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## ON THE SIZE OF SYSTEMS OF SETS EVERY t OF WHICH HAVE AN SDR, WITH AN APPLICATION TO THE WORST-CASE RATIO OF HEURISTICS FOR PACKING PROBLEMS\*

C. A. J. HURKENS† AND A. SCHRIJVER‡

Abstract. Let  $E_1, \dots, E_m$  be subsets of a set V of size n, such that each element of V is in at most k of the  $E_i$  and such that each collection of t sets from  $E_1, \dots, E_m$  has a system of distinct representatives (SDR). It is shown that  $m/n \leq (k(k-1)^r - k)/(2(k-1)^r - k)$  if t = 2r - 1, and  $m/n \leq (k(k-1)^r - 2)/(2(k-1)^r - 2)$  if t = 2r. Moreover it is shown that these upper bounds are the best possible. From these results the "worst-case ratio" of certain heuristics for the problem of finding a maximum collection of pairwise disjoint sets among a given collection of sets of size k is derived.

Key words. packing, system of distinct representatives, worst-case ratio, heuristics

AMS(MOS) subject classifications. 05C65, 05A05, 90C27

1. Introduction. We prove the following theorem, where m, n, k, and t are positive integers, with  $k \ge 3$ .

THEOREM 1. Let  $E_1, \dots, E_m$  be subsets of the set V of size n, such that we have the following:

- (1) (i) Each element of V is contained in at most k of the sets  $E_1, \dots, E_m$ ;
  - (ii) Any collection of at most t sets among  $E_1, \dots, E_m$  has a system of distinct representatives.

Then, we have the following:

(2) (i) 
$$\frac{m}{n} \leq \frac{k(k-1)^r - k}{2(k-1)^r - k}$$
 if  $t = 2r - 1$ ;  
(ii)  $\frac{m}{n} \leq \frac{k(k-1)^r - 2}{2(k-1)^r - 2}$  if  $t = 2r$ .

Note that by the König-Hall Theorem, condition (1)(ii) can be replaced by the following:

(3) For any  $s \le t$ , any s of the sets among  $E_1, \dots, E_m$  cover at least s elements of V.

We give a proof of Theorem 1 in § 2. We also show that the bounds given in (2) are best possible in the following sense.

THEOREM 2. For any fixed k, t (with  $k \ge 3$ ), there exist m, n and  $E_1, \dots, E_m \subseteq V$  (with |V| = n) satisfying (1) and having equality in the appropriate line of (2).

The proof of Theorem 2 is based on a construction using regular graphs of large girth (see  $\S$  3).

Finally, in § 4 we apply these results to derive the worst-case ratio of certain heuristic algorithms for the problem of finding a largest family of pairwise disjoint sets among a given family of sets of size k (this problem is NP-complete for any  $k \ge 3$ ).

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2. Proof of Theorem 1. To show Theorem 1, we first give a lemma. Let  $E_1, \dots, E_m$  be a collection of finite nonempty sets, which we order so that  $|E_1|, \dots, |E_h| \ge 2$  and  $|E_{h+1}| = \dots = |E_m| = 1$ , for some  $h \le m$ . We define a new collection as follows. Let

$$W:=E_{h+1}\cup\cdots\cup E_m.$$

Let for each  $i = 1, \dots, h, X_i$  be a set of size  $|E_i| - 2$ , disjoint from  $E_1 \cup \dots \cup E_m$  and so that if  $i \neq j$  then  $X_i \cap X_j = \emptyset$ . Let  $X_1 \cup \dots \cup X_h =: \{y_1, \dots, y_q\}$ . Then the *derived* collection of sets is formed by the following sets:

(5) 
$$(E_1 \setminus W) \cup X_1, \cdots, (E_h \setminus W) \cup X_h, \{y_1\}, \cdots, \{y_q\}$$

Furthermore, we define a collection  $E_1, \dots, E_m$  to have the *t*-SDR-*property* if any *t* sets among  $E_1, \dots, E_m$  have a system of distinct representatives.

LEMMA. For  $t \ge 3$ , if  $E_1, \dots, E_m$  has the t-SDR-property, then the derived collection (5) has the (t - 2)-SDR-property.

*Proof.* Suppose (5) does not have the (t-2)-SDR-property. Then there exists a collection II of p sets among (5) covering at most p-1 elements, for some  $p \le t-2$ . Assume we have chosen p minimal. This immediately implies the following:

(6) (i) 
$$|\cup \Pi| = p - 1;$$

(ii) Each element in  $\cup \Pi$  is covered by at least two sets in  $\Pi$ .

From (6)(ii) we directly have for any  $i = 1, \dots, h$  and  $x \in X_i$ :

(7) 
$$\{x\} \in \Pi \Leftrightarrow (E_i \setminus W) \cup X_i \in \Pi.$$

Without loss of generality, all sets  $(E_1 \setminus W) \cup X_1, \dots, (E_h \setminus W) \cup X_h$  belong to  $\Pi$ (as we can delete all sets  $E_j$  from  $E_1, \dots, E_h$  for which  $(E_j \setminus W) \cup X_j \notin \Pi$ ), and without loss of generality,  $(E_1 \cup \dots \cup E_h) \cap W = E_{h+1} \cup \dots \cup E_m$ .

Note the following:

(8) 
$$q = |X_1 \cup \cdots \cup X_h| = \sum_{i=1}^h (|E_i| - 2), \quad p = h + q,$$
$$\left| \bigcup_{i=1}^h (E_i \setminus W) \right| = |\cup \Pi| - q = (p-1) - q = h - 1.$$

So,

(9) 
$$\left| \bigcup_{i=1}^{m} E_{i} \right| = \left| \bigcup_{i=1}^{h} (E_{i} \cap W) \right| + \left| \bigcup_{i=1}^{h} (E_{i} \setminus W) \right| = (m-h) + (h-1) = m-1.$$

Moreover, by (6)(ii),  $\sum_{i=1}^{h} |E_i \setminus W| \ge 2 \cdot |\bigcup_{i=1}^{h} (E_i \setminus W)|$ , and hence

(10)  
$$m = h + \left| \bigcup_{i=1}^{h} (E_i \cap W) \right| \le h + \sum_{i=1}^{h} |E_i \cap W| = h + \sum_{i=1}^{h} |E_i| - \sum_{i=1}^{h} |E_i \setminus W|$$
$$\le h + \sum_{i=1}^{h} |E_i| - 2 \cdot \left| \bigcup_{i=1}^{h} (E_i \setminus W) \right| = h + 2h + \sum_{i=1}^{h} (|E_i| - 2) - 2(h - 1)$$
$$= h + 2h + q - 2(h - 1) = h + q + 2 = p + 2 \le t.$$

Inequalities (9) and (10) contradict the fact that  $E_1, \dots, E_m$  has the *t*-SDR-property.  $\Box$ 

*Proof of Theorem* 1. We prove Theorem 1 by induction on *t*.

Case 1. t = 1. Then we have that each of  $E_1, \dots, E_m$  is nonempty, and hence  $m \leq \sum_{i=1}^{m} |E_i| \leq kn$ , by (1)(i).

Case 2. t = 2. Then we have that each of  $E_1, \dots, E_m$  is nonempty, and that no two of the singletons among  $E_1, \dots, E_m$  are the same. Without loss of generality, let  $E_{h+1}, \dots, E_m$  be the singletons among  $E_1, \dots, E_m$ . Then  $m - h \leq n$ , and

(11) 
$$m+h=2h+(m-h) \leq \sum_{i=1}^{h} |E_i| + \sum_{i=h+1}^{m} |E_i| = \sum_{i=1}^{m} |E_i| \leq kn$$

(by (1)(i)). Hence  $2m = (m - h) + (m + h) \le (k + 1)n$ , and (2) follows.

Case 3.  $t \ge 3$ . Then consider the derived collection  $E'_1, \dots, E'_{m'}$  on  $V' := \bigcup_{i=1}^{m'} E'_i$  as in (5). Note that m' = h + q and n' := |V'| = n - |W| + q. Denote the right-hand side term in (2) by  $\varphi(k, t)$ .

As by the lemma above,  $E'_1, \dots, E'_{m'}$  has the (t-2)-SDR-property, and as trivially each element of V' is in at most k of the sets  $E'_1, \dots, E'_{m'}$  we have by induction that  $m' \leq \varphi(k, t-2)n'$ . That is,

(12) 
$$h+q \leq \varphi(k,t-2)(n-|W|+q).$$

Writing the terms in different order, we have

m = h + |W|

(13) 
$$\varphi(k,t-2) | W | + h - (\varphi(k,t-2)-1)q \leq \varphi(k,t-2)n.$$

Moreover, as  $E_1, \dots, E_m$  cover any element at most k times:

(14) 
$$|W| + 2h + q = |W| + 2h + \sum_{i=1}^{h} (|E_i| - 2) = |W| + \sum_{i=1}^{h} |E_i| = \sum_{i=1}^{m} |E_i| \le kn.$$

Hence,

(15)  

$$= \frac{1}{2\varphi(k,t-2)-1} (\varphi(k,t-2) | W | + h - (\varphi(k,t-2)-1)q) + \frac{\varphi(k,t-2)-1}{2\varphi(k,t-2)-1} (| W | + 2h + q)$$

$$\leq \frac{1}{2\varphi(k,t-2)-1} \varphi(k,t-2)n + \frac{\varphi(k,t-2)-1}{2\varphi(k,t-2)-1} kn$$

$$= \frac{(k+1)\varphi(k,t-2)-k}{2\varphi(k,t-2)-1} n = \varphi(k,t)n.$$

The last equality follows directly by substituting the corresponding right-hand side of (2).  $\Box$ 

3. Proof of Theorem 2. To prove Theorem 2 we use a result of Erdös and Sachs [1]:

(16) For every k and  $\gamma$  there exists a k-regular graph of girth  $\gamma$ .

As a consequence of (16) we have the following:

(17) For every k, s, and  $\gamma$  there exists a bipartite graph of girth at least  $\gamma$ , with color classes U and W, say, such that each vertex in U has degree k, and each vertex in W has degree s.

(To see that (17) follows from (16), let *H* be a 2ks-regular graph of girth  $\gamma$ . Consider any Eulerian orientation of the edges of *H* (i.e., one for which all indegrees and outdegrees equal ks). Split each vertex v into k + s vertices  $v_1, \dots, v_k, w_1, \dots, w_s$  and divide the arcs entering v equally over  $v_1, \dots, v_k$  and divide the arcs leaving v equally over  $w_1, \dots, w_s$ . Forgetting the orientations, we obtain a bipartite graph with the required properties.)

Now choose k, t. Let  $r := \lfloor \frac{1}{2}t \rfloor$ . Consider the tree T, with vertices 1, 2,  $\cdots$ , 1 +  $(k-1) + (k-1)^2 + \cdots + (k-1)^{r-1}$ , so that for i < j, vertices i and j are connected by an edge, if and only if  $(k-1)i \le j \le (k-1)i + (k-2)$ . So each vertex has degree k, except for vertex 1, which has degree k - 1, and for the vertices  $1 + (k-1) + \cdots + (k-1)^{r-2} + 1$ ,  $\cdots$ ,  $1 + (k-1) + \cdots + (k-1)^{r-1}$ , which have degree one.

First let t be even. Let G be a  $(k-1)^r$ -regular graph of girth t + 1 (cf. (16)). Let G have p vertices:  $v_1, \dots, v_p$ . Consider p copies  $T_1, \dots, T_p$  of T (denoting the copy of vertex i in  $T_j$  by  $i_j$ ). For each  $j = 1, \dots, p$ , partition the set of  $(k-1)^r$  edges of G incident to  $v_j$  (arbitrarily) into  $(k-1)^{r-1}$  classes of size k-1, and connect them to the  $(k-1)^{r-1}$  vertices  $i_j$  in  $T_j$  of degree one. So the final graph H = (W, F) has all degrees equal to k, except for the vertices  $1_1, \dots, 1_p$ , which have degree k-1. Let  $E_1, \dots, E_m$  be the collection  $F \cup \{\{1_1\}, \dots, \{1_p\}\}$ . This collection clearly satisfies (1)(i), and direct counting shows equality in (2)(ii). To see that the collection satisfies (1)(ii), let  $E_1, \dots, E_s$  form a subcollection with  $|E_1 \cup \dots \cup E_s| < s$  and s as small as possible. Suppose  $s \leq t$ . As  $E_1, \dots, E_s$  must form a connected hypergraph, it contains at most one singleton (since any path between  $1_i$  and  $1_j$  in H contains at least t-1 edges). So assume  $E_2, \dots, E_s$  are edges of H. Then they do not contain any circuit (as each  $T_i$  is a tree and as G has girth t + 1 > s). So  $|E_2 \cup \dots \cup E_s| \geq s$ , a contradiction.

Next let t be odd. Let G be a bipartite graph, of girth at least t + 1, so that in one color class U each vertex has degree  $(k - 1)^r$  and in the other color class W each vertex has degree k. Let  $U =: \{u_1, \dots, u_p\}$ . Consider again p copies  $T_1, \dots, T_p$  of T, as above. For  $j = 1, \dots, p$  partition the set of  $(k - 1)^r$  edges of G incident to  $u_j$  (arbitrarily) into  $(k - 1)^{r-1}$  classes of size k - 1, and connect them to the  $(k - 1)^{r-1}$  vertices  $i_j$  in  $T_j$  of degree one. Again, the final graph H = (W, F) has all degrees equal to k, except for the vertices  $1_1, \dots, 1_p$  that have degree k - 1. Let  $E_1, \dots, E_m$  be the collection  $F \cup \{\{1_1\}, \dots, \{1_p\}\}$ . Similarly, as above, we show that this collection satisfies (1) and has equality in (2)(i).

4. Application to the worst-case ratio of heuristics. The problem of finding a largest collection of pairwise disjoint sets among a given collection  $X_1, \dots, X_q$  of k-sets is NP-complete, for any  $k \ge 3$ . Call any collection of pairwise disjoint sets a *packing*.

For any fixed s, we can apply the following heuristic algorithm  $H_s$ . Start with the empty packing. If we have found a packing  $Y_1, \dots, Y_n$  from  $X_1, \dots, X_q$ , we could select  $p \leq s$  sets among  $Y_1, \dots, Y_n$ , and replace them by p + 1 sets from  $X_1, \dots, X_q$ , so that the arising collection is a packing with n + 1 sets. Repeating this, the algorithm terminates with a collection  $Y_1, \dots, Y_n$  so that

(18) For each  $p \leq s$ , the union of any p + 1 pairwise disjoint sets among  $X_1, \dots, X_q$  intersects at least p + 1 sets among  $Y_1, \dots, Y_n$ .

This defines heuristic  $H_s$ , which is, for any fixed s, a polynomial-time algorithm however it clearly need not lead to a largest packing. We might ask how far the packing found with  $H_s$  is from the largest packing.

To this end, consider a largest packing  $Z_1, \dots, Z_m$  from  $X_1, \dots, X_q$ . We claim that m/n satisfies the bounds given in (2), taking t := s + 1, and that these bounds are best possible. That is, the "worst-case ratio" of the heuristic is given in (2).

Indeed, let

(19) 
$$V := \{Y_1, \dots, Y_n\}$$
 and  $E_i := \{Y_i | Y_i \cap Z_i \neq 0\}$  for  $i = 1, \dots, m$ .

Then by (18),  $E_1, \dots, E_m$  satisfy (1), and hence we obtain the bounds given in (2).

In turn, it is not difficult to see that for any collection  $E_1, \dots, E_m$  of sets of size at most k, containing any point at most k times, we can assume they are of form (19) for certain packings  $Y_1, \dots, Y_n$  and  $Z_1, \dots, Z_m$  of k-sets. Thus starting with  $E_1, \dots, E_m$  as described in § 3 above, making these  $Y_1, \dots, Y_n, Z_1, \dots, Z_m$ , and taking  $\{X_1, \dots, X_q\} := \{Y_1, \dots, Y_n, Z_1, \dots, Z_m\}$ , we obtain a system of sets attaining the worst-case ratio. (That is because we may assume that  $H_s$  selects the sets  $Y_1, \dots, Y_n$  in the first *n* iterations.)

Note that we may assume even that the sets  $Y_1, \dots, Y_n, Z_1, \dots, Z_m$  form the collection of all cliques of size k in a graph. Hence, we cannot obtain a better worst-case ratio by restricting the collections of sets to collections of k-cliques.

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