** components is represented as a bottom design (d').

```
td_spec_impl := technique $d \to d', c'$
pre not bot$(d)$ = system []
post exists $v$ (exists $P$ (exists $Q$
        (last_prim$(d)$ = $(v :=$ prim $\sqsubseteq Q)$ and
        cset$(d') \cap$ cset$(d) = \emptyset$ and
        bot$(d') = d'$ and
        $c' = (v := P \sqsubseteq Q)$ and
        forall $w$ $(w \in$ cset$(d') \to w \in P)$ and
        forall $w$ $(w \in P \to w \in$ cset(last_prim_context$(d) \cup$ cset$(d')))$
        and $\Gamma_{\mathrm{bb}}(d') \cup \Gamma_{\mathrm{bb}}($last_prim_context$(d)) \vdash P \sqsubseteq Q)))$
```

We can use the binary operation $*$ of Section 3.3 for describing the addition of the new **prim** components to the current design.

```
td_step_1 := technique d
def d', c' := td_spec_impl(d);
    insert(d' * d, c')
```

Let us illustrate this with the example of the four-bit adder of Section 3.4.2. At some point of that development we said: "In order to decompose the 74183 single-bit full adder, we employ ... which are provided by new components 7408, 7432 and 7486 respectively." Then these new components were viewed as a mini-design d' and we concatenated d' with d and furthermore we said: "Secondly, we must insert a new version of 74183 which has 74183IMPL instead of **prim**." In fact this was an example of the execution of d', c' := td_spec_impl(d); insert($d' * d, c'$).

Lemma 3.4.6 (Partial correctness of td_step$_1$).

{TD_INV} d := td_step$_1$(d); {TD_INV}.

Proof. Assume that initially d is the design given in definition 3.4.4. We assume that this design satisfies TD_INV. If this design has no **prim** components then td_step$_1$ fails because the precondition of td_spec_impl does not hold. Therefore we can assume that there is at least one **prim** component and hence last_prim(d) = (x_n := **prim** $\sqsubseteq M_n$). Therefore td_spec_impl yields a bottom design (d') with new names, say

$$\begin{array}{llll} z_1 & := & \textbf{prim} \sqsubseteq & R_1 \\ & \ldots & & \\ z_k & := & \textbf{prim} \sqsubseteq & R_k \\ \textbf{system} & [] & & \end{array}$$

and it also yields a component ($x_n := P \sqsubseteq M_n$) for some P. This P contains only names from $\{x_1, \ldots, x_{n-1}, z_1, \ldots, z_k\}$ and it satisfies $[z_1 \sqsubseteq R_1], \ldots, [z_k \sqsubseteq R_k], [x_1 \sqsubseteq M_1], \ldots, [x_{n-1} \sqsubseteq M_{n-1}] \vdash P \sqsubseteq M_n$. Furthermore each z_i ($1 \leq i \leq k$) occurs in P. This bottom design is added to the current design, using $*$, and execution of 'insert' yields the design

$$\begin{array}{llll} z_1 & := & \textbf{prim} & \sqsubseteq\ R_1 \\ & \ldots & & \\ z_k & := & \textbf{prim} & \sqsubseteq\ R_k \\ x_1 & := & \textbf{prim} & \sqsubseteq\ M_1 \\ & \ldots & & \\ x_{n-1} & := & \textbf{prim} & \sqsubseteq\ M_{n-1} \\ x_n & := & P & \sqsubseteq\ M_n \\ y_1 & := & P_1 & \sqsubseteq\ Q_1 \end{array}$$

$$\begin{array}{llll} \dots & & & \\ y_m := & P_m & \sqsubseteq & Q_m \\ \textbf{system} & [S_1, \dots, S_l] & & \end{array}$$

which is pf $\wedge$ ds and bbc. We note that the top of the latter design equals precisely the top of the original design. Finally we must show that all names from $\{x_1, \dots, x_n, y_1, \dots, y_m, z_1, \dots, z_k\}$ which are not in cset(d_b) are 'part of' the system. For the names from $\{x_1, \dots, x_n, y_1, \dots, y_m\}$ this follows from the fact that TD_INV holds for the original design. For the names from $\{z_1, \dots, z_k\}$ this follows from the fact that x_n is 'part of' the system and the fact that $z_1, \dots, z_k$ occur in the new glass-box description of the component named x_n. □

td_step$_1$ raises two related problems. The first problem is that the developer has no choice in selecting the **prim** component to be implemented (although he can influence later choices by 'thinking in advance'). For the **prim** components which are in the top design d_t he has no influence at all on the order in which they are selected. The second problem is that it may be hard to make sure that the order of the **prim** components will match the order of the **prim** components in the bottom design d_b. We shall remedy these problems by describing the possibility that the developer modifies the order of the **prim** components. We formalise this by defining a technique td_step$_2$.

> td_step$_2$:= **technique** $d \rightarrow d'$
> **pre true**
> **post** $d' =_{\mathrm{pp}} d$

Let us illustrate this with the example of the four-bit adder of Section 3.4.2. Indeed, at a certain point we said: "We could select the last **prim**-component as the next candidate to be implemented – which typically is a kind of default in top-down development. However, we have no intention to decompose GND, so we permute some **prim** components, putting GND first." This was a typical execution of td_step$_2$.

Lemma 3.4.7 (Partial correctness of td_step$_2$).

$$\{\text{TD\_INV}\}\ d := \text{td\_step}_2(d);\ \{\text{TD\_INV}\}.$$

Proof. We assume that TD_INV holds for d and we must show that it holds also for d' where $d' =_{\mathrm{pp}} d$. Since we have top(d) $=_{\mathrm{pp}} d_t$ we also have top(d') $=_{\mathrm{pp}} d_t$. The fact that d' is bbc follows from the fact that d is bbc. Finally we note that the 'part of' relation does not depend on the order of the **prim** components. □

It is possible to execute td_step by choosing between td_step_1 and td_step_2. Using the non-deterministic choice construct $[]$ from the design-development language we define td_step as

```
td_step := technique $d$
def $d' := \text{td\_step}_1(d)$; $[]$ $d' := \text{td\_step}_2(d)$;
    $d'$
```

Now we can combine the results on td_step_1 and td_step_2.

Theorem 3.4.8 (Partial correctness of the top-down technique).

(i) $\{\text{TD\_INV}\}\ d := \text{td\_step}(d);\ \{\text{TD\_INV}\}$,

(ii) $\{\text{bot}(d_b) = d_b \textbf{ and } \text{top}(d_t) = d_t\}\ \ d := \text{td}(d_b, d_t);\ \ \{\text{POST}\}$.

Proof. (i) By the semantics of $[]$ and by lemmas 3.4.6 and 3.4.7. (ii) By (i) and remark 3.4.3, noting that $\text{TD\_INV} \wedge \text{bot}(d) \neq d_b$ implies TD_INV. □

Remark 3.4.9 (i) Let us consider the total-correctness question for td, which is as follows: can every execution sequence which is according to the top-down model but which is not ready yet, be completed to a full execution sequence? In other words: is it the case that a top-down development process essentially never can get stuck? The anser is positive: every execution sequence can be completed. But the main reason for this is the reflexivity of $\sqsubseteq$. From a practical point of view, this answer is of little interest. As soon as executability considerations or efficiency considerations (at the product level) play a role, either 'thinking in advance' or 'backtracking' are needed.

(ii) In the postconditions of the techniques td_step_0 and td_step_1 we have clauses
$\Gamma_{\text{bb}}(\text{last\_prim\_context}(d)) \vdash P \sqsubseteq Q$ and $\Gamma_{\text{bb}}(d') \cup \Gamma_{\text{bb}}(\text{last\_prim\_context}(d))$ $\vdash P \sqsubseteq Q$. It is interesting to note that if in these clauses we replace Γ_{bb} by Γ_{gb} the resulting top-down technique is still partially correct with respect to POST. We can interpret this fact as follows: as long as the developer works according to the top-down technique, it does not matter if he knows the difference between black-box correctness and glass-box correctness.

(iii) Our description of the top-down technique should be considered as open-ended. One can think of techniques td_step_3, td_step_4 etc. E.g. td_step_3 could describe the possibility of back-tracking where components are removed and where non-**prim** components are transformed into **prim** components. td_step_4 could describe the possibility of adding **prim** components which need not be 'part of' the system of the current design, but which happen to be present in d_b. A general form of the top-down technique could be based on a technique td_step given as follows.

```
td_step := technique $d$
def $[]_{i=1,2,\ldots,n}$ $d' :=$ td_step$_i(d)$;
    $d'$
```
□

Finally we consider a kind of completeness question: can every pf $\wedge$ ds $\wedge$ bbc design d where all components play a role in its system, be obtained by means of a top-down development? The answer is positive, for a given d is easily obtained from a $d_t = \text{top}(d)$ by a sequence of td_step$_1$ steps.

3.4.5 Bottom-up development

In order to obtain another invariant we take again the postcondition POST as a starting point. POST consists of three conjuncts and a candidate invariant is obtained by simply omitting the second conjunct. This yields bot(d) $= d_b$ and bbc(d), i.e. during the development process the bottom of the design remains constant and black-box correctness is adopted as a methodological principle. We need not strengthen this assertion by requiring that all components are built in terms of primitive components, since this is taken care of by the fact that we consider only designs which are pf $\wedge$ ds and hence wf. We adopt the following invariant:

$$\text{BU\_INV} \quad :\equiv \quad \text{bot}(d) = d_b \textbf{ and } \text{bbc}(d).$$

Now we can develop a design-program based on this invariant. The technique 'bu' given below has two input parameters (d_b and d_t). It is given by an explicit definition and it uses one variable (d). After execution of an initialisation statement and a repetition construct it yields the value of d as its result.

```
bu := technique $d_b, d_t$
def  $d := d_b$;
     while not top$(d) =_{\text{pp}} d_t$ do $d :=$ bu_step$(d)$; od;
     $d$
```

where bu_step satisfies $\{\text{BU\_INV}\wedge\text{top}(d) \neq_{\text{pp}} d_t\}$ $d :=$ bu_step(d); $\{\text{BU\_INV}\}$.

Remark 3.4.10 Under the given assumption for bu_step we have

$$\{\text{bot}(d_b) = d_b \textbf{ and } \text{top}(d_t) = d_t\} \;\; d := \text{bu}(d_b, d_t); \quad \{\text{POST}\}.$$

This can be proved as follows. First we verify that after execution of $d := d_b$;

the invariant BU_INV holds. We have $d = d_b$, so bot(d) $= d_b$. Since d_b is a bottom design it contains only **prim** components and therefore it is trivially bbc. As a next step we must show that execution of $d :=$ bu_step(d) does not violate BU_INV. This follows from the assumption for bu_step. Finally we note that top(d) $=_{\mathrm{pp}} d_t$ and BU_INV implies POST. □

We now turn our attention to bu_step. It will turn out that there exist several techniques which satisfy the assumption for bu_step. Therefore we shall investigate several techniques which we shall call bu_step$_0$, bu_step$_1$ etc.

Let us assume an operation 'add_system_element' which takes a design, d say, and a term P. If d has system $[S_1, \ldots, S_l]$, then add_system_element(d, P) yields the design which has the same components as d, but which has the sequence $[S_1, \ldots, S_l, P]$ as its system. We do not give a formal definition of 'add_system_element'.

The technique bu_step$_0$ describes the creation of a term and its addition to the system of the current design.

```
bu_step_0 := technique $d \to d'$
pre true
post exists $P$
        (forall $w$ ($w \in P \to w \in$ cset($d$)) and
        $d' =$ add_system_element($d, P$))
```

Lemma 3.4.11 (Partial correctness of bu_step$_0$).

{BU_INV} $d :=$ bu_step$_0$(d); {BU_INV}.

Proof. Assume that initially d is the design given in definition 3.4.4. We assume that this design satisfies BU_INV. Since P contains only names from $\{x_1, \ldots, x_n, y_1, \ldots, y_m\}$, evaluation of add_system_element(d, P)) yields the design

$$\begin{array}{llll}
x_1 & := & \mathbf{prim} \sqsubseteq & M_1 \\
 & \ldots & & \\
x_n & := & \mathbf{prim} \sqsubseteq & M_n \\
y_1 & := & P_1 \quad \sqsubseteq & Q_1 \\
 & \ldots & & \\
y_m & := & P_m \quad \sqsubseteq & Q_m \\
\multicolumn{4}{l}{\mathbf{system}\ [S_1, \ldots, S_l, P]}
\end{array}$$

which is pf ∧ ds and bbc. We note that the bottom of the latter design equals precisely the bottom of the initial design. □

Let us assume an operation 'add_component' which takes a design, d say, and a component c and which yields the design which has all components of d and which has furthermore c as its last component. We do not give a formal definition of 'add_component'. The technique bu_step$_1$ descibes the creation of a non-prim component and its addition to the current design.

```
bu_step$_1$ := **technique** $d \rightarrow d'$
**pre true**
**post exists** $v$ (**exists** $P$ (**exists** $Q$
        (**not** $v \in \text{cset}(d)$ **and**
        **forall** $w$ $(w \in P \rightarrow w \in \text{cset}(d))$ **and**
        $\Gamma_{\text{bb}}(d) \vdash P \sqsubseteq Q$ **and**
        $d' = \text{add\_component}(d, (v := P \sqsubseteq Q)))))$
```

Note that we did not give the relative order in which P and Q have to be constructed.

Lemma 3.4.12 (Partial correctness of bu_step$_1$).

{BU_INV} d := bu_step$_1(d)$; {BU_INV}.

Proof. Assume that initially d is the design given in definition 3.4.4. We assume that this design satisfies BU_INV. The result d' is the result of evaluating add_component with d as its first argument and with a component $(v := P \sqsubseteq Q)$ as its second argument. From the postcondition of bu_step$_1$ we have that v is a new name and that all names occurring in P are from $\{x_1, \ldots, x_n, y_1, \ldots, y_m\}$. Furthermore this component is correct in the black-box context of d. Evaluation of add_component$(d, (v := P \sqsubseteq Q))$ yields the design

$$\begin{array}{llll}
x_1 & := & \textbf{prim} \sqsubseteq & M_1 \\
 & \ldots & & \\
x_n & := & \textbf{prim} \sqsubseteq & M_n \\
y_1 & := & P_1 \quad \sqsubseteq & Q_1 \\
 & \ldots & & \\
y_m & := & P_m \quad \sqsubseteq & Q_m \\
v & := & P \quad \sqsubseteq & Q \\
\textbf{system} & [S_1, \ldots, S_l] & &
\end{array}$$

which is pf $\wedge$ ds and bbc. We note that the bottom of the latter design equals precisely the bottom of the initial design. □

It is possible to execute bu_step by choosing between bu_step$_0$ and bu_step$_1$. Using the non-deterministic choice construct [] from the design-development language we define bu_step as

```
    bu_step := **technique** $d$
    **def** $d' :=$ bu_step$_0(d)$; $[]$ $d' :=$ bu_step$_1(d)$;
        $d'$
```

Now we can combine the results on bu_step$_0$ and bu_step$_1$.

Theorem 3.4.13 (Partial correctness of the bottom-up technique).

(i) {BU_INV} $d :=$ bu_step(d); {BU_INV},

(ii) {bot(d_b) $= d_b$ **and** top(d_t) $= d_t$} $d :=$ bu(d_b, d_t); {POST}.

Proof. (i) By the semantics of $[]$ and by 3.4.11 and 3.4.12. (ii) By (i) and remark 3.4.10, noting that BU_INV $\wedge$ top(d) $\neq_{\mathrm{pp}} d_t$ implies BU_INV. □

Remark 3.4.14 (i) Let us consider the total-correctness question for bu, which is as follows: assume a given d_b and d_t; can every execution sequence which is according to bu(d_b, d_t) but which is not ready yet, be completed to a full execution sequence? The anser is negative because we can only *add* system elements and not remove them. Once we have too many system elements, we have no way left to arrive at the same system as d_t.

(ii) In the postcondition of the technique bu_step$_1$ we have a clause $\Gamma_{\mathrm{bb}}(d) \vdash P \sqsubseteq Q$. It is interesting to note that if in this clause we replace Γ_{bb} by Γ_{gb}, the resulting bottom-up technique is *not* partially correct with respect to POST. Instead its execution will yield a glass-box correct design.

(iii) Our description of the bottom-up technique should be considered as open-ended. One can think of techniques bu_step$_3$, bu_step$_4$ etc. E.g. bu_step$_3$ could describe the possibility of back-tracking, where components or system elements are removed. □

Finally we consider a kind of completeness question: can every pf $\wedge$ ds $\wedge$ bbc design d be obtained by means of a bottom-up development? The answer is positive, for a given d is easily built-up from a d_b = bot(d) by means of a number of bu_step$_1$ steps followed by a number of bu_step$_0$ steps.

As an illustration of the design program bu, we show a sequence of designs corresponding with a possible bottom-up development of the four-bit adder. The first design d_1 is d_b which is

```
    7400  :=  prim    ⊑  $(z = \overline{(a.b)})$
    GND   :=  prim    ⊑  $(g = 0)$
    system [ ]
```

d_2 is obtained by a bu_step$_1$:

$$\begin{array}{llll}
7400 & := & \mathbf{prim} & \sqsubseteq \; (z = \overline{(a.b)}) \\
\mathrm{GND} & := & \mathbf{prim} & \sqsubseteq \; (g = 0) \\
7404 & := & 7404\mathrm{IMPL} & \sqsubseteq \; (q = \overline{p}) \\
\multicolumn{4}{l}{\mathbf{system}\ [\,]}
\end{array}$$

d_3 is obtained by a bu_step$_1$:

$$\begin{array}{llll}
7400 & := & \mathbf{prim} & \sqsubseteq \; (z = \overline{(a.b)}) \\
\mathrm{GND} & := & \mathbf{prim} & \sqsubseteq \; (g = 0) \\
7404 & := & 7404\mathrm{IMPL} & \sqsubseteq \; (q = \overline{p}) \\
7408 & := & 7408\mathrm{IMPL} & \sqsubseteq \; (z = x.y) \\
\multicolumn{4}{l}{\mathbf{system}\ [\,]}
\end{array}$$

d_4 is obtained by a bu_step$_1$:

$$\begin{array}{llll}
7400 & := & \mathbf{prim} & \sqsubseteq \; (z = \overline{(a.b)}) \\
\mathrm{GND} & := & \mathbf{prim} & \sqsubseteq \; (g = 0) \\
7404 & := & 7404\mathrm{IMPL} & \sqsubseteq \; (q = \overline{p}) \\
7408 & := & 7408\mathrm{IMPL} & \sqsubseteq \; (z = x.y) \\
7432 & := & 7432\mathrm{IMPL} & \sqsubseteq \; (z = x + y) \\
\multicolumn{4}{l}{\mathbf{system}\ [\,]}
\end{array}$$

d_5 is obtained by a bu_step$_1$:

$$\begin{array}{llll}
7400 & := & \mathbf{prim} & \sqsubseteq \; (z = \overline{(a.b)}) \\
\mathrm{GND} & := & \mathbf{prim} & \sqsubseteq \; (g = 0) \\
7404 & := & 7404\mathrm{IMPL} & \sqsubseteq \; (q = \overline{p}) \\
7408 & := & 7408\mathrm{IMPL} & \sqsubseteq \; (z = x.y) \\
7432 & := & 7432\mathrm{IMPL} & \sqsubseteq \; (z = x + y) \\
7486 & := & 7486\mathrm{IMPL} & \sqsubseteq \; (z = x \oplus y) \\
\multicolumn{4}{l}{\mathbf{system}\ [\,]}
\end{array}$$

d_6 is obtained by a bu_step$_1$:

$$\begin{array}{llll}
7400 & := & \mathbf{prim} & \sqsubseteq \; (z = \overline{(a.b)}) \\
\mathrm{GND} & := & \mathbf{prim} & \sqsubseteq \; (g = 0) \\
7404 & := & 7404\mathrm{IMPL} & \sqsubseteq \; (q = \overline{p}) \\
7408 & := & 7408\mathrm{IMPL} & \sqsubseteq \; (z = x.y) \\
7432 & := & 7432\mathrm{IMPL} & \sqsubseteq \; (z = x + y) \\
7486 & := & 7486\mathrm{IMPL} & \sqsubseteq \; (z = x \oplus y) \\
74183 & := & 74183\mathrm{IMPL} & \sqsubseteq \; (s = a \oplus b \oplus c_i,\ c_o = a.b + c_i.(a \oplus b)) \\
\multicolumn{4}{l}{\mathbf{system}\ [\,]}
\end{array}$$

d_7 is obtained by a bu_step$_1$:

$$\begin{array}{llll}
7400 & := & \mathrm{prim} & \sqsubseteq \ (z = \overline{(a.b)}) \\
\mathrm{GND} & := & \mathrm{prim} & \sqsubseteq \ (g = 0) \\
7404 & := & \mathrm{7404IMPL} & \sqsubseteq \ (q = \overline{p}) \\
7408 & := & \mathrm{7408IMPL} & \sqsubseteq \ (z = x.y) \\
7432 & := & \mathrm{7432IMPL} & \sqsubseteq \ (z = x + y) \\
7486 & := & \mathrm{7486IMPL} & \sqsubseteq \ (z = x \oplus y) \\
74183 & := & \mathrm{74183IMPL} & \sqsubseteq \ (s = a \oplus b \oplus c_i,\ c_o = a.b + c_i.(a \oplus b)) \\
74283 & := & \mathrm{74283IMPL} & \sqsubseteq \ (\mathrm{int}(\vec{s}) = \mathrm{int}(\vec{a}) + \mathrm{int}(\vec{b})) \\
\mathbf{system} & [\ ] & &
\end{array}$$

and after a bu_step$_0$ we get d_8 which is

$$\begin{array}{llll}
7400 & := & \mathrm{prim} & \sqsubseteq \ (z = \overline{(a.b)}) \\
\mathrm{GND} & := & \mathrm{prim} & \sqsubseteq \ (g = 0) \\
7404 & := & \mathrm{7404IMPL} & \sqsubseteq \ (q = \overline{p}) \\
7408 & := & \mathrm{7408IMPL} & \sqsubseteq \ (z = x.y) \\
7432 & := & \mathrm{7432IMPL} & \sqsubseteq \ (z = x + y) \\
7486 & := & \mathrm{7486IMPL} & \sqsubseteq \ (z = x \oplus y) \\
74183 & := & \mathrm{74183IMPL} & \sqsubseteq \ (s = a \oplus b \oplus c_i,\ c_o = a.b + c_i.(a \oplus b)) \\
74283 & := & \mathrm{74283IMPL} & \sqsubseteq \ (\mathrm{int}(\vec{s}) = \mathrm{int}(\vec{a}) + \mathrm{int}(\vec{b})) \\
\mathbf{system} & [\ 74283\ ] & &
\end{array}$$

This concludes the bottom-up example.

3.5 Design Evolution

3.5.1 General

In Section 3.4 we investigated design creation, but it would be wrong to assume that in realistic software development the 'machine&user-context' in which the design is supposed to be valid is always a constant machine&user-context. In current software development practice it may very well be the case that about 50% of the development costs of a product are spent on so-called 'maintenance' (see e.g. [1] pages 540, 541). Traditionally, maintenance was classified into software *update* and software *repair*, where software repair includes a *corrective* aspect (see e.g. [1] page 536). In this chapter we shall not investigate this corrective aspect. From now on we shall use the term 'design evolution' (because 'maintenance' suggests that there might be something like 'wear', which of course is not the case for software products).

If we want to discuss design evolution then we must have an abstraction of the (variable) machine&user-context in which a (variable) design 'evolves'. We adopt the definitions of 3.4.1 where a machine&user-context w is a pair (d_m, d_s) of pf $\wedge$ ds designs where d_m is called the machine of w and d_s is called the system user of w. The design d is said to be valid in w if $\text{bbv}(d, d_m) \wedge \text{bbv}(d_s, d)$. This definition was motivated by viewing the **prim** components of d as a specification of all building blocks that are needed by d and by viewing the design d as a description of a product to be delivered to the user of d.

We consider design evolution to be the evolution of a design in a changing machine&user-context. The developer operates on a variable design which is part of a global state. Also part of this global state is a variable machine&user-context $w = (d_m, d_s)$. The machine&user-context is modified, let us say at certain points in time. We formalise this view by assuming that there are three variables d_m, d_s and d. We have the following intuitions for these variables: d_m = 'current machine,' d_s = 'current system user', d = 'current design'.

It is the task of the developer to (re-)establish the invariant INV, which depends both on the design d and on the machine&user-context (d_m, d_s) and which expresses that d is valid in the machine&user-context (d_m, d_s) and that d is black-box correct.

$$\text{INV} \equiv \text{bbv}(d, d_m) \text{ and } \text{bbv}(d_s, d) \text{ and } \text{bbc}(d).$$

The following scenario is adopted. We assume a state in which d_m, d_s and d are such that INV holds. We now assume that the machine&user-context of the next state has been modified and we say that an *external event* has happened. The developer must find a design d' such that after establishing the state modification $d := d'$; the invariant INV holds again. The developer may do this by acting according to some technique. Let us assume that the change of the machine&user-context is such that either the machine is modified or the system user is modified, but not both. We shall discuss both kinds of machine&user-context change separately. One might devise many techniques addressing the problem of design evolution; we only show some of the simplest techniques. In Section 3.5.2 we shall give a technique which deals with a changing machine. In Section 3.5.3 we shall give a technique which deals with a changing system user.

3.5.2 Changing machine

The following procedure can be used for modeling an external event:

change := **event** $d \rightarrow d'$
pre true
post true

After the external event d_m := change(d_m); has happened there is a new machine but there is still the old system user. The condition bbv(d, d_m) may be false but bbv(d_s, d) and bbc(d) hold. The developer must restore the invariant and therefore we look for a suitable statement s_m such that

{INV} d_m := change(d_m); s_m {INV}.

One possible technique is based on the idea of an *emulator*, which is nothing but a 'layer' interpolating between the new machine and the old design. Finding an emulator is described by a technique which takes designs d_m (the new machine) and d (the old design) and yields an emulator d_{em}.

emulator := **technique** $d_m, d \rightarrow d_{em}$
pre true
post bbc(d_{em}) **and** bbv(d_{em}, d_m) **and** bbv(d, d_{em})

Now if emulator(d_m, d) yields d_{em}, then we might say that $d_{em} \circ d_m$ is 'd-equivalent' with the old machine, by which we mean that bbv($d, d_{em} \circ d_m$) holds. So it is possible to make the old design 'run' upon $d_{em} \circ d_m$ by constructing the composition $d \circ (d_{em} \circ d_m)$. By lemma 3.3.7 (associativity of $\circ$) this is the same as $(d \circ d_{em}) \circ d_m$. If we assume that the developer has modification rights with respect to d but not with respect to the machine, he should replace the old design d by $(d \circ d_{em})$. This indicates that we can take the following statement for s_m:

d := $d \circ$ emulator(d_m, d);

Let us illustrate this with the example of the four-bit adder of Section 3.4.2. Suppose that the supply of 7400 nand-gates gets exhausted whereas there is a rich supply of 7402 nor-gates, say. Then we could have an emulator design as follows:

7402	:=	**prim**	$\sqsubseteq$	$(z = \overline{(a+b)})$
GND	:=	**prim**	$\sqsubseteq$	$(g = 0)$
7400	:=	7400IMPL	$\sqsubseteq$	$(z = \overline{(a.b)})$

system [7400,GND]

where 7400IMPL implements the functionality of the nand-gate using nor-gates and ground-connections only.

Theorem 3.5.1 {INV} d_m :=change(d_m); d := $d \circ$ emulator(d_m, d); {INV}

Proof. Assume that initially d satisfies INV. After d_m := change(d_m); has happened we have bbv(d_s, d) and bbc(d). Execution of the emulator technique yields some design d_{em} such that bbc(d_{em}) and bbv(d_{em}, d_m) and bbv(d, d_{em}) hold. We must investigate each conjunct of INV for the new design $d \circ d_{em}$.

- bbv($d \circ d_{em}, d_m$) holds because bbv(d_{em}, d_m) holds.
- bbv($d_s, d \circ d_{em}$) holds because bbv(d_s, d) holds.
- bbc($d \circ d_{em}$) follows by lemma 3.3.14 (ii) from bbc(d), bbc(d_{em}) and bbv(d, d_{em}). □

Remark 3.5.2 We shall briefly sketch an alternative solution for the statement s_m. It is based on the idea of (re-)starting a top-down development process. Let us assume that we have an operation 'remove_unused' which takes a design and which yields the design which is obtained by removing all components which are not 'part of' the system (in the sense of $\leq^d$ sys(d)). Actually, for top-down made designs this means that no implementations are thrown away. Let us also assume a technique 'determine_bottom' which for given machine d_m yields a bottom design d_b such that bbv(d_b, d_m). After the external event d_m := change(d_m); has happened we are in a state in which bbv(d_s, d) and bbc(d) holds. If in this state the statement d := remove_unused(d); d_b := determine_bottom(d_m); d_t := top(d); is executed then the invariant TD_INV as given in Section 3.4.4 holds (where d_t := top(d) means to fix the existing top, rather than throwing something away). When TD_INV holds, the developer can start executing the repetition statement which we know already from the top-down approach, viz. **while not** bot(d) = d_b **do** d := td_step(d); **od**;. If the latter statement terminates then INV holds again.

Under certain conditions it may be possible to derive an emulator from the result of this top-down development process. □

3.5.3 Changing system user

After the external event d_s := change(d_s); has happened there is a new system user but there is still the old machine. The conditions bbv(d, d_m) and bbc(d) hold but bbv(d_s, d) may be false. The developer must restore the invariant and therefore we look for a suitable statement s_s such that

{INV} d_s := change(d_s); s_s {INV}.

One possible technique is based on the idea of a ***simulator***. Finding a 'simulator' is described by the following technique:

```
simulator := technique $d_s, d \rightarrow d_{si}$
pre true
post bbc($d_{si}$) and bbv($d_{si}, d$) and bbv($d_s, d_{si}$)
```

Now we can take the following statement for s_s:

d := simulator(d_s, d) $\circ$ d;

Theorem 3.5.3 {INV} d_s :=change(d_s); d := simulator(d_s, d) $\circ$ d; {INV}.

Proof. Assume that initially d satisfies INV. After d_s := change(d_s); has happened we have bbv(d, d_m) and bbc(d). Execution of the simulator technique yields some design d_{si} such that bbc(d_{si}) and bbv(d_{si}, d) and bbv(d_s, d_{si}) hold. We must investigate each conjunct of INV for the new design $d_{si} \circ d$.

- bbv($d_{si} \circ d, d_m$) holds because bbv(d, d_m) holds.
- bbv($d_s, d_{si} \circ d$) holds because bbv(d_s, d_{si}) holds.
- bbc($d_{si} \circ d$) follows from lemma 3.3.14 (ii) from bbc(d_{si}), bbc(d) and bbv(d_{si}, d). □

Remark 3.5.4 We shall briefly sketch an alternative solution for the statement s_s. It is based on the idea of (re-)starting a bottom-up development process. Let us assume that we have an operation 'empty_system' which takes a design and yields the design which is obtained by making the system equal to []. Let us also assume a technique 'determine_top' which for given system user d_s yields a top design d_t such that bbv(d_s, d_t). After the external event d_s := change(d_s); has happened we are in a state in which bbv(d, d_m) and bbc(d) holds. If in this state the statement d := empty_system(d); d_t := determine_top(d_s); d_b := bot(d); is executed then the invariant BU_INV as given in Section 3.4.5 holds. Therefore the developer can start executing the repetition statement which we know already from the bottom-up approach, viz. **while not** top(d) $=_{\mathrm{pp}}$ d_t **do** d := bu_step(d); **od**;. If the latter statement terminates then INV holds again.

Under certain conditions it may be possible to derive a simulator from the result of this bottom-up development process. □

3.6 Design Partition

3.6.1 General

In this section we want to investigate parallel development. Often it is desirable to have several developers doing their work simultaneously such that their interaction is limited. The result of their joint effort should be a valid and correct design. The condition that a design d is valid and correct can be expressed by INV as given in Sections 3.4.3 and 3.5. The motivation behind this approach is that one wants to achieve a reduction in development time compared with the approach in which there is only one active developer at a time.

The design-development language of appendix B does not have a parallel composition construct for statements but some design-programs offer a possibility of parallelism at the 'implementation level'. Because the procedures can not have side-effects and because we have expression lists, there exists a kind of parallelism in the following sense: let p_1 and p_2 be procedures (techniques) and consider the statement

$$d_1, d_2 := p_1(d_1), p_2(d_2);$$

Now the execution of $p_1(d_1)$ can be done simultaneously with (and independently of) the execution of $p_2(d_2)$.

We could try to devise design-programs dealing with design creation and design evolution, making special 'parallel' versions of the techniques given in Sections 3.4 and 3.5. However, in order to keep matters simple, we shall in this chapter focus only on design-programs s which satisfy {INV} s {INV}. Such an s may range over a very large class of implementation techniques. E.g. s might correspond with optimisations of the current design e.g with respect to performance requirements which are not expressed as black-box descriptions; however, s can also correspond with an entire re-design.

One of the ideas behind the notion of a design is the locality principle that it should be possible to implement each component in a design without worrying about the implementation of the other components in the design. As one of the results of the formalisation of this idea (Section 2.5.4) we have the so-called bbc-preserving glass-box modifications, abbreviated as bbc-gb-mod. We write this as a binary predicate on pf $\wedge$ ds designs: bbc-gb-mod(d, d') means that d' is a bbc-preserving glass-box modification of d. Because for d and d' satisfying bbc-gb-mod(d, d') we have bbc$(d) \Rightarrow$ bbc(d') and since one design can contain many components, these modifications offer an opportunity for parallel development. This will be formalised in Section 3.6.2.

Modifications of black-box descriptions can not always be dealt with in a similar way: if the black-box description of one component is modified, this modification may destroy the correctness of several other components. Therefore we must somehow limit the scope of such modifications. For example, if the design d consists in fact of two unrelated parts, then a black-box modification in one part can not endanger the correctness of components which are in the other part. It follows that there is a possibility for parallel development if the current design d can be partitioned into d_1 and d_2 such that $d = d_1 * d_2$. This approach will be investigated in Section 3.6.3.

Similarly there is a possibility for parallel development if the current design d can be partitioned into d_1 and d_2 such that $d = d_1 \circ d_2$. This approach will be investigated in Section 3.6.4. We briefly mention several approaches based on $\sharp$ in Section 3.6.5.

The application of the techniques which we shall discuss in Sections 3.6.3 and 3.6.4 sometimes needs preparations in the sense that the developer must restructure (transform) the current design. (e.g. using lemmas 3.2.9 and 3.2.16 and remark 3.2.17). In many situations the application of the techniques from Sections 3.6.3 and 3.6.4 is doomed to fail unless suitable preparatory transformations are performed first. The theory of these preparatory transformations probably must be based on the use of algebraic operations on designs such as $*$ and $\circ$, but may require additional operations. We have not investigated this yet.

3.6.2 Splitting into components

In this section we want to formalise the idea that the locality principle of components can be exploited for parallel development. First we need some notation.

Definition 3.6.1 Let v be a name and d a pf $\wedge$ ds design such that $v \in$ cset(d). Then we define $d[v]$:= 'the unique component in d with name v'. $\square$

The technique called 'impl$_c$' given below describes the activity of one developer who gets a design d and the name of a component in this design (v). If the execution of this technique succeeds, then the developer has performed a bbc-gb-mod and the result is one component. Formally the technique does not exclude the possibility that the developer does nothing but taking an existing old component, but for the partial correctness of the parallel development technique this does not matter. As before, we use the design-development language of appendix B and we have variables of distinct sorts:

$d, \ldots$ for pf $\wedge$ ds designs, $c, \ldots$ for components and $v, \ldots$ for names.

```
impl$_c$ := **technique** $d, v \rightarrow c$
**pre**  $v \in \text{cset}(d)$
**post exists** $d'$ ($d'[v] = c$ **and** bbc-gb-mod$(d, d')$ **and**
                                **forall** $u$ (($u \in \text{cset}(d)$ **and** $u \neq v$) $\rightarrow$ $d'[u] = d[u]$) )
```

There is a very simple splitting to be done before the actual parallelism can start. Two distinct component names must be selected such that the corresponding components are non-**prim**. This is described by the technique 'split$_c$' given below.

```
split$_c$ := **technique** $d \rightarrow v_1, v_2$
**pre true**
**post** $v_1 \neq v_2$ **and**
      $v_1 \in \text{cset}(d)$ **and** $v_2 \in \text{cset}(d)$ **and**
      $v_1 \notin \text{cset}(\text{bot}(d))$ **and** $v_2 \notin \text{cset}(\text{bot}(d))$
```

As in Section 3.4.4 we assume that there is an operation 'insert' which serves for inserting a component by overwriting an existing component (which has the same name as the component to be inserted). The following technique describes the splitting, the parallel development and the insertion of the results.

```
pardev$_c$ := **technique** $d$
**def** $v_1, v_2$ := split$_c(d)$;
     $c_1, c_2$ := impl$_c(d, v_1)$, impl$_c(d, v_2)$;
     insert(insert($d, c_1$),$c_2$)
```

The partial correctness of this technique with respect to INV is stated in the following theorem.

Theorem 3.6.2 {INV} $d :=$ pardev$_c(d)$; {INV}.

Proof. Let initially d be a pf $\wedge$ ds design given as

```
$x_1$  :=   **prim** $\sqsubseteq$  $M_1$
       ...
$x_n$  :=   **prim** $\sqsubseteq$  $M_n$
$y_1$  :=   $P_1$     $\sqsubseteq$  $Q_1$
       ...
$y_m$  :=   $P_m$     $\sqsubseteq$  $Q_m$
**system** $[S_1, \ldots, S_l]$
```

The following technique describes the splitting, the parallel development and the recombination of the results.

```
pardev$_*$ := **technique** $d_m, d_s, d$
**def** $d_1, d_2$ := split$_*(d)$;
    $d'_m, d''_m$ := msplit(top$(d_m), d_1, d_2)$;
    $d'_s, d''_s$ := ssplit(bot$(d_s), d_1, d_2)$;
    $d_1, d_2$ := impl$(d'_m, d'_s, d_1)$, impl$(d''_m, d''_s, d_2)$;
    $d_1 * d_2$
```

The partial correctness of this technique with respect to INV is stated in the following theorem.

Theorem 3.6.3 {INV} $d :=$ pardev$_*(d_m, d_s, d)$; {INV}.

Proof. Use lemma 3.3.3 (ii), lemma 3.3.23 and lemma 3.3.16. □

3.6.4 Splitting according to $\circ$

Let again impl be a technique as given in Section 3.6.3 with the property that it does not violate the invariant INV and let us assume that this technique 'impl' describes the activity of one single developer.

There is a splitting to be done before the actual parallelism can start. This is described by the technique 'split$_\circ$' given below.

```
split$_\circ$ := **technique** $d \rightarrow d_1, d_2$
**pre true**
**post** $d_1 \circ d_2 = d$
```

As before it is possible to indicate which parts of the machine&user-context are relevant for d_1 and which parts of the machine&user-context are relevant for d_2. This is relatively simple, due to the fact that we can view $d_1 \circ d_2$ as a layered design with layers d_1 and d_2. We see directly that d_1 must be validated with respect to top(d_2) and the bottom of the system user, i.e. bot(d_s). Similarly we see directly that d_2 must be validated with respect to top(d_m) and bot(d_1).

It is tempting to propose a technique similar to pardev$'_\circ$ given below as a description of the splitting, the parallel development and the recombination of the results.

```
pardev$'_\circ$ := **technique** $d_m, d_s, d$
**def** $d_1, d_2$ := split$_\circ(d)$;
```

Note that this 'impl' may modify black-box descriptions; so 'impl' certainly is not restricted to just bbc-gb-mods. There is a splitting to be done before the actual parallelism can start. This is described by the technique 'split$_*$' given below.

```
split$_*$ := **technique** $d \rightarrow d_1, d_2$
**pre true**
**post** $d_1 * d_2 = d$
```

Based on the splitting of d into d_1 and d_2 it is possible to indicate which parts of the machine&user-context are relevant for d_1 and which parts of the machine&user-context are relevant for d_2. Of course in many practical situations it is not the case that the system user and the machine are formalised as designs. Nevertheless, in such situations we probably still have that certain parts of the machine&user-context are relevant for d_1 and that other parts of the machine&user-context are relevant for d_2.

In view of lemma 3.3.23, the developers need not know d_m and d_s completely; it is sufficient if they have access to the relevant parts of top(d_m) and bot(d_s). In fact top(d_m) only needs to be given up to top-equivalence (see definition 3.3.28). The splitting of the bottom of the system user d_s is a matter of counting **prim** components and system elements whereas the splitting of the top of the machine d_m might be more complicated. The corresponding techniques are given below. We define def$_\circ(d_1, d_2)$ $:\Leftrightarrow$ "the composition $d_1 \circ d_2$ is defined".

```
msplit := **technique** $d_m, d_1, d_2 \rightarrow d'_m, d''_m$
**pre** def$_\circ(d_1 * d_2, d_m)$ **and** top$(d_m) = d_m$
**post** $d'_m * d''_m =_{\text{top}} d_m$ **and** def$_\circ(d_1, d'_m)$ **and** def$_\circ(d_2, d''_m)$

ssplit := **technique** $d_s, d_1, d_2 \rightarrow d'_s, d''_s$
**pre** def$_\circ(d_s, d_1 * d_2)$ **and** bot$(d_s) = d_s$
**post** $d'_s * d''_s = d_s$ **and** def$_\circ(d'_s, d_1)$ **and** def$_\circ(d''_s, d_2)$
```

The condition top(d_m) = d_m in the precondition of 'msplit' says that d_m must be a top design. The condition bot(d_s) = d_s in the precondition of 'ssplit' says that d_s must be a bottom design. Note that if 'msplit' is invoked with arguments satisfying its precondition then by lemma 3.3.30 its execution need not fail. Note also that if 'ssplit' is invoked with arguments satisfying its precondition then by remark 3.3.32 its execution need not fail.

The following technique describes the splitting, the parallel development and the recombination of the results.

```
pardev$_*$ := **technique** $d_m, d_s, d$
**def** $d_1, d_2$ := split$_*(d)$;
    $d'_m, d''_m$ := msplit(top$(d_m), d_1, d_2)$;
    $d'_s, d''_s$ := ssplit(bot$(d_s), d_1, d_2)$;
    $d_1, d_2$ := impl$(d'_m, d'_s, d_1)$, impl$(d''_m, d''_s, d_2)$;
    $d_1 * d_2$
```

The partial correctness of this technique with respect to INV is stated in the following theorem.

Theorem 3.6.3 {INV} $d :=$ pardev$_*(d_m, d_s, d)$; {INV}.

Proof. Use lemma 3.3.3 (ii), lemma 3.3.23 and lemma 3.3.16. □

3.6.4 Splitting according to $\circ$

Let again impl be a technique as given in Section 3.6.3 with the property that it does not violate the invariant INV and let us assume that this technique 'impl' describes the activity of one single developer.

There is a splitting to be done before the actual parallelism can start. This is described by the technique 'split$_\circ$' given below.

```
split$_\circ$ := **technique** $d \rightarrow d_1, d_2$
**pre true**
**post** $d_1 \circ d_2 = d$
```

As before it is possible to indicate which parts of the machine&user-context are relevant for d_1 and which parts of the machine&user-context are relevant for d_2. This is relatively simple, due to the fact that we can view $d_1 \circ d_2$ as a layered design with layers d_1 and d_2. We see directly that d_1 must be validated with respect to top(d_2) and the bottom of the system user, i.e. bot(d_s). Similarly we see directly that d_2 must be validated with respect to top(d_m) and bot(d_1).

It is tempting to propose a technique similar to pardev$'_\circ$ given below as a description of the splitting, the parallel development and the recombination of the results.

```
pardev$'_\circ$ := **technique** $d_m, d_s, d$
**def** $d_1, d_2$ := split$_\circ(d)$;
```

```
$d'_m := \text{top}(d_2);$
$d''_m := \text{top}(d_m);$
$d'_s := \text{bot}(d_s);$
$d''_s := \text{bot}(d_1);$
$d_1, d_2 := \text{impl}(d'_m, d'_s, d_1), \text{impl}(d''_m, d''_s, d_2);$
$d_1 \circ d_2$
```

Remark 3.6.4 The proposition that we have

$$\{\text{INV}\}\ d := \text{pardev}'_\circ(d_m, d_s, d);\ \{\text{INV}\}$$

just does not hold in general.

The counter-example is as follows. In this counterexample we do not worry about the validation with repect to d_m and d_s but we focus on the interface between d_1 and d_2. Consider the algebraic system with preorder $\Re_1$ as before. Assume that the designs d_1 and d_2 which are obtained by 'split$_\circ$' respectively are given by

$$\begin{array}{llll} y & := & \textbf{prim} \sqsubseteq & 4 \\ \textbf{system} & [], & & \end{array} \qquad \begin{array}{lllll} x & := & 1 & \sqsubseteq & 1 \\ \textbf{system} & [x]. & & & \end{array}$$

Indeed bbv(d_1, d_2) because $[x \sqsubseteq 1] \vdash x \sqsubseteq 4$. In this case d_1 happens to be a bottom design and d_2 happens to be a top design, so after the execution of the assignments to d'_m, d''_m, d'_s and d''_s we have $d'_m = d_2$ and $d''_s = d_1$. Now one execution of 'impl' yields a design which is valid with respect to the design d_2 as given above and the other execution of 'impl' yields a design which is valid with respect to the design d_1 as given above. Assume that the designs yielded by these two executions respectively are given as

$$\begin{array}{llll} y & := & \textbf{prim} \sqsubseteq & 2 \\ \textbf{system} & [], & & \end{array} \qquad \begin{array}{lllll} x & := & 1 & \sqsubseteq & 3 \\ \textbf{system} & [x]. & & & \end{array}$$

Let us refer to the latter designs as the new value of d_1 and the new value of d_2 respectively. Note that bbv(new value of d_1, old value of d_2) holds because $[x \sqsubseteq 1] \vdash x \sqsubseteq 2$ and that bbv(old value of d_1, new value of d_2) holds because $[x \sqsubseteq 3] \vdash x \sqsubseteq 4$. But bbv(new value of d_1, new value of d_2) does not hold. Therefore the result of pardev$'_\circ$ is not bbc. □

We shall now propose a technique pardev$_\circ$ which is an improved version of pardev$'_\circ$. The idea is that either the bottom of d_1 or the top of d_2 should remain constant in order to avoid the problems shown in remark 3.6.4. We investigate the solution in which the top of d_2 remains constant. In fact it only needs to remain constant up to top-equivalence (see definition 3.3.28).

Therefore we need a slightly modified version of 'impl', which we shall call 'impl$^+$' which is like 'impl' as used above but has the additional property that its output design has the same top as its input design. We obtain the definition of 'impl$^+$' from the definition of 'impl' by simply adding a conjunct to the postcondition (note that this makes one of the other conjuncts redundant).

```
impl$^+$ := **technique** $d_m, d_s, d \to d'$
**pre**  bbv$(d, d_m)$ **and** bbv$(d_s, d)$ **and** bbc$(d)$
**post** bbv$(d', d_m)$ **and** bbv$(d_s, d')$ **and** bbc$(d')$ **and** top$(d') =_{\text{top}}$ top$(d)$
```

The technique 'pardev$_\circ$' given below describes the splitting, the parallel development and the recombination of the results.

```
pardev$_\circ$ := **technique** $d_m, d_s, d$
**def** $d_1, d_2$ := split$_\circ(d)$;
    $d'_m$ := top$(d_2)$;
    $d''_m$ := top$(d_m)$;
    $d'_s$ := bot$(d_s)$;
    $d''_s$ := bot$(d_1)$;
    $d_1, d_2$ := impl$(d'_m, d'_s, d_1)$, impl$^+(d''_m, d''_s, d_2)$;
    $d_1 \circ d_2$
```

The partial correctness of this technique with respect to INV is stated in the following theorem.

Theorem 3.6.5 {INV} $d :=$ pardev$_\circ(d_m, d_s, d)$; {INV}.

Proof. Use lemma 3.3.21 (iii), (iv) lemma 3.3.23 and lemma 3.3.14 (ii). □

Of course there is also a version of this technique based on the solution in which the bottom of d_1 remains constant.

3.6.5 Splitting according to $\sharp$

Splitting according to $\sharp$ for parallel development is not as straighforward as according to $*$ and $\circ$. We do not give a formalisation, but we sketch a few approaches.

The first approach is to adopt black-box correctness. However we must be careful for the proposition 'd_1, d_2 are bbc $\Rightarrow d_1 \sharp d_2$ is bbc' does not hold. But of course we could split a given d into d_1 and d_2 in the sense that $d_1 \sharp d_2$ is defined (and equal to d) and that both d_1 and d_2 are $\Gamma_{\text{bb}}(d_2)$-correct. Now at first sight it seems attractive to have two independent developments for

d_1 and d_2, preserving their $\Gamma_{\mathrm{bb}}(d_2)$-correctness and their $\sharp$-definedness. But on second thought we see that their interface encompasses d_2 completely: the black-box descriptions of d_2 must be kept fixed because these determine $\Gamma_{\mathrm{bb}}(d_2)$ and the glass-box descriptions of d_2 must be kept fixed because $\sharp$-definedness requires d_1 and d_2 to have that column in common. As it turns out, the technique works in principle, but there are so many constraints on d_2 that there is in fact no parallelism at all.

The second approach is to adopt glass-box correctness. Now we can exploit the fact that d_1, d_2 are gbc $\Rightarrow$ $d_1 \sharp d_2$ is gbc. There are two points worth noting. The first point is that the validation of $d_1 \sharp d_2$ with respect to the machine&user-context is determined by d_2 alone. The second point is that d_1 and d_2 must have one column in common. Therefore one could cast that column into the shape of a bottom design which is kept fixed during the parallel development.

The third approach is to employ the generalised designs sketched in Section 3.3.4. This does not give any new approach to parallelism, but there are generalisations of the splitting according to $*, \circ$ and $\sharp$ as discussed above.

3.7 Looking Back

Section 3.2 presents an extension of the $\lambda\pi$-calculus, which is necessary in order to define algebraic operations on designs. The interpretation (in $\lambda\pi$) of each algebraic operation gives rise to an extension of $\lambda\pi$ – each extension requiring the introduction of additional rules.

By defining binary operations $(*, \circ)$ and unary operations (bot, top) on designs, an *algebra of designs* is obtained. The algebraic laws that hold for these operations are investigated and – as it turns out – certain laws of great simplicity hold (see e.g. remark 3.3.12). Under certain conditions the result of applying a binary operation to two bbc designs is a bbc design again. For $*$ this condition is simply 'true' and for $\circ$ it is formulated as a binary predicate bbv; we have the following laws:

$$\begin{aligned} &\mathrm{bbc}(d_1) \wedge \mathrm{bbc}(d_2) \;\Leftrightarrow\; \mathrm{bbc}(d_1 * d_2), \\ &\mathrm{bbc}(d_1) \wedge \mathrm{bbc}(d_2) \wedge \mathrm{bbv}(d_1, d_2) \;\Leftrightarrow\; \mathrm{bbc}(d_1 \circ d_2). \end{aligned}$$

The transitivity of $\sqsubseteq$ suggests a binary operation $\sharp$. It has the the property that d_1, d_2 are bbc $\not\Rightarrow$ $d_1 \sharp d_2$ is bbc and the positive result that d_1, d_2 are gbc $\Rightarrow$ $d_1 \sharp d_2$ is gbc. Several generalisations of our theory are discussed – suggested by this $\sharp$ – which are interesting for a particular kind of stepwise refinement.

To describe models of the software development process, a simple design-development language is introduced, leading us to models of the development process as highly non-deterministic design-programs of an imperative nature.

A postcondition for design creation is formulated and from this postcondition two invariants can be derived in a natural way. The first invariant is investigated in Section 3.4.4 and it leads to the derivation of a design-program which corresponds to the top-down approach. The second invariant is investigated in Section 3.4.5 and it leads to the derivation of a design-program which corresponds to the bottom-up approach. It is remarkable that the top-down and bottom-up models of the development process can be derived in a systematic manner by applying (at the design-program level) an approach which comes from the field of classical sequential programming. The possibilities for deriving invariants and design-programs describing design creation by this approach have by no means been exhausted in Section 3.4. It certainly is interesting to investigate other possibilities.

As it turns out the formulation of the design-programs of Section 3.4 requires several ad-hoc operations on designs; we think this is mainly due to the fact that both top-down and bottom-up development processes take place within the scope of *one* variable design and that therefore one must be very explicit about names, contexts, components, etc. In remarks 3.4.9 (ii) and 3.4.14 (ii) an interesting difference between the top-down approach and the bottom-up approach is revealed: the top down approach always yields a black-box correct design (even if the developer does not know the difference between black-box correctness and glass-box correctness) but there exist two variants of the bottom-up approach where the first variant yields a black-box correct design and the second variant yields a glass-box correct design.

Using the algebra of designs, we discuss the *validation* of a given design with respect to a given machine&user-context and the related problems of design *evolution*. In Section 3.5 we discuss some simple models of the development process which deal with design evolution.

The fact that it is possible to split designs and reassemble them again, gives rise to models of the development process where two (or more) developers each operate on a part of a design such that when each of them has finished his part, their results are fitted together to yield a new design which is both bbc and valid.

Most of the basic ideas behind the models of the development process which are given in Sections 4, 5 and 6 are not really new, but their formalisation is very useful because it makes it possible to reason about the development process in a precise way. The relatively subtle (but important!) points addressed in remark 3.2.17, remark 3.3.35, remark 3.6.4 and remarks 3.4.9

(ii) and 3.4.14 (ii) demonstrate that one needs to be careful indeed when discussing the development process.

It should be noted that most sections in this chapter are open-ended. Appendix A and Section 3 are open-ended because one can always imagine new useful operations (both in terms of $\lambda\pi$ and in terms of designs). Appendix B is open-ended because one might extend the design-development language with almost any language construct from classical specification languages and classical programming languages. Sections 4, 5 and 6 are open-ended because one can always devise other design-programs; in fact we only investigate some of the simplest design-programs.

It is important that the concepts investigated in this chapter are applied in one or more case studies. The language COLD-K can be employed for such case studies. This is possible because COLD-K is based on an algebraic system (viz. the algebra CA of class descriptions) and has $\lambda\pi$ for parameterisation and because it has components and designs as built-in language constructs [8]. For a very first example using a technique from this chapter we refer to [9]. A large example will be presented in Chapters 4 and 5.

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Appendix A

$\lambda\pi$-Calculus with sequences

In order to define algebraic operations on designs it is useful to use sequences within $\lambda\pi$. For that purpose we shall make an extension of the calculus $\lambda\pi$ in this appendix. We shall not give the complete definition of the resulting calculus, but rather describe the modifications with respect to the calculus given in Chapter 2 in an incremental manner.

As a first step we shall adapt the definition of $\lambda\pi$-calculus for an algebraic system with preorder $\mathfrak{R}$ by adding *sequences* to the language of $\mathfrak{R}$. We shall denote sequences by using square brackets; e.g. the sequence with two elements P and Q is denoted by $[P, Q]$. We shall have projection functions π_i for $i = 1, 2, \ldots$. We shall avoid the problems that arise when a projection function π_i can be applied to a sequence whose length is less than i by adapting the type system. Furthermore we shall have a binary operation $*$ for *concatenation*. The laws that describe the operations on sequences will be given as rules of $\lambda\pi$.

We do not *define* these constructions in $\lambda\pi$ as one would do in classical lambda calculus. Some of the techniques one uses in classical lambda calculus for defining 'pairing' are based on having η-reduction and it is not clear (yet) what η reduction in $\lambda\pi$-calculus should look like.

Definition A.1.1 (i) Add to the inductive definition of the set of *type symbols*:

- if $\sigma_1, \ldots, \sigma_m$ are type symbols $(m \geq 0)$, then also is $(\sigma_1, \ldots, \sigma_m)$.

(ii) Add to the definition of the *alphabet*:

- *Symbols for operations on sequences*: [,], π_i (for $i = 1, 2, \ldots$), $*$.

(iii) Add to the inductive definition of the set $\Lambda_{\Re}$ of *lambda terms* for $\Re$, and their *type*:

- (i) if $P_1 \in \Lambda_{\Re}, \ldots, P_n \in \Lambda_{\Re}$ for $n \geq 0$ with types $\sigma_1, \ldots, \sigma_n$ then $[P_1, \ldots, P_n] \in \Lambda_{\Re}$ with type $(\sigma_1, \ldots, \sigma_n)$.

  (ii) if $P \in \Lambda_{\Re}$ with type $(\sigma_1, \ldots, \sigma_n)$ and $1 \leq i \leq n$ then $\pi_i(P) \in \Lambda_{\Re}$ with type σ_i. □

Definition A.1.2 (length). If a lambda term P has type $(\sigma_1, \ldots, \sigma_n)$ then the *length* of P is defined as n. □

Definition A.1.3 (Rules for sequences). We adopt the rules of $\lambda\pi$ ($\models_1$, $\models_2$, context, refl., trans., λI_1, λI_2, ap., π, $=I$, subst.) and we add the following rules:

$$(\pi_i) \qquad \frac{}{\Gamma \vdash \pi_i([P_1, \ldots, P_n]) = P_i}$$

$$(*_1) \qquad \frac{}{\Gamma \vdash [P_1, \ldots, P_n] * [Q_1, \ldots, Q_m] = [P_1, \ldots, P_n, Q_1, \ldots, Q_m]}$$

$$(\sqsubseteq\text{-seq}) \; \frac{\Gamma \vdash P_1 \sqsubseteq Q_1, \quad \ldots \quad , \Gamma \vdash P_n \sqsubseteq Q_n}{\Gamma \vdash [P_1, \ldots, P_n] \sqsubseteq [Q_1, \ldots, Q_n]}$$

□

We could add a rule for surjectivity of sequence construction, but since we do not need it we shall not add it. From now on we write $\lambda\pi$ to denote the extended version of the calculus.

Lemma A.1.4 (Properties of $*$). Let $P \equiv [P_1, \ldots, P_m]$, $Q \equiv [Q_1, \ldots, Q_n]$ and $R \equiv [R_1, \ldots, R_p]$.

(i) $\vdash (P * Q) * R = P * (Q * R)$,

(ii) $\vdash [] * P = P * [] = P$.

Proof. (i) We use rule $(*_1)$.

$$\begin{aligned}
\vdash (P * Q) * R &\equiv ([P_1, \ldots, P_m] * [Q_1, \ldots, Q_n]) * [R_1, \ldots, R_p] \\
&\overset{*_1}{=} [P_1, \ldots, P_m, Q_1, \ldots, Q_n] * [R_1, \ldots, R_p] \\
&\overset{*_1}{=} [P_1, \ldots, P_m, Q_1, \ldots, Q_n, R_1, \ldots, R_p] \\
&\overset{*_1}{=} [P_1, \ldots, P_m] * [Q_1, \ldots, Q_n, R_1, \ldots, R_p]
\end{aligned}$$

$$\stackrel{*_1}{=} [P_1, \ldots, P_m] * ([Q_1, \ldots, Q_n] * [R_1, \ldots, R_p])$$
$$\equiv P * (Q * R).$$

(ii) $\vdash [] * P \equiv [] * [P_1, \ldots, P_m] \stackrel{*_1}{=} [P_1, \ldots, P_m] \equiv P$ and finally $\vdash P * [] \equiv [P_1, \ldots, P_m] * [] \stackrel{*_1}{=} [P_1, \ldots, P_m] \equiv P$. □

Remark A.1.5 It is tempting to think that every term P of type $(\sigma_1, \ldots, \sigma_m)$ can be written as $[P_1, \ldots, P_m]$ for suitable $P_1, \ldots, P_m$ such that $\vdash P = [P_1, \ldots, P_m]$. This is not the case however. As an example consider the algebraic system with preorder $\Re_1 = (\mathbb{N}, \leq, \{+\}, \{0, 1, 2, \ldots\})$ and $P :\equiv (\lambda x \sqsubseteq 1.[0, 0])2$. An even simpler example is the term x^σ for $\sigma \equiv (\sigma_1, \ldots, \sigma_m)$. □

Remark A.1.6 Instead of sequence construction using [and] we could have chosen constructors [] and cons. In that case we might have chosen projection functions 'hd' and 'tl'. □

We can define the relations $\rightarrow$, $\twoheadrightarrow$ and $=_\pi$ in an obvious way by viewing the rules $(\pi, \pi_i, *_1)$ as basic reduction steps.

Definition A.1.7 (Reduction).

The relation $\rightarrow$ is defined inductively by:

1. $\Gamma \vdash R \sqsubseteq A \Rightarrow \Gamma \vdash (\lambda x \sqsubseteq A.B)R \rightarrow B[x := R]$,
2. $\Gamma \vdash \pi_i([P_1, \ldots, P_n]) \rightarrow P_i$ $\quad (1 \leq i \leq n)$,
3. $\Gamma \vdash [P_1, \ldots, P_n] * [Q_1, \ldots, Q_m] \rightarrow [P_1, \ldots, P_n, Q_1, \ldots, Q_m]$,
4. $\Gamma \vdash M \rightarrow N \Rightarrow \Gamma \vdash f_j(\ldots, M, \ldots) \rightarrow f_j(\ldots, N, \ldots)$,
5. $\Gamma \vdash P_i \rightarrow P_i' \Rightarrow \Gamma \vdash [P_1, \ldots, P_i, \ldots, P_n] \rightarrow [P_1, \ldots, P_i', \ldots, P_n]$ $\quad (1 \leq i \leq n)$,
6. $\Gamma \vdash P \rightarrow P' \Rightarrow \Gamma \vdash P * Q \rightarrow P' * Q$,
7. $\Gamma \vdash Q \rightarrow Q' \Rightarrow \Gamma \vdash P * Q \rightarrow P * Q'$,
8. $\Gamma \vdash P \rightarrow P' \Rightarrow \Gamma \vdash \pi_i(P) \rightarrow \pi_i(P')$,
9. $\Gamma \vdash M \rightarrow N \Rightarrow \Gamma \vdash ZM \rightarrow ZN$,
10. $\Gamma \vdash M \rightarrow N \Rightarrow \Gamma \vdash MZ \rightarrow NZ$,
11. $\Gamma \vdash P \rightarrow Q \Rightarrow \Gamma \vdash (\lambda x \sqsubseteq P.M) \rightarrow (\lambda x \sqsubseteq Q.M)$,
12. $\Gamma, [x \sqsubseteq P] \vdash M \rightarrow N$, $x \notin \Gamma \Rightarrow \Gamma \vdash (\lambda x \sqsubseteq P.M) \rightarrow (\lambda x \sqsubseteq P.N)$.

The relation $\twoheadrightarrow$ is defined inductively by:

1. $\Gamma \vdash M \rightarrow N \Rightarrow \Gamma \vdash M \twoheadrightarrow N$,

2. $\Gamma \vdash M \twoheadrightarrow M$,
3. $\Gamma \vdash M \twoheadrightarrow N,\ \Gamma \vdash N \twoheadrightarrow L \Rightarrow \Gamma \vdash M \twoheadrightarrow L$.

The relation $=_\pi$ is defined inductively by:

1. $\Gamma \vdash M \twoheadrightarrow N \Rightarrow \Gamma \vdash M =_\pi N$, provided $M \twoheadrightarrow N$ only by π-reductions, i.e. excluding the clauses 2. and 3. from the definition of $\rightarrow$,
2. $\Gamma \vdash M =_\pi N \Rightarrow \Gamma \vdash N =_\pi M$,
3. $\Gamma \vdash M =_\pi N,\ \Gamma \vdash N =_\pi L \Rightarrow \Gamma \vdash M =_\pi L$. □

And of course we have the following lemma, justifying $\rightarrow, \twoheadrightarrow$ and $=_\pi$.

Lemma A.1.8

(i) $\Gamma \vdash M \rightarrow N \Rightarrow \Gamma \vdash M = N$,

(ii) $\Gamma \vdash M \twoheadrightarrow N \Rightarrow \Gamma \vdash M = N$,

(iii) $\Gamma \vdash M =_\pi N \Rightarrow \Gamma \vdash M = N$.

Proof.

(i) By induction over the definition of $\rightarrow$.
(ii) By induction over the definition of $\twoheadrightarrow$.
(iii) By induction over the definition of $=_\pi$. □

This enriched $\lambda\pi$-calculus has reasonable properties: In $\lambda\pi$ with the extensions of definitions A.1.1 and A.1.3 every term strongly normalises (SN). This can be shown by adapting the computability argument of Chapter 2. Furthermore in $\lambda\pi$-calculus with the extensions of definitions A.1.1 and A.1.3, $\twoheadrightarrow$ satisfies the diamond property i.e. $\Gamma \vdash M \twoheadrightarrow M_1, M \twoheadrightarrow M_2 \Rightarrow \exists M_3 \cdot (\Gamma \vdash M_1 \twoheadrightarrow M_3, M_2 \twoheadrightarrow M_3)$. This can be shown by adapting the proof of the weak diamond property and using Newman's lemma again, just as was done for $\lambda\pi$ in Chapter 2.

In order to have an interpretation for the algebraic operations on designs, we extend $\lambda\pi$ once more by introducing two operations on closed terms, viz. $*$ and $\circ$. One can view $*$ as a generalisation of the concatenation of sequences ($*$). One can view $M \circ N$ as functional composition, where an automatic un-Currying takes place. We shall not give the complete definition of the resulting calculus, but rather describe the extensions with respect to the calculus described so-far in an incremental manner. We shall use the conventions that $\sigma_1 \rightarrow \ldots \rightarrow \sigma_n \rightarrow \tau$ abbreviates $\sigma_1 \rightarrow (\ldots(\sigma_n \rightarrow \tau)\ldots)$ and that $\lambda\vec{x} \sqsubseteq \vec{P}.T$ is shorthand for $\lambda x_1 \sqsubseteq P_1 \cdots \lambda x_n \sqsubseteq P_n.T$

Definition A.1.9 ($\Lambda_{\Re}$). (i) Add to the definition of the *alphabet* a clause:

- A *binary operation symbol*: $\circ$.

(ii) Add to the inductive definition of the set of *lambda terms* and their *type*:

- if $P \in \Lambda_{\Re}$ with type $\sigma_1 \to \ldots \to \sigma_m \to (\varsigma_1, \ldots, \varsigma_j)$ and $Q \in \Lambda_{\Re}$ with type $\tau_1 \to \ldots \to \tau_n \to (\eta_1, \ldots, \eta_k)$ then $P * Q \in \Lambda_{\Re}$ with type $\sigma_1 \to \ldots \to \sigma_m \to \tau_1 \to \ldots \to \tau_n \to (\varsigma_1, \ldots, \varsigma_j, \eta_1, \ldots, \eta_k)$,
- if $P \in \Lambda_{\Re}$ with type $\sigma_1 \to \ldots \to \sigma_m \to (\varsigma_1, \ldots, \varsigma_j)$ and $Q \in \Lambda_{\Re}$ with type $\tau_1 \to \ldots \to \tau_n \to (\sigma_1, \ldots, \sigma_m)$ then $P \circ Q \in \Lambda_{\Re}$ with type $\tau_1 \to \ldots \to \tau_n \to (\varsigma_1, \ldots, \varsigma_j)$. □

Definition A.1.10 (rules for $*, \circ$). We adopt the rules of $\lambda\pi$ ($\models_1$, $\models_2$, context, refl., trans., λI_1, λI_2, ap., π, $=I$, subst., π_i, $*_1$, $\sqsubseteq$ -seq) and we add the rules:

$$(*_2) \qquad \frac{}{\lambda\vec{x} \sqsubseteq \vec{P}.S * \lambda\vec{y} \sqsubseteq \vec{Q}.T = \lambda\vec{x}\vec{y} \sqsubseteq \vec{P}\vec{Q}.S * T} \; (S, T \text{ not of a type } \sigma \to \tau)$$

$$(\circ_1) \qquad \frac{}{M \circ (\lambda\vec{x} \sqsubseteq \vec{B}.N) \; = \; \lambda\vec{x} \sqsubseteq \vec{B}.(M \circ N)} \; (\vec{x} \notin M,\ N \text{ not of type } \sigma \to \tau)$$

$$(\circ_2) \qquad \frac{}{(\lambda\vec{x} \sqsubseteq \vec{M}.N) \circ [\vec{L}] \; = \; (\lambda\vec{x} \sqsubseteq \vec{M}.N)\overleftarrow{L}}$$

where it is understood that the conclusion of rule $\circ_2$ is an abbreviation of the following:
$(\lambda x_1 \sqsubseteq M_1. \cdots \lambda x_m \sqsubseteq M_m.N) \circ [L_1, \ldots, L_m] \; = \; (\lambda x_1 \sqsubseteq M_1. \cdots (\lambda x_m \sqsubseteq M_m.N)L_m \ldots)L_1$. □

Since we have extended the set of terms and added rules, we must reconsider the properties of $*$ and $\circ$ again:

Lemma A.1.11 (Properties of $*$).
Let $P \equiv \lambda\vec{x} \sqsubseteq \vec{P'}.[P_1, \ldots, P_m]$, $Q \equiv \lambda\vec{y} \sqsubseteq \vec{Q'}.[Q_1, \ldots, Q_n]$ and $R \equiv \lambda\vec{z} \sqsubseteq \vec{R'}.[R_1, \ldots, R_p]$.

(i) $\vdash (P * Q) * R = P * (Q * R)$,

(ii) $\vdash [] * P = P * [] = P$.

Proof. (i) We use the rules ($*_1$) and ($*_2$).

$$\vdash (P * Q) * R \equiv ((\lambda\vec{x} \sqsubseteq \vec{P'}.[P_1, \ldots, P_m]) * (\lambda\vec{y} \sqsubseteq \vec{Q'}.[Q_1, \ldots, Q_n])) * R$$

$$\begin{aligned}
&\overset{*2}{=} (\lambda\vec{x}\vec{y} \sqsubseteq \vec{P}'\vec{Q}'.([P_1,\dots,P_m] * [Q_1,\dots,Q_n])) * R \\
&\overset{*1}{=} (\lambda\vec{x}\vec{y} \sqsubseteq \vec{P}'\vec{Q}'.([P_1,\dots,P_m,Q_1,\dots,Q_n])) * R \\
&\overset{*2}{=} \lambda\vec{x}\vec{y}\vec{z} \sqsubseteq \vec{P}'\vec{Q}'\vec{R}'.([P_1,\dots,P_m,Q_1,\dots,Q_n] * [R_1,\dots,R_p]) \\
&\overset{*1}{=} \lambda\vec{x}\vec{y}\vec{z} \sqsubseteq \vec{P}'\vec{Q}'\vec{R}'.[P_1,\dots,P_m,Q_1,\dots,Q_n,R_1,\dots,R_p] \\
&\overset{*1}{=} \lambda\vec{x}\vec{y}\vec{z} \sqsubseteq \vec{P}'\vec{Q}'\vec{R}'.([P_1,\dots,P_m] * [Q_1,\dots,Q_n,R_1,\dots,R_p]) \\
&\overset{*2}{=} (\lambda\vec{x} \sqsubseteq \vec{P}'.[P_1,\dots,P_m]) * \lambda\vec{y}\vec{z} \sqsubseteq \vec{Q}'\vec{R}'.[Q_1,\dots,Q_n,R_1,\dots,R_p] \\
&\overset{*1}{=} P * \lambda\vec{y}\vec{z} \sqsubseteq \vec{Q}'\vec{R}'.([Q_1,\dots,Q_n] * [R_1,\dots,R_p]) \\
&\overset{*2}{=} P * ((\lambda\vec{y} \sqsubseteq \vec{Q}'.[Q_1,\dots,Q_n]) * \lambda\vec{z} \sqsubseteq \vec{R}'.[R_1,\dots,R_p]) \\
&\equiv P * (Q * R).
\end{aligned}$$

(ii) First we prove $[] * P = P$.

$$\begin{aligned}
\vdash [] * P &\equiv [] * \lambda\vec{x} \sqsubseteq \vec{P}'.[P_1,\dots,P_m] \\
&\overset{*2}{=} \lambda\vec{x} \sqsubseteq \vec{P}'.([] * [P_1,\dots,P_m]) \\
&\overset{*1}{=} \lambda\vec{x} \sqsubseteq \vec{P}'.[P_1,\dots,P_m] \equiv P.
\end{aligned}$$

Finally we prove $P * [] = P$.

$$\begin{aligned}
\vdash P * [] &\equiv (\lambda\vec{x} \sqsubseteq \vec{P}'.[P_1,\dots,P_m]) * [] \\
&\overset{*2}{=} \lambda\vec{x} \sqsubseteq \vec{P}'.([P_1,\dots,P_m] * []) \\
&\overset{*1}{=} \lambda\vec{x} \sqsubseteq \vec{P}'.[P_1,\dots,P_m] \equiv P.
\end{aligned}$$

□

Lemma A.1.12 (Properties of $\circ$). The following properties hold only under certain conditions. A sufficient condition is that both the left-hand side and the right-hand side of the stated equation are fully reducible (by which we mean that all candidate-redexes are redexes). We consider closed terms P, Q and R. Let $P \equiv \lambda x_1 \sqsubseteq P_1.\cdots\lambda x_m \sqsubseteq P_m.[A_1,\dots,A_j] \equiv \lambda\vec{x} \sqsubseteq \vec{P}.[\vec{A}]$. Let $Q \equiv \lambda y_1 \sqsubseteq Q_1.\cdots\lambda y_n \sqsubseteq Q_n.[B_1,\dots,B_m] \equiv \lambda\vec{y} \sqsubseteq \vec{Q}.[\vec{B}]$.
Let $R \equiv \lambda z_1 \sqsubseteq R_1.\cdots\lambda z_p \sqsubseteq R_n.[\vec{C}] \equiv \lambda\vec{z} \sqsubseteq \vec{R}.[\vec{C}]$.

(i) $\vdash (P \circ Q) \circ R = P \circ (Q \circ R)$,

(ii) $\vdash [] \circ P = P \circ [] = P$.

Proof. (i) $M[\vec{x} := \vec{B}]$ abbreviates $M[x_m := B_m]\dots[x_1 := B_1]$. We freely use π-equality.

$$\vdash (P \circ Q) \circ R \equiv ((\lambda\vec{x} \sqsubseteq \vec{P}.[\vec{A}]) \circ \lambda\vec{y} \sqsubseteq \vec{Q}.[\vec{B}]) \circ R$$

$$
\begin{aligned}
&\overset{\circ_1}{=} (\lambda\vec{y} \sqsubseteq \vec{Q}.((\lambda\vec{x} \sqsubseteq \vec{P}.[\vec{A}]) \circ [\vec{B}])) \circ R \\
&\overset{\circ_2}{=} (\lambda\vec{y} \sqsubseteq \vec{Q}.(\lambda\vec{x} \sqsubseteq \vec{P}.[\vec{A}])\overleftarrow{B}) \circ R \\
&\equiv (\lambda\vec{y} \sqsubseteq \vec{Q}.(\lambda\vec{x} \sqsubseteq \vec{P}.[\vec{A}])\overleftarrow{B}) \circ \lambda\vec{z} \sqsubseteq \vec{R}.[\vec{C}] \\
&\overset{\circ_1}{=} \lambda\vec{z} \sqsubseteq \vec{R}.((\lambda\vec{y} \sqsubseteq \vec{Q}.(\lambda\vec{x} \sqsubseteq \vec{P}.\ [\vec{A}])\overleftarrow{B}) \circ [\vec{C}]) \\
&\overset{\circ_2}{=} \lambda\vec{z} \sqsubseteq \vec{R}.((\lambda\vec{y} \sqsubseteq \vec{Q}.(\lambda\vec{x} \sqsubseteq \vec{P}.[\vec{A}])\overleftarrow{B})\overleftarrow{C}) =_\pi \ldots \\
&=_\pi \lambda\vec{z} \sqsubseteq \vec{R}.[\vec{A}[\vec{x} := \vec{B}][\vec{y} := \vec{C}]] \\
&\equiv \lambda\vec{z} \sqsubseteq \vec{R}.[\vec{A}[\vec{x} := \vec{B}[\vec{y} := \vec{C}]]] =_\pi \ldots \\
&=_\pi \lambda\vec{z} \sqsubseteq \vec{R}.((\lambda\vec{x} \sqsubseteq \vec{P}.[\vec{A}])\overleftarrow{B}[\vec{y} := \vec{C}]) \\
&\overset{\circ_2}{=} \lambda\vec{z} \sqsubseteq \vec{R}.((\lambda\vec{x} \sqsubseteq \vec{P}.[\vec{A}]) \circ [\vec{B}[\vec{y} := \vec{C}]]) \\
&\equiv \lambda\vec{z} \sqsubseteq \vec{R}.(P \circ [\vec{B}[\vec{y} := \vec{C}]]) =_\pi \ldots \\
&=_\pi \lambda\vec{z} \sqsubseteq \vec{R}.(P \circ ((\lambda\vec{y} \sqsubseteq \vec{Q}.[\vec{B}])\overleftarrow{C})) \\
&\overset{\circ_1}{=} P \circ \lambda\vec{z} \sqsubseteq \vec{R}.((\lambda\vec{y} \sqsubseteq \vec{Q}.[\vec{B}])\overleftarrow{C}) \\
&\overset{\circ_2}{=} P \circ (\lambda\vec{z} \sqsubseteq \vec{R}.((\lambda\vec{y} \sqsubseteq \vec{Q}.[\vec{B}]) \circ [\vec{C}])) \\
&\overset{\circ_1}{=} P \circ ((\lambda\vec{y} \sqsubseteq \vec{Q}.[\vec{B}]) \circ (\lambda\vec{z} \sqsubseteq \vec{R}.[\vec{C}])) \\
&\equiv P \circ (Q \circ R).
\end{aligned}
$$

(ii) $\vdash P \circ [] = P$ is nothing but rule ($\circ_2$). For the converse, note that P must be of type $\tau_1 \to \ldots \tau_n \to ()$. Since P is closed and fully reducible, it must be π convertible to a term of the form $\lambda\vec{x} \sqsubseteq \vec{P}.[]$. Therefore $\vdash [] \circ P = [] \circ \lambda\vec{x} \sqsubseteq \vec{P}.[] \overset{\circ_1}{=} \lambda\vec{x} \sqsubseteq \vec{P}.([] \circ []) \overset{\circ_2}{=} \lambda\vec{x} \sqsubseteq \vec{P}.[] \equiv P$. $\square$

Appendix B

Design-development language

In this appendix we shall define a simple language which can be employed for expressing certain models of the software development process. It is particularly tuned to the setting of our study of growing, combining and modifying designs. The models of the development process we aim at, are abstract in the sense that they leave room for creative freedom by a human developer.

Essentially the design-development language provides for procedures with input and output parameters. The procedures can be defined by either imperative programming-language constructs (assignment, **while**, sequential composition, etc.) or by first-order predicate logic constructs (**not**, **and**, **forall**, etc.). This language is kept small and simple. Its semantics is relational, i.e. the meaning of a procedure is just a relation on input-domain $\times$ output-domain.

The motivation for these choices is as follows. The imperative style is chosen because we view it as natural and intuitive to support a notion like 'current design'. The logical constructs and the relational semantics are required by the abstractness of our models. The imperative constructs are required because sometimes we want to be very specific in our models – like in 'first do this, then do that', or in case of repetitive models. The procedures themselves serve as a simple modularisation mechanism for our models. At the end of this apendix we shall discuss some more aspects of the use of this language. There we shall also show a few examples.

We assume the following sets

> *Var*: variables with typical elements $x, y, \ldots$
> *Pred$_n$*: n-ary predicate symbols with typical elements $P_n, \ldots$
> *Op$_n$*: n-ary operation symbols ($n \in \mathbb{N}$) with typical elements $O_n, \ldots$

We shall define the following sets

> *Dexp*: deterministic expressions with typical elements $e, \ldots$
> *Asn*: assertions with typical elements $A, \ldots$
> $Elist_k$ ($k \in \mathbb{N}^+$): expressions lists of length k with typical elements $l_k, \ldots$
> *Stat*: statements with typical elements $s, \ldots$
> $Proc_{n,m}$ ($n \in \mathbb{N}, m \in \mathbb{N}^+$): procedures with typical elements $p_{n,m}, \ldots$

We shall define mappings yielding the sets of 'free variables' (denoted as *FV*) and the so-called 'assign-set' (denoted as AS). The assign-set of a statement will contain those variables for which it follows on syntactical grounds that assignments are made to them.

$$\begin{aligned}
&FV : Dexp \to \mathcal{P}(Var)\\
&FV : Asn \to \mathcal{P}(Var)\\
&FV : Elist_k \to \mathcal{P}(Var)\\
&FV : Stat \to \mathcal{P}(Var)\\
&\mathrm{AS} : Stat \to \mathcal{P}(Var)
\end{aligned}$$

We give the definitions now, where it is understood that *Dexp*, *Asn*, $Elist_k$, *Stat* and $Proc_{n,m}$ are the smallest sets which are closed under the syntax rules given below, under the indicated restrictions related to *FV* and AS and where it is also understood that the mappings *FV* and the mapping AS are simultaneously inductively defined.

$$\begin{aligned}
e \;::=\;& x\\
\mid\;& O_n(e_1, \ldots, e_n)
\end{aligned}$$

$$\begin{aligned}
A \;::=\;& P_n(e_1, \ldots, e_n)\\
\mid\;& \mathbf{true}\\
\mid\;& \mathbf{not}\ A'\\
\mid\;& (A'\ \mathbf{or}\ A'')\\
\mid\;& (A'\ \mathbf{and}\ A'')\\
\mid\;& (A' \to A'')\\
\mid\;& (A' \leftrightarrow A'')\\
\mid\;& \mathbf{exists}\ x\ (A')\\
\mid\;& \mathbf{forall}\ x\ (A')
\end{aligned}$$

$$\begin{aligned}
l_k \;::=\;& x && (k = 1)\\
\mid\;& O_n(l'_n) && (k = 1)\\
\mid\;& l'_n, l''_m && (k = n + m)\\
\mid\;& p_{n,m}(l'_n) && (k = m)
\end{aligned}$$

s ::= (i.e. empty)
| $x_1, \dots, x_k$:= l_k; ($x_1, \dots, x_k$ distinct)
| $s' \ s''$
| $s' \, [] \, s''$
| **while** A **do** s' **od**;

$p_{n,m}$ ::= **proc** $x_1, \dots, x_n \to y_1, \dots, y_m$ **axiom** A

($FV(A) \subseteq \{\vec{x}\} \cup \{\vec{y}\}$, $x_1, \dots, x_n$, $y_1, \dots, y_m$ distinct)

| **proc** $x_1, \dots, x_n$ **def** $s \ l_k$

($k = m$, $FV(l_k) \subseteq \{\vec{x}\} \cup \mathrm{AS}(s)$, $FV(s) \subseteq \{\vec{x}\}$, $x_1, \dots, x_n$ distinct)

proc $x_1, \dots, x_n \to y_1, \dots, y_m$ **pre** A **post** A' abbreviates **proc** $x_1, \dots, x_n \to y_1, \dots, y_m$ **axiom** (A **and** A') provided $FV(A) \subseteq \{\vec{x}\}$, $FV(A') \subseteq \{\vec{x}\} \cup \{\vec{y}\}$, $x_1, \dots, x_n$, $y_1, \dots, y_m$ distinct.

$FV : Dexp \to \mathcal{P}(Var)$ is defined by

$FV(x) = \{x\}$
$FV(O_n(\vec{e})) = \bigcup_{i=1,\dots,n} FV(e_i)$

$FV : Asn \to \mathcal{P}(Var)$ is defined by

$FV(P_n(\vec{e})) = \bigcup_{i=1,\dots,n} FV(e_i)$
$FV(\textbf{true}) = \emptyset$
$FV(\textbf{not } A) = FV(A)$
$FV((A \textbf{ or } A')) = FV(A) \cup FV(A')$
$FV((A \textbf{ and } A')) = FV(A) \cup FV(A')$
$FV((A \to A')) = FV(A) \cup FV(A')$
$FV((A \leftrightarrow A')) = FV(A) \cup FV(A')$
$FV(\textbf{exists } x \ (A)) = FV(A) \setminus \{x\}$
$FV(\textbf{forall } x \ (A)) = FV(A) \setminus \{x\}$

$FV : Elist_k \to \mathcal{P}(Var)$ is defined by

$FV(x) = \{x\}$
$FV(O_n(l_n)) = FV(l_n)$
$FV(l_n, l'_m) = FV(l_n) \cup FV(l'_m)$
$FV(p_{n,m}(l_n)) = FV(l_n)$

$FV : Stat \to \mathcal{P}(Var)$ is defined by

$$\begin{aligned}
&FV(\,) = \emptyset\\
&FV(\vec{x} := l;) = FV(l)\\
&FV(s\; s') = FV(s) \cup (FV(s') - \mathrm{AS}(s))\\
&FV(s \,[]\, s') = FV(s) \cup FV(s')\\
&FV(\textbf{while } A \textbf{ do } s \textbf{ od};) = FV(A) \cup FV(s)
\end{aligned}$$

AS : $\mathit{Stat} \to \mathcal{P}(\mathit{Var})$ is defined by

$$\begin{aligned}
&\mathrm{AS}(\,) = \emptyset\\
&\mathrm{AS}(\vec{x} := l;) = \{\vec{x}\}\\
&\mathrm{AS}(s\; s') = \mathrm{AS}(s) \cup \mathrm{AS}(s')\\
&\mathrm{AS}(s \,[]\, s') = \mathrm{AS}(s) \cap \mathrm{AS}(s')\\
&\mathrm{AS}(\textbf{while } A \textbf{ do } s \textbf{ od};) = \emptyset
\end{aligned}$$

We assume a set D (the domain of the built-in data type) with typical elements $d, \ldots$ and we assume meaning functions

$$\begin{aligned}
&[\![\;]\!] : \mathit{Pred}_n \to \mathcal{P}(D^n)\\
&[\![\;]\!] : \mathit{Op}_n \to D^n \to D
\end{aligned}$$

The set Σ of *states*, with typical elements $\sigma, \ldots$ is defined by

$$\Sigma = \mathit{Var} \to D$$

We shall define the following meaning functions

$$\begin{aligned}
&[\![\;]\!] : \mathit{Dexp} \to \Sigma \to D\\
&[\![\;]\!] : \mathit{Asn} \to \mathcal{P}(\Sigma)\\
&[\![\;]\!] : \mathit{Elist}_k \to \Sigma \to \mathcal{P}(D^k)\\
&[\![\;]\!] : \mathit{Stat} \to \Sigma \to \mathcal{P}(\Sigma)\\
&[\![\;]\!] : \mathit{Proc}_{n,m} \to \mathcal{P}(D^{n+m})
\end{aligned}$$

We shall identify $D \times (D \times D)$ with $(D \times D) \times D$. We write $\sigma\{d/x\}$ for the modified state $\lambda\!\!\lambda\, y \cdot (\text{ if } y = x \text{ then } d \text{ else } \sigma(y))$.

The meaning of a deterministic expression yields for a given state the result value. The meaning of an assertion is the set of states in which it holds. The meaning of an expression list yields for a given state the set of possible result sequences. A statement is viewed as a non-deterministic state transformer. A procedure is viewed as a relation between an input sequence and a result sequence.

$[\![\;]\!] : \mathit{Dexp} \to \Sigma \to D$ is defined by

$$[\![x]\!](\sigma) = \sigma(x)$$

$$[\![O_n(\vec{e})]\!](\sigma) = [\![O_n]\!]([\![\vec{e}]\!](\sigma))$$

$[\![\ ]\!] : Asn \rightarrow \mathcal{P}(\Sigma)$ is defined by

$$\begin{aligned}
&[\![P_n(\vec{e})]\!] = \{\sigma \mid [\![\vec{e}]\!](\sigma) \in [\![P_n]\!]\}\\
&[\![\mathbf{true}]\!] = \Sigma\\
&[\![\mathbf{not}\ A]\!] = \Sigma \setminus [\![A]\!]\\
&[\![(A\ \mathbf{or}\ A')]\!] = [\![A]\!] \cup [\![A']\!]\\
&[\![(A\ \mathbf{and}\ A')]\!] = [\![A]\!] \cap [\![A']\!]\\
&[\![(A \rightarrow A')]\!] = \{\sigma \mid \sigma \notin [\![A]\!] \vee \sigma \in [\![A']\!]\}\\
&[\![(A \leftrightarrow A')]\!] = \{\sigma \mid \sigma \in [\![A]\!] \Leftrightarrow \sigma \in [\![A']\!]\}\\
&[\![\mathbf{exists}\ x\ (A)]\!] = \{\sigma \mid \exists d \cdot \sigma\{d/x\} \in [\![A]\!]\}\\
&[\![\mathbf{forall}\ x\ (A)\ ]\!] = \{\sigma \mid \forall d \cdot \sigma\{d/x\} \in [\![A]\!]\}
\end{aligned}$$

$[\![\ ]\!] : Elist_k \rightarrow \Sigma \rightarrow \mathcal{P}(D^k)$ is defined by

$$\begin{aligned}
&[\![x]\!](\sigma) = \{\sigma(x)\}\\
&[\![O_n(l)]\!](\sigma) = \{[\![O_n]\!](\vec{d}) \mid \vec{d} \in [\![l]\!](\sigma)\}\\
&[\![l, l']\!](\sigma) = \{\vec{d}\vec{d'} \mid \vec{d} \in [\![l]\!](\sigma) \wedge \vec{d'} \in [\![l']\!](\sigma)\}\\
&[\![p_{n,m}(l)]\!](\sigma) = \{\vec{d'} \mid \exists \vec{d} \in [\![l]\!](\sigma) \cdot \vec{d}\vec{d'} \in [\![p_{n,m}]\!]\}
\end{aligned}$$

$[\![\ ]\!] : Stat \rightarrow \Sigma \rightarrow \mathcal{P}(\Sigma)$ is defined by

$$\begin{aligned}
&[\![\ ]\!](\sigma) = \{\sigma\}\\
&[\![\vec{x} := l;]\!](\sigma) = \{\sigma\{\vec{d}/\vec{x}\} \mid \vec{d} \in [\![l]\!](\sigma)\}\\
&[\![s'\ s'']\!](\sigma) = \{\sigma'' \mid \exists \sigma' \cdot (\sigma' \in [\![s']\!](\sigma) \wedge \sigma'' \in [\![s'']\!](\sigma'))\}\\
&[\![s \,[\!]\, s']\!](\sigma) = [\![s]\!](\sigma) \cup [\![s']\!](\sigma)\\
&[\![\mathbf{while}\ A\ \mathbf{do}\ s\ \mathbf{od};]\!](\sigma) = \textstyle\bigcup_{i=0,1,\ldots} \Phi_i(\sigma)\\
&\text{where we define } \Phi_i : \Sigma \rightarrow \mathcal{P}(\Sigma) \text{ for } i = 0, 1, \ldots \text{ by}\\
&\Phi_0 = \lambda\!\!\lambda\, \sigma \cdot \{\sigma\} \cap [\![\mathbf{not}\ A]\!]\\
&\Phi_{i+1} = \lambda\!\!\lambda\, \sigma \cdot\\
&\{\sigma'' \mid \exists \sigma' \in \Phi_i(\sigma)\ \cdot\ (\sigma' \in [\![A]\!] \wedge \sigma'' \in [\![s]\!](\sigma') \wedge \sigma'' \in [\![\mathbf{not}\ A]\!])\}
\end{aligned}$$

$[\![\ ]\!] : Proc_{n,m} \rightarrow \mathcal{P}(D^{n+m})$ is defined by

$$\begin{aligned}
&[\![\mathbf{proc}\ \vec{x} \rightarrow \vec{y}\ \mathbf{axiom}\ A]\!] = \{\vec{d}\vec{d'} \mid \exists \sigma' \cdot (\sigma'\{\vec{d}/\vec{x}\}\{\vec{d'}/\vec{y}\} \in [\![A]\!])\}\\
&[\![\mathbf{proc}\ \vec{x}\ \mathbf{def}\ s\ l]\!] = \{\vec{d}\vec{d'} \mid \exists \sigma' \cdot (\exists \sigma \in [\![s]\!]\ (\sigma'\{\vec{d}/\vec{x}\}) \cdot (\vec{d'} \in [\![l]\!](\sigma)))\}
\end{aligned}$$

Now we shall discuss the use of this language. Typically the domain of the built-in data type is the set of all (pf $\wedge$ ds) designs, but we shall also use other domains, however without formally having defined this. We take

$$\begin{aligned}
&Op_1 = \{\text{top}, \text{bot}, \ldots\}\\
&Op_2 = \{\circ, *, \ldots\}\\
&Pred_1 = \{\text{bbc}, \text{gbc}, \ldots\}\\
&Pred_2 = \{=, =_{[\![\ ]\!]}, =_{\text{pp}}, =_{\text{top}}, \text{bbv}, \text{gbv}, \text{bbc-gb-mod}, \ldots\}
\end{aligned}$$

where $=$ denotes equality on D and where $=_{[\![\ ]\!]}$ is defined by $d_1 =_{[\![\ ]\!]} d_2 \;:\Leftrightarrow \vdash_{\lambda\pi} [\![d_1]\!] = [\![d_2]\!]$. The dots $(\ldots)$ mean that we shall feel free to extend the set of operations and predicates when necessary. Both $\circ$ and $*$ are written in infix notation. Also $=$ and $=_{[\![\ ]\!]}$ are written in infix notation.

If no confusion can arise we sometimes omit parentheses e.g. writing A **and** A' instead of (A **and** A'). We shall freely give names to procedures, writing $n := p$ where n is the name to be given to the procedure p. We do not allow recursion (yet). Sometimes we replace the keyword **proc** by **event** or **technique**.

We give three simple examples of design-programs written in this language. In these examples we have three procedures (techniques) which are named 'split', 'impl' and 'pardev' respectively. The procedure 'split' is given in a pre- and postcondition style. The result of executing split(d) can be any pair of designs d_1 and d_2 for which $d_1 * d_2 = d$ and in that sense 'split' is non-deterministic.

split := **technique** $d \to d_1, d_2$
pre bbc(d)
post $d_1 * d_2 = d$

The second example is about a procedure 'impl' which also is given in a pre- and postcondition style. The technique 'impl' should be considered as an instruction to a developer to perform a bbc-preserving glass-box modification (bbc-gb-mod) upon a bbc input design and to yield the design resulting from this modification.

impl := **technique** $d \to d'$
pre bbc(d)
post bbc-gb-mod(d, d')

The third example is about a procedure 'pardev' which is given as a very simple algorithm consisting of an assignment statement and a result expression. It invokes the two procedures of the other examples. The technique 'pardev' can be viewed as a model of the development process offering a possibility for parallel development. It says that first of all the input design d should be split using the technique 'split' and that the resulting parts of d should be temporarily stored in d_1 and d_2. Then the technique 'impl' should be applied to each of these parts (possibly in parallel) and the results should be fitted together again with $*$ to form the final result design.

```
pardev:= **technique** $d$
**def** $d_1, d_2$ := split($d$);
    impl($d_1$) $*$ impl($d_2$)
```

After these examples, we can give some more motivation for this procedure concept. Procedures have to yield a sequence of results because we want to view splitting as a procedure (as shown in the 'split' example). We need the non-determinism since we must be able to describe the creative freedom of the developer. The fact that procedures have no side-effects makes reasoning about them relatively easy.

In order to state properties of our design-programs we shall write $\{A_1\}s\{A_2\}$ iff the following holds:

$$\forall \sigma_1 \sigma_2 \cdot ((\sigma_1 \in [\![A_1]\!] \wedge \sigma_2 \in [\![s]\!] \sigma_1) \Rightarrow \sigma_2 \in [\![A_2]\!])$$

i.e. if we start in a state σ_1 in which A_1 holds and if execution of s stops in state σ_2 then A_2 holds in σ_2. Note that the execution of s starting in σ_1 can go wrong for it may be the case that that there is no $\sigma_2 \in [\![s]\!](\sigma_1)$. In this case we say that the execution of s fails. It follows that non-termination is viewed as a failure as well. We shall refer to $\{A_1\}s\{A_2\}$ as the *partial correctness* of s with respect to A_1 and A_2. As an easy example of this notation we have $\{\text{bbc}(d)\}\ d_1, d_2 := \text{split}(d);\ \{d_1 * d_2 = d\}$.

Appendix C

List of symbols

In this appendix we give a list of the symbols used. For each symbol the list contains a very short informal description. The list has been subdivided into a number of sub-lists. The first sub-list contains general mathematical symbols. The second sub-list contains the symbols which are introduced in Chapter 2. The third sublist contains the symbols which are introduced and/or used first in Section 2. In a similar way the fourth sub-list contains the symbols which are introduced and/or used first in Section 3, and so on. For some symbols the list contains a relevant page number – usually the defining occurrence of the symbol.

General mathematical symbols

$\Rightarrow, \Leftarrow$	Logical implication
$\Leftrightarrow$	Logical equivalence
$\wedge$, $\vee$	Conjunction, disjunction
$\forall$, $\exists$	Universal, existential quantification
$\equiv$	Syntactical equality
$=$	Equality
$:\equiv$	Abbreviation
$\{ \;\}$	Set construction
$\{ \mid \}$	Set comprehension
$\in$	Set membership
$\emptyset$	Empty set
$\cup$, $\cap$	Set union, set intersection
$\setminus$	Set difference
$\subseteq$, $\supseteq$	Set inclusion
$\mathcal{P}$	Powerset
C^*	The set of sequences with elements from C
$\times$	Cartesian product

$(,)$	Pair
$(,\ldots,)$	Tuple (= sequence)
$\mathbb{N}$, $\mathbb{N}^+$	Natural numbers, positive nat. numbers
$<, \leq$	Less than, less than or equal to
$+$	Addition
$0, 1, 2, \ldots$	Natural numbers
$f : A \to B$	f is a function from A to B
$\lambda\!\!\lambda$	Lambda notation for functions
$[:=]$	Substitution

Symbols from Chapter 2

$\Re$	Typical algebraic system with preorder	55
$\Re_1$	$\mathbb{N}$ as algebraic system	58
λ	Abstractor (symbol)	61
$\lambda\pi$	Lambda calculus based on rule (π)	60
$\Lambda_\Re$	Set of terms for $\lambda\pi$	62
0	Basic type symbol	61
$\to$	Constructor for type symbols	61
$\sigma, \sigma_1, \ldots, \tau, \tau_1, \ldots$	Typical type symbols	61
$x_0, x_1, \ldots$	Variables	62
x_i^τ	Typical variable of type τ	62
$\sqsubseteq$	Partial order (symbol)	62
$A, B, C, L, M, P, \ldots$	Typical lambda terms	63
x, y, z, u, v, w	Typical variables	63
$[\varphi]$	Formula φ viewed as an assumption	64
Γ, Δ	Typical contexts	64
$\vdash$	Derivation symbol	64
$(\models_1)$	Rule of $\lambda\pi$ (algebraic system oracle)	65
$(\models_2)$	Rule of $\lambda\pi$ (algebraic system oracle)	65
(context.)	Rule of $\lambda\pi$ (context rule)	65
(refl)	Rule of $\lambda\pi$ (reflexivity)	66
(trans.)	Rule of $\lambda\pi$ (transitivity)	66
(λI_1)	Rule of $\lambda\pi$ (lambda introduction)	66
(λI_2)	Rule of $\lambda\pi$ (lambda introduction)	66
$(ap.)$	Rule of $\lambda\pi$ (application)	66
(π)	Rule of $\lambda\pi$ (partial contraction rule)	67
$(=I)$	Rule of $\lambda\pi$ (= introduction)	68
(subst.)	Rule of $\lambda\pi$ (substitution)	68
$\diamond$	Diamond (Church Rosser) property)	80
SN	Strong normalisation property	76
$C_\Re$	The set of components	84

$c, c_1, \dots$	Typical components	84
$D_{\Re}$	The set of designs	86
$d, d_1, \dots$	Typical designs	86
$:=$, $\sqsubseteq$	Symbols used in concrete syntax of designs	86
prim, system	Symbols used in concrete syntax of designs	86
wf	Well-formed	87
$[\![\ ]\!]$	Meaning function for designs	91
gbc, bbc	Glass-box correct, black-box correct	87
bbc-gb-mod	bbc-preserving glass-box modification	??

Symbols concerning designs

$\vec{M}$	Abbreviated form of e.g. $M_1, \dots, M_n$	
$D'_{\Re}$	Designs where the system is a sequence	123
cset( )	Set of names of a design	124
sys( )	Set of names in the system of a design	124
arity( )	Arity of a design	125
$=_{[\![\ ]\!]}$	Semantic equivalence on designs	125
pf	**prims**-first	129
ds	Directly specified	129
$\Gamma_{\text{gb}}, \Gamma_{\text{bb}}$	Glass-box context, black-box context	132
$\mathcal{L}$	Typical sequence of abstractions	135
$\mathcal{S}$	Typical Sequence of substitutions	135

Symbols concerning algebraic operations on designs

$*$	Concatenation operation on designs (binary)	133
$\circ$	Composition operation on designs (binary)	136
e	Empty design	138
e_n	Identity design of arity (n, n)	138
gbv, bbv	Glass-box valid, black-box valid	141
$\bot, \top$	Min. and max. elements (cf. $\sqsubseteq$)	144
bot, top	Bottom and top operations on designs	144
$=_{\text{pp}}$	Equality modulo permutation of prims	147
$=_{\text{top}}$	Top-equivalence	148
$\natural$	Column-wise operation on designs	150
$\langle\ \rangle$	Embedding	153

Symbols concerning design creation

$0, 1, \cdot, +, \oplus, \overline{\ \ }$	Symbols of Boolean logic (in example)	156
GND,7400,7401, ...	TTL component names (in example)	156
$\vec{b}, \vec{s}$	Typical bit-sequences (in example)	156

$\mathrm{int}(\vec{b})$	Integer respresented by $\vec{b}$ (in example)	156
w	Typical machine&user-context	162
INV	General invariant to be (re-) established	164
POST	Postcondition of design creation	164
γ	Typical set of component names	165
$<_1^d$	One-step 'part of' relation	165
$<^d$	'Part of' relation	165
$\leq^d$	Occurs (directly or via $<^d$) in ...	165
TD_INV	Top-down invariant	166
td	Top-down technique	166
td_step	One-step top-down technique	166
last_prim	Operation yielding component	167
last_prim_context	Operation yielding 'context' design	167
td_impl	Technique implementing one component	168
insert	Operation for inserting a component	167
$\mathrm{td\_step}_0$, $\mathrm{td\_step}_1, \ldots$	Alternative techniques for td_step	167
td_spec_impl	Combined specification & implementation	169
BU_INV	Bottom-up invariant	173
bu	Bottom-up technique	173
bu_step	One-step bottom-up technique	173
$\mathrm{bu\_step}_0$, $\mathrm{bu\_step}_1, \ldots$	Alternative techniques for bu_step	174
add_system_element	Operation for adding a system element	174
add_component	Operation for adding a component	175

Symbols concerning design evolution

change	Procedure describing arbitrary event	179
emulator	Technique describing emulation	180
remove_unused	Operation to remove unused components	181
determine_bottom	Technique yielding a valid bottom design	181
simulator	Technique describing simulation	182
empty_system	Operation for making the system empty	182
determine_top	Technique yielding a valid top design	182

Symbols concerning design partition

$d[v]$	The unique component in d with name v	184
impl_c	One-developer bbc-gb-mod technique	184
split_c	Technique for splitting into components	185
pardev_c	Component-based parallel technique	185
impl	Arbitrary technique respecting INV	186
split_*	Technique for splitting according to $*$	187
$\mathrm{def}_\circ$	Definedness of composition	187

msplit	Technique for splitting machine	187
ssplit	Technique for splitting system user	187
pardev_*	Parallel technique based on $*$	187
$\text{split}_\circ$	Technique for splitting according to $\circ$	188
$\text{pardev}'_\circ$	Fallacious version of $\text{pardev}_\circ$	188
$\text{pardev}_\circ$	Parallel development technique based on $\circ$	190
impl^+	Technique as impl not affecting top	190

Symbols concerning $\lambda\pi$-calculus with sequences

$(, ,)$	Constructor for type symbols	195
$[, ,]$	Constructor for terms (sequence)	196
π_i	Constructor for terms (projection)	195
$*$	Constructor for terms (concatenation)	195
η	Classical notion of reduction	195
(π_i), $(*_1)$, ($\sqsubseteq$-seq)	Rules of $\lambda\pi$ with sequences	196
$\longrightarrow$	One step reduction	197
$\twoheadrightarrow$	Reduction	197
$=_\pi$	π-convertibility	198
$\overset{*_1}{=}$	Equality because of rule $(*_1)$	196
$\tau, \sigma, \eta, \varsigma$	Typical type symbols	195
$\circ$	Constructor for terms	199
$(*_2)$, $(\circ_1)$, $(\circ_2)$	Rules of $\lambda\pi$ with sequences	199
$\lambda\vec{x} \sqsubseteq \vec{M}.$	Abbreviating $\lambda x_1 \sqsubseteq M_1 . \ldots \lambda x_m \sqsubseteq M_m.$	199
$\overleftarrow{M}$	Abbreviating $M_n \ldots M_1$	199

Symbols concerning the design-development language

Var	Set of variables	202
$Pred_n$	Set of n-ary predicate symbols	202
Op_n	Set of n-ary operation symbols	202
$Dexp$	Set of deterministic expressions	203
Asn	Set of assertions	203
$Elist_k$	Set of expressions lists of length k	203
$Stat$	Set of statements	203
$Proc_{n,m}$	Set of procedures	203
$x, y, \ldots$	Typical elements of Var	202
$P_n, \ldots$	Typical elements of $Pred_n$	202
$O_n, \ldots$	Typical elements of Op_n	202
$e, \ldots$	Typical elements of $Dexp$	203
$A, \ldots$	Typical elements of Asn	203
$l_k, \ldots$	Typical elements of $Elist_k$	203

$s, \ldots$	Typical elements of *Stat*	203
$p_{n,m}, \ldots$	Typical elements of $Proc_{n,m}$	203
$FV(\ )$	Set of free variables of ...	203
AS( )	Assignset of ...	203
::=	Production symbol of syntax	203
|	Separator for alternatives in syntax	203
true, not, or, and	Keywords for logical connectives	203
$\square$	Symbol for non-deterministic choice	204
while, do, od	Keywords for repetition construct	204
;	Statement terminator	204
proc, axiom, def	Keywords for procedures	204
:=	Symbol for giving a name to a procedure	207
$[\![\ ]\!]$	Semantics of design-development language	205
D	Data domain (e.g. pf $\wedge$ ds designs)	205
$d \ldots$	Typical elements of the data domain	205
Σ	Set of states	205
$\sigma, \ldots$	Typical states	205
$\sigma\{d/x\}$	Modification of a state	205
Φ_i	Semantics of i steps in repetition	206
$\{A_1\}s\{A_2\}$	Partial correctness (design-program level)	208
pre, post	Alternative keywords for procedures	204
event, technique	Alternatives keywords for **proc**	207
split, pardev, impl	Examples of techniques	207

Chapter 4

Formal Specification of a Text Editor

4.1 Introduction

Many of the classical problems of software construction are caused by the absence of specifications or – if there are specifications – by the ambiguities in their formulation. Therefore the use of formal specification techniques is considered worthwhile. The ability to construct large formal specifications supports the applicability of the notions of component, black-box description and design of Chapter 2 and the correctness-preserving transformations of designs investigated in Chapter 3. This chapter deals with the construction of a formal specification and by way of example we specify a display-oriented text editor.

A text editor is a good example for illustrating formal specification techniques because of the following reasons. First, most software designers are familiar with at least one text editor, so the subject in itself may raise some interest and there is no need for an introduction into some less known application area. Secondly an editor is complex enough to give opportunity for illustrating many specification techniques and design principles.

In [1] the language COLD is proposed, in which one can formally express the design of a complete software system in various stages of its development. This language will appear in several user-oriented versions. There exists a kernel-language COLD-K [1,2,3] which contains all essential semantic features of the design language. We use this kernel language for expressing all formal definitions and axioms throughout our case study. This has the advantage that no ambiguities can arise and that a mechanical syntax-checking and type-checking can take place. However, sometimes when we want to state derivable properties of certain operations, we use a somewhat more liberal style of notation.

There are several related reasons that make the undertaking of performing this editor case study worthwhile. The first reason is that it shows the usefulness of formal specification techniques in general and of $\lambda\pi$-calculus and COLD-K in particular. The second reason is that we get an opportunity to annotate the specification and to formulate some guidelines which might be of help for other users of formal specification techniques. The third reason is that it will provide us with a starting-point for a large and typical application of the design methods studied in Chapter 3. The application consists of a systematic top-down development of the editor, which will be undertaken in Chapter 5.

The construction of a formal specification typically involves the following activities: (1) formalisation of application domain-specific concepts: in our case the most important concept being that of *text*, and (2) writing actual specifications of the system under consideration, which in our case is the *editor*. Typically one starts with (1) and then proceeds with (2). However during the writing of the actual specifications, one might discover the need of additional operations etc. and these should be added to the application domain-specific concepts formalised already. This is precisely what has happened during the construction of the specification presented in this chapter. The chapter is a rational reconstruction of this process; we have collected the application-domain specific concepts and put them *before* the description of the editor. A central role will be played by the concept of *text*. We shall devote some effort to an investigation of this concept by introducing a collection of algebraic operations on texts. In this way we develop a machinery which will turn out to be useful, without making any specification decisions about the actual text editor itself yet. Section 4.2 gives an overview of the entire editor specification. Text and algebraic operations on text are the subject of Section 4.3.

The formal specification presented in this chapter will focus on the functional aspects of a text editor, by which we mean those aspects that are not related to performance issues. Rather than creating a completely new editor we take over several concepts from the EMACS family of editors [4,5,6]. Of course, we have to make choices about the way our editor behaves and choose among the many different possible features. We do not claim that our choices are preferable over the many other possibilities. Instead, the points we want to make in this chapter are (1) that it is possible to make a precise description of the behaviour of an editor and (2) that writing a formal specification helps in constructing an editor that is based on a few well-understood concepts, rather than a large collection of ad-hoc features. In Section 4.4 we describe a display, some data types for interfacing and a file system. The actual text editor is the subject of Section 4.5. In Section 4.6 we discuss some related work. Section 4.7 is devoted to conclusions.

In Appendix A we provide an overview of all sorts, functions, predicates and procedures used in this chapter. For each symbol there is a short informal description. In Appendix B we give a number of standard class descriptions.

4.2 Overview of the Formal Specification

In this section we give an overview of the formal specification presented in the remainder of this chapter. This formal definition will consist of many definitions of sorts, functions, predicates, procedures and axioms. These definitions are grouped into a number of modules, or class descriptions as they are called in COLD-K. Of course we must avoid that this becomes an almost endless list of definitions without its goal being clear in advance. Therefore the overview presented in this section is goal-oriented in the sense that it begins with the top-level definition of the entire formal editor specification. This editor specification will not be finished until Section 4.5.14 where its top-level definition turns out to be a procedure called key. We show a fragment of the definition of key, where it is understood that the sort Char corresponds with a character-set.

```
PROC key: Char ->
PAR  c:Char
DEF  ( printable(c)      ?;  insert_character(c)
     | ord(c) = 0  {^@} ?;  set_mark
     | ord(c) = 1  {^A} ?;  beginning_of_line
       ...
       ord(c) = 20 {^T} ?;  insert-file
       ...
       etc.
     )
```

This procedure will be made available to the user of the editor and things must be arranged such that when the user hits some key on his keyboard, producing character c say, then key(c) is invoked which activates one of the editor operations `insert_character`(c), `set_mark`, `beginning_of_line`, `insert_file` etc. Most of these operations result in a modification of an edit-buffer and this modification at its turn can be observed on the screen of the user's video display unit. The procedure key is contained in a class description called KEYBIND_SPEC which imports two other class descriptions called

WITEFA_SPEC and MOREDOP_SPEC. The names of the latter class descriptions are derived from WIndow-and-TExt-FAcility and MORe-EDiting-OPerations. The window and text facility is the kernel of the editor whereas MOREDOP_SPEC provides a few more operations which can be described easily in terms of operations from this kernel. More precisely, WITEFA_SPEC provides a layer of general-purpose editing primitives, whereas MOREDOP_SPEC turns them into one particular editor – using the general-purpose primitives in a specialised way. These two class descriptions are presented in Sections 4.5.1-4.5.12 and 4.5.13 respectively. The window and text facility provides operations such as

```
PROC insert_character: Char ->
```

The insert_character operation has roughly speaking the effect that an edit-buffer corresponding with a notion of 'current text' is modified, that the screen of the display is updated and that the cursor moves one position to the right. When we try to explain the meaning of insert_character in more detail, it becomes clear that we need a lot of preparations. In particular we must have descriptions of the conceptual organisation of the edit-buffers, of the notion of 'current text' and of the relation between the contents of the edit-buffers and the visible contents of the screen. There are also operations that deal with file handling. For example we have

```
PROC insert_file: ->
```

which indicates a need to describe a file system as well. Since files are addressed by names, we must also be prepared to model the strings which are used for that. The above discussions explain why WITEFA_SPEC and MOREDOP_SPEC at their turn are based on several lower level class descriptions, amongst which:

- DISPLAY_SPEC: a model of the video display unit,
- 'SEQ_SPEC' and 'STRING_SPEC': sequences and as a special case of these, strings (the quotes are formally part of the names).
- FILE_SPEC: a file-system.

These are presented in Sections 4.4.2, 4.4.3 and 4.4.4 respectively. By way of example we shall have a closer look at DISPLAY_SPEC, which specifies the video display unit – at least as much of it as we need to specify the editor. The state space of DISPLAY_SPEC is spanned by two variable functions

```
FUNC screen: -> Text VAR
FUNC cursor: -> Nat # Nat VAR
```

where Text denotes a sort of texts and where Nat refers to the natural numbers. These numbers represent the vertical and horizontal co-ordinates of the cursor. All display operations are described by their effect on either **screen** or **cursor** or both. For example

```
PROC nl: -> MOD screen, cursor
```

serves for sending a new-line command to the display, thereby possibly modifying the screen and the cursor. The sort `Text` is in fact an application domain-specific concept and it plays a key role throughout this case study. Therefore the formal specification begins with a study of texts and operations on texts. This study comes *before* the specification of the display and the file system. We view a text as a sequence of lines. The sort of lines is denoted as `Line` and each line consists of a sequence of characters. We shall have the following sorts introduced formally:

```
SORT Line
SORT Text
```

and there are operations to select a line from a text and to select a character from a line.

```
FUNC sel: Text # Nat -> Line
FUNC sel: Line # Nat -> Char
```

Starting from this very simple model, a rich collection of operations on texts is defined, including a variety of cut and paste operations. For example, there is a **paste** operation such that $\mathtt{paste}(t, u, k, l)$ means to take a text t and to insert another text u into it immediately before the position with given coordinates (k, l).

```
FUNC paste: Text # Text # Nat # Nat -> Text
```

As a kind of inverse of paste there is an operation cut such that $\mathtt{cut}(t, k, l, m, n)$ means to take a text t and to cut out the piece of text beginning at position (k, l) and ending at position (m, n). It yields a pair (t_1, t_2) where $t_1 =$ 'remaining text' and $t_2 =$ 'deleted text'.

```
FUNC cut: Text # Nat # Nat # Nat # Nat -> Text # Text
```

We shall not only give the definitions of these and similar operations, but we shall also study some of their properties, which are quite elegant from an algebraic point of view. The operations are grouped into a number of class descriptions among which `LINE_SPEC`, `TEXT_SPEC`, `TEXT_OPS1_SPEC`, `TEXT_OPS2_SPEC`, `TEXT_OPS3_SPEC`, `STRING_SPEC`, and `PROFILE_SPEC`. This is done in Section 4.3. No real decisions about the editor are taken yet in Section 4.3; the only thing which happens is that those operations are introduced formally which are needed to discuss text editing. In fact, at the end of Section 4.3 there is hardly any clue whether we aim at an EMACS-like editor, an ED-like editor or a VI-like editor. We could even specify a MacPaint-like system, although our collection of operations is quite useless in that case.

This concludes the overview of the formal specification of this chapter. The overview has been written afterwards, in a goal-oriented fashion, but the actual presentation in the remainder of this chapter will be organised the other way around. This is because we want to formally introduce each sort and operation before it is used.

4.3 Text and Algebraic Operations On Text

4.3.1 Introduction

There will be several approaches to modeling texts, which differ both with respect to the style of description and the level of abstraction. E.g. consider the following text:

```
first line of text
second line
```

then we can model it as a sequence of lines where the first line has 18 characters viz. "`first line of text`" and where the second line has 11 characters, viz. "`second line`".

An equally valid, but somewhat more abstract approach is to model texts by focusing on their 'contour' only. In this approach the above text is modeled by just the sequence $\langle 18, 11\rangle$ alone. Of course there is a kind of forgetful mapping from the first model to the second model in the sense that the information conveyed by the actual characters in the text, is lost in the second model. Such approaches at distinct levels of abstraction will play a role in our formalisation of the notion of *text* and as it turns out there will be operations on texts at several levels of abstraction, such that the corresponding forgetful mapping behaves homomorphically.

In fact there are several more ways of modeling texts and we mention two of these now. First, we can describe texts as strings, provided we adopt some line-separator. This approach is, roughly speaking, at the same level of abstraction as the first approach mentioned above, where a text is modeled by a sequence of lines. Secondly we can describe texts just by their length which we could define as the length of the string representation. In a sense this is the most abstract non-trivial model, since it forgets about both the structure and the contents of the text.

The above discussion may seem a bit vague and it is the very purpose of our formalisation of the notion of text to cast such ideas into definitions and propositions which are described with mathematical rigour. In this formalisation we will introduce several ***algebraic operations*** on texts. For example there will be an operation called `cut` which serves for taking apart texts and an operation `paste` which serves for fitting together texts. As it turns out, there are many meaningful operations on texts and it is possible to build a rich collection of them. We shall build our collection with the guideline that an operation is included when we foresee that it can be used when specifying an editor. In case of doubt about the usefulness of an operation, we just include it, because there is not much harm in having too much operations; this is because we are not specifying any system to be built yet but we are just gathering a vocabulary for speaking about such a system.

A second guideline is that an operation is more likely to be useful when it has nice algebraic properties. When introducing a new operation, it is a good idea to consider the questions: does associativity hold? does commutativity hold? does idempotency hold? does a neutral element exist? etc. Note that it depends on the signature of an operation whether these questions are meaningful. Sometimes such a question is not meaningful but we can invent a suitable variation of it which does make sense. For the applicable properties we then investigate whether they hold or not.

In order to get started, we need specifications describing several data types of a mathematical nature. For each such data type there is one COLD-K class description. We do not start completely from the beginning, but instead we use data types such as `NAT_SPEC`, `CHAR_SPEC`, `SEQ_SPEC`, `TUPLE_SPEC`, `SET_SPEC`, `MAP_SPEC` etc. We adopt these from [7] and we only add a few simple constants such as `127` and `'a'`. Just for completeness, these class descriptions are given in Appendix B. This chapter is organised in such a way that when all formal COLD-K texts below are appended and Appendix B is put before that, we get a well-formed list of abbreviation-type components.

We start with the formal introduction of the sorts `Line` and `Text`. This is the subject of Section 4.3.2. Next, we introduce strings and we have a look at the relation between texts and strings. This is the subject of Sections 4.3.3 and 4.3.4. Furthermore we say a few words about printability, which is in Section 4.3.5.

After that, our systematic study of algebraic operations on texts begins. This study of algebraic operations covers the Sections 4.3.6 to 4.3.11.

4.3.2 Texts

We pay some attention to the question '*what is text?*'. It is tempting to say that a text is a sequence of characters, separated by new-line symbols. However, if you ask this question to someone without programming experience, he or she will probably tell you about characters on some two-dimensional medium like paper or a screen. Therefore it is more natural approach to say that a *text* is a sequence of lines where each line is a sequence of characters. This leads us to a formalisation of lines first.

We use SEQ_SPEC from Appendix B which is a parameterised description of sequences (sort Seq) with constructor operations empty and cons and also operations hd, tl, sel, cat, len, rev, bag for head, tail, selection, concatenation, length, reversal and bag construction respectively. Its parameter restriction only mentions a sort Item. LINE_SPEC is obtained as an instantiation of SEQ_SPEC where the Items are replaced by characters (sort Char) and where the resulting sequences are named Lines. The specification CHAR_SPEC provides among other things this sort Char. It is taken from Appendix B and it serves as an actual parameter here.

It is a peculiarity of COLD-K that when establishing such an instantiation the renaming takes place in the parameterised specification (SEQ_SPEC); the resulting renamed version of it is applied to the actual parameter (CHAR_SPEC).

```
LET LINE_SPEC :=
APPLY RENAME
        SORT Seq TO Line,
        SORT Item TO Char
IN SEQ_SPEC TO CHAR_SPEC;
```

In a similar way we introduce the sort Text as sequences of lines. It is important to realise that a text is something to be distinguished from the internal representation of a text inside a text editor.

```
LET TEXT_SPEC :=
APPLY RENAME
        SORT Seq          TO Text,
        SORT Item         TO Line,
        FUNC empty: -> Seq TO niltext
IN SEQ_SPEC TO LINE_SPEC;
```

Since we used an instantiation of SEQ_SPEC we have already the operations niltext, cons, hd, tl, sel, cat, len and rev. Note that niltext is the text having no lines at all.

We shall introduce some more operations now and they will be made part of a class description called TEXT_OPS1_SPEC. We want to add a remark about structuring our specification here. The purpose of TEXT_OPS1_SPEC and similar class descriptions to be introduced later is to *structure* our collection of sorts and operations. Each class description typically begins with a collection of imports, followed by a CLASS ... END type class description containing the newly introduced operations. The class descriptions serve as a grouping mechanism for sort definitions, function definitions, predicate definitions etc. Of course we aim at a *functional* grouping by which we mean that we put together those operations which are closely related with respect to their purpose and their technical contents.

```
LET TEXT_OPS1_SPEC :=
IMPORT NAT_SPEC  INTO
IMPORT CHAR_SPEC INTO
IMPORT LINE_SPEC INTO
IMPORT TEXT_SPEC INTO
CLASS
```

Sometimes is is desirable to consider non-niltext texts only; later we shall introduce a predicate ok on texts, such that ok(t) will imply that $t \neq$ niltext. It would be convenient to have a unique text with the intuition of 'empty text'. However, there are *two* candidates for this, viz. niltext and the text that consists of one empty line. Let us call the latter text zero.

We introduce some auxiliary functions, grouped into functions to construct texts (zero, addempty and addchar), and functions to take texts apart (first and rest). The function zero yields the text that consists of one empty line. The function addempty adds an empty line in front of a text. The function addchar adds one character at the beginning of the first line of a text. first and rest are unary functions on texts. The function first yields the first character of a text. The function rest yields a text that is the input text with its first character removed. Note that hd, tl and cons are overloaded in the sense that there are two operations called hd (viz. one one texts and one on lines) and similarly two tl and two cons operations. Later we shall come back to this overloading mechanism.

```
   FUNC zero: -> Text
   DEF  cons(empty,niltext)
```

```
FUNC addempty: Text -> Text
PAR  t:Text
DEF  cons(empty,t)

FUNC addchar: Char # Text -> Text
PAR  c:Char,t:Text
DEF  cons(cons(c,hd(t)),tl(t))

FUNC first:Text -> Char
PAR  t:Text
DEF  hd(hd(t))

FUNC rest: Text -> Text
PAR  t:Text
DEF  cons(tl(hd(t)),tl(t))
```

We have the obvious properties

$$\forall\, t : \mathtt{Text}, c : \mathtt{Char}\ (\ t \neq \mathtt{niltext} \Rightarrow \\ \mathtt{first}(\mathtt{addchar}(c,t)) = c \wedge \mathtt{rest}(\mathtt{addchar}(c,t)) = t\ )$$

and

$$\forall\, t : \mathtt{Text} \\ (\ \mathtt{hd}(\mathtt{addempty}(t)) = \mathtt{empty} \wedge \mathtt{tl}(\mathtt{addempty}(t)) = t\ )$$

where we took some obvious notational freedom with respect to COLD-K in the sense that we wrote $\forall$ rather than `FORALL`, $t \neq$ `niltext` instead of `NOT t = niltext` etc. This kind of syntactic sugar is formally not part of COLD-K and when syntax- and type-checking the formal texts of this chapter, we excluded the propositions which are based on this syntactic sugar. We only use this syntactic sugar when discussing properties of definitions and never to give the formal definitions themselves. Throughout this chapter the actual formal definitions are in COLD-K. These formal COLD-K texts are recognisable by their somewhat smaller font (`first` rather than `first`) and by the complete absence of mathematical type-setting (`=>` rather than $\Rightarrow$).

```
END;  {of TEXT_OPS1_SPEC}
```

As a next step we relate texts with *strings*, by which we mean sequences of characters. This is the subject of our next two sections.

4.3.3 Strings

We introduce the sort String as sequences of characters. Again we use SEQ_SPEC from Appendix B . In informal writing we freely denote strings using double quotes, e.g. "this is a string". We introduce the usual lexicographical ordering on strings as a predicate less. It is defined by recursion.

It is interesting to point out that the strictness principles of COLD-K are essential for this definition. In particular, consider the case where t = empty, then hd(t) is undefined and then by strictness so is ord(hd(t)). Therefore lss(ord(hd(s)),ord(hd(t))) is false and so is the equation hd(s) = hd(t). Therefore in that case less(s,t) is false.

```
LET STRING_SPEC :=
EXPORT

  SORT Char,
  SORT Nat,
  SORT String,
  FUNC empty:                    -> String,
  FUNC cons : Char # String      -> String,
  FUNC hd   : String             -> Char,
  FUNC tl   : String             -> String,
  FUNC len  : String             -> Nat,
  FUNC sel  : String # Nat       -> Char,
  FUNC cat  : String # String    -> String,
  FUNC rev  : String             -> String,
  PRED less : String # String

FROM

IMPORT APPLY RENAME
        SORT Seq  TO String,
        SORT Item TO Char
IN SEQ_SPEC TO CHAR_SPEC INTO

IMPORT NAT_SPEC  INTO
IMPORT CHAR_SPEC INTO
CLASS

  PRED less: String # String
  PAR  s:String,t:String
  DEF  s = empty AND NOT t = empty
       OR (lss(ord(hd(s)),ord(hd(t)))
      OR hd(s) = hd(t) AND less(tl(s),tl(t)))

END;
```

4.3.4 Relating Texts and Strings

Let us now discuss how texts and strings are related. By adopting some special separator character, control-j say, we can define a bijection between the sort String and the set of non-niltext texts not containing this special separator character. We call these texts *ok* and for this purpose we introduce a predicate ok. We formally introduce control-j as a constant ctr_j first and we use the operation chr: Nat -> Char from CHAR_SPEC for that.

```
LET TEXT_OPS2_SPEC :=
IMPORT NAT_SPEC        INTO
IMPORT CHAR_SPEC       INTO
IMPORT LINE_SPEC       INTO
IMPORT TEXT_SPEC       INTO
IMPORT TEXT_OPS1_SPEC INTO
IMPORT STRING_SPEC     INTO
CLASS

    FUNC ctr_j: -> Char
    DEF  chr(10)

    PRED ok: Text
    IND  FORALL c:Char,t:Text
         ( ok(zero:Text);
           ok(t) => ok(addempty(t));
           ok(t) AND NOT c = ctr_j => ok(addchar(c,t)) )
```

Note that ok$(t) \Rightarrow t \neq$niltext. We have the following induction principle for ok texts. To prove an assertion A for ok texts, it suffices to show A(zero) and *two* induction steps. The first induction step is $\forall t$: Text$(A(t) \Rightarrow A($addempty$(t))$. The second induction step is $\forall t$: Text$, c$: Char$(A(t) \land c \neq$ ctr_j $\Rightarrow A($addchar$(c, t))$.

We have the following decomposition principle for ok texts. Each ok text is of one of three possible forms, viz. zero, addempty(...) and addchar(...). We can distinguish these three possibilities also as {1} hd(t) = empty $\land$ tl(t) = niltext, {2} hd(t) = empty $\land$ tl$(t) \neq$ niltext and {3} hd$(t) \neq$ empty.

The functions text and string to be introduced below establish a bijection between the sort String and the set of ok texts. These functions text and string are similar to Meertens' Line and Unline functions [10]. Although it is possible to give inductive or axiomatic characterisations of these functions, we use recursive definitions. This has the advantage that it is easy to experiment with these definitions using any classical imperative or functional programming language. The function text converts a string to a text by interpreting ctr_j as a separator. The function string applied to a text t converts t to a string by putting a ctr_j between the lines. In the definition of the function string we distinguish three cases, where the cases {2} and {3} give rise to a recursive call of string. We easily verify that the guard {2} implies ok(tl(t)) and that the guard of {3} implies ok(rest(t)).

```
FUNC text: String -> Text
PAR  s:String
DEF  ( {1} s = empty ?;
           zero
     | {2} hd(s) = ctr_j ?;
           addempty(text(tl(s)))
     | {3} NOT (s = empty OR hd(s) = ctr_j) ?;
           addchar(hd(s),text(tl(s)))
     )

FUNC string: Text -> String
PAR  t:Text
DEF  ok(t) ?;
     ( {1} hd(t) = empty AND tl(t) = niltext ?;
           empty
     | {2} hd(t) = empty AND NOT tl(t) = niltext ?;
           cons(ctr_j,string(tl(t)))
     | {3} NOT hd(t) = empty ?;
           cons(first(t),string(rest(t)))
     )
```

We leave several options un-investigated here currently such as embedding Char $\setminus$ { ctr_j } into Char, or having a *litteral character* in the string representation.

4.3.5 Printability

Let us say a few words about the problems arising because in our formalisation texts may contain non-printable characters. There are two such problems. The first problem is obvious: non-printable characters cannot be printed or displayed as such by a display-oriented editor. Of course it is always possible to print some substitute, e.g. '~'.

The second problem is that inside a text editor some representation for texts must be chosen. If the text editor is only meant for manipulating ok texts, then its designer is in a comfortable position because he has the option of choosing the string-representation discussed before or some other related representation. We need not make such a choice here, but we introduce a suitable predicate denoting printability. We use the fact that in ASCII the printable characters are in the range ' '..'~'. We consider the TAB character as non-printable.

```
FUNC blank: -> Char
DEF  chr(32)

FUNC tilde: -> Char
DEF  chr(126)

PRED printable: Char
PAR  c:Char
DEF  leq(ord(blank),ord(c)) AND leq(ord(c),ord(tilde))
```

For later use we elaborate on the idea of replacing non-printables by some substitute, and in order to keep things simple, we choose `tilde` as a substitute indeed. Therefore we introduce by overloading three functions called `printify` with the intuition of 'make printable'.

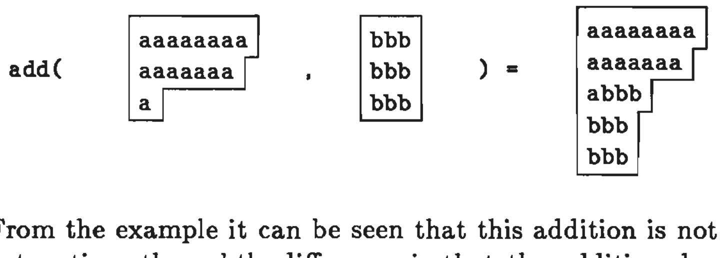
```
FUNC printify: Char -> Char
PAR  c:Char
DEF  ( printable(c)     ?; c
     | NOT printable(c) ?; tilde
     )

FUNC printify: Line -> Line
PAR  l:Line
DEF  ( l = empty     ?; empty
     | NOT l = empty ?; cons(printify(hd(l)),printify(tl(l)))
     )

FUNC printify: Text -> Text
PAR  t:Text
DEF  ( t = niltext     ?; niltext
     | NOT t = niltext ?; cons(printify(hd(t)),printify(tl(t)))
     )
```

We used the mechanism of overloading which means that we have several functions with the same identifier (`printify`) but with different argument types and/or result types. This is allowed in COLD-K and it has the advantage that we are not forced to invent new names ourselves (`printify_char`, `printify_line` and `printify_text`, say). When using overloaded functions or predicates, we must make sure that our expressions and assertions can be analysed in precisely one way. We give a simple example: `AXIOM printify(printify(niltext)) = niltext` is well-typed because there is only one way of analysing it, viz. by assuming that both `printify` invocations refer to the function `printify: Text -> Text`. Note that under this assumption the axiom becomes well-typed.

4.3.6 Natural Operations on Text

One of the best ways to get an understanding of the concept of text as formalised by the sort `Text`, is to look for algebraic operations operating on texts. In particular, we shall start with a simple binary operation of function-type `Text # Text` $\rightarrow$ `Text`. Of course it is not hard to define the insertion of a character into a text, which would be an operation of function-type `Text # Char # Nat`^{2} $\rightarrow$ `Text`, where we used `Nat`^{2} as an obvious shorthand for `Nat # Nat`. However, we avoid such heterogeneous operations, by which we mean those operations that deal both with texts and with characters. We must be prepared to let the domain `Nat`^{2} play a role when we define operations dealing with co-ordinate pairs.

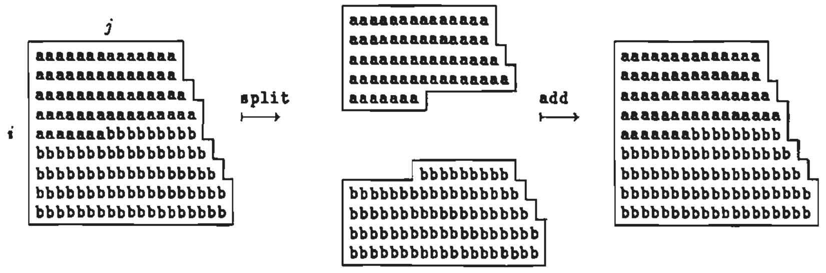
We shall introduce an operation called add below and we refer to it as *(natural) addition* because it can be viewed as a natural way of adding texts. Its definition is based on the idea that people are supposed to read texts line after line, scanning lines from left to right. To read the addition of two texts, one first reads the first text and after that, immediately proceeds with the second text. In Section 4.3.9 we shall also encounter other ways of adding texts for which we want to use names such as *vertical addition* and *horizontal addition*. Let us give an example showing the natural addition of two texts and postpone the formalisation for a moment.



From the example it can be seen that this addition is not the same as concatenation; the subtle difference is that the addition does not introduce a jump to a new line at the point where the two texts are joined.

Although the above addition operation seems too trivial for being interesting, it will serve as a kind of starting point for finding other operations which can be employed in the description of cut and paste capabilities for an editor. An interesting question which poses itself immediately is whether add has an *inverse*. Since addition is not injective there can not be an inverse in the strict sense, but we can propose an operation whose intuition is related to the idea of 'undoing the effect of an addition'. This operation will be called `split`.

The operation `split` describes the splitting of a text into two parts, where the splitting-point is just before some given position. We add one remark about the use of co-ordinates. Whenever we use pairs (i, j) where i and j are co-ordinates – or similar quantities – it is understood that i is the *vertical* co-ordinate and j is the *horizontal* co-ordinate. The following picture may convey some intuition for both `split` and `add`.



The fact that an arbitrary co-ordinate pair may indicate a position that is non-existing in the given text has as consequences that `split` is a partial operation and that the result of `split` need not be a pair of ok texts. Also add is a partial operation, which is the effect of an explicit guard testing the ok-ness of the input texts. We give the formal definitions below using recursion.

```
FUNC split: Text # Nat # Nat -> Text # Text
PAR  t:Text, i:Nat, j:Nat
DEF  ( {1} i = 0 AND j = 0 ?;
           zero, t
     | {2} i = 0 AND NOT j = 0 ?
           LET t1:Text,t2:Text;
           t1,t2 := split(rest(t),0,pred(j));
           addchar(first(t),t1), t2
```

```
        | {3} NOT i=0 ?;
              LET t1:Text,t2:Text;
              t1,t2 := split(tl(t),pred(i),j);
              cons(hd(t),t1), t2
        )

   FUNC add: Text # Text -> Text
   PAR  t1:Text, t2:Text
   DEF  ok(t1) AND ok(t2) ?;
        ( {1} hd(t1) = empty AND tl(t1) = niltext ?;
              t2
        | {2} hd(t1) = empty AND NOT tl(t1) = niltext ?;
              addempty(add(tl(t1),t2))
        | {3} NOT hd(t1) = empty ?;
              addchar(first(t1),add(rest(t1),t2))
        )
```

We state several properties of these operations. Addition of ok texts is associative, i.e.

$$\begin{array}{l}\forall\, t_1, t_2, t_3 : \mathtt{Text}\ \ (\ \mathsf{ok}(t_1) \wedge \mathsf{ok}(t_2) \wedge \mathsf{ok}(t_3) \Rightarrow \\ \quad \mathsf{add}(t_1, \mathsf{add}(t_2, t_3)) = \mathsf{add}(\mathsf{add}(t_1, t_2), t_3)\ )\end{array}$$

which can be shown by induction on t_1, using the induction principle of Section 4.3.4. Now we can also explain why we used the name **zero** for the text with one empty line: it behaves both as a left and a right neutral element with respect to the operation add:

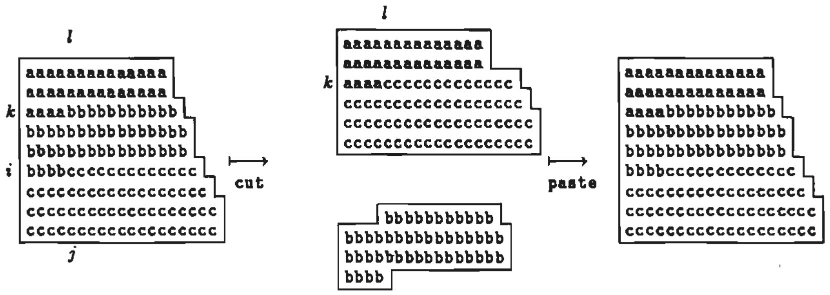
$$\begin{array}{l}\forall\, t : \mathtt{Text}\ \ (\ \mathsf{ok}(t) \Rightarrow \\ \quad \mathsf{add}(t, \mathtt{zero}) = t \wedge \mathsf{add}(\mathtt{zero}, t) = t\ )\end{array}$$

where the first equality is a consequence of the induction principle of Section 4.3.4 and where the second equality holds just by definition. The function add is the inverse of `split` in the following sense:

$$\begin{array}{l}\forall\, t : \mathtt{Text},\ c : \mathtt{Nat}^2\ \ (\ \mathsf{ok}(t) \wedge \mathtt{split}(t, c)! \Rightarrow \\ \quad \mathsf{add}(\mathtt{split}(t, c)) = t\ )\end{array}$$

which can be shown by induction on the vertical co-ordinate of the splitting point. The basis of the induction is $c = (0, j)$ which at its turn is shown by induction on j.

Next, we describe two operations which are somewhat more complicated, but which can be defined easily in terms of the operations add and `split`. These operations are called cut and paste and they can be viewed as a complementary pair of operations – just like **split** and **add**. The operation **cut** deletes a text-region from a given position (inclusive) unto another given position (exclusive) if possible. It yields a pair (t_1, t_2) where t_1 = 'remaining text' and t_2 = 'deleted text'. The operation **paste** takes a text t and inserts another text u into it immediately before the position with given co-ordinates (k, l) say, if possible. The following picture may convey some intuition for both **cut** and **paste**.



Formally we define **cut** and **paste** by using the simpler operations **split** and **add**. These **split** and **paste** are partial operations and as a consequence, so are **cut** and **paste**.

```
FUNC cut: Text # Nat # Nat # Nat # Nat -> Text # Text
PAR  t:Text, n:Nat, m:Nat, i:Nat, j:Nat
DEF  LET t1: Text,t2: Text; t1,t2   := split(t,i,j);
     LET t11:Text,t12:Text; t11,t12 := split(t1,n,m);
     add(t11,t2), t12

FUNC paste: Text # Text # Nat # Nat -> Text
PAR  t:Text,u:Text,k:Nat,l:Nat
DEF  LET t1: Text,t2: Text; t1,t2 := split(t,k,l);
     add(t1,add(u,t2))
```

For appropriate choices of the $\mathtt{Nat}^2$ argument, the **zero** text behaves also as a left and a right neutral element with respect to the operation **paste**.

$$\forall\, t : \mathtt{Text},\ c : \mathtt{Nat}^2\ \ (\, \mathrm{ok}(t) \wedge \mathtt{paste}(t, \mathtt{zero}, c)! \Rightarrow \\ \mathtt{paste}(t, \mathtt{zero}, c) = t\,)$$

which holds because **zero** is the left neutral element with respect to **add** and **add** is the inverse of **split**.

$$\forall\, t : \mathtt{Text},\ c : \mathtt{Nat}^2\ \ (\, \mathrm{ok}(t) \wedge c = (0,0)\ \Rightarrow \\ \mathtt{paste}(\mathtt{zero}, t, c) = t\,)$$

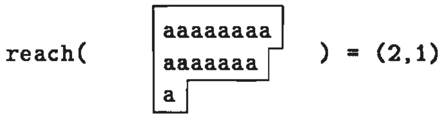
The latter proposition follows from the fact that **zero** is the left- and right neutral element with respect to add. The operation **paste** is the inverse of cut in the following sense:

$$\forall\, t : \texttt{Text},\ c, d : \texttt{Nat}^2\ \ (\, \texttt{ok}(t) \wedge \texttt{cut}(t, c, d)!\ \Rightarrow \\ \quad \texttt{paste}(\texttt{cut}(t, c, d), c) = t\ )$$

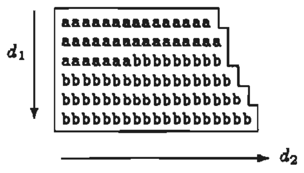
which follows from the fact that add is the inverse of `split` and furthermore from the property $\texttt{split}(t_1, c) = (t_{11}, \ldots) \Rightarrow \texttt{split}(\texttt{add}(t_{11}, t_2), c) = (t_{11}, t_2)$, which at its turn is shown by induction on the vertical and horizontal co-ordinates of the splitting point – just as before. In a certain way, `split` is also the inverse of add and cut is the inverse of **paste**, but before we can state this precisely, we first need another concept, viz. that of the *reach* of a text. This is the subject of our next section.

4.3.7 The Reach of a Text

The function reach applied to a text t results the pair (d_1, d_2) where d_1 is the co-ordinate of the last line of t and where d_2 is the number of characters in the last line of t. We give an example:



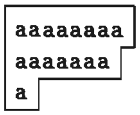
We call such a pair(d_1, d_2) a reach. We have chosen the term *reach* because of the intuition that reach(t) indicates 'how far you can get', starting at position $(0, 0)$ and trying to get as far downwards and rightwards as possible within this text t. In a certain sense reach(t) can be viewed as a 'size' of t. Note that d_1 and d_2 play a somewhat different role: d_1 is the co-ordinate of the last line whereas d_2 is the co-ordinate of the last character in the last line *plus one*. We sketch the intuition behind reach by a picture:



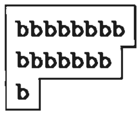
We give the formal definition using recursion below. There are three cases marked as `1'`, `2'` and `3'` to indicate that we employ a decomposition principle for texts different from that of Section 4.3.4.

```
FUNC reach: Text -> Nat # Nat
PAR  t:Text
DEF  ok(t) ?;
     ( {1'} tl(t) = niltext AND hd(t) = empty ?;
       0, 0
     | {2'} tl(t) = niltext AND NOT hd(t) = empty ?;
       LET d1:Nat,d2:Nat; d1,d2 := reach(rest(t)); (d1,succ(d2))
     | {3'} NOT tl(t) = niltext ?;
       LET d1:Nat,d2:Nat; d1,d2 := reach(tl(t));   (succ(d1),d2)
     )
```

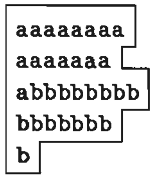
Note that reach(zero) = (0,0). The function add defined below can be viewed as the 'addition' of two reaches. Sometimes we refer to it as *natural addition*. Let us give an example first. We show two texts and their reaches, and after that we determine the reach of their addition simply by adding the texts and applying the reach operator.

let t1 = (

), so reach(t1) = (2,1)

let t2 = (

), so reach(t2) = (2,1)

add(t1,t2) = (

), so reach(add(t1,t2)) = (4,1)

The interesting observation now is that we could have derived the value of reach(add(t1,t2)) also from the reaches of t1 and t2 alone. This is precisely the purpose of having the function add on reaches.

```
FUNC add: Nat # Nat # Nat # Nat -> Nat # Nat
PAR  c1:Nat, c2:Nat, d1:Nat, d2:Nat
DEF  ( d1 = 0     ?; c1, add(c2,d2)
     | NOT d1 = 0 ?; add(c1,d1), d2
     )
```

Let us enter the data of the above example: $\mathtt{add}((2,1),(2,1)) = (4,1)$ indeed. Addition on reaches is associative and the pair $(0,0)$ behaves as a left- and right neutral element with respect to add as can be shown by a case-analysis using the definition of add. The following proposition confirms our intuition as sketched by the above example.

$$\forall\, t,u : \mathtt{Text} \;\; (\, \mathtt{ok}(t) \wedge \mathtt{ok}(u) \Rightarrow \\ \mathtt{reach}(\mathtt{add}(t,u)) = \mathtt{add}(\mathtt{reach}(t),\mathtt{reach}(u))\,)$$

This can be shown by induction on t using the induction principle of Section 4.3.4 and where for the step $t = \mathtt{addchar}(c,t')$ one distinguishes two cases: either $\mathtt{len}(t) = \mathtt{len}(u) = 1$, so c contributes to the reach of the added texts, or $\mathtt{len}(t) + \mathtt{len}(u) > 2$, in which case c does not contribute to it. Now we have the `reach` function, we can also formulate the proposition that in a certain sense `split` is the inverse of add.

$$\forall\, t,u : \mathtt{Text} \;\; (\, \mathtt{ok}(t) \wedge \mathtt{ok}(u) \Rightarrow \\ \mathtt{split}(\mathtt{add}(t,u),\mathtt{reach}(t)) = (t,u)\,)$$

which can be shown by induction on t using the induction principle of Section 4.3.4. We can also perform splitting on reaches.

```
FUNC split: Nat # Nat # Nat # Nat -> Nat # Nat # Nat # Nat
PAR  r:Nat, s:Nat, i:Nat, j:Nat
DEF  (i,j) , (sub(r,i), (lss(i,r)?; s | i = r ?; sub(s,j)))
```

This function `split` is useful for calculating the reaches of the texts obtained by splitting a text with a given reach when also the splitting-point is given.

$$\forall\, t,t_1,t_2 : \mathtt{Text},\; c : \mathtt{Nat}^2 \\ (\, \mathtt{ok}(t) \wedge \mathtt{ok}(t_1) \wedge \mathtt{ok}(t_2) \wedge \mathtt{split}(t,c) = (t_1,t_2) \Rightarrow \\ \mathtt{split}(\mathtt{reach}(t),c) = (\mathtt{reach}(t_1),\mathtt{reach}(t_2))\,)$$

which can be shown by referring to the definition of `split` on reaches and to the fact that $\mathtt{split}(t,c)$ yields the two texts t_1, t_2 that are obtained by splitting t with the splitting point at c. Hence the reach of t_1 equals c and we can get the reach of t_2 by a simple case-analysis. In any case, the vertical co-ordinate of t_2 is the difference of the vertical co-ordinates of $\mathtt{reach}(t)$ and c. If the splitting point is *before* the last line of t, then the length of the last line of t_2 is simply the length of the last line of t. If the splitting point is *in* the last line, then the horizontal co-ordinate of t_2 is the length of this last line minus the horizontal co-ordinate of the splitting point. We can also perform a kind of `cut` and `paste` on reaches.

```
FUNC cut: Nat # Nat # Nat # Nat # Nat # Nat
     -> Nat # Nat # Nat # Nat
PAR  t:Nat # Nat, n:Nat, m:Nat, i:Nat, j:Nat
DEF  LET t1: Nat # Nat,t2: Nat # Nat; t1, t2  := split(t,i,j);
     LET t11:Nat # Nat,t12:Nat # Nat; t11,t12 := split(t1,n,m);
     add(t11,t2) , t12

FUNC paste: Nat # Nat # Nat # Nat # Nat # Nat -> Nat # Nat
PAR  t:Nat # Nat,u:Nat # Nat,k:Nat,l:Nat
DEF  LET t1: Nat # Nat,t2: Nat # Nat; t1,t2 := split(t,k,l);
     add(t1,add(u,t2))
```

This function cut is useful for calculating the reaches of the texts obtained by cutting a text with a given reach.

$$\begin{array}{l}\forall\, t, t_1, t_2 : \texttt{Text},\ c, d : \texttt{Nat}^2 \\ \ (\ \text{ok}(t) \wedge \text{ok}(t_1) \wedge \text{ok}(t_2) \wedge \texttt{cut}(t, c, d) = (t_1, t_2) \Rightarrow \\ \quad \texttt{cut}(\texttt{reach}(t), c, d) = (\texttt{reach}(t_1), \texttt{reach}(t_2))\ )\end{array}$$

which follows from the fact that add and split on texts commute via reach with add and split on reaches, using the obvious similarity in both definitions of cut. In the same way one can show the following proposition, which expresses that the latter function paste can be used for calculating the reach of the text obtained by pasting two texts with given reaches.

$$\begin{array}{l}\forall\, t, u : \texttt{Text},\ c : \texttt{Nat}^2\ \ (\ \text{ok}(t) \wedge \text{ok}(u) \Rightarrow \\ \quad \texttt{reach}(\texttt{paste}(t, u, c)) = \texttt{paste}(\texttt{reach}(t), \texttt{reach}(u), c)\ )\end{array}$$

We can summarize some of the above propositions by saying that reach: Text $\rightarrow$ Nat^2 is a homomorphic mapping from the algebra of texts to the algebra of reaches. So, we can view the algebra of reaches as a simplified version of the algebra of texts. In general, reach(t) does not contain enough information for reconstructing t, but still it tells us something about the behaviour of t under the application of split, add, cut and paste operations.

Let us formulate that in a certain sense cut is the inverse of paste.

$$\begin{array}{l}\forall\, t, u : \texttt{Text},\ c : \texttt{Nat}^2\ \ (\ \text{ok}(t) \wedge \text{ok}(u) \wedge \texttt{paste}(t, u, c)!\ \Rightarrow \\ \quad \texttt{cut}(\texttt{paste}(t, u, c), c, \texttt{add}(c, \texttt{reach}(u))) = (t, u)\ )\end{array}$$

which follows from the definitions of cut and paste and from the homomorphism property of reach. Sometimes it is necessary to compare two reaches, so we introduce predicates lss: Nat^2 # Nat^2 and leq: Nat^2 # Nat^2.

```
   PRED lss: Nat # Nat # Nat # Nat
   PAR  i1:Nat, i2:Nat, j1:Nat, j2:Nat
   DEF  lss(i1,j1) OR (i1 = j1 AND lss(i2,j2))

   PRED leq: Nat # Nat # Nat # Nat
   PAR  i1:Nat,i2:Nat,j1:Nat,j2:Nat
   DEF  lss(i1,j1) OR ( i1 = j1 AND leq(i2,j2) )
```

Notice that a position in a given text can be described precisely by the reach of the text *preceding* this position; hence reaches can also be viewed as *position indicators*.
So if lss$((c_1, c_2), (d_1, d_2))$, then this can be interpreted as 'the position (c_1, c_2) comes *before* the position (d_1, d_2)'.

```
END; {of TEXT_OPS2_SPEC}
```

Thus TEXT_OPS2_SPEC adds to the notions of texts and strings of Sections 4.3.2 and 4.3.3 several conversion operations (Sections 4.3.4 and 4.3.5), a range of adding and splitting operations (Section 4.3.6) and their images under the operation reach (Section 4.3.7).

4.3.8 The Profile of a Text

When we use co-ordinate pairs for indicating positions in a text – as we do in split, cut and paste – then we have to be careful. The point is that our co-ordinate pairs are based on natural numbers and thus may indicate positions that are non-existing in a given text. We develop some machinery so that later, in the context of a text editor, we can deal with these problems formally. This will happen in Section 4.5.5, when we shall postulate a so-called text-invariant. If we have a text t, then we might collect all information that is relevant for 'addressing' in t. For this purpose we introduce so-called *profiles*. The profile of a text t is a sequence of natural numbers, one for each line of t such that the i-th element in the sequence is precisely equal to the length of the i-th line in t. We start with an example and then do the formalisation.

```
                 aaaaaaaa
      profile(   aaaaaaa     ) = <8,7,1>
                 a
```

Now we formally introduce a sort Profile. An object of sort Profile is nothing but a sequence (Seq) of natural numbers (Nat).

```
LET PROFILE :=
APPLY RENAME
      SORT Seq            TO Profile,
      SORT Item           TO Nat,
      FUNC empty: -> Seq TO nilprofile
IN SEQ_SPEC TO NAT_SPEC;
```

And of course we must define the profile corresponding with a given text, which we do by introducing a function `profile`. Therefore we write another class description called `PROFILE_SPEC` which imports a number of class descriptions introduced before. We want to add a remark about such imports here. In general, when writing specifications, we need not adopt some minimality principle. Instead of that we begin each new class description with a relatively rich collection of imports – so that we have enough notations at our disposal. There is no harm in importing too much, whereas it is a nuisance if the imports turn out to be insufficient.

```
LET PROFILE_SPEC :=
IMPORT PROFILE        INTO
IMPORT NAT_SPEC       INTO
IMPORT CHAR_SPEC      INTO
IMPORT LINE_SPEC      INTO
IMPORT TEXT_SPEC      INTO
IMPORT TEXT_OPS1_SPEC INTO
IMPORT TEXT_OPS2_SPEC INTO
CLASS

   FUNC profile: Text -> Profile
   PAR  t:Text
   DEF  ok(t) ?;
        ( {1} hd(t) = empty AND tl(t) = niltext ?;
              cons(0,nilprofile)
        | {2} hd(t) = empty AND NOT tl(t) = niltext ?;
              cons(0,profile(tl(t)))
        | {3} NOT hd(t) = empty ?;
              LET p:Profile; p := profile(rest(t));
              cons(succ(hd(p)),tl(p))
        )

END;
```

It is possible to define a kind of addition called add and operations like `split`, `cut` and `paste` on profiles. For suitable choice of these operations, `profile`: Text $\rightarrow$ Nat* becomes a homomorphic mapping from the algebra of texts to the algebra of profiles. Furthermore one can define a function reach: $\mathtt{Nat}^* \rightarrow \mathtt{Nat}^2$ such that it is a homomorphic mapping from the algebra of profiles to the algebra of reaches. The following picture sketches this situation.

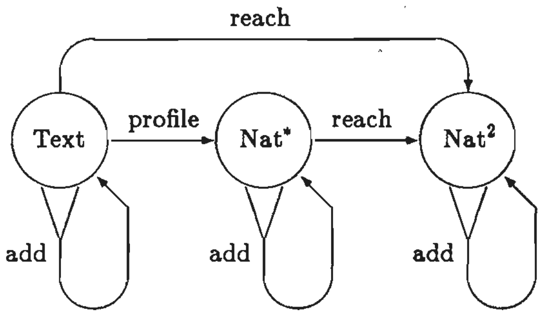


Fig 4.1. The algebras of texts, profiles and reaches.

So, we can view the algebra of profiles as being intermediate between the algebra of texts and the algebra of reaches. We leave these things for later investigation, because we do not need them really yet.

```
LET TEXT_OPS3_SPEC :=
IMPORT NAT_SPEC        INTO
IMPORT CHAR_SPEC       INTO
IMPORT LINE_SPEC       INTO
IMPORT TEXT_SPEC       INTO
IMPORT TEXT_OPS1_SPEC  INTO
IMPORT TEXT_OPS2_SPEC  INTO
IMPORT STRING_SPEC     INTO
IMPORT PROFILE_SPEC    INTO
CLASS
```

We define a predicate called **intext**, thereby showing a typical application of profiles. We use the standard selection function **sel**. If we have a given text t and a co-ordinate pair (i, j) then intext(t, i, j) holds if either (i, j) indicates a position that exists in t or (i, j) corresponds with the very end of a line – which is an acceptable position for splitting a text into parts, and therefore for pasting by the function **paste**.

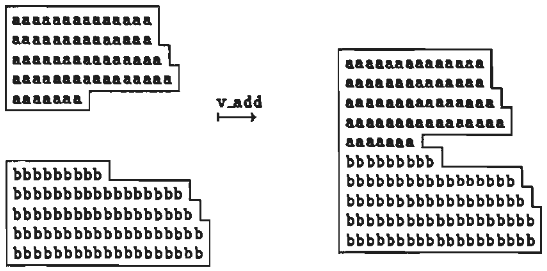
```
   PRED intext: Text # Nat # Nat
   PAR  t:Text, i:Nat, j:Nat
   DEF  LET m:Nat,n:Nat; m,n := reach(t);
        leq(i,m) AND leq(j,sel(profile(t),i))
```

In particular, we have `intext(zero,(0,0))`.

4.3.9 Vertical and Horizontal Composition of Text

In this section we introduce two ways of adding texts for which we want to use the names *vertical addition* and *horizontal addition*. These operations become important when one wants to describe multi-window capabilities of a text editor. Although multi-window capabilities are not included in the text editor described in this chapter, we believe that it is instructive to have a look at the necessary operations at an algebraic level.

We introduce an operation called v_add below and we refer to it as *vertical addition*. Actually there is nothing new here since vertical addition is nothing but *concatenation*.



We give the formal definition below.

```
FUNC v_add: Text # Text -> Text
PAR  t1:Text, t2:Text
DEF  cat(t1,t2)
```

We state a few well-known properties. Vertical addition is associative.

$$\forall\, t_1, t_2, t_3 : \texttt{Text} \quad (\ \texttt{v\_add}(t_1,\texttt{v\_add}(t_2, t_3)) = \texttt{v\_add}(\texttt{v\_add}(t_1, t_2), t_3)\ )$$

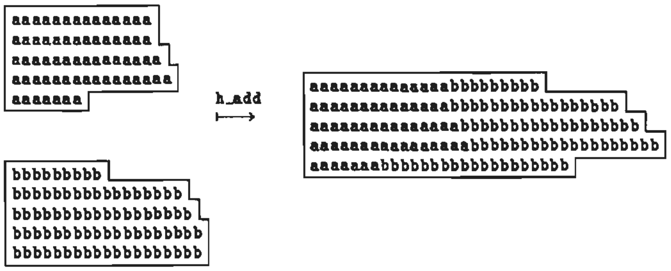
niltext behaves both as a left and a right neutral element with respect to the operation v_add.

$$\forall\, t : \texttt{Text} \quad (\ \texttt{v\_add}(t,\texttt{niltext}) = t \wedge \texttt{v\_add}(\texttt{niltext}, t) = t\ )$$

The len function from texts to natural numbers behaves homomorphically as stated below.

$$\forall\, t_1, t_2 : \texttt{Text} \quad (\ \texttt{len}(\texttt{v\_add}(t_1, t_2)) = \texttt{add}(\texttt{len}(t_1), \texttt{len}(t2))\ )$$

We introduce another operation called **h_add** below and we refer to it as *horizontal addition.*



We give the formal definition below.

```
FUNC h_add: Text # Text -> Text
PAR  t1:Text, t2:Text
DEF  ( t1 = niltext AND t2 = niltext       ?; niltext
     | NOT (t1 = niltext AND t2 = niltext) ?;
       cons(cat(hd(t1),hd(t2)),h_add(tl(t1),tl(t2)))
     )
```

From this definition it can be seen that when two texts have different lengths, their horizontal addition is undefined. Horizontal addition is associative.

$$\forall\, t_1, t_2, t_3 : \texttt{Text}\ (\ \texttt{len}(t_1) = \texttt{len}(t_2) = \texttt{len}(t_3) \Rightarrow \\ \texttt{h\_add}(t_1, \texttt{h\_add}(t_2, t_3)) = \texttt{h\_add}(\texttt{h\_add}(t_1, t_2), t_3)\ )$$

which follows by induction on the length of the texts involved and using the well-known associativity of line-concatenation. For each length l, there is a left- and right neutral element and we introduce the notation empties(l) for it:

```
FUNC empties: Nat -> Text
PAR  n:Nat
DEF  ( n = 0?;     niltext
     | NOT n = 0?; cons(empty,empties(pred(n)))
     )
```

so we can denote this property as follows:

$$\forall\, t : \texttt{Text}, l : \texttt{Nat} \ (\texttt{len}(t) = l \Rightarrow \\ \texttt{h\_add}(t,\texttt{empties}(l)) = t \wedge \texttt{h\_add}(\texttt{empties}(l), t) = t \ )$$

which follows again by induction on l, using the fact that **empty:Line** is neutral with respect to concatenation. The set of texts of a given length is closed under horizontal addition:

$$\forall\, t_1, t_2 : \texttt{Text}, l : \texttt{Nat} \ (\texttt{len}(t_1) = \texttt{len}(t_2) = l \Rightarrow \\ \texttt{len}(\texttt{h\_add}(t_1, t_2)) = l \ )$$

which follows again by induction on l. Horizontal and vertical addition are related by the interchange law. We have met this interchange law before at another level, viz. at the level of composition mechanisms for designs. We refer to Chapter 3 lemma 3.3.11 and remark 3.3.12

$$\forall\, t_1, t_2, t_3, t_4 : \texttt{Text} \ (\texttt{len}(t_1) = \texttt{len}(t_2) \ \wedge \ \texttt{len}(t_3) = \texttt{len}(t_4) \Rightarrow \\ \texttt{v\_add}(\texttt{h\_add}(t_1, t_2),\texttt{h\_add}(t_3, t_4)) \\ = \texttt{h\_add}(\texttt{v\_add}(t_1, t_3),\texttt{v\_add}(t_2, t_4)) \ )$$

which follows by induction on $\texttt{len}(t_1)$.

4.3.10 Operations for Searching

In this section we do some preparations such that later we can specify the search-command of an editor. The formalisation of searching is done in terms of strings, which is most natural. In typical editing sessions one searches for a word or for a short term. However, there is no clear intuition for searching an entire sentence or an entire formula that extends over more than one line. So we describe searching such that it becomes conceptually clear for ctr_j-less strings. We adopt the same formalisation then for all strings. After that, we tranfer the formalisation by homomorphic mappings to the algebra of texts. We introduce a predicate match such that $\texttt{match}(t, s, i)$ holds in the situation where in string t there is an occurrence of a substring s at position i.

```
PRED match: String # String # Nat
PAR  t:String, s:String, i:Nat
DEF  s = empty OR
     hd(s) = sel(t,i) AND match(t,tl(s),succ(i))
```

By way of preparation for a kind of sentinel technique, we introduce the following slightly modified version of of match, which we call match'.

```
PRED match': String # String # Nat
PAR  t:String, s:String, i:Nat
DEF  i = len(t) OR match(t,s,i)
```

Now we define the result of a search operation as the least position at which a match – in the sense of match' – occurs.

```
FUNC search: String # String -> Nat
PAR  t:String, s:String
DEF  SOME i:Nat
     ( match'(t,s,i) AND
       FORALL j:Nat ( match'(t,s,j) => leq(i,j) ) )
```

Note that if in t there is no occurrence of s, then search(t,s) 'falls through' in the sense that it yields len(t), which is the sentinel technique announced before. Now we transfer the above formalisation to the algebra of texts by employing the mappings string and text.

```
FUNC search: Text # Text -> Nat # Nat
PAR  t:Text, s:Text
DEF  LET t1:String, t2:String;
     t1,t2 := split(string(t),search(string(t),string(s)));
     reach(text(t1))
```

where we used an obvious split operation on strings.

```
FUNC split: String # Nat -> String # String
PAR  s:String, i:Nat
DEF  ( i = 0     ?;
       empty, s
     | NOT i = 0 ?;
       LET t1:String, t2:String; t1,t2 := split(tl(s),pred(i));
       cons(hd(s),t1), t2
     )
```

The approach followed here could be condidered as a part of our methodology of writing formal specifications: choose among the different possible representations the one which yields the simplest and most elegant definitions. If the remainder of the specification is in terms of another representation, just use suitable mappings to transfer the results from one representation to another.

```
END; {of TEXT_OPS3_SPEC}
```

Thus, TEXT_OPS3_SPEC adds to TEXT_OPS2_SPEC some operations for profiling texts (Section 4.3.8), for vertical and horizontal composition of texts (Section 4.3.9) and for searching (Section 4.3.10).

4.3.11 Procrustean Operations

In this section we shall introduce some machinery for reducing the size of texts that are *too large* for some purpose and also some machinery for stretching texts that are *too small*. However, we not stick to this Procrustean terminology and later in this section we introduce operations called `look` and `fill` where the names of the operations have been derived from their typical applications.

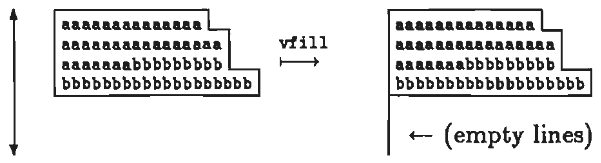
One of the most important problems associated with the specification and design of an editor is the fact that in general the text being edited is too large to fit on the screen of the display. The text itself may be too long and some of the lines may be too wide. One – currently rather obsolete – solution is to make the editor line-oriented. This solves the problem of the text being too long and then the problem of the lines being too wide can be solved by e.g. wrap-around. A better solution is adopted in the so-called display-oriented editors where the screen contains one or more rectangular *windows*. These windows can be used for 'looking' to the text(s) being edited. In order to keep things simple, we restrict ourselves to the case where there is just *one* window. This window corresponds with a 'subtext' of the text being edited. The cursor on the screen corresponds with the 'current position' (= dot) in the text being edited. The editor maintains a kind of 'window-invariant' viz. that always this dot and a suitable subtext surrounding the dot are visible in the window.

A second important problem is that the text may be shorter than the vertical size of the window; also some of the lines in the text may be shorter than the horizontal size of the window. The obvious solution is *filling*, by which we mean the positions within the window for which there is no character in the text are displayed as blank.

In this chapter we choose to discuss a display-oriented editor, but as before, we postpone the description of the actual editor and we develop some machinery for formulating the ideas of *looking* and *filling* as mentioned above. This machinery takes the shape of a collection of algebraic operations on texts. We start with *filling* and we put everything dealing with filling in a class description called FILL_SPEC. Again we begin with an ample collection of imports.

```
LET FILL_SPEC :=
IMPORT NAT_SPEC       INTO
IMPORT CHAR_SPEC       INTO
IMPORT LINE_SPEC       INTO
IMPORT TEXT_SPEC       INTO
IMPORT TEXT_OPS1_SPEC INTO
IMPORT TEXT_OPS2_SPEC INTO
IMPORT TEXT_OPS3_SPEC INTO
IMPORT STRING_SPEC     INTO
IMPORT PROFILE_SPEC    INTO

CLASS
```

We have to provide for filling in both vertical and horizontal directions. This observation leads us directly to two functions `vfill` and `hfill`. The function `vfill` serves for filling in the vertical direction. Somehow it seems natural to choose our definitions such that the text gets 'upwards adjusted', so that filling takes place at the *end* of the text.



We give the formal definition of `vfill` below. We can use the function `empties` which we introduced in Section 4.3.9. Recall that `empties`(l) is the text having precisely l empty lines.

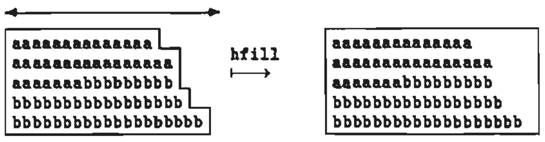
```
   FUNC vfill: Text # Nat -> Text
   PAR  t:Text, n:Nat
   DEF  LET i:Nat; i := len(t);
        LET te:Text;
        te := empties( geq(n,i)?; sub(n,i) | lss(n,i)?; 0 );
        cat(t,te)
```

The function `hfill` serves for filling in the horizontal direction. As an auxiliary for the description of `hfill` we have a procedure which describes the construction of lines consisting of blanks only. We refer to the constant `blank` for the ' ' character which we introduced earlier.

```
   FUNC blanks: Nat -> Line
   PAR  j:Nat
```

```
DEF  ( j = 0 ?;     empty
     | NOT j = 0 ?; cons(blank,blanks(pred(j)))
     )
```

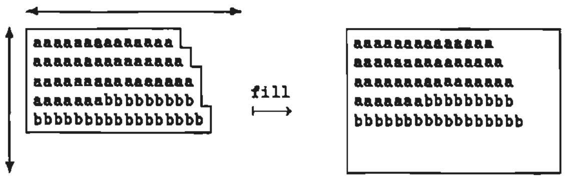
Somehow it seems natural to choose our definitions such that the text gets 'left adjusted', so that filling takes place at the *end* of the lines.



We give the formal definition of `hfill` below.

```
FUNC hfill: Text # Nat -> Text
PAR  t:Text, n:Nat
DEF  ( t = niltext ?;
       niltext
     | NOT t = niltext ?;
       LET l:Line; l := hd(t);
       LET i:Nat;  i := len(l);
       LET j:Nat;  j := ( geq(n,i)?; sub(n,i) | lss(n,i)?; 0 );
       cons(cat(l,blanks(j)),hfill(tl(t),n))
     )
```

And we combine these to get a function `fill` which does both vertical and horizontal filling.



Note that `vfill` must be applied first, so that a subsequent application of `hfill` will make sure that the empties introduced by `vfill` get the desired length. It just does not work the other way around. We give the formal definition of `fill` below.

```
   FUNC fill: Text # Nat # Nat -> Text
   PAR  t:Text, i:Nat, j:Nat
   DEF  hfill(vfill(t,i),j)

END; {of FILL_SPEC}
```

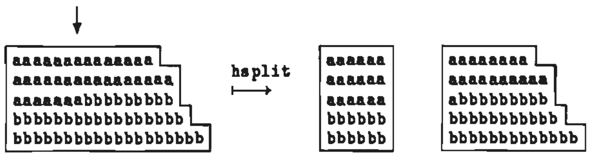
This completes our formalisation of filling and as a next step we introduce some machinery for the description of *looking*. As stated before, the application we have in mind is to describe how a window is used for 'looking' to a text. We put everything related to looking in a class description called LOOK_SPEC.

```
LET LOOK_SPEC :=
IMPORT NAT_SPEC        INTO
IMPORT CHAR_SPEC       INTO
IMPORT LINE_SPEC       INTO
IMPORT TEXT_SPEC       INTO
IMPORT TEXT_OPS1_SPEC INTO
IMPORT TEXT_OPS2_SPEC INTO
IMPORT TEXT_OPS3_SPEC INTO
IMPORT STRING_SPEC     INTO
IMPORT PROFILE_SPEC    INTO
CLASS
```

To begin with, we have again an auxiliary. It describes the splitting at line-level.

```
   FUNC hsplit: Line # Nat -> Line # Line
   PAR  s:Line, n:Nat
   DEF  ( (n = 0)  ?;
          (empty,s)
        | NOT n = 0?;
          LET s1:Line, s2:Line; s1,s2 := hsplit(tl(s),pred(n));
          (cons(hd(s),s1),s2)
        )
```

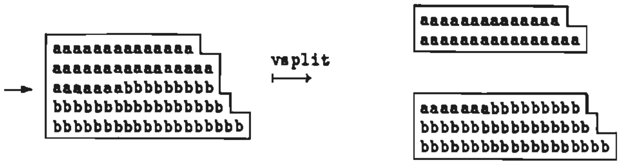
The way we define the algebraic operations for 'looking' is somewhat similar to the way we defined the cut operation before. There it turned out that it was convenient to define the more elementary split operation first. We have to provide for splitting both in horizontal and vertical direction. These considerations lead us directly to two functions hsplit and vsplit. The function hsplit serves for splitting in the horizontal direction.



We give the formal definition of hsplit below. We use recursion for propagating the effect of hsplit:Line → Line^2 over an entire text.

```
FUNC hsplit: Text # Nat -> Text # Text
PAR  t:Text, n:Nat
DEF  ( t = niltext ?; niltext, niltext
     | NOT t = niltext ?;
       LET l1:Line,l2:Line; l1,l2 := hsplit(hd(t),n);
       LET t1:Text,t2:Text; t1,t2 := hsplit(tl(t),n);
       cons(l1,t1), cons(l2,t2)
     )
```

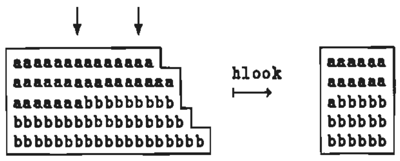
Note that h_add is the inverse of hsplit. The function vsplit serves for splitting in the vertical direction.



We give the formal definition of vsplit below.

```
FUNC vsplit: Text # Nat -> Text # Text
PAR  t:Text, n:Nat
DEF  ( n = 0 ?;      niltext, t
     | NOT n = 0 ?;
       LET t1:Text,t2:Text; t1,t2 := vsplit(tl(t),pred(n));
       cons(hd(t),t1), t2
     )
```

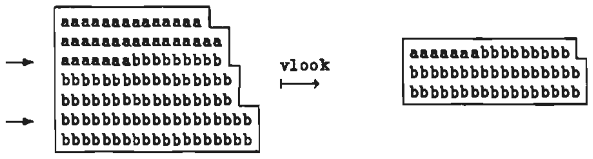
Note that v_add is the inverse of vsplit. Splitting horizontally twice yields what we call a horizontal look (function hlook).



We give the formal definition of hlook below.

```
FUNC hlook: Text # Nat # Nat-> Text
PAR  t:Text, j1:Nat, j2:Nat
DEF  LET t1:Text, t2:Text;  t1 ,t2  := hsplit(t,j2);
     LET t11:Text,t12:Text; t11,t12 := hsplit(t1,j1);
     t12
```

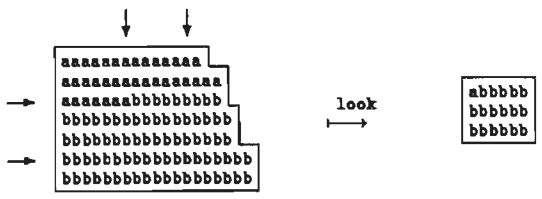
Splitting vertically twice yields what we call a vertical look (function vlook).



We give the formal definition of vlook below.

```
FUNC vlook: Text # Nat # Nat -> Text
PAR  t:Text, i1:Nat, i2:Nat
DEF  LET t1:Text, t2:Text;  t1 ,t2  := vsplit(t,i2);
     LET t11:Text,t12:Text; t11,t12 := vsplit(t1,i1);
     t12
```

And we combine them in an obvious way to get a function look which performs both a vertical and a horizontal 'subsetting' of a given text. look(t, c, d) will denote the contents of a window whose leftmost uppermost corner is at c and whose rightmost lowermost corner is determined by d.



We give the formal definition of look below. We prefer to define it such that the application of vlook is done first, followed by hlook because in this way the result of look is more often defined.

```
   FUNC look: Text # Nat # Nat # Nat # Nat -> Text
   PAR  t:Text, i1:Nat, i2:Nat, j1:Nat, j2:Nat
   DEF  hlook(vlook(t,i1,j1),i2,j2)

END; {of LOOK_SPEC}
```

Both class descriptions FILL_SPEC and LOOK_SPEC are intended to play a role in formulating precisely how the text being edited should be shown on the screen of a display device. This will be done in Section 4.5.12.

4.4 Interfacing an Editor with its Environment

4.4.1 Introduction

In this section we describe several class descriptions that are important for interfacing the editor with its environment. First of all we construct our own model of a physical display device. This is the subject of Section 4.4.2. After that we introduce special kinds of sequences and strings. This is the subject of Section 4.4.3. Finally we describe a simple file-system, which is the subject of Section 4.4.4.

4.4.2 DISPLAY: an Abstract Display

In this section we shall look for a formal description of a *display device*. The device that we have in mind is a classical display of the so-called video-display-unit type – a VT102 or so, say. We do not investigate all the details of one or more concrete displays and instead we start with an ***abstract display.*** We abstract from the character-sequences that should be sent to the display for 'clear screen', 'cursor motion' etc. In the abstract display the sending of these concrete character sequences is replaced by abstract commands `cl`, `cm` etc. The choice of the names and the functionality of the abstract commands have been inspired by to the so-called ***termcap*** facility (abbreviating ***terminal capabilities***) as used on UNIX systems. We decided that we shall not try to include all possible features and capabilities that various displays may have. Instead we restrict our description to those display capabilities that are minimally required by the EMACS editor described in [4].

The corresponding class description is called `DISPLAY_SPEC` and it is given below. There are two variable functions which span the state-space. The first function is called `screen` and it corresponds with the observable contents on the screen of the display, which is the text that you see when you sit in front of the display. The screen has a size given by two constants: `li` and `co`, abbreviating *li*nes and *co*lumns.

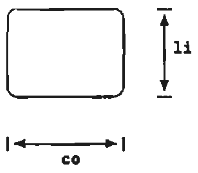


Fig 4.2 The window of the display.

The second function is called `cursor` and it corresponds with the observable position of the cursor on the screen of the display. There are a few operations that are not considered executable, but which we export just for reasoning purposes. In particular, these are the procedure `displ_op` and the functions `screen` and `cursor`.

```
LET DISPLAY_SPEC :=
EXPORT

  SORT Char,
  SORT Nat,
  SORT Text,
  FUNC li: -> Nat,
  FUNC co: -> Nat,
  PROC cr: -> ,
  PROC nl: -> ,
  PROC bc: -> ,
  PROC ce: -> ,
     PROC cl: -> ,
     PROC nd: -> ,
     PROC up: -> ,
     PROC cm: Nat # Nat -> ,
     PROC print: Char -> ,
     PROC displ_op: -> ,
     FUNC screen: -> Text ,
     FUNC cursor: -> Nat # Nat

   FROM
IMPORT NAT_SPEC       INTO
IMPORT CHAR_SPEC      INTO
IMPORT LINE_SPEC      INTO
IMPORT TEXT_SPEC      INTO
IMPORT TEXT_OPS1_SPEC INTO
IMPORT TEXT_OPS2_SPEC INTO
IMPORT STRING_SPEC    INTO
IMPORT PROFILE_SPEC   INTO
IMPORT FILL_SPEC      INTO
CLASS

     FUNC screen: -> Text VAR
     FUNC cursor: -> Nat # Nat VAR

     FUNC li: -> Nat                              % number of lines
     FUNC co: -> Nat                              % number of columns

     PROC cr: -> MOD screen, cursor               % carriage return
     PROC nl: -> MOD screen, cursor               % newline
     PROC bc: -> MOD screen, cursor               % backwards cursor
     PROC ce: -> MOD screen, cursor               % clear to end-of-line
     PROC cl: -> MOD screen, cursor               % erase display
     PROC nd: -> MOD screen, cursor               % move cursor right
     PROC up: -> MOD screen, cursor               % move cursor up
     PROC cm: Nat # Nat -> MOD screen, cursor % cursor motion
     PROC print: Char   -> MOD screen, cursor % character processing
```

Of course we have to specify the effect of the procedures upon the screen (function screen) and upon the cursor (function cursor). This effect is such that a certain invariant is maintained and we refer to it as *display-invariant*. We believe that it is a good methodological principle to formulate this invariant first. We introduce predicates DI1, DI2 and DI3.

- DI1 states that the screen is rectangular and that its size is given by li and co.

- DI2 states that the cursor is on the screen.
- DI3 states that the screen contains only printable characters.

We add one remark related to DI3. Of course we know that many text editors offer the possibility of entering non-printable characters into texts and that such characters still somehow get displayed on the screen (e.g. as tilde), but this matter is not settled within the display but within the editor. We use the predicate printable on Char from Section 4.3.5.

```
PRED DI1:
DEF  len(screen) = li AND
     FORALL i:Nat ( lss(i,li) =>
     ( len(sel(screen,i)) = co ) )

PRED DI2:
DEF  LET c1:Nat,c2:Nat; c1,c2 := cursor;
     lss(c1,li) AND lss(c2,co)

PRED DI3:
DEF  FORALL i:Nat, j:Nat
     ( lss(i,li) AND lss(j,co) => printable(sel(sel(screen,i),j)) )
```

The invariance of DI1 $\wedge$ DI2 $\wedge$ DI3 can be expressed as follows:

```
PROC displ_op: ->
DEF  ( cr
     | nl
     | bc
     | ce
     | cl
     | nd
     | up
     | cm(SOME i:Nat, j:Nat ())
     | print(SOME c:Char (printable(c)))
     )

AXIOM INIT => DI1 AND DI2 AND DI3
AXIOM DI1 AND DI2 AND DI3 =>
      [ displ_op ] DI1 AND DI2 AND DI3
```

Next we turn our attention to the semantical description of the display operations which amounts to giving the axioms of the CLASS ... END part of DISPLAY_SPEC. To begin with, we have definedness axioms and termination axioms.

```
    AXIOM geq(li,1) AND geq(co,1)

    AXIOM < cr > TRUE;
          < nl > TRUE;
          < bc > TRUE;
          < ce > TRUE;
          < cl > TRUE;
          < nd > TRUE;
          < up > TRUE

    AXIOM FORALL i:Nat, j:Nat
          ( < cm(i,j) > TRUE )

    AXIOM FORALL c:Char
          ( printable(c) =>
          < print(c) > TRUE )
```

The effect of each procedure is described by a postcondition. Before giving these postconditions, we introduce by overloading two simple auxiliary functions called `blank_text`. The first function called `blank_text` yields one-line texts consisting of blank characters only. The second function `blank_text` takes two arguments i, j say, and then it yields the text with blanks of dimensions i and j. We use the function `blanks: Nat -> Line` from `FILL_SPEC`.

```
  FUNC blank_text: Nat -> Text
  PAR  n:Nat
  DEF  cons(blanks(n),niltext)

  FUNC blank_text: Nat # Nat -> Text
  PAR  i:Nat, j:Nat
  DEF  ( i = 0?;       niltext
       | NOT i = 0 ?; cons(blanks(j),blank_text(pred(i),j))
       )
```

We introduce several postcondition predicates, one for each procedure. These predicates have formal parameters `s'` for the previous `screen`, `c'` for the previous `cursor`, `s` for `screen` and `c` for `cursor`. The first predicate describes the effect of 'carriage return'. This case is simple: the second co-ordinate of the cursor becomes 0.

```
  PRED post_cr: Text # Nat # Nat # Text # Nat # Nat
  PAR  s':Text, c':Nat # Nat, s:Text, c: Nat # Nat
  DEF  LET i:Nat,j:Nat; i,j := c';
       c = (i,0);
       s = s'
```

After we shall have defined this (and several similar) predicates, we state axiomatically that post_cr is the postcondition of cr indeed. We shall not write all axioms in full detail, since it should be clear from the axiom for cr what the remaining axioms should look like.

The second predicate describes what happens if a 'new-line' is sent to the display. At first sight this case seems simple too: the first co-ordinate of the cursor must be incremented by 1. However there is a complication because the cursor may already be on the last line of the screen. In the latter situation so-called scrolling takes place, i.e. one line of blanks is added to the end of the screen, whereas the first line of the screen is removed.

```
PRED post_nl: Text # Nat # Nat # Text # Nat # Nat
PAR  s':Text, c':Nat # Nat, s:Text, c: Nat # Nat
DEF  LET i:Nat,j:Nat; i,j := c';
     c = (( lss(i,pred(li))?; succ(i) | i = pred(li)?; i),j) AND
     s = ( lss(i,pred(li))?; s'
         | i = pred(li)   ?; tl(v_add(s',blank_text(co)))
         )
```

The predicates for 'backwards cursor', 'clear to end-of-line', 'erase display', 'move cursor right', 'move cursor up' and 'cursor motion' speak for themselves.

```
PRED post_bc: Text # Nat # Nat # Text # Nat # Nat
PAR  s':Text, c':Nat # Nat, s:Text, c: Nat # Nat
DEF  LET i:Nat,j:Nat; i,j := c';
     s = s';
     c = (i,(gtr(j,0)?; pred(j) | j = 0?; 0))

PRED post_ce: Text # Nat # Nat # Text # Nat # Nat
PAR  s':Text, c':Nat # Nat, s:Text, c: Nat # Nat
DEF  LET i:Nat,j:Nat; i,j := c';
     LET tt:Text,kk:Text; tt,kk := cut(s',i,j,i,co);
     s = paste(tt,blank_text(sub(co,j)),i,j);
     c = c'

PRED post_cl: Text # Nat # Nat # Text # Nat # Nat
PAR  s':Text, c':Nat # Nat, s:Text, c: Nat # Nat
DEF  s = blank_text(li,co);
     c = (0,0)

PRED post_nd: Text # Nat # Nat # Text # Nat # Nat
PAR  s':Text, c':Nat # Nat, s:Text, c: Nat # Nat
DEF  LET i:Nat,j:Nat; i,j := c';
```

```
     c = (i, (lss(j,pred(co))?; succ(j) | j = pred(co)?; j));
     s = s'

PRED post_up: Text # Nat # Nat # Text # Nat # Nat
PAR  s':Text, c':Nat # Nat, s:Text, c: Nat # Nat
DEF  LET i:Nat,j:Nat; i,j := c';
     c = ((gtr(i,0)?; pred(i) | i = 0?; 0), j);
     s = s'

PRED post_cm: Text # Nat # Nat # Nat # Nat # Text # Nat # Nat
PAR  s':Text,c':Nat # Nat,i:Nat,j:Nat,s:Text,c:Nat # Nat
DEF  s = s';
     c = ( (lss(i,li)?; i | geq(i,li)?; pred(li)),
           (lss(j,co)?; j | geq(j,co)?; pred(co)) )
```

And finally we have one important predicate dealing with printable-character processing.

```
PRED post_print: Text # Nat # Nat # Char # Text # Nat # Nat
PAR  s':Text, c':Nat # Nat, c:Char, s:Text, c: Nat # Nat
DEF  printable(c) =>

     ( LET i:Nat,j:Nat; i,j := c';
       LET tt:Text,kk:Text; tt,kk := cut(s',i,j,i,succ(j));
       c = (i,(lss(j,pred(co))?; succ(j) | j=pred(co)?; j));
       s = paste(tt,addchar(c,zero),i,j) )
```

We state axiomatically that post_cr is the postcondition of cr.

```
AXIOM {cr}
FORALL screen':Text, cursor':Nat # Nat
( screen = screen' AND cursor = cursor' =>
  [ cr ] post_cr(screen',cursor',screen,cursor) )
```

We do not write all the axioms in full detail, since it should be clear from the axiom for cr what they should look like.

```
% AXIOM nl,bc,ce,cl,nd,up  as cr

AXIOM {cm} FORALL screen':Text, cursor':Nat # Nat, i:Nat, j:Nat
( screen = screen' AND cursor = cursor' =>
  [ cm(i,j) ] post_cm(screen',cursor',i,j,screen,cursor) )

AXIOM {print} FORALL screen':Text, cursor':Nat # Nat, c:Char
( screen = screen' AND cursor = cursor' =>
      [ print(c) ] post_print(screen',cursor',c,screen,cursor) )

END;
```

Now the invariance of the display-invariant under the display operations should be derivable in the sense that $\mathtt{DI1} \wedge \mathtt{DI2} \wedge \mathtt{DI3} \Rightarrow$ `[ displ_op ]` $\mathtt{DI1} \wedge \mathtt{DI2} \wedge \mathtt{DI3}$. This can be shown by verifying its invariance under each operation separately. By way of example, we shall present this for one of them and we choose `cr`.

Assume that in a given state $\mathtt{DI1} \wedge \mathtt{DI3} \wedge \mathtt{DI3}$ holds and that in this state we have executed `cr`. It is important to observe that `DI1` and `DI3` depend only on the screen and not on the cursor. `DI2` at its turn depends only on the cursor and not on the screen. From the axiom about `cr` and the postcondition predicate of `cr`, we can see that in this new state `screen` = `PREV screen` and this immediately establishes `DI1` and `DI3`. So now we have to check for `DI2`. The vertical co-ordinate of the cursor has kept its old value (in the postcondition predicate denoted as `i`) which thus is less than `li`, as was to be shown. Finally we consider the horizontal co-ordinate of the cursor. We see that it has become 0, so we should check $0 < \mathtt{co}$ which holds by the axiom saying $\mathtt{li} \geq 1 \wedge \mathtt{co} \geq 1$.

This concludes the description of the abstract display. In the specification of the editor, we shall have to describe the relation between the text being edited and the dot on one hand and the screen and the cursor of the display on the other hand. We introduce such a relation, called the *window-invariant* in Section 4.5.12.

4.4.3 'SEQ' and 'STRING': Data types for Interfacing

Before we can specify editing operations, we must decide about the way in which we have to invoke our editing operations. It is hard to avoid that these operations will take arguments and we must be precise about the types of these arguments. A frequently occurring data type is the sort of strings, or more precisely, some data type *representing* strings.

In our window-and-text facility to be described in Sections 4.5.1-4.5.12 we shall introduce so-called buffers which have names associated with them and we shall need some kind of strings for that. For example, there will be a procedure yank_`buffer` which takes a string argument for addressing a buffer to be 'yanked' and a procedure `current_buffer_name` which yields a string. Also certain operations for file handling will deal with string arguments. Now we want to be able to choose a suitable representation of these strings by means of conventional data reification techniques. For example, the string `"hello, world"` could be represented by the address of a memory location containing the `'h'` where subsequent locations contain the `'e'`, the `'l'` and so on until some termination value indicates the end of the string.

Of course it would not be wise to make such representation choices now already during the specification phase, but in order to avoid problems in a later design phase, we must make sure that our specification does not exclude certain reasonable choices.

Consider the above representation and assume that the memory contains `"hello, world"` starting at address 1024 but also at address 1037, say. Now there is a problem with equality: the test $1024 = 1037$ yields `FALSE`, although both string representations are equivalent in the sense that they represent the same sequence of characters.

There is also a problem with the dynamic allocation of strings. In `STRING_SPEC` we only have functions such as `FUNC empty: -> String` and `FUNC cons: Char # String -> String`. This implies that formally an implementation of this `cons` is not allowed to have some side-effect such as storing characters in memory locations. The two problems sketched above indicate that, at least for certain applications, we better not adopt the sort `String`, but a slightly different sort, which we shall name `'String'`. The quotes in `'String'` are formally part of the identifier. For the latter sort we introduce an `eq` predicate which is meant for an equivalence relation as indicated above and we introduce a procedure called `cons` rather than a function. Furthermore we explicitly introduce the possibility that a representation invariant has to be maintained. We shall refer to the objects of this sort `'String'` as *implementable strings.*

Of course the introduction of this special type of strings is a complication for our task of formally specifying an editor, especially because the use of the sort `'String'` will propagate throughout large parts of our formal specification. However we think that data reification is an important topic and therefore we try to deal with it explicitly, rather than running away from the problem. Readers who prefer not to get involved with this topic, should just skip the current section and continue reading at Section 4.4.4, replacing everywhere `'String'` by `String`, and replacing `eq` by equality ($=$).

The purpose of having `'String'` rather than `String` is *not* to add an implementation bias. Instead we like to refer to our approach as specification for implementability. First we introduce a somewhat more general concept, by specifying a data type of *implementable sequences.* These are very much like sequences, except that they allow for dynamic implementations. They are given by the class description `'SEQ_SPEC'`.

The parameter restriction is not the usual ITEM, but a somewhat more general form 'ITEM', which provides an eq predicate rather than just equality. The advantage of 'ITEM' over ITEM is that with 'ITEM' it is possible to have implementable sequences of implementable sequences of ..., rather than only implementable sequences of items with equality.

```
LET 'ITEM' :=
IMPORT ITEM INTO
CLASS

  SORT 'Item'                          FREE
  PRED eq       : 'Item' # 'Item'      FREE
  PRED item_inv:                       FREE
  FUNC f        : 'Item' -> Item       FREE

  AXIOM item_inv =>
        FORALL i:'Item',j:'Item'
        ( eq(i,j) <=> f(i) = f(j) )

END;
```

The body of the class description 'SEQ_SPEC' introduces a sort 'Seq'. The specification 'SEQ_SPEC' is based on the use of an abstraction function f – by overloading – and a representation invariant seq_inv. Notice that we do not really specify the invariant; we just allow the implementation to maintain one. An implementer who does not need this, may just define seq_inv DEF TRUE. Furthermore in 'SEQ_SPEC' we have procedures cons and cat rather than functions. There is an eq predicate which should be viewed as the *external notion of equality* on 'Seq'.

Not all sorts and operations are considered *executable*, but some of the non-executable sorts and operations are exported nevertheless because it is convenient to have them for reasoning purposes. In particular, the sort Seq, the predicate seq_inv and the function f are needed for reasoning purposes only. The remaining exported sorts and operations are considered executable. Formally, COLD-K does not provide a program-execution model and when we make statements about executability, these do not refer to any formal notion of COLD-K.

```
LET 'SEQ_SPEC' :=

LAMBDA X : 'ITEM' OF
```

```
EXPORT

  SORT Seq,
  PRED seq_inv:,
  FUNC f      : 'Seq'              -> Seq,

  SORT 'Seq',
  SORT Nat,
  SORT 'Item',
  FUNC empty  :                    -> 'Seq',
  PROC cons   : 'Item' # 'Seq'     -> 'Seq',
  FUNC hd     : 'Seq'              -> 'Item',
  FUNC tl     : 'Seq'              -> 'Seq',
  PRED eq     : 'Seq' # 'Seq'            ,
  FUNC sel    : 'Seq' # Nat        -> 'Item',
  PROC cat    : 'Seq' # 'Seq'      -> 'Seq',
  PROC rev    : 'Seq'              -> 'Seq'

FROM

IMPORT X                     INTO
IMPORT NAT_SPEC              INTO
IMPORT APPLY SEQ_SPEC TO X INTO

CLASS

  SORT 'Seq' VAR

  FUNC empty:                     -> 'Seq'
  PROC cons : 'Item' # 'Seq' -> 'Seq' MOD 'Seq'
  FUNC hd   : 'Seq'               -> 'Item'
  FUNC tl   : 'Seq'               -> 'Seq'
  PRED eq   : 'Seq' # 'Seq'

  FUNC f: 'Seq' -> Seq
  PRED seq_inv: VAR

  AXIOM {INVARIANCE}

  INIT AND item_inv => seq_inv;
  item_inv AND seq_inv => [ p ] seq_inv

  PROC p: ->
  DEF  ( FLUSH cons(SOME i:'Item' (), SOME s:'Seq' ())
       | FLUSH cat (SOME s:'Seq'  (), SOME t:'Seq' ())
       | FLUSH rev (SOME s:'Seq'  ())
       )

 AXIOM {ABSTRACTION}

    item_inv AND seq_inv =>
    FORALL t:'Seq' ( f(t)! );

    item_inv AND seq_inv =>
    FORALL s:'Seq', t:'Seq', i:'Item', j:'Item'
    ( f(empty) = empty;
    ( [ LET u:'Seq'; u := cons(i,s) ] f(u) = cons(f(i),f(s)) );
      NOT f(s) = empty => f(hd(s)) = hd(f(s));
      NOT f(s) = empty => f(tl(s)) = tl(f(s));
      eq(s,t) <=> f(s) = f(t) )

  AXIOM {TERMINATION}

  item_inv AND seq_inv =>
  FORALL i:'Item', s:'Seq' ( < cons(i,s) > TRUE )

  PROC cat: 'Seq' # 'Seq' -> 'Seq' MOD 'Seq'
  PROC rev: 'Seq'         -> 'Seq' MOD 'Seq'
  FUNC sel: 'Seq' # Nat   -> 'Item'

  AXIOM {ABSTRACTION}

  item_inv AND seq_inv =>
  FORALL s:'Seq', t:'Seq', n:Nat
  ( [ LET u:'Seq';  u := cat(s,t) ] f(u) = cat(f(s),f(t));
    [ LET u:'Seq';  u := rev(s)   ] f(u) = rev(f(s));
    [ LET i:'Item'; i := sel(s,n) ] f(i) = sel(f(s),n) )

  AXIOM {TERMINATION}

  item_inv AND seq_inv =>
  FORALL s:'Seq',t:'Seq'
  ( < FLUSH cat(s,t) > TRUE;
    < FLUSH rev(s)   > TRUE )

END;
```

Now implementable strings are nothing but implementable sequences of characters. We do not need the full generality offered by 'ITEM', but that does not matter. We also introduce the usual lexicographical ordering on sequences as a predicate `less`. Again we can distinguish between non-executables and executables. In particular, `String`, `f`: `'String'` → `String` and `string_inv` need not be executable. The remaining exported sorts and operations are considered executable.

```
LET 'STRING_SPEC' :=
EXPORT

  SORT String,
  FUNC f: 'String' -> String,
  PRED string_inv:,

  SORT Char,
  SORT Nat,
  SORT 'String',
  FUNC empty:                        -> 'String',
  PROC cons : Char # 'String'        -> 'String',
  FUNC hd   : 'String'               -> Char,
  FUNC tl   : 'String'               -> 'String',
  PRED eq   : 'String' # 'String'       ,
  FUNC sel  : 'String' # Nat         -> Char,
  PROC cat  : 'String' # 'String'    -> 'String',
  PROC rev  : 'String'               -> 'String',
  PRED less : 'String' # 'String'

FROM
IMPORT NAT_SPEC    INTO
IMPORT CHAR_SPEC   INTO
IMPORT STRING_SPEC INTO

IMPORT APPLY RENAME
      SORT Seq      TO String,
      SORT 'Seq'    TO 'String',
      SORT Item     TO Char,
      SORT 'Item'   TO Char,
      PRED seq_inv: TO string_inv
IN 'SEQ_SPEC' TO
IMPORT CHAR_SPEC INTO
CLASS
      PRED eq: Char # Char PAR c:Char,d:Char DEF c = d
      PRED item_inv: DEF TRUE
      FUNC f: Char -> Char PAR c:Char DEF c
END

INTO
```

```
CLASS

  PRED less: 'String' # 'String'

  AXIOM string_inv =>
        FORALL  s:'String',t:'String'
        ( less(s,t) <=> less(f(s),f(t)) )

END;
```

4.4.4 FILE: a File system

Both the possibility to edit existing texts and the possibility to store the results of an edit-session are indispensable for text editing. We assume that the basic mechanisms for text storage and retrieval are provided by some *file system*. We cannot completely specify an editor unless we also have a description of the available file system. The class description `FILE_SPEC` below models a simple file system with just enough operations for our purposes.

We use names (sort `'String'`) as file-identifications. Since the file system can in fact be independent of any particular string implementation, we have parameterised the file system specification over implementations of `'STRING_SPEC'`. The predicate `valid` describes the *valid* names of the file system, corresponding with the situation that for certain names there exist files in the system, whereas for other names there is no file (yet).

We restrict ourselves to character-files and we model the contents of a file as a string. This is described by the function `file` which for a given name yields the contents of the corresponding file. We allow for an arbitrary number of files in the file system, but in any state at most two files can be active, viz. one for input and one for output. So `read` and `eof` operate on the file that has been reset most recently. Similarly `write` operates on the most recently rewritten file. In the description below this idea of having at most two active files is modelled by the functions `infile` and `outfile` which correspond with the *names* of the input file and the output file respectively.

We suppose that the operations `rewrite`, `reset`, `read`, `write` and `eof` are executable. We also export `valid`, `file` and `pos`, but this is just because we might want to reason with them; they need not be executable.

```
LET FILE_SPEC :=
LAMBDA X : 'STRING_SPEC' OF
EXPORT
```

```
  SORT String,
  PRED valid : String,
  FUNC file  : String     -> String,
  FUNC pos   : String     -> Nat,

  SORT Nat,
  SORT Char,
  SORT 'String',
  PROC rewrite: 'String'  -> ,
  PROC reset  : 'String'  -> ,
  PROC read   :           -> Char,
  PROC write  : Char      -> ,
  PRED eof    :

FROM
IMPORT X             INTO
IMPORT NAT_SPEC      INTO
IMPORT CHAR_SPEC     INTO
IMPORT STRING_SPEC   INTO
CLASS

  FUNC file : String -> String VAR
  FUNC pos  : String -> Nat    VAR

  PRED valid: String
  PAR  s:String
  DEF  file(s)!

  FUNC infile :      -> String VAR
  FUNC outfile:      -> String VAR

  PROC rewrite: 'String' ->  MOD file,outfile

  AXIOM FORALL s:'String' ( string_inv => (
        < rewrite(s) > TRUE;
        [ rewrite(s) ]
        ( outfile = f(s);
          file(f(s)) = empty;
          FORALL t:'String',u:String
          ( NOT f(t) = f(s) =>
              file(f(t)) = u <=> PREV file(f(t)) = u ) ) ) )

  PROC reset : 'String' -> MOD pos,infile
  AXIOM FORALL s:'String' ( string_inv AND valid(f(s)) => (
        < reset(s) > TRUE;
        [ reset(s) ]
        ( infile = f(s);
          pos(f(s)) = zero;
          FORALL t:'String',i:Nat
          ( NOT f(t) = f(s) =>
              pos(f(t)) = i <=> PREV pos(f(t)) = i ) ) ) )

  PROC read : -> Char MOD pos

  AXIOM string_inv AND lss(pos(infile),len(file(infile))) => (
        < read > TRUE;
        [ LET c:Char; c := read ]
        ( c = sel(file(infile),PREV pos(infile));
          pos(infile) = succ(PREV pos(infile));
          FORALL t:String,i:Nat
          ( NOT t = infile =>
              pos(infile) = i <=> PREV pos(t) = i ) ) )

  PROC write : Char -> MOD file

  AXIOM FORALL c:Char ( string_inv AND valid(outfile) => (
        < write(c) > TRUE;
        [ write(c) ]
        ( file(outfile) = cat((PREV file(outfile)),cons(c,empty));
          FORALL t:String,u:String
          ( NOT t = outfile =>
            ( file(outfile) = u <=> PREV file(outfile) = u )))))

  PRED eof:

  AXIOM string_inv AND valid(infile) AND pos(infile)! =>
        ( eof <=> pos(infile) = len(file(infile)) )

END;
```

We tried to keep the specification of the file system as simple as possible. In particular, our description does not cover the phenomenon of output buffering and the related problems among which the need for a so-called *flush* operation.

Note also that we did not exclude the situation that `infile` and/or `outfile` are undefined – typically after system initialisation. In that case nothing is specified about the effect of `read` and `write`. For example if `infile` is undefined then the assertion `lss(pos(infile),len(file(infile)))` is false, so `read` can do anything – e.g. abort.

4.5 Text Editing

4.5.1 Introduction

We adopt the basic idea underlying display-oriented editors where the user of an editor need not give *print* or *display* commands himself. Instead, the editor more or less automatically makes sure that parts of one or more texts are displayed within windows on the screen of a display. We also adopt the idea that the user can edit several texts simultaneously. In combination with cut and paste capabilities, this makes it possible to move pieces of text from one text to another. It should be possible for the user to give names to texts. As a third idea we adopt the algebraic operations `cut` and `paste` of Section 4.3.6 as a starting point for the cut and paste capabilities of the editor. These ideas are also realised in the EMACS editor [4], However, in [4] the algebraic aspects are somewhat hidden and instead there are simply operations `copy-region-to-buffer` and `yank-buffer` offering cut and paste capabilities.

We shall describe a kind of abstract machine which we refer to as *window-and-text facility*. The corresponding class description is named `WITEFA_SPEC` – for `WI`ndow and `TE`xt `FA`cility). This facility allows for the manipulation of texts and it makes sure that a suitable part of one text is displayed on the screen of a display which serves as a window. It supports a notion of 'current text' and a mechanism for associating names with texts. For the time being the window management part of the window-and-text facility is kept simple: there is just one window with with dimensions `li` and `co` which thus covers the entire screen of the display (see Section 4.4.2). It should be stressed however that our approach would be suited for the inclusion of multi-window features: the algebraic operations `h_add` and `v_add` of Section 4.3.9 would be a good starting point for describing the partition of a multi-window screen.

We avoid the introduction of a notion of *buffer*, which in our view is best postponed until an implementation phase. We have operations whose names are based on a buffer-oriented terminology, but we do this only for maintaining a kind of name-compatibility with EMACS. In the formal specification of these operations we do not use the buffer-oriented terminology. Instead of that we have the notion of *marked text*, which is the subject of our next section.

The remainder of this section is organised as follows. In Section 4.5.2 we formally introduce marked texts. Section 4.5.3 is about formally importing the machinery developed before into the editor specification. Section 4.5.4 is about modelling the state-space. In Section 4.5.5 we formulate a so-called text-invariant. The definition of the editor operations will cover the Sections 4.5.6 to 4.5.10. The termination axioms are given for all operations together in Section 4.5.11. In Section 4.5.12 we discuss the connection between the editor and the display. In Section 4.5.13 we present a simple editor which can be built on top of our window-and-text facility. Finally in Section 4.5.14 we discuss the connection between the editor and the keyboard.

4.5.2 Marked Texts

A marked text is a composite object that consists of a text and a collection of co-ordinate pairs. These co-ordinate pairs will be called ***markers***. We expect that we need the following markers.

- Dot: a kind of 'current' location in the text.
- Mark: a marker that can be put on any position in the text. This mark can be used for selecting a text-region for cut- and paste operations. In the typical applications, the mark indicates the beginning of this text region and the dot indicates its end. This application of the mark will be worked out in Section 4.5.7.

Furthermore we have considered the possibility of having the following additional markers, but at least for the time being we do not need these urgently, so we shall not include them in our specification: 1. bob: marker that is always at the beginning of the text. 2. eob: marker that is always at the end of the text. 3. searcher: a marker that plays a role in search- and replace commands. One could even go further and consider the introduction of a stack of markers (in [6] this is called a 'mark ring'), allowing for the manipulation of a large number of markers.

The following picture sketches a marked text.

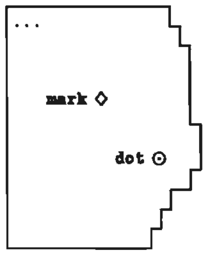


Fig 4.3. Marked text.

We do not introduce the idea of *locations* for storing marked texts directly, but instead we focus now on the contents of such locations. We formalise markers as tuples $\langle i, j \rangle$. This is done by introducing a class-description called COPA_SPEC (COPA for CO-ordinate PAir). We have projection functions **v** for the vertical co-ordinate and **h** for the horizontal co-ordinate.

```
LET COPA_SPEC :=
APPLY APPLY RENAME
       SORT Item1                    TO Nat,
       SORT Item2                    TO Nat,
       SORT Tup                      TO Copa,
       FUNC proj1: Tup -> Item1      TO v,
       FUNC proj2: Tup -> Item2      TO h,
       FUNC tup: Item1 # Item2 -> Tup TO copa
IN TUP2_SPEC TO NAT_SPEC TO NAT_SPEC;
```

And now we can define a class description MTEXT_SPEC, which serves for formalising marked texts as triples of the form $\langle t, d, m \rangle$ where t is the text and where d and m are the dot and the mark respectively.

```
LET MTEXT_SPEC :=
IMPORT APPLY APPLY APPLY RENAME
       SORT Item1                            TO Text,
       SORT Item2                            TO Copa,
       SORT Item3                            TO Copa,
       SORT Tup                              TO MText,
       FUNC proj1: Tup -> Item1              TO text,
       FUNC proj2: Tup -> Item2              TO dot,
       FUNC proj3: Tup -> Item3              TO mark,
       FUNC tup: Item1 # Item2 # Item3 -> Tup TO mtext
IN TUP3_SPEC TO TEXT_SPEC TO COPA_SPEC TO COPA_SPEC INTO
```

```
IMPORT NAT_SPEC  INTO
IMPORT COPA_SPEC INTO
CLASS

       FUNC dot: MText -> Nat # Nat
       PAR  b: MText
       DEF  v(dot(b)), h(dot(b))

       FUNC mark: MText -> Nat # Nat
       PAR  b: MText
       DEF  v(mark(b)), h(mark(b))
```

This provides us with an extensional and functional specification of marked texts. We introduce a number of modification operations on marked texts – not in the sense of modifications on states of course.

```
       FUNC modtext: MText # Text -> MText
       PAR  m:MText, t:Text
       DEF  mtext(t,dot(m),mark(m))

       FUNC moddot: MText # Copa -> MText
       PAR  m:MText, c:Copa
       DEF  mtext(text(m),c,mark(m))

       FUNC modmark: MText # Copa -> MText
       PAR  m:MText, c:Copa
       DEF  mtext(text(m),dot(m),c)

END;
```

4.5.3 Stating the Application Domain

Most software products have a certain functionality associated with some specific application-domain. In our case this is text editing and therefore we have collected a number of definitions which are specifically related to the notion of text. This collection of definitions should be viewed as an *application-domain specific notational framework*, rather than as a product specification. It is convenient to have a name for this application-domain specific notational framework. Therefore we introduce a class description called APP_DOM_SPEC which encompasses many of the class descriptions given before.

```
LET APP_DOM_SPEC :=

IMPORT BOOL_SPEC       INTO
IMPORT NAT_SPEC        INTO
IMPORT CHAR_SPEC       INTO
IMPORT LINE_SPEC       INTO
IMPORT TEXT_SPEC       INTO
IMPORT TEXT_OPS1_SPEC INTO
IMPORT TEXT_OPS2_SPEC INTO
IMPORT TEXT_OPS3_SPEC INTO
IMPORT STRING_SPEC     INTO
IMPORT PROFILE_SPEC    INTO
IMPORT FILL_SPEC       INTO
IMPORT LOOK_SPEC       INTO
MTEXT_SPEC;
```

4.5.4 Spanning the State-space

Our window-and-text facility manages a finite collection of marked texts. At the specification level, we describe this collection as a map. We shall use several operations from MAP_SPEC as given in Appendix B such as empty: → Map, app: Map # Item1 → Item2 and dom: Map → Set1 which yields a set of Item1. Sometimes we write $m[i]$ for app(m,i). Of course we import the application-domain specific notational framework as collected in APP_DOM_SPEC.

```
LET WITEFA_SPEC :=

IMPORT APP_DOM_SPEC  INTO
IMPORT 'STRING_SPEC' INTO
IMPORT DISPLAY_SPEC  INTO

IMPORT (APPLY FILE_SPEC TO 'STRING_SPEC') INTO

IMPORT APPLY APPLY RENAME
         SORT Item1 TO String,
         SORT Item2 TO MText
IN MAP_SPEC TO STRING_SPEC TO MTEXT_SPEC INTO

IMPORT APPLY RENAME
         SORT Item TO String,
         SORT Set  TO Set1
IN SET_SPEC TO STRING_SPEC INTO
```

```
CLASS
```

The collection of marked texts in WITEFA_SPEC is formally described by a variable function mtexts. In this way marked texts will have names associated with them, where names are just strings.

```
   FUNC mtexts: -> Map VAR
```

Just to keep things concrete, let us briefly sketch one possible use of this variable map. In Section 4.5.13 we shall describe a simple editor on top of our window-and-text facility. In this editor we shall have three marked texts, denoted as mtexts["main"], mtexts["mini"] and mtexts["kill"]. In EMACS terminology these would be called the main-buffer, the mini-buffer and the kill-buffer respectively. The marked text named "main" contains essentially the text being edited, whereas the marked text named "mini" allows the user to input file names and search strings. The marked text named "kill" always contains the most recently deleted text.

Now we proceed with spanning the state-space. There is a notion of *current marked text* which formally is described by introducing a variable function.

```
   FUNC current: -> String VAR
```

The variables mtexts and current need not be considered executable. All operations to be described later *are* considered executable.

At first sight it seems attractive to postulate an initialisation condition by using the built-in initially assertion INIT of COLD-K, but this may be hard to implement and therefore we prefer the use of an initialisation procedure. Instead of INIT ⇒ ... we must write INIT ⇒ [init(...)] ...

We want to avoid the situation where current:String becomes undefined. Therefore the initialisation procedure gets a 'String' argument which represents the name of the first marked text. This marked text consists of the zero text with $(0,0)$ as dot and mark. After the initialisation has been invoked, the user of the window-and-text facility can introduce more names – and hence more marked texts. The procedure init defined here has modification rights with respect to mtexts, current and a predicate named WTI. The motivation for this WTI and its definition will be provided later in Section 4.5.5. The procedure init also has use rights with respect to the procedure displ_op which was defined in Section 4.4.2. In the description of the postcondition of init below we refer to the function f: 'String' -> String which was defined in Section 4.4.3.

```
   PROC init: 'String' ->
```

```
MOD  mtexts,current,WTI USE displ_op

AXIOM INIT => FORALL s:'String' ( < init(s) > TRUE )

AXIOM INIT =>
FORALL s:'String'
( [ init(s) ]
    current = f(s) AND
    mtexts = add(empty,f(s),mtext(zero,copa(0,0),copa(0,0))) )
```

We have classified the operations into the following groups:

- operations for dot and mark control,
- operations for text modification,
- operations for marked-text management,
- operations for searching,
- operations for string conversion.

4.5.5 The Text Invariant

We must make a decision about the behaviour of the editor. There are a number of marked texts, and one of them, is the 'current' marked text. Within this current marked text there is a point of interest, indicated by *dot*. Probably this is close to the piece of text about which the user of the editor is thinking and where he is going to do his next insertion or deletion. The user can move this dot and the cursor in the sense that the cursor on the screen somehow mirrors the position of the dot. The question now is if one wants to allow the situation where the dot indicates a position that is non-existent in the text of the current marked text. We believe that a consequent and elegant approach is to avoid this situation entirely. Consider the following text and suppose that dot is at the position corresponding with the A at the end of the first line.

```
this-is-a-very-long-line-of-text-rather-long-indeed-A
short-line
this-is-a-very-long-line-of-text-rather-long-indeed-C
this-is-a-very-long-line-of-text-rather-long-indeed-D
this-is-a-very-long-line-of-text-rather-long-indeed-E
```

Now according to our approach it should not be possible to move the dot one line downwards while staying within the same column. Indeed, suppose that it would be possible, then it is unclear what it means to do an insert operation next.

We formulate the above ideas by introducing a so-called text-invariant. The text-invariant implies that for each marked text both the dot and the mark correspond with positions that exist in the text of that marked text; furthermore this text should be ok.

Considering ok texts only has the advantage that when this window-and-text facility has to be implemented, its designer is in a comfortable position in the sense that he has the option of choosing the string-representation of texts. Another advantage exists already in the specification phase: all operations such as `cut` and `paste` of Section 4.3.6 which are well-defined for ok texts only, are usable. Now we easily formulate this part of our text-invariant and we introduce it as a predicate `TI1`.

```
PRED TI1:
DEF  FORALL s:String, mt:MText
     ( mt = app(mtexts,s) =>
       (LET d: Nat # Nat; d := dot(mt);
        LET m: Nat # Nat; m := mark(mt);
        intext(text(mt),d) AND intext(text(mt),m) AND ok(text(mt))))
```

A second constraint on the state-space arises because we have a notion of `current` and this should be the name of an existing marked text.

```
PRED TI2:
DEF  app(mtexts,current)!

PRED TI:
DEF  TI1 AND TI2
```

We must formalise the requirement that the operations of the `WITEFA_SPEC` respect `TI`. As a first attempt, we could express this requirement by saying that `TI` is an invariant in the following weak sense, which we might refer to as *repetition-invariant*. Let `witefa_op` denote an invocation of one of the operations of `WITEFA_SPEC`.

```
% AXIOM INIT => [ init(SOME s:'String'()) ] [ (witefa_op)* ] TI
% (not adopted)
```

This first attempt fails, because it does not guarantee that `TI` is preserved by an interleaving of `witefa_op` invocations with invocations of other operations. As a second attempt, we could express this requirement as follows.

```
% AXIOM INIT => [ init(SOME s:'String'()) ] TI;
%       TI   => [ witefa_op ] TI
% (not adopted)
```

We shall refer to this formulation by saying that TI is a *classical invariant*. This attempt fails too, because we must allow an implementation to have auxiliary variables and hence to maintain a stronger invariant WTI, say. The problem is that from the fact that WTI is a classical invariant one cannot conclude that TI is so too.

Therefore we introduce as a part of the specification another auxiliary predicate WTI (for Window and Text Invariant) which must be a classical invariant and which at its turn must imply TI. One might be tempted to think that the classical invariance of WTI under witefa_op follows already automatically if we simply do *not* add WTI to the modification rights of the operations from the the witefa_op group. However, this would be too strong, because it would forbid an implementation of the procedures to change the truth-value of the invariant *from false to true* – just by luck. So we must add WTI to the modification rights of the operations from the the witefa_op group and we must state its invariance by an axiom.

```
PRED WTI: VAR

AXIOM {WTI1} INIT => [ init(SOME s:'String'()) ] WTI;
             WTI  => [ witefa_op ] WTI

AXIOM {WTI2} WTI => TI
```

We adopt the latter solution and we might refer to its construction by saying that TI is an *observational invariant* with respect to witefa_op. The intuition behind this term is as follows. Define an experiment as the execution of some interleaving of witefa_op invocations with invocations of other operations. Now this construction guarantees that after any experiment, it can be observed that TI holds, although in fact something stronger may hold, which need not be observable.

Actually witefa_op can be defined in terms of the operations from the next sections. Its definition refers to all but one procedures of WITEFA_SPEC, the only exception being init.

```
PROC witefa_op: ->
DEF  ( FLUSH bolp
     | FLUSH eolp
     | forward_character
     | backward_character
     | next_line
     | previous_line
     | beginning_of_line
     | end_of_line
```

```
    | beginning_of_buffer
    | end_of_buffer
    | set_mark
    | exchange_dot_and_mark
    | insert_file(SOME s:'String' (valid(f(s))))
    | insert_character(SOME c:Char printable(c))
    | newline
    | yank_buffer(SOME s:'String' (app(mtexts,f(s))!))
    | delete_next_character
    | erase_region
    | erase_buffer(SOME s:'String' ())
    | copy_region_to_buffer(SOME s:'String' ())
    | FLUSH current_buffer_name
    | write_named_file(SOME s:'String' ())
    | switch_to_buffer(SOME s:'String' ())
    | search_forward(SOME s:'String' ())
    | FLUSH buffer_to_string (SOME s:'String' (app(mtexts,f(s))!))
    )
```

In the specification of the operations, we must take care not to contradict the assertion that TI is an observational invariant. Even better, we can try to make a kind of 'almost-invariance' of TI derivable in the sense that init establishes TI and that WTI $\Rightarrow$ [witefa_op] TI. We decided to do so and furthermore we shall show below that init establishes TI. We shall later show by way of example this almost invariance for *one* of the operations, which will be copy_region_to_buffer of Section 4.5.7.

Let us show now already that INIT $\Rightarrow$ $\forall$s:'String' [init(s)] TI is derivable from the given postcondition of init. This postcondition states that current = f(s) $\wedge$ mtexts = add(empty,f(s), mtext(zero,copa(0,0), copa(0,0))). For TI1 we must check something for all marked texts mt in the range of mtexts, but we see that there is only one, which is the marked text mtext(zero,copa(0,0),copa(0,0)). We easily verify that intext(zero,(0,0)) and ok(zero) holds. Now we are left with TI2 which requires that current is in the domain of mtexts. Since current = f(s), it is even the only element in the domain. This shows that init establishes TI.

The presentation of signatures and the the pre- and postcondition style axioms of the operations will be based on the classification of the operations into several groups as sketched in Section 4.5.4. This will cover the Sections 4.5.6 to 4.5.10. The termination axioms are given for all operations together in Section 4.5.11.

4.5.6 Operations for Dot and Mark control

To begin with, we give the signature of the operations from this group.

```
PROC bolp                    : -> Bool    MOD WTI
PROC eolp                    : -> Bool    MOD WTI
PROC forward_character       : -> MOD mtexts,WTI USE displ_op
PROC backward_character      : -> MOD mtexts,WTI USE displ_op
PROC next_line               : -> MOD mtexts,WTI USE displ_op
PROC previous_line           : -> MOD mtexts,WTI USE displ_op
PROC beginning_of_line       : -> MOD mtexts,WTI USE displ_op
PROC end_of_line             : -> MOD mtexts,WTI USE displ_op
PROC beginning_of_buffer     : -> MOD mtexts,WTI USE displ_op
PROC end_of_buffer           : -> MOD mtexts,WTI USE displ_op
PROC set_mark                : -> MOD mtexts,WTI USE displ_op
PROC exchange_dot_and_mark   : -> MOD mtexts,WTI USE displ_op
```

We describe the operations in pre- and postcondition style. TI must be respected, which for the operations of this group amounts to the restriction of TI1 that we may not move dot or mark to a position that is non-existing in the text. We have several similar axioms and in order to save space the following premiss serves as a common clause for the specification of several operations:

```
AXIOM WTI => (
LET mtext':MText; mtext' = app(mtexts,current);
LET i:Nat,j:Nat;  i,j := dot(mtext');
```

Our first operations are two Boolean procedures named bolp and eolp abbreviating *beginning-of-line predicate* and *end-of-line predicate.*

```
[ LET b:Bool; b := bolp ]
( b = true <=> j = 0 );

[ LET b:Bool; b := eolp ]
( b = true <=> len(sel(text(mtext'),i)) = j );
```

We used the selection operation sel: Text -> Line which – somewhat implicitly – was constructed in Section 4.3.2. The four next operations deal with moving the dot rightwards, leftwards, downwards and upwards respectively. Each operation has two clauses: one for the normal case and one for the situation where it has no effect because TI1 must be respected. The axioms below show a difference between our editor and EMACS [4]. The difference lies in the effect of trying to move rightwards when the dot is at the end of a line. In that case our editor does nothing whereas in EMACS the dot moves to the beginning of the next line. Similar differences exist for leftward, downward and upward movements.

In view of the fact that mtexts is a map based on MAP_SPEC of Appendix B , we can use the operation add: Map # String -> MText for describing how the marked text addressed by current is overwritten.

It is interesting to analyse the specification of previous_line below for the particular case that the vertical co-ordinate of the dot is 0, i.e. $i = 0$. In that case pred(i) is undefined and therefore intext(...,pred(i),j) is false. We see that the postcondition mtexts = PREV mtexts applies – so nothing happens.

```
intext(text(mtext'),i,succ(j)) =>
[ forward_character ]
( LET new_dot:  Copa;  new_dot   := copa(i,succ(j));
  LET new_mtext:MText; new_mtext := moddot(mtext',new_dot);
      mtexts = add((PREV mtexts),current,new_mtext) );

NOT intext(text(mtext'),i,succ(j)) =>
[ forward_character ]
( mtexts = (PREV mtexts) );

gtr(j,0) =>
[ backward_character ]
( LET new_dot:Copa; new_dot := copa(i,pred(j))
      {rest as for forward_character} );

NOT gtr(j,0) =>
[ backward_character ]
( mtexts = (PREV mtexts) );

intext(text(mtext'),succ(i),j) =>
[ next_line ]
( LET new_dot:Copa; new_dot := copa(succ(i),j)
      {rest as for forward_character} );

NOT intext(text(mtext'),succ(i),j) =>
[ next_line ]
( mtexts = (PREV mtexts) );

intext(text(mtext'),pred(i),j) =>
[ previous_line ]
( LET new_dot:Copa; new_dot := copa(pred(i),j)
      {rest as for forward_character} );
```

```
NOT intext(text(mtext').pred(i).j) =>
[ previous_line ]
( mtexts = (PREV mtexts) );
```

The next operations deal with moving the dot to the extreme positions in a line. This is easy because there is no danger of moving dot to a non-existing position.

```
[ beginning_of_line ]
( LET new_dot:Copa; new_dot := copa(i,0)
      {rest as for forward_character} );

[ end_of_line ]
( LET new_dot:Copa; new_dot := copa(i,len(sel(text(mtext'),i)))
      {rest as for forward_character} );
```

The next operations deal with moving the dot to the extreme positions within the buffer. Again there is no danger of moving dot to a non-existing position.

```
[ beginning_of_buffer ]
( LET new_dot:Copa; new_dot := copa(0,0)
      {rest as for forward_character} );

[ end_of_buffer ]
( LET new_dot:Copa; new_dot := copa(reach(text(mtext')))
      {rest as for forward_character} );
```

The following operations deal with controlling the mark. WTI and hence TI1 hold in the begin-state. Therefore we know that both dot and mark are at existing positions and this remains the case if we put mark at dot or if we exchange dot and mark. This shows that there is no danger of violating TI1.

```
[ set_mark ]
( LET new_mark: Copa;  new_mark  := dot(mtext');
  LET new_mtext:MText; new_mtext := modmark(mtext',new_mark);
      mtexts = add((PREV mtexts),current,new_mtext) );
```

```
[ exchange_dot_and_mark ]
( LET new_dot :Copa; new_dot  := mark(mtext');
  LET new_mark:Copa; new_mark := dot(mtext');
  LET new_mtext:MText;
  new_mtext := modmark(moddot(mtext',new_dot),new_mark);
      mtexts = add((PREV mtexts),current,new_mtext) )

)
```

4.5.7 Operations for Text Modification

These operations serve for changing the text of some marked text and most of them modify the *current* marked text. To begin with, we give the signature of the operations from this group.

```
PROC insert_file: 'String' ->
MOD  mtexts,WTI USE displ_op, reset, read

PROC insert_character     : Char     -> MOD mtexts,WTI USE displ_op
PROC newline              :          -> MOD mtexts,WTI USE displ_op
PROC yank_buffer          : 'String' -> MOD mtexts,WTI USE displ_op
PROC delete_next_character:          -> MOD mtexts,WTI USE displ_op
PROC erase_region         :          -> MOD mtexts,WTI USE displ_op
PROC erase_buffer         : 'String' -> MOD mtexts,WTI USE displ_op
PROC copy_region_to_buffer: 'String' -> MOD mtexts,WTI USE displ_op
```

We describe the operations in pre- and postcondition style and we must make sure that TI is respected. Again we have several similar axioms and we have a common clause for the specification of several operations.

```
AXIOM WTI =>

FORALL s:'String', c:Char (
```

Most operations of this group have the effect that the buffer indicated by `current` becomes modified. In order to define the postcondition of the operation `insert_file`, we need a specification of the underlying file system and so we have imported `FILE_SPEC` as described in Section 4.4.4. Recall that for `s:'String'` we have `f(s)` as the corresponding string and `file(f(s))` as the string contents of the corresponding file. Finally `text(file(f(s)))` denotes the contents of this file viewed as a text. Of course all of this only holds under the assumption that this file exists, which is if `valid(f(s))` holds. The main description tool for the operations `insert_file`, `insert_character`,

newline and yank_buffer is the paste operator on texts as described in Section 4.3.6. Some postconditions look complicated because we must re-calculate the co-ordinates of the dot; furthermore the co-ordinates of the mark may need re-calculation depending on the relative position of mark with respect to dot. In order to describe these re-calculations we use the add and paste operations on reaches from Section 4.3.7. The new dot is given as copa(add(d,reach(t))) but it is interesting to note that we could write copa(paste(d,reach(t),d)) alternatively because
$\forall x, y : \mathtt{Nat}^2$ (paste(x,y,x) = add(x,y)) is a simple property of the algebra of reaches (cf. the definitions of paste, split and add).

```
valid(f(s)) =>
[ insert_file(s) ]
( LET t:Text; t := text(file(f(s)));

  LET mtext':MText; mtext' := app((PREV mtexts),current);
  LET d: Nat # Nat; d := dot(mtext');
  LET m: Nat # Nat; m := mark(mtext');

  LET new_text:Text; new_text := paste(text(mtext'),t,d);
  LET new_dot :Copa; new_dot  := copa(add(d,reach(t)));
  LET new_mark:Copa;
      new_mark := ( lss(m,d)    ?; copa(m)
                  | NOT lss(m,d)?; copa(paste(m,reach(t),d))
                  );
  LET new_mtext:MText;
      new_mtext := mtext(new_text,new_dot,new_mark);
      mtexts = add((PREV mtexts),current,new_mtext) ) ;

printable(c) =>
[ insert_character(c) ]
( LET t:Text; t := addchar(c,zero)
      {rest as for insert_file(s)} );

[ newline ]
( LET t:Text; t := addempty(zero)
      {rest as for insert_file(s)} );

is_in(f(s),dom(mtexts)) =>
[ yank_buffer(s) ]
( LET t:Text; t := text(app((PREV mtexts),f(s)))
      {rest as for insert_file(s)} )

)
```

The main description tool for the operations delete_next_character and erase_region is the cut operator on texts as described in Section 4.3.6. Note that cut always yields a pair (t_1, t_2) where t_1 = 'remaining text' and t_2 = 'deleted text'. We apply cut, retaining its first result and throwing away its second result. Again the co-ordinates of the mark need re-calculation when cutting takes place at dot. As before, we have one premiss serving the specification of several operations.

```
AXIOM WTI =>

FORALL s:'String', c:Char
(  LET mtext':MText; mtext' := app(mtexts,current);

   LET i:Nat,j:Nat; i,j  := dot(mtext');
   LET d: Nat # Nat; d   := dot(mtext') ;
   LET m: Nat # Nat; m   := mark(mtext');
```

Again several operations have two clauses: one for the normal case and one for the case where the operation has no effect. For delete_next_character and erase_region it is not obvious what should happen with mark; we adopt some ad-hoc solution.

```
intext(text(mtext'),i,succ(j)) =>
[ delete_next_character ]
( LET new_text:Text,u:Text;
  new_text,u := cut(text(mtext'),i,j,i,succ(j));
  LET new_dot :Copa; new_dot := copa(d);
  LET new_mark:Copa; new_mark :=
      ( leq(m,d)     ?;
        copa(m)
      | NOT leq(m,d)?;
        LET x: Nat # Nat,y:Nat # Nat; x,y := cut(m,i,j,i,succ(j));
        copa(x)
      );
  LET new_mtext:MText;
  new_mtext := mtext(new_text,new_dot,new_mark);
      mtexts = add((PREV mtexts),current,new_mtext) );

NOT intext(text(mtext'),i,succ(j)) =>
[ delete_next_character ]
( mtexts = (PREV mtexts) );

lss(m,d) =>
[ erase_region ]
( LET new_text:Text,u:Text; new_text,u := cut(text(mtext'),m,d);
     LET new_dot :Copa; new_dot  := copa(m);
     LET new_mark:Copa; new_mark := copa(m)
         {rest as for delete_next_character} );

   NOT lss(m,d) =>
   [ erase_region ]
   ( mtexts = (PREV mtexts) );
```

There are also a few operations that take a 'String' argument addressing a certain buffer to be modified. Again it is not always clear what should happen with mark and dot. In the postcondition of copy_region_to_buffer we use again the cut operation, but now its first result is thrown away and its second result is retained.

```
   [ erase_buffer(s) ]
   ( mtexts
     = add((PREV mtexts),f(s),mtext(zero,copa(0,0),copa(0,0))) );

   lss(m,d) =>
   [ copy_region_to_buffer(s) ]
   ( LET t:Text,new_text:Text;
     t,new_text := cut(text(mtext'),m,d);
     mtexts
     = add((PREV mtexts),f(s),mtext(new_text,copa(0,0),copa(0,0))) );

   NOT lss(m,d) =>
   [ copy_region_to_buffer(s) ]
   ( mtexts = (PREV mtexts) )

   )
```

Let us show now that WTI $\Rightarrow \forall$ s:'String' [copy_region_to_buffer(s)] TI is derivable from the given postcondition of copy_region_to_buffer, as promised earlier. This postcondition states that if mark $\geq$ dot, nothing happens, in which case we are done. So let us assume that mark $<$ dot. We know that mark and dot are *in* the old current text, so consider the expression cut(text(mtext'),m,d) where mtext' is the previous marked text and where m,d are its mark and dot respectively. Then this expression yields two ok texts. The second of these is denoted as new_text and it is added to mtexts together with (0,0) for mark and dot. Note that it is added by the 'overwriting' add of MAP_SPEC. Now TI1 holds because intext(new_text,(0,0)) for arbitrary ok values of new_text. Furthermore TI2 holds because the range of mtexts could only grow, whereas current remained unaffected. This shows that copy_region_to_buffer preserves TI when WTI holds as a precondition.

4.5.8 Operations for Marked-text Management

To begin with, we give the signature of the operations from this group. We considered a procedure list_buffers, but we decided not to include it because it is not clear how we should make it elegantly deliver its output. Similarly we decided not to include a procedure delete_buffer: 'String' → because of the question of what should happen when the current marked text is deleted, which is problematic in view of TI2.

```
PROC current_buffer_name: -> 'String'
MOD  WTI

PROC write_named_file: 'String' ->
MOD  WTI
USE  rewrite, write

PROC switch_to_buffer: 'String' ->
MOD  mtexts, current, WTI
USE  displ_op
```

The operations are described by their postconditions. As before, we have one premiss serving the specification of several operations.

```
AXIOM WTI =>

FORALL s:'String' (
```

We give the description of the operations below. In order to define the postcondition of write_named_file we refer again to the file system of Section 4.4.4. The operation switch_to_buffer may create a new marked text – not in the sense of dynamic object creation of course. We take the ad-hoc value zero for its text component and then TI1 dictates the value (0,0) for dot and mark.

```
[ LET t:'String'; t := current_buffer_name ]
( f(t) = current );

[ write_named_file(s) ]
( valid(f(s));
  file(f(s)) = string(text(app(mtexts,current)));
  FORALL t:'String'
  ( NOT f(t) = f(s) => valid(f(s)) <=> PREV valid(f(s)) );
```

```
    FORALL t:'String',u:String
    ( NOT f(t) = f(s) => file(f(s)) = u <=> PREV file(f(s)) = u ) );

  [ switch_to_buffer(s) ]
  ( current = f(s);
    is_in(f(s),dom(PREV mtexts))     => mtexts = (PREV mtexts);
    NOT is_in(f(s),dom(PREV mtexts)) => mtexts =
        add((PREV mtexts),f(s),mtext(zero,copa(0,0),copa(0,0))) )

  )
```

4.5.9 Operations for Searching

We describe just one operation for searching. It takes a `'String'` argument which is interpreted as the string representation of the search text. The main description tool is the `search` operation of Section 4.3.10 which was introduced precisely for this purpose.

```
  PROC search_forward: 'String' -> MOD mtexts,WTI USE displ_op

  AXIOM WTI =>

  FORALL  s:'String' (
   [ search_forward(s) ]
   ( LET mtext':MText; mtext' := app((PREV mtexts),current);
     LET first_part:Text,second_part:Text;
     first_part, second_part := split(text(mtext'),dot(mtext'));
     LET i:Nat,j:Nat; i,j := search(second_part,text(f(s)));

     (i,j) = reach(second_part)      => mtexts = (PREV mtexts);

     NOT (i,j) = reach(second_part) =>
           ( LET new_dot:  Copa;
             new_dot := copa(add(reach(first_part),i,j));
             LET new_mtext:MText;
             new_mtext := mtext(text(mtext'),new_dot,mark(mtext'));
                 mtexts = add((PREV mtexts),current,new_mtext) ) ) )
```

In the above precondition the clause `(i,j) = reach(second_part)` denotes the situation where the search text does not occur.

4.5.10 Operations for String Conversion

We expect that there will arise applications for a conversion of the text of a marked text into a string. One such application will be presented in Section 4.5.13 when we shall describe a simple editor, where the contents of the marked text named "mini" is converted to a string. The converse can be programmed by using the hd and tl operations on strings and the insert_character operation. We introduce an operation buffer_to_string taking a 'String' argument. This argument is interpreted as the name of the marked text whose text at its turn is to be converted to 'String'. The operation introduced here has *no* modification rights, except for WTI of course. We use the abstraction function f: 'String' → String and the function string, converting texts into strings.

```
PROC buffer_to_string: 'String' -> 'String'
MOD  WTI
```

The effect of this operation is described again by a pre- and postcondition style axiom.

```
AXIOM WTI =>

FORALL s:'String' (
 app(mtexts,f(s))! =>
 [ LET t:'String'; t := buffer_to_string(s) ]
 ( f(t) = string(text(app(mtexts,f(s)))) ) )
```

4.5.11 Termination Axioms

We specify the termination of the procedures of the witefa_op group in the sense that for each procedure invocation there is a state that can be reached as its final state. This is a rather weak form of termination; but if furthermore we happen to know that the algorithmic constructs used in its implementation are deterministic, then it implies the strong form of termination in the sense that every procedure invocation terminates. Actually we relativize the termination axioms by the assertion WTI which was introduced in Section 4.5.5. As a naive attempt, we could try to write the termination axiom as AXIOM WTI ⇒ < witefa_op > TRUE, but this axiom is far too weak for being interesting; it just states that at least one operation can terminate for some argument value. So we have to provide one assertion for each operation.

```
AXIOM WTI => (
< FLUSH bolp > TRUE;
< FLUSH eolp > TRUE;
< forward_character > TRUE;
< backward_character > TRUE;
< next_line > TRUE;
< previous_line > TRUE;
< beginning_of_line > TRUE;
< end_of_line > TRUE;
< beginning_of_buffer > TRUE;
< end_of_buffer > TRUE;
< set_mark > TRUE;
< exchange_dot_and_mark > TRUE;

FORALL s:'String' ( valid(f(s)) => < insert_file(s) > TRUE );
FORALL c:Char     < insert_character(c) > TRUE;

< newline > TRUE;

FORALL s:'String' ( app(mtexts,f(s))! => < yank_buffer(s) > TRUE );

< delete_next_character > TRUE;
< erase_region > TRUE;

FORALL s:'String' < erase_buffer(s) > TRUE;
FORALL s:'String' < copy_region_to_buffer(s) > TRUE;

< FLUSH current_buffer_name > TRUE;

FORALL s:'String' < write_named_file(s) > TRUE;
FORALL s:'String' < switch_to_buffer(s) > TRUE;
FORALL s:'String' < search_forward(s) >   TRUE;

FORALL s:'String'
( app(mtexts,f(s))! => < FLUSH buffer_to_string(s) > TRUE ) )
```

4.5.12 Connecting the Editor with the Display

The class descriptions FILL_SPEC and LOOK_SPEC of Section 4.3.11 provide enough machinery for formulating the 'window-invariant' of an editor. This invariant describes the relation between the current marked text on the one hand and the screen and the cursor of the display on the other hand. Note that WITEFA_SPEC imports APP_DOM_SPEC which encompasses LOOK_SPEC and FILL_SPEC. The main purpose of this section is to introduce one additional axiom that applies to the window-and-text facility.

We suppose that we have *one window* the size of which is given by the values of li and co and we leave the treatment of multi-window features as a generalisation for later. We start by introducing an auxiliary function size.

```
FUNC size: -> Nat # Nat
DEF  li,co
```

We introduce several more auxiliaries for notational purposes.

```
FUNC text: -> Text
DEF  text(app(mtexts,current))

FUNC dot: -> Nat # Nat
DEF  dot(app(mtexts,current))
```

Informally the window-invariant states that the window should correspond with a 'look' to the text, if necessary filled with blanks, such that the dot is visible as the cursor.

We do not need a separate concept of 'window'. Instead of that, it is sufficient to deal with the *position* and the *size* of a window. The position of a window can be given by a co-ordinate pair (o_1, o_2) that indicates the position of the leftmost-uppermost corner of the window. We refer to this as the *screen-origin* or just as the *origin* The following picture sketches part of the situation.

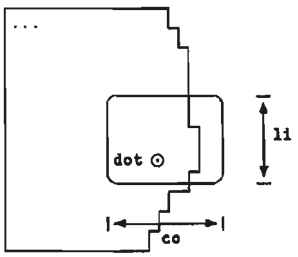


Fig 4.4. The relation between window and text.

We introduce a function p_sub which describes pair-wise subtraction of co-ordinates. It serves for calculating the position of the cursor from given dot and given screen-origin. We also define a function p_add which describes pair-wise addition of co-ordinates.

```
FUNC p_sub: Nat # Nat # Nat # Nat -> Nat # Nat
PAR  d1:Nat, d2:Nat, o1:Nat, o2:Nat
DEF  sub(d1,o1), sub(d2,o2)

FUNC p_add: Nat # Nat # Nat # Nat -> Nat # Nat
PAR  o1:Nat, o2:Nat, s1:Nat, s2:Nat
DEF  add(o1,s1), add(o2,s2)
```

Now we must formulate our window-invariant which we define as a predicate `WI`. We must also deal with the problem that the text may contain non-printables. Fortunately we earlier (Section 4.3.5) introduced the `printify: Text -> Text` operation, so this problem gets solved by introducing an application of `printify` in `WI`.

```
PRED WI:
DEF  LET origin: Nat # Nat; origin := p_sub(dot,cursor);
     LET filled: Text; filled := fill(text,p_add(origin,size));
     screen = printify(look(filled,origin,p_add(origin,size)))
```

The requirement that `WI` is an observational invariant should be considered as a part of the specification of our window-and-text facility. This requirement should be added to the axioms `WTI1` and `WTI2` given in Section 4.5.5 and we do so now by writing another axiom labeled `WTI3`.

```
AXIOM {WTI3} WTI => WI
```

We must add two remarks about this last axiom. The first remark is that we have chosen for a certain editor-behaviour that is based on similar concepts for the window-oriented subsetting of texts for the horizontal and for the vertical direction. When the text is too large in the vertical direction, it is shortened by the application of `vlook`. When the text is too large in the horizontal direction, it is shortened by the application of `hlook`. The user of the editor perceives this as a kind of leftward text-movement when typing long lines. Probably it is dependent on the kind of texts being edited whether this approach is convenient or not.

The second remark is that our specification still leaves a certain amount of implementation freedom, where some of this freedom is unacceptable from an ergonomic point of view. In particular, nothing has been said about the stability of the position of the window with respect to the text. For example there is no formal requirement that always should be tried to re-use the previous origin value unless this turns out impossible.

```
END; {of WITEFA_SPEC}
```

This concludes the specification of the window-and-text facility.

4.5.13 MOREDOP: More Editing Operations

In this section we shall present a simple editor which can be built on top of the window-and-text facility presented above. This is just a small example editor and more powerful and user-friendly editors can be built in a similar way. The editing primitives of WITEFA_SPEC are of a general-purpose nature in the sense that we can imagine that they are useful for *any* editor. Now we address the construction of one particular editor with characteristics that it has a fixed number of dedicated buffers and a particular kill/yank mechanism. Therefore we start with the definition of some additional operations which are specific for this editor. They are put together in a class description MOREDOP_SPEC (for MORe EDiting OPerations). Our simple editor uses precisely three marked texts:

- "mini" which serves for inputting search texts and file names,
- "kill" which will be used in combination with the operations yank_buffer and copy-region-to-buffer, thereby providing *cut and paste* capabilities at the user-level,
- "main" which essentially contains the text being edited.

The user can switch from "main" to "mini" by escape and from "mini" back again to "main" by either escape or return.

We introduce three 'String' constants which serve as the fixed names for the three marked texts. The procedure startup serves for initialisation. The remaining operations are typical editing commands.

```
LET MOREDOP_SPEC :=
IMPORT NAT_SPEC       INTO
IMPORT BOOL_SPEC      INTO
IMPORT CHAR_SPEC      INTO
IMPORT WITEFA_SPEC    INTO
CLASS

   PROC mini: -> 'String'
   DEF  cons('m',cons('i',cons('n',cons('i',empty))))

   PROC main: -> 'String'
   DEF  cons('m',cons('a',cons('i',cons('n',empty))))
```

```
  PROC kill: -> 'String'
  DEF  cons('k',cons('i',cons('l',cons('l',empty))))

  PROC startup: ->
  DEF  init(mini);
       switch_to_buffer(kill);
       switch_to_buffer(main)

  PROC escape: ->
  DEF  ( eq(current_buffer_name,mini)     ?; switch_to_buffer(main)
       | NOT eq(current_buffer_name,mini) ?; switch_to_buffer(mini)
       )

  PROC return: ->
  DEF  ( eq(current_buffer_name,mini)     ?; switch_to_buffer(main)
       | NOT eq(current_buffer_name,mini) ?; newline
       )

  PROC delete_to_killbuffer: ->
  DEF  copy_region_to_buffer(kill);
       erase_region

  PROC yank_from_killbuffer: ->
  DEF  yank_buffer(kill)

  PROC search_forward: ->
  DEF  search_forward(buffer_to_string(mini))

  PROC insert_file: ->
  DEF  insert_file(buffer_to_string(mini))

  PROC write_named_file: ->
  DEF  write_named_file(buffer_to_string(mini))

  PROC delete_previous_character: ->
  DEF  ( bolp = true  ?; SKIP
       | bolp = false ?;
                backward_character; delete_next_character
       )

END;
```

It is interesting to compare the definition of `delete_previous_character` with the way one would program it on top of EMACS [4] where we assume `skip` to be defined by `(defun (skip (progn)))`:

```
   (defun (delete-previous-character
          (if (bolp)
              (skip)
              (progn (backward-character) (delete-next-character)))))
```

4.5.14 KEYBIND: Connecting the Editor with the Keyboard

In order to complete the simple editor, we need a procedure key, say which takes characters as its argument and which invokes operations described by WITEFA_SPEC and the additional editing operations from Section 4.5.13. Essentially the procedure key is defined by one large case-statement. In order to keep things simple, we omitted features such as escape-prefixes [4].

```
LET KEYBIND_SPEC  :=
EXPORT

   SORT Nat,
   SORT Char,
   SORT Text,
   PROC startup:        -> ,
   PROC key     : Char -> ,
   FUNC screen :        -> Text,
   FUNC cursor :        -> Nat # Nat

FROM
IMPORT NAT_SPEC     INTO
IMPORT CHAR_SPEC    INTO
IMPORT WITEFA_SPEC  INTO
IMPORT MOREDOP_SPEC INTO
CLASS

   PROC key: Char ->
   PAR  c:Char
   DEF  ( printable(c)      ?;  insert_character(c)
        | ord(c) = 0  {^@} ?;  set_mark
        | ord(c) = 1  {^A} ?;  beginning_of_line
        | ord(c) = 2  {^B} ?;  backward_character
        | ord(c) = 4  {^D} ?;  delete_next_character
        | ord(c) = 5  {^E} ?;  end_of_line
        | ord(c) = 6  {^F} ?;  forward_character
        | ord(c) = 13 {^M} ?;  return
        | ord(c) = 14 {^N} ?;   next_line
        | ord(c) = 16 {^P} ?;   previous_line
        | ord(c) = 19 {^S} ?;   search_forward
        | ord(c) = 20 {^T} ?;   insert_file
        | ord(c) = 21 {^U} ?;   write_named_file
        | ord(c) = 23 {^W} ?;   delete_to_killbuffer
        | ord(c) = 24 {^X} ?;   beginning_of_buffer
        | ord(c) = 25 {^Y} ?;   yank_from_killbuffer
        | ord(c) = 26 {^Z} ?;   end_of_buffer
        | ord(c) = 27 {ESC}?;   escape
        | ord(c) = 127{DEL}?;   delete_previous_character
        )

END;
```

This concludes the construction of the formal specification of the editor.

4.6 Related Work

Meyer et. al. [12] describe a strategy for displaying structured objects such as programs on a screen of limited size. They develop a formal model of the screen allocation, called 'calculus of windows'. Feldman [13] describes a text editor in a functional style using FP. Just as in our formalisation, text is considered to be a sequence of lines, each of which is a sequence of characters. Gutknecht [15] describes the text editor LARA. This seems to be a kind of combination between a formatting system and an editor, based on the what-you-see-is-what-you-get principle. If we compare this with our approach, it can be seen that Gutknecht puts much more (supposed) knowledge about the structure of documents into his editor than we did.

Sufrin [14] gives an elaborate specification of a display-oriented text editor. Unlike we do, he considers text as a sequence of characters with new-line symbols. His texts are somewhat similar to our marked texts in the sense that they have one pointer to a position in the text associated with them. This is done by defining text such that each text consists of two sequences, which should be appended and the splitting-point corresponds with the dot position.

Partsch [16] describes the specification and transformation process of a simple line-oriented editor using a sugared version of CIP-L. First of all, we should note that Partsch is in a comfortable position since he assumes a lot of syntactic sugar. Since COLD-K is a kernel language it is somewhat more Spartan and we assumed many features *not* to be available (yet). The text editor in [16] is single-buffer and line-oriented. There is nothing comparable with our window-invariant WI. Furthermore he does not deal with file handling. However it should be remarked that he describes *undo* facilities, some of which in our case are absent. A text is considered as a sequence of lines, just as in our formalisation. In the formal model of the editor, the text being edited is called the "current text file" and it is modelled as a triple $\langle t_1, l, t_2 \rangle$, where t_1 is the text *before* the current line, l is the current line and t_2 is the text *after* the current line.

Guttag and Horning [17] present a formal state-based model of a simple display, comparable with our DISPLAY_SPEC. They also present a formalisation of 'text' including pictures, views and line/paragraph-breaking facilities. In particular, a text is viewed as a sequence of paragraphs where each paragraph at its turn is a formatted English string.

4.7 Looking Back

In this section, we shall in restrospect summarise the main lines of the work presented in this chapter. One of the main purposes of constructing the formal specification presented in this chapter was to illustrate the use of formal specification techniques. In particular, we wanted to show how the language COLD-K can be used as a tool for describing complex systems. Let us explicitly point out some of the interesting points encountered during this case study.

The standard class descriptions of Appendix B. provide us with the possibility to use standard mathematical data types such as natural numbers, sequences, maps etc. It is hard to imagine how we could specify an editor *without* using such data types. These standard class descriptions have been written in a certain axiomatic style and this style is different from the style of specification used during the rest of our editor specification. This observation is encouraging, for it suggests that the ability to re-invent the specification of all these standard mathematical data types need not be part of the skills of the avarage software developer willing to use formal specification techniques. This is why we put BOOL_SPEC, NAT_SPEC etc. in an appendix. These class descriptions are indispensable but atypical.

After the introduction of Section 4.1 we investigated in Section 4.3 the important concept of text and algebraic operations on text. At several occasions we used the mathematical data type of Seq from Appendix B . We introduced several operations on texts and instead of using an axiomatic approach, we just *defined* these operations. In particular, we used recursive definitions which are easily executable. This means that in an early phase of the specification construction some experiments can be done. The possibility to do such experiments can be considered as a tool for establishing a close correspondence between the necessary intuition about these operations and the formal definitions. Of course this may not be the only tool and therefore we immediately mention the tool of *reasoning* about the formal definitions. We searched for suitable algebraic laws for our operations. It was our experience that this helps in establishing a set of well-understood operations with useful notations. Let us mention one example: when we first defined the operation named `hsplit` in Section 4.3.11 we thought of it as 'vertical split' because there is a vertical splitting-line. Later, when looking for its inverse, this turned out to be `h_add` of Section 4.3.9. But then we immediately saw the inconsistency in the terminology and we changed from 'vertical split' to 'horizontal split'.

We showed that elementary and almost trivial operations form the basis for powerful cut and paste operations, which indeed correspond with cut and paste capabilities as found in editors. Similarly the elementary `hsplit` and `vsplit` form the basis for more complicated operations that play a role in the necessary subsetting of texts as required due to the physical limitations of the display device.

We 'discovered' a wealth of algebraic systems related to text such as the algebras of strings, reaches and profiles. We investigated the mappings between them which turned out to have interesting properties. As has been shown, the knowledge of these algebraic systems and the mappings between them can be used for specification purposes. We also believe that this knowledge is equally fruitful when *implementing* a text processing system.

In Section 4.4 we described several class descriptions which are important for interfacing the editor with its environment. In Section 4.4.2 we described a display, which can be viewed as a component that is not really part of the actual editor, but which is indispensable for the editor. We think that this phenomenon to include such descriptions is typical for the specification of many realistic and relevant systems. An important methodological point was demonstrated by introducing the display-invariant $\mathtt{DI1} \wedge \mathtt{DI2} \wedge \mathtt{DI3}$ first. In this way we had a consistency-criterion for checking the pre- and postcondition style specifications that were written after that.

We introduced the sorts `'Seq'` and `'String'`, deliberately facing the problems that if we want to provide the implementation of a data type ourselves, then this data type probably comes with an eq-predicate rather than with just equality. Of course one is lucky if some data type is built-in into the programming language and if it comes with a usable equality; but we wanted to show the specification techniques required when this is not the case.

In Section 4.4.4 we introduced a file system, which also can be viewed as a component that is not really part of the actual editor, but that still is indispensable. The most important guideline when introducing this file system, was to restrict ourselves. The complete description of a file system is a serious enterprise in itself and we had to cut-off many possibilities.

In Section 4.5 we started describing the actual text editor. The study of algebraic operations on texts turned out to be fruitful. For example, the `copy-region-to-buffer` and `yank-buffer` operations are just direct applications of certain algebraic operators. We were able to use the Procrustean operations of Section 4.3.11 for formalising the display-management of the editor in a compact and abstract way.

In Section 4.5.5 we discussed several notions of 'invariant' and finally chose for the option of observational invariant. Still, the other notions of invariance are useful in their own right; e.g. the definition of observational invariant refers to the existence of a classical invariant. Furthermore we had a kind of 'almost invariance' of `TI` in the sense that `init` establishes `TI` and that `WTI` $\Rightarrow$ `[ witefa_op ] TI`. The latter property constitutes already a consistency check on our specification. In Sections 4.5.6 to 4.5.10 we demonstrated another style of pre- and postcondition-style axioms, by writing the postconditions in-situ in the axioms.

The formal specification of Section 4.5 covers a number of important aspects of a text editor, although the editor described is relatively poor in its bells and whistles. However it is far from trivial and its functionality makes it a usable editor. We were able to cover several interesting features also present in other text editors and in particular in EMACS.

It is important to notice that the application-domain specific framework would have enabled several other choices for the editor behaviour as well. E.g. the question whether the dot should or should not stay within the boundaries of the current text could be discussed. Similarly we had got all notations and techniques for introducing an arbitrary number of markers at hand.

We deliberately did not model the current text as a pair of sequences $\langle t_1, t_2 \rangle$ where t_1 is the text before the current position and t_2 is the text after it. Essentially this is the model adopted by [14] and [16]. This model more or less collapses if one wants many markers or a marker that goes outside the text. The price we had to pay for our 'marked-text-as-triples' approach is that occasionally we had to describe clumsy re-calculations of co-ordinate pairs.

We showed how $\lambda\pi$-calculus and COLD-K can be used as a tool for the description of a relatively large and complex software system. By doing so we illustrated a number of general-purpose specification techniques. This chapter presents one more example of a large formal specification and as such, it can be viewed as is a contribution to the advancement of formal specification techniques in general. Note that the ability to construct large formal specifications supports the applicability of the notions of component, black-box description and design (Chapter 2) and the correctness-preserving transformations of designs investigated in Chapter 3. In Chapter 5 we apply the techniques of Chapter 2 and 3 in an implementation activity where the current chapter serves as a starting point. We postpone a more complete evaluation of the editor case study until the end of this implementation activity.

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Appendix A

List of Symbols

In this appendix we give a list of the sorts, functions, predicates and procedures used. For each symbol the list contains a short informal description. The list has been subdivided into a number of sub-lists. The first sub-list contains the symbols used from B. The second sub-list contains the symbols that are introduced in Section 4.3. The third sub-list contains the symbols that are introduced in Section 4.4 etc.

Symbols from the standard class descriptions

`Bool`	Booleans
`true` : $\rightarrow$ `Bool`	constant true
`false`: $\rightarrow$ `Bool`	constant false
`not` : `Bool` $\rightarrow$ `Bool`	negation
`and` : `Bool`^{2} $\rightarrow$ `Bool`	conjunction
`or` : `Bool`^{2} $\rightarrow$ `Bool`	disjunction
`Nat`	natural numbers
`zero`: $\rightarrow$ `Nat`	constant 0
`succ`: `Nat` $\rightarrow$ `Nat`	successor
`pred`: `Nat` $\rightarrow$ `Nat`	predecessor
`lss` : `Nat`^{2}	less than
`leq` : `Nat`^{2}	less or equal
`gtr` : `Nat`^{2}	greater than
`geq` : `Nat`^{2}	greater or equal
`add` : `Nat`^{2} $\rightarrow$ `Nat`	addition
`sub` : `Nat`^{2} $\rightarrow$ `Nat`	subtraction
`mul` : `Nat`^{2} $\rightarrow$ `Nat`	multiplication
`div` : `Nat`^{2} $\rightarrow$ `Nat`	division
`0`: $\rightarrow$ `Nat`	constant 0
`1`: $\rightarrow$ `Nat`	constant 1

`2: → Nat`	constant 2
`etc.`	
`Char`	ASCII characters
`ord: Char → Nat`	conversion function
`chr: Nat → Char`	conversion function
`'a': → Char`	constant 'a'
`'i': → Char`	constant 'i'
`etc.`	
`Tup`	2-tuples (= pairs)
`tup  : Item1 # Item2 → Tup`	pairing
`proj1: Tup           → Item1`	taking first field
`proj2: Tup           → Item2`	taking second field
`Tup`	3-tuples (=triples)
`tup  : Item1 # Item2 # Item3 → Tup`	triple construction
`proj1: Tup                   → Item1`	taking first field
`proj2: Tup                   → Item2`	taking second field
`proj3: Tup                   → Item3`	taking third field
`Set`	finite sets
`is_in: Item # Set`	element predicate
`empty: → Set`	empty set
`Seq`	finite sequences
`empty:            → Seq`	empty sequence
`cons : Item # Seq → Seq`	sequence construction
`hd   : Seq        → Item`	head
`tl   : Seq        → Seq`	tail
`len  : Seq        → Nat`	length
`sel  : Seq # Nat  → Item`	selection
`cat  : Seq # Seq  → Seq`	concatenation
`rev  : Seq        → Seq`	reversal
`bag  : Seq        → Bag`	conversion to bag
`Map`	finite mappings
`empty:                     → Map`	empty mapping
`add  : Map # Item1 # Item2 → Map`	'overwriting' addition
`rem  : Map # Item1         → Map`	removal
`app  : Map # Item1         → Item2`	map-application
`dom  : Map                 → Set1`	domain
`ran  : Map                 → Set2`	range

Symbols concerning Text and Algebraic Operations on Texts

Line	lines (= sequences of chars)
Text	texts (= sequences of lines)
niltext : → Text	the text with 0 lines
zero : → Text	the text with one empty line
addempty: Text→ Text	put an empty line before ..
addchar : Char # Text → Text	put a character in front of ..
first : Text → Char	first character
rest : Text → Text	text except for its first char
String	strings (= sequences of chars)
empty: → String	empty string
cons : Char # String → String	string construction
hd : String → Char	head
tl : String → String	tail
len : String → Nat	length
sel : String # Nat → Char	selection
cat : String # String → String	concatenation
less : String # String	lexicographical ordering
ctr_j : → Char	control-j
ok : Text	non-niltext and no control-j's
text : String → Text	conversion
string : Text → String	conversion
blank : → Char	constant ' '
tilde : → Char	constant '~'
printable: Char	printable (no control chars)
printify : Char → Char	make printable
printify : Line → Line	make printable
printify : Text → Text	make printable
split: Text # Nat^2 → Text^2	inverse of natural addition
add : Text^2 → Text	natural addition of texts
cut : Text # Nat^2 # Nat^2 → Text^2	cutting a piece out of a text
paste: Text^2 # Nat^2 → Text	pasting one text into another
reach: Text → Nat^2	position immediately after
add : Nat^2 # Nat^2→ Nat^2	addition of reaches
split: Nat^2 # Nat^2→ Nat^2 # Nat^2	splitting of reaches
cut : Nat^2 # Nat^2 # Nat^2 → Nat^2 # Nat^2	cutting of reaches
paste: Nat^2 # Nat^2 # Nat^2 → Nat^2	pasting of reaches
lss : Nat^2 # Nat^2	stricly less than
leq : Nat^2 # Nat^2	less or equal
Profile	profiles (= sequences of Nat)
profile : Text → Profile	the profile of a text

`intext : Text # Nat`^{2}	position being in boundaries
`nilprofile: → Profile`	profile of niltext
`v_add : Text`^{2} `→ Text`	vertical addition
`h_add : Text`^{2} `→ Text`	horizontal addition
`empties : Nat → Text`	text with empty lines
`match : String`^{2} `# Nat`	string matching
`match': String`^{2} `# Nat`	string matching with sentinel
`search: String`^{2} `→ Nat`	search operation
`search: Text`^{2} `→ Nat`^{2}	search operation
`split : String # Nat → String`^{2}	splitting a string
`vfill : Text # Nat → Text`	vertical filling
`blanks: Nat → Line`	line with blank characters
`hfill : Text # Nat → Text`	horizontal filling
`fill : Text # Nat`^{2} `→ Text`	filling in two directions
`hsplit: Line # Nat → Line # Line`	horizontal splitting
`hsplit: Text # Nat → Text`^{2}	horizontal splitting
`vsplit: Text # Nat → Text`^{2}	vertical splitting
`hlook : Text # Nat`^{2} `→ Text`	horizontal look
`vlook : Text # Nat`^{2} `→ Text`	vertical look
`look : Text # Nat`^{2} `# Nat`^{2} `→ Text`	text subsetted by 'window'

Symbols concerning Interfacing the Editor with its Environment

`li: → Nat`	number of lines
`co: → Nat`	number of columns
`PROC cr: →`	carriage return
`PROC nl: →`	newline
`PROC bc: →`	backwards cursor
`PROC ce: →`	clear to end-of-line
`PROC cl: →`	erase display
`PROC nd: →`	move cursor right
`PROC up: →`	move cursor up
`PROC cm: Nat`^{2} `→`	cursor motion
`PROC print: Char →`	printable character processing
`PROC displ_op: →`	arbitrary display command
`screen: → Text`	observable contents of screen
`cursor: → Nat`^{2}	cursor position
`DI1:`	display-invariant (conjunct 1)
`DI2:`	display-invariant (conjunct 2)
`DI3:`	display-invariant (conjunct 3)
`blank_text: Nat → Text`	single-line text of blanks
`blank_text: Nat`^{2} `→ Text`	text with blanks

`post_cr: Text # Nat`^{2}` # Text # Nat`^{2}	postcondition-predicate
`post_nl: Text # Nat`^{2}` # Text # Nat`^{2}	postcondition-predicate
`post_bc: Text # Nat`^{2}` # Text # Nat`^{2}	postcondition-predicate
`post_ce: Text # Nat`^{2}` # Text # Nat`^{2}	postcondition-predicate
`post_cl: Text # Nat`^{2}` # Text # Nat`^{2}	postcondition-predicate
`post_nd: Text # Nat`^{2}` # Text # Nat`^{2}	postcondition-predicate
`post_up: Text # Nat`^{2}` # Text # Nat`^{2}	postcondition-predicate
`post_cm: Text # Nat`^{2}` # Nat`^{2}` # Text # Nat`^{2}	postcondition-predicate
`post_print: Text # Nat`^{2}` # Char # Text # Nat`^{2}	postcondition-predicate
`'Item'`	data type representing items
`eq : 'Item' # 'Item'`	equivalence relation
`item_inv:`	invariant needed for 'Item'
`seq_inv :`	invariant needed for 'Seq'
`f : 'Item' → Item`	abstraction function
`f : 'Seq' → Seq`	abstraction function
`'Seq'`	implementable sequences
`empty: → 'Seq'`	representation of empty seq.
`PROC cons: 'Item' # 'Seq' → 'Seq'`	constructor procedure
`hd : 'Seq' → 'Item'`	head
`tl : 'Seq' → 'Seq'`	tail
`eq : 'Seq' # 'Seq'`	external notion of equality
`sel: 'Seq' # Nat → 'Item'`	selection
`PROC cat: 'Seq' # 'Seq' → 'Seq'`	concatenation
`PROC rev: 'Seq' → 'Seq'`	reversal
`'String'`	implementable strings
`f: 'String' → String`	abstraction function
`string_inv:`	invariant needed for 'String'
`empty: → 'String'`	representation of empty string
`PROC cons : Char # 'String' → 'String'`	constructor procedure
`hd : 'String' → Char`	head
`tl : 'String' → 'String'`	tail
`eq : 'String' # 'String'`	external notion of equality
`sel: 'String' # Nat → Char`	selection
`PROC cat: 'String' # 'String' → 'String'`	concatenation
`PROC rev: 'String' → 'String'`	reversal
`less: 'String' # 'String'`	lexicographic ordering
`valid : String`	existence of a file
`file : String → String`	contents of file
`pos : String → Nat`	position of read-pointer
`PROC rewrite: 'String' →`	open file for writing
`PROC reset : 'String' →`	open file for reading

`PROC read : → Char`	reading from file
`PROC write : Char →`	writing to file
`eof:`	end-of-file predicate

Symbols concerning Text Editing

`Copa`	co-ordinate pairs
`v: Copa → Nat`	vertical co-ordinate
`h: Copa → Nat`	horizontal co-ordinate
`copa:` $\texttt{Nat}^2$ `→ Copa`	pairing
`MText`	marked texts
`text : MText → Text`	select text field
`dot : MText → Copa`	select dot field
`mark : MText → Copa`	select mark field
`mtext: Text # Copa # Copa → MText`	marked text construction
`dot : MText→` $\texttt{Nat}^2$	dot of a marked text
`mark : MText→` $\texttt{Nat}^2$	mark of a marked text
`modtext : MText # Text → MText`	modification operation
`moddot : MText # Copa → MText`	modification operation
`modmark: MText # Copa → MText`	modification operation
`mtexts : → Map VAR`	collection of marked texts
`current: → String VAR`	name 'current' marked text
`PROC init: 'String' →`	initialisation procedure
`TI1:`	text-invariant (conjunct 1)
`TI2:`	text-invariant (conjunct 2)
`TI:`	text-invariant
`WTI: VAR`	window and text invariant
`PROC witefa_op: →`	arbitrary operation of ...
`PROC bolp : → Bool`	'witefa' operation
`PROC eolp : → Bool`	'witefa' operation
`PROC forward_character : →`	'witefa' operation
`PROC backward_character : →`	'witefa' operation
`PROC next_line : →`	'witefa' operation
`PROC previous_line : →`	'witefa' operation
`PROC beginning_of_line : →`	'witefa' operation
`PROC end_of_line : →`	'witefa' operation
`PROC beginning_of_buffer : →`	'witefa' operation
`PROC end_of_buffer : →`	'witefa' operation
`PROC set_mark : →`	'witefa' operation
`PROC exchange_dot_and_mark : →`	'witefa' operation
`PROC insert_file : 'String' →`	'witefa' operation

PROC insert_character : Char →	'witefa' operation
PROC newline : →	'witefa' operation
PROC yank_buffer : 'String' →	'witefa' operation
PROC delete_next_character: →	'witefa' operation
PROC erase_region : →	'witefa' operation
PROC erase_buffer : 'String' →	'witefa' operation
PROC copy_region_to_buffer: 'String' →	'witefa' operation
PROC current_buffer_name: → 'String'	'witefa' operation
PROC write_named_file: 'String' →	'witefa' operation
PROC switch_to_buffer: 'String' →	'witefa' operation
PROC search_forward : 'String' →	'witefa' operation
PROC buffer_to_string: 'String' → 'String'	'witefa' operation
size: → Nat^2	the pair (li,co)
text: → Text	current text
dot : → Nat^2	current dot
p_sub: Nat^2 # Nat^2 → Nat^2	pair-wise subtraction
p_add: Nat^2 # Nat^2 → Nat^2	pair-wise addition
WI:	window-invariant
PROC mini: → 'String'	representation of "mini"
PROC main: → 'String'	representation of "main"
PROC kill: → 'String'	representation of "kill"
PROC startup : →	editor operation
PROC escape : →	editor operation
PROC return : →	editor operation
PROC delete_to_killbuffer : →	editor operation
PROC yank_from_killbuffer : →	editor operation
PROC search_forward : →	editor operation
PROC insert_file : →	editor operation
PROC write_named_file : →	editor operation
PROC delete_previous_character: →	editor operation
PROC key: Char →	top-level operation of editor

Appendix B

Standard Class Descriptions

```
DESIGN

% This is a specification of the data type of booleans with
% inductive definitions for the non-constructor operations.
% The inductive definitions have the shape of truth tables.

LET BOOL_SPEC :=

EXPORT

  SORT Bool,
  FUNC true :               -> Bool,
  FUNC false:               -> Bool,
  FUNC not   : Bool         -> Bool,
  FUNC and   : Bool # Bool -> Bool,
  FUNC or    : Bool # Bool -> Bool,
  FUNC imp   : Bool # Bool -> Bool,
  FUNC eqv   : Bool # Bool -> Bool,
  FUNC xor   : Bool # Bool -> Bool

FROM
CLASS

  SORT Bool
  FUNC true : -> Bool
  FUNC false: -> Bool

  AXIOM {BOOL1} true!;
        {BOOL2} false!;
        {BOOL3} NOT true = false
  PRED is_gen: Bool
  IND  is_gen(true);
       is_gen(false)

  AXIOM   FORALL b:Bool
  {BOOL4} is_gen(b)

  FUNC not: Bool -> Bool
  IND  not(true ) = false;
       not(false) = true

  FUNC and: Bool # Bool -> Bool
  IND  and(false,false) = false;
       and(false,true ) = false;
       and(true ,false) = false;
       and(true ,true ) = true

  FUNC or: Bool # Bool -> Bool
  IND  or(false,false) = false;
       or(false,true ) = true;
       or(true ,false) = true;
       or(true ,true ) = true

  FUNC imp: Bool # Bool -> Bool
  IND  imp(false,false) = true;
       imp(false,true ) = true;
       imp(true ,false) = false;
       imp(true ,true ) = true

  FUNC eqv: Bool # Bool -> Bool
  IND  eqv(false,false) = true;
       eqv(false,true ) = false;
       eqv(true ,false) = false;
       eqv(true ,true ) = true

  FUNC xor: Bool # Bool -> Bool
  IND  xor(false,false) = false;
       xor(false,true ) = true;
       xor(true ,false) = true;
       xor(true ,true ) = false

END;
```

```
% This is a specification of the data type of natural numbers
% with inductive definitions for the non-constructor operations.

LET NAT_SPEC' :=

EXPORT

  SORT Nat,
  FUNC zero: -> Nat,
  FUNC succ: Nat -> Nat,
  FUNC pred: Nat -> Nat,
  PRED lss: Nat # Nat,
  PRED leq: Nat # Nat,
  PRED gtr: Nat # Nat,
  PRED geq: Nat # Nat,
  FUNC add: Nat # Nat -> Nat,
  FUNC sub: Nat # Nat -> Nat,
  FUNC mul: Nat # Nat -> Nat,
  FUNC div: Nat # Nat -> Nat,
  FUNC mod: Nat # Nat -> Nat,
  FUNC exp: Nat # Nat -> Nat,
  FUNC log: Nat # Nat -> Nat,
  FUNC max: Nat # Nat -> Nat,
  FUNC min: Nat # Nat -> Nat

FROM
CLASS

  SORT Nat
  FUNC zero: -> Nat
  FUNC succ: Nat -> Nat

  AXIOM
  {NAT1} zero!;
         FORALL m:Nat,n:Nat (
  {NAT2} succ(m)!;
  {NAT3} NOT succ(m) = zero;
  {NAT4} succ(m) = succ(n) => m = n )

  PRED is_gen: Nat
  IND  FORALL m:Nat
       ( is_gen(zero);
         is_gen(m) => is_gen(succ(m)) )
```

```
AXIOM  FORALL n:Nat
{NAT5} is_gen(n)

FUNC pred: Nat -> Nat
IND  FORALL n:Nat
     ( pred(succ(n)) = n )

PRED lss: Nat # Nat
IND  FORALL m:Nat,n:Nat
     ( lss(m,succ(m));
       lss(m,n) => lss(m,succ(n)) )

PRED leq: Nat # Nat
IND  FORALL m:Nat,n:Nat
     ( leq(m,m);
       leq(m,n) => leq(m,succ(n)) )

PRED gtr: Nat # Nat
IND  FORALL m:Nat,n:Nat
     ( gtr(succ(m),m);
       gtr(m,n) => gtr(succ(m),n) )

PRED geq: Nat # Nat
IND  FORALL m:Nat,n:Nat
     ( geq(m,m);
       geq(m,n) => geq(succ(m),n) )

FUNC add: Nat # Nat -> Nat
IND  FORALL m:Nat,n:Nat
     ( add(m,zero) = m;
       add(m,succ(n)) = succ(add(m,n)) )

FUNC sub: Nat # Nat -> Nat
IND  FORALL m:Nat,n:Nat
     ( sub(m,zero) = m;
       gtr(m,n) => sub(m,succ(n)) = pred(sub(m,n)) )

FUNC mul: Nat # Nat -> Nat
IND  FORALL m:Nat,n:Nat
     ( mul(m,zero) = zero;
       mul(m,succ(n)) = add(mul(m,n),m) )
```

```
  FUNC div: Nat # Nat -> Nat
  IND  FORALL m:Nat,n:Nat,q:Nat,r:Nat
       ( m = add(mul(n,q),r) AND lss(r,n) => div(m,n) = q )

  FUNC mod: Nat # Nat -> Nat
  IND  FORALL m:Nat,n:Nat,q:Nat,r:Nat
       ( m = add(mul(n,q),r) AND lss(r,n) => mod(m,n) = r )

  FUNC exp: Nat # Nat -> Nat
  IND  FORALL m:Nat,n:Nat
       ( exp(m,zero) = succ(zero);
         exp(m,succ(n)) = mul(m,exp(m,n)) )

  FUNC log: Nat # Nat -> Nat
  IND  FORALL m:Nat,n:Nat,p:Nat
       ( leq(exp(m,p),n) AND lss(n,exp(m,succ(p))) => log(m,n) = p )

  FUNC max: Nat # Nat -> Nat
  IND  FORALL m:Nat,n:Nat
       ( geq(m,n) => max(m,n) = m;
         leq(m,n) => max(m,n) = n )

  FUNC min: Nat # Nat -> Nat
  IND  FORALL m:Nat,n:Nat
       ( leq(m,n) => min(m,n) = m;
         geq(m,n) => min(m,n) = n )

END;
```

```
% This is a specification of the data type of natural numbers
% based on NAT_SPEC', providing notations for some specific numbers.

LET NAT_SPEC :=

IMPORT NAT_SPEC' INTO

CLASS

  FUNC 0   : -> Nat DEF zero
  FUNC 1   : -> Nat DEF succ(zero)
  FUNC 2   : -> Nat DEF succ(succ(zero))
  FUNC 3   : -> Nat % etc.
  FUNC 4   : -> Nat
  FUNC 5   : -> Nat
  FUNC 6   : -> Nat
  FUNC 7   : -> Nat
  FUNC 8   : -> Nat
  FUNC 9   : -> Nat

  FUNC 10   : -> Nat
  FUNC 11   : -> Nat
  FUNC 12   : -> Nat
  FUNC 13   : -> Nat
  FUNC 14   : -> Nat
  FUNC 15   : -> Nat
  FUNC 16   : -> Nat
  FUNC 17   : -> Nat
  FUNC 18   : -> Nat
  FUNC 19   : -> Nat

  FUNC 20   : -> Nat
  FUNC 21   : -> Nat
  FUNC 22   : -> Nat
  FUNC 23   : -> Nat
  FUNC 24   : -> Nat
  FUNC 25   : -> Nat
  FUNC 26   : -> Nat
  FUNC 27   : -> Nat
  FUNC 28   : -> Nat
  FUNC 29   : -> Nat
  FUNC 30   : -> Nat
  FUNC 31   : -> Nat
  FUNC 32   : -> Nat
  FUNC 72   : -> Nat
  FUNC 74   : -> Nat
  FUNC 89   : -> Nat
  FUNC 97   : -> Nat
  FUNC 98   : -> Nat
  FUNC 99   : -> Nat
  FUNC 100  : -> Nat
  FUNC 101  : -> Nat
  FUNC 102  : -> Nat
  FUNC 103  : -> Nat
  FUNC 104  : -> Nat
  FUNC 105  : -> Nat
  FUNC 106  : -> Nat
  FUNC 107  : -> Nat
  FUNC 108  : -> Nat
  FUNC 109  : -> Nat
  FUNC 110  : -> Nat
  FUNC 111  : -> Nat
  FUNC 112  : -> Nat
  FUNC 113  : -> Nat
  FUNC 114  : -> Nat
  FUNC 115  : -> Nat
  FUNC 116  : -> Nat
  FUNC 117  : -> Nat
  FUNC 118  : -> Nat
  FUNC 119  : -> Nat
  FUNC 120  : -> Nat
  FUNC 121  : -> Nat
  FUNC 122  : -> Nat

  FUNC 126  : -> Nat
  FUNC 127  : -> Nat
  FUNC 128  : -> Nat
  FUNC 1024 : -> Nat

END;
```

```
% This is a specification of the data type of ASCII characters.

LET CHAR_SPEC' :=

EXPORT

  SORT Char,
  SORT Nat,
  FUNC ord: Char -> Nat,
  FUNC chr: Nat -> Char,
  FUNC minchar: -> Char,
  FUNC maxchar: -> Char,
  PRED lsschar: Char # Char

FROM
IMPORT NAT_SPEC INTO
CLASS

  SORT Char
  FUNC min: -> Nat DEF 0
  FUNC max: -> Nat DEF 127

  FUNC ord: Char -> Nat
  FUNC chr: Nat -> Char
  {ord is an embedding and
   chr is its inverse, i.e. a conversion function}

  PRED dom: Nat
  IND  FORALL m:Nat
       ( leq(min,m) AND leq(m,max) => dom(m) )

  PRED is_gen: Char
  IND  FORALL m:Nat ( dom(m) => is_gen(chr(m)) )

  AXIOM {EMBEDDING}

  {1} FORALL m:Nat (  chr(m)! <=> dom(m) );
  {2} FORALL m:Nat ( dom(m) => (ord(chr(m)) = m) );
  {3} FORALL c:Char ( is_gen(c) )

  FUNC minchar: -> Char DEF chr(min)

  FUNC maxchar: -> Char DEF chr(max)
  PRED lsschar: Char # Char
  PAR  c:Char, d:Char
  DEF  lss(ord(c),ord(d))

END;
```

```
% This is a specification of the data type of ASCII characters
% based on CHAR_SPEC', providing notations for some specific chars.

LET CHAR_SPEC :=

IMPORT CHAR_SPEC' INTO
IMPORT NAT_SPEC   INTO
CLASS

  FUNC bell: -> Char DEF chr(7)
  FUNC tab:  -> Char DEF chr(9)

  FUNC 'a': -> Char DEF chr(97)
  FUNC 'b': -> Char DEF chr(98)
  FUNC 'c': -> Char DEF chr(99)
  FUNC 'd': -> Char DEF chr(100)
  FUNC 'e': -> Char DEF chr(101)
  FUNC 'f': -> Char DEF chr(102)
  FUNC 'g': -> Char DEF chr(103)
  FUNC 'h': -> Char DEF chr(104)
  FUNC 'i': -> Char DEF chr(105)
  FUNC 'j': -> Char DEF chr(106)
  FUNC 'k': -> Char DEF chr(107)
  FUNC 'l': -> Char DEF chr(108)
  FUNC 'm': -> Char DEF chr(109)
  FUNC 'n': -> Char DEF chr(110)
  FUNC 'o': -> Char DEF chr(111)
  FUNC 'p': -> Char DEF chr(112)
  FUNC 'q': -> Char DEF chr(113)
  FUNC 'r': -> Char DEF chr(114)
  FUNC 's': -> Char DEF chr(115)
  FUNC 't': -> Char DEF chr(116)
  FUNC 'u': -> Char DEF chr(117)
  FUNC 'v': -> Char DEF chr(118)
  FUNC 'w': -> Char DEF chr(119)
  FUNC 'x': -> Char DEF chr(120)
  FUNC 'y': -> Char DEF chr(121)
  FUNC 'z': -> Char DEF chr(122)

END;
```

```
LET ITEM :=
CLASS
  SORT Item FREE
END;


LET ITEM1 :=
CLASS
  SORT Item1 FREE
END;


LET ITEM2 :=
CLASS
  SORT Item2 FREE
END;


LET ITEM3 :=
CLASS
  SORT Item3 FREE
END;
```

```
% This is a specification of a strict linear ordering.

LET SLO :=

CLASS

  SORT Item   FREE
  PRED r: Item # Item

  AXIOM  FORALL i:Item,j:Item,k:Item (
  {SLO1} NOT r(i,i);
  {SLO2} r(i,j) AND r(j,k) => r(i,k);
  {SLO3} r(i,j) OR r(j,i) OR (i = j) )

END;
```

```
% This is an axiomatic specification of the 2-tuple data type
% with inductive definitions for the non-constructor operations.

LET TUP2_SPEC :=

LAMBDA X : ITEM1 OF
LAMBDA Y : ITEM2 OF
EXPORT

  SORT Tup,
  SORT Item1,
  SORT Item2,
  FUNC tup  : Item1 # Item2 -> Tup,
  FUNC proj1: Tup           -> Item1,
  FUNC proj2: Tup           -> Item2

FROM
IMPORT X INTO
IMPORT Y INTO
CLASS

  SORT Tup   DEP Item1,Item2
  FUNC tup: Item1 # Item2 -> Tup

  AXIOM  FORALL i1:Item1,j1:Item1,i2:Item2,j2:Item2 (
  {TUP1} tup(i1,i2)!;
  {TUP2} tup(i1,i2) = tup(j1,j2) => i1 = j1 AND i2 = j2 )

  PRED is_gen: Tup
  IND  FORALL i1:Item1,i2:Item2
       ( is_gen(tup(i1,i2)) )

  AXIOM  FORALL t:Tup
  {TUP3} is_gen(t)

  FUNC proj1: Tup -> Item1
  IND  FORALL i1:Item1,i2:Item2
       ( proj1(tup(i1,i2)) = i1 )

  FUNC proj2: Tup -> Item2
  IND  FORALL i1:Item1,i2:Item2
       ( proj2(tup(i1,i2)) = i2 )

END;
```

```
% This is an axiomatic specification of the 3-tuple data type
% with inductive definitions for the non-constructor operations.

LET TUP3_SPEC :=

LAMBDA X : ITEM1 OF
LAMBDA Y : ITEM2 OF
LAMBDA Z : ITEM3 OF
EXPORT

  SORT Tup,
  SORT Item1,
  SORT Item2,
  SORT Item3,
  FUNC tup  : Item1 # Item2 # Item3 -> Tup,
  FUNC proj1: Tup                   -> Item1,
  FUNC proj2: Tup                   -> Item2,
  FUNC proj3: Tup                   -> Item3

FROM
IMPORT X INTO
IMPORT Y INTO
IMPORT Z INTO
CLASS

  SORT Tup   DEP Item1,Item2,Item3
  FUNC tup: Item1 # Item2 # Item3 -> Tup

  AXIOM
  FORALL i1:Item1,j1:Item1,i2:Item2,j2:Item2,i3:Item3,j3:Item3 (

  {TUP1} tup(i1,i2,i3)!;

  {TUP2} tup(i1,i2,i3) = tup(j1,j2,j3)
          => i1 = j1 AND i2 = j2 AND i3 = j3 )

  PRED is_gen: Tup
  IND  FORALL i1:Item1,i2:Item2,i3:Item3
       ( is_gen(tup(i1,i2,i3)) )

  AXIOM  FORALL t:Tup
  {TUP3} is_gen(t)
  FUNC proj1: Tup -> Item1
  IND  FORALL i1:Item1,i2:Item2,i3:Item3
       ( proj1(tup(i1,i2,i3)) = i1 )

  FUNC proj2: Tup -> Item2
  IND  FORALL i1:Item1,i2:Item2,i3:Item3
       ( proj2(tup(i1,i2,i3)) = i2 )

  FUNC proj3: Tup -> Item3
  IND  FORALL i1:Item1,i2:Item2,i3:Item3
       ( proj3(tup(i1,i2,i3)) = i3 )

END;
```

```
% This is an axiomatic specification of the data type of finite sets
% with inductive definitions for the non-constructor operations.

LET SET_SPEC :=

LAMBDA X:ITEM OF
EXPORT

  SORT Item,
  SORT Nat,
  SORT Set,
  PRED is_in : Item # Set,
  FUNC empty :              -> Set,
  FUNC ins   : Item # Set -> Set,
  FUNC rem   : Item # Set -> Set,
  FUNC union : Set  # Set -> Set,
  FUNC isect : Set  # Set -> Set,
  FUNC diff  : Set  # Set -> Set,
  PRED subset: Set  # Set,
  FUNC card  : Set        -> Nat

FROM
IMPORT X INTO
IMPORT NAT_SPEC INTO
CLASS

  SORT Set   DEP Item
  PRED is_in: Item # Set
  FUNC empty:             -> Set
  FUNC ins  : Item # Set -> Set

  AXIOM
  {SET1} empty!;
  {SET2} FORALL i:Item,s:Set ( ins(i,s)! )

  AXIOM  FORALL i:Item,j:Item,s:Set (
  {SET3} NOT is_in(i,empty);
  {SET4} is_in(i,ins(j,s)) <=> i = j OR is_in(i,s);
  {SET5} ins(i,ins(j,s)) = ins(j,ins(i,s));
  {SET6} ins(i,ins(i,s)) = ins(i,s) )

  PRED is_gen: Set
  IND  FORALL i:Item,s:Set
       ( is_gen(empty);
         is_gen(s) => is_gen(ins(i,s)) )
  AXIOM  FORALL s:Set
  {SET7} is_gen(s)

  FUNC rem: Item # Set -> Set
  IND  FORALL i:Item,j:Item,s:Set
       ( rem(i,empty) = empty:Set;
         rem(i,ins(i,s)) = rem(i,s);
         NOT i = j => rem(i,ins(j,s)) = ins(j,rem(i,s)) )

  FUNC union: Set # Set -> Set
  IND  FORALL i:Item,s:Set,t:Set
       ( union(s,empty) = s;
         union(s,ins(i,t)) = ins(i,union(s,t)) )

  FUNC isect: Set # Set -> Set
  IND  FORALL i:Item,s:Set,t:Set
       ( isect(s,empty) = empty;
         isect(ins(i,s),ins(i,t)) = ins(i,isect(s,t));
         NOT is_in(i,s) => isect(s,ins(i,t)) = isect(s,t) )

  FUNC diff: Set # Set -> Set
  IND  FORALL i:Item,s:Set,t:Set
       ( diff(s,empty) = s;
         diff(s,ins(i,t)) = rem(i,diff(s,t)) )

  PRED subset: Set # Set
  IND  FORALL i:Item,s:Set,t:Set
       ( subset(s,s);
         subset(s,t) => subset(s,ins(i,t)) )

  FUNC card: Set -> Nat
  IND  FORALL i:Item,s:Set
       ( card(empty) = zero;
         NOT is_in(i,s) => card(ins(i,s)) = succ(card(s)) )

END;
```

```
% This is an axiomatic specification of the data type of finite
% bags, with inductive definitions for the non-constructor
% operations.

LET BAG_SPEC :=

LAMBDA X:ITEM OF
EXPORT

  SORT Item,
  SORT Nat,
  SORT Set,
  SORT Bag,
  PRED is_in : Item # Bag,
  FUNC empty :              -> Bag,
  FUNC ins   : Item # Bag -> Bag,
  FUNC rem   : Item # Bag -> Bag,
  FUNC union : Bag  # Bag -> Bag,
  FUNC isect : Bag  # Bag -> Bag,
  FUNC diff  : Bag  # Bag -> Bag,
  PRED subbag: Bag  # Bag,
  FUNC mult  : Item # Bag -> Nat,
  FUNC set   : Bag        -> Set

FROM
IMPORT X INTO
IMPORT NAT_SPEC INTO
IMPORT APPLY SET_SPEC TO X INTO
CLASS

  SORT Bag   DEP Item
  PRED is_in: Item # Bag
  FUNC empty:             -> Bag
  FUNC ins  : Item # Bag -> Bag

  AXIOM
  {BAG1} empty:Bag!;
  {BAG2} FORALL i:Item,b:Bag ( ins(i,b)! )

  AXIOM  FORALL i:Item,j:Item,b:Bag,c:Bag (
  {BAG3} NOT is_in(i,empty:Bag);
  {BAG4} is_in(i,ins(j,b)) <=> i = j OR is_in(i,b);
  {BAG5} ins(i,ins(j,b)) = ins(j,ins(i,b));
  {BAG6} ins(i,b) = ins(i,c) => b = c )
```

```
PRED is_gen: Bag
IND  FORALL i:Item,b:Bag
     ( is_gen(empty);
       is_gen(b) => is_gen(ins(i,b)) )

AXIOM  FORALL b:Bag
{BAG7} is_gen(b)

FUNC rem: Item # Bag -> Bag
IND  FORALL i:Item,j:Item,b:Bag
     ( rem(i,empty) = empty:Bag;
       rem(i,ins(i,b)) = b;
       NOT i = j => rem(i,ins(j,b)) = ins(j,rem(i,b)) )

FUNC union: Bag # Bag -> Bag
IND  FORALL i:Item,b:Bag,c:Bag
     ( union(b,empty) = b;
       union(b,ins(i,c)) = ins(i,union(b,c)) )

FUNC isect: Bag # Bag -> Bag
IND  FORALL i:Item,b:Bag,c:Bag
     ( isect(b,empty) = empty;
       isect(ins(i,b),ins(i,c)) = ins(i,isect(b,c));
       NOT is_in(i,b) => isect(b,ins(i,c)) = isect(b,c) )

FUNC diff: Bag # Bag -> Bag
IND  FORALL i:Item,b:Bag,c:Bag
     ( diff(b,empty) = b;
       diff(b,ins(i,c)) = rem(i,diff(b,c)) )

PRED subbag: Bag # Bag
IND  FORALL i:Item,b:Bag,c:Bag
     ( subbag(b,b);
       subbag(b,c) => subbag(b,ins(i,c)) )

FUNC mult: Item # Bag -> Nat
IND  FORALL i:Item,j:Item,b:Bag
     ( mult(i,empty) = zero;
       mult(i,ins(i,b)) = succ(mult(i,b));
       NOT i = j => mult(i,ins(j,b)) = mult(i,b) )
```

```
  FUNC set: Bag -> Set
  IND  FORALL i:Item,b:Bag
       ( set(empty) = empty;
         set(ins(i,b)) = ins(i,set(b)) )

END;
```

```
% This is an axiomatic specification of the data type of finite
% sequences, with inductive definitions for the non-constructor
% operations.

LET SEQ_SPEC :=

LAMBDA X : ITEM OF
EXPORT

  SORT Item,
  SORT Nat,
  SORT Bag,
  SORT Seq,
  FUNC empty:             -> Seq,
  FUNC cons : Item # Seq -> Seq,
  FUNC hd   : Seq         -> Item,
  FUNC tl   : Seq         -> Seq,
  FUNC len  : Seq         -> Nat,
  FUNC sel  : Seq # Nat   -> Item,
  FUNC cat  : Seq # Seq   -> Seq,
  FUNC rev  : Seq         -> Seq,
  FUNC bag  : Seq         -> Bag

FROM
IMPORT X INTO
IMPORT NAT_SPEC INTO
IMPORT APPLY BAG_SPEC TO X INTO
CLASS

  SORT Seq   DEP Item
  FUNC empty:             -> Seq
  FUNC cons : Item # Seq -> Seq

  AXIOM
  {SEQ1} empty:Seq!;
  {SEQ2} FORALL i:Item,s:Seq ( cons(i,s)! )

  AXIOM  FORALL i:Item,j:Item,s:Seq,t:Seq  (
  {SEQ3} NOT cons(i,s) = empty;
  {SEQ4} cons(i,s) = cons(j,t) => i = j AND s = t )

  PRED is_gen: Seq
  IND  FORALL i:Item,s:Seq
       ( is_gen(empty),
         is_gen(s) => is_gen(cons(i,s)) )
  AXIOM   FORALL s:Seq
  {SEQ5}  is_gen(s)

  FUNC hd: Seq -> Item
  IND  FORALL i:Item,s:Seq
       ( hd(cons(i,s)) = i )

  FUNC tl: Seq -> Seq
  IND  FORALL i:Item,s:Seq
       ( tl(cons(i,s)) = s )

  FUNC len: Seq -> Nat
  IND  FORALL i:Item,s:Seq
       ( len(empty) = zero;
         len(cons(i,s)) = succ(len(s)) )

  FUNC sel: Seq # Nat -> Item
  IND  FORALL i:Item,j:Item,s:Seq,n:Nat
       ( sel(cons(i,s),zero) = i;
         sel(s,n) = j => sel(cons(i,s),succ(n)) = j )

  FUNC cat: Seq # Seq -> Seq
  IND  FORALL i:Item,s:Seq,t:Seq
       ( cat(empty,s) = s;
         cat(cons(i,s),t) = cons(i,cat(s,t)) )

  FUNC rev: Seq -> Seq
  IND  FORALL i:Item,s:Seq
       ( rev(empty) = empty;
         rev(cons(i,s)) = cat(rev(s),cons(i,empty)) )

  FUNC bag: Seq -> Bag
  IND  FORALL i:Item,s:Seq
       ( bag(empty) = empty;
         bag(cons(i,s)) = ins(i,bag(s)) )

END;
```

```
% This is an axiomatic specification of the data type of finite
% maps, with inductive definitions for the non-constructor
% operations.

LET MAP_SPEC :=

LAMBDA X : ITEM1 OF
LAMBDA Y : ITEM2 OF
EXPORT

  SORT Item1,
  SORT Item2,
  SORT Set1,
  SORT Set2,
  SORT Map,
  FUNC empty:                       -> Map,
  FUNC add   : Map # Item1 # Item2 -> Map,
  FUNC rem   : Map # Item1          -> Map,
  FUNC app   : Map # Item1          -> Item2,
  FUNC dom   : Map                  -> Set1,
  FUNC ran   : Map                  -> Set2

FROM
IMPORT X INTO
IMPORT Y INTO
IMPORT APPLY RENAME SORT Set TO Set1, SORT Item TO Item1
             IN SET_SPEC
       TO X
INTO
IMPORT APPLY RENAME SORT Set TO Set2, SORT Item TO Item2
             IN SET_SPEC
       TO Y
INTO
CLASS

  SORT Map   DEP Item1,Item2
  FUNC empty:                       -> Map
  FUNC add   : Map # Item1 # Item2 -> Map
  FUNC app   : Map # Item1          -> Item2

  AXIOM
  {MAP1} empty:Map!;
  {MAP2} FORALL m:Map,i:Item1,v:Item2
         ( add(m,i,v)! )
```

```
 AXIOM  FORALL i:Item1,j:Item1,v:Item2,w:Item2,m:Map  (
 {MAP3} NOT app(empty,i)!;
 {MAP4} app(add(m,i,v),j) = w <=>
         ( (i = j AND v = w) OR (NOT i = j AND app(m,j) = w) );
 {MAP5} NOT i = j => add(add(m,i,v),j,w) = add(add(m,j,w),i,v);
 {MAP6} add(add(m,i,v),i,w) = add(m,i,w) )

 PRED is_gen: Map
 IND  FORALL m:Map,i:Item1,v:Item2
      ( is_gen(empty);
        is_gen(m) => is_gen(add(m,i,v)) )

 AXIOM  FORALL m:Map
 {MAP7} is_gen(m)

 FUNC rem: Map # Item1 -> Map
 IND  FORALL m:Map,i:Item1,j:Item1,v:Item2
      ( rem(empty,i) = empty;
        rem(add(m,i,v),i) = rem(m,i);
        NOT i = j => rem(add(m,i,v),j) = add(rem(m,j),i,v) )

 FUNC dom: Map -> Set1
 IND  FORALL m:Map,i:Item1,v:Item2
      ( dom(empty) = empty;
        dom(add(m,i,v)) = ins(i,dom(m)) )

 FUNC ran: Map -> Set2
 IND  FORALL m:Map,i:Item1,v:Item2
      ( ran(empty) = empty;
        ran(add(m,i,v)) = ins(v,ran(rem(m,i))) )

END;
```

Chapter 5

Systematic Design of a Text Editor

5.1 Introduction

This chapter describes the implementation of a display-oriented text editor. It serves for illustrating the notions of component, black-box description and design as described in Chapter 2. We shall follow one of the models of the software development process studied in Chapter 3, viz. the top-down model. We take the formal specification presented in Chapter 4 and we make it part of an initial design. This initial design serves as the starting point for a top-down development process to meet conditions of verification, validation and executability.

We shall use the language COLD-K [1] for our case study. The notions of component, black-box description and design as described in Chapter 2 are available in this language. One of the purposes of this chapter is to illustrate the use of design principles based on formal techniques. In particular, we want to show how the theory of Chapter 2 and Chapter 3 can be used as a tool for developing complex systems. We need a language as a vehicle and this will be COLD-K.

We begin with constructing a top design and a bottom design first and not before Section 5.5.1 our initial design will be constructed. After that the top-down development process starts. We describe this process as a sequence of modification steps, affecting one variable design. Since in the top-down development process most modifications amount to the *addition* of formal text to this variable design, we only give the newly added formal texts after each step. This chapter is organised such way that it is possible to concatenate the keyword `DESIGN` followed by the abbreviation-type components from Chapter

4 and of Sections 5.3.2 to 5.3.5 and all formal texts contained in Sections 5.5 and 5.6, thereby obtaining a final design which corresponds with the result of the development process and which can be syntax- and type-checked.

The presentation of this process corresponds more or less with the actual development process of the case study. Of course it is also a rational reconstruction in the sense that sometimes we had to do a little backtracking which is not reflected in this chapter. Also some thinking-in-advance and 'throw-away' preliminary implementation activities took place. These activities are typical for a top-down development process, but it is not always possible to describe them precisely. This means that sometimes we state "in order to implement ..., we postulate another component ..." although the motivation for certain details is incomplete. We did our best to avoid a 'deus ex machina' effect.

We adopt the methodological principles of conservativity, origin consistency and visbility consistency as discussed in [2]. The basic idea of *conservativity* is that when constructing a class description of the form `IMPORT` P `INTO` Q, the axioms of Q should not impose new restrictions upon the sorts, functions, predicates and procedures introduced in P. The principle of *origin consistency* serves for avoiding the situation where there are two or more defining occurrences of one name. The principle of visibility consistency amounts to a restriction for class descriptions of the form `IMPORT` P `INTO` Q when p is a procedure $p \in \Sigma(P)$. The restriction is that if some side-effect of p is exported by Q, then so it must be by P.

We adopt the implementation relation based on signature inclusion and theory inclusion, as argued in [2]. We shall implicitly assume the monotonicity of the COLD-K import, export and renaming operators – in fact without formal justification. We also adopt the condition prim*s first* and the condition *directly specified* as discussed in Chapter 3. Finally we adopt the principles of *black-box correctness* (Chapter 2) and *black-box validation* (Chapter 3) – based on the exclusive use of specifications. COLD-K does not provide a program-execution model and therefore we have no formal criteria for deciding if a certain COLD-K text is executable or not. We aim at a manual translation from COLD-K to a classical imperative programming language (C) and with this idea in mind we shall sometimes (in informal speaking) distinguish between executable operations and non-executable operations: operations which need not be translated or which can not be translated in a straightforward manner are said to be non-executable.

This chapter is organised as follows. In Section 5.2 we discuss the top of the editor design. This is easy in view of the preparatory work of Chapter 4. In Section 5.3 we discuss the bottom of the editor design. This amounts to the introduction of several new specifications. These describe instances, attributes, blocks and tables which will be assumed as available primitives. Section 5.4 contains a summary of the top-down approach. In Section 5.5 we begin a top-down development process. We implement the three components which constitute the system of the editor design; these are KEYBIND, MOREDOP and WITEFA. During the implementation of these system components, several new components are postulated. In Section 5.6 the top-down development process is continued; the newly postulated components are implemented which leads again to the introduction of new components etc. This goes on until at the end of Section 5.6 only the primitives of Section 5.3 are left.

In Section 5.7 we discuss some related work. Sections 5.8 and 5.9 are devoted to conclusions and evaluation. In Appendix A we introduce a lower design layer. In Appendix B we give a list of symbols used in this chapter. In Appendix C we give the C program resulting from the composition of the editor design and the design of Appendix A. Finally, in Appendix D we provide a 'reference chart'.

5.2 The Top of the Editor Design

In the specification presented in Chapter 4 we did not mention the structure of the design in which the various class descriptions of the formal specification fit. Now it is time to do so and, more precisely, we must cast these class descriptions into a design d_t. Recall from Chapter 3 that the top of a design d is the design in which only those components are retained whose names occur in the system of d; it is denoted as top(d). We must indicate a design d_t which is considered as the top of a design d_{editor}, which at its turn must be constructed during the subsequent development process. The design d_t is shown below. We assume that at the position of the dots in this design we have LET-constructs introducing the names BOOL_SPEC, NAT_SPEC, CHAR_SPEC ... WITEFA_SPEC, MOREDOP_SPEC and KEYBIND_SPEC.

```
DESIGN
   ...

   COMP WITEFA : WITEFA_SPEC;
   COMP MOREDOP: MOREDOP_SPEC;
   COMP KEYBIND: KEYBIND_SPEC

SYSTEM WITEFA,MOREDOP,KEYBIND
```

The above top design represents a certain view upon the editor design to be constructed in this chapter. More precisely, it is the view that is of interest to the *user* of the editor design. This top design is a kind of *contract* that specifies precisely the components of the design that are made available. In this case there are three available components, viz. WITEFA, MOREDOP and KEYBIND. These are specified by WITEFA_SPEC, MOREDOP_SPEC and KEYBIND_SPEC respectively. When the design will be finished, it may have many more components, but these are outside the scope of the user.

Let us have a look at each of the three components mentioned in the system and let us explain why each is part of the system. We begin with KEYBIND_SPEC, which is the most obvious component. It provides procedures `PROC key: Char ->` , `PROC startup: ->` and the functions `FUNC screen: -> Text` and `FUNC cursor: -> Nat # Nat` which are precisely enough when using the editor directly for editing texts.

A less obvious way to use the editor design is to employ it as just one layer of a larger composite design where new features have been added to the editor. A simple example of this would be to have another keybinding e.g. by introducing a procedure `PROC key_2: Char ->` , say. In order to write this key_2 one needs both the operations provided by the component WITEFA (e.g. backward_character) and those provided by MOREDOP (e.g. delete_previous_character). One could also go one step further and add a layer providing for dynamic keybinding and programmability e.g. as available with MLisp in EMACS [4]. In the latter case one may decide to ignore the KEYBIND component entirely. So there are in fact two categories of users. The first category consists of the users who simply connect key to their physical keyboard and then start typing. The users of the second category are real software developers constructing another 'higher level' design $d_{top\text{-}layer}$, say which is put on top of the editor design d_{editor} – which could be done with the operator $\circ$ from Chapter 3. This explains why we included WITEFA and MOREDOP in addition to KEYBIND as part of the system.

This provides us with the following condition which is part of the postcondition of the software development process:

$$\text{top}(d_{editor}) = d_t$$

where the equality on designs is to be considered modulo the relative order of components.

5.3 The Bottom of the Editor Design

5.3.1 Introduction

Before we undertake a software development process, we must first introduce our primitive components. In the Sections 5.3.2 and 5.3.3 we introduce *instances* and *attributes*, which are the basic ingredients of an *attribute-oriented* approach, by which we mean that it is easy to add attributes to objects. This approach has been worked out by Jonkers in [3]. In the Sections 5.3.4 and 5.3.5 we introduce the data types of tables and blocks. We cast the specifications of these primitive components into the form of a bottom design. This will be done in Section 5.3.6

5.3.2 Specifying Instances

```
LET INST_SPEC :=
EXPORT
  SORT Inst,
  FUNC nil   : -> Inst,
  PROC create: -> Inst
FROM
CLASS

  SORT Inst  VAR
  FUNC nil: -> Inst

  AXIOM
  {INST1} INIT => nil! AND FORALL a:Inst ( a = nil )

  PROC create: -> Inst  MOD Inst

  AXIOM
  {INST2} < create > TRUE;
  {INST3} [ LET a:Inst; a := create ]
            a! AND (PREV NOT a!) AND
            FORALL b:Inst ( (PREV NOT b!) => b = a )

END;
```

5.3.3 Specifying Attributes

```
LET ATTR_SPEC :=
LAMBDA X : CLASS SORT Inst FREE END OF
LAMBDA Y : CLASS SORT Item FREE END OF

EXPORT
  SORT Inst,
  SORT Item,
  FUNC attr    : Inst        -> Item,
  PROC set_attr: Inst # Item ->
FROM
IMPORT X INTO
IMPORT Y INTO
CLASS

  FUNC attr    : Inst        -> Item   VAR
  PROC set_attr: Inst # Item ->        MOD attr

  AXIOM    FORALL i:Inst,v:Item (
  {ATTR1} < set_attr(i,v) > TRUE;
  {ATTR2} [ set_attr(i,v) ]
          attr(i) = v AND
          FORALL j:Inst,w:Item ( NOT j = i =>
          ( attr(j) = w <=> PREV attr(j) = w ) ) )

END;
```

5.3.4 Specifying Tables

In the description of the editor (Chapter 4) there is a variable map from strings to marked texts. Somehow this is going to be reflected in the implementation of the editor and therefore we introduce a specification of *tables*. These tables can be used for efficiently dealing with a mapping from implementable strings (sort 'String') to the representation of marked texts. Recall that we have already the class description MAP_SPEC which provides the sort Map of finite mappings with functions empty: -> Map for the empty mapping and add: Map # Item1 # Item2 -> Map for 'overwriting' addition. Furthermore there are operations rem: Map # Item1 -> Map for removal, app: Map # Item1 -> Item2 for map application, dom: Map -> Set1 for domain and ran: Map -> Set2 for range. These mappings are described as an algebraic data type – just as `Nat`, say. All objects of sort `Map` exist already in the initial state and none of the operations may have side-effects. This means that formally it is not allowed to use certain conventional data reification techniques when implementing these mappings.

Therefore we introduce another sort of so-called *tables* which will be specified with implementability in mind. We must warn the reader here that this will give rise to several pages of technicalities. In the description `TABLE_SPEC` given below, a table will be modeled as a modifiable mapping by which we mean that associated with every table there is a variable map – in the sense of the sort `Map`. In this way the maps from `MAP_SPEC` will play a role in the formal specification, serving as auxiliaries for the description of something different, viz. tables. The tables themselves correspond with a variable sort `Table`. Initially precisely one table exists and optionally more tables can be created dynamically when needed. We employ the attribute-oriented approach mentioned in Section 5.3.1. Tables are introduced as a kind of instances and the variable maps associated with them are introduced as a kind of attributes.

The specification `TABLE_SPEC` is parameterised with respect to the domain sort and the range sort of the tables. The simplest way of describing this would be to adopt a definition beginning with something like `LET TABLE_SPEC := LAMBDA X:ITEM1 OF LAMBDA Y:ITEM2 OF` ... etc. We adopt `ITEM2` indeed, but we do not adopt `ITEM1` because of two complications which we want to take into account.

The first complication is necessary when conventional data reification techniques are to be allowed. Typical data reification techniques for tables are linked lists, sorted lists, open hashing, closed hashing, binary search trees and balanced trees. Most of these techniques can not be applied when the sort `Item1` comes without any relations or operations – the only exception being the linked list technique. In order to sort a list we need a binary relation, `less` say, on `Item1`. The same holds for binary search trees and balanced trees. Both open hashing and closed hashing require that the `Item1` values can be converted to natural numbers or integers. Of course we need not choose among these techniques here, but we must make suitable preparations not to exclude almost *all* options. Therefore we require that the sort `Item1` comes with a binary predicate `less`. For this kind of applications, we have `SLO` in our library of standard class descriptions – abbreviating Strict Linear Order.

There is a second complication which leads us to not adopting `SLO`, but a slightly different version of it called `'SLO'`. This is needed because we want to allow for using a sort with an equivalence predicate `eq` such as `'String'` for the domain-sort of the tables. E.g. consider two 'String' objects s_1 and s_2 with $s_1 \neq s_2$ but eq(s_1,s_2); then a table-lookup using s_1 should yield the same result as a table-lookup using s_2. Furthermore we allow for a representation invariant item1_inv associated with the domain sort. The specification of 'SLO' takes SLO as a starting point and the formal relation between 'Item1' and Item1 is given by an abstraction function f. When comparing 'SLO' with SLO we see that essentially, the former allows for a wider class of implementations of the concept 'strict linear order'; but strictly formally speaking, it is the other way around. The point is that the role of the unquoted names is not the same in 'SLO' and SLO.

The formal specification below begins with 'SLO' which imports a renamed version of SLO. Recall from our standard library that SLO provides a sort Item with a binary relation r which is axiomatically stated to be a strict linear ordering – the kind of relation often denoted by <. Therefore RENAME SORT Item TO Item1 IN SLO provides a sort Item1 with a binary relation r: Item1 # Item1 which is again a strict linear ordering. The CLASS ... END part of 'SLO' is an extension of SLO introducing the sort 'Item1' with binary predicates eq and less. Furthermore it introduces a predicate item1_inv and an abstraction function f. There is just one axiom which serves for stating the precise nature of the eq and less predicates: the function f is required to behave homomorphically such that = and r are the images of eq and less respectively. The axiom is relativised by item1_inv. Note that 'SLO' exports everything it contains, including Item1 and r. As explained above 'SLO' serves as parameter restriction. Similarly ITEM2 which only requires the presence of a sort Item2 serves as a parameter restriction for a second formal parameter.

```
LET 'SLO1' :=
IMPORT
  RENAME
    SORT Item TO Item1
  IN SLO
INTO
CLASS

  SORT 'Item1'                       FREE
  PRED eq        : 'Item1' # 'Item1' FREE
  PRED less      : 'Item1' # 'Item1' FREE
  PRED item1_inv:                    FREE
  FUNC f         : 'Item1' -> Item1  FREE
  AXIOM item1_inv =>
        FORALL i:'Item1',j:'Item1'
        ( eq(i,j) <=> f(i) = f(j);
          less(i,j) <=> r(f(i),f(j)) )

END;


LET TABLE_SPEC :=

LAMBDA X : 'SL01' OF
LAMBDA Y :  ITEM2  OF
```

This completes the introduction of the formal parameters of TABLE_SPEC and now we turn our attention to its body. There will be local definitions introducing TABLE_INST_SPEC, MAP_FROM_SL01_TO_ITEM2 and TABLE_MAP_SPEC which serve for introducing tables as renamed instances and for associating variable maps with these tables. Within the CLASS ... END part of the specification we shall define the table operations among which the procedures new: -> Table and add: Table # 'Item1' # Item2 -> . We must be prepared to allow techniques which require a representation invariant. E.g. when a sorted list is used to represent tables, the invariant might state that the list is a-cyclic and sorted; when a binary tree is used, the invariant might state that all nodes in a left-hand side subtree have Item1 values which are less than the Item1 values in the corresponding right-hand side subtree. The actual choice is for the implementer and here we just introduce a predicate table_inv for which we state axiomatically that the table procedures preserve it as an invariant. For the definition of the operations such as add this also implies that we have to introduce a case-analysis. The purpose of the case-analysis is to describe what can happen when the invariant does not hold.

TABLE_SPEC has an explicit export list serving as a compact overview of the sorts and operations provided. The sorts Map, Item1, the function map: Table -> Map and the function app: Map # Item1 -> Item2 are needed for reasoning purposes only. The remaining exported sorts and operations are considered executable. All table operations are defined as PROC allowing for a maximum of implementation freedom – although maybe this is not needed for certain of them, such as is_in_dom.

```
EXPORT

  SORT Map,
  SORT Item1,
  FUNC map: Table -> Map,
  FUNC app: Map # Item1 -> Item2,

  SORT Bool,
  SORT Table,
  SORT Item2,
  SORT 'Item1',
  PRED table_inv: ,
  PROC new      :                           -> Table,
  PROC add      : Table # 'Item1' # Item2 -> ,
  PROC rem      : Table # 'Item1'          -> ,
  PROC app      : Table # 'Item1'          -> Item2,
  PROC is_in_dom: 'Item1' # Table          -> Bool

FROM
```

Now the local definitions follow. The first one defines TABLE_INST_SPEC which provides the sort Table with FUNC nil: -> Table and PROC create: -> Table. The second definition is based on MAP_SPEC which is a parameterised class description. Recall from our standard library that it has two parameters with parameter restrictions ITEM1 and ITEM2 requiring the presence of sorts Item1 and Item2 respectively. It provides the algebraic data type of maps, i.e. all maps that can be constructed by the constructor operations empty and add. Before we can employ this MAP_SPEC it must be instantiated which we do in this case by taking X and Y as actual parameters. Since we are within the scope of the abstractions LAMBDA X:'SL01' OF LAMBDA Y:ITEM2 OF ... we easily verify that X exports a sort Item1 and that Y exports a sort Item2.

```
LET TABLE_INST_SPEC :=
  RENAME
    SORT Inst TO Table
  IN INST_SPEC;

LET MAP_FROM_SL01_TO_ITEM2 :=
  APPLY APPLY
    MAP_SPEC
  TO X TO Y;
```

The third local definition serves for associating maps with tables. We employ

ATTR_SPEC from Section 5.3.3 which is parameterised with respect to the sorts of the instances and attributes. In this case the role of instances is played by Table whereas the role of attributes is played by Map. Furthermore we must get rid of the instance-and-attribute-oriented names and replace them by table-and-map-oriented names. As usual the renaming is done in the parameterised description and as a result we get a parameterised description whose parameter restrictions are renamed versions of CLASS SORT Inst FREE END and ITEM respectively. When suitably renamed, these require the presence of a sort Table and a sort Map respectively. This shows that we can take TABLE_INST_SPEC and MAP_FROM_SL01_TO_ITEM2 as actual parameters. In this way we get an 'attribute' function map: Table -> Map and an 'assignment' procedure set_map: Table # Map -> .

```
LET TABLE_MAP_SPEC :=
  APPLY APPLY
    RENAME
      SORT Inst                            TO Table,
      SORT Item                            TO Map,
      FUNC attr    : Inst         -> Item  TO map,
      PROC set_attr: Inst # Item ->        TO set_map
    IN ATTR_SPEC
  TO TABLE_INST_SPEC TO MAP_FROM_SL01_TO_ITEM2;
```

We import the formal parameters X and Y and the locally defined class descriptions. The auxiliary operation p corresponds with an arbitrary invocation of one of the table operations.

```
IMPORT X                         INTO
IMPORT Y                         INTO
IMPORT BOOL_SPEC                 INTO
IMPORT MAP_FROM_SL01_TO_ITEM2    INTO
IMPORT TABLE_INST_SPEC           INTO
IMPORT TABLE_MAP_SPEC            INTO
CLASS

  PRED table_inv: VAR

  AXIOM {INVARIANCE}
  INIT AND item1_inv => table_inv;
  item1_inv AND table_inv => [ p ] table_inv

  PROC p: ->
  DEF  ( FLUSH new
       | add(SOME t:Table(),SOME i:'Item1'(),SOME j:Item2())
```

```
        | rem(SOME t:Table(),SOME i:'Item1'())
        | FLUSH app(SOME t:Table(),SOME i:'Item1'())
        | FLUSH is_in_dom(SOME i:'Item1'(),SOME t:Table())
        )
```

Now the actual definitions of the table operations can be given. The procedure new yields a fresh table, denoted as t, which is obtained by t := create. Furthermore this fresh table gets the empty map associated with it and this is established by the expression set_map(t,empty) where we use the procedure set_map from TABLE_MAP_SPEC.

```
  PROC new: -> Table
  DEF  LET t:Table; t := create;
       set_map(t,empty);
       t
```

Very much in the same style we model add. As a first approximation we could define add(t,i,j) by something like DEF set_map(t,add(map(t),i,j)). Although this gives the basic idea there is a difficulty because we must take into account that the invariant need not hold. When the invariant does not hold, it may become true (just by luck) and furthermore the map attribute may get any value, as described by USE set_map END. Otherwise the map attribute is updated using set_map again. Also some care is needed here because this map attribute is a finite mapping from Item1 to Item2 whereas our add procedure takes an 'Item1' argument. So we have to use the abstraction function f, writing set_map(t,add(map(t),f(i),j)) rather than set_map(t,add(map(t),i,j)).

```
  PROC add: Table # 'Item1' # Item2 ->
  PAR  t:Table,i:'Item1',j:Item2
  DEF  ( NOT table_inv ?;
         MOD table_inv USE set_map END
       | table_inv ?;
         set_map(t,add(map(t),f(i),j))
       )

  PROC rem: Table # 'Item1' ->
  PAR  t:Table,i:'Item1'
  DEF  ( NOT table_inv ?;
         MOD table_inv USE set_map END
       | table_inv ?;
         set_map(t,rem(map(t),f(i)))
       )
  PROC app: Table # 'Item1' -> Item2
  PAR  t:Table,i:'Item1'
  DEF  ( NOT table_inv ?;
         SOME i:Item2 ()
       | table_inv ?;
         app(map(t),f(i))
       )

  PROC is_in_dom: 'Item1' # Table -> Bool
  PAR  i:'Item1',t:Table
  DEF  ( NOT table_inv ?;
         SOME b:Bool ()
       | table_inv ?;
         ( app(map(t),f(i))!     ?; true
         | NOT app(map(t),f(i))! ?; false
         )
      )

END;
```

This concludes the specification of tables.

5.3.5 Specifying Blocks

In order to store texts within our editor, we need a facility that manages an unbounded amount of storage and that on request releases finite portions of storage in the form of a kind of blocks of storage locations. Below we give a specification of such a facility. To keep things simple, we did not include possibilities for returning portions of storage that are no longer needed. The function cont applied to a block b and a natural number $n <$ size(b) yields the *contents* of the n-th location in this block. The procedure store can be applied to a block b, a natural number $n <$ size(b) and an item i. Its effect is to *store* the value i in the n-th location in this block. The numbers 0...size(b) $-$ 1 serve as addresses.

In the class description BLOCK_SPEC below we have four axioms describing the initial state and the procedures alloc, grow and store respectively. The first axiom states that initially there are no blocks yet. The second axiom describes both the termination of alloc(n) and its postcondition which defines the size of the newly 'allocated' block and which states that precisely one new block is created. The third axiom describes the termination of grow(b,n) and its postcondition. Formally this axiom must be relativised by the premiss size(b)! which of course holds in the states reachable from the initial state by using alloc and grow. The last axiom gives the termination and the postcondition of store(b,n,i). Again this axiom is relativised and the premiss is n < size(b).

```
LET BLOCK_SPEC :=
LAMBDA X:ITEM OF
EXPORT

  SORT Block,
  SORT Nat,
  SORT Item,
  FUNC size   : Block                -> Nat,
  FUNC cont   : Block # Nat          -> Item,
  PROC store  : Block # Nat # Item ->,
  PROC alloc  : Nat                  -> Block,
  PROC grow   : Block # Nat          ->

FROM
IMPORT X        INTO
IMPORT NAT_SPEC INTO
CLASS

  SORT Block VAR
  FUNC size   : Block       -> Nat VAR
  PROC alloc  : Nat         -> Block MOD Block
  PROC grow   : Block # Nat ->       MOD size

  AXIOM INIT => NOT EXISTS  a:Block ()

  AXIOM FORALL n:Nat (
        < alloc(n) > TRUE;
        [ LET b:Block; b := alloc(n) ]
          size(b) = n AND
          (PREV NOT b!) AND FORALL c:Block
                            ( (PREV NOT c!) => c = b ) )

  AXIOM FORALL b:Block, n:Nat  ( size(b)! =>
        ( < grow(b,n) > TRUE;
          [ grow(b,n) ]
          ( size(b) = add(n,PREV size(b));
            FORALL c:Block ( NOT b = c AND (PREV size(c)!) =>
            ( size(c) = PREV size(c) ) ) ) ) )

  FUNC cont: Block # Nat -> Item    VAR
  PROC store : Block # Nat # Item ->  MOD cont

  AXIOM    FORALL b:Block,n:Nat,i:Item
           ( lss(n,size(b) ) => (
           < store(b,n,i) > TRUE;
           [ store(b,n,i) ]
           ( cont(b,n) = i;
             FORALL c:Block, m:Nat, j:Item
             ( lss(m,size(c)) =>
               ( NOT n = m OR NOT b = c =>
                 cont(c,m) = j <=> PREV cont(c,m) = j) ) ) ) )

END;
```

5.3.6 The Bottom of the Editor Design

We expect the underlying machine on which we have to build our editor to provide us with implementations of the following COLD-K class descriptions.

- Data-types immediately available from a typical programming language such as Pascal or C. Let us assume that we have general purpose data types BOOL_SPEC, NAT_SPEC, CHAR_SPEC, INST_SPEC, ATTR_SPEC.
- Implementable sequences described by 'SEQ_SPEC', modifiable maps, described by TABLE_SPEC, and so-called blocks described by BLOCK_SPEC.
- A display device. Let us assume that this is given by the class description DISPLAY_SPEC. Also we assume a simple file system: FILE_SPEC.

The primitive components from the second group are considered not available in the programming language at hand. Therefore we can expect that at some point in time we might implement some of them ourselves. On the other hand, these components are of a general-purpose nature and they do not reflect that we are aiming at an editor. These give rise to the following primitive components:

'SEQ' implementing 'SEQ_SPEC',
TABLE implementing TABLE_SPEC,
BLOCK implementing BLOCK_SPEC.

Now it is time to cast the specifications of all primitive components into the shape of a design d_b. Recall from Chapter 3 that the bottom of a design d is the design with the empty system and in which only those components are retained that have no glass-box description – i.e. implementation module. The bottom of a design d is denoted as bot(d). We must cast the specifications of the primitive components into a design d_b which is considered as the bottom of a design d_{editor}, which at its turn must be constructed during the subsequent development process. The design d_b is shown below. We assume that at the position of the dots in this design we have a number of LET abbreviations, which introduce the names BOOL_SPEC, NAT_SPEC, CHAR_SPEC etc.

```
DESIGN
  ...
  COMP BOOL    : BOOL_SPEC;
  COMP NAT     : NAT_SPEC;
  COMP CHAR    : CHAR_SPEC;
  COMP INST    : INST_SPEC;
  COMP ATTR    : ATTR_SPEC;

  COMP 'SEQ'   : 'SEQ_SPEC';
  COMP TABLE   : TABLE_SPEC;
  COMP BLOCK   : BLOCK_SPEC;

  COMP DISPLAY: DISPLAY_SPEC;
  COMP FILE    : FILE_SPEC

SYSTEM NONE
```

This provides us with the following condition which is part of the postcondition of the software development process:

$$\text{bot}(d_{editor}) = d_b$$

In Appendix A we shall consider a design d_{basic} whose system can be plugged into d_b and hence also in d_{editor}. Of course the contents of this appendix are supposed to be outside the scope of our design d_{editor}. In particular, the principle of *black-box validation* does not allow the use of knowledge of implementation details from d_{basic}.

5.4 Summary of the Top-down Approach

At this point we have not started the actual construction of our software system yet and instead of that we have fixed the boundaries of the design d_{editor} to be created. This was done in Sections 5.2 and 5.3 where we have chosen the user-view d_t and the machine-view d_b respectively. Let us assume that we can find a system-user who would like to use the system components specified by d_t. Let us also assume that somehow we can establish the availability of a machine providing the primitive components specified by d_b. Under these assumptions the validation conditions mentioned in Chapter 3 are fulfilled.

We shall describe the principles of the top-down approach informally now. For the formal treatment we refer to Chapter 3. The top-down approach is based on the so-called ***top-down invariant*** which is defined as the conjunction of the following conditions.

1. $\text{top}(d_{editor}) = d_t$ where the equality on designs is to be considered modulo the relative order of components. Intuitively this means that the user-view on the editor design is kept fixed.
2. d_{editor} is black-box correct. Roughly speaking, this corresponds with the situation where all black-box descriptions are *true* in the sense that the corresponding glass-box descriptions (if present) are implementations indeed. Formally this is described best by referring to the implementation relation $\sqsubseteq$.
3. All components of d_{editor} are directly or indirectly used in the system of d_{editor} – except possibly those occurring also in d_b, i.e. except possibly those which happen to be available anyway.

A suitable initial state for the development process can be established simply by assigning d_t to d_{editor}. Let us check that this makes the top-down invariant hold indeed. First, the top of this design d_t equals d_t, which is a property of top designs. Secondly the design d_t is black-box correct since its components do not have a glass-box description yet and hence are correct by definition. Finally, all components occur in the system. In our particular case these components are `WITEFA`, `MOREDOP`, `KEYBIND`. This shows that assigning d_t to d_{editor} makes the top-down invariant hold.

After this assignment a top-down development process can really begin and this is the subject of Section 5.5. First we add the primitive components of d_b to the editor design. Next, we must select some primitive component and make sure that it gets implemented according to the principle of black-box correctness. In our particular case we shall select `KEYBIND` as the first component to be implemented and since its specification is already in algorithmic style this is easy. Next we treat `MOREDOP` which is equally easy and after that there is only one component to be selected left which is `WITEFA`. To implement `WITEFA` is more complicated and it requires the introduction of several new primitive components. After `WITEFA` has been implemented we are done with the system components and this stage will be reached at the end of Section 5.5. Although at that stage all system components have been implemented, the design is still not finished yet because there are several newly postulated primitive components which were needed for `WITEFA`. In Section 5.6 these are implemented one after the other. Again this requires the introduction of new primitive components and in this way the top-down development process proceeeds until only those primitive components are left that occur in the bottom design d_b. This stage will be reached at the end of Section 5.6.

At that stage we can finish the development process. The condition bot(d_{editor}) = d_b serves as the termination condition of the development process. Recall that the clause bot(d_{editor}) refers to the so-called bottom of the design d_{editor} and that intuitively this bottom is a machine-view, containing information about the primitive components. The top-down development process can be viewed as the execution of a design program of the form given below, where *td_step* satisfies the assumption that it does not violate the top-down invariant.

```
technique $(d_b, d_t)$
def  $d_{editor} := d_t$;
     while not bot$(d_{editor}) = d_b$ do $d_{editor} :=$ td_step$(d_{editor})$; od;
     $d_{editor}$
```

The precise syntax and semantics of such design programs is given in Chapter 3. Here we restrict ourselves to a short informal explanation. The header **technique** (d_b, d_t) indicates that this is a design program where d_b and d_t act as parameters. The definition of the design program follows after the keyword **def** and it consists of an assigment statement to the variable design d_{editor} and a loop of iterated application of *td_step*. The final d_{editor} serves as the result of this design program.

5.5 Implementing the System Components

5.5.1 Introduction

We begin with d_t, but we add immediately the primitive components from d_b to it. This yields the design consisting of the keyword DESIGN and the LET abbreviations given before followed by the following components:

```
COMP BOOL   : BOOL_SPEC;
COMP NAT    : NAT_SPEC;
COMP CHAR   : CHAR_SPEC;
COMP INST   : INST_SPEC;
COMP ATTR   : ATTR_SPEC;

COMP 'SEQ'  : 'SEQ_SPEC';
```

```
      COMP TABLE   : TABLE_SPEC;
      COMP BLOCK   : BLOCK_SPEC;

      COMP DISPLAY: DISPLAY_SPEC;
      COMP FILE    : FILE_SPEC;

{@}   COMP WITEFA : WITEFA_SPEC;
{@}   COMP MOREDOP: MOREDOP_SPEC;
{@}   COMP KEYBIND: KEYBIND_SPEC;
```

and having as a system WITEFA,MOREDOP,KEYBIND. The black-box descriptions (the xxx_SPECs) of this design will remain unchanged during the the development process and so does its system. However, some of these components will get a glass-box description later and as a reminder for this, they have been marked now by {@}. The symbol {@} should be read as: "this is going to be replaced later". Since there are no glass-box descriptions yet, the above initial design is black-box correct.

5.5.2 Implementing KEYBIND

In this section we give the implementation of the component KEYBIND which contains a procedure key, taking characters as its argument and invoking operations provided by WITEFA and MOREDOP. Furthermore it exports the initialisation procedure startup – which we can assume to be provided by MOREDOP since MOREDOP_SPEC exports startup. It also exports screen and cursor which we can assume to be provided by WITEFA since WITEFA_SPEC imports APP_DOM_SPEC which at its turn imports DISPLAY_SPEC.

The KEYBIND_IMPL given below is almost the same as KEYBIND_SPEC. The subtle difference is that we use component names in the import clauses rather than abbreviation names. Note that earlier KEYBIND_SPEC was used as a *black-box description*. Therefore it was necessary to import NAT_SPEC, BOOL_SPEC etc. because importing NAT or BOOL would violate the principle of *direct specification*. Recall from Chapter 3 definition 3.2.13, that a design is *directly specified* if it is wellformed and no black-box description contains the name of a component. Now we are in a different position. We are going to construct a *glass-box description* KEYBIND_IMPL and now we are allowed to refer to component names. In order to get an executable implementation, we must in fact do so.

The procedure key is described by one large case-statement and we consider this as already executable. The procedure key calls the procedures from WITEFA and MOREDOP: insert_character, set_mark, ..., delete_ previous_character. Most procedures called by key come from WITEFA, but the following procedures come from MOREDOP: return, search_forward, insert_file, write_named_file, delete_to_killbuffer, yank_from_killbuffer, escape and delete_previous_character.

```
LET KEYBIND_IMPL :=
EXPORT
   SORT Nat,
   SORT Char,
   SORT Text,
   PROC startup:       -> ,
   PROC key    : Char -> ,
   FUNC screen :       -> Text,
   FUNC cursor :       -> Nat # Nat
FROM
IMPORT NAT     INTO
IMPORT CHAR    INTO
IMPORT WITEFA  INTO
IMPORT MOREDOP INTO
CLASS

   PROC key: Char ->
   PAR  c:Char
   DEF  ( printable(c)      ?;  insert_character(c)
        | ord(c) = 0   {^@} ?;  set_mark
        | ord(c) = 1   {^A} ?;  beginning_of_line
        | ord(c) = 2   {^B} ?;  backward_character
        | ord(c) = 4   {^D} ?;  delete_next_character
        | ord(c) = 5   {^E} ?;  end_of_line
        | ord(c) = 6   {^F} ?;  forward_character
        | ord(c) = 13 {^M} ?;  return
        | ord(c) = 14 {^N} ?;  next_line
        | ord(c) = 16 {^P} ?;  previous_line
        | ord(c) = 19 {^S} ?;  search_forward
        | ord(c) = 20 {^T} ?;  insert_file
        | ord(c) = 21 {^U} ?;  write_named_file
        | ord(c) = 23 {^W} ?;  delete_to_killbuffer
        | ord(c) = 24 {^X} ?;  beginning_of_buffer
        | ord(c) = 25 {^Y} ?;  yank_from_killbuffer
        | ord(c) = 26 {^Z} ?;  end_of_buffer
        | ord(c) = 27 {ESC}?;  escape
        | ord(c) = 127{DEL}?;  delete_previous_character
        )

END;
```

Now we are in a position where the primitive component `{@} COMP KEYBIND: KEYBIND_SPEC` can be replaced by a component having both a black-box description and a glass-box description, viz. by `COMP KEYBIND: KEYBIND_SPEC := KEYBIND_IMPL`.

```
% COMP KEYBIND : KEYBIND_SPEC := KEYBIND_IMPL;
% this is to replace an earlier primitive component.
```

We must add some explanation to this because we had to solve a technical complication regarding the presentation of the (dynamic) development process in this (static) chapter. When we were in the specification phase (Chapter 4) this was no problem because at each stage we only added formal texts *after* the texts obtained already. We organise our text in such a way that there is a simple mechanical operation of separating the formal texts from the informal introductions and annotations. Then the concatenation of the keyword `DESIGN`, the `LET` abbreviations from Chapter 4, Sections 5.3.2 to 5.3.5 and the remaining formal texts from this chapter (Section 5.5.1 to 5.6.13) will constitute *one* design which can be syntax- and type-checked. This one design encompasses all the formal texts employed in various stages of the top-down development process. The technical complication is that one component may occur initially just as primitive whereas in a later stage it is replaced by another component, having both a black-box description and a glass-box description. Of course we can not have a well-formed design where one component-name occurs twice and therefore we include such 'second defining ocurrences' of components in the design, but we only include them as comment. Just as above, these situations will be marked by the sentence "`% this is to replace an earlier primitive component.`"

In fact, we have now carried out our first top-down design-transformation step d_{editor} := td_step(d_{editor}); as described in Section 5.4. The resulting design reads the same as the one given in Section 5.5.1 but for two modifications: first, the "`LET KEYBIND_IMPL := ... ;`"-text is added at an appropriate place, and second, the line "`COMP KEYBIND : KEYBIND_SPEC;`" is replaced by "`COMP KEYBIND : KEYBIND_SPEC := KEYBIND_IMPL;`".

5.5.3 Implementing MOREDOP

In this section we give the implementation of the component `MOREDOP` which deals with a few additional operations which are added on top of `WITEFA`. The description given below is the same as `MOREDOP_SPEC`, except for the fact that we use component names instead of abbreviation-names.

```
LET MOREDOP_IMPL  :=
IMPORT NAT        INTO
IMPORT BOOL       INTO
IMPORT CHAR       INTO
IMPORT WITEFA     INTO
CLASS

   PROC mini: -> 'String'
   DEF  cons('m',cons('i',cons('n',cons('i',empty))))

   PROC main: -> 'String'
   DEF  cons('m',cons('a',cons('i',cons('n',empty))))

   PROC kill: -> 'String'
   DEF  cons('k',cons('i',cons('l',cons('l',empty))))

   PROC startup: ->
   DEF  init(mini);
        switch_to_buffer(kill);
        switch_to_buffer(main)

   PROC escape: ->
   DEF  ( eq(current_buffer_name,mini)     ?; switch_to_buffer(main)
        | NOT eq(current_buffer_name,mini) ?; switch_to_buffer(mini)
        )

   PROC return: ->
   DEF  ( eq(current_buffer_name,mini)     ?; switch_to_buffer(main)
        | NOT eq(current_buffer_name,mini) ?; newline
        )

   PROC delete_to_killbuffer: ->
   DEF  copy_region_to_buffer(kill);
        erase_region

   PROC yank_from_killbuffer: ->
   DEF  yank_buffer(kill)

   PROC search_forward: ->
   DEF  search_forward(buffer_to_string(mini))

   PROC insert_file: ->
   DEF  insert_file(buffer_to_string(mini))

   PROC write_named_file: ->
   DEF  write_named_file(buffer_to_string(mini))
   PROC delete_previous_character: ->
   DEF  ( bolp = true  ?; SKIP
        | bolp = false ?;
                 backward_character;
                 delete_next_character
        )

END;
```

Again we indicate the completion of the second top-down design transformation step by simply writing the following two lines:

```
% COMP MOREDOP : MOREDOP_SPEC := MOREDOP_IMPL;
% this is to replace an earlier primitive component.
```

We implemented two components and we were lucky in the sense that we could do so in a rather trivial way. This was possible because the black-box descripions of these components were already in the right form. The next component to be dealt with is WITEFA and in view of its complexity and the axiomatic style of its description, we can expect that the construction of its implementation will be less trivial. Indeed, almost of the entire rest of this chapter concerns the implementation of WITEFA and the implementation of components that we shall postulate and that are directly or indirectly used by WITEFA.

5.5.4 Transforming the State space

We must choose how to represent marked texts. This choice is crucial for the efficiency of the resulting editor, both in terms of execution time and memory usage. Making this choice is also an important point in the software development process, since many procedures to be developed later will be based on this choice.

Before presenting our choice, we present a few informal efficiency considerations. We assume that there is no a-priori bound on the size of the texts that can be edited: neither the number of lines nor the number of characters within one line should be bounded. Of course a real program execution environment imposes limits upon the amount of available memory, but this is not formally described by the specifications of our primitive components and we do not worry about that. Instead of that we want that no 'a priori' bounds get built-in into our design. So choosing a fixed-size two-dimensional array as a representation with the rows of the array representing lines would be wrong in this respect.

Better candidate representations are those based upon the use of sequences. For example, we might represent a text by one or more sequences of line-representations, where a line-representation at its turn is a sequence of characters. This certainly is a workable approach, but we do not choose it for the following reason. The problem is that the sequences themselves need some kind of representation, such as linked lists. The straightforward representation of sequences by linked list would use some form of cells, where each cell contains a 'next cell' address. This means that for each character stored in the editor, there is an overhead of one address. Even if we assume that a character and an address require the same amount of storage, we waste 50% of the available storage. Of course we could try to improve upon the representation of the sequences but then the approach might loose some of its elegancy.

After these considerations we shall present our choice, but we would like to add immediately that there is no claim whatsoever that this would be the best choice. The amount of possible choices here is enormous and the question of which is the best one has many aspects. We just took one which seems reasonable.

We use for each text one block, which is a large extensible one-dimensional array, as already introduced in Section 5.3.5. Within this block, the text is stored in a way that is related to the string representation of text, viz. by using some separator between the lines. For efficiency reasons we shall not make the block grow in steps of one location at a time, but instead we shall use larger increments. From this it follows that often there is some *free space* in the block. We decide that this free space should be contiguous rather than scattered over the block and that it will take the shape of a kind of *gap* which may be at any position within the block. Of course we must also somehow keep track of the mark and the dot, which simply become natural numbers indicating a position in the block. A similar representation has been used also in the implementation of the EMACS editor descibed in [4].

In order to get elegant algorithms taking care for the window-invariant `WI`, we also want to keep track of the dot, *viewed as a co-ordinate pair*. Since we have operations such as `exchange_dot_and_mark` and `end_of_buffer`, it is a consequence that we must also keep track of the mark viewed as a co-ordinate pair and of the reach of the entire text. Apart from storing a marked text as a block-with-positions, we explicitly store the reach of the text before dot, the reach of the text before mark and the reach of the entire text.

So we store the text, the dot and the mark as block-with-positions and we redundantly store their homomorphic images under the reach operation as well. This means that many edit procedures have to do their work twice: first of all they operate on the block-with-positions data and then secondly they perform the same operation on reaches. For the operations on block-with-positions data we have to be careful with respect to efficiency, but fortunately the operations on reaches such as add, split, cut, and paste pose less efficiency problems. Actually we just take their definition, which can be considered executable already.

We start the formalisation of our choice now. As discussed before, we need some sort of objects that can represent MText objects. For this purpose we introduce a sort Buf, whose objects will be called *buffers*. Buf is nothing but a renamed version of Inst with several attributes. Each buffer has a *block* associated with it. Several Nat attributes are needed as 'pointers' to certain positions in the block.

- block: Buf → Block VAR
- dot : Buf → Nat VAR
- mark : Buf → Nat VAR
- gap1 : Buf → Nat VAR
- gap2 : Buf → Nat VAR

Finally we need a number of pair-wise attributes. For example, we need an attribute dot: Buf → Nat^2 which keeps track of the dot viewed as a co-ordinate pair.

- dot : Buf → Nat^2 VAR
- mark : Buf → Nat^2 VAR
- reach: Buf → Nat^2 VAR

Sometimes we use a terminology based on a spatial view of buffers. E.g. we shall speak about moving leftwards or going downwards when we mean to decrease the value of a variable that indicates a position.

5.5.5 Specifying Pair-wise Attributes

For dealing with co-ordinate pairs it is convenient to have pair-wise attributes.

```
LET ATTR2_SPEC :=
LAMBDA X : CLASS SORT Inst FREE END OF
LAMBDA Y : CLASS SORT Item1 FREE END OF
LAMBDA Z : CLASS SORT Item2 FREE END OF

EXPORT
  SORT Inst,
  SORT Item1,
  SORT Item2,
  FUNC attr    : Inst                  -> Item1 # Item2,
  PROC set_attr: Inst # Item1 # Item2  ->
FROM
IMPORT X INTO
IMPORT Y INTO
IMPORT Z INTO
CLASS

  FUNC attr    : Inst                  -> Item1 # Item2 VAR
  PROC set_attr: Inst # Item1 # Item2  ->               MOD attr

  AXIOM   FORALL i:Inst,v:Item1,w:Item2 (
  {ATTR1} < set_attr(i,v,w) > TRUE;
  {ATTR2} [ set_attr(i,v,w) ]
          attr(i) = (v,w) AND
          FORALL j:Inst,x:Item1, y:Item2 ( NOT j = i =>
          ( attr(j) = (x,y) <=> PREV attr(j) = (x,y) ) ) )

END;
```

5.5.6 ATTR2: a Postulated Component

Since we have already a specification ATTR2_SPEC, we can directly postulate the following component:

```
{@}  COMP ATTR2 : ATTR2_SPEC;
```

We best employ the meta-operator COPY when using ATTR2. The effect of this operator is to make a textual copy of its argument. This will guarantee that no undesired aliasing takes place because each copy of ATTR2 can be viewed as a fresh programming variable. This is due to the so-called *origin* mechanism of COLD-K. For a discussion of the COPY operator we refer to [3] and for an explanation of the origin mechanism we refer to [1]. Here we only sketch how the use of COPY in combination with the origin mechanism works on a simplified example. Let us – by way of example – assume that ATTR0 is defined by LET ATTR0 := CLASS PRED attr: VAR END which means that it describes a two-valued programming variable. Now to get two instances of such variables we can write LET V := COPY(ATTRO); LET W := COPY(ATTRO); which is by definition of COPY nothing but LET V := CLASS PRED **attr**: VAR END; LET W := CLASS PRED **attr**: VAR END;. Since the latter text introducing V and W contains two defining occurrences of **attr**, these have different origins; we can imagine a language implementation attaching these origins and then we get e.g. $\mathtt{attr}_1$ and $\mathtt{attr}_2$ respectively which means that they act as two different programming variables. When syntax- and type-checking we replace COPY(ATTR2) by ATTR2. Similar remarks apply for ATTR.

5.5.7 Transforming the State space (continued)

Now we are ready to introduce a class description BUF describing the sort Buf and the associated attributes. We give the formal definition below.

```
LET BUF_INST :=
RENAME
  SORT Inst TO Buf
IN INST;

LET BUF :=

% local definitions follow:

LET CHAR_BLOCK :=
  APPLY RENAME
        SORT Item TO Char
  IN BLOCK TO CHAR;

LET BUF_BLOCK :=
  APPLY APPLY RENAME
        SORT Inst                        TO Buf,
        SORT Item                        TO Block,
        FUNC attr    : Inst        -> Item TO block,
        PROC set_attr: Inst # Item ->      TO set_block
  IN COPY(ATTR) TO BUF_INST TO CHAR_BLOCK;

LET BUF_DOT :=
  APPLY APPLY RENAME
        SORT Inst                        TO Buf,
        SORT Item                        TO Nat,
        FUNC attr    : Inst        -> Item TO dot,
        PROC set_attr: Inst # Item ->      TO set_dot
  IN COPY(ATTR) TO BUF_INST TO NAT;

LET BUF_MARK :=
  APPLY APPLY RENAME
        SORT Inst                          TO Buf,
        SORT Item                          TO Nat,
        FUNC attr    : Inst         -> Item TO mark,
        PROC set_attr: Inst # Item ->      TO set_mark
  IN COPY(ATTR) TO BUF_INST TO NAT;

LET BUF_GAP1 :=
  APPLY APPLY RENAME
        SORT Inst                          TO Buf,
        SORT Item                          TO Nat,
        FUNC attr    : Inst         -> Item TO gap1,
        PROC set_attr: Inst # Item ->      TO set_gap1
  IN COPY(ATTR) TO BUF_INST TO NAT;

LET BUF_GAP2 :=
  APPLY APPLY RENAME
        SORT Inst                          TO Buf,
        SORT Item                          TO Nat,
        FUNC attr    : Inst         -> Item TO gap2,
        PROC set_attr: Inst # Item ->      TO set_gap2
  IN COPY(ATTR) TO BUF_INST TO NAT;

% Now the pair-wise attributes follow:

LET BUF_DOT_2 :=
  APPLY APPLY APPLY RENAME
        SORT Inst                                            TO Buf,
        SORT Item1                                           TO Nat,
        SORT Item2                                           TO Nat,
        FUNC attr    : Inst                -> Item1 # Item2 TO dot,
        PROC set_attr: Inst # Item1 # Item2 ->             TO set_dot
  IN COPY(ATTR2) TO BUF_INST TO NAT TO NAT;

LET BUF_MARK_2 :=
  APPLY APPLY APPLY RENAME
        SORT Inst                                            TO Buf,
        SORT Item1                                           TO Nat,
        SORT Item2                                           TO Nat,
        FUNC attr    : Inst                -> Item1 # Item2 TO mark,
        PROC set_attr: Inst # Item1 # Item2 ->             TO set_mark
  IN COPY(ATTR2) TO BUF_INST TO NAT TO NAT;

LET BUF_REACH_2 :=
  APPLY APPLY APPLY RENAME
        SORT Inst                                         TO Buf,
        SORT Item1                                        TO Nat,
        SORT Item2                                        TO Nat,
        FUNC attr    : Inst               -> Item1 # Item2 TO reach,
        PROC set_attr: Inst # Item1 # Item2 ->         TO set_reach
  IN COPY(ATTR2) TO BUF_INST TO NAT TO NAT;

% end of local definitions

IMPORT NAT INTO
IMPORT CHAR_BLOCK  INTO
IMPORT BUF_INST    INTO
IMPORT BUF_BLOCK   INTO
IMPORT BUF_DOT     INTO
IMPORT BUF_MARK    INTO
IMPORT BUF_GAP1    INTO
IMPORT BUF_GAP2    INTO

IMPORT BUF_DOT_2   INTO
IMPORT BUF_MARK_2  INTO
       BUF_REACH_2;
```

As already indicated, the basic idea behind this representation is as follows. An object Buf has a block associated with it, which contains all characters of the text represented by *b*. Within this block there is a so-called *gap*. The gap of b corresponds with the positions gap1(*b*) (inclusive) to gap2(*b*) (exclusive). The gap contains the free space in the block and when the dot is near the gap, it is possible to do insertions easily and efficiently. The characters in the block correspond with the string representation, but the positions in the gap are not used. The following picture sketches a buffer with its gap.

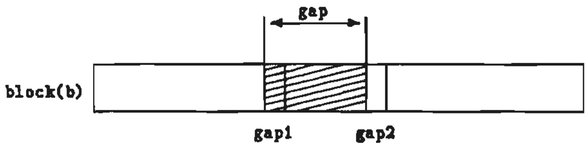

Fig 5.1. Buffer with gap.

Recall the following variables spanning the state space of WITEFA, introduced in Chapter 4.

- `FUNC mtexts  : → Map    VAR`
- `FUNC current: → String VAR`

These are going to be replaced by other 'new' variables. We want to introduce a table from strings (sort `'String'`) to buffers. Therefore we want to instantiate `TABLE` with an implementation of `'STRING_SPEC'` and with `BUF` thereby getting the possibility to create tables. Actually we need only one table, but that does not matter. We introduce new variables, which serve for taking over the role of the original ones; these are the following:

- `FUNC table   : →  Table    VAR`
- `FUNC current: →  'String' VAR`

For purposes such as searching in a text or copying texts, we introduce a simple programming variable of sort `Nat`. We call it `mover`, since it is not a relatively static pointer, such as mark or dot, but it is supposed to *move* during the execution of a search operation, or when it is used as a loop-counter. We also need a pair-wise version of `mover`.

- `mover: →  Nat  VAR`
- `mover: →` $\mathtt{Nat}^2$ `VAR`

Finally it will turn out to be convenient if we have one more simple programming variable of sort `Nat`. We intend to use it as a loop-counter in operations such as `yank_buffer` and `copy_region_to_buffer`. There is no need for a pair-wise version of `counter`.

- `FUNC counter: →  Nat  VAR`

As soon as we try to provide these new variables, we discover the need for two new components: `SVAR` providing simple programming variables and `'STRING'` providing an implementation of the sort `'String'`. Therefore we have an intermezzo introducing two new components.

5.5.8 Specifying Simple Programming Variables

We need a class description specifying a simple programming variable. With a ***simple programming variable*** we mean a variable nullary function of type → `Item`. Of course there must be an update procedure. We do not include some form of variable creation.

```
LET SVAR_SPEC :=

LAMBDA X : ITEM OF
EXPORT

  SORT Item,
  FUNC val:       -> Item,
  PROC upd: Item ->

FROM
IMPORT X INTO
CLASS

  FUNC val:       -> Item VAR
  PROC upd: Item -> MOD val

  AXIOM FORALL i:Item
  ( < upd(i) > TRUE;
    [ upd(i) ] val = i )

END;
```

5.5.9 SVAR: a Postulated Component

We postulate a new component SVAR with black-box description SVAR_SPEC. Just as for ATTR2 and ATTR, we best employ the meta-operator COPY for using SVAR. When syntax- and type-checking we replace this COPY(SVAR) by SVAR.

```
{@}  COMP SVAR : SVAR_SPEC;
```

5.5.10 'STRING': another Postulated Component

Since we have already a specification 'STRING_SPEC', we can directly postulate the following component:

```
{@}  COMP 'STRING' : 'STRING_SPEC';
```

5.5.11 Transforming the State space (continued)

After this intermezzo we can give the formal definition of the new variables which are added to the state space of WITEFA. First of all we instantiate TABLE.

```
LET TABLE_STRING_BUF :=
APPLY APPLY RENAME
        SORT Map        TO Buf_Map,
        SORT Item1      TO String,
        SORT 'Item1'    TO 'String',
        PRED item1_inv: TO string_inv,
        SORT Item2      TO Buf
IN TABLE TO 'STRING' TO BUF;
```

This instantiation of TABLE provides us with operations such as PROC new: -> Table, PROC add:Table # 'String' # Buf -> and PROC app: Table # 'String' -> Buf.

Now we introduce our new variables which is achieved by renaming and instantiating SVAR. This use of SVAR can be viewed as the COLD-K way of declaring variables.

Since these variables will serve as representations of (marked) texts, we collect them in a class description called TEXT_VARS. We import the application domain specific notational framework of APP_DOM_SPEC. We also import the data type of maps from strings to marked texts, but this need not be executable; we only use it for reasoning purposes.

```
LET TEXT_VARS :=

IMPORT BUF          INTO
IMPORT CHAR         INTO
IMPORT BOOL         INTO
IMPORT 'STRING'     INTO
IMPORT APP_DOM_SPEC INTO

IMPORT APPLY APPLY RENAME
        SORT Item1 TO String,
        SORT Item2 TO MText
IN MAP_SPEC TO STRING_SPEC TO MTEXT_SPEC INTO

IMPORT APPLY RENAME
         SORT Item TO String,
         SORT Set  TO Set1
IN SET_SPEC TO STRING_SPEC INTO

IMPORT APPLY RENAME
        SORT Item         TO Table,
        FUNC val: -> Item TO table,
        PROC upd: Item -> TO upd_table
IN COPY(SVAR) TO TABLE_STRING_BUF INTO
```

```
IMPORT APPLY RENAME
        SORT Item         TO 'String',
        FUNC val: -> Item TO current,
        PROC upd: Item -> TO upd_current
IN COPY(SVAR) TO 'STRING' INTO

IMPORT APPLY RENAME
        SORT Item         TO Nat,
        FUNC val: -> Item TO mover,
        PROC upd: Item -> TO set_mover
IN COPY(SVAR) TO NAT INTO

IMPORT APPLY RENAME
        SORT Item         TO Nat,
        FUNC val: -> Item TO mover1,
        PROC upd: Item -> TO set_mover1
IN COPY(SVAR) TO NAT INTO

IMPORT APPLY RENAME
        SORT Item         TO Nat,
        FUNC val: -> Item TO mover2,
        PROC upd: Item -> TO set_mover2
IN COPY(SVAR) TO NAT INTO

IMPORT
CLASS

  FUNC mover: -> Nat # Nat
  DEF  (mover1,mover2)

  PROC set_mover: Nat # Nat ->
  PAR  i:Nat,j:Nat
  DEF  set_mover1(i);
       set_mover2(j)

END INTO

IMPORT APPLY RENAME
        SORT Item         TO Nat,
        FUNC val: -> Item TO counter,
        PROC upd: Item -> TO set_counter
IN COPY(SVAR) TO NAT INTO

IMPORT TABLE_STRING_BUF INTO
```

```
CLASS
```

Note that we have two alternative ways of dealing with the pair-wise version of the mover. The first approach is to write directly in terms of `mover1` and `mover2` and alternatively we could use `mover`: $\rightarrow$ `Nat`^{2}.

Our next task is to describe precisely the relation between the old variables and the newly introduced ones. Therefore we aim at the definition of an abstraction function `f: Buf` $\rightarrow$ `MText`. We introduce the abstraction in two phases, using the algebra of strings as an intermediate level. This is natural since the linear structure of blocks is closely related to the structure of strings. We shall have several abstraction functions which are called `f` – by overloading. The functions `f` taking a `Block` argument take *all* characters into account, including those in the gap. The functions `f` taking a `Buf` argument skip the characters in the gap.

```
FUNC f: Block # Nat # Nat -> String
PAR  b:Block,m:Nat,n:Nat
%    the string obtained from cont(b,m)..cont(b,n-1)
DEF  ( m = n     ?; empty
     | NOT m = n ?; cons(cont(b,m),f(b,succ(m),n))
     )

PRED in_gap: Buf # Nat
PAR  b:Buf,n:Nat
%    n is a position in the gap of b
DEF  geq(n,gap1(b)) AND lss(n,gap2(b))
```

The next function `f` is meant for arguments b, m and n such that $m \leq n$ where m and n are not in the gap of b. This `f` is defined by a case analysis where we distinguish two cases. Assume the above restriction on b, m, n. The first case is characterised by `lss(`n`,gap1(`b`)) OR geq(`m`,gap2(`b`))` and thus either $m \leq n <$ `gap1(`b`)` (both parameters are to the left of the gap) or `gap2(`b`)` $\leq m \leq n$ (both parameters are to the right of the gap). The second case is characterised by `NOT(lss(`n`,gap1(`b`)) OR geq(`m`,gap2(`b`)))` and thus $m <$ `gap1(`b`)` $\leq$ `gap2(`b`)` $\leq n$ which means that the positions $m, \ldots, n-1$ encompass the gap.

```
FUNC f: Buf # Nat # Nat -> String
PAR  b:Buf,m:Nat,n:Nat
%    for leq(m,n) and not in_gap(b,m) and not in_gap(b,n):
%    string from positions m...n-1 except for those in the gap
DEF  ( (lss(n,gap1(b)) OR geq(m,gap2(b)))       ?;
       f(block(b),m,n)
     | NOT( lss(n,gap1(b)) OR geq(m,gap2(b))) ?;
```

```
          LET s1:String; s1 := f(block(b),m,gap1(b));
          LET s2:String; s2 := f(block(b),gap2(b),n);
          cat(s1,s2)
        )

FUNC f: Buf # Nat -> String
PAR  b:Buf,n:Nat
%    string from positions 0..n-1 except for those in gap
DEF  f(b,0,n)

FUNC f: Buf -> String
PAR  b:Buf
%    string obtained from the entire block, skipping the gap
DEF  f(b,size(block(b)))
```

These functions f enable us to view a part of a buffer or an entire buffer as a string. As a next step we shall push the overloading somewhat further and we introduce another collection of functions f which enable us to view a part of a buffer or an entire buffer as a *text*.

```
FUNC f: Block # Nat # Nat -> Text
PAR  b:Block,m:Nat,n:Nat
DEF  text(f(b,m,n))

FUNC f: Buf # Nat # Nat -> Text
PAR  b:Buf,m:Nat,n:Nat
DEF  text(f(b,m,n))

FUNC f: Buf # Nat -> Text
PAR  b:Buf,n:Nat
DEF  text(f(b,n))
```

In the defining expression of the following function we need a type-cast because otherwise the Text expression `text(f(b))` would become ambiguous (assuming `b:Buf`). The intended interpretation is that `f(b)` is a string; the other interpretation is that `f` refers to a function `f: Buf -> MText` to be introduced later, which would correspond with `f(b)` being a `MText`. The guideline of where to use these type-casts and where not to is very simple: use no type-casts except where a real ambiguity arises. When it turns out that many type-casts are needed, one may conclude that the usefulness of overloading has reached its limit.

```
FUNC f: Buf -> Text
PAR  b:Buf
DEF  text((f(b)):String)
```

```
FUNC f: Buf -> MText
PAR  b:Buf
DEF  LET do: Nat # Nat; do := reach(f(b,(dot(b)):Nat));
     LET ma: Nat # Nat; ma := reach(f(b,(mark(b)):Nat));
     mtext(f(b),copa(do),copa(ma))
```

If we would just aim at the definition of the last function f: Buf → MText, then some of the other functions could easily be eliminated. However we prefer this step-by-step approach, because some of the functions which now seem to be auxiliaries, might turn out useful in their own right later, e.g. for formulating loop-invariants. The last function f is our abstraction function showing how Buf objects represent marked texts.

We must extend the abstraction described by f: Buf → MText to maps. Recall that Buf_Map is the sort of maps from strings to buffers. Map is the sort of maps from strings to marked texts.

```
FUNC f: Buf_Map -> Map
PAR  bm:Buf_Map
DEF  SOME m:Map
     ( FORALL s:String, mt:MText
       ( app(m,s) = mt <=> f(app(bm,s)) = mt ) )
```

We *define* the variables required by the specification (Chapter 4) in terms of newly introduced variables. Recall that by means of the imports of TEXT_VARS we have a variable function current which is of type 'String'; it is used below to get the String expression f(current).

```
FUNC current: -> String
DEF  f(current)

FUNC mtexts: -> Map
DEF  f(map(table))
```

Using these newly defined variables, we can easily define the functions text and dot which are formally required here because they occur (as auxiliaries) in the specification of Chapter 4.

```
FUNC text: -> Text DEF text(app(mtexts,current))
FUNC dot: -> Nat # Nat DEF dot(app(mtexts,current))
```

Now we turn our attention to formulating a suitable invariant assertion. The assertions table_inv and string_inv do not require special attention since they will hold automatically. Still we must formally have them as conjuncts of our invariant. This is condition {1} below. The value of the 'String' variable current must be an entry in table. This is condition {2} below. Furthermore we must regulate the relative positions of dot, mark, gap1, gap2 and block-size. This is condition {3} below. Finally the co-ordinate pair representations of dot and mark should correspond with their natural number representations and the reach attribute should contain the reach of the text represented by the buffer. This is condition {4} below.

All these conditions are collected in a predicate called TI'. We can view it as a transformed version of the text-invariant TI from our specification. At the same time it is the representation invariant for the chosen representation of marked texts. We repeat TI1, TI2 etc. here as well because formally the principle of signature incusion requires us to have them 'somewhere' in the implementation of WITEFA.

```
PRED TI1: % as in Section 4.5.5
PRED TI2: % as in Section 4.5.5
PRED TI:  % as in Section 4.5.5
PROC witefa_op: -> % as in Section 4.5.5


PRED TI':
DEF  {1} string_inv;
         table_inv;
     {2} app(map(table),current)!;
     {3} FORALL s:String( app(mtexts,s)! =>
         ( LET b:Buf; b := app(map(table),s);
           leq(dot(b),size(block(b)));
           leq(mark(b),size(block(b)));
           leq(gap1(b),size(block(b)));
           leq(gap2(b),size(block(b)));
           leq(gap1(b),gap2(b));
           NOT in_gap(b,dot(b));
           NOT in_gap(b,mark(b))));
     {4} FORALL s:String( app(mtexts,s)! =>
         ( LET b:Buf; b := app(map(table),s);
           (dot(b))  : Nat # Nat = reach(f(b,(dot(b)):Nat));
           (mark(b)) : Nat # Nat = reach(f(b,(mark(b)):Nat));
           (reach(b)): Nat # Nat = reach(f(b)) ) )
```

It should be possible to show that

$$\text{TI'} \Rightarrow \text{TI}.$$

Recall from Chapter 4 that TI $\Leftrightarrow$ TI1 $\wedge$ TI2 and that TI1 requires that for every string s in the domain of mtexts with corresponding marked text mt the following holds:

> intext(text(mt),d) $\wedge$ intext(text(mt),m) $\wedge$ ok(text(mt))
> where d and m denote the dot and the mark of mt respectively.

TI2 requires that app(mtexts,current)!. We shall show that TI' $\Rightarrow$ TI now.

Assume TI' and let s be an arbitrary string in the domain of mtexts. Then app(mtexts,s) = f(app(map(table),s)). If we denote the buffer app(map(table),s) by b, then TI' gives us the following facts about b.

- dot (b) $\leq$ size(block(b))
- mark(b) $\leq$ size(block(b))
- gap1(b) $\leq$ size(block(b))
- gap2(b) $\leq$ size(block(b))
- gap1(b) $\leq$ gap2(b)
- dot (b) $\notin$ gap(b)
- mark(b) $\notin$ gap(b)

where we used some obvious shorthand. Therefore f(b,dot(b):Nat) is a prefix text of f(b) so its reach is an existing position in f(b). This shows intext(text(mt),d) where d is the dot of mt. The same can be done for mark. Furthermore notice that f(b) is ok since it is defined by means of the conversion function text: String $\rightarrow$ Text. This shows TI1. Finally TI2 follows from app(map(table),current)!. Hence we have shown that TI' $\Rightarrow$ TI.

We also add a few simple operations and predicates which are directly connected with the chosen representation. The names of the operations eobp, eolp etc. abbreviate end-of-buffer-predicate, end-of-line-predicate etc. We write {C}OR for *conditional* OR as a hint related to the executability of certain assertions.

We shall use some informal terminology but we show by means of a few examples how this can be formalised when needed. Let b be a given buffer. Then the phrase "i is a position at end-of-buffer" means f(b,i):Text = f(b). The phrase "i is a position at end-of-line" means i_2 = sel(profile(f(b)),i_1) where (i_1, i_2) = reach(f(b,i)). The phrase "i is a position at beginning-of-line" means $i_2 = 0$ where (i_1, i_2) = reach(f(b,i)). The phrase "i is a position at beginning-of-buffer" means reach(f(b,i)) = $(0,0)$.

```
FUNC right: Buf # Nat -> Nat
PAR  b:Buf,i:Nat
```

```
%    next position (going rightwards), skipping gap if necessary
DEF  ( succ(i) = gap1(b)     ?; gap2(b)
     | NOT succ(i) = gap1(b) ?; succ(i)
     )

PRED eobp: Buf # Nat
PAR  b:Buf,i:Nat
%    i is is a position at end-of-buffer
DEF  i = size(block(b))

PRED eolp: Buf # Nat
PAR  b:Buf,i:Nat
%    i is a position at end-of-line
DEF  eobp(b,i) {C}OR cont(block(b),i) = ctr_j

PRED eobp: Buf
PAR  b:Buf
%    dot is at end-of-buffer
DEF  eobp(b,dot(b))

PRED eolp: Buf
PAR  b:Buf
%    dot is at end-of-line
DEF  eolp(b,dot(b))

FUNC left: Buf # Nat -> Nat
%    next position (going leftwards), skipping gap if necessary
PAR  b:Buf,i:Nat
DEF  ( i = gap2(b)     ?; pred(gap1(b))
     | NOT i = gap2(b) ?; pred(i)
     )

PRED bobp: Buf # Nat
PAR  b:Buf,i:Nat
%    i is a position at beginning-of-buffer
DEF  i = 0 OR gap1(b) = 0 AND i = gap2(b)

PRED bobp: Buf
PAR  b:Buf
%    dot is at beginning-of-buffer
DEF  bobp(b,dot(b))

PRED bolp: Buf # Nat
PAR  b:Buf,i:Nat
%    i is a position at beginning-of-line
```

```
   DEF  bobp(b,i) {C}OR cont(block(b),left(b,i)) = ctr_j

   PRED bolp: Buf
   PAR  b:Buf
   %    dot is at beginning-of-line
   DEF  bolp(b,dot(b))

END; {of TEXT_VARS}
```

We must make one remark about the use of the attributes called mover. When we use both mover:Nat and mover:$\texttt{Nat}^2$, then we want for a given buffer b the following assertion to hold.

$$\texttt{reach(f(}b\texttt{,mover:Nat))} = \texttt{mover:Nat}^2,$$

and we shall refer to this situation by saying that *the pair-wise version of mover precisely follows the mover.*

We would like to start programming the WITEFA operations, but first we need another intermezzo. Just as in the specification phase (Chapter 4) we prefer to postpone dealing with the fact that the physical display device must mirror the contents of the current buffer. We do so by postulating a suitable component. This is the subject of the next two sections.

5.5.12 Specifying a Window-Invariant Package

We postulate a component which takes care for the window invariant WI from Chapter 4. It provides two procedures called mod_text_restore and mod_dot_restore. The first procedure is intended for being used in the initialisation phase of an editor or after a series of modifications affecting the current text and dot; then mod_text_restore establishes the window invariant WI. The second procedure is intended for being used after a series of modifications moving the dot alone, but still thereby possibly disturbing the window invariant WI.

Below we give a class description WI_PACKAGE_SPEC, specifying these two procedures. Clearly they must operate on the *representation* of the marked texts and therefore we can only expect them to work if the corresponding representation invariant TI' holds. Note that TEXT_VARS is the class description containing the definition of TI'. Although at first sight it seems natural to import TEXT_VARS into WI_PACKAGE_SPEC, this can not be done because of the condition 'directly specified' from Chapter 3. Instead, we employ a class description TEXT_VARS_SPEC. It can be verified that TEXT_VARS $\sqsubseteq$ TEXT_VARS_SPEC.

```
LET TEXT_VARS_SPEC :=
IMPORT APP_DOM_SPEC INTO
IMPORT APPLY APPLY RENAME
  SORT Item1 TO String,
  SORT Item2 TO MText
IN MAP_SPEC TO STRING_SPEC TO MTEXT_SPEC INTO
CLASS

  SORT Table VAR
  FUNC mtexts: -> Map VAR
  FUNC current: -> String VAR
  PROC upd: Table -> MOD mtexts
  PRED TI': DEP Table, mtexts, current

END;
```

In WI_PACKAGE_SPEC we have a local invariant wi_package_inv which is necessary in order to allow for additional variables which may be needed for the implementation of WI_PACKAGE. Such variables might be created dynamically and/or need initialisation; hence there must be an initialisation procedure which is init_wi_package. We repeat the definition of WI because it is employed in the axioms; p_sub, p_add and size at their turn are repeated because they occur in WI – and also for reasons of signature inclusion.

```
LET WI_PACKAGE_SPEC :=
IMPORT TEXT_VARS_SPEC INTO
IMPORT DISPLAY_SPEC   INTO
CLASS

  FUNC p_sub: Nat # Nat # Nat # Nat -> Nat # Nat % as in Section 4.5.12
  FUNC p_add: Nat # Nat # Nat # Nat -> Nat # Nat % as in Section 4.5.12
  FUNC size: -> Nat # Nat % as in Section 4.5.12
  PRED WI: % as in Section 4.5.12

  PRED wi_package_inv: VAR

  PROC init_wi_package: ->
  MOD  wi_package_inv

  PROC mod_text_restore: ->
  MOD  wi_package_inv
  USE  displ_op

  PROC mod_dot_restore: ->
  MOD  wi_package_inv
  USE  displ_op

  PROC mod_dot: -> % auxiliary
  DEF  LET mt':Map;
       mt' := mtexts;
       USE upd:Table -> END;
       TI' AND text(app(mtexts,current)) = text(app(mt',current)) AND
       (mark(app(mtexts,current))):Copa = mark(app(mt',current)) AND
       FORALL s:String
       ( NOT s = current => app(mtexts,s) = app(mt',s) ) ?

  AXIOM [ init_wi_package ] wi_package_inv;
        wi_package_inv =>
        [ mod_text_restore | mod_dot_restore ] wi_package_inv

  AXIOM TI' AND wi_package_inv =>
        [ mod_text_restore ] WI

  AXIOM TI' AND wi_package_inv =>
        WI => [ mod_dot ] [ mod_dot_restore ] WI

  AXIOM {TERMINATION}

        < init_wi_package > TRUE;
        TI' AND wi_package_inv => < mod_text_restore > TRUE;
        TI' AND wi_package_inv => < mod_dot_restore  > TRUE
```

We can consider the conjunction of `WI` and `wi_package_inv` as a strengthened version of `WI` and in view of the similarity with the strengthening of `TI` to `TI'` we introduce the notation `WI'`.

```
  PRED WI':
  DEF  WI AND wi_package_inv

END;
```

One might be tempted to think that `wi_package_inv` needs not be in the modification lists of `mod_text_restore` and of `mod_dot_restore`. For these procedures are supposed not to violate `wi_package_inv` and this is guaranteed indeed if `wi_package_inv` cannot be modified. However, this would be too strong, because it forbids an implementation to change `wi_package_inv` from false to true – just by luck.

5.5.13 WI_PACKAGE: a postulated component

We postulate a component having WI_PACKAGE_SPEC as its black-box description. This is an important component in the sense that it hides all problems related to the fact that our editor is *display-oriented*. Note that in particular mod_text_restore works as a kind of magic button: just activate it and then the display is completely taken care of. We take a certain risk by postulating this component because it is not entirely clear in advance that sufficiently efficient algorithms for mod_text_restore and mod_dot_restore will be found.

```
{@}  COMP WI_PACKAGE : WI_PACKAGE_SPEC;
```

5.5.14 Programming the Editing Operations

Recall that the specification requires a classical invariant WTI which should logically imply WI and TI. We shall propose such an invariant now.

```
LET WITEFA_IMPL :=

IMPORT DISPLAY    INTO
IMPORT TEXT_VARS  INTO
IMPORT WI_PACKAGE INTO
IMPORT (APPLY FILE TO 'STRING') INTO
CLASS

   PRED WTI:
   DEF  WI' AND TI'
```

Since we know already that WI' ⇒ WI and TI' ⇒ TI, the fact that WTI logically implies WI and TI is immediate from the definition of WTI. Now we have this invariant we can already write the init procedure and one of the witefa_op procedures, viz. switch_to_buffer, which serves as a kind of buffer initialisation. We begin with init. By means of the imports of TEXT_VARS we have 'assignment' procedures upd_table and up_current respectively. These are used in the body of init below where upd_table(new) updates the 'COLD variable' table, giving it the value new and where upd_current(s) updates the 'COLD variable' current, giving it the 'String' value s. The procedure create_buffer serves as an auxiliary for both init and switch_to_buffer. Arbitrarily we choose the value 1024 for the initial block-size of newly created buffers.

```
   PROC init: 'String' ->
```

```
PAR  s:'String'
DEF  upd_table(new);
     create_buffer(s);
     upd_current(s);
     init_wi_package;
     mod_text_restore

PROC create_buffer: 'String' ->
PAR  s:'String'
DEF  LET b:Buf; b := create;
     add(table,s,b);
     set_block(b,alloc(1024));
     set_gap1(b,0);
     set_gap2(b,1024);
     set_dot(b,1024);
     set_mark(b,1024);
     % pair-wise:
     set_dot(b,0,0);
     set_mark(b,0,0);
     set_reach(b,0,0)
```

The specification (Chapter 4) requires that WTI is a classical invariant. Therefore, among other things we must verify INIT $\Rightarrow$ $\forall s$: 'String' [init(s)] WTI. We shall briefly show this now and we begin with with TI'. This amounts to checking all conjuncts of TI', which are labelled {1}, {2} etc.

Conjunct {1} requires string_inv and table_inv. Now string_inv holds because 'STRING_SPEC' has been obtained from 'SEQ_SPEC' and in the corresponding instantiation it was defined that item_inv :$\Leftrightarrow$ TRUE; therefore INIT $\wedge$ item_inv $\Rightarrow$ string_inv applies. We apply this reasoning once more: table_inv holds because TABLE_STRING_BUF has been obtained from TABLE and in the corresponding instantiation it was defined that item1_inv :$\Leftrightarrow$ string_inv; therefore INIT $\wedge$ item1_inv $\Rightarrow$ table_inv applies. After having verified this we need not worry about string_inv and table_inv any more, for there are absolutely no operations that could make them false.

Conjunct {2} requires the definedness of the current buffer. This is established by upd_table(new), create_buffer(s) and upd_current(s). Conjuncts {3} and {4} regulate the various buffer attributes. In this case there is just one buffer and create_buffer makes its gap cover the entire block. It puts the dot and the mark at the end of the block which is $(0,0)$ when viewed as a co-ordinate pair. This establishes {3} and {4}.

We must also verify that wi_package_inv holds, which precisely is the effect of init_wi_package.

Finally we must show that WI holds and this is established by the invocation of mod_text_restore. The procedure mod_text_restore only works if both TI' and wi_package_inv hold but this is the case indeed thanks to the fact that
mod_text_restore comes after all other expressions of init. This concludes our verification of INIT $\Rightarrow \forall s$: 'String' [init(s)] WTI and we proceed with another editor operation.

```
PROC switch_to_buffer: 'String' ->
PAR  s:'String'
DEF  upd_current(s);
     ( is_in_dom(s,table) = true     ?; SKIP
     | NOT is_in_dom(s,table) = true ?; create_buffer(s)
     );
     mod_text_restore
```

We must verify that switch_to_buffer preserves WTI which is
WTI $\Rightarrow \forall s$: 'String' [switch_to_buffer(s)] WTI.

Conjunct {1} of TI' holds since we assumed WTI as a precondition. {2} holds because either s is already an entry in table or s is added to table. In the former case it holds by the definition of is_in_dom; in the latter case create_buffer guarantees that the buffer corresponding to s represents a defined text. For {3} and {4} we reason differently for the old buffers (where we must refer to the precondition WTI) and for the newly created buffer (where we reason just as for init(s) before).

Finally the invocation of mod_text_restore preserves wi_package_inv and it establishes WI. This shows that switch_to_buffer preserves WTI.

Apart from verifying the preservation of WTI, we must also verify the satisfaction of the pre- and postcondition style axioms from the specification (Chapter 4). By way of example we show this for switch_to_buffer(s). Its postcondition requires current = f(s) which is made true by the expression upd_current(s). The remainder of its postcondition is based on a case-analysis. If s is an existing entry in table already, then mtexts = PREV mtexts must hold, which is the effect of SKIP indeed. If s is a new entry, then the marked text corresponding with s should be

```
mtext(zero,copa(0,0),copa(0,0))
```

which should have been added to PREV mtexts. We easily verify that this is the marked text represented by the buffer created and initialised by create_buffer. This shows that switch_to_buffer satisfies its pre- and postcondition style axiom.

Later we shall program the remaining `WITEFA` operations. Just like `switch_to_buffer` these must also have the properties of preserving `WTI` and satisfying their pre- and postcondition style axioms. We shall not treat these at the same level of detail as `switch_to_buffer`. We postpone a discussion of the termination axioms until Section 5.5.15.

We proceed by implementing the operations for dot and mark control as specified in Chapter 4 Section 4.5.6. To this end we first introduce another predicate on buffers which is not part of `TI'`, but which should be viewed as an ***additional nice buffer property.*** This predicate is called `ready` and roughly speaking, it holds for a buffer if the gap is located at the dot, so that the buffer is ready for an insert operation at dot. The following buffer is not ready:

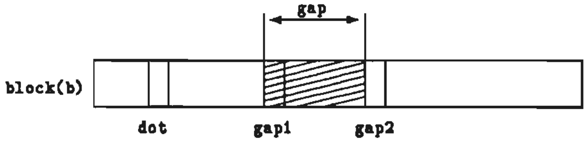

Fig 5.2. Buffer which is not 'ready'.

But the buffer below *is* ready:

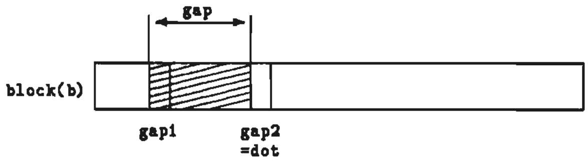

Fig 5.3. Buffer which is 'ready'.

We do not add the requirement that all buffers are ready to `TI'` because this would make dot-movements much more complicated and hence probably also slower.

```
PRED ready: Buf
PAR  b:Buf
DEF  dot(b) = gap2(b)
```

So if a buffer b with non-empty gap is ready, then inserting a character c in it, as necessary for the `insert_character` operation, becomes very simple:

```
    store(block(b),gap1(b),c);
    set_gap1(b,gap1(b) + 1 )
```

where we omittted the necessary updating of the pair-wise mark, dot and reach attributes. If we want to do an insert operation in a state in which the buffer in question is not ready, then it should be made so. Therefore we introduce a procedure make_ready operating on buffers and having use-rights with respect to store, set_dot, set_mark, set_gap1 and set_gap2. This procedure must have the following properties which of course need only be guaranteed if TI' holds:

1. make_ready(b) only affects attributes of b,
2. [make_ready(b)] ready(b),
3. [make_ready(b)] (f(b)):MText = PREV f(b),
   so essentially the represented marked text does not change.

We start the development of make_ready now. We must deal with two cases depending on the relative position of the gap and the dot. Therefore we use two auxiliaries called make_ready1 and make_ready2. The task of make_ready1 is to move the gap downwards to let it meet the dot. The task of make_ready2 is to move the gap upwards until the new gap2 equals the dot. Let us give the definition of make_ready now and after that deal with the auxiliaries.

```
PROC make_ready: Buf ->
PAR  b:Buf
DEF  ( dot(b) = gap2(b)      ?; SKIP
     | lss(dot(b),gap1(b)) ?; make_ready1(b)
     | gtr(dot(b),gap2(b)) ?; make_ready2(b)
     )
```

For make_ready1 we use a loop and as a the loop-body we employ a procedure gap_down. It must leave the text represented by the buffer unaffected.

$$\begin{array}{l} \forall\, b\text{:Buf}\ (\ \text{TI'} \wedge \text{gap1}(b) > 0 \Rightarrow \\ \quad [\ \text{gap\_down}(b)\ ] \\ \quad (\ (\text{f}(b))\text{:Text} = \text{PREV f}(b) \\ \quad\quad \wedge\ \text{dot}(b)\text{:Nat} = \text{PREV dot}(b) \\ \quad\quad \wedge\ \text{mark}(b)\text{:Nat} = \text{PREV mark}(b\ )\ ) \end{array}$$

After the loop of make_ready1 has been executed, the new position of the mark must be calculated. This calculation is done by a case-analysis on the relative position of the mark. Depending on the position of the mark, it may be necessary to move the mark over the gap, which means to increase the value of mark by gap2(b) − gap1(b). After execution of the gap_down loop, there are three possible situations: the first situation is characterised by mark(b) $<$ dot(b) and since the gap has reached dot(b) by moving downwards, we are certain that the gap has not crossed mark(b). The second situation is characterised by mark(b) $\geq$ 'old value of gap2(b)' which means that mark(b) was already at the high end of the buffer compared with the gap and since the gap has moved downwards we are certain again that the gap has not crossed mark(b). The third situation is when neither mark(b) $<$ dot(b) nor mark(b) $\geq$ 'old value of gap2(b)' holds, which is precisely when the assertion dot(b) $\leq$ mark(b) $<$ gap1(b) was valid *before* moving the gap. In this third situation the value of mark must be increased and after that the mark will have crossed the gap in the sense that gap2(b) $\leq$ mark(b).

```
PROC make_ready1: Buf ->
PAR  b:Buf
DEF  LET old_gap2:Nat;
     old_gap2 := gap2(b);
     ( NOT dot(b) = gap1(b) ?;
           gap_down(b)
     ) *;  dot(b) = gap1(b) ?;

     ( lss(mark(b),(dot(b)):Nat) ?; SKIP
     | geq(mark(b),old_gap2)     ?; SKIP
     | NOT (lss(mark(b),(dot(b)):Nat) OR geq(mark(b),old_gap2)) ?;
       set_mark(b,add(mark(b),sub(gap2(b),gap1(b))))
     );
     set_dot(b,gap2(b))
```

The task of make_ready2 is to move the gap upwards until the value of gap2 equals the dot. For make_ready2 we use a loop and in the loop-body we employ a procedure gap_up with the property:

```
∀ b:Buf ( TI' ∧ gap2(b) < size(block(b))) ⇒
    [ gap_up(b) ]
    ( (f(b)):Text = PREV f(b)
      ∧ dot(b):Nat = PREV dot(b)
      ∧ mark(b):Nat = PREV mark(b ) )
```

Again we have a loop and a calculation of the mark. Depending on the position of the mark, it may be necessary to move the mark over the gap, which now means to *de*crease the value of mark by gap2(*b*) $-$ gap1(*b*). Again we distinguish three situations where the third situation is precisely when the assertion gap2(b) $\leq$ mark(b) $<$ dot(b) was valid *before* moving the gap. In this third situation the value of mark must be decreased and after that the mark will have crossed the gap in the sense that mark(b) $<$ gap1(b).

```
PROC make_ready2: Buf ->
PAR  b:Buf
DEF  LET old_gap1:Nat;
     old_gap1 := gap1(b);
     (NOT dot(b) = gap2(b) ?;
          gap_up(b)
     ) *; dot(b) = gap2(b) ?;
     ( lss(mark(b),old_gap1) ?; SKIP
     | geq(mark(b),dot(b))   ?; SKIP
     | NOT (lss(mark(b),old_gap1) OR geq(mark(b),dot(b))) ?;
       set_mark(b,sub(mark(b),sub(gap2(b),gap1(b))))
     )
```

We give the definitions of gap_down and gap_up below.

```
PROC gap_down: Buf ->
PAR  b:Buf
%    move the gap one position downwards
DEF  store(block(b),pred(gap2(b)),cont(block(b),pred(gap1(b))));
     set_gap1(b,pred(gap1(b)));
     set_gap2(b,pred(gap2(b)))

PROC gap_up: Buf ->
PAR  b:Buf
%    move the gap one position upwards
DEF  store(block(b),gap1(b),cont(block(b),gap2(b)));
     set_gap1(b,succ(gap1(b)));
     set_gap2(b,succ(gap2(b)))
```

We add one remark about the algorithm for make_ready as developed above. As an alternative, it would have been possible to put the adaptation of dot and mark *in* gap_up and gap_down. This alternative would make the presentation of the algorithm somewhat smoother. On the other hand, the current algorithm is slightly more efficient, which hopefully is worthwhile when editing large texts.

This concludes the work on making buffers ready and we proceed with a few more editor operations, among which those operations which deal with dot-movements. Actually this turns out to be quite a lot of work, which seems of a somewhat ad-hoc nature. We shall have to introduce many auxiliaries. In the definition of right_dot below we need a type-cast for the ambiguous expression dot(b); this is because later there will be another function right: Buf # Nat # Nat -> Nat # Nat – which will appear in connection with search_forward. No function left: Buf # Nat # Nat -> Nat # Nat is defined, however.

```
PROC right_dot: Buf ->
PAR  b:Buf
%    to be used only if dot is not at end-of-line
DEF  set_dot(b,right(b,(dot(b)):Nat));
     LET d1:Nat,d2:Nat; d1,d2 := dot(b); set_dot(b,d1,succ(d2))

PROC forward_character: ->
DEF  LET b:Buf; b := app(table,current);
     ( eolp(b)     ?; SKIP
     | NOT eolp(b) ?;
       right_dot(b);
       mod_dot_restore
     )

PROC left_dot: Buf ->
PAR  b:Buf
%    to be used only if dot is not at beginning-of-line
DEF  set_dot(b,left(b,dot(b)));
     LET d1:Nat,d2:Nat; d1,d2 := dot(b); set_dot(b,d1,pred(d2))

PROC backward_character: ->
DEF  LET b:Buf; b := app(table,current);
     ( bolp(b)     ?; SKIP
     | NOT bolp(b) ?;
       left_dot(b);
       mod_dot_restore
     )

PROC bolp: -> Bool
DEF  LET b:Buf; b := app(table,current);
     ( bolp(b)     ?; true
     | NOT bolp(b) ?; false
     )

PROC eolp: -> Bool
DEF  LET b:Buf; b := app(table,current);
     ( eolp(b)     ?; true
     | NOT eolp(b) ?; false
     )
```

Next, we turn our attention to the procedure `next_line` and the first thing to do is introducing an auxiliary function, also called `next_line`. It serves for finding out whether it is possible to move the dot to the same horizontal position in the next line; if so, then it calculates this new position. The following picture shows the situation where the dot is neither in the first line nor in the last line of the text.

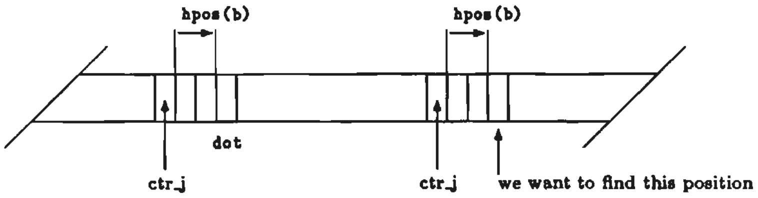

Fig 5.4. Operation of `next_line`.

This function `next_line` requires a little bit of searching and stepping through the block. There is some searching rightwards for finding the end of the current line and hence the beginning of the next line. Then some further stepping to the right is needed for getting to the position with the appropriate horizontal co-ordinate. Essentially this searching and stepping is programmed using *recursion*. This need not be very efficient, but at least the recursion depth is limited by the line length. The function `next_line` yields a pair consisting of a Boolean value and a natural number. This pair is (`true`,the 'next-line position'), if this exists; it is (`false`,0) otherwise. We must explain the term 'next-line position'. For a given buffer b we say that n is the 'next-line position' if

$$\begin{aligned}&n_1 = d_1 + 1 \land n_2 = d_2\\&\text{where } (n_1, n_2) = \texttt{reach(f(}b, n\texttt{))}\\&\quad\text{and } (d_1, d_2) = \texttt{dot}(b).\end{aligned}$$

Actually the value 0 is just arbitrary. We use some auxiliaries which will be defined immediately afterwards. In the definition of `next_line` below we employ a function `end_of_line` yielding two results which are denoted as `j` and `k`, although `k` is not needed here. Later we shall encounter another usage of `end_of_line` where its second result (`k`) is actually used.

```
FUNC next_line: Buf -> Bool # Nat
PAR  b:Buf
DEF  LET j:Nat,k:Nat;
     j,k := end_of_line(b,dot(b));
     ( eobp(b,j)     ?; (false,0)
     | NOT eobp(b,j) ?; right_stepping(b,right(b,j),hpos(b))
     )
```

We used several auxiliaries which will be defined now. The function `right_stepping` applied to a buffer b and natural numbers j and h yields a pair consisting of a Boolean value and a natural number. This pair is (true,the position obtained by going h steps rightwards within the same line starting at position j), if possible; it is (false,0) otherwise. Again we used a phrase which is easily formalised; we say that n is the position obtained by going h steps rightwards within the same line starting at position j if

$$\begin{aligned}&n_1 = j_1 \land n_2 = j_2 + h\\ &\text{where } (n_1, n_2) = \texttt{reach(f}(b,n)\texttt{)}\\ &\quad\text{and } (j_1, j_2) = \texttt{reach(f}(b,j)\texttt{)}.\end{aligned}$$

The procedure right_stepping is defined as follows.

```
FUNC right_stepping: Buf # Nat # Nat -> Bool # Nat
PAR  b:Buf,j:Nat,h:Nat
DEF  ( h = 0     ?; (true,j)
     | NOT h = 0 ?; ( eolp(b,j)     ?;
                      (false,0)
                    | NOT eolp(b,j) ?;
                      right_stepping(b,right(b,j),pred(h))
                    )
     )
```

The function hpos simply yields the horizontal co-ordinate of the dot for a given buffer. The function end_of_line applied to a buffer b and a position i yields the next position – going rightwards – for which the 'end-of-line' predicate holds and the number of rightward steps needed to reach this position. It is defined recursively and both result values play a role in the recursion, although this is not the motivation for end_of_line having two results.

```
FUNC hpos: Buf -> Nat
PAR  b:Buf
DEF  LET d1:Nat,d2:Nat; d1,d2 := dot(b);
     d2

FUNC end_of_line: Buf # Nat -> Nat # Nat
PAR  b:Buf,i:Nat
DEF  ( eolp(b,i)     ?; (i,0)
     | NOT eolp(b,i) ?;
       LET n:Nat,m:Nat; n,m := end_of_line(b,right(b,i));
       (n,succ(m))
     )
```

The following function is called previous_line and it is an auxiliary for the procedure with the same name. Again we give a sketch.

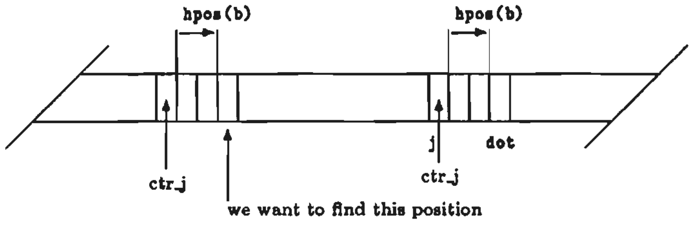

Fig 5.5. Operation of `previous_line`.

This function yields a pair consisting of a Boolean value and a natural number. This pair is (`true`,the 'previous-line position'), if this exists; it is (`false`,0) otherwise.

```
FUNC previous_line: Buf -> Bool # Nat
PAR  b:Buf
DEF  LET j:Nat; j:= beginning_of_line(b,dot(b));
     ( bobp(b,j)     ?;
       (false,0)
     | NOT bobp(b,j) ?;
       right_stepping(b,beginning_of_line(b,left(b,j)),hpos(b))
     )
```

We need just one more auxiliary. It serves for finding the next position – going leftwards – for which the beginning-of-line predicate holds.

```
FUNC beginning_of_line: Buf # Nat -> Nat
PAR  b:Buf,i:Nat
DEF  ( bolp(b,i)     ?; i
     | NOT bolp(b,i) ?; beginning_of_line(b,left(b,i))
     )
```

We implement the remaining procedures for dot and mark control. In the definition of `end_of_line` below, we see another usage of the earlier `end_of_line` function where we *do* need its second result – which reveals the motivation for that `end_of_line` having two results.

```
PROC beginning_of_line:  ->
DEF  LET b:Buf; b := app(table,current);
     set_dot(b,beginning_of_line(b,dot(b)));
     LET d1:Nat,d2:Nat; d1,d2 := dot(b); set_dot(b,d1,0);
     mod_dot_restore
```

```
PROC end_of_line:  ->
DEF  LET b:Buf; b := app(table,current);
     LET n:Nat,m:Nat; n,m := end_of_line(b,dot(b));
     set_dot(b,n);
     LET d1:Nat,d2:Nat; d1,d2 := dot(b);
     set_dot(b,d1,add(d2,m));
     mod_dot_restore

PROC next_line:  ->
DEF  LET b:Buf; b := app(table,current);
     LET bb:Bool, i:Nat;
     bb,i := next_line(b);
     ( bb = false     ?; SKIP
     | NOT bb = false ?;
       set_dot(b,i);
       LET d1:Nat,d2:Nat; d1,d2 := dot(b); set_dot(b,succ(d1),d2);
       mod_dot_restore
     )

PROC previous_line:  ->
DEF  LET b:Buf; b := app(table,current);
     LET bb:Bool, i:Nat;
     bb,i := previous_line(b);
     ( bb = false     ?; SKIP
     | NOT bb = false ?;
       set_dot(b,i);
       LET d1:Nat,d2:Nat; d1,d2 := dot(b); set_dot(b,pred(d1),d2);
       mod_dot_restore
     )

FUNC beginning_of_buffer: Buf -> Nat
PAR  b:Buf
DEF  ( gap1(b) = 0     ?; gap2(b)
     | NOT gap1(b) = 0 ?; 0
     )

PROC beginning_of_buffer: ->
DEF  LET b:Buf; b := app(table,current);
     set_dot(b,beginning_of_buffer(b));
     set_dot(b,0,0);
     mod_dot_restore

FUNC end_of_buffer: Buf -> Nat
PAR  b:Buf
DEF  size(block(b))
```

```
PROC end_of_buffer: ->
DEF  LET b:Buf; b := app(table,current);
     set_dot(b,end_of_buffer(b));
     set_dot(b,reach(b));
     mod_dot_restore

PROC set_mark: ->
DEF  LET b:Buf; b := app(table,current);
     set_mark(b,(dot(b)):Nat);
     set_mark(b,(dot(b)):Nat # Nat)
     {no screen updating needed}

PROC exchange_dot_and_mark: ->
DEF  LET b:Buf; b := app(table,current);
     LET i:Nat; i := dot(b);
     LET j:Nat; j := mark(b);
     LET d1:Nat,d2:Nat; d1,d2 := dot(b);
     LET m1:Nat,m2:Nat; m1,m2 := mark(b);
     set_dot(b,j);
     set_mark(b,i);
     set_dot(b,m1,m2);
     set_mark(b,d1,d2);
     mod_dot_restore
```

Next comes the implementation of the operations for text modification as specified in Chapter 4 Section 4.5.7. Again we introduce an additional nice buffer property. We shall refer to the value of gap2(b) − gap1(b) as the *size of the gap*. We shall devote some effort to solving the problem that the size of the gap may become 0 which implies that there is no more space for insertions. We introduce a simple predicate space which holds for a buffer b in case the size of the gap is non-zero. Just as ready, this predicate space is not part of TI', although of course it is a desirable buffer property.

```
PRED space: Buf
PAR  b:Buf
DEF  gtr(gap2(b),gap1(b))
```

Inserting in a buffer that has no space requires that space should be created and for this purpose we introduce a procedure make_space of type Buf # Nat → and having use-rights with respect to store,set_gap1,set_gap2 and grow. This procedure should have the following properties which must hold for $n > 0$ under the assumption TI' ∧ ready(b):

1. make_space(b,n) only affects attributes of b,
2. [make_space(b,n)] space(b) $\wedge$ ready(b),
3. [make_space$(b$,n)] (f(b)):MText $=$ PREV f(b).

The second parameter of make_space describes the space-increment. It serves for exploiting estimates of the required amount of space. We expect a typical implementation of BLOCK to show a reasonable performance if grow is invoked every now and then for a large increment; we expect it to be wasteful and disproportionately slow if it is invoked many times for very small increments. We arrange matters such that the procedure make_space can also be used in cases where the buffer did not run out of space yet.

The introduction of new space in a buffer takes place at the high end of the buffer. We can view this as the introduction of a *second gap*, whose size is equal to the amount of new space. From the point of view that we want to respect TI' this is one gap too many and hence this second gap has to move downwards to meet the other gap. The technique of moving this second gap is essentially the same as used in make_ready1. When the two gaps have met, which is when mover $=$ gap2(b), the invariant TI' and the readiness of the buffer can be restored. This is done by merging the two gaps which amounts to making gap2(b) and dot(b) equal to the position immediately after the merged gaps.

```
PROC make_space: Buf # Nat ->
PAR  b:Buf, n:Nat
DEF  set_mover(size(block(b)));
     grow(block(b),n);

     ( NOT mover = gap2(b) ?;
           second_gap_down(b,n)
     ) *;  mover = gap2(b) ?;

     set_gap2(b,add(gap2(b),n));
     set_dot(b,gap2(b));

     ( lss(mark(b),gap1(b))     ?; SKIP
     | NOT lss(mark(b),gap1(b)) ?; set_mark(b,add(mark(b),n))
     )
```

The following picture sketches the situation.

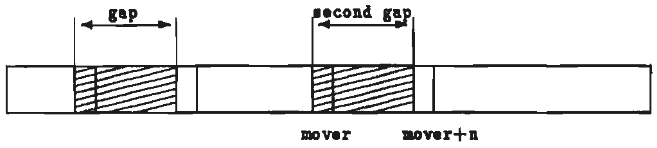

Fig 5.6. Merging two gaps.

The procedure second_gap_down serves for moving the second gap one position downwards. It has two arguments where the first argument indicates a buffer and where the second argument indicates the size of the second gap.

```
PROC second_gap_down: Buf # Nat ->
PAR  b:Buf,n:Nat
DEF  store(block(b),pred(add(mover,n)),cont(block(b),pred(mover)));
     set_mover(pred(mover))
```

This concludes the work on making space in buffers. As discussed before, it is simple to insert a non-ctr_j character in a ready buffer with space, leaving the readiness of the buffer invariant. In order to update the pair-wise attributes, we use the natural addition on reaches, which is the function add: $\mathtt{Nat}^2 \# \mathtt{Nat}^2 \to \mathtt{Nat}^2$ and furthermore we use other operations on reaches such as paste. The following procedure is meant for inserting a non-ctr_j character in a ready buffer with space. When the dot comes *before* the mark, the new pair-wise mark attribute is calculated as paste(mark(b),0,1,dot(b)). In such calculations we avoid using the abstraction functions f which are meant for reasoning purposes only – some of them would be very inefficient indeed.

```
PROC insert_character: Buf # Char ->
PAR  b:Buf,c:Char
%    only if space(b)
DEF  store(block(b),gap1(b),c);
     set_gap1(b,succ(gap1(b)));

     ( lss((mark(b)):Nat,dot(b))     ?;
       SKIP
     | NOT lss((mark(b)):Nat,dot(b)) ?;
       set_mark(b,paste(mark(b),0,1,dot(b)))
     );
     set_reach(b,paste(reach(b),0,1,dot(b)));
     set_dot(b,add(dot(b),0,1))
```

Also the introduction of a new line in a ready buffer with space can be done in a simple way, leaving the readiness of the buffer invariant.

```
PROC newline: Buf ->
PAR  b:Buf
%    only if space(b)
DEF  store(block(b),gap1(b),ctr_j);
     set_gap1(b,succ(gap1(b)));

     ( lss((mark(b)):Nat,dot(b))     ?;
       SKIP
     | NOT lss((mark(b)):Nat,dot(b)) ?;
       set_mark(b,paste(mark(b),1,0,dot(b)))
     );
     set_reach(b,paste(reach(b),1,0,dot(b)));
     set_dot(b,add(dot(b),1,0))
```

Now we can program some of the insertion procedures. The first one is `insert_file` and to keep things simple, we use the `insert_character` and `newline` procedures described above. If it is necessary to insert large files frequently, then it might be worthwhile to reconsider this algorithm and to remove the pasting from the main loop.

```
PROC insert_file: 'String' ->
PAR  s:'String'
DEF  LET b:Buf; b := app(table,current);
     make_ready(b);
     reset(s);
     ( NOT eof?; ( space(b)     ?; SKIP
                 | NOT space(b) ?; make_space(b,1024)
                 );
                 LET c:Char; c := read;
                 ( c = ctr_j     ?; newline(b)
                 | NOT c = ctr_j ?; insert_character(b,c)
                 )
     ) *;  eof?;
     mod_text_restore

PROC insert_character: Char ->
PAR  c:Char
DEF  LET b:Buf; b := app(table,current);
     make_ready(b);
     ( space(b)     ?; SKIP
     | NOT space(b) ?; make_space(b,128)
     );
     insert_character(b,c);
```

```
      mod_text_restore
```

After this, the newline procedure is easy.

```
 PROC newline: ->
 DEF  LET b:Buf; b := app(table,current);
      make_ready(b);
      ( space(b)     ?; SKIP
      | NOT space(b) ?; make_space(b,128)
      );
      newline(b);
      mod_text_restore
```

Now we turn our attention to the procedure yank_buffer. We shall use the term *yank buffer*, by which we mean the buffer corresponding with the 'String' argument of yank_buffer. All characters in the yank buffer must be copied into the current buffer, and as can be expected, a loop will take care for this. At first sight this loop seems rather obvious, but actually its construction requires some care in order not to exclude the case $b = y$, where b denotes the current buffer and y denotes the yank-buffer. If $b = y$ then the current buffer is inserted into itself. In order to develop the loop, we shall indicate what will be left unaffected by the loop-body and we shall state the loop-invariant.

The loop-body does not affect the marked texts represented by the buffers in table. The invariant is that an initial fragment of the text in the yank buffer is present in the gap of the current buffer b. Both the counter and the mover variables indicate the size of this initial fragment albeit in a different way; more precisely:

$$\texttt{f(block(}b\texttt{),gap1(}b\texttt{),counter)} = \texttt{f(}y\texttt{,mover)} \wedge \texttt{counter} \leq \texttt{gap2(}b\texttt{)}$$

Note that two functions f are involved. The first f takes the characters in the gap into account, whereas the second function f is defined to ignore the characters in the gap, which in the latter case is the gap of the yank buffer. Finally the ready and space properties of the current buffer are part of the invariant.

The initialisation for this loop is done by putting the mover at the first non-gap position of y and by making the counter equal to gap1(b). Furthermore of course b must be made ready. To make enough space, make_space is invoked and its second argument is the amount of free space needed for the contents of the yank buffer y. The latter amount equals the size of the block of y minus the size of its gap.

After the contents of the yank buffer has been copied into the gap of b, the value of gap1, which is the lower bound of the gap can be increased in order to make the copied characters really part of the text represented by b. Note that immediately after the contents of y has been copied, the reach of y has not changed yet, even not when $b = y$. Finally the pair-wise mark, dot and reach attributes must be modified by pasting the reach of the yank buffer into them. These modifications of pair-wise attributes should be conducted as a kind of simultaneous assignment due to the possible aliasing in the sense that $b = y$.

```
PROC yank_buffer: 'String' ->
PAR  s:'String'
DEF  LET b:Buf; b := app(table,current);
     LET y:Buf; y := app(table,s);

     make_ready(b);
     make_space(b,sub(size(block(y)), sub((gap2(y),gap1(y)))));

     set_mover((gap1(y) = 0 ?; gap2(y) | NOT gap1(y) = 0 ?; 0));
     set_counter(gap1(b));

     ( NOT mover = size(block(y)) ?;
           store(block(b),counter,cont(block(y),mover));
           set_mover(right(y,mover:Nat));
           set_counter(succ(counter))
     )*;   mover = size(block(y)) ?;
     set_gap1(b,counter);

     LET r:Nat # Nat; r := reach(y);
     ( lss(mark(b),(dot(b)):Nat)     ?;
       SKIP
     | NOT lss(mark(b),(dot(b)):Nat) ?;
       set_mark(b,paste(mark(b),r,dot(b)))
     );
     set_reach(b,paste(reach(b),r,dot(b)));
     set_dot(b,add(dot(b),r));

     mod_text_restore
```

Now we develop `copy_region_to_buffer`. Let b denote the current buffer. First it must be tested if `mark(`b`)` $<$ `dot(`b`)` for if not, then no action is required. Otherwise, the next thing to be done is to check if the target buffer exists already. If it does not exist, then it should be created and appropriately initialised.

Let c denote the target buffer. The next step is to enter the characters of the region of b into the target buffer c. This entering starts at position 0 in c. It is possible to calculate the required amount of space in c in advance. More precisely, if the region is larger than the block-size of c, then a space-increment of `dot(`b`)` $-$ `mark(`b`)` $-$ `size(block(`c`))` is needed.

There is a simple loop for copying the text from the region to the target buffer c. The loop-body leaves the value of the following expression unaffected:

`f(block(`c`),0,counter)` $+$ `f(block(`b`),mover,dot(`b`))`

where the $+$ denotes natural addition of texts. The first summand is the text already copied and the second summand is the text to be copied yet. Furthermore the loop-body only affects attributes of c. The assertion `mark(`b`)` $\leq$ `mover` $\leq$ `dot(`b`)` serves as the loop-invariant. Notice again that some care is needed not to exclude the case $b = c$ for in that case we must invoke `mod_text_restore`.

```
PROC copy_region_to_buffer: 'String' ->
PAR  s:'String'
DEF  LET b:Buf; b := app(table,current);

     ( NOT lss(mark(b),(dot(b)):Nat) ?;
       SKIP
     | lss(mark(b),(dot(b)):Nat)     ?;

       ( is_in_dom(s,table) = true     ?; SKIP
       | NOT is_in_dom(s,table) = true ?; create_buffer(s)
       );
       LET c:Buf; c := app(table,s);

       ( leq(sub(dot(b),mark(b)),size(block(c)))     ?; SKIP
       | NOT leq(sub(dot(b),mark(b)),size(block(b))) ?;
         grow(block(c),sub(sub(dot(b),mark(b)),size(block(c))))
       );
       set_mover((mark(b)):Nat);
       set_counter(0);
       ( NOT mover = (dot(b)):Nat ?;
             store(block(c),counter,cont(block(b),mover));
             set_mover(right(b,mover:Nat));
             set_counter(succ(counter))
       ) *;  mover = (dot(b)):Nat ?;
       set_gap1(c,counter);
       set_gap2(c,size(block(c)));
       set_dot(c,0);
```

```
      set_mark(c,0);

      LET x:Nat # Nat,y:Nat # Nat;
      x,y := cut(reach(b),mark(b),dot(b));
      set_reach(c,y);
      set_dot(c,0,0);
      set_mark(c,0,0);

      ( b = c ?; mod_text_restore | NOT b = c ?; SKIP )

    )
```

Next, we shall proceed with a few operations which are concerned with deleting pieces of text.

```
PROC delete_next_character: ->
DEF  LET b:Buf; b := app(table,current);
     LET i:Nat,j:Nat; i,j := dot(b);

     ( (cont(block(b),dot(b)) = ctr_j OR
        dot(b) = size(block(b))) ?;
       SKIP
     | NOT (cont(block(b),dot(b)) = ctr_j OR
            dot(b) = size(block(b))) ?;
       make_ready(b);
       ( mark(b) = (dot(b)):Nat     ?; set_mark(b,succ(dot(b)))
       | NOT mark(b) = (dot(b)):Nat ?; SKIP
       );
       set_dot(b,succ(dot(b)));
       set_gap2(b,succ(gap2(b)));

       ( leq(mark(b),(dot(b)):Nat)     ?; SKIP
       | NOT leq(mark(b),(dot(b)):Nat) ?;
         LET x:Nat # Nat, y:Nat # Nat;
         x,y := cut(mark(b),i,j,i,succ(j));
         set_mark(b,x)
       );

       LET p:Nat # Nat, q:Nat # Nat;
       p,q := cut(reach(b),i,j,i,succ(j));
       set_reach(b,p);

       mod_text_restore

     )
```

```
PROC erase_region: ->
DEF  LET b:Buf; b := app(table,current);

     ( geq(mark(b),(dot(b)):Nat)     ?;
       SKIP
     | NOT geq(mark(b),(dot(b)):Nat) ?;
       make_ready(b);
       set_gap1(b,mark(b));
       set_mark(b,(dot(b)):Nat);

       LET p:Nat # Nat, q:Nat # Nat;
       p,q := cut(reach(b),mark(b),dot(b));
       set_reach(b,p);
       set_dot(b,(mark(b)):Nat # Nat);

       mod_text_restore

     )

PROC erase_buffer: 'String' ->
PAR  s:'String'
DEF  ( is_in_dom(s,table) = true     ?; SKIP
     | NOT is_in_dom(s,table) = true ?; create_buffer(s)
     );
     LET b:Buf; b := app(table,s);
     set_gap1(b,0);
     set_gap2(b,size(block(b)));
     set_dot(b,gap2(b));
     set_mark(b,gap2(b));

     set_dot(b,0,0);
     set_mark(b,0,0);
     set_reach(b,0,0);

     ( eq(s,current)     ?; mod_text_restore
     | NOT eq(s,current) ?; SKIP
     )
```

Now we turn to the implementation of the operations for marked-text management as specified in Chapter 4 Section 4.5.8, (with an exception for `switch_to_buffer` which is dealt with already before).

```
PROC current_buffer_name: -> 'String'
DEF  current
```

For the procedure `write_named_file` we employ a loop and the mover serves as a kind of loop-counter. The loop-body leaves the value of following expression unaffected:

```
file(f(s)) + f(b,mover,size(block(b)))
```

and also the fact that mover is not within the gap. The loop-body does not affect other files than the file indicated by s.

```
PROC write_named_file: 'String' ->
PAR  s:'String'
DEF  LET b:Buf; b := app(table,current);
     rewrite(s);
     set_mover((gap1(b) = 0 ?; gap2(b) | NOT gap1(b) = 0 ?; 0));

     ( NOT mover = size(block(b)) ?;
           write(cont(block(b),mover));
           set_mover(right(b,mover:Nat))
     ) *;  mover = size(block(b)) ?
```

Now we turn to the implementation of the operations for searching as specified in Chapter 4 Section 4.5.9. In the implementation of `search_forward`, we employ a simple loop. We need an auxiliary predicate `match`. We write {C}OR for *conditional* OR and {C}AND for *conditional* AND as a hint related to the executability of this recursive predicate.

```
PRED match: Buf # 'String' # Nat
PAR  b:Buf,s:'String',i:Nat
DEF  eq(s,empty) {C}OR
     ( lss(i,size(block(b)))
       {C}AND hd(s) = cont(block(b),i)
       {C}AND match(b,tl(s),right(b,i)) )
```

The intuition behind this predicate `match` is that match(b, s, i) holds if the text represented by the len(s) non-gap positions in b starting at i equals the text represented by s. Of course this definition of `match` corresponds only with the intuition if i is not in the gap of b. To put it more formally:
$i \notin$ gap(b) $\Rightarrow$ match(b, s, i) $\Leftrightarrow$ match(f(b), f(s), len(f(b, i))) where the second `match` is the one from the specification (Chapter 4).

```
PRED match': Buf # 'String' # Nat
PAR  b:Buf,s:'String',i:Nat
DEF  i = size(block(b)) {C}OR match(b,s,i)
```

For searching a string s in a buffer b, the main assertion of the loop-invariant is

$$\forall i\texttt{:Nat}\ (\ \texttt{dot}(b) \leq i < \texttt{mover} \wedge i \notin \texttt{gap}(b) \Rightarrow \texttt{NOT match'}(b,s,i)\ )$$

where we used some obvious shorthand. Furthermore we make sure that the pair-wise version of mover precisely follows mover:Nat. This is necessary for in case of a successful search the pair-wise dot attribute must be updated. We have the initialisation mover $=$ dot(b) and the termination condition match'(b,s,mover).

```
PROC search_forward: 'String' ->
PAR  s:'String'
DEF  LET b:Buf; b := app(table,current);
     set_mover((dot(b)):Nat # Nat);
     set_mover((dot(b)):Nat);

     ( NOT match'(b,s,mover) ?;
            set_mover(right(b,mover:Nat # Nat));
            set_mover(right(b,mover:Nat))
     ) *;   match'(b,s,mover) ?;

     ( geq(mover,size(block(b)))     ?;
       SKIP
     | NOT geq(mover,size(block(b))) ?;
       set_dot(b,mover:Nat # Nat);
       set_dot(b,mover:Nat);

       mod_dot_restore
     )
```

where we used a simple auxiliary function right. This function right depends on mover: Nat.

```
FUNC right: Buf # Nat # Nat -> Nat # Nat
PAR  b:Buf,i:Nat,j:Nat
DEF  ( cont(block(b),mover) = ctr_j     ?; (succ(i),0)
     | NOT cont(block(b),mover) = ctr_j ?; (i,succ(j))
     )
```

We expect that improvements could be made with respect to the efficiency of this search-algorithm. Especially the use of the Knuth-Morris-Pratt search algorithm [5] could be considered.

Finally we turn to the implementation of the operations for string conversion as specified in Chapter 4 Section 4.5.10. We use recursion which is acceptable if the procedure below is only used for obtaining relatively short strings such as words to be searched for or such as file names.

```
  PROC buffer_to_string: 'String' -> 'String'
  PAR  s:'String'
  DEF  LET b:Buf; b := app(table,s);
       buffer_to_string(b,  ( gap1(b) = 0     ?; gap2(b)
                            | NOT gap1(b) = 0 ?; 0
                            ) )

  FUNC buffer_to_string: Buf # Nat -> 'String'
  PAR  b:Buf,i:Nat
  DEF  ( i = size(block(b))     ?;
         empty
       | NOT i = size(block(b)) ?;
         cons(cont(block(b),i),buffer_to_string(b,right(b,i)))
       )

END;
```

5.5.15 Termination

In the specification (Chapter 4) there is a termination axiom for `init`, requiring

$$\texttt{INIT} \Rightarrow \forall s:\texttt{'String'}\ (\ <\ \texttt{init}(s)\ >\ \texttt{TRUE}\ )$$

and furthermore there is one large axiom dealing with the termination of all operations from the `witefa_op` group.

```
    WTI ⇒ (
        < FLUSH bolp > TRUE;
        < FLUSH eolp > TRUE;
        < FLUSH forward_character > TRUE;
        < FLUSH backward_character > TRUE;
        etc.
```

We did not mention the satisfaction of our implementation with respect to these axioms yet and we shall by way of example discuss this for `init` now. We must unfold the definition of `init`(s) as presented in Section 5.5.14. This yields

```
    upd_table(new);
    create_buffer(s);
    upd_current(s);
    init_wi_package;
    mod_text_restore
```

To show that this sequential composition can terminate, we must consecutively verify the termination of each of its sub-expressions separately. We begin with the first sub-expression which is upd_table(new). We unfold the specification of new as given in Section 5.3.4.

```
    t := create;
    set_map(t,empty);
    t
```

The first step to be verified is t := create which follows from the assertion

```
    {INST2} < create > TRUE
```

given in Section 5.3.2. The next sub-expression is set_map(t,empty) which we must analyse, starting with t and empty. This t is defined because it is the result of create whereas empty is a *defined* function from the class description MAP_SPEC which was specified algebraically. So set_map is invoked with defined arguments and hence its termination is a consequence (noting that set_map is a renamed version of set_attr) of the assertion

```
    {ATTR1} < set_attr(i,v) > TRUE
```

given in Section 5.3.3 where we take t for i and empty for v. This shows that the evaluation of new can terminate. Also upd_table(new) can terminate; to see this, notice that upd_table is a renamed version of upd, so we can use the axiom

```
    < upd(i) > TRUE
```

of Section 5.5.8. In this way we can proceed until we have checked each sub-expression of the sequential composition of init. We do not present the rest here.

Similar verification steps as indicated above for init apply in fact to all WITEFA operations. Some of these operations have been programmed with repetition constructs and in these cases the classical technique of variant functions applies. We do not present these verification steps in this chapter.

This concludes WITEFA_IMPL whence we can implement the last system component WITEFA – just as we did earlier with KEYBIND and MOREDOP.

```
% COMP WITEFA : WITEFA_SPEC := WITEFA_IMPL;
% this is to replace an earlier primitive component.
```

At this point of the development, d_{editor} equals (apart from lots of LET-constructs) the design sketched below.

```
DESIGN

COMP BOOL    : BOOL_SPEC;
COMP NAT     : NAT_SPEC;
COMP CHAR    : CHAR_SPEC;
COMP INST    : INST_SPEC;
COMP ATTR    : ATTR_SPEC;
COMP 'SEQ'   : 'SEQ_SPEC';
COMP TABLE   : TABLE_SPEC;
COMP BLOCK   : BLOCK_SPEC;
COMP DISPLAY : DISPLAY_SPEC;
COMP FILE    : FILE_SPEC;    % from d_b

COMP ATTR2      : ATTR2_SPEC;
COMP SVAR       : SVAR_SPEC;
COMP 'STRING'   : 'STRING_SPEC';
COMP WI_PACKAGE : WI_PACKAGE_SPEC; % newly postulated

COMP WITEFA  : WITEFA_SPEC   := WITEFA_IMPL;
COMP MOREDOP : MOREDOP_SPEC  := MOREDOP_IMPL;
COMP KEYBIND : KEYBIND_SPEC  := KEYBIND_IMPL

SYSTEM WITEFA,MOREDOP,KEYBIND
```

This is a typical point of the top-down development: all system components have been implemented, but this was only achieved by introducing a number of newly postulated primitive components, to wit: ATTR2, SVAR, 'STRING' and WI_PACKAGE. Therefore the development process is by far not finished yet: we have to embark on the implementation of these primitive components now. It is characteristic for the top-down approach that no implementation effort has been spent on un-used components: each of the three implemented components was actually part of the system and hence was formally part of the *visible external interface* of the design.

5.6 Implementing the Internal Components

During the implementation of the system components, several new components have been postulated: ATTR2, SVAR, 'STRING' and WI_PACKAGE. We call them *internal* components, in view of the fact that they are not part of the system of the design d_{editor}. In this section the top-down development process is continued; the internal components are implemented which leads again to the introduction of new components etc. This goes on until only the primitives of Section 5.3 are left.

5.6.1 Implementing WI_PACKAGE

We turn our attention to mod_text_restore and mod_dot_restore, but it will take about 10 pages of preparations before arriving at the actual definition of these procedures. We adopt the following two-phase approach for the execution of mod_text_restore and mod_dot_restore.

- First phase: decide if a new cursor or a new screen contents is needed. If so, then derive from the buffer named current the desired cursor value (let us call this the *concept cursor*) and store the desired screen contents in a two-dimensional array variable (the *concept screen*).
- Second phase: if necessary, transfer the contents of the concept screen to the actual screen of the physical display device and make the actual cursor equal to the concept cursor.

The main reason for adopting this two-phase approach is that it gives rise to a very desirable separation of concerns. Indeed, the task to be performed by mod_text_restore is rather complex: first of all there is a non-trivial structure-clash between the representation of the text in the buffer named current and the two-dimensional layout of the text to be visualised on the screen. Secondly the text must be subject to applications of the look, fill and printify operations, as prescribed by WI. Also mod_text_restore has to establish suitable values for cursor and origin. In addition to this we assume that we are faced with a physical display device with limited capabilities and whose communication link is a potential bottleneck in the execution speed of the editor. Although the screen has a two-dimensional layout, we assume that the transfer of text towards the screen is done by so-called *serial* communication. This amounts to the restriction that all text transfer goes by means of commands such as ce (clear to end-of-line), cm (cursor motion) and print (send one single character). Avoiding gross inefficiencies in the display-handling is a major complication of the task of mod_text_restore.

We must pay a price for this solution of achieving a separation of concerns. The price is the time to be spent on copying characters from the buffer named `current` into the two-dimensional array variable (the concept screen). Immediately after such copying has taken place, one or more of these characters is transfered again from the concept screen towards the display and clearly it would be a short-cut to take the characters directly from the buffer.

These considerations heavily depend on the nature of the computing machinery available. When there is enough computing power available whereas the connection with the display is a classical and relatively slow serial link we need not worry about the price of filling an array. This could be the case when a VT102 is connected to a VAX, say. But when the computer power is very restricted whereas the display is easily accessible, e.g. because it is directly built into the computer hardware, then we better adopt another solution. This might be the case when using an APPLE-II, say. Since the available computing machinery comes from a rapidly evolving technology, it is almost impossible to give a timeless solution.

The relevant information is passed from the first phase to the second phase by two variables. We present them in informal notation first.

- `ccursor`: $\rightarrow$ `Nat`^{2} `VAR`
- `cscreen`: $\rightarrow$ array $[0..\texttt{li}-1, 0..\texttt{co}-1]$ of `Char` `VAR`

Before proceeding with `WI_PACKAGE`, we insert an intermezzo which is about two-dimensional arrays.

5.6.2 Specifying Two-dimensional Arrays

To prepare for the updating of the screen, it is convenient to use a two-dimensional array. In this section we specify two-dimensional arrays. To keep things simple, indexing in an array always starts at zero.

```
LET DOMAIN1 :=
EXPORT
  SORT Nat,
  FUNC size1: -> Nat
FROM
IMPORT NAT_SPEC INTO
CLASS
  FUNC size1: -> Nat
  AXIOM size1!
END;
```

```
LET DOMAIN2 :=
EXPORT
  SORT Nat,
  FUNC size2: -> Nat
FROM
IMPORT NAT_SPEC INTO
CLASS
  FUNC size2: -> Nat
  AXIOM size2!
END;

LET ARRAY2_SPEC :=
LAMBDA X: DOMAIN1 OF
LAMBDA Y: DOMAIN2 OF
LAMBDA Z: ITEM    OF

EXPORT
  SORT Nat,
  SORT Item,
  SORT Array2,
  FUNC size1 :                            -> Nat,
  FUNC size2 :                            -> Nat,
  PROC create:                            -> Array2,
  FUNC val   : Array2 # Nat # Nat         -> Item,
  PROC upd   : Array2 # Nat # Nat # Item ->
FROM

IMPORT X        INTO
IMPORT Y        INTO
IMPORT Z        INTO
IMPORT NAT_SPEC INTO

CLASS

  SORT Array2 VAR
  PROC create: -> Array2  MOD Array2

  AXIOM
  {ARRAY1} < create > TRUE;
  {ARRAY2} INIT => NOT EXISTS  a:Array2 ();
  {ARRAY3} [ LET a:Array2; a := create ]
           a! AND (PREV NOT a!) AND
           FORALL b:Array2 ( (PREV NOT b!) => b = a )

  FUNC val: Array2 # Nat # Nat-> Item     VAR
  PROC upd: Array2 # Nat # Nat # Item ->  MOD val

  AXIOM     FORALL a:Array2,m:Nat,n:Nat,i:Item
            ( lss(m,size1) AND lss(n,size2) => (
  {ARRAY4} < upd(a,m,n,i) > TRUE;
  {ARRAY5} [ upd(a,m,n,i) ]
            ( val(a,m,n) = i;
              FORALL b:Array2,k:Nat,l:Nat,j:Item
              ( lss(k,size1) AND lss(l,size2) =>
                ( NOT k = m OR NOT l = n OR NOT a = b =>
                  val(b,k,l) = j <=> PREV val(b,k,l) = j) ) ) ) )

END;
```

5.6.3 ARRAY2: a Postulated Component

We postulate the following component:

```
{@}  COMP ARRAY2: ARRAY2_SPEC;
```

5.6.4 Implementing WI_PACKAGE (continued)

After this intermezzo we can introduce the necessary variables.

```
LET CCURSOR :=
IMPORT APPLY RENAME
        SORT Item         TO Nat,
        FUNC val: -> Item TO ccursor1,
        PROC upd: Item -> TO set_ccursor1
IN COPY(SVAR) TO NAT INTO

IMPORT APPLY RENAME
        SORT Item         TO Nat,
        FUNC val: -> Item TO ccursor2,
        PROC upd: Item -> TO set_ccursor2
IN COPY(SVAR) TO NAT INTO

CLASS

  FUNC ccursor: -> Nat # Nat
  DEF  (ccursor1,ccursor2)

  PROC set_ccursor: Nat # Nat ->
  PAR  i:Nat,j:Nat
  DEF  set_ccursor1(i);
       set_ccursor2(j)

END;
```

We introduce a class description called CSCREEN which at its turn contains a local definition CS_ARRAY which provides us with arrays of dimensions li and co. It is interesting to have a look at the application expression below where a renamed version of ARRAY2 is applied to its three actual parameters; these are DISPLAY, DISPLAY (again) and CHAR. Let us check that this application expression is correct: the renamed version of ARRAY2 has three LAMBDA parameters, each with its associated *parameter restriction*. The first parameter restriction requires the presence of a sort Nat and a defined function li; the second parameter restriction requires again Nat and a defined co whereas the third parameter restriction only mentions a sort Char. Now we verify that indeed, DISPLAY provides both li and co – whence it can appear twice. In fact DISPLAY provides much more than just Nat, li and co, but that does not matter. Finally, of course CHAR provides Char. To put it somewhat more formally: DISPLAY $\sqsubseteq$ 'renamed DOMAINi' $(i = 1,2)$ where $\sqsubseteq$ refers to the formal implementation relation.

```
LET CSCREEN :=

LET CS_ARRAY :=
APPLY APPLY APPLY RENAME
        SORT Item          TO Char,
        FUNC size1: -> Nat TO li,
        FUNC size2: -> Nat TO co
IN ARRAY2 TO DISPLAY TO DISPLAY TO CHAR;

IMPORT CS_ARRAY INTO
APPLY RENAME
        SORT Item         TO Array2,
        FUNC val: -> Item TO cscreen,
        PROC upd: Item -> TO upd_cscreen
IN COPY(SVAR) TO CS_ARRAY;
```

The purpose of using SVAR in connection with ARRAY2 is to provide for precisely one array. This yields a specialisation of the somewhat more general ARRAY2, since ARRAY2 allows for a multitude of arrays, all having the same dimensions. In general, these arrays can be created dynamically, e.g. by LET a:Array2; a := create but because we need only *one* array, we shall use create just once and keep the Array2 object returned by create in the simple programming variable FUNC cscreen: -> Array2. So we have a two-dimensional variable array cscreen and we can index with i,j:Nat by writing val(cscreen,i,j) and an assignment is denoted as upd(cscreen,i,j,c). This notation for indexing should be compared with the conventional cscreen[i,j] or cscreen[i][j] of Pascal and C respectively. The notation for assignment should be compared with the conventional cscreen[i,j] := c of Pascal or cscreen[i][j] = c; of C. Of course the module CSCREEN needs initialisation, which is upd_cscreen(create:Array2) (cf. the code of init_wi_package in Section 5.6.7).

We collect the variables for interfacing the two phases as sketched before in a class description called CONCEPT_VARS. We add to this an abstraction function f: Array2 $\rightarrow$ Text which serves for reasoning purposes. In order to define this abstraction function, we need auxiliaries denoted as f and g; since this g is useful in its own right, we export it as well. In the definition of CONCEPT_VARS below we employ two auxiliary class descriptions CSCREEN_SPEC and ABSTRACT_CSCREEN. The latter class description actually contains the definitions of the functions f and g. The reader easily recognises that a generalisation-specialisation approach is used in connection with ABSTRACT_CSCREEN; the precise reason for this will become clear in the next section. At this point it is sufficient to to notice that in CONCEPT_VARS the generalisation-specialisation approach does no harm, since the associated LAMBDA can be eliminated easily. It can be verified that CSCREEN $\sqsubseteq$ CSCREEN_SPEC.

```
LET CSCREEN_SPEC :=
CLASS

  SORT Nat
  SORT Char
  FUNC li: -> Nat
  FUNC co: -> Nat
  SORT Array2 VAR
  FUNC cscreen: -> Array2 VAR
  PROC create: -> Array2  MOD Array2
  PROC upd_cscreen: Array2 -> MOD cscreen
  FUNC val: Array2 # Nat # Nat -> Char VAR
  PROC upd: Array2 # Nat # Nat # Char ->  MOD val

END;

LET ABSTRACT_CSCREEN :=
LAMBDA CS:CSCREEN_SPEC OF
EXPORT
```

```
   SORT Nat,
   SORT Line,
   SORT Text,
   SORT Array2,
   FUNC f: Array2       -> Text,
   FUNC g: Array2 # Nat -> Line
FROM

IMPORT NAT_SPEC  INTO
IMPORT LINE_SPEC INTO
IMPORT TEXT_SPEC INTO
IMPORT CS        INTO

CLASS

  FUNC g: Array2 # Nat # Nat -> Line
  PAR  a:Array2,n:Nat,i:Nat
  %    the line from characters i..co-1 in the n-th line of a
  DEF  ( i = co     ?; empty
       | NOT i = co ?; cons(val(a,n,i),g(a,n,succ(i)))
       )

  FUNC g: Array2 # Nat -> Line
  PAR  a:Array2,n:Nat
  %    the n-th line in a
  DEF  g(a,n,0)

  FUNC f: Array2 # Nat -> Text
  PAR  a:Array2,n:Nat
  %    the text from lines n..li-1 in a
  DEF  ( n = li     ?; niltext
       | NOT n = li ?; cons(g(a,n),f(a,succ(n)))
       )

  FUNC f: Array2 -> Text
  PAR  a:Array2
  %    the text represented by a
  DEF  f(a,0)

END;

LET CONCEPT_VARS :=
IMPORT CCURSOR INTO
IMPORT CSCREEN INTO
APPLY ABSTRACT_CSCREEN TO CSCREEN;
```

Note that CONCEPT_VARS is an abbreviation at component level, rather than local within WI_PACKAGE_IMPL. Later it will appear that we happen to have an opportunity for re-using CONCEPT_VARS.

Now we turn our attention to the first phase of the WI_PACKAGE procedures. We present some considerations which are still independent from the representation of marked texts. We start by having another look at the window invariant which was defined as a predicate WI and which was specified to be an observational invariant, i.e. it must be a consequence of the classical invariant WTI. We use + and − as shorthand for the *pair-wise* addition and subtraction operations on co-ordinate pairs. Let us write ψ as a shorthand for each of the printify functions. WI is given as:

```
LET origin: Nat²; origin := dot − cursor;
LET filled: Text; filled := fill(text,origin + size);
screen = ψ(look(filled,origin,origin + size))
```

We note that for a given dot and text, there is a certain degree of freedom in choosing cursor and origin. Although the clause origin := dot − cursor suggests that the cursor must be chosen first, and that after that the origin can be calculated, it can also be done the other way around. Actually we prefer the latter approach for our implementation. Therefore we introduce the origin by a quantifier. We rewrite origin = dot − cursor into cursor = dot − origin and in this way we get an alternative but equivalent definition of WI.

```
∃ origin: Nat²
( cursor = dot − origin;
  LET filled: Text; filled := fill(text,origin + size);
  screen = ψ(look(filled,origin,origin + size)) )
```

Our screen-update strategy will be based on the idea of not moving the origin unless this is really necessary. Therefore the editor must have a mechanism for recalling the previous origin value. More precisely, the editor will keep track of the previous origin value for each buffer separately. This is achieved by introducing a variable function origin: Buf → Nat².

If we want to argue that WI holds then we we may do so by showing that the value of this variable origin for the current buffer makes the body of the quantified assertion within WI true. In this way we get another predicate which we shall name WI''.

```
cursor =  dot − origin(b);
LET filled: Text; filled := fill(text,origin(b) + size);
```

$$\texttt{screen} = \psi(\texttt{look(filled,origin(}b\texttt{),origin(}b\texttt{)} + \texttt{size))}$$

where b denotes the current buffer. Clearly `WI''` $\Rightarrow$ `WI`. We also give the precise definition in COLD-K:

```
PRED WI'':
DEF  LET b:Buf; b := app(table,current);
     cursor = p_sub(dot(b),origin(b));
     LET filled: Text; filled := fill(f(b),p_add(origin(b),size));
     screen = printify(look(filled,origin(b),p_add(origin(b),size)))
```

So now `WI` has been transformed into `WI''` which essentially consists of two conditions of the following form:

$$\texttt{cursor} = \texttt{dot(}b\texttt{)} - \texttt{origin(}b\texttt{)} \qquad \text{... c1)}$$
$$\texttt{screen} = \psi(\texttt{look(filled,origin(}b\texttt{),origin(}b\texttt{)} + \texttt{size))} \qquad \text{... c2)}$$

We introduce the notations $<_p$ and $\leq_p$ for pair-wise comparison of co-ordinates with the meaning

$$(o_1, o_2) <_p (s_1, s_2) \;:\Leftrightarrow\; o_1 < s_1 \wedge o_2 < s_2$$
$$(o_1, o_2) \leq_p (s_1, s_2) \;:\Leftrightarrow\; o_1 \leq s_1 \wedge o_2 \leq s_2$$

We consider `WI''` in conjunction with the display invariant which among other things states that `cursor` $<_p$ `size`. If we combine this with `c1`, we obtain `dot(`b`)` $<_p$ `origin(`b`)` + `size`. Furthermore, when the subtraction `dot(`b`)` $-$ `origin(`b`)` is to yield a defined value, we must have `origin(`b`)` $\leq_p$ `dot(`b`)`. In this way we get another condition, to be named `c3`, with the property that `c1` $\Rightarrow$ `c3`. This `c3` depends on `dot(`b`)` and `origin(`b`)`. The important observation regarding this `c3` is that once `c3` holds, it is always possible to make `WI''` hold by 'assignments' to `screen` and `cursor`.

$$\texttt{c3} :\Leftrightarrow \texttt{origin(}b\texttt{)} \leq_p \texttt{dot(}b\texttt{)} <_p \texttt{origin(}b\texttt{)} + \texttt{size}$$

Both `mod_dot_restore` and `mod_text_restore` have to deal with two possible situations: either `c3` holds, which means that the old `origin(`b`)` value is still usable, or `c3` does not hold, which means that a modification of `origin(`b`)` is inevitable.

Now we can give a first attempt in formulating our screen-update strategy. We must get two procedures which serve for establishing `WI''` after a modification of the dot and/or the text. The first procedure serves for establishing `WI''` after a modification of the dot. We expect that updating the screen will move the cursor, so we first adjust the screen and after that try to get the cursor right. The definition below still is far from complete, and an expression such as MOD origin(*b*); c3 ? is meant as shorthand for "modify origin(*b*) such that c3 holds".

```
%   PROC mod_dot_restore: ->
%   DEF  ( c3 ?;
%          MOD cursor         ; cursor = dot(b) - origin(b) {c1} ?;
%        | NOT c3 ?;
%          MOD origin(b); c3 ?;
%          MOD screen, cursor ; screen = printify( ... )    {c2} ?;
%          MOD cursor         ; cursor = dot(b) - origin(b) {c1} ?
%        )
```

So now, after dot has been modified, we can restore WI'' by invoking mod_dot_restore.

The second procedure serves for establishing WI'' after a modification of both the dot and the text or of just the text alone. In general, we expect that both the screen and the cursor must be updated, even in case the old origin is still usable. Note that it is reasonable to expect that the cursor needs updating, for even if it can essentially stay the same, then it probably still will get messed-up by the screen updating.

```
%   PROC mod_text_restore: ->
%   DEF  ( c3 ?;
%          SKIP
%        | NOT c3 ?;
%          MOD origin(b); c3 ?;
%        );
%        MOD screen, cursor; screen = printfy( ... )    {c2} ?;
%        MOD cursor        ; cursor = dot(b) - origin(b) {c1} ?
```

Before we program mod_dot_restore and mod_text_restore, we have an intermezzo for postulating a component dealing with *display handling*, by which we mean updating the cursor and the screen of the physical device for given desired cursor value and desired screen contents.

5.6.5 Specifying Display Handling

We introduce two procedures which take data from CONCEPT_VARS, or more precisely, which take data from the concept cursor and the concept screen; they transfer these data to the physical display device. The procedure update_cursor makes the cursor equal to the concept cursor without affecting the screen. The procedure update_screen makes the screen equal to the text represented by the concept screen, thereby possibly disturbing the cursor. There is also an initialisation procedure and an invariant. The latter is called display_handling_inv.

We would like to add a remark about the module structure of the specification given below. DISPLAY_HANDLING_SPEC is constructed in terms of specification modules only (such as NAT_SPEC, DISPLAY_SPEC, and in fact also CCURSOR_SPEC, CSCREEN_SPEC) and no reference is made to component names (where we mean to those with COMP). To achieve this, we exploit the fact that ABSTRACT_CSCREEN is parameterised. This is the reason for the parameterisation: in the previous section ABSTRACT_CSCREEN was applied to CSCREEN whereas here it is applied to CSCREEN_SPEC. In the next section DISPLAY_HANDLING_SPEC will be used as a black-box description; the fact that it is constructed in terms of specification modules only, guarantees that the condition 'directly specified' of Chapter 3 is satisfied.

```
LET CCURSOR_SPEC :=
CLASS

  SORT Nat
  FUNC ccursor: -> Nat # Nat VAR
  PROC set_ccursor: Nat # Nat -> MOD ccursor

END;

LET CONCEPT_VARS_SPEC :=
IMPORT CCURSOR_SPEC INTO
IMPORT CSCREEN_SPEC INTO
APPLY ABSTRACT_CSCREEN TO CSCREEN_SPEC;

LET DISPLAY_HANDLING_SPEC :=
IMPORT NAT_SPEC INTO
IMPORT DISPLAY_SPEC INTO
IMPORT CONCEPT_VARS_SPEC INTO
CLASS

  PRED display_handling_inv: VAR

  PROC init_display_handling: ->
  MOD  display_handling_inv

  PROC update_cursor: ->
  MOD  display_handling_inv USE displ_op

  PROC update_screen: ->
  MOD  display_handling_inv USE displ_op
  PRED pre:
  DEF  LET i:Nat, j:Nat; i,j := ccursor;
       lss(i,li) AND lss(j,co) AND f(cscreen)!

  AXIOM [ init_display_handling ] display_handling_inv

  AXIOM display_handling_inv AND pre  =>
        [ update_cursor | update_screen ] display_handling_inv

  AXIOM display_handling_inv AND pre =>
        [ update_cursor ] cursor = ccursor AND screen = PREV screen

  AXIOM display_handling_inv AND pre =>
        [ update_screen ] screen = f(cscreen)

  AXIOM {TERMINATION}

        < init_display_handling > TRUE;
        display_handling_inv AND pre => < update_cursor > TRUE;
        display_handling_inv AND pre => < update_screen > TRUE

END;
```

One might be tempted to think that display_handling_inv needs not to be in the modification lists of update_cursor and of update_screen. For these procedures are supposed not to violate display_handling_inv and this is guaranteed indeed if display_handling_inv cannot be modified. However, this would be too strong, because it forbids an implementation of the procedures to change display_handling_inv *from false to true.*

5.6.6 DISPLAY_HANDLING: a Postulated Component

After having specified display handling, we formally introduce a component for it.

```
{@}  COMP DISPLAY_HANDLING : DISPLAY_HANDLING_SPEC;
```

5.6.7 Implementing WI_PACKAGE (continued)

As argued before, our screen-update strategy requires a variable function origin: Buf $\rightarrow$ Nat^2. We formally introduce this below.

```
LET ORIGIN :=
  APPLY APPLY APPLY RENAME
        SORT Inst                                              TO Buf,
        SORT Item1                                             TO Nat,
        SORT Item2                                             TO Nat,
        FUNC attr    : Inst -> Item1 # Item2 TO origin,
        PROC set_attr: Inst # Item1 # Item2 -> TO set_origin
  IN COPY(ATTR2) TO BUF_INST TO NAT TO NAT;
```

We shall need two simple programming variables of sort Nat. They will be used as loop-counters for processing two-dimensional arrays.

```
LET XX :=
APPLY RENAME
        SORT Item         TO Nat,
        FUNC val: -> Item TO xx,
        PROC upd: Item -> TO upd_xx
IN COPY(SVAR) TO NAT;

LET YY :=
APPLY RENAME
        SORT Item         TO Nat,
        FUNC val: -> Item TO yy,
        PROC upd: Item -> TO upd_yy
IN COPY(SVAR) TO NAT;
```

After these preparations we can present the import structure of the implementation of WI_PACKAGE.

We use TEXT_VARS of Section 5.5.11, DISPLAY_HANDLING of Section 5.6.6, ORIGIN of Section 5.6.7, XX, YY defined above and finally CONCEPT_VARS of Section 5.6.4.

```
LET WI_PACKAGE_IMPL :=

IMPORT TEXT_VARS        INTO
IMPORT DISPLAY_HANDLING INTO
IMPORT ORIGIN           INTO
IMPORT XX               INTO
IMPORT YY               INTO
IMPORT CONCEPT_VARS     INTO

CLASS
```

At some point in the execution of mod_text_restore and mod_dot_restore it will be necessary to build-up the contents of the concept-screen. Clearly this contents must be derived from the buffer named current. We shall develop a procedure cscreen_build taking care for this. This procedure can be viewed as the heart of a display-oriented editor, because it must transform the 'block-with-positions' representation of a text into the desired two-dimensional, fixed-size screen-contents. Furthermore it may not be too inefficient because it will be invoked after every insert- or delete-command during execution of an editor. It will take a few pages of work before we shall have completed a detailed definition of this procedure.

We plan to make this cscreen_build such that it modifies cscreen,xx,yy, mover, but nothing else and such that

```
cscreen! ∧ c3 ∧ TI' ⇒
[ cscreen_build ]
  f(cscreen)
  = ψ(look(fill(t,origin(b)+size),origin(b),origin(b)+size))
   where t = f(b,size(block(b)))
```

and where b denotes app(table(current)). We shall use a loop for copying the relevant characters from the current buffer one by one into the cscreen array. We get a loop-invariant replacing the sub-term size(block(b)) in the postcondition given above by mover:Nat. So this loop-invariant says that the concept-screen contains printified and 'Procrustesed' version of an initial fragment of the text in the current buffer. Furthermore we add the assertion that mover:Nat^2 corresponds to mover:Nat to the loop-invariant.

We could initially establish this loop-invariant by making f(cscreen) equal to a text with blanks only and putting mover $=0$. We adopt the idea of filling the cscreen array with blanks indeed, but we can improve with respect to idea of putting mover $=$ 0. The point is that everything *before* the origin becomes irrelevant due to the application of the look operator. Therefore we make the mover point initially to the position in the buffer which corresponds with the beginning of the line containing the origin. The initialisation for establishing this loop-invariant uses two procedures:

- clear_cscreen which puts blanks at all positions of cscreen,
- mover_to_origin_line which puts the mover to the beginning of the line containing the origin.

Let us also discuss the termination condition for the loop of cscreen_build already. Obviously the condition mover $\geq$ size(block(b)) would do the job, but again it is easy to do better than that. The point is that everything *after* origin(b) + size becomes irrelevant due to the application of the look operator. Therefore we adopt the following termination condition

mover $\geq$ size(block(b)) $\vee$
vertical co-ordinate of mover $\geq$ vertical co-ordinate of (origin(b) + size)

where b denotes app(table(current)). The procedure clear_cscreen serves for putting blanks at all positions of cscreen. It uses xx and yy as loop-counters.

```
PROC clear_cscreen: ->
DEF  upd_xx(0);
     ( NOT xx = li ?;
          upd_yy(0);
          ( NOT yy = co ?;
               upd(cscreen,xx,yy,blank);
               upd_yy(succ(yy))
          ) *;  yy = co ?;
          upd_xx(succ(xx))
     ) *;  xx = li ?
```

The procedure mover_to_origin_line should make the mover move towards the beginning of the line containing the origin. We use an auxiliary v_eq to be used in formulating the stop-criterion of the loop of mover_to_origin_line:

```
PRED v_eq: Nat # Nat # Nat # Nat
PAR  m1:Nat,m2:Nat,o1:Nat,o2:Nat
DEF  m1 = o1
```

Now we program mover_to_origin_line and we begin with an efficiency consideration. We exploit that, for large texts, the origin is likely to be much closer to the position of the dot than to the begin of the text. Note that the dot, due to our invariant WI'' (implying c3), never precedes the origin. Therefore we always have to carry out a backward scan, starting from the dot. For this scan we shall employ several 'assignments', where set_mover(left(b,mover)) changes mover:Nat whereas set_mover1(pred(mover1)) affects mover:Nat # Nat. Note that we have mover1:Nat which is the same as the vertical coordinate of mover:Nat # Nat. For the introduction of these movers we refer to TEXT_VARS in Section 5.5.11.

For mover_to_origin_line we employ a loop, whose invariant includes the assertion that the vertical co-ordinate of mover:Nat^2 (i.e. mover1) corresponds to mover:Nat. Formally this is mover1 $=$ m_1 where (m_1, m_2) $=$ reach(f(b,mover:Nat)) and where b is the relevant buffer. The horizontal co-ordinate mover2 is irrelevant during execution of the loop and afterwards it will be made equal to 0. Furthermore the assertion that the mover does not precede the origin is part of the invariant as well. More precisely, we have the assertion $o_1 \leq m_1$ where (o_1, o_2) = `origin(`b`)` and where (m_1, m_2) = `reach(f(`b`,mover:Nat))`. When the beginning of the line containing the origin has not been reached yet, it is easy to maintain the invariant by letting `mover:Nat` go one position leftwards and adapting `mover1` correspondingly. For adapting `mover1`, two situations arise: either the leftward step crossed a line-boundary line, in which case `mover1` must be decremented, or the leftward step took place within the same line, in which case `mover1` must keep its value.

```
PROC mover_to_origin_line: Buf ->
PAR  b:Buf
DEF  set_mover((dot(b)):Nat);
     set_mover((dot(b)):Nat # Nat);
     ( NOT ( v_eq(mover,origin(b)) AND bolp(b,mover) )?;
           set_mover(left(b,mover));
           ( eolp(b,mover)     ?; set_mover1(pred(mover1))
           | NOT eolp(b,mover) ?; SKIP
           )
     ) *;  ( v_eq(mover,origin(b)) AND bolp(b,mover) )?;
     set_mover2(0)
```

We introduce some more simple auxiliaries for comparing co-ordinate pairs.

```
PRED v_geq: Nat # Nat # Nat # Nat
PAR  m1:Nat,m2:Nat,o1:Nat,o2:Nat
DEF  geq(m1,o1)

PRED h_geq: Nat # Nat # Nat # Nat
PAR  m1:Nat,m2:Nat,o1:Nat,o2:Nat
DEF  geq(m2,o2)

PRED h_lss: Nat # Nat # Nat # Nat
PAR  m1:Nat,m2:Nat,o1:Nat,o2:Nat
DEF  lss(m2,o2)
```

To keep the actual definition of `cscreen_build` readable, we need two more auxiliaries. The first is a predicate called `built` and it describes the termination condition as discussed before.

```
PRED built: Buf
PAR  b:Buf
DEF  geq(mover,size(block(b))) OR v_geq(mover,p_add(origin(b),size))

FUNC size: -> Nat # Nat
```

```
DEF  li,co
```

The predicate `in_window` serves for finding out if the mover is at a position which contributes to the term
ψ(`look(fill(t,origin(`b`) + size),origin(`b`),origin(`b`) + size))` from the postcondition of `cscreen_build`. In fact `in_window` only tests the horizontal co-ordinate of `mover`, which is safe if we know that the vertical co-ordinate of mover is within the limits given by `origin(`b`)` and `origin(`b`)` + `size`. For this reason we must formally add one more conjunct to the loop-invariant of `cscreen_build`, viz.

vertical co-ordinate of `origin(`b`)` $\leq$ vertical co-ordinate of `mover`

We easily define `in_window` using `h_geq` and `h_lss`.

```
PRED in_window: Buf
PAR  b:Buf
DEF  h_geq(mover,origin(b)) AND h_lss(mover,p_add(origin(b),size))
```

This concludes the preparations for `cscreen_build`. Its outline is

```
 "initialisation"
 ( NOT built(b) ?;
       ( NOT in_window(b) ?; SKIP
       | in_window(b)     ?; "copy char unless it is ctr_j"
       );
       "increment mover"
 ) *;  built(b) ?
```

Finally we fill all the details.

```
PROC cscreen_build: Buf ->
PAR  b:Buf
DEF  clear_cscreen;
     mover_to_origin_line(b);
     ( NOT built(b) ?;
          ( NOT in_window(b) ?; SKIP
          | in_window(b)     ?;
            LET c:Char; c := cont(block(b),mover);
            ( c = ctr_j     ?;
              SKIP
            | NOT c = ctr_j ?;
              upd(cscreen,p_sub(mover,origin(b)),printify(c))
            )
          );
```

```
            ( cont(block(b),mover) = ctr_j     ?;
              set_mover1(succ(mover1));
              set_mover2(0)
            | NOT cont(block(b),mover) = ctr_j ?;
              set_mover2(succ(mover2))
            );
            set_mover(right(b,mover:Nat))
      ) *;  built(b) ?
```

We introduce predicates p_lss and p_leq for the pair-wise comparison of co-ordinates ($<_p$ and $\leq_p$) and the operations p_add and p_sub for pair-wise addition and subtraction. We also introduce the condition c3 as discussed before in Section 5.6.4.

```
PRED p_lss: Nat # Nat # Nat # Nat
PAR  o1:Nat, o2:Nat, s1:Nat, s2:Nat
DEF  lss(o1,s1) AND lss(o2,s2)

PRED p_leq: Nat # Nat # Nat # Nat
PAR  o1:Nat, o2:Nat, s1:Nat, s2:Nat
DEF  leq(o1,s1) AND leq(o2,s2)

FUNC p_sub: Nat # Nat # Nat # Nat -> Nat # Nat
PAR  d1:Nat, d2:Nat, o1:Nat, o2:Nat
DEF  sub(d1,o1), sub(d2,o2)

FUNC p_add: Nat # Nat # Nat # Nat -> Nat # Nat
PAR  o1:Nat, o2:Nat, s1:Nat, s2:Nat
DEF  add(o1,s1), add(o2,s2)

PRED c3: Buf
PAR  b:Buf
DEF  p_leq(origin(b),dot(b)) AND
     p_lss(dot(b),p_add(origin(b),size))
```

The following detailed version of mod_dot_restore uses the old origin value, if possible. If this is not possible, then it applies the following strategy both for the vertical and horizontal co-ordinate of the origin: *first try zero*, but if zero is not acceptable, then choose the origin co-ordinate such that the cursor gets centered on the screen. In order to implement this idea of centering we need two auxiliaries:

```
FUNC half_li: -> Nat
DEF  div(li,2)
```

```
FUNC half_co: -> Nat
DEF  div(co,2)

PROC mod_dot_restore: ->
DEF  LET b:Buf; b := app(table,current);
     ( c3(b)     ?;
       SKIP
     | NOT c3(b) ?;
       LET d1:Nat,d2:Nat; d1,d2 := dot(b);
       set_origin(b,( lss(d1,li) ?; 0
                    | NOT lss(d1,li) ?; sub(d1,half_li)
                    ),
                    ( lss(d2,co) ?; 0
                    | NOT lss(d2,co) ?; sub(d2,half_co)
                    ) );
       cscreen_build(b);
       update_screen
     );
     set_ccursor(p_sub(dot(b),origin(b)));
     update_cursor
```

This mod_dot_restore has the property to restore WI in the sense that TI' $\wedge$ wi_package_inv $\Rightarrow$ WI $\Rightarrow$ [mod_dot] [mod_dot_restore] WI. Just for completeness we also show what it means to 'modify dot' at the *representation level*.

```
PROC mod_dot: ->
DEF  LET t:Text;
     t := f(app(table,current));
     USE set_dot: Buf # Nat ->, set_dot: Buf # Nat # Nat ->,
         set_mover: Nat ->, set_mover: Nat # Nat ->,
         set_gap1, set_gap2
     END;
     f(app(table,current)) = t AND TI' ?
```

Along the same lines we obtain a detailed version of mod_text_restore.

```
PROC mod_text_restore: ->
DEF  LET b:Buf; b := app(table,current);
     ( c3(b)     ?;
       SKIP
     | NOT c3(b) ?;
       LET d1:Nat,d2:Nat; d1,d2 := dot(b);
       set_origin(b,( lss(d1,li) ?; 0
                    | NOT lss(d1,li) ?; sub(d1,half_li)
                    ),
                    ( lss(d2,co) ?; 0
                    | NOT lss(d2,co) ?; sub(d2,half_co)
                    ) )
        );
        cscreen_build(b);
        update_screen;
        set_ccursor(p_sub(dot(b),origin(b)));
        update_cursor
```

To complete WI_PACKAGE_IMPL, we must give the definitions of wi_package_inv and the initialisation procedure init_wi_package, and repeat the definitions of WI and WI'.

```
  PRED wi_package_inv:
  DEF  cscreen! AND display_handling_inv

  PROC init_wi_package: ->
  DEF  upd_cscreen(create:Array2);
       init_display_handling

  PRED WI : % as in Section 4.5.12
  PRED WI': % as in Section 5.5.12

END;

% COMP WI_PACKAGE : WI_PACKAGE_SPEC := WI_PACKAGE_IMPL;
% this is to replace an earlier primitive component.
```

5.6.8 Implementing DISPLAY_HANDLING

Now it is time to develop algorithms for the procedures update_cursor and update_screen. Updating the cursor poses no problem, but updating the screen is not trivial at all. The development of update_screen will be guided by efficiency considerations.

We assume that the sending of command-messages to the display is relatively slow and that it is a possible bottleneck in the execution speed of the editor. Of course this assumption depends on the precise nature of the communication mechanism between the computer which executes the editor program and the display device. If we consider a VT102 type terminal and a 1200 Baud serial communication link our assumption is quite right.

The simplest algorithm, of rewriting the entire screen every time when update_screen is called, is considered prohibitively expensive in terms of the number of display command-messages issued. Therefore it seems worthwhile trying to economize on the use of these messages. The key observation is that during a typical edit session, most keystroke-command invocations require in fact only a modification of a *small* fragment of the contents of the screen. In such situations the editor only needs to direct the cursor towards the place where the modifications must take place and then overwrite the old screen contents there.

In order to do so, the editor needs keep track of the contents of the actual screen and of the actual cursor position. For this purpose we introduce a so-called *shadow-administration* of the screen and the cursor. This administration takes the shape of a two-dimensional array of characters for the shadow-screen and a programming variable of type $\texttt{Nat}^2$ for the shadow-cursor. In view of the similarity with the variables of CONCEPT_VARS, it seems appropriate to do a re-use at textual level.

```
LET SHADOW_VARS :=
RENAME
  FUNC ccursor1     :             -> Nat       TO scursor1,
  FUNC ccursor2     :             -> Nat       TO scursor2,
  FUNC ccursor      :             -> Nat # Nat TO scursor,
  PROC set_ccursor1: Nat          ->           TO set_scursor1,
  PROC set_ccursor2: Nat          ->           TO set_scursor2,
  PROC set_ccursor : Nat # Nat ->              TO set_scursor,
  FUNC cscreen      :             -> Array2    TO sscreen,
  PROC upd_cscreen : Array2       ->           TO upd_sscreen
IN COPY(CONCEPT_VARS);
```

We shall need two simple programming variables of sort Nat. They will be used as loop-counters for processing two-dimensional arrays.

```
LET II :=
APPLY RENAME
        SORT Item         TO Nat,
        FUNC val: -> Item TO ii,
        PROC upd: Item -> TO upd_ii
IN COPY(SVAR) TO NAT;

LET JJ :=
APPLY RENAME
        SORT Item         TO Nat,
        FUNC val: -> Item TO jj,
        PROC upd: Item -> TO upd_jj
IN COPY(SVAR) TO NAT;
```

The definition of display_handling_inv is nothing but a formalisation of the idea of maintaining a shadow-administration. We also immediately give the corresponding initialisation procedure init_display_handling for establishing this invariant. The initialisation uses a procedure clear_sscreen which is invoked by init_display_handling. This procedure clear_sscreen serves for putting blanks at all positions of sscreen. It uses ii and jj as loop-counters and it is very similar to the procedure clear_cscreen as introduced in Section 5.6.7. In init_display_handling we use the procedure cl from DISPLAY to clear the entire screen.

```
LET DISPLAY_HANDLING_IMPL :=

IMPORT NAT            INTO
IMPORT CHAR           INTO
IMPORT DISPLAY        INTO
IMPORT TEXT_OPS2_SPEC INTO
IMPORT CONCEPT_VARS   INTO
IMPORT SHADOW_VARS    INTO
IMPORT II             INTO
IMPORT JJ             INTO

CLASS

  PRED pre: % as in Section 5.6.5

  PRED display_handling_inv:
  DEF  cursor = scursor AND screen = f(sscreen)

  PROC init_display_handling: ->
  DEF  cl;
       upd_sscreen(create:Array2);
       clear_sscreen;
       set_scursor(0,0)

  PROC clear_sscreen: ->
  DEF  upd_ii(0);
       ( NOT ii = li ?;
             upd_jj(0);
             ( NOT jj = co ?;
                   upd(sscreen,ii,jj,blank);
                   upd_jj(succ(jj))
             ) *;  jj = co ?;
             upd_ii(succ(ii))
       ) *;  ii = li ?
```

The procedure update_cursor is easy. We use the procedure cm from DISPLAY for moving the cursor.

```
PROC update_cursor: ->
DEF  ( scursor = ccursor     ?; SKIP
     | NOT scursor = ccursor ?;
       cm(ccursor);
       set_scursor(ccursor)
     )
```

The procedure update_screen is more complicated. The complication comes from the fact that we want to avoid rewriting the entire screen when certain parts of it are in fact still right. We decompose this procedure in a kind of top-down approach (at procedure level rather than at component level). The procedure update_screen contains a loop with ii serving as a loop-counter. Its invariant is given by

$$\texttt{display\_handling\_inv} \wedge \texttt{ii} \leq \texttt{li} \wedge$$
$$\forall\, i\texttt{:Nat} \; (\; i < \texttt{ii} \Rightarrow (\; \texttt{sel(screen,}i\texttt{)} \; = \; \texttt{g(cscreen,}i\texttt{)} \;) \;)$$

So the relation between screen and its shadow-administration sscreen is always maintained and so is the relation between cursor and scursor. The correspondence between the real screen and the concept screen (cscreen) is enforced line after line.

```
PROC update_screen: ->
DEF  upd_ii(0);
     ( NOT ii = li ?;
           update_line;
           upd_ii(succ(ii))
     ) *;  ii = li ?
```

where the procedure update_line is to be detailed below.
This update_line should enforce the relation

```
sel(screen,ii) = g(cscreen,ii) = g(sscreen,ii)
```

i.e. it should make sure that the ii-th line arrives both on the screen and in the shadow-administration. Furthermore scursor must be kept up-to-date and neither ii nor the contents of the other lines may be affected. We use a loop with jj serving as a loop-counter. Its invariant is given by

$$\texttt{display\_handling\_inv} \wedge \texttt{jj} \leq \texttt{co} \wedge$$
$$\forall\, j\texttt{:Nat} \; (\; j < \texttt{jj} \Rightarrow$$
$$(\; \texttt{sel(sel(screen,ii),}j\texttt{)} \; = \; \texttt{val(cscreen,ii,}j\texttt{)} \;) \;)$$

to which we should add that neither ii nor the contents of the other lines may be affected.

```
PROC update_line: ->
DEF  upd_jj(0);
     ( NOT jj = co ?;
            update_character;
            upd_jj(succ(jj))
     ) *;   jj = co ?
```

where the procedure update_character is to be detailed below.
This update_character should enforce the relation

```
sel(sel(screen,ii),jj)
   = val(cscreen,ii,jj)
   = val(sscreen,ii,jj)
```

whereby it should keep scursor up-to-date and where all positions with co-ordinates (i, j) for $i \neq$ ii $\vee j <$ jj remain unchanged. We do not require that *only* position (ii,jj) is affected. This is because we issue ce commands instead of writing blanks, as will be explained below.

We briefly explain update_character. First it is tested if the contents of the screen at position (ii,jj) happens to have the desired value already, for in that case no action is required. We expect it to have the desired value often, but if it has not, then the next step is to make sure that the cursor gets at position (ii,jj).

When a blank must be written, a "clear to end-of-line" command is issued and the corresponding modifications in the shadow-administration are made. Note that when this happens, everything on the screen with co-ordinates in $\{(i, j) \mid i =$ ii $\wedge j \geq$ jj$\}$ might change. The motivation for using "clear to end-of-line" commands is is that we hope to gain efficiency because we expect many lines in cscreen to have trailing blanks.

When a non-blank character must be written, the shadow-administration must be kept up-to-date again. We have to be careful with the updating of scursor, because when a character is written, the horizontal co-ordinate of the cursor is incremented by one, except when it is already at co $-$ 1. In the latter case the horizontal co-ordinate keeps its old value.

```
PROC update_character: ->
DEF  ( val(sscreen,ii,jj) = val(cscreen,ii,jj)      ?;
       SKIP
     | NOT val(sscreen,ii,jj) = val(cscreen,ii,jj) ?;
       % if the cursor is not at (ii,jj) then put it there:
          ( scursor = (ii,jj)     ?; SKIP
          | NOT scursor = (ii,jj) ?;
            cm(ii,jj);
            set_scursor(ii,jj)
          );
          ( val(cscreen,ii,jj) = blank     ?;
            ce;
            sce
          | NOT val(cscreen,ii,jj) = blank ?;
            print(val(cscreen,ii,jj));
            upd(sscreen,ii,jj,val(cscreen,ii,jj));
            ( jj = pred(co)     ?; set_scursor(ii,jj)
            | NOT jj = pred(co) ?; set_scursor(ii,succ(jj))
            )
          )
        )
```

We used an auxiliary sce to be given below. It serves for modifying sscreen corresponding to the effect of a "clear to end-of-line" (ce) operation on the real screen. It has been programmed using recursion, which has the advantage that we do not need another programming variable. We could also use this sce to simplify clear_sscreen, but this would disturb the analogy between clear_sscreen and clear_cscreen; of course we could redesign clear_cscreen, but we prefer to avoid such unnecessary backtraking in the development process. After all, the true reason for introducing sce lies here, where we need it as a 'shadow'-version of ce.

```
  PROC sce: ->
  DEF  sce(scursor)

  PROC sce: Nat # Nat ->
  PAR  i:Nat,j:Nat
  DEF  ( j = co     ?; SKIP
       | NOT j = co ?;
         upd(sscreen,i,j,blank);
         sce(i,succ(j))
       )

END;

% COMP DISPLAY_HANDLING:DISPLAY_HANDLING_SPEC:=DISPLAY_HANDLING_IMPL;
% this is to replace an earlier primitive component.
```

We conclude this section whith a remark. Although we tried to do better than the simplest algorithm of rewriting the entire screen every time when update_screen is called, the resulting solution certainly is not optimal yet. One reason for this lies in the relatively restricted set of terminal capabilities which we assumed by adopting DISPLAY. Another reason is that we consider the search for highly sophisticated screen updating-algorithms outside the scope of this case study.

5.6.9 Implementing ATTR2

It is not hard to give an implementation of ATTR2_SPEC using ATTR.

```
LET ATTR2_IMPL :=
LAMBDA X : CLASS SORT Inst  FREE END OF
LAMBDA Y : CLASS SORT Item1 FREE END OF
LAMBDA Z : CLASS SORT Item2 FREE END OF

EXPORT
  SORT Inst,
  SORT Item1,
  SORT Item2,
  FUNC attr    : Inst                    -> Item1 # Item2,
  PROC set_attr: Inst # Item1 # Item2  ->
FROM
IMPORT X INTO
IMPORT Y INTO
IMPORT Z INTO

IMPORT
  APPLY APPLY RENAME SORT Item TO Item1 IN COPY(ATTR) TO X TO Y
INTO

IMPORT
  APPLY APPLY RENAME SORT Item TO Item2 IN COPY(ATTR) TO X TO Z
INTO

CLASS

  FUNC attr: Inst -> Item1 # Item2
  PAR  i:Inst
  DEF  (attr(i),attr(i))

  PROC set_attr: Inst # Item1 # Item2  ->
  PAR  i:Inst,v:Item1,w:Item2
  DEF  set_attr(i,v);
       set_attr(i,w)
```

```
END;

% COMP ATTR2 : ATTR2_SPEC := COPY(ATTR2_IMPL);
% this is to replace an earlier primitive component.
```

5.6.10 Implementing SVAR

We give an implementation of SVAR_SPEC where the procedure upd has been transformed into an algorithmic definition. We need not import any other components. The definitions below are considered executable because it is very obvious how they could be mapped onto classical imperative programming-language constructs. We could translate FUNC val: -> Item VAR into a declaration var val:Item; in Pascal or just Item val; in C. An assignment upd(i) would be translated into val := i in Pascal or val = i; in C.

```
LET SVAR_IMPL :=
LAMBDA X : ITEM OF

IMPORT X INTO
CLASS

  FUNC val: -> Item VAR

  PROC upd: Item ->
  PAR  i:Item
  DEF  MOD val END; val = i ?

  AXIOM FORALL i:Item ( < upd(i) > TRUE )

END;

% COMP SVAR : SVAR_SPEC := COPY(SVAR_IMPL);
% this is to replace an earlier postulated component.
```

5.6.11 Implementing 'STRING'

We give an implementation using 'SEQ'.

```
LET 'STRING_IMPL' :=

IMPORT CHAR INTO
```

```
IMPORT
APPLY RENAME
        SORT Seq      TO String,
        SORT Item     TO Char,
        SORT 'Seq'    TO 'String',
        SORT 'Item'   TO Char,
        PRED seq_inv: TO string_inv
IN 'SEQ' TO
        IMPORT CHAR INTO
        CLASS
          FUNC f: Char -> Char PAR c:Char        DEF c
          PRED eq: Char # Char PAR c:Char,d:Char DEF c = d
          PRED item_inv:       PAR NONE          DEF TRUE
        END

INTO
```

and the only thing we must program explicitly here is the operation `less`. The definition for `less` given below is derived from its specification (`less: 'String' # 'String'`) and the definition of `less: String # String` given in Chapter 4. The main goal of the transformation is to avoid evaluating subexpressions such as hd(s) and `tl`(s) when s is empty.

```
CLASS

  PRED less: 'String' # 'String'
  PAR  s:'String', t:'String'
  DEF  eq(s,empty) {C}AND NOT eq(t,empty)
       {C}OR ( NOT eq(s,empty) {C}AND NOT eq(t,empty)
               {C}AND ( lss(ord(hd(s)),ord(hd(t))) {C}OR
                        hd(s) = hd(t) {C}AND less(tl(s),tl(t)) ) )


END;

% COMP 'STRING' : 'STRING_SPEC' := 'STRING_IMPL';
% this is to replace an earlier primitive component.
```

5.6.12 Implementing ARRAY2

We can give an implementation using BLOCK.

```
LET ARRAY2_IMPL :=
LAMBDA X: DOMAIN1 OF
LAMBDA Y: DOMAIN2 OF
LAMBDA Z: ITEM    OF

IMPORT X   INTO
IMPORT Y   INTO
IMPORT Z   INTO
IMPORT NAT INTO

IMPORT APPLY RENAME
    SORT Block TO Array2
IN COPY(BLOCK) TO Z INTO

CLASS

  PROC create: -> Array2
  DEF  alloc(mul(size1,size2))

  FUNC val: Array2 # Nat # Nat -> Item
  PAR  a:Array2,i:Nat,j:Nat
  DEF  cont(a,add(mul(i,size1),j))

  PROC upd: Array2 # Nat # Nat # Item ->
  PAR  a:Array2,i:Nat,j:Nat,x:Item
  DEF  store(a,add(mul(i,size1),j),x)
END

% COMP ARRAY2 : ARRAY2_SPEC := COPY(ARRAY2_IMPL);
% this is to replace an earlier primitive component.
```

After this last design-transformation step d_{editor} := td_step(d_{editor}); we have arrived at the situation that bot(d_{editor}) = d_b, i.e. no further steps are required and the top-down development process is completed.

5.6.13 Arriving at an Editor Design

This conclude our top-down development process and we formally state the system of the design d_{editor}.

```
SYSTEM WITEFA,MOREDOP,KEYBIND
```

A rough sketch of the design obtained in this way is given in Appendix D. In a separate document [12], a number of diagrams are given which display the structure of the resulting design in detail.

5.7 Related work

Partsch [7] describes the specification and transformation process of a line-oriented editor using a sugared version of CIP-L. It is interesting to compare this with the specification and implementation described before.

A text is considered as a sequence of lines, just as in our formalisation. In the formal model of the editor, the text being edited is called the "current text file" and it is modelled as a triple $\langle t_1, l, t_2 \rangle$, where t_1 is the text *before* the current line, l is the current line and t_2 is the text *after* the current line.

This conceptual model is in fact not changed during the subsequent transformation process. This should be contrasted with our design, where there is more difference between the conceptual model of the text being edited (viz. an ok sequence of lines with two co-ordinate pairs) and the chosen representation (viz. an array with a gap and a bunch of pointers and co-ordinate pairs). We feel that, with our choice, we were in a better position for studying issues of data reification, modularisation and information hiding.

It is interesting to compare our approach to data reification with Jones' approach presented in [11]. As an example we take `'SEQ_SPEC'` and `'SEQ_IMPL'` from Chapter 4 and Appendix A respectively. The specification introduces the sort `'Seq'` and an abstraction function `f: 'Seq' -> Seq`. Jones proposes similar functions, but calls them ***retrieve functions*** (usually denoted as `retr`). In Jones' approach there are standardised ***proof obligations*** associated with each abstraction function. The main difference with our approach lies in the formal status of the proof obligations; in `'SEQ_SPEC'` the proof obligations are written down explicitly as a part of the specification. These are the axioms labelled `{ABSTRACTION}`, e.g. containing the assertion `item_inv AND seq_inv` $\Rightarrow$ `∀s,t:'Seq', i,j:'Item' ( [ LET u:'Seq'; u := cons(i,s) ] f(u) = cons(f(i),f(s)) )`. In Jones' approach one would have similar assertions being part of the proof obligation. When these obligations are fullfilled, i.e. when the proof is given, this shows in Jones' terminology that `cons: 'Item' # 'Seq' -> 'Seq'` *models* `cons: Item # Seq -> Seq`, where strictly, this statement is with respect to the given abstraction function (`f: 'Seq' -> Seq`). In our approach, the principle of black-box correctness gives rise to the statement $\Gamma \vdash$ `'SEQ_IMPL'` $\sqsubseteq$ `'SEQ_SPEC'` where Γ provides information about `BOOL`, `NAT`, `INST`, `ATTR` etc. Showing this statement amounts to proving the axioms in `'SEQ_SPEC'` from the definitions in `'SEQ_IMPL'`. So essentially the same specification techniques and proof obligations are used, but the difference lies in the formal status of the abstraction functions and the proof obligations.

Let us also compare our work with the EMACS editor [4]. We maintained a kind of compatibility with EMACS at the level of procedure names. Also the chosen representation of texts is somewhat similar to the representation chosen in EMACS. Without doubt, from the viewpoint of its amount of features and also of its efficiency the EMACS editor is superior to the editor constructed in this chapter. However, no attempts have been made to give a formal specification of it and as a consequence many features are of an ad-hoc nature. In this context we like to point out that our approach has by no means been pushed to its limits by this case study, and that certainly extensions and improvements to the current design can be made. The documentation of this EMACS takes the shape of comments in the C program text; but here we touch the weak spot of the EMACS design. For example, its so-called ultra-hot screen management package contains an explicit warning which suggests that the reader should not even try to understand this module, let alone make a modification in it. This is quite unacceptable from a methodological point of view and we feel that in this respect we did a better job.

5.8 Looking Back

In this section, we shall in restrospect summarise the main lines of the work presented in this chapter. For an evaluation and conclusions we refer to the next section (5.9). One of the main purposes of constructing the design presented in this chapter was to illustrate the notions of component, black-box description and design as described in Chapter 2 and the correctness preserving transformations studied in Chapter 3. Furthermore we wanted to show how the language COLD-K can be used as a tool for developing complex systems. Therefore, let us explicitly point out some of the interesting points encountered during this case study.

In Section 5.2 we established the top d_t of the editor design which was easy in view of the preparatory work of Chapter 4. This lead to the following conjunct of the post-condition of the development process:

$$\mathrm{top}(d_{editor}) = d_t$$

In Section 5.3 we established the bottom of the editor design. We had to enter a kind of specification phase in order to describe the available primitives. This lead to another conjunct of the post-condition of the development process:

$$\mathrm{bot}(d_{editor}) = d_b$$

In Section 5.5 we began a top-down development process, starting with the system components KEYBIND, MOREDOP and WITEFA. When implementing KEYBIND and MOREDOP we were lucky in the sense that we could do so in a rather trivial way. This was possible because the black-box descripions of these components were already in the right form.

The next component to be dealt with was WITEFA and due to its complexity and the axiomatic style of its description, it was far less trivial to implement. Another complication was the fact that we aimed at a high efficiency in terms of memory usage. Implementation of WITEFA began with choosing a suitable representation of marked texts. This was a crucial step in the development process and it was guided by efficiency considerations, both in terms of execution time and memory usage. Making this representation choice can be viewed as a data reification step. Programming of the various editing operations of the window-and-text facility had to be postponed in order to introduce the basic machinery of abstraction functions and representation invariants which was needed for this data reification.

The knowledge of the algebraic systems related to the concept of text as investigated in Chapter 4 turned out useful. We decided to store texts as block-with-positions which was introduced as a variant of the string representation of texts. We decided to the store dot and mark in two ways. First of all they were stored as natural numbers and secondly their homomorphic images under the reach operation were stored as well – redundantly. The collection of algebraic systems studied in Chapter 4 immediately provided a collection of alternative ideas for this representation: we could have adopted the string representation or we could have added profiles redundantly.

In section 5.6 the top-down development process was continued. The newly postulated components were implemented which lead again to the introduction of new components etc. This went on until at the end of Section 5.6 only the primitives given in Section 5.3 were left.

Let us discuss the well-known advantages and disadvantages of top-down development in the context of this Section 5.6. The main advantage is the possibility of having a separation of concerns. When implementing WITEFA we needed not worry about the problems of satisfying the window-invariant WI, except for the simple fact that at the end of every buffer modification we inserted a mod_dot_restore or mod_text_restore invocation. The fact that at that phase of the development process there was no implementation of mod_dot_restore or mod_text_restore available made it impossible to use hidden implementation details. This is an illustration of Chapter 3 remark 3.4.9.(ii). The disadvantage of top-down development is that one has to take a certain risk by postulating components; for WI_PACKAGE it was not entirely clear in advance that sufficiently efficient algorithms for `mod_text_restore` and `mod_dot_restore` could be found.

The next component to be implemented was `WI_PACKAGE` and again we found a workable decomposition of its functionality by adopting a two-phase approach for the execution of its procedures. The interface was given by the data structures of `CONCEPT_VARS` which includes a two-dimensional array. We had to postulate a component
`DISPLAY_HANDLING` thereby getting the possibility of not worrying (yet) about the actual screen-updating. Later, when implementing `DISPLAY_HANDLING` we were able to demonstrate how the postcondition specification and the efficiency requirements guided us quite elegantly to an algorithm which performs much better than the simplest algorithm, of rewriting the entire screen every time.

Furthermore we had to implement some data type components such as `ATTR2`, `SVAR` etc. Finally at the end of Section 5.6 we arrived at the situation

$$\text{bot}(d_{editor}) = d_b \text{ and } \text{top}(d_{editor}) = d_t$$

where furthermore the black-box correctness of d_{editor} could be assumed.

Our approach with respect to the degree of formalisation was based on the so-called rigorous approach; the black-box descriptions and the glass-box descriptions are in a formal language, but there are no formal proof objects as such. This means that there is no absolute certainty about the black-box correctness of d_{editor} but the degree of formalisation allows for a fast and convincing analysis of the correctness of a certain component when the suspicion arises that there is a weak spot.

In the Appendix A we give a design d_{basic} such that the composition of d_{editor} and d_{basic} is black-box valid. This means that we can construct the design

$$d_{editor} \circ d_{basic}$$

whose black-box correctness follows from the black-box correctness of d_{editor} and d_{basic} and the black-box validation of their composition by Chapter 3 lemma 3.3.14 (ii). Very much in the same way one could conceive another design, $d_{top\text{-}layer}$ say, where we could *add* more sophisticated features to the editor, such as dynamic keybinding and programmability. Such a layer would yield another example of the $\circ$ operator when we would construct $d_{top\text{-}layer} \circ d_{editor}$. We had to restrict ourselves, and we undertook the construction of d_{basic} but we spent no effort on $d_{top\text{-}layer}$.

The design $d_{editor} \circ d_{basic}$ was the starting point for an activity of system integration and code generation. This was performed manually, which was doable, but which in fact also can be automated. In Appendix C we give the C program resulting from the composition of the editor design and the design of Appendix A. In Appendix C we briefly discuss some of the technical points which arose during this manual process of code generation.

5.9 Evaluation

After all the effort spent on the editor case-study, it is time to evaluate the entire approach and to draw some conclusions.

It is interesting to compare the relative sizes of the specification part of the editor (Chapter 4) and the implementation part (this chapter), which do not differ very much. At first sight this might seem strange, for one might expect specifications to be abstract and hence compact, whereas implementations tend to be complicated by efficiency considerations and executability constraints. So, we seem to have the situation $|\text{spec}| \approx |\text{impl}|$ whereas one might expect $|\text{spec}| \ll |\text{impl}|$. In order to explain this, let us analyse the situation in more detail. First of all, the chapter on the specification part of the editor (Chapter 4) is *more* than just the specification of one editor: it also contains a general-purpose library of standard modules (`BOOL_SPEC`, `NAT_SPEC`, `CHAR_SPEC` etc.), a formalisation of the application domain which is text editing and also models of the *interfaces* of the editor with its environment, viz. the display and the file-system. If we consider these things as a vocabulary which is not a part of the actual editor-specification, we get a much more restricted notion of 'specification'. Assuming the latter notion of specification, we have that $|\text{spec}| \approx \frac{1}{4}|\text{(Chapter 4)}|$. But it would not be fair to conclude that thus the specification is much smaller than the implementation indeed, because when we have a closer look to the structure of *this* chapter, we see that it contains much more than just 'implementations'. First of all we have again the phenomenon that the interfaces of the implementations are made explicit (`INST_SPEC`, `ATTR_SPEC`, `BLOCK_SPEC` etc.). Secondly this chapter contains much more than one monolithic implementation, but a large part of it consists of *intermediate specifications*, i.e. black-box descriptions of internal components (`ATTR2_SPEC`, `WI_PACKAGE_SPEC`, `DISPLAY_HANDLING_SPEC` etc.). It is also worth noting that formally all texts from Chapter 4 are part of the final composite design $d_{editor} \circ d_{basic}$. Certain operations from the specification are already executable and are used in that way indeed (e.g. cut and paste operations on reaches). To give some figures: $|d_{editor}| \approx 4000$ lines whereas the C translation of the final composite design $|C(d_{editor} \circ d_{basic})| \approx 2000$ lines. Many parts of the COLD texts which are translated yield, roughly speaking, one line C text for each line of COLD text. In terms of 'number of lines', d_{basic} does not contribute much to the total composite design. To conclude our analysis, we see that $|\,\mathrm{spec}\,| \approx |\,\mathrm{impl}\,|$ is a rather imprecise statement, due to the presence of general-purpose descriptions and intermediate-level descriptions. Our designs contain much more than just a top-level specification of the product and its implementation: they contain much additional descriptions, making various notions and interfaces *explicit*, which is necessary for reasoning about program-correctness and for making the resulting product maintainable.

Now we turn our attention to the evaluation of the usefulness of COLD-K for the development of a complex system – as this editor is. The fact that COLD-K is a wide-spectrum language turned out very useful: we used both axiomatic descriptions (e.g. in the library) and algorithmic definitions (cut and paste operations on texts, implementation of the editor operations etc.). Furthermore we used both static descriptions (e.g. in the library and for all operations on texts) and state-based descriptions (e.g. for the display, the file-system, the editor operations, blocks, buffers etc.). In this way we were able to choose among several styles of descriptions, depending on the particular problem at hand, and also depending on matters of naturalness and even of taste. This can be considered as an advantage of COLD-K over more restricted formalisms such as algebraic specification languages.

We were able to mimick both the algorithmic constructs and the data-types of languages such as Pascal and C very well, although of course a code-generator would have been of help. The task of translating COLD-K texts to C did not take much time (5% of the total time spent on this case study, say) but there are additional advantages of having an automatic code-generator, viz. the standardisation of the translation and the possibility to do prototyping. There are no fundamental technical obstacles for the construction of such a code-generator for a reasonable subset of COLD-K (or some other COLD version). This should be viewed as an advantage, which in the future could lead to a further increase in attractiveness of the approach where the theory of Chapter 2 and Chapter 3 is instantiated by COLD-K. A weak point of the current manual approach is that maintaining the consistency of the COLD-K text and the C text must be done manually as well. When making design modifications or when correcting design errors (yes, we made some), a very error-prone situation arises.

Let us have a look at the syntax of COLD-K, which at some points is not optimal with respect to user-friendliness. For example instead of `add(p,q)` one would like to write `p + q` and also one would like to have short notations for the frequently occurring renaming and application expressions of parameterised data types, writing `SEQ[Nat]` instead of `APPLY RENAME SORT Item TO Nat IN SEQ TO NAT`. Language versions of COLD are on their way where this will be remedied but at the time this case study was done, no such version was available yet. Anyway it was worthwhile to undertake one or more serious applications with COLD-K before embarking on the construction of yet another language version. Furthermore, as this case study shows, the fact that certain syntactic sugar is missing is not essential for the applicability of the language. It might lead to writing 50% more text, say, and some clumsy notations but that is no serious obstacle for applicability, of course. The availability of powerful abstraction mechanisms and structuring mechanisms is much more essential.

As a next topic we shall review the resulting editor, viewed as a product plus its documentation. The editor design takes reasonable efficiency considerations into account, based on the assumptions that display communication is expensive and that any a-priori restriction to the size of the texts would be unacceptable. Furthermore we assumed that the ratio
$|\mathit{text\ stored}|/|\mathit{memory\ used}|$ should be close to 1 rather than $\frac{1}{2}$, and this has been achieved by using dynamic memory allocation primitives and the technique of having a movable gap in each buffer. Also the implementations of strings and tables are quite efficient; in fact the tables even could have been simpler for the restricted use made of them when employing the editor via `KEYBIND` (there are only 3 names). If we would undertake the construction of a higher-level design $d_{\mathit{top\text{-}layer}}$, the complications of our table implementation could turn out worthwhile. There are also certain operations which have a relatively poor performance, but in these cases it is also easy to see why this is the case and how to improve it when needed. An example of this is the `insert_file` operation which takes 43 seconds to insert a 100K text-file on a SUN 3/50. As it turns out, this operation spends 70% of its time to paste $(0,1)$ reaches in the pair-wise mark and reach attributes. But there is an obvious black-box correctness preserving glass-box modification (bbc-preserving gb-mod in the terminology of Chapter 2) to remedy this: modify `WITEFA_IMPL` to make `insert_file` operate on the buffer directly instead of via `insert_character` and use two auxiliary counters of sort `Nat` to keep track of the reach of the inserted text; in this way the new mark and the new reach can be found by just one application of `paste` for each.

Let us have a look at the efficiency of some typical editor operations: to insert a character at the end of a short (i.e. $<$ `co` $- 1$) line takes 58 ms, which is fast enough to respond to manual typing. To insert a character at the beginning of a typical text line (= 40 characters, say) takes about 100 ms; the 100 ms is when using a VT100 emulation on a SUN whereas this time is three or more times higher when using a 1200 Bd terminal connection. In the latter case the terminal connection has become the bottleneck. To perform a `write_named_file` operation takes 2.5 seconds for a 100K text-file. We 'profiled' the editor during a short typical edit session where a 70 lines document was updated by addition of a few lines of text and several other small modifications. As it turned out, the editor spent 87% of time on `mod_text_restore` and 1.5% on `mod_dot_restore`. These `mod_..._restore`'s invoke `cscreen_build` and `update_screen` and the editor spent 61% and 25% of its time on the latter two procedures respectively. The execution speed of most typical editor operations depends neither on the number of lines in the text being edited nor on the position of the dot – obvious exceptions being `search_forward`, `insert_file` and `write_named_file`. So when inserting characters, it does not matter whether the dot is at the begining of a short text or in the middle of a 100K text. This advantage is due to our choice of the buffer data structures and the `mover_to_origin_line` algorithm. The resulting editor has been used on several occasions and it turns out to be usable, though of course it is not the fastest and most sophisticated editor currently available. We conclude that the editor is quite usable from an efficiency point of view, but this does not mean that it is a true product which is ready for sales and distribution – this was not the purpose of the case study after all. Especially when considering the moderate assumptions on the display-capabilities and the fact that we took no special efforts to push the execution speed to its limits, we can from an efficiency viewpoint regard the editor experiment as relatively successful.

We must also review the documentation of the editor, which consists of Chapter 4 and this chapter. It provides for a formal description of all functional aspects of the editor design as well as for informal explanations. The strong point of our approach is that all interfaces with the environment of the editor have been made explicit and that there are specifications at several levels. This documentation (Chapter 4 and this chapter) is quite voluminous and maybe it is hard work for a reader to get completely through it, but it is important to realise that these chapters contain a complete description of the editor design, including all details of its interfaces and with very accurate specifications of all data-types involved; furthermore we dealt with various exceptional cases which tend to complicate the design, such as the case of the cursor reaching the end of a line or the bottom of the screen, the case of trying to move the dot rightwards when at the end of a text line etc. Altogether, this editor is a relatively large and complex software system. It can be expected that due to its completeness and its component structure, the editor design will be 'robust' for various forms of design evolution and efficiency improvements. The theory of correctness-preserving transformations on designs from Chapter 2 and Chapter 3 is applicable here: in particular, since the principle of black-box correctness has been adopted, many efficiency improvements can take the shape of black-box correctness preserving glass-box modifications – meaning that there is a 'locality principle' which can yield a significant reduction of the verification task. It can be expected that this editor provides an excellent starting point for further optimisations and extensions. By way of example we mention two such extensions: to add multi-window features to the editor and to add a layer of dynamic keybinding and programmability.

We shall now explicitly point out two places in the entire documentation which can be considered as successul and elegant. These are the following:

- the formalisation of 'text', which lead to a rich collection of algebraic operations on texts and to the 'discovery' of elegant algebraic laws and of several important homomorphic mappings,
- the description of the buffer data-structure which we could describe completely formally together with its invariant properties ($\mathtt{gap1}(b) \leq \mathtt{gap2}(b)$, $\mathtt{dot}(b) \not\in \mathtt{gap}(b)$ etc.) and several more subtle buffer properties such as ready and space.

They show how formal descriptions and efficiency considerations can go hand in hand. Note how the ready and space properties play a central role in the efficiency of dot-movements and text-modifications.

We must admit that the current documentation includes neither a user-manual nor diagrams showing some hierarchical decomposition of the editor; but this should not be viewed as a weak point of our approach. On the contrary, the formal specification of the editor provides an excellent starting point for writing a user manual – which is something different from the design documentation. The fact that the entire design is available as a formal text, makes it possible to have certain types of diagrams generated automatically, and indeed this has been done using van den Bos en van Ommering's graphical language POLAR. The reader is refered to [12] for the actual diagrams – which are quite helpful. These diagrams show how the various bits and pieces are put together, by means of a compact presentation-in-the-large of both d_{editor} and d_{basic}. This should be compared with the situation often encountered in current industrial practice, where the diagrams come *instead of* essential design documentation.

To conclude, let us once more explicitly recall the main achievement of the editor case study which is that it provides us with a large and realistic example of the notion of *design* developed in Chapter 2 and several applications of the theory concerning design transformations from Chapter 3.

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Appendix A

A Lower Design Layer

A.1.1 Introduction

In this appendix we describe the development of a design d_{basic} which provides for a few *basic* data types. In the context of the development of an editor, it can be viewed as a design-layer *below* the editor design. It provides for implementations of 'SEQ' and TABLE. The 'interfaces' of d_{basic} have been chosen such that its system fits into the primitive components of d_{editor}. In the terminology of Chapter 3 this can be stated more formally as: "the pair (d_{editor}, d_{basic}) is black-box valid".

A.1.2 The Top of the Lower Design Layer

The design top(d_{basic}), is shown below. At the position of the dots in this design we assume LET-constructs introducing the names BOOL_SPEC, NAT_SPEC, CHAR_SPEC etc. This top design contains all essential information for checking the (black-box) validation of d_{basic} with respect to d_{editor}. Therefore, this top design should match the bottom design given in Section 5.3.6.

```
DESIGN
    ...
    COMP BOOL    : BOOL_SPEC;
    COMP NAT     : NAT_SPEC;
    COMP CHAR    : CHAR_SPEC;
    COMP INST    : INST_SPEC;
    COMP ATTR    : ATTR_SPEC;

    COMP BLOCK   : BLOCK_SPEC;
    COMP DISPLAY: DISPLAY_SPEC;
    COMP FILE    : FILE_SPEC;

    COMP 'SEQ'   : 'SEQ_SPEC';
    COMP TABLE   : TABLE_SPEC

SYSTEM BOOL,NAT,CHAR,INST,ATTR,'SEQ',TABLE,BLOCK,DISPLAY,FILE
```

A.1.3 The Bottom of the Lower Design Layer

The design bot(d_{basic}), i.e. the bottom of our lower design layer is given below. At the position of the dots in the bottom design below we assume a number of LET abbreviations, introducing the names BOOL_SPEC, NAT_SPEC, CHAR_SPEC etc. This bottom design also includes the components BLOCK, DISPLAY, and FILE which play no role in this design except for the fact that they are simply passed on to next higher design layer. This bottom does *not* include components 'SEQ', and TABLE because it is precisely the purpose of the design d_{basic} to provide implementations for these.

```
  DESIGN
    ...
    COMP BOOL    : BOOL_SPEC;
    COMP NAT     : NAT_SPEC;
    COMP CHAR    : CHAR_SPEC;
    COMP INST    : INST_SPEC;
    COMP ATTR    : ATTR_SPEC;

    COMP BLOCK   : BLOCK_SPEC;
    COMP DISPLAY: DISPLAY_SPEC;
    COMP FILE    : FILE_SPEC

  SYSTEM NONE
```

A.1.4 Implementing the System Components

A.1.4.1 Introduction

We begin a simple development process and we start with the top design given in Section A.1.2. This yields the design consisting of the keyword DESIGN and a large number of LET abbreviations including those for BOOL_SPEC, NAT_SPEC, CHAR_SPEC, INST_SPEC, ATTR_SPEC, BLOCK_SPEC, DISPLAY_SPEC, FILE_SPEC, 'SEQ_SPEC', and TABLE_SPEC given before, followed by the following components:

```
     COMP BOOL    : BOOL_SPEC;
     COMP NAT     : NAT_SPEC;
     COMP CHAR    : CHAR_SPEC;
     COMP INST    : INST_SPEC;
     COMP ATTR    : ATTR_SPEC;

     COMP BLOCK   : BLOCK_SPEC;
     COMP DISPLAY: DISPLAY_SPEC;
     COMP FILE    : FILE_SPEC;

 {@}  COMP 'SEQ'  : 'SEQ_SPEC';
 {@}  COMP TABLE  : TABLE_SPEC;
```

and having as a system BOOL,NAT,CHAR,INST,ATTR,'SEQ', TABLE, BLOCK, DISPLAY, FILE. The black-box descriptions of this design remain unchanged from now on. Again the symbol {@} means "this is going to be replaced later". Just as we did with d_{editor}, we have mixed the formal texts with informal descriptions such that we can present the various stages of the development process in an incremental fashion and such that there is a mechanical operation of extracting all formal texts to get one design which can be syntax- and type-checked. Trivially, the above initial design is black-box correct. For this simple design there will be no difference between a top-down and a bottom-up development process. Note that we just have to implement 'SEQ' and TABLE and these can be independent. When TABLE would use 'SEQ', or conversely, there would be a difference, but as it happens, this will not be the case.

A.1.4.2 Implementing Sequences

Recall the data type of *implementable sequences* whose specification was given already in Chapter 4. Below an implementation of these sequences is given. It is based on the idea of linked lists and we use an attribute-oriented approach. The application of the export operator below is not strictly required, but the reader may appreciate it here because the export signature serves as a summary of what is required by the specification.

```
LET 'SEQ_IMPL' :=

LAMBDA X : 'ITEM' OF
EXPORT
```

```
  SORT Seq,
  PRED seq_inv:,
  FUNC f       : 'Seq'             -> Seq,

  SORT 'Seq',
  SORT Nat,
  SORT 'Item',
  FUNC empty   :                   -> 'Seq',
  PROC cons    : 'Item' # 'Seq'    -> 'Seq',
  FUNC hd      : 'Seq'             -> 'Item',
  FUNC tl      : 'Seq'             -> 'Seq',
  PRED eq      : 'Seq' # 'Seq'         ,
  FUNC sel     : 'Seq' # Nat       -> 'Item',
  PROC cat     : 'Seq' # 'Seq'     -> 'Seq',
  PROC rev     : 'Seq'             -> 'Seq'

FROM
```

Next we shall introduce several local definitions. 'S_INST' introduces the sort 'Seq' as a renamed version of Inst. SEQ_ITEM introduces sequences of items. More precisely, it exports the sorts Item, Nat, Bag, Seq, the predicate empty and the operations cons, hd, tl, len, sel, cat, rev and bag.

```
LET 'S_INST' :=
  RENAME
    SORT Inst TO 'Seq'
  IN INST;

LET SEQ_ITEM :=
  APPLY
    SEQ_SPEC
  TO X;
```

'S_INST_ITEM' and 'S_INST_NEXT' serve for associating item and next attributes to 'Seq' objects respectively. Let us analyse the structure of 'S_INST_ITEM' below in detail. It consists of a renamed version of ATTR which is applied to two actual parameters, viz. 'S_INST' and X. This renamed version of ATTR is a parameterised description which requires a sort 'Seq' for its first parameter and a sort 'Item' for its second parameter; it exports the sorts 'Seq' and 'Item', the function item: 'Seq' -> 'Item' and the procedure set_item: 'Seq' # 'Item' -> . The definition of 'S_INST_NEXT' is very similar to that of 'S_INST_ITEM'.

```
LET 'S_INST_ITEM' :=
  APPLY APPLY
    RENAME
      SORT Inst                            TO 'Seq',
      SORT Item                            TO 'Item',
      FUNC attr    : Inst          -> Item TO item,
      PROC set_attr: Inst # Item ->        TO set_item
    IN COPY(ATTR)
  TO 'S_INST' TO X;

LET 'S_INST_NEXT' :=
  APPLY APPLY
    RENAME
      SORT Inst                            TO 'Seq',
      SORT Item                            TO 'Seq',
      FUNC attr    : Inst          -> Item TO next,
      PROC set_attr: Inst # Item ->        TO set_next
    IN COPY(ATTR)
  TO 'S_INST' TO 'S_INST';

% end of local definitions

IMPORT X              INTO
IMPORT NAT            INTO
IMPORT 'S_INST'       INTO
IMPORT SEQ_ITEM       INTO
IMPORT 'S_INST_ITEM'  INTO
IMPORT 'S_INST_NEXT'  INTO

CLASS
```

The invariant seq_inv presented below states that each 'Seq' object represents a *defined* sequence; in view of the given abstraction function f this implies that the graph of the next attribute is cycle-free.

```
  PRED seq_inv:
  DEF  FORALL s:'Seq' ( f(s)! )

  FUNC f: 'Seq' -> Seq
  PAR  n:'Seq'
  DEF  ( n = nil     ?; empty
       | NOT n = nil ?; cons(f(item(n)),f(next(n)))
       )

  FUNC empty: -> 'Seq'
  DEF  nil
  PROC cons: 'Item' # 'Seq' -> 'Seq'
  PAR  c:'Item',s:'Seq'
  DEF  LET t:'Seq'; t := create;
       set_item(t,c);
       set_next(t,s);
       t

  FUNC hd: 'Seq' -> 'Item'
  PAR  s:'Seq'
  DEF  item(s)

  FUNC tl: 'Seq' -> 'Seq'
  PAR  s:'Seq'
  DEF  next(s)

  PRED eq: 'Seq' # 'Seq'
  PAR  s:'Seq', t:'Seq'
  DEF  s = t {C}OR
       NOT(s = nil) AND NOT(t = nil)
       {C}AND eq(hd(s),hd(t)) {C}AND eq(tl(s),tl(t))

  FUNC sel: 'Seq' # Nat -> 'Item'
  PAR  s:'Seq', n:Nat
  DEF  ( n = 0     ?; hd(s)
       | NOT n = 0 ?; sel(tl(s),pred(n))
       )

  PROC cat: 'Seq' # 'Seq' -> 'Seq'
  PAR  s:'Seq', t:'Seq'
  DEF  ( s = nil     ?; t
       | NOT s = nil ?; cons(hd(s),cat(tl(s),t))
       )

% If both s and t are sequences, then revap(s,t)
% appends "the reverse of s" to t.

  PROC revap: 'Seq' # 'Seq' -> 'Seq'
  PAR  s:'Seq', t: 'Seq'
  DEF  ( s = nil     ?; t
       | NOT s = nil ?; revap(tl(s),cons(hd(s),t))
       )

  PROC rev: 'Seq' -> 'Seq'
  PAR  s:'Seq'
  DEF  revap(s,empty)

END;

% COMP 'SEQ' : 'SEQ_SPEC' :* 'SEQ_IMPL'
% this is to replace an earlier primitive component.
```

A.1.4.3 Implementing Tables

Below an implementation of tables is given. It is based on the use of binary trees. We *replace* the original Map attribute (function map: Table -> Map from TABLE_SPEC) by a Node attribute, where each Node object at its turn is attributed by 'Item1', Item2 attributes and two Node attributes. Our implementation should be compared with [6] Algorithm 4.52. We shall easily avoid Wirth's **var** parameters by using procedures with one or more result parameters. We considered Jonkers' four-step transformation technique [3] which would mean to (1) add new attributes and relate them to the old attributes, (2) add assignments to the new attributes, (3) use the new attributes and (4) to remove the original attributes. This would be a natural continuation of the attribute-oriented approach adopted already in the specification TABLE_SPEC. We did not employ this transformation technique however, because the last step of removing the original Map attribute could not be justified in terms of the formal implementation relation mentioned in Section 5.1 – although from a practical point of view it seems acceptable. Again the export operator below is not strictly needed.

```
LET TABLE_IMPL :=

LAMBDA X : 'SLO1' OF
LAMBDA Y : ITEM2  OF

EXPORT

  SORT Map,
  SORT Item1,
  FUNC map: Table -> Map,
  FUNC app: Map # Item1 -> Item2,

  SORT Bool,
  SORT Table,
  SORT Item2,
  SORT 'Item1',
  PRED table_inv: ,
  PROC new       :                              -> Table,
  PROC add       : Table # 'Item1' # Item2 -> ,
  PROC rem       : Table # 'Item1'              -> ,
  PROC app       : Table # 'Item1'              -> Item2,
  PROC is_in_dom: 'Item1' # Table              -> Bool

FROM
```

Now we get several local definitions. T_INST below introduces the sort Table. MAP_FROM_SLO1_TO_ITEM2 introduces the sort Map of finite maps from Item1 to Item2. NODE_INST introduces the sort of Node of so-called nodes. SET_OF_NODE introduces sets of these nodes – with associated set operations ins, is_in, union etc. We introduce SET_OF_NODE because when formulating the representation invariant table_inv, we need to speak about the set of nodes in a (sub)tree. NODE_SLO1, NODE_ITEM2, NODE_LEFT, and NODE_RIGHT associate various attributes with these nodes and as a result, every node has the following attributes: item1, item2, left and right. Finally TABLE_ROOT associates a so-called root-node with every table.

```
LET T_INST :=
  RENAME
    SORT Inst TO Table
  IN INST;

LET MAP_FROM_SLO1_TO_ITEM2 :=
  APPLY APPLY
    MAP_SPEC
  TO X TO Y;

LET NODE_INST :=
  RENAME
    SORT Inst TO Node
  IN INST;

LET SET_OF_NODE:=
  APPLY
    RENAME
      SORT Item TO Node,
      SORT Set  TO Nodeset
    IN SET_SPEC
  TO NODE_INST;

LET NODE_SLO1 :=
  APPLY APPLY
```

```
    RENAME
      SORT Inst                           TO Node,
      SORT Item                           TO 'Item1',
      FUNC attr    : Inst         -> Item TO item1,
      PROC set_attr: Inst # Item ->       TO set_item1
    IN COPY(ATTR)
  TO NODE_INST TO X;

LET NODE_ITEM2 :=
  APPLY APPLY
    RENAME
      SORT Inst                           TO Node,
      SORT Item                           TO Item2,
      FUNC attr    : Inst         -> Item TO item2,
      PROC set_attr: Inst # Item ->       TO set_item2
    IN COPY(ATTR)
  TO NODE_INST TO Y;

LET NODE_LEFT :=
  APPLY APPLY
    RENAME
      SORT Inst                           TO Node,
      SORT Item                           TO Node,
      FUNC attr    : Inst         -> Item TO left,
      PROC set_attr: Inst # Item ->       TO set_left
    IN COPY(ATTR)
  TO NODE_INST TO NODE_INST;

LET NODE_RIGHT:=
  APPLY APPLY
    RENAME
      SORT Inst                           TO Node,
      SORT Item                           TO Node,
      FUNC attr    : Inst         -> Item TO right,
      PROC set_attr: Inst # Item ->       TO set_right
    IN COPY(ATTR)
  TO NODE_INST TO NODE_INST;

LET TABLE_ROOT :=
  APPLY APPLY
    RENAME
      SORT Inst                           TO Table,
      SORT Item                           TO Node,
      FUNC attr    : Inst         -> Item TO root,
      PROC set_attr: Inst # Item ->       TO set_root
    IN COPY(ATTR)
  TO T_INST TO NODE_INST;

% end of local definitions

IMPORT X                      INTO
IMPORT Y                      INTO
IMPORT BOOL                   INTO
IMPORT T_INST                 INTO
IMPORT MAP_FROM_SLO1_TO_ITEM2 INTO
IMPORT NODE_INST              INTO
IMPORT SET_OF_NODE            INTO
IMPORT NODE_SLO1              INTO
IMPORT NODE_ITEM2             INTO
IMPORT NODE_LEFT              INTO
IMPORT NODE_RIGHT             INTO
IMPORT TABLE_ROOT             INTO
CLASS

  FUNC nodes: Table -> Nodeset
  PAR  t:Table
  DEF  nodes(root(t))

  FUNC nodes: Node -> Nodeset
  PAR  n:Node
  DEF  ( n = nil     ?; empty
       | NOT n = nil ?; ins(n,union(nodes(left(n)),nodes(right(n))))
       )

  PRED table_inv: DEF

  {1} FORALL t:Table,u:Table
      ( NOT t = u => isect(nodes(t),nodes(u)) = empty );

  {2} FORALL m:Node,n:Node
      ( NOT n = nil =>
        ( is_in(m,nodes(left(n))) => less(item1(m),item1(n));
          is_in(m,nodes(right(n))) => less(item1(n),item1(m)) ) )

  FUNC map: Table -> Map
  PAR  t:Table
  DEF  SOME p:Map
       ( FORALL i:Item1, j:Item2
         ( app(p,i) = j <=>
           EXISTS n:Node
```

```
            ( is_in(n,nodes(root(t)));
              f(item1(n)) = i;
              item2(n) = j ) ) )
```

Now we can implement the procedure new which was specified by LET `t:Table; t := create; set_map(t,empty); t`. We preserve the sequential structure of this definition, but the assignment `set_map(t,empty)` is replaced by an assignment of `nil` to the root-node attribute. Because we *defined* the map attribute in terms of the root-node attribute, this replacement is justified when we can show that assigning `nil` to the root-node makes the map attribute empty indeed. The map of a table is the map of its root-node and because `nodes(nil)` is the empty set, we get (by definition of `map: Node -> Map`) a map p with the property $\forall i,j\ (\mathrm{app}(p,\mathtt{f}(i)) = j \Leftrightarrow \exists n : \mathtt{Node}\ (n \in \emptyset \wedge \ldots))$. This is the empty map.

```
PROC new: -> Table
DEF  LET t:Table; t := create;
     set_root(t,nil);
     t
```

The remaining procedures are dealt with in a very similar way. Some of these were specified by a case-analysis on the invariant `table_inv`, such that here we need not worry about what happens when the invariant does not hold: an arbitrary modification of all attributes is allowed in that case. In the other case, the specification prescribes that a certain assignment to the map attribute must take place. E.g. for `add(t,i,j)` this is `set_map(t,add(map(t),f(i),j))`. Again this is established by an assignment to the root-node.

```
PROC add: Table # 'Item1' # Item2 ->
PAR  t:Table,i:'Item1',j:Item2
DEF  set_root(t,add(root(t),i,j))

PROC add: Node # 'Item1' # Item2 -> Node
PAR  n:Node,i:'Item1',j:Item2
DEF  ( n = nil?;
       LET m:Node; m := create;
       set_item1(m,i);
       set_item2(m,j);
       set_left(m,nil);
       set_right(m,nil);
       m
     | NOT n = nil AND eq(i,item1(n)) ?;
       set_item2(n,j);
```

```
          n
        | NOT n = nil AND less(i,item1(n)) ?;
          set_left(n,add(left(n),i,j));
          n
        | NOT n = nil AND less(item1(n),i) ?;
          set_right(n,add(right(n),i,j));
          n
        )

  PROC rem: Table # 'Item1' ->
  PAR  t:Table,i:'Item1'
  DEF  set_root(t,delete(root(t),i))

  PROC delete: Node # 'Item1' -> Node
  PAR  n:Node, i:'Item1'
  DEF  ( n = nil ?;
          n
        | NOT n = nil AND less(i,item1(n)) ?;
          set_left(n,delete(left(n),i));
          n
        | NOT n = nil AND less(item1(n),i) ?;
          set_right(n,delete(right(n),i));
          n
        | NOT n = nil AND eq(i,item1(n)) ?;

            ( left(n) = nil ?;
              right(n)
            | right(n) = nil ?;
              left(n)
            | NOT left(n) = nil AND NOT right(n) = nil ?;
              LET m:Node,j:'Item1',k:Item2;
              m,j,k := del(left(n));
              set_left(n,m); set_item1(n,j); set_item2(n,k);
              n
            )
         )
```

We used an auxiliary function `del` to delete the right-most node in a tree, i.e. the one with the largest `item1` value; its result consists of (1) the modified tree and (2) the `item1`, `item2` values of the deleted node. Somewhat more formally, `del(n)` could be specified as follows LET `i:'Item1', j:Item2; i,j := rightmost(n); set_map(n,rem(map(n),f(i))); n,i,j`. The function `rightmost` is defined recursively below.

```
FUNC rightmost: Node -> 'Item1' # Item2
PAR  n:Node
DEF  NOT n = nil ?;
     ( right(n) = nil ?; item1(n), item2(n)
     | NOT right(n) = nil ?; rightmost(right(n))
     )

PROC del: Node -> Node # 'Item1' # Item2
PAR  n:Node
DEF  ( NOT right(n) = nil ?;
       LET m:Node,i:'Item1',j:Item2;
       m,i,j := del(right(n));
       set_right(n,m);
       n,i,j
     | right(n) = nil ?;
       left(n), item1(n), item2(n)
     )

PROC app: Node # 'Item1' -> Item2
PAR  n:Node,i:'Item1'
DEF  ( eq(i,item1(n)) ?;
       item2(n)
     | less(i,item1(n)) ?;
       app(left(n),i)
     | less(item1(n),i) ?;
       app(right(n),i)
     )

PROC app: Table # 'Item1' -> Item2
PAR  t:Table,i:'Item1'
DEF  app(root(t),i)

PROC is_in_dom: 'Item1' # Node -> Bool
PAR  i:'Item1',n:Node
DEF  ( n = nil ?;
       false
     | NOT n = nil AND eq(i,item1(n)) ?;
       true
     | NOT n = nil AND less(i,item1(n)) ?;
       is_in_dom(i,left(n))
     | NOT n = nil AND less(item1(n),i) ?;
       is_in_dom(i,right(n))
     )
```

```
  PROC is_in_dom: 'Item1' # Table -> Bool
  PAR  i:'Item1',t:Table
  DEF  is_in_dom(i,root(t))

END
```

The above implementation contains several functions and predicates which need not be executable but which are needed for reasoning purposes: both nodes and both map functions, `rightmost` and of course `table_inv`.

```
% COMP TABLE : TABLE_SPEC := TABLE_IMPL
% this is to replace an earlier primitive component.
```

The structure of the lower design layer d_{basic} is given by its initial design (Section A.1.4.1) and the two replacements discussed in Sections A.1.4.2 and A.1.4.3. We conclude by mentioning its system.

```
SYSTEM BOOL,NAT,CHAR,INST,ATTR,'SEQ',TABLE,BLOCK,DISPLAY,FILE
```

Appendix B

List of Symbols

In this appendix we give a list of the sorts, functions, predicates and procedures used. The list does not include symbols from the C program of the editor, nor does it include the symbols introduced already in Chapter 4. The symbols from Appendix A are also not mentioned unless they are introduced in Section 5.3. If a symbol occurs both in a black-box description and in a glass-box description, then it is mentioned only once and its place in the list is derived from its black-box description.

For each symbol the list contains a very short informal description. The list has been subdivided into a number of sub-lists. The first sub-list contains the symbols that are introduced in Section 5.3. The second sub-list contains the symbols that are introduced in Section 5.5. The third sub-list contains the symbols that are introduced in Section 5.6.

Symbols concerning the bottom of the editor design

`Inst,`	instances (atoms)
`nil: → Inst,`	instance constant
`PROC create: → Inst`	creation of a new instance
`attr: Inst → Item,`	attribute
`PROC set_attr: Inst # Item →`	attribute modification
`Table,`	modfiable tables
`table_inv: ,`	table invariant
`map: Table → Map,`	Map attribute of table
`PROC set_attr: Table # Map →`	modification of Map attribute
`PROC p: →`	arbitrary table operation
`PROC new: → Table`	empty table creation

PROC add: Table # 'Item1' # Item2 →	addition of an entry
PROC rem: Table # 'Item1' →	removal of an entry
PROC app: Table # 'Item1' → Item2	table look-up
PROC is_in_dom: 'Item1' # Table → Bool	'is in domain' test
Block,	memory blocks
size: Block → Nat,	number of locations
cont: Block # Nat → Item,	contents of a location
PROC store: Block # Nat # Item →,	storing a value in a location
PROC alloc: Nat → Block,	creation of a new block
PROC grow: Block # Nat →	increment of block size
attr: Inst → Item1 # Item2	pair-wise attribute
PROC set_attr: Inst # Item1 # Item2 →	pair-wise modification

Symbols concerning system components

Buf	(marked text) buffers
block: Buf → Block	block attribute
PROC set_block: Buf # Block →	block attribute modification
dot: Buf → Nat	dot attribute
PROC set_dot: Buf # Nat →	dot attribute modification
mark: Buf → Nat	mark attribute
PROC set_mark: Buf # Nat →	mark attribute modification
gap1: Buf → Nat	lower bound of gap
PROC set_gap1: Buf # Nat →	modification of lower bound
gap2: Buf → Nat	upper bound of gap
PROC set_gap2: Buf # Nat →	modification of upper bound
dot: Buf → Nat^2	pair-wise dot attribute
PROC set_dot: Buf # Nat^2 →	modification of pair-wise dot
mark: Buf → Nat^2	mark attribute
PROC set_mark: Buf # Nat^2 →	modification of pair-wise mark
reach: Buf → Nat^2	reach attribute
PROC set_reach: Buf # Nat^2 →	modification of reach attribute
val: → Item,	value of programming variable
PROC upd: Item →	assignment
Buf_Map	maps from strings to buffers
table: → Table	table from 'String' to buffers
PROC upd_Table: Table →	assignment
current: → 'String'	name of current buffer
PROC upd_current: 'String'→	assignment
mover: → Nat	general purpose 'loop-counter'
PROC set_mover: Nat →	assignment

mover1,: → Nat	mover (vertical co-ordinate)
PROC set_mover1,: Nat →	assignment
mover2: → Nat	mover (horizontal co-ordinate)
PROC set_mover2: Nat →	assignment
mover: → Nat^2	the pair (mover1,mover2)
PROC set_mover: Nat^2 →	pair-wise assignment
counter: → Nat	general purpose loop-counter
PROC set_counter: Nat →	assignment
f: Block # Nat^2 → String	abstraction function
in_gap: Buf # Nat	test if position is in gap
f: Buf # Nat^2 → String	abstraction function
f: Buf # Nat → String	abstraction function
f: Buf → String	abstraction function
f: Block # Nat^2 → Text	abstraction function
f: Buf # Nat^2 → Text	abstraction function
f: Buf # Nat → Text	abstraction function
f: Buf → Text	abstraction function
f: Buf → MText	abstraction function
f: Buf_Map → Map	abstraction function
TI':	strengthened text-invariant
right: Buf # Nat → Nat	next position (rightwards)
eobp: Buf # Nat	end-of-buffer predicate
eolp: Buf # Nat	end-of-line predicate
eobp: Buf	dot is at end-of-buffer
eolp: Buf	dot is at end-of-line
left: Buf # Nat → Nat	next position (leftwards)
bobp: Buf # Nat	beginning-of-buffer predicate
bobp: Buf	dot is at beginning-of-buffer
bolp: Buf # Nat	beginning-of-line predicate
bolp: Buf	dot is at beginning-of-line
WI':	strengthened window-invariant
wi_package_inv:	invariant of wi_package
PROC init_wi_package: →	initialisation procedure
PROC mod_text_restore: →	restore WI (text modified)
PROC mod_dot_restore: →	restore WI (dot modified)
PROC mod_dot: →	arbitrary dot modification
ready: Buf	buffer is ready for insert
PROC make_ready: Buf →	make buffer ready
PROC make_ready1: Buf →	auxiliary for make_ready

`PROC make_ready2: Buf →`	auxiliary for `make_ready`
`PROC gap_down: Buf →`	move gap one position down
`PROC gap_up: Buf →`	move gap one position up
`PROC right_dot: Buf →`	move dot one position right
`PROC left_dot: Buf →`	move dot one position left
`next_line: Buf → Bool # Nat`	auxiliary for `next_line`
`right_stepping: Buf #` $\texttt{Nat}^2$ `→ Bool # Nat`	going rightwards (if possible)
`hpos: Buf → Nat`	horizontal position
`end_of_line: Buf # Nat →` $\texttt{Nat}^2$	auxiliary for `end_of_line`
`previous_line: Buf → Bool # Nat`	auxiliary for `previous_line`
`beginning_of_line: Buf # Nat → Nat`	auxiliary for `beginning_of_line`
`end_of_buffer: Buf → Nat`	auxiliary for `end_of_buffer`
`space: Buf`	test for free space
`PROC make_space: Buf # Nat →`	introduction of more free space
`PROC second_gap_down: Buf # Nat →`	move free space down
`PROC insert_character: Buf # Char →`	auxiliary for `insert_character`
`PROC newline: Buf →`	auxiliary for `newline`
`match: Buf # 'String' # Nat`	search-string matching
`match': Buf # 'String' # Nat`	match with sentinel
`right: Buf #` $\texttt{Nat}^2$ `→` $\texttt{Nat}^2$	pair-wise rightward move
`buffer_to_string: Buf # Nat → 'String'`	aux. `buffer_to_string`

Symbols concerning internal components

`Array2,`	two-dimensional arrays
`PROC create: → Array2,`	array creation
`val   : Array2 #` $\texttt{Nat}^2$ `→ Item,`	indexing in an array
`PROC upd   : Array2 #` $\texttt{Nat}^2$ `# Item →`	assignment
`ccursor1: → Nat`	concept cursor (vertical)
`PROC set_ccursor1: Nat →`	assignment
`ccursor2: → Nat`	concept cursor (horizontal)
`PROC set_ccursor2: Nat →`	assignment
`ccursor: →` $\texttt{Nat}^2$	concept cursor
`PROC set_ccursor:` $\texttt{Nat}^2$ `→`	assignment
`cscreen: → Array2`	concept screen
`PROC upd_cscreen: Array2 →`	assignment
`g: Array2 #` $\texttt{Nat}^2$ `→ Line`	abstraction function
`g: Array2 # Nat → Line`	abstraction function
`f: Array2 # Nat → Text`	abstraction function
`f: Array2 → Text`	abstraction function
`WI''`	transformed window-invariant
`display_handling_inv:`	display handling invariant
`PROC init_display_handling: →`	initialisation procedure

`PROC update_cursor: →`	transfer of concept to display
`PROC update_screen: →`	transfer of concept to display
`origin: Buf → Nat # Nat`	leftmost uppermost corner
`PROC set_origin: Buf # Nat # Nat →`	assignment
`xx: → Nat`	simple programming variable
`PROC upd_xx: Nat →`	assignment
`yy: → Nat`	simple programming variable
`PROC upd_yy: Nat →`	assignment
`PROC clear_cscreen: →`	fill concept screen with blanks
`v_eq:` Nat^2 `#` Nat^2	equality on vertical co-ordinate
`PROC mover_to_origin_line: Buf →`	put mover at line with origin
`v_geq:` Nat^2 `#` Nat^2	comparison (vertical)
`h_geq:` Nat^2 `#` Nat^2	comparison (horizontal)
`h_lss:` Nat^2 `#` Nat^2	comparison (horizontal)
`built: Buf`	build-up of cscreen done
`size: →` Nat^2	the pair (li,co)
`in_window: Buf`	test if position is in window
`PROC cscreen_build: Buf →`	build-up concept screen
`p_lss:` Nat^2 `#` Nat^2	comparison of co-ordinate pairs
`p_leq:` Nat^2 `#` Nat^2	comparison of co-ordinate pairs
`p_sub:` Nat^2 `#` Nat^2 `→` Nat^2	comparison of co-ordinate pairs
`p_add:` Nat^2 `#` Nat^2 `→` Nat^2	addition of co-ordinate pairs
`c3: Buf`	'necessary condition'
`half_li: → Nat`	li divided by two
`half_co: → Nat`	co divided by two
`ii: → Nat`	simple programming variable
`PROC upd_ii: Nat →`	assignment
`jj: → Nat`	simple programming variable
`PROC upd_jj: Nat →`	assignment
`scursor1: → Nat`	shadow cursor (vertical)
`scursor2: → Nat`	shadow cursor (horizontal)
`scursor: →` Nat^2	shadow cursor
`PROC set_scursor1: Nat →`	assignment
`PROC set_scursor2: Nat →`	assignment
`PROC set_scursor :` Nat^2 `→`	assignment
`sscreen: → Array2`	shadow administration screen
`PROC upd_cscreen: Array2 →`	assignment
`PROC clear_sscreen: →`	fill shadow screen with blanks
`PROC update_line: →`	update line on screen
`PROC update_character: →`	update character on screen
`PROC sce: →`	'shadow' clear to end of line
`PROC sce:` Nat^2 `→`	'shadow' clear to end of line

Appendix C

The C Program of the Editor

After the development processes of the editor and the lower design layer have been finished, the resulting glass-box descriptions have been translated into the C programming language [8] manually. Let us devote a few pages of explanation to this translation. First we explain the translation of the data structures. We assumed the data types of BOOL, NAT, and CHAR as built-in into C, and we dealt with them by a few simple macro definitions. E.g. for BOOL we only needed the following.

```
#define Bool  int
#define true  1
#define false 0
```

We employed a straightforward technique for dealing with the attribute-oriented approach based on INST and ATTR. This was done by collecting *all* attributes of a given object sort and declare that sort as a pointer to a structure. The latter structure contains one field for each attribute function. By introducing macro definitions – one for each attribute function – it becomes possible to use the same functional notation as in COLD-K again. We show this for the object sort Node (from Appendix A) having four attribute functions: item1: Node → Item1, item2: Node → Item2 etc.

```
#define Node struct attrs_Node *
struct attrs_Node {
        Item1 attr_item1;
        Item2 attr_item2;
        Node  attr_left;
        Node  attr_right;
};
```

```
    #define item1(N) ((N)->attr_item1)
    #define item2(N) ((N)->attr_item2)
    #define left(N)  ((N)->attr_left )
    #define right(N) ((N)->attr_right)
```

Next we explain the translation of the algorithms. Whenever possible we sticked to a straightforward line-by-line translation, although we did no attempt to devise a completely standardised translation scheme yet. We show a simple example and for that purpose we recall the procedure make_space from WITEFA_IMPL:

```
  PROC make_space: Buf # Nat ->
  PAR  b:Buf, n:Nat
  DEF  set_mover(size(block(b)));
       grow(block(b),n);

       ( NOT mover = gap2(b) ?;
             second_gap_down(b,n)
       ) *;  mover = gap2(b) ?;

       set_gap2(b,add(gap2(b),n));
       set_dot(b,gap2(b));

       ( lss(mark(b),gap1(b))     ?; SKIP
       | NOT lss(mark(b),gap1(b)) ?; set_mark(b,add(mark(b),n))
       )
```

We show its C translation below. It refers to obvious macros NOT and SKIP. Recall that in C the symbol = denotes assignment whereas == denotes equality!

```
   void make_space(b,n)
   Buf b;
   Nat n;
   {
           mover = (size(block(b)));
           grow(block(b),n);
           while ( NOT (mover == gap2(b)) ) {
                   second_gap_down(b,n);
           }
           gap2(b) = add(gap2(b),n);
           dot(b) = gap2(b);
```

```
            if ( lss(mark(b),gap1(b)) ) {
                    SKIP
            }
            else {
                    mark(b) = add(mark(b),n);
            }
    }
```

The above example is rather unproblematic. At other places we faced technical complications such as operations yielding composite results (e.g. `next_line: Buf -> Bool # Nat` and `right_stepping: Buf # Nat # Nat -> Bool # Nat` from `WITEFA_IMPL`). For the latter complication we employed two different solutions, depending on the question whether the operation is defined recursively – which is not the case for `next_line` but which *is* the case for `right_stepping`. For a non-recursive operation we made several copies of its direct translation, viz. one for each element of the result. For a recursive operation, this technique of making copies could lead to gross inefficiencies so we had to simulate a call-by-reference mechanism using the C operators `*` (indirection) and `&` (address-of). Other complications are related to (1) name clashes due to overloading, (2) using macros for efficiency reasons, (3) COLD-K renamings and (4) modularisation of the C program text. Although it might be somewhat more work to devise universal solutions for these problems, we easily solved them for the particular cases at hand.

We had to provide small C modules providing implementations of `BLOCK`, `DISPLAY` and `FILE`. These components have no COLD-K glass-box descriptions and the C modules should be viewed as reasonable approximations of `BLOCK_SPEC`, `DISPLAY_SPEC` and `FILE_SPEC`. For `BLOCK` we had to use the C library functions `malloc` and `realloc` [9]. For `DISPLAY` we adopted the command sequences of a VT102 terminal [10], which can also be emulated on SUN 3/50 or Philips P2000-C computer systems. For `FILE` we had to use the so-called standard I/O library as partially described in [8] and which becomes available by inclusion of the line

```
#include <stdio.h>
```

For each instance of `SVAR` we declared a simple programming variable in C. We by-passed the implementation of `ARRAY2`, just using C arrays. We inserted a `fflush(outfile);` instruction in the translation of `write_named_file`. We added a top-level initialisation and main-loop to connect the `key` procedure with the standard input. This is done by means of the C function called `main` which has a local variable `c` of sort `Char` and which has the program text given below. The user must type a control-c character to terminate an edit session – whence the termination condition of the loop, `ord(c) != 3`.

```
MOREDOP_startup();
c = getchar();
while (ord(c) != 3) {
        key(c);
        c = getchar();
}
```

The translation process took several days, but the time needed to do this was only a small fraction of the total time spent on the editor (5%, say). In the C program texts, we inserted small comments related to the translation; we did *not* insert comment to explain the program as such, because after all we have the designs d_{editor} and d_{basic} for that. Although the manual translation process introduced several small mistakes, we did not spend much time in debugging. Of course an automatic code-generator would be useful, but the current manual aproach was already very satisfactory. The choice of the target-language (e.g. C versus Pascal) is a relatively unimportant detail of the editor case study and in fact the choice for C was only made when most of the design work was already done. We could have chosen Pascal as well, although in retrospect, we learned to appreciate several pragmatic issues related to C such as the `cpp` macro preprocessor and the powerful `malloc/realloc` memory-management functions. The decisive point in favor of C was its growing popularity in industrial contexts.

The complete C program texts are provided in [12]. Since the above discussion contains already an overview of most relevant technical issues, we do not include all C program texts here; it is always possible to look-up details in [12]. Compilation of these C program texts by `cc -O` yields an executable editor. Just by way of example, we include the C program text of one component and we have chosen to show the translation of `TABLE_IMPL`.

```
/* table_impl.c */

#define guard(N) if (!(N)) { printf("FATAL ERROR (guard false)"); }
#define Nat int

/* TABLE_IMPL
   'Item1' --> Item1,
   eq: 'Item1' # 'Item1' --> eq_Item1,
   add: Table # 'Item1' # 'Item2' ->  --> add_Table,
   add: Node # 'Item1' # 'Item2' ->  --> add_Node,
*/


#define Node struct attrs_Node *
struct attrs_Node {
        Item1 attr_item1;
        Item2 attr_item2;
        Node  attr_left;
        Node  attr_right;
};
#define item1(N) ((N)->attr_item1)
#define item2(N) ((N)->attr_item2)
#define left(N)  ((N)->attr_left )
#define right(N) ((N)->attr_right)
Node create_Node() {
        char *malloc();
        return((Node)malloc(sizeof(struct attrs_Node)));
}

#define Table struct attrs_Table *
struct attrs_Table {
        Node attr_root;
};
#define root(N) ((N)->attr_root)
Table create_Table() {
        char *malloc();
        return((Table)malloc(sizeof(struct attrs_Table)));
}

Table new()
{
        Table t;

        t = create_Table();
        root(t) = nil;
        return(t);
}

Node add_Node(n,i,j)
Node n;
Item1 i;
Item2 j;
{
        Node m;

        if (n == nil) {
                m = create_Node();
                item1(m) = i;
                item2(m) = j;
                left(m)  = nil;
                right(m) = nil;
                return(m);
        }
        else if (eq_Item1(i,item1(n))) {
                item2(n) = j;
                return(n);
        }
        else if (less(i,item1(n))) {
                left(n) = add_Node(left(n),i,j);
                return(n);
        }
        else if (less(item1(n),i)) {
                right(n) = add_Node(right(n),i,j);
                return(n);
        }
}

void  add_Table(t,i,j)
Table t;
Item1 i;
Item2 j;
{
        root(t) = add_Node(root(t),i,j);
}

del(n,return1,return2,return3)
Node n;
Node  *return1;
Item1 *return2;
Item2 *return3;
```

```
{
        if (right(n) != nil) {
                  Node m; Item1 i; Item2 j;
                  del(right(n),&m,&i,&j);
                  right(n) = m;
                  *return1 = n; *return2 = i; *return3 = j;
                  return;
        }
        else if (right(n) == nil) {
                  *return1 = left(n);
                  *return2 = item1(n);
                  *return3 = item2(n);
                  return;
        }
}

Node delete(n,i)
Node n;
Item1 i;
{
         if (n == nil) {
                  return(n);
         }
         else if (less(i,item1(n))) {
                  left(n) = delete(left(n),i);
                  return(n);
         }
         else if (less(item1(n),i)) {
                  right(n) = delete(right(n),i);
                  return(n);
         }
         else if (eq_Item1(i,item1(n))) {

                  if (left(n) == nil) {
                          return(right(n));
                  }
                  else if (right(n) == nil) {
                          return(left(n));
                  }
                  else if (left(n) != nil AND right(n) != nil) {
                          Node m; Item1 j; Item2 k;
                          del(left(n),&m,&j,&k);
                          left(n) = m; item1(n) = j; item2(n) = k;
                          return(n);
                  }
        }
}

void rem(t,i)
Table t;
Item1 i;
{
        root(t) = delete(root(t),i);
}


rightmost(n,return1,return2)
Node n;
Item1 *return1;
Item2 *return2;
{
        guard(n != nil);
        if (right(n) == nil) {
                *return1 = item1(n); *return2 = item2(n);
                return;
        }
        else if (right(n) != nil) {
                rightmost(right(n),return1,return2);
                return;
        }
}


Item2  app_Node(n,i)
Node n;
Item1 i;
{
        if (eq_Item1(i,item1(n))) {
                return(item2(n));
        }
        else if (less(i,item1(n))) {
                return(app_Node(left(n),i));
        }
        else if (less(item1(n),i)) {
                return(app_Node(right(n),i));
        }
}

Item2  app(t,i)
Table t;
```

```
Item1 i;
{
        return(app_Node(root(t),i)) ;
}

Bool isindom_Node(i,n)
Item1 i;
Node n;
{
        if (n == nil) {
                return(false);
        }
        else if (eq_Item1(i,item1(n))) {
                return(true);
        }
        else if (less(i,item1(n))) {
                return(isindom_Node(i,left(n)));
        }
        else if (less(item1(n),i)) {
                return(isindom_Node(i,right(n)));
        }
}

Bool is_in_dom(i,t)
Item1 i;
Table t;
{
        return(isindom_Node(i,root(t)));
}
```

Appendix D: Reference Chart

For each component of d_{editor} its most important sorts and operations are given below.

```
DESIGN
   COMP BOOL    : BOOL_SPEC; { Bool, true, false, not, and, or
   COMP NAT     : NAT_SPEC  ; { Nat, zero, succ, lss, leq, add, sub, 0, 1, etc.
   COMP CHAR    : CHAR_SPEC; { Char, ord, chr, 'a', 'i', etc.
   COMP INST    : INST_SPEC; { Inst, nil, create
   COMP ATTR    : ATTR_SPEC; { attr, set_attr
   COMP 'SEQ'   : 'SEQ_SPEC'; { 'Seq', empty, cons, hd, tl, eq, sel, cat, rev
   COMP TABLE   : TABLE_SPEC; { Table, new, add, rem, app, is_in_dom
   COMP BLOCK   : BLOCK_SPEC; { Block, size, cont, store, alloc, grow
   COMP DISPLAY: DISPLAY_SPEC; { li, co, cr, nl, bc, ce, cl, nd, up, cm, print
   COMP FILE    : FILE_SPEC     ; { valid, file, pos, rewrite, reset, read, write, eof
   COMP ARRAY2 : ARRAY2_SPEC := ARRAY2_IMPL;
         { Array2, create, val, upd
   COMP 'STRING' : 'STRING_SPEC' := 'STRING_IMPL';
         { 'String', empty, cons, hd, tl, eq, sel, cat, rev, less
   COMP SVAR : SVAR_SPEC := SVAR_IMPL;
         { val, upd
   COMP ATTR2 : ATTR2_SPEC := ATTR2_IMPL;
         { attr, set_attr
   COMP DISPLAY_HANDLING:DISPLAY_HANDLING_SPEC:=DISPLAY_HANDLING_IMPL;
         { init_display_handling, update_cursor, update_screen,
   COMP WI_PACKAGE : WI_PACKAGE_SPEC := WI_PACKAGE_IMPL;
         { init_wi_package, mod_text_restore, mod_dot_restore
   COMP WITEFA : WITEFA_SPEC := WITEFA_IMPL;
         { bolp, eolp, forward_character, backward_character, next_line, previous_line,
           beginning_of_line, end_of_line, beginning_of_buffer, end_of_buffer, set_mark,
           exchange_dot_and_mark, insert_file, insert_character, newline, yank_buffer,
           delete_next_character, erase_region, erase_buffer, copy_region_to_buffer, current_buffer_name,
           write_named_file, switch_to_buffer, search_forward, buffer_to_string, buffer_to_string
   COMP MOREDOP: MOREDOP_SPEC := MOREDOP_IMPL;
         { mini, main, kill, startup, escape, return, delete_to_killbuffer, yank_from_killbuffer,
           search_forward, insert_file, write_named_file, delete_previous_character
   COMP KEYBIND: KEYBIND_SPEC  := KEYBIND_IMPL
         { key
SYSTEM WITEFA,MOREDOP,KEYBIND
```

Curriculum vitae

De schrijver van dit proefschrift werd op 4 augustus 1954 geboren te Sittard. In 1972 behaalde hij aan het St.-Michiellyceum te Geleen het diploma Gymnasium-β. Hij studeerde vervolgens electrotechniek aan de Technische Hogeschool Eindhoven, alwaar hij in 1979 het ingenieursexamen aflegde. Afstudeerhoogleraar was prof.dr.ir. J.P.M. Schalkwijk. In de tweede helft van 1979 was hij verbonden aan CSELT in Turijn als wetenschappelijk onderzoeker op het gebied van video-codering. Na het vervullen van de militaire dienstplicht trad hij in 1981 in dienst van Philips Telecommunicatie Industrie te Hilversum, later APT. Hier heeft hij gewerkt als software-ontwikkelaar, onder andere in de context van het TCP-16 project betreffende een gedistribueerd computer-systeem voor de besturing van telefooncentrales. Sinds 1 april 1984 is hij werkzaam bij het Philips Natuurkundig Laboratorium, meer in het bijzonder in de sector Technische Informatica, als wetenschappelijk onderzoeker op het gebied van software-ontwerptechnieken.

Samenvatting

Het onderzoek dat in het proefschrift beschreven wordt, heeft formalisering van ontwerpmethoden voor complexe systemen als onderwerp. Het bestaat uit twee delen: een eerste deel met een theoretisch karakter en een tweede deel waarin een aantal van de resultaten van het eerste deel aan een complex ontwerpprobleem worden getoetst.

Centraal in het eerste deel staat het begrip 'ontwerp van een systeem'. In het algemeen is een systeem opgebouwd uit modules, en bestaan er relaties tussen die modules onderling (meestal in de vorm van een hiërarchische opbouw) en tussen modules en specificaties daarvan – een relatie die wij implementatie-relatie noemen. Daarbij staan wij binnen één ontwerp toe dat modules en hun specificaties naast elkaar bestaan met een verschillende graad van detaillering. Het begrip ontwerp (design) blijkt gefundeerd te kunnen worden op een speciale versie van λ-calculus, de $\lambda\pi$-calculus, die in het proefschrift wordt ontwikkeld. Op basis daarvan is het tevens mogelijk diverse methodologisch inzichtelijke correctheidsbegrippen voor ontwerpen te introduceren. Deze begrippen, die betrekkig hebben op 'information hiding', worden op deze wijze van een wiskundig-logische grondslag voorzien. Aangetoond wordt dat er verbanden bestaan tussen correctheidsbegrippen voor ontwerpen enerzijds en reductiestrategiën voor $\lambda\pi$-calculus anderzijds.

Uitgaande van het formele ontwerpbegrip kunnen correctheidsbehoudende transformaties op ontwerpen gedefiniëerd worden. Een belangrijke rol speelt het feit dat ontwerpen in het algemeen samengesteld worden uit componenten en dat, onder zekere voorwaarden, correctheid van het geheel uit die van de delen kan worden afgeleid. Diverse ontwerpstrategieën, waaronder de welbekende 'top-down' methode, blijken strikt formeel gekarakteriseerd te kunnen worden met de in het proefschrift geïntroduceerde begrippen. Met behulp van deze aanpak kan een zeer duidelijk onderscheid gemaakt worden tussen de statische en de dynamische aspecten van het proces van software-ontwikkeling, hetgeen bijdraagt tot een beter inzicht in het proces van software-ontwikkeling.

Het tweede gedeelte is een case-study. Het omvat de specificatie en een daaruit met de top-down methode ontwikkelde realistische implementatie van een tekst-verwerker. Beide zijn uitgevoerd met behulp van de ontwerptaal COLD die mede op de resultaten uit het eerste deel is gebaseerd. Bij de implementatie van de tekst-verwerker worden enerzijds de abstracte specificaties getransformeerd tot algoritmen en worden anderzijds de data-typen verfijnd door het kiezen van geschikte representaties via meerdere nivo's van

verfijning. Deze studie laat zien dat de gebruikte technieken leiden tot een systematische aanpak, een goed gestructureerd en tevens efficiënt product en een redelijk toegankelijke documentatie. Tevens toont ze aan dat er nog mogelijkheden te over zijn voor verdere uitbouwing van theorie, gereedschappen en door toepassing verkregen ervaringen.

STELLINGEN

behorende bij het proefschrift

A FORMALISATION OF DESIGN METHODS

A λ-calculus Approach to System Design
with an Application to Text Editing

van

L.M.G. Feijs

Eindhoven,
23 februari 1990.

1. Het idee van Nassi-Schneidermann diagrammen, n.l. dat doosjes-in-doosjes tekeningen gebruikt kunnen worden om met 'flow-of-control' operatoren opgebouwde programma's weer te geven, kan mutatis mutandis van nut zijn om met 'module-composition' operatoren opgebouwde modules als tekeningen weer te geven.

   [R.D. van den Bos, L.M.G. Feijs en R.C. van Ommering, POLAR, a Picture-oriented language for abstract representations, gepresenteerd op de 2^e Meteor workshop, Mierlo, 11-13 Sept 1989.]

2. De met INGRES geassocieerde vraagtaal QUEL kan geduid worden met technieken uit de denotationele semantiek, waarbij elke deel-betekenisfunctie een extra argument heeft, te weten een bedeling van tuples aan tuple-variabelen. In het bijzonder, voor een relatie geïntroduceerd met **create** A (naam = string, adres = string) en een tuple-variabele geïntroduceerd met **range of** t **is** A, kan de semantiek van een vraag als

   $$[\![\textbf{retrieve}\ (t.\text{naam})\ \textbf{where not}\ t.\text{adres} = \text{`geldrop'}]\!]$$

   op compositionele wijze beschreven worden door

   $$\{\,\vec{u} \in \text{string}^1 \mid \exists_{\Gamma:\mathcal{V}\to\text{string}^2}\ \begin{array}{l} dom(\Gamma) = \{\,t\,\} \wedge \\ [\![(t.\text{naam})]\!](\Gamma) = \vec{u} \wedge \\ [\![\text{not } t.\text{adres} = \text{`geldrop'}]\!](\Gamma) \wedge \forall_{t\in dom(\Gamma)}\,\Gamma(t) \in A\,\} \end{array}$$

   waarbij $\mathcal{V}$ de verzameling van tuple-variabelen voorstelt en waarbij de beide deel-betekenisfuncties $[\![\ \ ]\!]$ op voor de hand liggende wijze gedefiniëerd zijn.

   [W.E. Baats, L.M.G. Feijs, J.H.A. Gelissen, A formal specification of INGRES, in: M. Wirsing, J.A. Bergstra (Eds.), Algebraic Methods: Theory, Tools and Applications, LNCS 394, Springer-Verlag blz. 207-245 (1989).]

3. De gedachte dat typecorrectheid van expressies in programmeertalen en specificatietalen iets te maken heeft met dimensiecontrole zoals die bij natuurkundige vergelijkingen gebruikelijk is, leeft wel op latente wijze maar wordt slechts zelden uitgewerkt. Nochtans is dit zeer wel mogelijk.

   Zo kunnen in een programmaspecificatie die gedeeltelijk in de mechanica geïnterpreteerd kan worden, types L en T gebruikt worden voor lengte en tijd respectievelijk en L/T voor snelheid; een en ander kan zo ingericht worden dat er wel operaties als $+ : L \times L \to L$, $+ : T \times T \to T$, $/ : L \times T \to L/T$ en $* : L/T \times T \to L$ zijn, maar bijvoorbeeld geen $* : L \times L \to L$.

4. Een 'Spartaans' maar krachtig formalisme verdient vaak de voorkeur boven formalismen waarvan de schijnbare uitdrukkingskracht steunt op een ad hoc collectie van elk op zichzelf aantrekkelijke handigheden.

5. Het verdient aanbeveling dat software-ontwikkelaars tenminste een derde deel van hun werktijd aan studie en opleiding besteden – ook, en zelfs juist, wanneer de werkdruk dit bij voortduring niet lijkt toe te laten.

6. De volgende gereedschappen vormen een minimale maar tevens zéér nuttige collectie ter ondersteuning van een als 'breed-spectrumtaal' aangeduid formalisme zoals COLD: controleprogramma's voor syntax- en typecorrectheid, een modulebibliotheek, een codegenerator voor een onproblematische deeltaal alsmede gereedschap voor het maken en hanteren van grafische representaties.

7. Voor de ontwerper van electronische schakelingen is er in de loop der jaren een omgangstaal opgebouwd met veel woorden die doelmatig zijn door hun compactheid of in de zin dat ze niet op storende wijze overbelast zijn. Voor de 'software engineering' is een passende omgangstaal ook gewenst maar nog slechts in mindere mate opgebouwd.

   In de eerste categorie vinden we b.v. trafo voor transformator, elco voor electrolytische condensator, super voor super-heterodyne ontvanger, modem voor modulator-demodulator, mux voor multiplexer, pf (lees: puf) voor picoFarad, tor voor transistor, R voor weerstand, C voor condensator etc. en in de tweede b.v. kangoeroeschakeling, varkensneusje, Eurokaart en totempaal-uitgang.

8. Bron-coderingstechnieken gebaseerd op een twee-componentenstrategie waarbij een signaal $S(t)$ wordt gesplitst in twee signalen $S_1(t)$ and $S_2(t)$ zodat $S_1(t)+S_2(t) = S(t)$, dragen steeds het risico met zich mee dat de splitsing niet leidt tot twee geheel onafhankelijke signalen en dat dus de som van de bit-rates gebruikt voor de codering van S_1 en S_2 groter is dan strikt noodzakelijk voor S.

   [L. Feijs, L. Chiariglione. Image coding by means of a two-component source-coding scheme. CSELT Rapporti tecnici - Vol. VIII - N. 2, June 1980.]

9. Zolang bij het ontwerpen van computer beeldschermen nog uitgegaan wordt van beeldherhalingsfrequenties die voor het menselijk oog waarneembaar zijn (knipperend beeld), verdient het aanbeveling om gebruik te maken van heldere tekens op een zwarte achtergrond.

10. Het in de context van technische systemen vaak gebruikte begrip 'upward-compatibility' heeft zonder nadere toelichting geen preciese betekenis. Een uitstekend voorbeeld is te verkrijgen door te stellen dat er een upward-compatibility relatie bestaat tussen de systemen van DUPLO en LEGO bouwblokken en dan details van het hiermede opgeroepen beeld te toetsen aan de werkelijkheid.

    [H. Wieneck, The world of LEGO toys, Harry N. Abrams, Inc., Publishers, New York. ISBN 0-8109-2362-9.]

11. De constatering dat een van de voornaamste toepassingen van het spel 'LIFE' betrekking heeft op het genereren van een bewegend patroon om beeldschermen van werkstations voor inbranden te behoeden, is leerrijk vanuit de optiek van wetenschapsmethodologie.

    [What is Life ?, in: E.R. Berlekamp, J.H. Conway, R.K. Guy, Winning ways, Vol. 2, Chapter 25, pp. 817-850.]