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# ON THE SIZE OF SYSTEMS OF SETS EVERY $t$ OF WHICH HAVE AN SDR, WITH AN APPLICATION TO THE WORST-CASE RATIO OF HEURISTICS FOR PACKING PROBLEMS* 

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#### Abstract

Let $E_{1}, \cdots, E_{m}$ be subsets of a set $V$ of size $n$, such that each element of $V$ is in at most $k$ of the $E_{i}$ and such that each collection of $t$ sets from $E_{1}, \cdots, E_{m}$ has a system of distinct representatives (SDR). It is shown that $m / n \leqq\left(k(k-1)^{r}-k\right) /\left(2(k-1)^{r}-k\right)$ if $t=2 r-1$, and $m / n \leqq\left(k(k-1)^{r}-2\right)$ ) $\left(2(k-1)^{r}-2\right)$ if $t=2 r$. Moreover it is shown that these upper bounds are the best possible. From these results the "worst-case ratio" of certain heuristics for the problem of finding a maximum collection of pairwise disjoint sets among a given collection of sets of size $k$ is derived.


Key words. packing, system of distinct representatives, worst-case ratio, heuristics
AMS(MOS) subject classifications. $05 \mathrm{C} 65,05 \mathrm{~A} 05,90 \mathrm{C} 27$

1. Introduction. We prove the following theorem, where $m, n, k$, and $t$ are positive integers, with $k \geqq 3$.

Theorem 1. Let $E_{1}, \cdots, E_{m}$ be subsets of the set $V$ of size $n$, such that we have the following:
(1) (i) Each element of $V$ is contained in at most $k$ of the sets $E_{1}, \cdots, E_{m}$;
(ii) Any collection of at most $t$ sets among $E_{1}, \cdots, E_{m}$ has a system of distinct representatives.

Then, we have the following:

$$
\begin{array}{ll}
\text { (i) } \frac{m}{n} \leqq \frac{k(k-1)^{r}-k}{2(k-1)^{r}-k} & \text { if } t=2 r-1 ;  \tag{2}\\
\text { (ii) } \frac{m}{n} \leqq \frac{k(k-1)^{r}-2}{2(k-1)^{r}-2} & \text { if } t=2 r .
\end{array}
$$

Note that by the König-Hall Theorem, condition (1)(ii) can be replaced by the following:
(3) For any $s \leqq t$, any $s$ of the sets among $E_{1}, \cdots, E_{m}$ cover at least $s$ elements of $V$.

We give a proof of Theorem 1 in § 2 . We also show that the bounds given in (2) are best possible in the following sense.

Theorem 2. For any fixed $k, t$ (with $k \geqq 3$ ), there exist $m, n$ and $E_{1}, \cdots, E_{m} \subseteq$ $V($ with $|V|=n)$ satisfying (1) and having equality in the appropriate line of (2).

The proof of Theorem 2 is based on a construction using regular graphs of large girth (see § 3).

Finally, in $\S 4$ we apply these results to derive the worst-case ratio of certain heuristic algorithms for the problem of finding a largest family of pairwise disjoint sets among a given family of sets of size $k$ (this problem is NP-complete for any $k \geqq 3$ ).

[^1]2. Proof of Theorem 1. To show Theorem 1, we first give a lemma. Let $E_{1}, \cdots$, $E_{m}$ be a collection of finite nonempty sets, which we order so that $\left|E_{1}\right|, \cdots,\left|E_{h}\right| \geqq 2$ and $\left|E_{h+1}\right|=\cdots=\left|E_{m}\right|=1$, for some $h \leqq m$. We define a new collection as follows. Let
\[

$$
\begin{equation*}
W:=E_{h+1} \cup \cdots \cup E_{m} \tag{4}
\end{equation*}
$$

\]

Let for each $i=1, \cdots, h, X_{i}$ be a set of size $\left|E_{i}\right|-2$, disjoint from $E_{1} \cup \cdots \cup E_{m}$ and so that if $i \neq j$ then $X_{i} \cap X_{j}=\varnothing$. Let $X_{1} \cup \cdots \cup X_{h}=:\left\{y_{1}, \cdots, y_{q}\right\}$. Then the derived collection of sets is formed by the following sets:

$$
\begin{equation*}
\left(E_{1} \backslash W\right) \cup X_{1}, \cdots,\left(E_{h} \backslash W\right) \cup X_{h},\left\{y_{1}\right\}, \cdots,\left\{y_{q}\right\} \tag{5}
\end{equation*}
$$

Furthermore, we define a collection $E_{1}, \cdots, E_{m}$ to have the $t$-SDR-property if any $t$ sets among $E_{1}, \cdots, E_{m}$ have a system of distinct representatives.

Lemma. For $t \geqq 3$, if $E_{1}, \cdots, E_{m}$ has the $t$-SDR-property, then the derived collection (5) has the ( $t-2$ )-SDR-property.

Proof. Suppose (5) does not have the ( $t-2$ )-SDR-property. Then there exists a collection $\Pi$ of $p$ sets among (5) covering at most $p-1$ elements, for some $p \leqq t-2$. Assume we have chosen $p$ minimal. This immediately implies the following:
(i) $|\cup \Pi|=p-1$;
(ii) Each element in $\cup \Pi$ is covered by at least two sets in $\Pi$.

From (6)(ii) we directly have for any $i=1, \cdots, h$ and $x \in X_{i}$ :

$$
\begin{equation*}
\{x\} \in \Pi \Leftrightarrow\left(E_{i} \backslash W\right) \cup X_{i} \in \Pi . \tag{7}
\end{equation*}
$$

Without loss of generality, all sets $\left(E_{1} \backslash W\right) \cup X_{1}, \cdots,\left(E_{h} \backslash W\right) \cup X_{h}$ belong to II (as we can delete all sets $E_{j}$ from $E_{1}, \cdots, E_{h}$ for which $\left(E_{j} \backslash W\right) \cup X_{j} \notin \Pi$ ), and without loss of generality, $\left(E_{1} \cup \cdots \cup E_{h}\right) \cap W=E_{h+1} \cup \cdots \cup E_{m}$.

Note the following:

$$
\begin{align*}
& q=\left|X_{1} \cup \cdots \cup X_{h}\right|=\sum_{i=1}^{h}\left(\left|E_{i}\right|-2\right), \quad p=h+q,  \tag{8}\\
& \left|\bigcup_{i=1}^{h}\left(E_{i} \backslash W\right)\right|=|\cup \Pi|-q=(p-1)-q=h-1 .
\end{align*}
$$

So,

$$
\begin{equation*}
\left|\bigcup_{i=1}^{m} E_{i}\right|=\left|\bigcup_{i=1}^{h}\left(E_{i} \cap W\right)\right|+\left|\bigcup_{i=1}^{h}\left(E_{i} \backslash W\right)\right|=(m-h)+(h-1)=m-1 . \tag{9}
\end{equation*}
$$

Moreover, by (6)(ii), $\sum_{i=1}^{h}\left|E_{i} \backslash W\right| \geqq 2 \cdot\left|\cup_{i=1}^{h}\left(E_{i} \backslash W\right)\right|$, and hence

$$
\begin{align*}
m & =h+\left|\bigcup_{i=1}^{h}\left(E_{i} \cap W\right)\right| \leqq h+\sum_{i=1}^{h}\left|E_{i} \cap W\right|=h+\sum_{i=1}^{h}\left|E_{i}\right|-\sum_{i=1}^{h}\left|E_{i} \backslash W\right| \\
& \leqq h+\sum_{i=1}^{h}\left|E_{i}\right|-2 \cdot\left|\bigcup_{i=1}^{h}\left(E_{i} \backslash W\right)\right|=h+2 h+\sum_{i=1}^{h}\left(\left|E_{i}\right|-2\right)-2(h-1)  \tag{10}\\
& =h+2 h+q-2(h-1)=h+q+2=p+2 \leqq t .
\end{align*}
$$

Inequalities (9) and (10) contradict the fact that $E_{1}, \cdots, E_{m}$ has the $t$-SDR-property.

Proof of Theorem 1. We prove Theorem 1 by induction on $t$.
Case 1. $t=1$. Then we have that each of $E_{1}, \cdots, E_{m}$ is nonempty, and hence $m \leqq \sum_{i=1}^{m}\left|E_{i}\right| \leqq k n$, by (1)(i).

Case 2. $t=2$. Then we have that each of $E_{1}, \cdots, E_{m}$ is nonempty, and that no two of the singletons among $E_{1}, \cdots, E_{m}$ are the same. Without loss of generality, let $E_{h+1}, \cdots, E_{m}$ be the singletons among $E_{1}, \cdots, E_{m}$. Then $m-h \leqq n$, and

$$
\begin{equation*}
m+h=2 h+(m-h) \leqq \sum_{i=1}^{h}\left|E_{i}\right|+\sum_{i=h+1}^{m}\left|E_{i}\right|=\sum_{i=1}^{m}\left|E_{i}\right| \leqq k n \tag{11}
\end{equation*}
$$

(by (1)(i)). Hence $2 m=(m-h)+(m+h) \leqq(k+1) n$, and (2) follows.
Case 3. $t \geqq 3$. Then consider the derived collection $E_{1}^{\prime}, \cdots, E_{m^{\prime}}^{\prime}$ on $V^{\prime}:=$ $\cup_{i=1}^{m^{\prime}} E_{i}^{\prime}$ as in (5). Note that $m^{\prime}=h+q$ and $n^{\prime}:=\left|V^{\prime}\right|=n-|W|+q$. Denote the right-hand side term in (2) by $\varphi(k, t)$.

As by the lemma above, $E_{1}^{\prime}, \cdots, E_{m^{\prime}}^{\prime}$ has the $(t-2)$-SDR-property, and as trivially each element of $V^{\prime}$ is in at most $k$ of the sets $E_{1}^{\prime}, \cdots, E_{m^{\prime}}^{\prime}$ we have by induction that $m^{\prime} \leqq \varphi(k, t-2) n^{\prime}$. That is,

$$
\begin{equation*}
h+q \leqq \varphi(k, t-2)(n-|W|+q) . \tag{12}
\end{equation*}
$$

Writing the terms in different order, we have

$$
\begin{equation*}
\varphi(k, t-2)|W|+h-(\varphi(k, t-2)-1) q \leqq \varphi(k, t-2) n . \tag{13}
\end{equation*}
$$

Moreover, as $E_{1}, \cdots, E_{m}$ cover any element at most $k$ times:

$$
\begin{equation*}
|W|+2 h+q=|W|+2 h+\sum_{i=1}^{h}\left(\left|E_{i}\right|-2\right)=|W|+\sum_{i=1}^{h}\left|E_{i}\right|=\sum_{i=1}^{m}\left|E_{i}\right| \leqq k n . \tag{14}
\end{equation*}
$$

Hence,

$$
\begin{align*}
m= & h+|W| \\
= & \frac{1}{2 \varphi(k, t-2)-1}(\varphi(k, t-2)|W|+h-(\varphi(k, t-2)-1) q)  \tag{15}\\
& +\frac{\varphi(k, t-2)-1}{2 \varphi(k, t-2)-1}(|W|+2 h+q) \\
\leqq & \frac{1}{2 \varphi(k, t-2)-1} \varphi(k, t-2) n+\frac{\varphi(k, t-2)-1}{2 \varphi(k, t-2)-1} k n \\
= & \frac{(k+1) \varphi(k, t-2)-k}{2 \varphi(k, t-2)-1} n=\varphi(k, t) n .
\end{align*}
$$

The last equality follows directly by substituting the corresponding right-hand side of (2).
3. Proof of Theorem 2. To prove Theorem 2 we use a result of Erdös and Sachs [1]:
(16) For every $k$ and $\gamma$ there exists a $k$-regular graph of girth $\gamma$.

As a consequence of (16) we have the following:
(17) For every $k, s$, and $\gamma$ there exists a bipartite graph of girth at least $\gamma$, with color classes $U$ and $W$, say, such that each vertex in $U$ has degree $k$, and each vertex in $W$ has degree $s$.
(To see that (17) follows from (16), let $H$ be a $2 k s$-regular graph of girth $\gamma$. Consider any Eulerian orientation of the edges of $H$ (i.e., one for which all indegrees and outdegrees equal $k s)$. Split each vertex $v$ into $k+s$ vertices $v_{1}, \cdots, v_{k}, w_{1}, \cdots, w_{s}$ and divide the arcs entering $v$ equally over $v_{1}, \cdots, v_{k}$ and divide the arcs leaving $v$ equally over $w_{1}, \cdots$, $w_{s}$. Forgetting the orientations, we obtain a bipartite graph with the required properties.)

Now choose $k, t$. Let $r:=\left\lfloor\frac{1}{2} t\right\rfloor$. Consider the tree $T$, with vertices $1,2, \cdots, 1+$ $(k-1)+(k-1)^{2}+\cdots+(k-1)^{r-1}$, so that for $i<j$, vertices $i$ and $j$ are connected by an edge, if and only if $(k-1) i \leqq j \leqq(k-1) i+(k-2)$. So each vertex has degree $k$, except for vertex 1 , which has degree $k-1$, and for the vertices $1+(k-1)+\cdots+$ $(k-1)^{r-2}+1, \cdots, 1+(k-1)+\cdots+(k-1)^{r-1}$, which have degree one.

First let $t$ be even. Let $G$ be a $(k-1)^{r}$-regular graph of girth $t+1$ (cf. (16)). Let $G$ have $p$ vertices: $v_{1}, \cdots, v_{p}$. Consider $p$ copies $T_{1}, \cdots, T_{p}$ of $T$ (denoting the copy of vertex $i$ in $T_{j}$ by $i_{j}$ ). For each $j=1, \cdots, p$, partition the set of $(k-1)^{r}$ edges of $G$ incident to $v_{j}$ (arbitrarily) into $(k-1)^{r-1}$ classes of size $k-1$, and connect them to the $(k-1)^{r-1}$ vertices $i_{j}$ in $T_{j}$ of degree one. So the final graph $H=(W, F)$ has all degrees equal to $k$, except for the vertices $1_{1}, \cdots, 1_{p}$, which have degree $k-1$. Let $E_{1}, \cdots$, $E_{m}$ be the collection $F \cup\left\{\left\{1_{1}\right\}, \cdots,\left\{1_{p}\right\}\right\}$. This collection clearly satisfies (1)(i), and direct counting shows equality in (2)(ii). To see that the collection satisfies (1)(ii), let $E_{1}, \cdots, E_{s}$ form a subcollection with $\left|E_{1} \cup \cdots \cup E_{s}\right|<s$ and $s$ as small as possible. Suppose $s \leqq t$. As $E_{1}, \cdots, E_{s}$ must form a connected hypergraph, it contains at most one singleton (since any path between $1_{i}$ and $1_{j}$ in $H$ contains at least $t-1$ edges). So assume $E_{2}, \cdots, E_{s}$ are edges of $H$. Then they do not contain any circuit (as each $T_{i}$ is a tree and as $G$ has girth $t+1>s$ ). So $\left|E_{2} \cup \cdots \cup E_{s}\right| \geqq s$, a contradiction.

Next let $t$ be odd. Let $G$ be a bipartite graph, of girth at least $t+1$, so that in one color class $U$ each vertex has degree $(k-1)^{r}$ and in the other color class $W$ each vertex has degree $k$. Let $U=:\left\{u_{1}, \cdots, u_{p}\right\}$. Consider again $p$ copies $T_{1}, \cdots, T_{p}$ of $T$, as above. For $j=1, \cdots, p$ partition the set of $(k-1)^{r}$ edges of $G$ incident to $u_{j}$ (arbitrarily) into $(k-1)^{r-1}$ classes of size $k-1$, and connect them to the $(k-1)^{r-1}$ vertices $i_{j}$ in $T_{j}$ of degree one. Again, the final graph $H=(W, F)$ has all degrees equal to $k$, except for the vertices $1_{1}, \cdots, 1_{p}$ that have degree $k-1$. Let $E_{1}, \cdots, E_{m}$ be the collection $F \cup\left\{\left\{1_{1}\right\}, \cdots,\left\{1_{p}\right\}\right\}$. Similarly, as above, we show that this collection satisfies (1) and has equality in (2)(i).
4. Application to the worst-case ratio of heuristics. The problem of finding a largest collection of pairwise disjoint sets among a given collection $X_{1}, \cdots, X_{q}$ of $k$-sets is NPcomplete, for any $k \geqq 3$. Call any collection of pairwise disjoint sets a packing.

For any fixed $s$, we can apply the following heuristic algorithm $H_{s}$. Start with the empty packing. If we have found a packing $Y_{1}, \cdots, Y_{n}$ from $X_{1}, \cdots, X_{q}$, we could select $p \leqq s$ sets among $Y_{1}, \cdots, Y_{n}$, and replace them by $p+1$ sets from $X_{1}, \cdots, X_{q}$, so that the arising collection is a packing with $n+1$ sets. Repeating this, the algorithm terminates with a collection $Y_{1}, \cdots, Y_{n}$ so that

For each $p \leqq s$, the union of any $p+1$ pairwise disjoint sets among $X_{1}, \cdots$, $X_{q}$ intersects at least $p+1$ sets among $Y_{1}, \cdots, Y_{n}$.

This defines heuristic $H_{s}$, which is, for any fixed $s$, a polynomial-time algorithmhowever it clearly need not lead to a largest packing. We might ask how far the packing found with $H_{s}$ is from the largest packing.

To this end, consider a largest packing $Z_{1}, \cdots, Z_{m}$ from $X_{1}, \cdots, X_{q}$. We claim that $m / n$ satisfies the bounds given in (2), taking $t:=s+1$, and that these bounds are best possible. That is, the "worst-case ratio" of the heuristic is given in (2).

Indeed, let

$$
\begin{equation*}
V:=\left\{Y_{1}, \cdots, Y_{n}\right\} \quad \text { and } \quad E_{i}:=\left\{Y_{j} \mid Y_{j} \cap Z_{i} \neq 0\right\} \quad \text { for } i=1, \cdots, m . \tag{19}
\end{equation*}
$$

Then by (18), $E_{1}, \cdots, E_{m}$ satisfy (1), and hence we obtain the bounds given in (2).
In turn, it is not difficult to see that for any collection $E_{1}, \cdots, E_{m}$ of sets of size at most $k$, containing any point at most $k$ times, we can assume they are of form (19) for certain packings $Y_{1}, \cdots, Y_{n}$ and $Z_{1}, \cdots, Z_{m}$ of $k$-sets. Thus starting with $E_{1}, \cdots$, $E_{m}$ as described in § 3 above, making these $Y_{1}, \cdots, Y_{n}, Z_{1}, \cdots, Z_{m}$, and taking $\left\{X_{1}, \cdots, X_{q}\right\}:=\left\{Y_{1}, \cdots, Y_{n}, Z_{1}, \cdots, Z_{m}\right\}$, we obtain a system of sets attaining the worst-case ratio. (That is because we may assume that $H_{s}$ selects the sets $Y_{1}, \cdots, Y_{n}$ in the first $n$ iterations.)

Note that we may assume even that the sets $Y_{1}, \cdots, Y_{n}, Z_{1}, \cdots, Z_{m}$ form the collection of all cliques of size $k$ in a graph. Hence, we cannot obtain a better worst-case ratio by restricting the collections of sets to collections of $k$-cliques.

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