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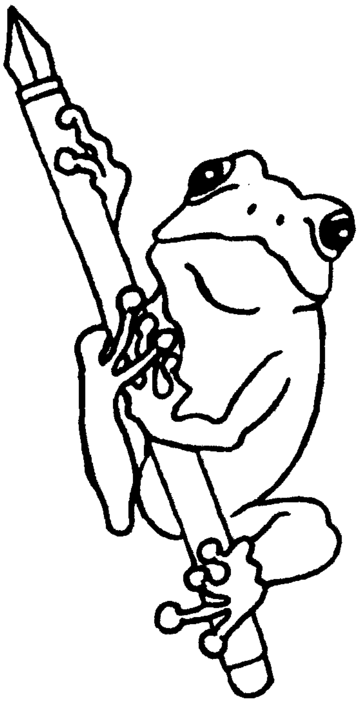

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Abstract

We review the theory of strongly elliptic operators on Lie groups and describe some new simplifications.

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1 Introduction

Analysis on Lie groups, and on more general manifolds, is largely governed by properties of strongly elliptic, or subelliptic, operators. It is a remarkable fact that the theory of these operators extends naturally from the commutative Euclidean group to a general, non-commutative, Lie group. Our purpose is to describe in the simplest and most direct fashion the Lie group version of the theory of strongly elliptic operators. In particular we demonstrate that each such operator generates a continuous semigroup, the 'heat' semigroup, in each continuous representation of the group. Subsequently we comment on the more interesting but more complicated subelliptic situation. Despite appearances the strongly elliptic theory remains largely commutative but the subelliptic theory contains a genuine non-commutative element.

There are two distinct but related approaches to the Lie group theory. Either one can examine the operators in a general representation of the group and use the left regular representation on the L_p-spaces as a calculational aid or one can begin with the L_p-theory and subsequently extend the key properties to a general representation. Moreover, in each of these approaches one can concentrate initially on the resolvents of the operators or on the semigroups they generate. In the language of partial differential equations these options correspond to emphasizing the Green function or the heat kernel. We adopt the second approach and begin with the L_p-theory since this allows closer comparison with the usual Euclidean theory. Moreover, we focus on the semigroup kernel since this leads directly to the 'Gaussian' bounds which are of great practical utility.

There are three principal steps in the L_p-theory of a d-dimensional (connected) Lie group G. First one considers the corresponding problem on the Euclidean group $\mathbf{R}^d$. Then one can construct the semigroup kernels by Fourier transformation and derive 'Gaussian' bounds on the kernels and their derivatives. This will be discussed in detail in Section 2. Secondly, one observes that G is locally diffeomorphic to $\mathbf{R}^d$ under the exponential map. Therefore the Euclidean kernel can be used as a local approximation to the Lie group kernel. Then, by a uniform iteration method, one constructs a function on $G \times G$ which corresponds to the global kernel. This is the key technical idea which is developed in Section 3. Thirdly, one uses the Gaussian properties of the local approximant to establish that the function constructed by this series expansion is indeed a semigroup kernel corresponding to the strongly elliptic operator and that it satisfies Gaussian bounds etc.. By this means one constructs a continuous semigroup whose generator is the closure of the original strongly elliptic operator and the Gaussian properties allow the extension of the semigroup to a general representation of the Lie group. In Section 4 we comment on the further extension of these results to subelliptic operators.

The iterative method which lies at the heart of this approach is a version of the parametrix method. It is analogous to the well known 'time-dependent' perturbation expansion although in this context one does not have a perturbation in any conventional sense. There are two remarkable features of the parametrix method. First it leads to a global solution starting from the initial local approximation. Technically this arises because the terms in the series expansion are given by convolution of terms localized in a fixed compact region. In particular larger distances only arise in higher order terms. Secondly, the expansion has extremely good convergent properties. The kernel is a function over $\mathbf{R}_+ \times G \times G$ with the variable $t \in \mathbf{R}_+$ interpretable as a time parameter. But the expansion is uniformly convergent over $G \times G$ for all $t > 0$. In most time-dependent problems the perturbation series are usually only convergent for small times but in the current context one obtains convergence for all times as a consequence of the Gaussian bounds on the local Euclidean approximant and its derivatives.

2 The Euclidean group

Let $L_p(\mathbf{R}^d)$ denote the usual L_p-spaces with respect to Lebesgue measure and $\partial_i = \partial/\partial x_i$, $i \in \{1, \ldots, d\}$, the corresponding partial derivatives. We adopt multi-index notation $\partial^\alpha = \partial_{i_1} \ldots \partial_{i_n}$, and $\xi^\alpha = \xi_{i_1} \ldots \xi_{i_n}$ for $\xi \in \mathbf{R}^d$, where $\alpha = (i_1, \ldots, i_n)$ with $i_j \in \{1, \ldots, d\}$. Further the length of α is denoted by $|\alpha| = n$. Then we consider m-th order partial differential operators

$$H = \sum_{\alpha;\, |\alpha| \leq m} c_\alpha \, \partial^\alpha$$

with coefficients $c_\alpha \in \mathbf{C}$ where m is an even integer.

The operator H is defined to be **strongly elliptic** if there is a $\mu > 0$ such that

$$\mathrm{Re}\left((-1)^{m/2} \sum_{\alpha;\, |\alpha| = m} c_\alpha \, \xi^\alpha \right) \geq \mu \, |\xi|^m \tag{1}$$

for all $\xi \in \mathbf{R}^d$ where $|\cdot|$ denotes the modulus. The largest value of μ for which (1) is valid is called the **ellipticity constant** of H. There are two other equivalent definitions which are worth mentioning.

First it is easy to establish that H is strongly elliptic with ellipticity constant μ if, and only if, for each $\lambda \in \langle 0, \mu \rangle$ there is a $\nu \geq 0$ such that

$$\mathrm{Re}\Big(\sum_{\alpha;\, |\alpha| \leq m} c_\alpha (i\xi)^\alpha \Big) \geq \lambda\, |\xi|^m - \nu \tag{2}$$

for all $\xi \in \mathbf{R}^d$.

Secondly, H is strongly elliptic if, and only if, there is a $\lambda > 0$ and $\nu \geq 0$ such that

$$\mathrm{Re}(\varphi, H\varphi) \geq \lambda\, N_{m/2}(\varphi)^2 - \nu\, \|\varphi\|_2^2 \tag{3}$$

for all $\varphi \in C_c^\infty(\mathbf{R}^d)$ where

$$N_k(\varphi) = \sup_{\alpha;\, |\alpha| = k} \|\partial^\alpha \varphi\|_2$$

and $\|\cdot\|_2$ denotes the L_2-norm. This third characterization (3) of strong ellipticity is called the **Gårding inequality** and the ellipticity constant of H is the least upper bound of the λ for which the inequality is satisfied uniformly on C_c^∞.

The equivalence of these last two conditions is a consequence of Fourier theory. If $\varphi \mapsto \tilde{\varphi} = \mathcal{F}\varphi$ denotes the Fourier transform on L_2 then $(\widetilde{\partial_j \varphi})(\xi) = i\,\xi_j\, \tilde{\varphi}(\xi)$. Therefore

$$\mathrm{Re}(\varphi, H\varphi) - \lambda\, \|\partial^\beta \varphi\|_2^2 + \nu\, \|\varphi\|_2^2 \;=\; \int_{\mathbf{R}^d} d\xi\, |\tilde{\varphi}|^2 \Big(\mathrm{Re} \sum_{\alpha;\, |\alpha| \leq m} c_\alpha (i\xi)^\alpha - \lambda((i\xi)^\beta)^2 + \nu \Big)$$

for all $\varphi \in L_2(\mathbf{R}^d)$ and all multi-indices β. But if (2) is valid then the right hand side is positive uniformly for β with $|\beta| = m/2$. Hence (2) implies (3). The converse implication also follows from this identity by a simple limiting argument.

Each strongly elliptic operator H, after appropriate closure, generates a strongly continuous semigroup S on each of the L_p-spaces with a well-behaved

'Gaussian' kernel K. This follows because H corresponds to a multiplication operator on the Fourier space;

$$(\widetilde{H\varphi})(\xi) = h(\xi)\,\tilde{\varphi}(\xi)$$

where

$$h(\xi) = \sum_{\alpha;\,|\alpha|\leq m} c_\alpha (i\xi)^\alpha \quad .$$

But then $S_t = e^{-tH}$ acts by convolution,

$$S_t\,\varphi = K_t * \varphi \quad ,$$

with the kernel K given by

$$K_t(x) = (2\pi)^{-d/2} \int_{\mathbf{R}^d} d\xi\, e^{-i\xi\cdot x} e^{-th(\xi)} \quad . \tag{4}$$

As strong ellipticity of H ensures that $\operatorname{Re} h(\xi) \geq \lambda\,|\xi|^m - \nu$ the kernel automatically has a Gaussian type decrease.

Proposition 2.1 *There exist $a, b > 0$ and $\omega \geq 0$ such that*

$$|K_t(x)| \leq a\, t^{-d/m} e^{\omega t} e^{-b(|x|^m/t)^{1/(m-1)}}$$

for all $x \in \mathbf{R}^d$ and all $t \geq 0$.

The proof is a simple exercise in contour integration. If one shifts the integral over $\mathbf{R}^d$ in (4) by replacing ξ with $\xi - i\eta$ where $\eta \in \mathbf{R}^d$ then

$$|K_t(x)| \leq (2\pi)^{-d/2} \int_{\mathbf{R}^d} d\xi\, e^{-\eta\cdot x} e^{-t \operatorname{Re} h(\xi - i\eta)} \quad .$$

But one then estimates that

$$\operatorname{Re} h(\xi - i\eta) \geq 2^{-1} \operatorname{Re} h(\xi) - \sigma\,|\eta|^m - \nu \geq \lambda\,|\xi|^m - \sigma\,|\eta|^m - \omega$$

for suitable $\lambda > 0$ and $\sigma, \nu, \omega \geq 0$. Therefore

$$|K_t(x)| \leq a\, e^{\omega t} e^{-\eta\cdot x} e^{\sigma t |\eta|^m}$$

and the required bounds follow by optimizing over η. For example, one sets $\eta = (|x|/(\sigma t))^{1/(m-1)} |x|^{-1} x$.

Similar estimates follow for the derivatives $\partial^\alpha K_t$ of the kernel. These have a Fourier representation analogous to (4) but with an additional factor $(i\xi)^\alpha$ in the integrand. Now for each α and $\varepsilon > 0$ there is a $k_{\alpha,\varepsilon} > 0$ such that

$$|\xi^\alpha| \leq (t\,|\xi|^m)^{|\alpha|/m} t^{-|\alpha|/m} \leq k_{\alpha,\varepsilon}\, t^{-|\alpha|/m} e^{\varepsilon t|\xi|^m} \quad .$$

Therefore a simple modification of the foregoing argument gives the following result.

Corollary 2.2 *There exist $b > 0$ and $\omega \geq 0$, and for each multi-index α an $a_\alpha > 0$, such that*

$$|(\partial^\alpha K_t)(x)| \leq a_\alpha\, t^{-(d+|\alpha|)/m} e^{\omega t} e^{-b(|x|^m/t)^{1/(m-1)}}$$

for all $x \in \mathbf{R}^d$ and all $t \geq 0$.

It is also evident that one has bounds

$$|x|^n |K_t(x)| \leq a_n\, t^{-(d-n)/m} e^{\omega t} e^{-b(|x|^m/t)^{1/(m-1)}}$$

for all $n \in \{0, 1, 2, \ldots\}$. Thus differentiation introduces an additional singular factor $t^{-1/m}$ but multiplication by $|x|$ introduces a factor $t^{1/m}$ which effectively removes the singularity. Hence an n-th order partial differential operator

$$L_n = \sum_{\alpha;\,|\alpha| \leq n} f_\alpha\, \partial^\alpha$$

with C^∞-coefficients f_α defined on a neighbourhood of the origin is defined to be of **actual order** n', with $n' \in \{0, 1, 2, \ldots\}$ if the coefficients satisfy bounds

$$|f_\alpha(x)| \leq c\,|x|^{(|\alpha|-n')\vee 0}$$

in an open neighbourhood of the origin or, equivalently, if $(\partial^\beta f_\alpha)(0) = 0$ for all multi-indices β with $|\beta| < (|\alpha| - n') \vee 0$. It follows that the composition of two partial differential operators with actual order n'_1 and n'_2 is a partial differential operator with actual order $n'_1 + n'_2$.

Proposition 2.3 *Let L_n be an operator of actual order n'. Then there exist a neighbourhood Ω of the origin and $a, b > 0$ and $\omega \geq 0$ such that*

$$|(L_n K_t)(x)| \leq a\, t^{-(d+n')/m} e^{\omega t} e^{-b(|x|^m/t)^{1/(m-1)}}$$

for all $x \in \Omega$ and all $t \geq 0$.

This is a direct consequence of the foregoing discussion. It gives the key estimates in the subsequent parametrix approximation.

The resolvents $(\lambda I + H)^{-1}$ can be constructed from the semigroup S for sufficiently large positive λ by

$$(\lambda I + H)^{-1} = \int_0^\infty dt\, e^{-\lambda t} S_t \quad .$$

Therefore the resolvents act by convolution with kernels r_λ which are given by the Laplace transforms of the K_t,

$$r_\lambda = \int_0^\infty dt\, e^{-\lambda t} K_t \quad . \tag{5}$$

Hence one can convert bounds on K_t and its derivatives into bounds on r_λ and its derivatives. In the sequel we need L_1-bounds.

Corollary 2.4 *Let L_n be an operator of actual order n' with $n' \leq m - 1$. Then there exist a neighbourhood Ω of the origin and constants $a > 0$ and $\rho \geq 0$ such that*

$$\|1_\Omega L_n r_\lambda\|_1 \leq a\, \lambda^{-1+n'/m}$$

for all $\lambda \geq \rho$.

Proof It follows directly from Proposition 2.3 and (5) that

$$\begin{aligned} \|1_\Omega L_n r_\lambda\|_1 &\leq a \int_\Omega dx \int_0^\infty dt\, e^{-(\lambda-\omega)t} t^{-(d+n')/m} e^{-b(|x|^m/t)^{1/(m-1)}} \\ &\leq a \int_0^\infty dt \int_{\mathbf{R}^d} dy\, e^{-(\lambda-\omega)t} t^{-n'/m} e^{-b|y|^{m/(m-1)}} \\ &\leq a'(\lambda-\omega)^{-1+n'/m} \quad . \end{aligned}$$

Therefore one can choose $\rho = 2\omega$ and $a = 2a'$. □

3 General groups

Let G be a connected d-dimensional Lie group which, for simplicity, we assume to be unimodular and $g \mapsto |g|$ a modulus on G, i.e., the shortest length, measured by a fixed Riemannian measure on G, of an absolutely continuous path from g to the identity e. This modulus is, via the exponential map, locally equivalent with the modulus on $\mathfrak{g}$, i.e., there exists a $c > 0$ such that

$$c^{-1}|a| \leq |\exp a| \leq c|a| \tag{6}$$

for all $a \in \mathfrak{g}$ sufficiently close to 0. Further let dg denote the Haar measure on G and $L_p(G)$ the corresponding L_p-spaces. Then G acts continuously by left translations,

$$(L(g)f)(h) = f(g^{-1}h) \quad ,$$

on each of the L_p-spaces. If $p \in [1, \infty\rangle$ the action is strongly continuous and if $p = \infty$ it is weakly* continuous. All subsequent topological properties are correspondingly understood with respect to the strong topology, or the weak* topology if $p = \infty$.

Next let $a_1, \ldots, a_d$ be a (vector space) basis of the Lie algebra $\mathfrak{g}$ of G. Then $t \mapsto \exp(-ta_i)$ is a one-parameter subgroup of G and the corresponding left translations $t \mapsto L(\exp(-ta_i))$ form a continuous one-parameter group. Let A_i denote the generator of this group. For example, if $G = \mathbf{R}^d$ and dg is Lebesgue measure then choosing the usual Cartesian basis one has $A_i = -\partial_i$.

Now we consider m-th order operators

$$H = \sum_{\alpha;\, |\alpha| \leq m} c_\alpha A^\alpha$$

with $c_\alpha \in \mathbf{C}$, m an even integer and $A^\alpha = A_{i_1} \ldots A_{i_n}$ for $\alpha = (i_1, \ldots, i_n)$. The domain of H in L_p is the subspace $L_{p;m} = \bigcap_{|\alpha| \leq m} D(A^\alpha)$ of m-times left differentiable functions. It is not difficult to establish that $L_{p;m}$ is dense in L_p and hence H is densely defined. Then the adjoint of H is densely defined and hence H is closable. Moreover, the subspace $L_{p;\infty} = \bigcap_{m \geq 0} L_{p;m}$ of C^∞-vectors is a core for each H.

The operators H are the direct analogue of those examined in the previous section for the Euclidean group and H is again defined to be strongly elliptic if the coefficients c_α satisfy either the bounds (1) or the equivalent bounds (2). Alternatively, this definition is equivalent to the Gårding inequalities (3) for all $\varphi \in C_c^\infty(G)$, but with

$$N_k(\varphi) = \sup_{\alpha;\, |\alpha| = k} \|A^\alpha \varphi\|_2 \quad ,$$

although this last equivalence is now not so evident. (It will be discussed at the end of the section.)

Our aim is to establish that the closure of each strongly elliptic operator H generates a continuous semigroup S, on each L_p-space, with a kernel K satisfying Gaussian type bounds. We approach this problem by first constructing a family of functions K which formally corresponds to the semigroup kernel. This construction starts by local approximation with the kernel of the analogous operator on the Euclidean group, i.e., the kernel discussed in the previous section. Secondly, we verify that the K do indeed have the correct properties for a semigroup kernel and that the generator of the semigroup is the closure of the original strongly elliptic operator.

The starting point of the construction is the observation that the kernel K, if it exists, should be a solution of the parabolic equation

$$(\partial_t + H)K_t = 0$$

for $t > 0$ with the initial condition $K_t \to \delta$ as $t \to 0$. Alternatively if one defines $K_t = 0$ for $t \leq 0$ then $(t,g) \mapsto K_t(g)$ from $\mathbf{R} \times G$ into $\mathbf{C}$ should be the fundamental solution for the heat operator $\partial_t + H$, i.e.,

$$((\partial_t + H)K_t)(g) = \delta(t)\,\delta(g) \tag{7}$$

for all $t \in \mathbf{R}$ and $g \in G$. Now the parametrix method expresses K as a perturbation expansion in terms of a localized version of the corresponding kernel for the Euclidean group. The perturbation parameter is the 'time' variable t and the expansion is a direct analogue of 'time-dependent' perturbation theory. The surprise is that the perturbation expansion for the semigroup kernel is convergent for all $t > 0$.

The local approximation procedure starts with the exponential map.

Let $\Omega \subset G$ be an open relatively compact neighbourhood of the identity $e \in G$ and W_0 an open ball in $\mathfrak{g}$ centered at the origin such that $\exp|_{W_0} : W_0 \to \Omega$ is an analytic diffeomorphism. Set $a_x = \sum_{i=1}^{d} x_i a_i$, for $x \in \mathbf{R}^d$, and $W = \{x \in \mathbf{R}^d : a_x \in W_0\}$. Then for $\varphi: \Omega \to \mathbf{C}$ define $\hat{\varphi}: W \to \mathbf{C}$ by $\hat{\varphi}(x) = \varphi(\exp(a_x))$. If Ω is small enough the image of Haar measure under this map is absolutely continuous with respect to Lebesgue measure. In particular, there exists a positive C^∞-function σ on W, bounded from below by a strictly positive constant, such that all derivatives are bounded on W and such that

$$\int_\Omega dg\, \varphi(g) = \int_W dx\, \sigma(x)\, \hat{\varphi}(x)$$

for all $\varphi \in L_1(\Omega\,; dg)$. We normalize the Haar measure dg such that $\sigma(0) = 1$.

The key feature of the exponential map is the existence of C^∞-vector fields $X_1, \ldots, X_d$ on W with the property

$$(X_i\hat{\varphi})(x) = (\widehat{A_i\varphi})(x) = (A_i\varphi)(\exp(a_x)) \tag{8}$$

for all $\varphi \in C_c^\infty(\Omega)$, where the $A_1, \ldots, A_d$ are generators of left translations. Moreover,

$$X_i\hat{\varphi} = -\partial_i\hat{\varphi} + Y_i\hat{\varphi} \tag{9}$$

for $\varphi \in C_c^\infty(\Omega)$ where the Y_i are C^∞-vector fields of actual order zero, i.e., $Y_i = \sum_{i=1}^d f_i\partial_i$ and the $f_i \in C_c^\infty(W)$ have a first-order zero at the origin. But then

$$\widehat{H\varphi} = \widehat{H}\hat{\varphi} + \widehat{H}'\hat{\varphi} \tag{10}$$

where $\widehat{H} = \sum_\alpha c_\alpha(-\partial)^\alpha$ is the operator corresponding to H on $\mathbf{R}^d$ and $\widehat{H}'$ is an operator of actual order at most $m-1$.

Next let $\widetilde{K}_t$ denote the kernel associated with $\widehat{H}$ on $\mathbf{R}^d$, with $\widetilde{K}_t = 0$ if $t \leq 0$, let $\chi \in C_c^\infty(\Omega)$ satisfy $\chi(e) = 1$ and define k_t by $\hat{k}_t = \widetilde{K}_t\hat{\chi}$. It follows immediately from (10) that

$$\begin{aligned} ((\partial_t + H)k_t)\hat{}\,(x) &= ((\partial_t + \widehat{H})(\widetilde{K}_t\hat{\chi}))(x) + (\widehat{H}'(\widetilde{K}_t\hat{\chi}))(x) \\ &= \delta(t)\,\delta(x) + \hat{l}_t(x) \end{aligned}$$

where $\hat{l}_t$ has the form (with a finite sum)

$$\hat{l}_t(x) = \sum_i (L^{(i)}\widetilde{K}_t)(x)\,\hat{\chi}_i(x)$$

with $\hat{\chi}_i \in C_c^\infty(W)$ and the $L^{(i)}$ are operators of actual order at most $m-1$. Therefore k_t and l_t have compact support and satisfy the heat equation

$$((\partial_t + H)k_t)(g) = \delta(t)\,\delta(g) + l_t(g)\quad . \tag{11}$$

Moreover, the bounds of Proposition 2.3 and the local equivalence of the moduli on G and $\mathbf{R}^d$, given by (6), ensure that one has bounds

$$|k_t(g)| \;\leq\; a\,t^{-d/m}e^{\omega t}e^{-b(|g|^m/t)^{1/(m-1)}} \tag{12}$$

$$|l_t(g)| \;\leq\; a\,t^{-(d+m-1)/m}e^{\omega t}e^{-b(|g|^m/t)^{1/(m-1)}} \tag{13}$$

for some $a, b > 0$ and $\omega \geq 0$ and all $t > 0$ and $g \in G$. In addition $k_t = l_t = 0$ for $t \leq 0$. Therefore one can construct a solution of the heat equation (7) by iteration of the approximate equation (11).

Proposition 3.1 *Define $K_t^{(n)}$ recursively by $K_t^{(0)} = k_t$ and*

$$K_t^{(n)} = -\int_0^t ds\, K_{t-s}^{(n-1)} * l_s \quad .$$

It follows that the series

$$K_t = \sum_{n\geq 0} K_t^{(n)}$$

is L_p-convergent to a limit $K_t \in L_{p;\infty}$ for all $p \in [1, \infty]$ and $t > 0$. The limit K satisfies the heat equation (7), with the convention $K_t = 0$ for $t \leq 0$. Moreover, $t \mapsto K_t$ is continuous from $\langle 0, \infty\rangle$ into $L_1^\rho(G)$ for all $\rho \geq 0$, where $L_1^\rho(G) = L_1(G; e^{\rho|g|}dg)$ is the weighted space with norm $\|\varphi\|_1^\rho = \int dg\, e^{\rho|g|}|\varphi(g)|$. Finally, there are $b > 0$ and $\omega \geq 0$, and for each multi-index α an $a_\alpha > 0$, such that

$$|(A^\alpha K_t)(g)| \leq a_\alpha\, t^{-(d+|\alpha|)/m} e^{\omega t} e^{-b(|g|^m/t)^{1/(m-1)}}$$

for all $g \in G$ and $t > 0$.

We sketch the main features of the proof.

First, it suffices to prove that the series for K is L_1-, and L_∞-, convergent because the L_p-convergence is then an immediate consequence. The L_1-convergence is particularly easy because the estimates (12) and (13) imply that

$$\|k_t\|_1 \leq a\, e^{\omega t} \quad , \quad \|l_t\|_1 \leq a\, t^{-1+1/m} e^{\omega t} \tag{14}$$

for suitable $a > 0$ and $\omega \geq 0$. Therefore, as

$$\|K_t^{(n)}\|_1 \leq \int_0^t ds\, \|K_{t-s}^{(n-1)}\|_1 \|l_s\|_1 \quad ,$$

one can argue by recursion that one has bounds

$$\|K_t^{(n)}\|_1 \leq a\, (b^n t^n / n!)^{1/m} e^{\omega t}$$

for all $n \geq 0$ and all $t > 0$. Thus the series is L_1-convergent for all $t > 0$.

The L_∞-convergence is slightly more complicated. It relies on the L_1-bounds (14), the analogous L_∞-bounds

$$\|k_t\|_\infty \leq a\, t^{-d/m} e^{\omega t} \quad , \quad \|l_t\|_\infty \leq a\, t^{-d/m-1+1/m} e^{\omega t}$$

and the recursion relations

$$\|K_t^{(n)}\|_\infty \leq \int_0^t ds \left(\|K_{t-s}^{(n-1)}\|_1 \|l_s\|_\infty\right) \wedge \left(\|K_{t-s}^{(n-1)}\|_\infty \|l_s\|_1\right)$$

(Here we use the unimodularity of G to simplify the calculation). One can then use these relations to bound successively the L_∞-norms of the $K_t^{(n)}$. The specific bound is that

$$\|K_t^{(n)}\|_\infty \leq a\, t^{-d/m} (b^n t^n / n!)^{1/m} e^{\omega t}$$

for suitable a, b, ω, uniformly for all $t > 0$ and $n \geq 0$. (The proof is by induction on n, the general induction process starts when $-d+n > -(m-1)$.) Hence one obtains uniform convergence of the series for K_t and bounds

$$|K_t(g)| \leq a\, t^{-d/m} e^{\omega t} \tag{15}$$

for all $g \in G$ and $t > 0$.

Secondly, similar estimates allow one to verify that K_t satisfies the heat equation (7) and the continuity properties.

Thirdly, consider the Gaussian bounds with $|\alpha| = 0$. It follows from (12) that

$$\sup_{g \in G} e^{\rho|g|} |k_t(g)| \leq a\, t^{-d/m} e^{\omega(1+\rho^m)t} \tag{16}$$

for all $\rho, t > 0$ with redefined values of $a > 0$ and $\omega \geq 0$. Moreover, a standard estimate gives

$$\int_G dg\, e^{\rho|g|} |k_t(g)| \leq a\, e^{\omega(1+\rho^m)t} \quad .$$

Similarly

$$\sup_{g \in G} e^{\rho|g|} |l_t(g)| \leq a\, t^{-(d+m-1)/m} e^{\omega(1+\rho^m)t}$$

and

$$\int_G dg\, e^{\rho|g|} |l_t(g)| \leq a\, t^{-1+1/m} e^{\omega(1+\rho^m)t} \quad . \tag{17}$$

Now one can use the estimates (16)—(17) to bound

$$\sup_{g\in G} e^{\rho|g|}|K_t^{(n)}(g)|$$

by a simple modification of the foregoing recursive arguments. The resulting bounds differ from the case $\rho = 0$ only by an additional factor $e^{\omega\rho^m t}$. Therefore one obtains bounds on $e^{\rho|g|}|K_t(g)|$ which differ from the earlier bounds (15) by the additional factor $e^{\omega\rho^m t}$. Explicitly,

$$|K_t(g)| \le a\, t^{-d/m} e^{\omega(1+\rho^m)t} e^{-\rho|g|}$$

for all $\rho, t > 0$. Optimizing with respect to ρ then gives the desired Gaussian bounds.

The proof of the C^∞ property of K_t and the bounds for the derivatives $A^\alpha K_t$ is analogous and we omit further details. □

The expansion used to construct the kernel K has a notable localization feature. The zero-order approximant $K^{(0)}$ is supported by Ω. Moreover, since the first-order approximant $K^{(1)}$ involves a convolution it is supported by Ω^2. Then by recursion $K^{(n)}$ is supported by Ω^n. Thus the large distance behaviour of the kernel is captured by the higher-order terms.

Let $U = \{U(g) : g \in G\}$ be a continuous representation of the Lie group G by bounded operators $U(g)$ on the Banach space $\mathcal{X}$ and assume U is weakly continuous, or weakly* continuous if $\mathcal{X}$ has a predual. Now let A_i denote the generator of the continuous one-parameter group $t \mapsto U(\exp(-ta_i))$. Then applying the previous definitions the strongly elliptic operator $H = \sum_{|\alpha|\le m} c_\alpha A^\alpha$ with domain $\mathcal{X}_m = \bigcap_{|\alpha|\le m} D(A^\alpha)$ becomes a densely defined, closable operator on $\mathcal{X}$.

Since the representation U is continuous one has bounds

$$\|U(g)\| \le M e^{\rho|g|}$$

with $M \ge 1$ and $\rho \ge 0$. Therefore, since the kernel K_t satisfies Gaussian bounds, $K_t \in L_1^\rho(G)$ and one can define bounded operators S_t on $\mathcal{X}$ by

$$S_t = U(K_t) = \int_G dg\, K_t(g)\, U(g) \quad .$$

Note that $t \mapsto S_t\xi$ is continuous from $\langle 0, \infty\rangle$ into $\mathcal{X}$ for all $\xi \in \mathcal{X}$, since $t \mapsto K_t$ is continuous from $\langle 0,\infty\rangle$ into $L_1^\rho(G)$. Because of the bounds $\|K_t^{(n)}\|_1^\rho \le$

$a\,(b^n t^n/n!)^{1/m} e^{\omega(1+\rho^m)t}$ it follows that $\lim_{t\downarrow 0} S_t\xi = \lim_{t\downarrow 0} U(K_t^{(0)})\xi$, if one of the two limits exists. But $(\widetilde{K}_t)_{t>0}$ is a bounded approximation of the identity and hence

$$\lim_{t\downarrow 0} U(K_t^{(0)})\xi = \lim_{t\downarrow 0} \int_W dx\,\sigma(x)\,\widetilde{K}_t(x)\,\hat{\chi}(x)\,U(\exp(a_x))\xi = \xi \quad .$$

Therefore $\lim_{t\downarrow 0} S_t\xi = \xi$ strongly if U is strongly continuous and weakly* if U is weakly* continuous.

We will first apply this to the L_p^ρ-, and $L_{p;n}^\rho$-, spaces with respect to the left regular representation. Here $L_p^\rho = L_p(G\,; e^{\rho|g|}dg)$ is the weighted space and $L_{p;n}^\rho = \bigcap_{|\alpha|\le n} D(A^\alpha)$ the associated space of n times left differentiable functions with respect to the weighted measure. Then

$$S_t\varphi = K_t * \varphi$$

and it follows that $S_t L_p^\rho \subseteq L_{p;\infty}^\rho \subseteq D(H)$. Moreover, if $p \in [1,\infty\rangle$ and $q \in \langle 1,\infty]$ is conjugate to p then

$$\begin{aligned} -\int_{\mathbf{R}} dt\,(\partial_t\tau)(t)\,(\psi, S_t\varphi) + \int_{\mathbf{R}} dt\,\tau(t)\,(\psi, HS_t\varphi) \\ = -\int_{\mathbf{R}} dt\,(\partial_t\tau)(t)\,(\psi, S_t\varphi) + \int_{\mathbf{R}} dt\,\tau(t)\,(H^*\psi, S_t\varphi) = 0 \end{aligned}$$

for all $\varphi \in L_p^\rho$, $\tau \in C_c^\infty(\langle 0,\infty\rangle)$ and $\psi \in C_c^\infty(G)$. But then by continuity it is valid for all $\psi \in L_q^\rho$. On the other hand the map $t \mapsto HS_t\varphi$ is continuous if $\varphi \in L_{p;m}^\rho$. Therefore it follows from the lemma of Du Bois–Reymond that $t \mapsto (\psi, S_t\varphi)$ is differentiable and $\frac{d}{dt}(\psi, S_t\varphi) + (\psi, HS_t\varphi) = 0$ for all $\varphi \in L_{p;m}^\rho$, $\psi \in L_q^\rho$ and $t > 0$. Then

$$\frac{d}{dt} S_t\varphi + HS_t\varphi = 0 \tag{18}$$

strongly for all $\varphi \in L_{p;m}^\rho$ by an application of the mean value theorem.

The family $S = \{S_t\}_{t>0}$ forms a semigroup if, and only if, K is a convolution semigroup. But the definition of K seems unsuited for direct verification of this property and so we have to approach it indirectly. We will argue that it follows from the lower semiboundedness of $\operatorname{Re} H$ on L_2. In some situations this latter property is an obvious implication of the strong ellipticity condition. For example, if $m = 2$ and H is expressed in the form

$$H = \sum_{i,j=1}^{d} c_{ij} A_i A_j + c_0$$

with the matrix of leading coefficients $C = (c_{ij})$ symmetric then strong ellipticity is equivalent to strict positivity of the real part of C. Therefore $\mathrm{Re}\, H$ is lower semibounded. Lower semiboundedness of the real part of a general second-order operator can then be deduced from this special case as the first-order terms are a small perturbation of the leading terms. But it appears that the proof of semiboundedness for higher order operators is somewhat more complicated to establish.

Proposition 3.2 *Each symmetric strongly elliptic operator H on $L_2(G\,;dg)$ is essentially self-adjoint and lower semibounded.*

It suffices to establish that the range of $(\lambda I+\overline{H})$ is equal to L_2 and its inverse is bounded for all large positive λ. For this we use a resolvent version of the foregoing parametrix techniques.

Let $\chi, \chi' \in C_c^\infty(G)$, $\mathrm{supp}\,\chi' \subset \Omega$, $\chi(e) = 1$ and $\chi' = 1$ on $\mathrm{supp}\,\chi$. Then for all $\varphi \in C_c^\infty(G)$ and $\psi \in L_2(G)$ one has for all $r \in C_c^\infty(G)$ with $\mathrm{supp}\, r \subseteq \mathrm{supp}\,\chi$

$$\begin{aligned} \int_G dg\, r(g)\,(\psi, (\lambda I + H)L(g)\varphi) &= (\psi, (\lambda I + H)(r * \varphi)) \\ &= \int_G dg\, ((\lambda I + H)r)(g)\,(\psi, L(g)\varphi)\,\chi'(g) \\ &= \int_G dg\, r(g)\,((\lambda I + H)\tau)(g) \qquad (19) \end{aligned}$$

where $\tau(g) = (\psi, L(g)\varphi)\,\chi'(g)$. Since $C_c^\infty(G)$ is dense in $L_1(G)$ it follows by continuity that (19) is valid for all $r \in L_1(G)$ with $\mathrm{supp}\, r \subseteq \mathrm{supp}\,\chi$. Now let r_λ be the function on G with support contained in Ω such that $\hat{r}_\lambda = \widetilde{R}_\lambda\hat{\chi}$ where $\widetilde{R}_\lambda$ denotes the kernel of the resolvent of $(\lambda I + \widehat{H})^{-1}$ on $\mathbf{R}^d$. Then one has in the sense of distributions

$$\begin{aligned} &(\psi, (\lambda I + H)(r_\lambda * \varphi)) \\ &\quad= \int_W dx\, \sigma(x)\, \widetilde{R}_\lambda(x)\,\hat{\chi}(x)\,((\lambda I + \widehat{H} + \widehat{H}')\hat{\tau})(x) \\ &\quad= \int_W dx\, \sigma(x)\,((\lambda I + \widehat{H} + \widehat{H}')(\widetilde{R}_\lambda\,\hat{\chi}))(x)\,\hat{\tau}(x) \\ &\quad= \int_W dx\, \sigma(x)\,\delta(x)\,\hat{\chi}(x)\,\hat{\tau}(x) + \int_W dx\, \sigma(x)\,\hat{s}_\lambda(x)\,(\psi, L(\exp(a_x))\varphi) \quad , \end{aligned}$$

where $\hat{s}_\lambda$ has the form

$$\hat{s}_\lambda(x) = \sum_i (L^{(i)}\tilde{R}_\lambda)(x)\,\hat{\chi}_i(x)\,\hat{\chi}'(x) \quad .$$

Once again the $\hat{\chi}_i \in C_c^\infty(W)$ and the $L^{(i)}$ are operators of actual order at most $m-1$. But the estimates of Corollary 2.4 imply that $\|r_\lambda\|_1 \leq a\,\lambda^{-1}$ and $\|s_\lambda\|_1 \leq a\,\lambda^{-1/m}$ for large λ. So

$$(\psi, (\lambda I + H)(r_\lambda * \varphi)) = (\psi, \varphi) + (\psi, s_\lambda * \varphi)$$

Therefore, if R_λ and S_λ denote the operators of convolution with r_λ and s_λ, respectively, then $\|R_\lambda\varphi\|_2 \leq a\,\lambda^{-1}\|\varphi\|_2$ and $\|S_\lambda\varphi\|_2 \leq a\,\lambda^{-1/m}\|\varphi\|_2$. Hence

$$(\lambda I + \overline{H})R_\lambda\varphi = \varphi + S_\lambda\varphi \tag{20}$$

for all $\varphi \in C_c^\infty(G)$ and by density it follows that $R_\lambda L_2 \subseteq D(\overline{H})$ and (20) is valid for all $\varphi \in L_2$. Thus if λ is sufficiently large that $a\,\lambda^{-1/m} < 1$ then $(I + S_\lambda)$ has a bounded inverse and

$$\varphi = (\lambda I + \overline{H})R_\lambda(I + S_\lambda)^{-1}\varphi \quad .$$

This establishes that the range of $(\lambda I + \overline{H})$ is equal to L_2 and hence $\overline{H}$ is self-adjoint. But it then follows that

$$\varphi = (I + S_\lambda^*)^{-1}R_\lambda^*(\lambda I + \overline{H})\varphi$$

and hence

$$\|\varphi\|_2 \leq a\lambda^{-1}(1 - a\lambda^{-1/m})^{-1}\|(\lambda I + \overline{H})\varphi\|_2 \quad .$$

Therefore $(\lambda I + \overline{H})$ has a bounded inverse. Thus $\overline{H}$ is lower semibounded by spectral theory. □

Now it is straightforward to prove that K is a convolution semigroup.

Since $\mathrm{Re}\,H$ is a symmetric strongly elliptic operator on L_2 it follows from Proposition 3.2 that it is lower semibounded on L_2, i.e., there is a $\nu \geq 0$ such that

$$\mathrm{Re}(\varphi, H\varphi) \geq -\nu\,\|\varphi\|_2^2$$

for all $\varphi \in L_{2;m}$. Next observe that if $\varphi_t \in D(H)$ satisfies the Cauchy equation

$$\frac{d}{dt}\varphi_t + H\varphi_t = 0 \tag{21}$$

for all $t > 0$ then

$$\frac{d}{dt}\|\varphi_t\|_2^2 = -2\,\mathrm{Re}(\varphi_t, H\varphi_t) \leq 2\nu\,\|\varphi_t\|_2^2 \quad .$$

Therefore $t \mapsto e^{-\nu t}\|\varphi_t\|_2$ is a decreasing function. Now suppose $\varphi_t^{(1)}$ and $\varphi_t^{(2)}$ both satisfy (21) and $\varphi_t^{(1)} \to \varphi$, $\varphi_t^{(2)} \to \varphi$ as $t \to 0$. Then $\varphi_t^{(1)} - \varphi_t^{(2)}$ also satisfies the equation but $\varphi_t^{(1)} - \varphi_t^{(2)} \to 0$ as $t \to 0$. Therefore, as a consequence of the foregoing decrease property, $\varphi_t^{(1)} = \varphi_t^{(2)}$, i.e., the solution of (21) is uniquely determined by the initial data $\varphi = \varphi_0$.

Now let $\varphi \in L_{2;m}$. Then $\varphi_t = S_{t+s}\varphi = K_{t+s} * \varphi$ satisfies (21) with initial data $\varphi_0 = S_s\varphi$. Moreover, $\varphi_t = S_tS_s\varphi$ satisfies the equation with the same initial data. Therefore

$$(S_{t+s} - S_tS_s)\varphi = 0$$

for all $\varphi \in L_{2;m}$, and by continuity, for all $\varphi \in L_2$. This establishes that S is a semigroup on L_2. But this implies that K_t is a convolution semigroup. Therefore S is also a semigroup on the other L_p^ρ-spaces and in any representation in a Banach space.

It follows from (18) that the generator H_S of S is an extension of H on L_p^ρ. Now $L_{p;\infty}^\rho$ is a dense S-invariant subspace and hence a core of H_S. Therefore H_S must be the closure of H.

At this point we have essentially established the main result.

Theorem 3.3 *Let H be an m-th order strongly elliptic operator acting on the Banach space $\mathcal{X}$, associated with a continuous representation U. Then the closure of H generates a continuous semigroup on $\mathcal{X}$ with a kernel $K_t \in C^\infty(G)$ satisfying bounds*

$$|(A^\alpha K_t)(g)| \leq a_\alpha\, t^{-(d+|\alpha|)/m} e^{\omega t} e^{-b(|g|^m/t)^{1/(m-1)}}$$

for all $g \in G$ and $t > 0$.

The kernel K_t was constructed in Proposition 3.1 and the foregoing argument established that $\{K_t\}_{t>0}$ is a convolution semigroup and the corresponding operators $S = \{S_t\}_{t\geq 0}$ defined by $S_t x = \int_G dg\, K_t(g)\, U(g)x$ form a continuous semigroup on $\mathcal{X}$. Let $\rho > 0$ be so large that $\|S_t\| \leq Me^{\rho t}$ uniformly for all $t > 0$, for some $M > 0$. Let H^L and S^L be the operator and semigroup corresponding to the left regular representation in L_1^ρ. Let $\varphi \in L_{1;\infty}^\rho$ and $x \in \mathcal{X}$. Then for all $t > 0$ one has

$$S_t U(\varphi)x = U(K_t * \varphi)x = U(S_t^L \varphi)x \quad .$$

Hence by the Duhamel formula

$$\begin{aligned} S_t U(\varphi)x - U(\varphi)x &= U(S_t^L\varphi - \varphi)x \\ &= -U\Big(\int_0^t ds\, H^L S_s^L \varphi\Big)x = -\int_0^t ds\, \Big(U(S_s^L H^L \varphi)x\Big) \quad . \end{aligned}$$

Therefore

$$\begin{aligned} \Big\|t^{-1}\Big(S_t U(\varphi)x - U(\varphi)x\Big) - U(H^L\varphi)x\Big\| &= \Big\|t^{-1}\int_0^t ds\, U(S_s^L H^L\varphi - H^L\varphi)x\Big\| \\ &\leq \sup_{0<s\leq t} \|S_s^L H^L \varphi - H^L\varphi\|_1^\rho \, \|x\| \quad . \end{aligned}$$

Since S^L is a continuous semigroup, it follows that $U(\varphi)x$ is in the domain of the generator H_S of S and

$$H_S U(\varphi)x = U(H^L\varphi)x = HU(\varphi)x \quad .$$

Taking for φ a smooth bounded approximation of the identity one deduces that H_S extends H. But the dense subspace of smooth elements of the representation, $\bigcap_\alpha D(A^\alpha)$, is invariant under S. Therefore $H_S = \overline{H}$ and the proof of the theorem is completed. □

One can also prove that H is closed on $L_p(G\,;dg)$ for $p \in \langle 1,\infty\rangle$. But this is not generally the situation for $p = 1$ or $p = \infty$. This is, however, not surprising since even in the Euclidean case, $G = \mathbf{R}^d$, it is well known that the Laplacian is not closed on L_1 or L_∞, i.e., the domain of the closed Laplacian contains functions which are not twice-differentiable in the natural sense. The fact that H is closed on L_2 does, however, improve the statement of Proposition 3.2. The operator H is self-adjoint.

Strong ellipticity was expressed as a restriction (1) on the coefficients c_α of the operator H. But it follows from the definition that $H - \lambda A^{\alpha_*} A^\alpha$ is also strongly elliptic for each multi-index $\alpha = (i_1, \ldots, i_{m/2})$, with $\alpha_* = (i_{m/2}, \ldots, i_1)$, and for each $\lambda \in \langle 0, \mu\rangle$ where μ is the ellipticity constant.

Therefore $\operatorname{Re} H - \lambda A^{\alpha *} A^{\alpha}$ is lower semibounded on $L_2(G\,; dg)$ by Proposition 3.2. This, however, implies that

$$\operatorname{Re}(\varphi, H\varphi) \geq \lambda \, \|A^{\alpha}\varphi\|_2^2 - \nu \, \|\varphi\|_2^2$$

for some $\nu \geq 0$ and all $\varphi \in C_c^{\infty}(G)$. Hence taking the supremum over α with $|\alpha| = m/2$ one deduces that H satisfies the Gårding inequality (3) on $L_2(G\,; dg)$. Thus strong ellipticity implies the Gårding inequality. But the converse implication can be established by evaluating the Gårding inequality with approximate local eigenfunctions of the operators A_i. Adopting the notation introduced prior to Proposition 3.1 one chooses $\varphi \in C_c^{\infty}(\Omega)$ with $\hat{\varphi}(x) = e^{i\xi \cdot x}\chi(x)$ where χ is a C^{∞}-function with support in a ball of radius r centred at the origin. Then using (8) and (9) the Gårding inequality for φ for large ξ and small r yields the strong ellipticity condition. Therefore *strong ellipticity is equivalent to the Gårding inequality.* This conclusion is of significance for the broader discussion of subelliptic operators.

4 Subelliptic operators

The theory of subelliptic operators is formulated in a similar fashion to the foregoing strongly elliptic theory but the vector space basis $a_1, \ldots, a_d$ of $\mathfrak{g}$ is replaced by an **algebraic basis** $a_1, \ldots, a_{d_1}$, i.e., a set of linearly independent elements whose Lie algebra spans $\mathfrak{g}$. Moreover, the main structural properties of the subelliptic operators are very similar to their strongly elliptic counterparts. Nevertheless, there are striking differences between the subelliptic and strongly elliptic theories. There is no Euclidean counterpart of an algebraic basis and the Euclidean group serves no useful role as a local approximation in the subelliptic setting. One can, however, exploit the parametrix method with the Euclidean group replaced by a non-commutative group which has a 'simple' algebraic structure related to the structure of the algebraic basis. We briefly outline some of the new and distinguishing features of the subelliptic theory. We begin with some geometric and algebraic features which are independent of the subelliptic operators but only relate to the underlying algebraic basis $a_1, \ldots, a_{d_1}$.

First, if the group G is viewed as a manifold the a_i correspond to directions on the manifold and one can define a distance relative to paths restricted to the directions of the algebraic basis. The ball of radius δ measured with respect to this subelliptic distance then behaves like δ^D as $\delta \to 0$ where D is an integer interpretable as the **local dimension** of the manifold relative to the basis. It is this dimension rather than the group dimension which determines the small time singularity of the subelliptic heat kernels. It can be calculated from the basis as follows.

Let $\mathfrak{g}_1$ denote the linear span of the algebraic basis $a_1, \ldots, a_{d_1}$ and $\mathfrak{g}_j$ the span of the algebraic basis together with the corresponding multiple commutators of order less than or equal to j. Then $\mathfrak{g}_1 \subset \mathfrak{g}_2 \subset \ldots \subset \mathfrak{g}_r = \mathfrak{g}$ where r is an integer and the smallest number with this property is referred to as the **rank** of the algebraic basis. Next set $V_1 = \mathfrak{g}_1$, and $V_j = \mathfrak{g}_j \backslash \mathfrak{g}_{j-1}$ for $j \in \{2, 3, \ldots\}$. This gives the direct sum decomposition

$$\mathfrak{g} = V_1 \oplus V_2 \oplus \cdots \oplus V_r$$

of the Lie algebra and

$$D = \sum_{j=1}^{r} j \dim V_j \quad .$$

Clearly, $D \geq d$ with equality if, and only if, $r = 1$. In particular this establishes that locally the subelliptic distance cannot be equivalent to the distance defined with a vector space basis. Nevertheless at large distances the two measures of separation are equivalent.

Secondly, one can associate a 'simpler' Lie algebra $\mathfrak{g}_0$ with $\mathfrak{g}$ by a contraction process which streamlines the algebraic structure. Define linear maps $\gamma_t : \mathfrak{g}_1 \mapsto \mathfrak{g}_1$ for each $t > 0$ such that $\gamma_t(a_i) = t a_i$ for $i \in \{1, 2, \ldots, d_1\}$. Then extend $a_1, \ldots, a_{d_1}$ to a basis of $\mathfrak{g}_2$ by adding elements $a_{d_1+1}, \ldots, a_{d_2}$ which are commutators of basis elements $a_1, \ldots, a_{d_1}$ and in addition extend γ_t to $\mathfrak{g}_2$ such that $\gamma_t(a_i) = t^2 a_i$ for $i \in \{d_1 + 1, \ldots, d_2\}$. Next extend $a_1, \ldots, a_{d_2}$ to a basis of $\mathfrak{g}_3$ by adding elements $a_{d_2+1}, \ldots, a_{d_3}$ which are double commutators of basis elements $a_1, \ldots, a_{d_1}$ and extend γ_t to $\mathfrak{g}_3$ such that $\gamma_t(a_i) = t^3 a_i$ for $i \in \{d_2 + 1, \ldots, d_3\}$. After repeating this process a finite number of times one obtains a vector space basis of $\mathfrak{g}$ and linear maps $\gamma_t : \mathfrak{g} \mapsto \mathfrak{g}$. Then $\mathfrak{g}_0$ is the **contraction** of $\mathfrak{g}$ with respect to γ, i.e., $\mathfrak{g}_0$ is the vector space $\mathfrak{g}$ equipped with the Lie bracket

$$[a, b]_0 = \lim_{t \downarrow 0} \gamma_t^{-1} [\gamma_t(a), \gamma_t(b)] \quad .$$

It follows that $\mathfrak{g}_0$ is a homogeneous Lie algebra with dilations $(\gamma_t)_{t>0}$ and $a_1, \ldots, a_{d_1}$ is an algebraic basis of $\mathfrak{g}_0$ with rank r and local dimension D.

Let G_0 be the connected, simply connected, Lie group with the Lie algebra $\mathfrak{g}_0$. It is the group G_0 which acts as the local approximation to G in the subelliptic theory. The simplifying feature of G_0 is the existence of the dilations γ_t, $t > 0$, which allow scaling arguments to extend local properties globally.

As an illustrative example suppose $a_1, \ldots, a_{d_1}$ is a vector space basis of $\mathfrak{g}$ then $\mathfrak{g}_1 = \mathfrak{g}$, $r = 1$, $D = d$ and $\gamma_t(a) = ta$ for all $a \in \mathfrak{g}$. Therefore $\gamma_t^{-1}[\gamma_t(a), \gamma_t(b)] = t[a,b] \to 0$ as $t \to 0$ and $\mathfrak{g}_0$ is abelian. Thus in this case $G_0 = \mathbf{R}^d$ and we return to the previous situation.

Now we turn to the definition and discussion of subelliptic operators.

Again we consider m-th order operators

$$H = \sum_{\alpha;\, |\alpha| \leq m} c_\alpha A^\alpha$$

on the spaces $L_p(G\,; dg)$ but now the multi-indices $\alpha = (i_1, \ldots, i_n)$ are formed from indices $i_j \in \{1, \ldots, d_1\}$ in the subelliptic directions. Now, however, the order of the products is more important since reordering a product A^α by use of the Lie algebraic relations introduces operators A_i in directions which are not contained in the algebraic basis. Therefore it is difficult to define a subelliptic analogue of strong ellipticity directly in terms of the coefficients c_α of H. But one can adopt the definition in terms of the Gårding inequality. Thus H is defined to be **subelliptic** on G if

$$\mathrm{Re}(\varphi, H\varphi) \geq \mu\, N_{m/2}(\varphi)^2 - \nu\, \|\varphi\|_2^2$$

for all $\varphi \in C_c^\infty(\mathbf{R}^d)$ where

$$N_k(\varphi) = \sup_{\alpha;\, |\alpha| = k} \|A^\alpha \varphi\|_2$$

and the supremum is restricted to multi-indices formed from the subelliptic directions.

The theory of subelliptic operators can now be developed in close analogy with the strongly elliptic theory. There are three main steps.

First, H is subelliptic on G if, and only if, it is subelliptic on the contracted group G_0. In the strongly elliptic case $G_0 = \mathbf{R}^d$ and this statement reiterates the equivalence of strong ellipticity and the Gårding inequality.

Secondly, one establishes that the closure of H generates a continuous semigroup S on the L_p-spaces over G_0 with a kernel K satisfying Gaussian bounds,

$$|(A^\alpha K_t)(g)| \le a_\alpha \, t^{-(D+|\alpha|)/m} e^{\omega t} e^{-b(|g|_1^m/t)^{1/(m-1)}}$$

for all $g \in G_0$ and $t > 0$ where $|\cdot|_1$ is the modulus associated with the subelliptic distance and the derivatives are in the subelliptic directions. The analogous step in the strongly elliptic case consisted of constructing S and K for the group $\mathbf{R}^d$. This was achieved in Section 2 by utilizing Fourier techniques. These methods are not applicable to G_0 and the proof is quite different. But the homogeneous structure on G_0, the existence of the dilations γ, is critical and can be viewed as the vestige of the Euclidean structure.

Finally, the properties of H on G_0 are extended to H on G by parametrix arguments. The main thrust of the reasoning is unchanged from the strongly elliptic case but the details are somewhat different.

Notes and remarks

Recent work on semigroups, kernels and kernel bounds, together with detailed references, is described in the books [Dav], [Rob] and [VSC]. The Lie group theory of strongly elliptic operators began with the unpublished thesis of Langlands [Lan] which established that these operators generate semigroups with smooth universal kernels. Langlands used parametrix arguments for the resolvent. His arguments, with some simplifications, are described in [Rob]. The current streamlined approach based on parametrix arguments for the kernel is abstracted from [ElR2] and [ElR1] but the essential convergence properties are taken from [BrR]. The derivation of Gaussian bounds via the parametrix expansion and weighted spaces differs from the now standard method introduced by Davies and described in the above books although the methods are related. The approach to subelliptic operators sketched in Section 4 is fully developed in [ElR3]. The local approximation by a nilpotent group is analogous to the method of Rothschild and Stein [RoS] but has the advantage that the dimension of the approximating group is the same as the original group, there are no additional dimensions.

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