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A direct numerical method for a class
of queueing problems

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0. Introduction

In the last years a lot of attention is given to queueing problems with
state dependent service and/or service in batches (see Rosenshine [2]).
Only for some of these problems it is possible to derive explicit expres-
sions for the stationary probabilities or other interesting quantities.
In this paper a numerical procedure is given for a rather general class of
these problems. The most important restrictions are that the arrivals has
to be one for one (but may be state dependent) and that the service mecha-
nism is negative exponential.
In a case where explicit expressions are available the computing time re-
quired for the calculation of these expressions was compared with the com-
puting time required for the direct numerical method. It turned out that
there was not much difference.

1. The model

Let (P,c) be a Markov chain with costs on the nonnegative integers $\mathbb{N}$, (P is
a Markov chain and c a nonnegative function on $\mathbb{N}$). The transition probabi-
lities are denoted by p_{ij}. Let f be an arbitrary real valued function on $\mathbb{N}$;
the function Pf is defined by $(Pf)(n) = \sum_{i=0}^{\infty} p_{ni} f(i)$, if this sum exists. If
c is interpreted as the expected costs in the first period, then $P^{n-1}c$ gives
the expected costs in the n-th period and $\sum_{n=0}^{\infty} P^{n}c$ the total expected costs.
The following assumptions are made:

i) $p_{ni} = 0$ for $i > n + 1$ and for all $n \in \mathbb{N}$.

$p_{nn+1} > 0$ for all $n \geq 1$.

$p_{00} = 1$, $c(0) = 0$.

ii) There is an integer N and positive numbers a and b such that $p_{nn+1} \leq a$
and $p_{nn} \leq 1 - 2a - b$ for $n \geq N$.

iii) For each n there is an $n_0 \geq n$ such that, starting in n_0, it is possible to reach state 0 without visiting $[n_0 + 1, \infty)$, that means, there is a sequence $n_0, n_1, n_2, \ldots, n_k$ with $n_0 > n_1 > n_2 > \ldots > n_k = 0$ such that $P_{n_i n_{i+1}} > 0$ for $i = 0, 1, \ldots, k-1$.

iv) There is an r, $\dfrac{a}{a + b} < r < 1$, such that $r^n c(n)$ is bounded on $\mathbb{N}$.

The state 0 is absorbing and the costs here are 0. We are interested in the total expected costs starting in some state n, $\sum\limits_{\ell=0}^{\infty} (P^\ell c)(n)$. In the next lemma the existence of this sum is proved and it is shown that $r^n \sum\limits_{\ell=0}^{\infty} (P^\ell c)(n)$ is bounded on $\mathbb{N}$.

For convenience we shall denote the space of all real valued functions f on $\mathbb{N}$ such that $r^n f(n)$ is bounded on $[k, \infty)$ by B_r^k and $\sup\limits_{n \geq k} |r^n f(n)|$ by $\| f \|_k$.

<u>Lemma.</u> The total costs starting in n, $\sum\limits_{\ell=0}^{\infty} (P^\ell c)(n)$ exist and $\sum\limits_{\ell=0}^{\infty} P^\ell c \in B_r^0$.

<u>Proof.</u> Let $N_0 > N$ be such that it is possible to reach state 0 from state $N_0 - 1$ without visiting $[N_0, \infty)$. First the expected costs until the first visit to $[0, N_0 - 1]$ are considered. Let $\widetilde{P} f$ for f an arbitrary function on $[N_0, \infty)$ be given by $(\widetilde{P} f)(n) = \sum\limits_{i=N_0}^{\infty} P_{ni} f(i)$.

Then $\sum\limits_{\ell=0}^{\infty} (\widetilde{P}^\ell c)(n)$, if existing, are the expected costs starting in n until the first visit to $[0, N_0 - 1]$. By assumption iv) the function c is an element of $B_r^{N_0}$. Let v be an arbitrary element of $B_r^{N_0}$, then for $n \geq N_0$

$$r^n | (\widetilde{P} f)(n)| = r^n \Big| \sum_{i=N_0}^{\infty} P_{ni} f(i) \Big| \leq r^n \sum_{i=N_0}^{\infty} P_{ni} \frac{\| f \|_{N_0}}{r^i} =$$

$$= \| f \|_{N_0} \Big(\frac{P_{nn+1}}{r} + P_{nn} + \sum_{i=N_0}^{n-1} P_{ni} r^{n-i} \Big) \leq$$

$$\leq \| f \|_{N_0} \Big\{ r + (1 - r)(P_{nn} + \frac{1 + r}{r} P_{nn+1}) \Big\} \leq$$

$$\leq \Big\{ r + (1 - r)(1 - 2a - b + \frac{1 + r}{r} a \Big\} \| f \|_{N_0} \ .$$

Hence $\widetilde{P}f$ is also an element of $B_r^{N_0}$ and $\| \widetilde{P}f \|_{N_0} \leq \rho \| f \|_{N_0}$ with

$$\rho := r + (1 - r)(1 - 2a - b + \frac{1 + r}{r} a) .$$

Since $1 > r > \frac{a}{a + b}$ the constant ρ is between 0 and 1 and therefore $\sum_{\ell=0}^{\infty} \widetilde{P}^{\ell} f$ is also an element of $B_r^{N_0}$ with $\| \sum_{\ell=0}^{\infty} \widetilde{P}^{\ell} f \|_{N_0} \leq \frac{1}{1 - \rho} \| f \|_{N_0}$. Hence

$$\sum_{\ell=0}^{\infty} (\widetilde{P}^{\ell} c)(n) \leq \frac{1}{r^n} \frac{1}{1 - \rho} \| c \|_{N_0} \quad \text{for } n \geq N_0 .$$

Once in $[0, N_0 - 1]$ there is a positive probability to reach state 0 without coming again in $[N_0, \infty)$. Therefore the total costs starting in n exist for each n and $\sum_{\ell=0}^{\infty} P^{\ell} c - \sum_{\ell=0}^{\infty} \widetilde{P}^{\ell} c$ is bounded on $[N_0, \infty)$. This implies $\sum_{\ell=0}^{\infty} P^{\ell} c \in B_r^0$. □

Now let $v := \sum_{\ell=0}^{\infty} P^{\ell} c$, then for $n \in \mathbb{N}$

(1) $\qquad v(n) = c(n) + \sum_{i=0}^{n+1} P_{ni} v(i) = c(n) + P_{nn+1} v(n + 1) + \sum_{i=0}^{n} P_{ni} v(i)$, or

(2) $\qquad v(n + 1) - v(n) = \frac{1}{P_{nn+1}} \{ \sum_{i=0}^{n-1} P_{ni} (v(n) - v(i)) - c(n) \} .$

Let $u(n) := v(n + 1) - v(n)$, $n \in \mathbb{N}$, then for $n \geq 1$

(3) $\qquad u(n) = \frac{1}{P_{nn+1}} \sum_{k=0}^{n-1} u(k) \sum_{i=0}^{k} P_{ki} - \frac{c(n)}{P_{nn+1}} .$

Hence $u(n)$ satisfies the following equation in x,

(4) $\qquad x(n) = \frac{1}{P_{nn+1}} \sum_{k=0}^{n-1} x(k) \sum_{i=0}^{k} P_{ni} - \frac{c(n)}{P_{nn+1}} , \quad n \geq 1 .$

Each solution of this equation is determined by $x(0)$. Let f be the solution with $f(0) = 0$ and let g be the solution of the homogeneous equation

$$x(n) = \frac{1}{P_{nn+1}} \sum_{k=0}^{n-1} x(k) \sum_{i=0}^{k} P_{ni}$$

with $g(0) = 1$. Then the general solution of (4) is $x = f + dg$ with d a con-

stant. Since u is a solution of (4), $u = f + d_u g$ with $d_u = u_0 = v(1)$. For g we have

$$g(n) = \frac{1}{P_{nn+1}} \sum_{k=0}^{n-1} g(k) \sum_{i=0}^{k} P_{ni} \geq \frac{g(n-1)}{P_{nn+1}} \sum_{i=0}^{n-1} P_{ni} .$$

Hence, if $n \geq N$, $g(n) \geq g(n-1)\frac{a+b}{a}$. Let N_0 be as in the proof of the lemma, then $g(N_0 - 1) > 0$. By the lemma $v(n)r^n$ is bounded, $v(n) \leq \frac{\|v\|_0}{r^n}$. The posibility of v yields

$$|u(n)| \leq \max(v(n+1),v(n)) \leq \frac{\|v\|_0}{r^{n+1}}$$

and since $r > \frac{a}{a+b}$ this implies $\lim_{n\to\infty} \frac{u(n)}{g(n)} = 0.$

This result can be used to construct an approximation procedure for $v(1)$. We had $u(n) = f(n) + v(1)g(n)$, hence $\frac{u(n)}{g(n)} = \frac{f(n)}{g(n)} + v(1)$. Since $\lim_{n\to\infty} \frac{u(n)}{g(n)} = 0$ this implies $v(1) = -\lim_{n\to\infty} \frac{f(n)}{g(n)}$.

It is easy to calculate $f(n)$ and $g(n)$. The quantity $-\frac{f(n)}{g(n)}$ can be used as approximation of $v(1)$. The error which is made is $\frac{|u(n)|}{g(n)} \leq \frac{\|v\|_0}{g(n)r^{n+1}}$.

Notice that $v(1) = u(0) = -\frac{f(k)}{g(k)}$ is also found if $u(k) = 0$ is substituted in (3), or $v(k+1) = v(k)$ in (2) or $v(k) = c(k) + P_{kk+1}v(k) + \sum_{i=0}^{k} P_{ki}v(i)$ in (1). This means that $-\frac{f(k)}{g(k)}$ is the approximation of $v(1)$ which is gotten if the transition $k \to k+1$ is replaced by a transition $k \to k$.

2. The quality of the approximation

In section 1 we found that the error of $-\frac{f(n)}{g(n)}$ as approximation of $v(1)$ is not larger than $\frac{\|v\|_0}{r^{n+1}}$. However, it is rather difficult to express this bound in the parameters of the process. If there is an integer q such that $p_{ni} = 0$ for $i < n - q$ it is possible to derive bounds for the error in $f(n)$ and $g(n)$. Let N_1 be such that $g(n) > 0$ for $n \geq N_1$. Define

$$f^*(n) := \frac{f(n)}{g(n)} , \quad c^*(n) = \frac{c(n)}{P_{nn+1}g(n)} , \quad p^*_{nk} = \frac{g(k)\sum_{i=0}^{k} P_{ni}}{g(n)P_{nn+1}} \quad \text{for } n,k \geq N_1.$$

Then

$$f^*(n) = \sum_{k=n-q}^{n-1} f^*(k) p_{nk}^* - c^*(n) \quad \text{for } n \geq N_1 + q .$$

Further $\sum_{k=0}^{n-1} p_{nk}^* = 1.$

Hence $f^*(n)$ for $n \geq N_1 + q$ can be interpreted as the total expected costs in a Markov process with transition probabilities p_{ij}^* which is stopped as soon as a state n with $n < N_1 + q$ is reached. In each state $n \geq N_1 + q$ the costs are $-c^*(n)$, the costs of stopping the process in $n < N_1 + q$ are $f^*(n)$.

It is easy to see now that for $n > N_2 \geq N_1 + q$ the costs $f^*(n)$ can be written as $f^*(n) = C_{2n} + R_{2n}$, where C_{2n} are the expected costs until a state $k < N_2$ is reached and R_{2n} the rest of the costs.

Let $p_k(n)$ be the probability that k is the first state in $[0, N_2 - 1]$, then

$$R_{2n} = \sum_{k=N_2-q}^{N_2-1} p_k(n) f^*(k) .$$

Hence

$$f^*(n) = C_{2n} + \sum_{k=N_2-q}^{N_2-1} p_k(n) f^*(k)$$

and

$$\sum_{k=N_2-q}^{N_2-1} p_k(n) \{f^*(n) - f^*(k)\} = C_{2n} .$$

Let $f^*(n)$ attain its maximum on $[N_2 - q, N_2 - 1]$ in k_h and its minimum in k_ℓ.
Then

$$f^*(n) - f^*(k_h) \leq C_{2n} \quad \text{and} \quad f^*(n) - f^*(k_\ell) \geq C_{2n} .$$

The process moves in each transition at least one step to the left, hence

$$0 \geq C_{2n} \geq - \sum_{j=N_2}^{n} c^*(j) .$$

Since $v(1) = -\lim_{n \to \infty} f^*(n)$ this gives the following bounds on $v(1)$

$$-f^*(k_h) \le v(1) \le -f^*(k_\ell) + \sum_{j=N_2}^{\infty} c^*(j) \; .$$

In most cases it is not so difficult to give a bound on $\sum_{j=N_2}^{\infty} c^*(j)$. We had

$$c^*(j) = \frac{c(j)}{p_{jj+1}g(j)} = \frac{c(j)}{\sum_{k=0}^{j-1} g(k) \sum_{i=0}^{k} p_{ji}} \le \frac{c(j)}{g(j-1) \sum_{i=0}^{j-1} p_{ji}} \; .$$

Let N_2 be chosen such that $N_2 > N_0$ (see the proof of the lemma), then

$$c^*(j) \le \frac{c(j)}{g(j-1)(a+b)} \qquad \text{for } j \ge N_2$$

while

$$g(j) \ge g(N_2 - 1)\left(\frac{a+b}{a}\right)^{j+1-N_2} \qquad \text{for } j \ge N_2 \; .$$

If $c(j)$ is bounded for instance we get the following result,

$$\sum_{j=N_2}^{\infty} c^*(j) \le \frac{B}{bg(N_2 - 1)} \; ,$$

where B is an upper bound of $c(j)$.

Remark. The interpretation of $f^*(n)$ as the total expected costs in a Markov chain is perhaps also of some analytical use. If for instance $q = 1$ and $p_{nn-1} > 0$ for all $n > 0$ then $g(n) > 0$ for all n and $p^*_{nn-1} = 1$ for $n > 0$. Hence $f^*(n) = f^*(n-1) - c^*(n)$ for all $n > 0$ and

$$v(1) = \sum_{1}^{\infty} c^*(n) = \sum_{1}^{\infty} \frac{c(n)}{g(n)p_{nn+1}} \; ,$$

where $g(\cdot)$ is given by $g(n) = \frac{p_{nn-1}}{p_{nn+1}} g(n-1)$.

3. Applications

This method can be applied to those queueing systems where the customers arrive one for one and where the service mechanism is negative exponential. State dependent service- and arrival rates and batch service, or other types of not one for one service are allowed. In these cases the embedded Markov chain which describes the number of customers at the moments of arrival has the prescribed structure. The stochastic process $X(t)$ of the number of customers at time t is a regenerative process (see Ross [3]). The cycle is the epoch between two subsequent times that an arriving customer comes in an empty system. The quotient of the expected cycle costs, EC, and the expected cycle time, ET, is in general equal to the average costs. Let c(n) be the expected costs per time in state n. If we set $c(i) = 0$ for $i \neq k$ and c(k) equal to the expected time in state k (until the first transition), then the average "costs", $\frac{EC}{ET}$, are equal to the probability that the system is in state k. The quantities EC and ET can be calculated with aid of the above described method.

A special case where also analytical results are known is that of a queue with Poisson arrival, states dependent batch service, infinite many services, and negative exponential service time. This queue is equivalent with an inventory system with Poisson arrival of customers, an (s,q) order strategy and negative exponentially distributed lead time (see also Wijngaard and van Winkel [4]). Galliher, Morse and Simond [1] derived explicit expressions for the stationary probabilities. We compared the calculation of these expressions with the direct numerical method and found about the same computing times. Notice that in this case the results of section 2 could be applied.

References

[1] Galliher, Morse and Simond, (1959), The dynamics of two classes of continuous - review inventory systems, Operations Res. 7, (362-384).

[2] Rosenshine, (1975), Queueing theory: The state of the art, AIIE transactions 7 (257-267).

[3] Ross (1970), Applied probability models with optimization applications, Holden-Day, San Francisco.

[4] Wijngaard, van Winkel (1974), Average number of back order in a continuous review (s,S) inventory system with exponentially distributed demand, presented at EURO I, Brussel.