Solution to problem 85-10 : An identity

An Identity

Problem 85-10, *by* M. S. KLAMKIN (University of Alberta) *and* O. G. RUEHR (Michigan Technological University).

Let

$$S(x, y, z, m, n, r) \equiv \frac{x^{m+1}}{m!} \sum_{j=0}^{n} \sum_{k=0}^{r} \frac{y^j z^k (j+k+m)!}{j!k!}.$$

Show that if $x+y+z=1$, then

$$S(x, y, z, m, n, r)+S(y, z, x, n, r, m)+S(z, x, y, r, m, n)=1.$$

Solution by A. J. BOSCH *and* F. W. STEUTEL (Eindhoven University of Technology).

First let $x>0$, $y>0$, $z>0$ (and $x+y+z=1$) and consider the following probabilistic model.

Three urns labelled I, II and III contain m, n and r balls respectively. We perform independent drawings by choosing I, II and III with probabilities x, y and z and taking one ball from the urn chosen (without replacement). Let P_{I} be the probability defined by

$P_{\mathrm{I}}=P$ (I is the first urn to be chosen when empty). Then by elementary combinatorics we have

$$P_{\mathrm{I}}=x \sum_{j=0}^{n} \sum_{k=0}^{r} \frac{(m+j+k)!}{m!j!k!} x^m y^j z^k.$$

The probabilities P_{II} and P_{III} are defined similarly. Since eventually an empty urn will be chosen, we have $P_{\mathrm{I}}+P_{\mathrm{II}}+P_{\mathrm{III}}=1$. Finally, a polynomial in x and y that is identically one for $x>0$, $y>0$, $x+y<1$ is also identically one without restriction.

Also solved by S. LJ. DAMJANOVIC (TANJUG Telecommunication Center, Belgrade, Yugoslavia), A. A. JAGERS (Technische Hogeschool Twente, Enschede, The Netherlands), W. B. JORDAN (Scotia, NY), GARY PARKER (Assurance·vie Desjardins) *and the proposers*.

Editorial note: Both Damjanovic and the proposers found the generalization

$$(*) \qquad \sum_{i=1}^{n} x_i \sum_{m_1=0}^{p_1} \frac{x_1^{m_1}}{m_1!} \sum_{m_2=0}^{p_2} \frac{x_2^{m_2}}{m_2!} \cdots \sum_{m_n=0}^{p_n} \frac{x_n^{m_n}}{m_n!} \delta_{m_i, p_i} (m_1+m_2+\cdots+m_n)!=1,$$

using the following generating function:

$$(**) \qquad \sum_{i=1}^{n} x_i \left\{ \frac{1}{\prod_{j=1, j\neq i}^{n}(1-u_j)} \right\} \left\{ \frac{1}{1-\sum_{s=1}^{n} u_s x_s} \right\} = \frac{1}{\prod_{i=1}^{n}(1-u_i)}.$$

The assertion is proved by repeated use of the following elementary identities:

$$\sum_{p=0}^{\infty} u^p \sum_{m=0}^{p} \frac{x^m}{m!}(m+k)! = \frac{1}{(1-u)} \frac{1}{(1-xu)^{k+1}},$$

$$\frac{1}{1-u} \sum_{q=0}^{\infty} v^q \sum_{s=0}^{q} \frac{y^s (s+k)!}{s!(1-xu)^{s+k+1}} = \frac{k!}{(1-u)(1-v)(1-xu-yv)^{k+1}}.$$

Finally to show that (∗∗) is an elementary algebraic identity, let

$$P_N=\prod_{i=1}^{N}(1-u_i),\quad L_N=\sum_{i=1}^{N}x_i,\quad R_N=\sum_{i=1}^{N}u_ix_i.$$

Then, if $L_N=1$, we have

$$\frac{1}{P_N}=\frac{L_N-1+1-R_N}{P_N(1-R_N)}=\sum_{i=1}^{N}\frac{x_i(1-u_i)}{P_N(1-R_N)}$$

$$=\sum_{i=1}^{N}x_i\left\{\frac{1}{\prod_{j=1,j\neq 1}^{N}(1-u_j)}\right\}\left\{\frac{1}{1-\sum_{s=1}^{N}u_sx_s}\right\},$$

which is (∗∗).

A simpler derivation of (∗) can be easily obtained by extending the probabilistic argument in the featured solution to n urns.