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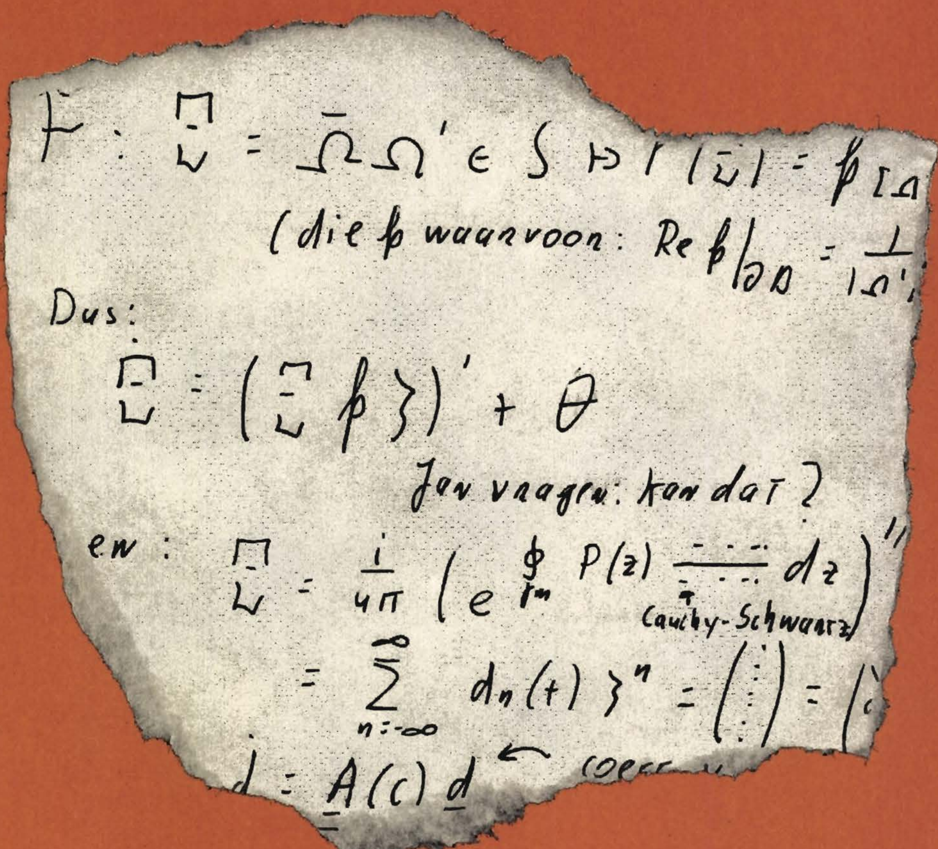

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MOVING BOUNDARY PROBLEMS IN RELATION WITH EQUATIONS OF LÖWNER-KUFAREEV TYPE

PROEFSCHRIFT

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Chapter 1

Introduction

One of the difficulties of writing mathematical texts is the paradox that a strict proof of an assertion is generally not its most straightforward deduction. Details may become important, a down-to-earth point of view is sometimes too limited and unavoidable side branches in a proof can confuse the reader. That is the reason why this chapter –in contrast with the Chapters 2-5– is written in a somewhat loose way. We do not avoid a didactic tone appealing to "mathematical intuition" when we summarize the content of this thesis and try to explain its aim. Nor do we go into details when we sketch the physical background and history of some of the problems. Finally we apologize for summarizing the contents of the Chapters 2-5 in a non-subsequential order.

1.1 General Overview of the thesis

This thesis is concerned with moving boundary problems. Such problems appear for example in fluid mechanics when a clump of matter is moving freely according to its hydrodynamic velocity. In particular, we will consider problems where this velocity mainly depends on the geometric shape of the matter at that time. We call such a momentaneous relation between the velocity field and the geometric shape quasi-static and present a mathematical formulation of quasi-static moving boundary problems in Section 3.1. After restricting ourselves to two-dimensional problems and after introducing a time-dependent conformal mapping from a reference domain to the domain occupied by the matter, we are led in Section 3.2 to what we call a quasi-linear Löwner-Kufareev equation. (We prefer to spell the latter name with two e's although it can also be written as "Kufarev".) This equation can be viewed as a non-linear version of the time-honored Löwner-Kufareev equation ([57, 49]) which was studied in the theory of subordination chains ([70, 9]) and which played a role in the proof of Bieberbach's conjecture ([14, 47]). The importance of the quasi-linear Löwner-Kufareev equation can be

illustrated by quoting an open question formulated during the 1994 Conference on Complex Analysis and Free Boundary Problems in St. Petersburg ([1]):
"A framework for solving moving boundary problems is as follows. ... Then the kinematic boundary condition for the normal velocity V_n of the free boundary $\partial\Omega(t)$ becomes the Löwner-Kufareev type equation... Can this formulation help with the solvability question and with analysis of the geometric properties of the moving boundary?..."
We try to make a contribution to the answer of this question in Section 5.4 where we prove the local solvability of quasi-linear Löwner-Kufareev equations under general conditions. (This section is published in a different form in [43].) The proof is based on some estimates on solutions of linear Löwner-Kufareev equations and an iteration technique. Should your main interest lie in the general relation between quasi-linear Löwner-Kufareev equations and quasi-static moving boundary problems, we suggest you to read Sections 3.1, 3.2, 5.1 and 5.4 only.
Apart from these general considerations we will study two particular moving boundary problems in more detail: the one for Stokes flow driven by surface tension and the one for Hele-Shaw flow. We postpone the introduction to these problems to the next section. It was R.W. Hopper who derived an equation – which we call Hopper's equation– for the first problem in his inspiring article "Plane Stokes flow driven by capillarity on a free surface" ([35]). In Section 3.3, which is based on [30], we show that if a time-dependent conformal mapping satisfies Hopper's equation, then it is a solution for Stokes flow driven by surface tension. In Section 3.4 we show that a time-dependent conformal mapping satisfies the moving boundary problem for Hele-Shaw flow if and only if this mapping satisfies an equation which resembles Hopper's equation. This leads, at the risk of confusion, to the introduction of the name "Hopper equations": equations which resemble the original one in a way made clear in Section 4.1. We realize that these names do not honour other mathematicians who recognized the significance of a formulation of certain moving boundary problems in terms of conformal mappings (see e.g. [78, 30]).
The core of this thesis is the study of Hopper equations in Chapters 4 and 5. This study is based on the theory of ordinary and partial differential equations, on complex function theory and on some functional analysis. The result is a set of propositions and theorems on the properties of solutions and on the existence and uniqueness of some classes of solutions. We also show a deep relationship between Hopper equations and certain partial differential equations, which we will call extended Löwner-Kufareev equations.
Before we summarize the contents of this study of Hopper equations in Section 1.3, we shall first discuss the physical background of the moving boundary problems mentioned.

1.2 Physical background

Most moving boundary problems in mathematical physics have the following constituents: the equations of motion, the dynamic boundary condition and the kinematic boundary condition. We present a short introduction to the moving boundary problem for Stokes flow driven by surface tension and multi-poles and for Hele-Shaw flow in the Subsections 1.2.2 and 1.2.3, respectively. We first discuss two types of equations of motion in Subsection 1.2.1.

1.2.1 Stokes' and Darcy's equations

The motion of a viscous Newtonian fluid can be described by the well-known Navier-Stokes equations and the continuity equation ([55, 87]). If the inertial and gravitational forces are negligible and the fluid is incompressible, these equations reduce to Stokes' equations:

$$\eta\Delta\underline{v}(\underline{x},t) = \nabla p(\underline{x},t) \qquad\qquad \nabla.\underline{v}(\underline{x},t) = 0$$

where η is the viscosity constant and where $\underline{v}(\underline{x},t)$ and $p(\underline{x},t)$ are the hydrodynamic velocity and the hydrostatic pressure at a point $\underline{x}$ at time t. From now on, we shall assume that the unit system is scaled such that $\eta = 1$. We return to the topic in which circumstances Stokes' equations are a good physical approximation in Subsection 1.2.2.
As time derivatives do not appear in Stokes' equations, we can suppress the variable t in the notation and write

$$\Delta\underline{v}(\underline{x}) = \nabla p(\underline{x}) \qquad\qquad \nabla.\underline{v}(\underline{x}) = 0 \tag{1.1}$$

We note that these equations make sense on any open subdomain of $I\!R^n$ for any $n \in I\!N$. Nevertheless, it is important to realize that the Laplacian of a vector field must be treated with care in order to give it a coordinate-independent meaning (see e.g. [2]). As long as Cartesian coordinates are used, no confusion can arise, but as we will also use cylindrical coordinates in Sections 2.2 and 2.4, it is convenient to rewrite Stokes' equations in a different notation:

$$g^{ij}(\underline{x})\nabla_i\nabla_j v^k(\underline{x}) = g^{ik}(\underline{x})\nabla_i p(\underline{x}) \qquad k = 1,..,n \tag{1.2a}$$

$$\nabla_i v^i(\underline{x}) = 0 \tag{1.2b}$$

where g is the first fundamental tensor, where ∇_i denotes the covariant derivative with respect to the variable x^i and where Einstein's summation convention is understood. The components of the stress tensor $\underline{\underline{T}}$ are given in this notation by

$$T^i_{\ j}(\underline{x}) = -p(\underline{x})\delta^i_j + \nabla_j v^i(\underline{x}) + g^{ik}(\underline{x})g_{lj}(\underline{x})\nabla_k v^l(\underline{x}) \qquad i,j = 1,..,n \tag{1.3}$$

We now turn to other equations of motions. It has been experimentally verified that a fluid in a porous medium can be described by Darcy's law ([61, 68, 16]):

$$\underline{v}(\underline{x}, t) = -\nu \nabla p(\underline{x}, t)$$

where ν is the effective permeability. From now on, we will assume that the unit system is scaled such that $\nu = 1$. If the fluid is incompressible, the continuity equation reduces to

$$\nabla . \underline{v}(\underline{x}, t) = 0$$

We will refer to these equations of motions as Darcy's equations. The reader who is interested in the circumstances in which Darcy's equations are a good physical approximation is referred to [61]. As time derivatives do not appear in Darcy's equations, we can suppress the variable t in the notation and write

$$\underline{v}(\underline{x}) = -\nabla p(\underline{x}) \qquad \nabla . \underline{v}(\underline{x}) = 0 \tag{1.4}$$

It is possible to represent solutions of Stokes' and Darcy's equations in two dimensions by analytic functions. We explain what we mean by this statement on the basis of Darcy's equations on an open domain $G \subset \mathbb{R}^2$. It immediately follows from the equations (1.4) that the function p is harmonic. If we assume that G is simply connected, there is a harmonic conjugate of p on G and this implies that an analytic function χ exists on G such that

$$p = -\text{Re } \chi \qquad v = \overline{\chi'} \tag{1.5}$$

where we use complex notation. Conversely, it is easily checked that if χ is an analytic function on an open domain $G \subset \mathbb{R}^2$, then the functions p and v defined by the relations (1.5) satisfy Darcy's equations. We therefore state that a solution of Darcy's equations can be represented by one analytic function. A comparable result holds for Stokes' equations: it can be shown that a solution of Stokes' equations in two dimensions can be represented by two analytic functions. We include a proof of this well-known result in Section 2.1.
These representations turn out to be very useful later on as they constitute an essential ingredient in the derivation of Hopper equations. This leads to the question whether we can generalize the concept of representing functions. Is it possible to represent solutions of Stokes' equations in three dimensions in terms of generalized analytic functions? We will answer this question affirmatively in Chapter 2. More precisely, we will show in Section 2.2 that every axially symmetric solution of Stokes' equations can be represented by two so-called deformed

analytic functions and we will show in Section 2.3 that every solution of Stokes' equations in three dimensions can be represented by two left monogenic functions (the latter section was published in a somewhat different form in [44]). In spite of these results, it turns out that a generalization of Hopper equations for more than two dimensions is very hard and may even be impossible. As we will not go further into this matter, the quick reader may skip Sections 2.2 and 2.3.

We end this section by a very short introduction to multi-poles (see e.g. [24, 51, 59]). Let G be an open domain in $\mathbb{R}^2$ containing $\underline{0}$ and let p and $\underline{v}$ satisfy Stokes' or Darcy's equations on $G\backslash\{\underline{0}\}$. A source at $\underline{0}$ of strength Q_1 (if $Q_1 < 0$ we speak of a sink) can be modelled by

$$\nabla.\underline{v}(\underline{x}) = Q_1\delta(\underline{x})$$

where δ denotes Dirac's delta function. One checks that this relation in complex notation corresponds to

$$v(z,\overline{z}) - \frac{Q_1}{2\pi\overline{z}} \rightarrow \text{const.} \qquad \text{if } r = |z| \rightarrow 0 \tag{1.6}$$

In the same way, a multi-pole of order $n \in \mathbb{N}\backslash\{1\}$ of strength $Q_n > 0$ directed along $(\cos\phi \;\; \sin\phi)^T$ can be modelled by

$$v(z,\overline{z}) - \frac{Q_n e^{i\phi}}{2\pi\overline{z}^n} \rightarrow \text{const.} \qquad \text{if } r = |z| \rightarrow 0 \tag{1.7}$$

1.2.2 Stokes flow driven by surface tension and multi-poles

Consider a fluid with a surface tension coefficient $\gamma \geq 0$ which occupies an open domain $G \subset \mathbb{R}^n$ with a compact C^2-surface ∂G. Let the outer domain be occupied by a fluid without surface tension under a uniform hydrostatic pressure p_0. On the basis of both experiments and theory ([93, 74, 8]), a good model for the dynamic boundary condition is

$$\underline{\underline{T}}(\underline{x})\underline{n}(\underline{x}) + \gamma\kappa(\underline{x})\underline{n}(\underline{x}) = -p_0\underline{n}(\underline{x}) \qquad \underline{x} \in \partial G$$

where $\underline{\underline{T}}$ denotes the stress tensor and where $\kappa(\underline{x})$ and $\underline{n}(\underline{x})$ denote the mean curvature and the outward-pointing normal vector at a point $\underline{x} \in \partial G$ ([83]).
We discuss this boundary condition in combination with Stokes' equations. We note that if $\gamma \neq 0$, it is possible to scale the unit system such that $\gamma = 1$. We also note that as only derivatives of the pressure p appear in Stokes' equations, we may assume $p_0 = 0$ without loss of generality. We thus obtain the following boundary value problem:

$$\Delta\underline{v}(\underline{x}) = \nabla p(\underline{x}) \qquad \nabla.\underline{v}(\underline{x}) = 0 \qquad \underline{x} \in G \tag{1.8a}$$

$$\underline{\underline{T}}(\underline{x})\underline{n}(\underline{x}) = -\gamma\kappa(\underline{x})\underline{n}(\underline{x}) \qquad \underline{x} \in \partial G \tag{1.8b}$$

where $\gamma = 0, 1$. A solution of this problem for $\gamma = 0$ (no surface tension) and $n = 2$ (two-dimensional flow) is given by:

$$p(x, y) = 0 \qquad \underline{v}(x, y) = \omega(y \quad -x)^T + (v_1 \quad v_2)^T$$

where ω, v_1 and v_2 are arbitrary constants. Such a solution is called a rigid-body motion. It is not difficult to show that rigid-body motions are the only solutions of the boundary value problem (1.8) with $\gamma = 0$ ([72]). As Stokes' equations are linear, this implies that the solution of boundary value problem (1.8) is determined up to a rigid-body motion. The existence of solutions is treated in e.g. [54].
In order to obtain a model of the moving boundary problem for Stokes flow driven by surface tension, the kinetic boundary condition has to be formulated. It is physically clear what this condition should be if the fluid can move freely: the boundary moves according to the hydrodynamic velocity and behaves as a membrane. However, we postpone a precise mathematical formulation in terms of Lagrange coordinates to Section 3.1 because some technical details are involved. Here, we suffice with the following loose formulation:

$$\Delta\underline{v}(\underline{x}, t) = \nabla p(\underline{x}, t) \qquad \nabla.\underline{v}(\underline{x}, t) = 0 \qquad t \in I, \underline{x} \in G_t \tag{1.9a}$$
$$\underline{\underline{T}}(\underline{x}, t)\underline{n}(\underline{x}, t) = -\gamma\kappa(\underline{x}, t)\underline{n}(\underline{x}, t) \qquad t \in I, \underline{x} \in \partial G_t \tag{1.9b}$$
$$\underline{V}(\underline{x}, t) = \underline{v}(\underline{x}, t) \qquad t \in I, \underline{x} \in \partial G_t \tag{1.9c}$$

where $\underline{V}$ denotes the velocity of the boundary. The moving boundary problem for Stokes flow driven by surface tension and multi-poles can be formulated in the same way. (That is the reason why we did not put $\gamma = 1$: although the only solutions of problem (1.9) with $\gamma = 0$ are the rigid-body motions, the solutions of the moving boundary problem for Stokes flow driven by multi-poles are not trivial.) The moving boundary problem (1.9) is called quasi-static because the dynamics only come in via the kinetic boundary condition. A moving boundary problem like this one is often studied in combination with initial data given by a domain G_0. In most applications, one is mainly interested in the shape evolution $t \to G_t$ of the solution of such an initial value problem; the pressure and velocity fields are of minor importance. We return to this point in Section 3.1.

Although this thesis aims to be a mathematical study, we make some comments on the question under which physical circumstances this model is a good approximation. The most important reduction that is made is the disregard of the gravitational and inertial forces. These forces are negligible indeed if the dimensionless Suratman and Bond number are small (see [35, 12], see also [52]). However, a lot of other approximations are implicitly made: it is assumed that the fluid is Newtonian and perfectly incompressible, that the viscosity and surface tension coefficients are constant, that the London-Van der Waals forces can be neglected, etc. Under which circumstances these approximations can be justified inevitably remains somewhat arbitrary. Nevertheless, the analytic solutions of the moving boundary problem (1.9) obtained so far are in good agreement with experimental data ([37, 36, 94]). A good account of numerical solutions and their significance for industrial applications can be found in [90].

1.2.3 Hele-Shaw flow

Consider an incompressible fluid in a porous medium which occupies an open domain $G \subset I\!R^n$ where a source of strength Q_1 is placed at $\underline{x} = \underline{0}$. If we assume that this fluid is in hydrodynamic equilibrium with the fluid in the outer domain under a uniform pressure p_0 and if we neglect the viscosity terms, we get the following model:

$$\underline{v}(\underline{x}) = -\nabla p(\underline{x}) \qquad \nabla.\underline{v}(\underline{x}) = Q_1\delta(\underline{x}) \qquad \underline{x} \in G \tag{1.10a}$$

$$p(\underline{x}) = p_0 \qquad \underline{x} \in \partial G \tag{1.10b}$$

As only derivatives of the pressure p appear in Darcy's equations, we may put $p_0 = 0$ without loss of generality. It is not difficult to show that this boundary value problem has a unique solution by rewriting it as a Dirichlet problem for the pressure p (see also Section 3.4).

If the fluid can move freely, we obtain the following quasi-static moving boundary problem by formulating the kinetic boundary condition as in the previous subsection:

$$\underline{v}(\underline{x},t) = -\nabla p(\underline{x},t) \qquad \nabla.\underline{v}(\underline{x},t) = Q_1(t)\delta(\underline{x}) \qquad t \in I, \underline{x} \in G_t \tag{1.11a}$$

$$p(\underline{x},t) = 0 \qquad t \in I, \underline{x} \in \partial G_t \tag{1.11b}$$

$$\underline{V}(\underline{x},t) = \underline{v}(\underline{x},t) \qquad t \in I, \underline{x} \in \partial G_t \tag{1.11c}$$

We note that the strength of the source is admitted to be time-dependent. This problem is referred to as the moving boundary problem for Hele-Shaw flow. Its study has a long history ([39, 79, 77]); its applications concern ground-water flows and oil production ([16, 67, 62]).

1.3 Introduction to Hopper and extended Löwner-Kufareev equations

In Section 1.1 we stated that the study of Hopper equations in Chapters 4 and 5 is the core of this thesis. Before we turn to the results of this study, we sketch the features of Hopper's equation, the equation for a time-dependent conformal mapping Ω which solves the moving boundary problem for Stokes flow driven by surface tension. We do not present this equation at this place because this would

require an introduction to a lot of notational matters (we suggest the reader to take a glance at equation (3.17) at page 52). The right-hand side of this equation is an expression where the mapping Ω and its derivatives appear; the highest time derivative is of order one. The left-hand side is an unknown time-dependent analytic function θ. (In the original article [35], in equation 22, the left-hand side is denoted by ψ.) We stress that this function cannot be considered a given function; it can only be determined a posteriori. However, we know an important property of this function θ: it is analytic on its domain for all fixed times. The origin of this property lies in the possibility to represent a two-dimensional Stokes flow by analytic functions.
At first, Hopper's equation does not seem to determine the time evolution of the mapping Ω: if we substitute a particular mapping Ω at a fixed time, we obtain a relation between two unknown functions, namely θ and $\frac{\partial \Omega}{\partial t}$. However, the knowledge that the function θ is analytic *does* seem to determine the time evolution of the mapping Ω in some way or another. In Hopper's own words ([35]): "... the requirement that $\psi(\zeta, t)$ be a function analytic ..., determines uniquely $\dot{\Omega}(\sigma, t)$ and therefore $\Omega(\zeta, t)$. ..."
This statement is based on Hopper's observation that if we make the Ansatz that Ω is a time dependent-polynomial –i.e. we substitute a parameterised time-dependent polynomial in Hopper's equation– we obtain the time evolution of Ω by requiring the function θ to be analytic –i.e. we get a set of differential equations for the coefficients in the parameterisation. De Graaf calls this the "cancellation of singularities" ([30]) as the singularities in the various terms on the right-hand side of Hopper's equation must compensate each other to make both sides of the equation analytic. It is shown in Hopper's article that such a procedure also works if we make the Ansatz that Ω is a partial fraction mapping.
Inspired by these results, we show in Section 4.1 that the substitution of a properly parameterised rational function in a Hopper equation leads to a set of differential equations for the parameters by calculating Cauchy integrals and applying the residue theorem. We note that this result is a generalisation of the aforementioned results in the sense that polynomial and partial fraction mappings are particular types of rational mappings and in the sense that Hopper's equation is a particular type of Hopper equation. Moreover, some of the differential equations found in this way by substituting a partial fraction mapping in a Hopper equation can be considered algebraic equations, i.e. they lead to conserved quantities, and are therefore easier to handle then the equations found by Hopper (see e.g. [7] where it is also shown that both sets of equations are equivalent).
Does the plain fact that a Hopper equation for a rational mapping is equivalent to a set of differential equations prove the existence of rational solutions? No, first it has to be shown that this set of equations is solvable. We do so after employing some theory of ordinary differential equations and some complex function theory in Section 4.2: Theorem 4.18 on page 78 states that the aforementioned set of

differential equations has a unique maximal solution under general conditions. As a result we find that the moving boundary problems for Stokes flow driven by surface tension and for Hele-Shaw flow have rational solutions. We note that this proves Hopper's conjecture on the existence of polynomial and partial fraction solutions ([35]).
Although these results are satisfactory, one may expect global solvability in some cases, for e.g. Stokes flow driven by surface tension and for Hele-Shaw flow with a source, i.e. $Q_1 > 0$. However, a rigorous proof of global solvability of the set of differential equations obtained by substituting a rational function in the corresponding Hopper equations turns out to be complicated because of technical difficulties. This is illustrated in Section 4.3, where we prove this global solvability for a special class of partial fraction mappings. This section also illustrates that the algebraic relations corresponding to the aforementioned conserved quantities behave in a way one may expect.

Let us take a breath for a moment and consider the results obtained so far. We may get the vague notion that Hopper equations and time-dependent rational mappings are designed for each other. It is at least somewhat bewildering that the considerations in Chapter 4 are independent of the precise form of the Hopper equation and, remember, are also independent of the values of the analytic function in the left-hand side. Together with these wonders, doubts may appear as it has not yet been shown for example that a rational mapping satisfying a Hopper equation cannot evolve into a non-rational mapping. Moreover: is there anything to state on the properties of non-rational solutions? To put things even more insubstantially: can we mathematically understand the concept of a Hopper equation?
In order to answer these questions to some extent, we introduce the extended Löwner-Kufareev equation: a first order partial differential equation of a certain type for functions which depend on a complex and a real variable. The name of this equation is based on the fact that the characteristics of this equation are determined by an ordinary differential equation of exactly the same type as the ordinary differential equation which underlies the non-extended Löwner-Kufareev equation. We show in Section 5.2 that the singularities of a solution of an extended Löwner-Kufareev equation move along these characteristics and cannot simply appear, disappear or change their nature (that is: a pole of order two remains a pole of order two, a branch point remains a branch point, etc.).
These results can be used to deduce properties of solutions of Hopper equations if we assume in advance that these solutions are very smooth, i.e. analytically extendable. More precisely, we show in Subsection 5.3.2 that a smooth solution Ω of a Hopper equation corresponds to a solution of an extended Löwner-Kufareev equation and we then show that this implies a rule for the propagation of singularities of the mapping Ω. In particular, we find that a smooth solution which is

rational at some fixed time must be rational for all other times. The implications for the moving boundary problems for Stokes flow driven by surface tension and multi-poles and for Hele-Shaw flow with a source are as follows. First of all, singularities of smooth solutions tend to move to infinity in accordance with the belief that the domains occupied by the fluids tend to become circular. Secondly, singularities of smooth solutions cannot simply appear, disappear or change their nature; this assertion can be considered as a guide on how to guess a proper Ansatz. Finally, rational solutions exist and are unique in the class of smooth solutions.

We stress that the relation between solutions of Hopper equations and of extended Löwner-Kufareev equations only appears if the solutions of the Hopper equations are assumed to be analytically extendable. However, it is shown in Subsection 5.3.3 that some of the properties mentioned above can be deduced in another way by making assumptions of a different type. What we actually show is: a mapping that is almost linear, which satisfies a Hopper equation and which is rational at some fixed time, is rational for all other times. We prove this assertion on the basis of the results obtained in Chapter 4 and on a lemma that states under which conditions a linearized version of a Hopper equation has a unique solution. This lemma itself is proved in Appendix C on the basis of a theorem on strongly continuous semi-groups of operators.

Chapter 2

Representations of Stokes Flows

In this chapter we consider solutions of Stokes' equations:

$$\triangle \underline{v} = \nabla p \cdot \qquad \nabla . \underline{v} = 0 \tag{2.1}$$

in two and three dimensions (see also Subsection 1.2.1). It is well-known that a solution of Stokes' equations in two dimensions can be represented by a pair of analytic functions ([48, 45]). For the convenience of the reader we show in Section 2.1 how this result can be obtained. The exposition is such that the generalisations in the subsequent sections become better understandable: we show in Section 2.2 that an axially symmetric solution of Stokes' equations in three dimensions can be represented by a pair of what we will call deformed analytic functions and we show in Section 2.3 that the general solution of Stokes' equations in three dimensions can be represented by a pair of left monogenic functions. We finally show in Section 2.4 how the representations of the Sections 2.1 and 2.2 can be used to rewrite boundary value problem (1.8).

2.1 Two-dimensional Stokes flow

Let $G \subset I\!R^2$ be an open, simply connected domain and let $\underline{v}$ and p on G satisfy Stokes' equations (2.1). As $\underline{v}$ is solenoidal and as G is simply connected, a function ψ on G exists, called the stream function, such that

$$\underline{v} = (v_1 \;\; v_2)^T = (\psi_y \;\; -\psi_x)^T$$

Let D and $\overline{D}$ denote the Cauchy-Riemann and the anti-Cauchy-Riemann operators times two:

$$D = 2\partial_{\overline{z}} = \partial_x + i\partial_y \qquad \overline{D} = 2\partial_z = \partial_x - i\partial_y$$

The equations (2.1) are equivalent to the following equations in complex notation:

$$D\overline{D}v = Dp \qquad v = -Di\psi \tag{2.2}$$

As the function p is smooth, a real function σ exists on G, which we will call the auxiliary function, such that

$$p = \triangle\sigma = D\overline{D}\sigma \tag{2.3}$$

It is clear that σ is not unique: if g is an arbitrary real harmonic function on G, then the function

$$\tilde{\sigma} = \sigma + g \tag{2.4}$$

is also an auxiliary function, i.e. $\tilde{\sigma}$ satisfies $\triangle\tilde{\sigma} = p$. We define the complex valued function F on G by

$$F = \sigma + i\psi \tag{2.5}$$

It follows immediately from the relations (2.2),(2.3) and (2.5) that F satisfies

$$DD\overline{D}F = 0 \tag{2.6}$$

It can be shown ([28]) that every smooth complex valued function f on a simply connected domain $G \subset \mathbb{R}^2$ satisfying

$$D^n\overline{D}^m f = (2)^{n+m}\partial_{\overline{z}}^n\partial_z^m f = 0 \qquad n, m \in \mathbb{N} \tag{2.7}$$

can be written as

$$f = \sum_{k=1}^{n} \overline{z}^{k-1} g_k + \sum_{k=1}^{m} z^{k-1}\overline{h_k} \tag{2.8}$$

where $g_k, k = 1, \ldots, n$ and $h_k, k = 1, \ldots, m$ are analytic functions; if $n = 0$ or $m = 0$, similar results hold. For $n = 2$ and $m = 1$ this implies that there is an analytic function φ and a complex valued, harmonic function χ such that

$$F = \overline{z}\varphi + \chi \tag{2.9}$$

It follows immediately from relation (2.4) that the function $\tilde{F} = \tilde{\sigma} + i\psi$ can be written as:

$$\tilde{F} = \overline{z}\varphi + \chi + g \tag{2.10}$$

Now we perform what physicists may call a gauge transformation: we choose the function g such that $\tilde{\chi} = \chi + g$ is an analytic function (take $g = i(-\operatorname{Im}\chi + \hat{\chi})$, where $\hat{\chi}$ denotes a harmonic conjugate of $\operatorname{Re}\chi$). We omit the tildes and find that F can be written as in formula (2.9) where φ and χ are now both analytic functions. We conclude that every Stokes flow on an open two-dimensional, simply connected domain can be represented by a pair of analytic functions.

The relations (2.2),(2.3),(2.5) and (2.9) lead to the following expressions for p and $\underline{v}$:

$$\begin{aligned} p &= 4\varphi_{1,x} \\ v_1 &= x\varphi_{2,y} - y\varphi_{1,y} - \varphi_1 + \chi_{2,y} \\ v_2 &= y\varphi_{1,x} - x\varphi_{2,x} - \varphi_2 - \chi_{2,x} \end{aligned} \tag{2.11}$$

where $\varphi_1 = \text{Re } \varphi$, $\varphi_2 = \text{Im } \varphi$, etc.. These relations can be written in complex notation as

$$p = 4\text{Re } \varphi' \qquad v = -\varphi + z\overline{\varphi'} + \overline{\chi'} \tag{2.12}$$

We also obtain the following expression for the matrix T corresponding to the stress tensor (see expression (1.3)):

$$T = \begin{pmatrix} -2\sigma_{yy} & 2\sigma_{xy} \\ 2\sigma_{xy} & -2\sigma_{xx} \end{pmatrix} \tag{2.13}$$

Remark 2.1 It is easily checked that if φ and χ are analytic functions on some open domain $G \subset \mathbb{R}^2$, then the functions p, v_1 and v_2 defined by the relations (2.12) satisfy Stokes' equations.

Remark 2.2 One can obtain the same representations of Stokes flow in terms of analytic functions in different ways [30, 82, 45, 64]. One possible way to get the representation starts with the remark that the two rows of the matrix T corresponding to the stress tensor can be considered to be solenoidal vector fields. This implies the existence of functions τ_1 and τ_2 on G such that

$$T = \begin{pmatrix} -\tau_{1,y} & \tau_{1,x} \\ \tau_{2,y} & -\tau_{2,x} \end{pmatrix} \tag{2.14}$$

The symmetry of the stress tensor –corresponding to the conservation of angular momentum [55]– implies the identity $\tau_{1,x} = \tau_{2,y}$ and this in turn implies the existence of a function ρ on G, called the Airy function, such that

$$\tau_1 = \rho_y \qquad \tau_2 = \rho_x \tag{2.15}$$

The expressions (2.12) then follow after some other considerations [45].
We notice that the Airy function times two is a special auxiliary function:

$$p = -\frac{1}{2}\text{Trace}(T) = -\frac{1}{2}(-\rho_{yy} - \rho_{xx}) = \triangle(\rho/2) \tag{2.16}$$

Substituting the relations (2.15) into identity (2.14) and comparing the result with identity (2.13), we come to the conclusion that we gauged the auxiliary function such that it is twice the Airy function. We notice that it is also possible to gauge the auxiliary function for example such that χ is anti-analytic or such that χ is purely imaginary.

Remark 2.3 The constant and linear parts of the Taylor series of the functions φ and χ are not interesting from the dynamical point of view. To illustrate this, we consider the special case:

$$\varphi = az + b \qquad \chi = cz + d \tag{2.17}$$

with $a, b, c, d \in \mathbb{C}$ and calculate the pressure and the components of the velocity by means of expressions (2.11). We find

$$p = 4a_1 \qquad v_1 = 2a_2 y - b_1 + c_1 \qquad v_2 = -2a_2 x - b_2 - c_2$$

So, linear representing functions correspond to rigid-body motions (see also Subsection 1.2.2). These relations also show that the functions φ and χ are not uniquely determined by the pressure and the velocity. However, one checks that if we identify Stokes flows which differ only by a constant pressure and a rigid body motion and if we identify analytic functions which differ only by a linear function, then the correspondence between Stokes flows and pairs of analytic functions is one-to-one.

Example 2.4 We consider a disc of radius R of viscous fluid with surface tension coefficient γ and density ρ rotating around the origin of the coordinates with a uniform angular velocity ω. We have the following solution of boundary value problem (1.8):

$$p(x, y) = \gamma / R \qquad \underline{v}(x, y) = (\omega y \quad -\omega x)^T$$

So, the pressure is constant and the particles in the disc accelerate without the presence of a force in contradiction to Newton's laws. This can happen because Stokes' equations have been derived from the Navier–Stokes equations by making some approximating assumptions (see Section 1.2). Without neglecting the inertial force, the pressure in the rotating disc is found to be:

$$p(x, y) = \gamma / R + \frac{1}{2}\omega^2 \rho (x^2 + y^2 - R^2)$$

We now remark that if ω is relatively large, the approximations leading to Stokes' equations are not allowed as the inertial force is proportional to ω^2.

2.2 Axially symmetric Stokes flow

In this section we will study axially symmetric solutions of Stokes' equations on axially symmetric domains $G \subset \mathbb{R}^3$. We will show in Subsection 2.2.2 that such solutions can be represented by what we will call deformed analytic functions. We will explain in Subsection 2.2.1 what deformed analytic functions are and which properties they possess.

2.2.1 Deformed analytic functions

Definitions 2.5 A domain $G \subset \mathbb{R}^2$ is called symmetric if $(r,z) \in G$ implies $(-r,z) \in G$. A function f on a symmetric domain G is called symmetric if $f(r,z) = f(-r,z)$ for all $(r,z) \in G$. A function f on a symmetric domain G is called anti-symmetric if $f(r,z) = -f(-r,z)$ for all $(r,z) \in G$. A pair $p = (p_1 \ p_2)^T$ of real differentiable functions p_1 and p_2 on an open symmetric domain G is called a deformed analytic function if p_1 is symmetric, p_2 is anti-symmetric and if these functions satisfy the following equations on G:

$$\partial_r p_1 = \partial_z p_2 \qquad \partial_z p_1 = -(\partial_r + \frac{1}{r})p_2 \tag{2.18}$$

We will call these equations the deformed Cauchy-Riemann equations. We define the deformed Cauchy-Riemann operator D by

$$D = \begin{pmatrix} \partial_r & -\partial_z \\ \partial_z & \partial_r + \frac{1}{r} \end{pmatrix}$$

and we define the operator $\tilde{D}$ by

$$\tilde{D} = \begin{pmatrix} \partial_z & -(\partial_r + \frac{1}{r}) \\ \partial_r & \partial_z \end{pmatrix}$$

Finally, we define the second order differential operators K_1 and K_2 by

$$K_1 = (\partial_r + \frac{1}{r})\partial_r + \partial_z^2 \qquad K_2 = \partial_r(\partial_r + \frac{1}{r}) + \partial_z^2$$

Remarks 2.6 The subset $I_G = \{(0,z) \in G\}$ of a symmetric domain G is not empty, otherwise G would not be connected. In the following we will assume $\underline{0} \in I_G$. We notice that for every simply connected and symmetric domain, the set I_G is connected.
It is clear that a deformed analytic function p satisfies $Dp = 0$ by definition. Notice that at infinity (i.e. for $r \to \pm\infty$), the operator D corresponds to the Cauchy-Riemann operator $\partial_{\overline{z}}$ times two, with (r,z) identified with (x,y). In the same way, the operator $\tilde{D}$ corresponds to $2i\partial_{\overline{z}}$. The operator

$$\overline{D} = \begin{pmatrix} \partial_r + \frac{1}{r} & \partial_z \\ -\partial_z & \partial_r \end{pmatrix}$$

corresponds to two times the anti-Cauchy-Riemann operator ∂_z, but we will use this operator in the following only once. The reader can check for himself that a theory of differentiation and primitivation of deformed analytic functions –as we will present in this subsection– would fail if $\tilde{D}$ is regardless replaced by $\overline{D}$.

Example 2.7 Let $G \subset I\!R^2$ be an open symmetric domain such that $I_G \neq I\!R$. Let $a \in I\!R$ be such that $(0, a) \notin I_G$. The function p defined by

$$p(r, z) = \begin{pmatrix} (r^2 + (z-a)^2)^{-\frac{1}{2}} \\ r^{-1}\left(1 - (z-a)(r^2 + (z-a)^2)^{-\frac{1}{2}}\right) \end{pmatrix}$$

is deformed analytic on G.

Lemma 2.8 *Let $p = (p_1 \;\; p_2)^T$ be a deformed analytic function on an open, symmetric domain $G \subset I\!R^2$. Then $p_1, p_2 \in C^\infty(G)$ and p_1 and p_2 satisfy*

$$K_1 p_1 = K_2 p_2 = 0$$

Proof
As p_1 is a differentiable function on G, it can be considered as a distribution in $\mathcal{D}'(G)$ (see e.g. [95]). Considering derivations in the sense of distributions, we find

$$K_1 p_1 = ((\partial_r + 1/r)\partial_r + \partial_z^2)p_1 = \partial_z\left((\partial_r + 1/r)p_2 + \partial_z p_1\right) = 0$$

The operator K_1 corresponds to the Laplacian in three dimensions in cylindrical coordinates with the derivatives with respect to the azimuthal variable ϕ omitted. So, K_1 is an elliptic operator and it follows that $p_1 \in C^\infty(G)$ [80]. It is then easily checked that $p_2 \in C^\infty(G)$ and $K_2 p_2 = 0$. □

Lemma 2.9 *Let $G \subset I\!R^2$ be an open, simply connected and symmetric domain. If a symmetric function p_1 satisfies $K_1 p_1 = 0$ on G, then a unique function p_2 exists on G such that $p = (p_1 \;\; p_2)^T$ is a deformed analytic function. If an anti-symmetric function p_2 satisfies $K_2 p_2 = 0$ on G, then a function p_1 exists on G, unique up to a constant, such that $p = (p_1 \;\; p_2)^T$ is a deformed analytic function.*

Proof
We only prove the first assertion; the second assertion can be proved in a similar way.
Existence. Let the symmetric function p_1 satisfy $K_1 p_1 = 0$ on G. One checks that the 1-form ω on G defined by

$$\omega(r, z) = -r\partial_z p_1(r, z)dr + r\partial_r p_1(r, z)dz$$

is closed, i.e. $d\omega = 0$. Because of this, and as G is simply connected, we can define a function p_2 on G by

$$p_2(r, z) = \frac{1}{r}\int_{(0,0)}^{(r,z)} \omega(\rho, \xi)$$

without specifying the path of integration. One checks that p_2 is well defined on I_G. One also checks that the function p_2 is anti-symmetric, differentiable and satisfies

$$(\partial_r + \frac{1}{r})p_2 = -\partial_z p_1 \qquad \partial_z p_2 = \partial_r p_1$$

Uniqueness. Let p_1 be a symmetric differentiable function on G and let p_2 and $\tilde{p}_2$ both satisfy equations (2.18) on G. It follows that the function Q on G defined by

$$Q(r,z) = r(p_2(r,z) - \tilde{p}_2(r,z))$$

satisfies

$$\partial_r Q = \partial_z Q = 0$$

and is therefore identical to a constant C. This constant C equals 0 as $Q(0,0) = 0$, and it follows that $\tilde{p}_2$ is identical to p_2. □

Lemma 2.10 *Let p be a deformed analytic function on an open, simply connected and symmetric domain $G \subset \mathbb{R}^2$. Then $\tilde{D}p$ is deformed analytic on G. Moreover, a deformed analytic function P exists on G such that $\tilde{D}P = p$.*

We call $\frac{1}{2}\tilde{D}p$ the derivative of p and we will denote this function as p'. We call $2P$ a primitive of p. One easily checks that two primitives of a deformed regular function differ by a constant in the first entry.

Proof

The first assertion immediately follows from Lemma 2.8 as

$$D\tilde{D} = \begin{pmatrix} 0 & -K_2 \\ K_1 & 0 \end{pmatrix}$$

The proof of the second assertion runs as follows. With use of the deformed Cauchy-Riemann equations (2.18), it can be shown that the 1-forms ω_1 and ω_2 on G defined by

$$\omega_1(r,z) = p_2(r,z)dr + p_1(r,z)dz$$
$$\omega_2(r,z) = -rp_1(r,z)dr + rp_2(r,z)dz$$

are closed, i.e. $d\omega_1 = d\omega_2 = 0$. Because of this, and as G is simply connected, we can define functions P_1 and P_2 on G by

$$P_1(r,z) = \int_{(0,0)}^{(r,z)} \omega_1(\rho,\xi) \qquad P_2(r,z) = \frac{1}{r}\int_{(0,0)}^{(r,z)} \omega_2(\rho,\xi)$$

without specifying the path of integration. We notice that P_2 is well defined on I_G, that P_1 is symmetric, that P_2 is anti-symmetric and that P_1 and P_2 satisfy

$$\partial_r P_1 = \partial_z P_2 = p_2 \qquad \partial_z P_1 = -(\partial_r + \frac{1}{r})P_2 = p_1$$

This implies that the function $P = \frac{1}{2}(p_1 \;\; p_2)^T$ is deformed analytic and satisfies $\tilde{D}P = p$. □

The three lemmas given above show that deformed analytic functions possess properties which resemble the properties of analytic functions. These properties will be used in the next subsection. We close this subsection by showing that also a kind of Weierstrass-approach to deformed analytical functions exists; that is: we will show that a deformed analytic function is analytic in the sense that it can be written as a kind of Taylor-series.

Definitions 2.11 The polynomials l_n and m_n for $n \in \mathbb{N}_0$ are defined on $\mathbb{R}^2$ by

$$l_n(r,z) = \sum_{k=0}^{[n/2]} \frac{(-1)^k n!}{(n-2k)!(k!)^2 4^k} r^{2k} z^{n-2k}$$

$$m_n(r,z) = \sum_{k=0}^{[(n-1)/2]} \frac{(-1)^{k+1} n!}{(n-2k-1)!(k!)^2 2(k+1)4^k} r^{2k+1} z^{n-2k-1} \qquad n \neq 0$$

$$m_0(r,z) = 0$$

The functions l_{-n} and m_{-n} for $n \in \mathbb{N}$ on $\mathbb{R}^2/\{(0,0)\}$ are defined by

$$l_{-n}(r,z) = \frac{2n-1}{4} l_{n-1}(r,z)|r|(r^2+z^2)^{-n}$$

$$m_{-n}(r,z) = \frac{n(2n-1)}{4(n-1)} m_{n-1}(r,z)|r|(r^2+z^2)^{-n} \qquad n \neq 1$$

$$m_{-1}(r,z) = 0$$

Finally we define for all $n \in \mathbb{Z}$

$$T_n = (l_n \;\; m_n)^T$$

Proposition 2.12 *The function T_n with $n \in \mathbb{N}_0$ is, up to a multiplicative constant, the only deformed analytic function homogeneous on $\mathbb{R}^2$ of order n. Its derivative is given by*

$$T_n' = nT_{n-1} \tag{2.19}$$

Moreover, we have for all $n, k \in \mathbb{N}_0$ and all $R > 0$:

$$\int_{|(r,z)|=R} T_n \cdot T_{-(k+1)} \, ds = (2 - \delta_{n0})\delta_{kn} \tag{2.20}$$

Proof
The polynomial $l_n, n \in \mathbb{N}_0$ is homogeneous of order n and satisfies $K_1 l_n = 0$. (The definition of $l_n, n \in \mathbb{N}$ is obtained from Laplace's representation of harmonic

polynomials in $\mathbb{R}^3$, see e.g. [60]). As K_1 is the Laplacian in three dimensions with the derivatives with respect to the azimuthal variable ϕ omitted, and as there is –up to a multiplicative constant– only one harmonic polynomial of order $n \in \mathbb{N}_0$ on $\mathbb{R}^3$ independent of the variable ϕ, the polynomial l_n is –up to a multiplicative constant– the only homogeneous polynomial of order n satisfying $K_1 l_n = 0$. The first assertion now follows from Lemma 2.9 and the easily established relation

$$m_n(r,z) = -\frac{1}{r}\partial_z \int_0^r \rho\, l_n(\rho,z)\, d\rho \qquad n \in \mathbb{N}_0$$

Relation (2.19) can be checked by differentiation.
The polynomials $P_n, n \in \mathbb{N}_0$ and $R_n, n \in \mathbb{N}_0$ defined by

$$\begin{aligned} P_n(t) &= l_n(\sqrt{1-t^2}, t) && t \in [-1,1] \\ R_n(t) &= -2(1-t^2)^{-1/2} m_{n+1}(\sqrt{1-t^2}, t) && t \in [-1,1] \end{aligned}$$

are the Legendre and the Jacobi polynomials (with $\alpha = \beta = 0$) normalized in the standard way (see e.g. [84]). We have the following orthogonality relations:

$$\begin{aligned} \int_{-1}^1 P_n(t)P_k(t)\, dt &= \frac{2}{2n+1}\,\delta_{nk} \\ \int_{-1}^1 R_n(t)R_k(t)(1-t^2)\, dt &= \frac{8(n+1)}{(2n+3)(n+2)}\,\delta_{nk} \end{aligned}$$

for all $n,k \in \mathbb{N}_0$. Noticing that the functions l_{-n} and m_{-n} for $n \in \mathbb{N}$ are homogeneous of order $-n$, we get from the first orthogonality relation for all $n,k \in \mathbb{N}_0$ and all $R > 0$:

$$\begin{aligned} &\int_{|(r,z)|=R} l_n l_{-(k+1)}\, ds \\ &= \int_{-\pi}^{\pi} l_n(R\cos\phi, R\sin\phi) l_{-(k+1)}(R\cos\phi, R\sin\phi) R\, d\phi \\ &= \frac{2k+1}{4} R^{n-(k+1)+1} \int_{-\pi}^{\pi} l_n(\cos\phi, \sin\phi) l_k(\cos\phi, \sin\phi) |\cos\phi|\, d\phi \\ &= \frac{2k+1}{2} R^{n-k} \int_{-1}^{1} l_n(\sqrt{1-t^2}, t) l_k(\sqrt{1-t^2}, t)\, dt = \delta_{nk} \end{aligned}$$

In the same way, we get for all $n,k \in \mathbb{N}_0$ and all $R > 0$:

$$\int_{|(r,z)|=R} m_n m_{-(k+1)}\, ds = \delta_{nk}(1 - \delta_{n0})$$

These last two identities lead to identity (2.20). □

Proposition 2.13 *Let p be a deformed analytic function on the open disc B_R, $R > 0$. The function p can be represented by a series*

$$p = \sum_{k=0}^{\infty} c_k T_k \qquad (2.21)$$

that converges uniformly on each disc $B_{R'}$ with $R' < R$. The coefficients are given by:

$$(c_k \;\; 0)^T = \frac{1}{k!} p^{(k)}(0,0) \qquad k \in \mathbb{N}_0 \qquad (2.22)$$

Proof
We start by making some estimates. As $|P_k(t)| \leq 1$ for all $k \in \mathbb{N}_0$ and all $t \in [-1, 1]$, we have

$$|l_k(r,z)| \leq (r^2 + z^2)^{k/2} \qquad k \in \mathbb{N}_0 \qquad (2.23)$$

This leads to the following (non-sharp) estimate for $m_k, k \in \mathbb{N}_0$:

$$|m_k(r,z)| = \left| \frac{1}{r} \int_0^r \rho \partial_z l_k(\rho, z) \, d\rho \right| \leq \frac{k}{r} \int_0^r \rho |l_{k-1}(\rho, z)| \, d\rho \leq k(r^2 + z^2)^{k/2} \quad (2.24)$$

and to the following estimate for $l_{-k}, k \in \mathbb{N}$:

$$|l_{-k}(r,z)| \leq \frac{2k-1}{4}(r^2 + z^2)^{(k-1)/2} |r| (r^2 + z^2)^{-k} \leq \frac{k}{2}(r^2 + z^2)^{-k/2} \qquad (2.25)$$

Let $p = (p_1 \;\; p_2)^T$ be a deformed analytic function on B_R, let $R' < R$ and choose R'' such that $R' < R'' < R$. As p_1 satisfies $K_1 p_1 = 0$ on B_R, this function p_1 corresponds to a harmonic function $\tilde{p}$ on the ball $\tilde{B}_R = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 < R\}$. The function $\tilde{p}$ can be represented by a series of homogeneous, harmonic polynomials that converges uniformly on $\tilde{B}_{R''}$ ([60]). This corresponds to a representation of p_1 on $B_{R''}$ by a series

$$p_1 = \sum_{k=0}^{\infty} c_k l_k \qquad \text{with} \qquad c_k = \int\limits_{|(r,z)|=R''} p_1 \, l_{-(k+1)} \, ds$$

It follows straightforwardly from inequality (2.23) that

$$|c_k| \leq \pi M(k+1) \, (R'')^{-k} \qquad k \in \mathbb{N}_0$$

where M denotes the maximum of $|p_1|$ on $\overline{B}_{R''}$. It follows from inequality (2.25) that for all $(r,z) \in B_{R'}$

$$|c_k l_k(r,z)| < \pi M(k+1)(R'/R'')^k = a_k \qquad k \in \mathbb{N}_0$$
$$|k c_k l_{k-1}(r,z)| < \pi M k(k+1) R'^{-1} (R'/R'')^k = b_k \qquad k \in \mathbb{N}$$

where the positive numbers $a_k, k \in \mathbb{N}_0$ and $b_k, k \in \mathbb{N}$ are such that the series $\sum_{k=0}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ both converge. Hence, we can apply Weierstrass' criterion for uniform convergence of a series of functions ([46]) and find that the series $\sum_{k=0}^{\infty} c_k l_k$ and $\sum_{k=1}^{\infty} k c_k l_k$ converge uniformly on $B_{R'}$. By means of this result, some standard theorems from analysis and Lemma 2.9, we then get for all $(r,z) \in B_{R'}$:

$$p_2(r,z) = \frac{1}{r} \int_{(0,0)}^{(r,z)} -\rho\partial_\xi p_1(\rho,\xi)\, d\rho + \rho\partial_\rho p_1(\rho,\xi)\, d\xi$$

$$= -\frac{1}{r}\int_0^r \rho\partial_z \left(\sum_{k=0}^{\infty} c_k l_k(\rho,z)\right) d\rho = -\frac{1}{r}\int_0^r \rho \left(\sum_{k=1}^{\infty} k c_k l_{k-1}(\rho,z)\right) d\rho$$

$$= \sum_{k=1}^{\infty} c_k \left(-\frac{1}{r}\partial_z \int_0^r \rho l_k(\rho,z) d\rho\right) = \sum_{k=0}^{\infty} c_k m_k(r,z)$$

One then argues, using inequality (2.24) and the arguments given above, that the series in the right-hand side converges uniformly on $B_{R'}$. We conclude that the series in relation (2.21) also converge uniformly. Expression (2.22) follows straightforwardly from relation (2.19). □

Remark 2.14 One checks that the coefficients in relation (2.21) are also given by the following Cauchy-type integrals:

$$c_k = \frac{1}{2-\delta_{k0}} \int_{|(r,z)|=R'} p \cdot T_{-(k+1)}\, ds$$

However, a Cauchy-like approach to the theory of deformed analytic function does not exist; the relation above for example does not hold if the path of integration is not circular.

2.2.2 Axially symmetric solutions of Stokes' equations

In this subsection we show that it is possible to represent axially symmetric solutions of Stokes' equations in terms of deformed analytic functions. First we explain what we mean by axially symmetric solutions of Stokes' equations.
A domain $G \in \mathbb{R}^3$ is called axially symmetric if we can choose cylindrical coordinates (r,ϕ,z) such that -after re-coordination of G- we have: $(r,\phi,z) \in G$ if and only if $(r,0,z) \in G$ for all $\phi \in (-\pi,\pi]$. Solutions of Stokes' equations (2.1) on an axially symmetric domain G are called axially symmetric if the function p and the components v_r, v_ϕ and v_z of $\underline{v}$ with respect to the orthonormal basis

$\{\partial_r, \frac{1}{r}\partial_\phi, \partial_z\}$ do not depend on the variable ϕ. Stokes' equations then get the following form (see also equations (1.2)):

$$v_{r,zz} - v_{z,rz} = p_{,r} \tag{2.26a}$$
$$v_{\phi,rr} + v_{\phi,zz} + (v_\phi/r)_{,r} = K_2 v_\phi = 0 \tag{2.26b}$$
$$v_{z,rr} - v_{r,rz} + (v_{z,r} - v_{r,z})/r = p_{,z} \tag{2.26c}$$
$$v_{r,r} + v_r/r + v_{z,z} = 0 \tag{2.26d}$$

Since the component v_ϕ decouples, we come to the following definition.

Definition 2.15 Functions p, v_r and v_z on an open symmetric domain $G \subset I\!R^2$ are said to satisfy Stokes' equations with axial symmetry if they are sufficiently smooth, if p and v_z are symmetric, if v_r is anti-symmetric and if they satisfy the equations (2.26a), (2.26c) and (2.26d).

Theorem 2.16 *If p, v_r and v_z satisfy Stokes' equations with axial symmetry on an open, simply connected and symmetric domain G, then deformed analytic functions φ and χ exist such that*

$$\begin{aligned} p &= -4\varphi_{1,z} \\ v_r &= r\varphi_{2,r} - z\varphi_{2,z} - \varphi_2 + \chi_{2,z} \\ v_z &= r\varphi_{1,r} - z\varphi_{1,z} + \varphi_1 + \chi_{1,z} \end{aligned} \tag{2.27}$$

Conversely, if φ and χ are deformed analytic functions on an open domain G, the functions p, v_r and v_z given by the expressions (2.27) *satisfy Stokes' equations.*

Proof
The second assertion is easily proved by differentiation. The first assertion is proved as follows. Let p, v_r and v_z satisfy Stokes' equations with axial symmetry on an open, simply connected and symmetric domain $G \subset I\!R^2$. One easily shows that it follows from relation (2.26d) that an anti-symmetric function ψ exists on G such that

$$v_r = \partial_z \psi \qquad v_z = -(\partial_r + \frac{1}{r})\psi \tag{2.28}$$

Substitution of these relations into equations (2.26a) and (2.26c) leads to

$$\partial_z(K_2\psi) = \partial_r p \qquad -(\partial_r + \frac{1}{r})(K_2\psi) = \partial_z p \tag{2.29}$$

Hence, if we define

$$q = K_2\psi \tag{2.30}$$

we find that $(p \;\; q)^T$ is a deformed analytic function on G. It follows from Lemma 2.10 that functions φ_1 and φ_2 exist on G such that the function $\varphi = (\varphi_1 \;\; \varphi_2)^T$ satisfies

$$D\varphi = 0 \qquad \tilde{D}\varphi = -\frac{1}{2}(p \;\; q)^T$$

Lemma 2.8 then implies

$$\begin{aligned} K_2(\psi + r\varphi_1 + z\varphi_2) &= q + \left(\partial_r(\partial_r + \tfrac{1}{r}) + \partial_z^2\right)(r\varphi_1 + z\varphi_2) \\ &= q + 2(\partial_r\varphi_1 + \partial_z\varphi_2) = 0 \end{aligned}$$

So, a deformed analytical function $\chi = (\chi_1 \ \ \chi_2)^T$ on G exists such that

$$\psi = -r\varphi_1 - z\varphi_2 + \chi_2$$

We remark that it follows from Lemma 2.10 that a deformed analytic function $\tilde{\chi}$ exists such that

$$\tilde{\chi}' = \chi' - \varphi$$

The relations (2.27) are now obtained by omitting the tilde and straightforward substitutions and differentiations. □

Remarks 2.17 If we identify deformed analytic functions which only differ by the first two terms in expansion (2.21) and if we also identify axially symmetric solutions Stokes' equations which only differ by a uniform pressure and a uniform velocity in the z-direction, then the correspondence between pairs (φ, χ) of deformed analytic functions and axially symmetric solutions of Stokes' equations is one-to-one (see also Remark 2.3).
We further notice that if we substitute $\tilde{\varphi} = -i\varphi$ into relations (2.11), omit the tilde and rearrange some terms, the resulting relations are completely analogous to relations (2.27).

We end this subsection by a discussion about the generalization of the concept of the Airy-function for axially symmetric Stokes flows. The easiest way to prove the existence of the Airy-function in the two-dimensional case was explained in Remark 2.2. We cannot copy the line of reasoning in the axially symmetric case as the rows of the matrix corresponding to the stress tensor cannot be considered as solenoidal vector fields. The reason for this is that not all Christoffel symbols vanish and, loosely speaking, the components of this tensor (with respect to the basis $\{\partial_r, \frac{1}{r}\partial_\phi, \partial_z\}$ and its dual) mix up when one calculates $T^i_{\ j;i}$ (where $i, j = 1, 2, 3$ correspond to the variables r, ϕ and z; see also identity (2.64)). However, we can introduce a kind of Airy-function based on the similarity of the identities given in Section 2.1 and the following identities.
Let σ be a function on G such that

$$K_1\sigma = p \tag{2.31}$$

It follows from relations (2.29), (2.30) and (2.31) that the function F on G defined by

$$F = (\sigma \ \ \psi)^T$$

satisfies $DD\overline{D}F = 0$ (see Remark 2.6). A particular solution of equation (2.31) is given by

$$\sigma = -z\varphi_1 + r\varphi_2 + \chi_1$$

as can be checked by differentiation. This particular auxiliary function σ has the property that the function F can be written as

$$F = u\varphi + \chi$$

where

$$u = \begin{pmatrix} -z & r \\ -r & -z \end{pmatrix}$$

The comparison of this result with relation (2.9) inspires us to define the Airy-function in this case as

$$\rho = \sigma/2 = (-z\varphi_1 + r\varphi_2 + \chi_1)/2$$

We wonder whether this function has any physical interpretation.

2.3 Three-dimensional Stokes flow

In this section we show that a solution of Stokes' equations on a domain $G \subset \mathbb{R}^3$ can be represented by a pair of left monogenic functions if this domain G has a property which we will call x-normality. In Subsection 2.3.1 we give a short introduction to the notation of Clifford analysis ([13, 22, 21]), recapitulate the definition of a left monogenic function and show how Stokes' equations can be written as a third order differential equation. In Subsection 2.3.2 we give the general solution of a class of differential equations, one of which is the mentioned third order differential equation. We show in Subsection 2.3.3 how this result can be used to represent Stokes flows in three dimensions in terms of two left monogenic functions.

2.3.1 Clifford Analysis

The Clifford algebra $Cl_{0,2}$ is algebraically isomorphic to the algebra $\mathbb{H}(\mathbb{R})$ of quaternions over $\mathbb{R}$, although its structure is slightly richer (it turns out to be a graded algebra). In stead of using the quaternionic symbols i, j and k we write e_1, e_2 and ι. So, the set $Cl_{0,2}$ is a real four-dimensional vector space spanned by the basis $\{1, e_1, e_2, \iota\}$. The Clifford product is denoted by juxtaposition. The Clifford product of two elements in $Cl_{0,2}$ can be calculated by the following rules:

- the Clifford product is linear
- 1 is the identity

- the products of the elements e_1, e_2 and ι is given by

$$e_1^2 = e_2^2 = \iota^2 = -1$$

$$e_1 e_2 = -e_2 e_1 = \iota \qquad \iota e_1 = -e_1 \iota = e_2 \qquad e_2 \iota = -\iota e_2 = e_1$$

An element $A = A_0 + A_1 e_1 + A_2 e_2 + A_{12}\iota \in Cl_{0,2}$; $A_0, A_1, A_2, A_{12} \in I\!R$ can be decomposed in what is called the scalar part, the vectorial part and the bivectorial part:

$$A = (A)_0 + (A)_1 + (A)_2$$
$$(A)_0 = A_0 \qquad (A)_1 = A_1 e_1 + A_2 e_2 \qquad (A)_2 = A_{12}\iota$$

The operation of grade involution is defined by

$$\overline{A} = (A)_0 - (A)_1 + (A)_2$$

The $Cl_{0,2}$-valued differential operators D and $\overline{D}$ are defined by

$$D = \partial_x + e_1 \partial_y + e_2 \partial_z \qquad \overline{D} = \partial_x - e_1 \partial_y - e_2 \partial_z$$

The operator D is called the generalized Cauchy-Riemann operator. We notice that

$$D\overline{D} = \overline{D}D = \Delta \tag{2.32}$$

A differentiable $Cl_{0,2}$-valued function f on an open domain $G \subset I\!R^3$ is called (anti-)left monogenic if it satisfies

$$Df = 0 \qquad (\overline{D}f = 0)$$

One checks that this equation is equivalent with the following, so-called generalized Cauchy-Riemann equations:

$$\begin{aligned} f_{0,x} - f_{1,y} - f_{2,z} &= 0 \\ f_{1,x} + f_{0,y} + f_{12,z} &= 0 \\ f_{2,x} - f_{12,y} + f_{0,z} &= 0 \\ f_{12,x} + f_{2,y} - f_{1,z} &= 0 \end{aligned}$$

We will now rewrite Stokes' equations (2.1) in three dimensions in the notation of Clifford analysis by means of an auxiliary function. Let $G \subset I\!R^3$ be an open 2-connected domain and let p and $\underline{v}$ be smooth functions satisfying Stokes' equations on G. As $\underline{v}$ is a solenoidal vector-field, a vector-field $\underline{\psi}$ on G exists such that

$$\underline{v} = -\nabla \wedge \underline{\psi} \tag{2.33}$$

where $\nabla \wedge \underline{\psi}$ denotes the curl of $\underline{\psi}$. This vector-field $\underline{\psi}$ is determined up to a gradient of a function and it is easily shown that this function can be chosen such that

$$\nabla \cdot \underline{\psi} = 0 \tag{2.34}$$

The vector-field $\underline{\psi}$ is still not unique. If f is a real harmonic function on G, then the vector-field

$$\underline{\tilde{\psi}} = \underline{\psi} + \nabla f \tag{2.35}$$

also satisfies the relations (2.33) and (2.34) with $\underline{\psi}$ replaced by $\underline{\tilde{\psi}}$.
As the function p on G is smooth, a real function σ on G exists, which we will call the auxiliary function, such that

$$p = \Delta\sigma \tag{2.36}$$

It is clear that σ is not unique: if g is an arbitrary real harmonic function on G, then the function

$$\tilde{\sigma} = \sigma + g \tag{2.37}$$

is also an auxiliary function, i.e. $\tilde{\sigma}$ satisfies $\Delta\tilde{\sigma} = p$.
Next we identify the vectors $\underline{v}$ and $\underline{\psi}$ with the $Cl_{0,2}$-valued functions v and ψ in the following way:

$$v = v_1 + v_2e_1 + v_3e_2 \qquad\qquad \psi = \psi_1 + \psi_2e_1 + \psi_3e_2$$

The equation $\Delta\underline{v} = \nabla p$ and the relations (2.33), (2.34) and (2.36) can then be written as

$$D\overline{D}v = Dp \qquad v = -D\,\iota\,\psi \qquad p = D\overline{D}\sigma \tag{2.38}$$

It follows immediately from these relations that the function F defined by

$$F = \sigma + \iota\psi \tag{2.39}$$

satisfies

$$DD\overline{D}F = 0 \tag{2.40}$$

(see also equation (2.6)).
Inspired by the results of the previous sections, we ask the following two questions. First, is it possible to give the general solution of equation (2.40) in terms of left monogenic functions? This question is answered affirmatively in the next subsection. We are then able to represent Stokes flows in terms of three left monogenic functions. However, we already remarked that there is some arbitrariness in the functions ψ and σ. This leads to the second question: is it possible to choose the functions ψ and σ such that the mentioned representation of Stokes flows can further be reduced? This question is answered affirmatively in Subsection 2.3.3 where we show that Stokes flows can be represented by two left monogenic functions under general conditions.

2.3.2 Solutions of $D^n\overline{D}^m f = 0$

In the previous subsection we met the partial differential equation (2.40), where F is a $Cl_{0,2}$-function on an open domain $G \subset I\!R^3$. In this subsection we consider the following, more general equation:

$$D^n\overline{D}^m f = 0 \qquad n, m \in I\!N_0 \tag{2.41}$$

It turns out that if we lay geometric restrictions on the domain G, then the general solution of this equation can be represented in terms of left monogenic functions (see also [44]).
As we already remarked in the previous subsection, left monogenic functions are a kind of generalization of analytic functions. In order to make things better understandable, we will explain in which sense the propositions in this subsection are generalizations of well-known results on analytic functions. For example, it is well known that every analytic function has a primitive. In the notation of Subsection 2.1, this means that for every function f on a simply-connected domain G that satisfies $Df = 0$, a function F exists such that

$$\overline{D}F = f \qquad DF = 0$$

This function F can be constructed by a line integral. However, a generalization of the concept of primitive functions for left monogenic functions on a domain $G \subset I\!R^3$ cannot be based on line integrals. The reason for this is that, roughly speaking, left monogenic functions correspond to closed 2-forms and surface integrals are therefore preferred above line integrals; see for instance the generalized Cauchy theorem ([22]). This explains why we will construct left monogenic primitives in a somewhat different way and why we have to put restrictions on the geometry of the underlying domains.

Definition 2.18 A domain $G \subset I\!R^3$ is called x-normal if a plane $V : x = a$ exists such that:

i). $V \cap G$ is simply connected

ii). every line segment connecting a point in G with its orthogonal projection on V lies entirely in G

We note that an x-normal domain is 2-connected. We also note that for every axially symmetric domain which is homeomorphic to a ball, one can choose coordinates such that G is x-normal.

Proposition 2.19 *Let f be a left monogenic function on an open x-normal domain $G \subset I\!R^3$. The function $\overline{D}f$ is left monogenic. Moreover, a left monogenic function F on G exists such that*

$$\overline{D}F = f \tag{2.42}$$

We call $\overline{D}f$ the derivative of f and F a primitive of f.

Proof

The first assertion follows immediately from identity (2.32) –which implies that D and $\overline{D}$ commute– and the remark that every left monogenic function on a domain G is in $C^\infty(G)$ ([13]). The proof of the other assertion runs as follows. Let

$$f = p + q_1 e_1 + q_2 e_2 + r \tag{2.43}$$

be a left monogenic function on G. We will prove the existence of functions P, Q_1, Q_2 and R on G such that the function F defined by

$$F = (P + Q_1 e_1 + Q_2 e_2 + R\iota)/2 \tag{2.44}$$

satisfies

$$DF = 0 \qquad \overline{D}F = f \tag{2.45}$$

Let $V : x = a$ be the plane to which respect G is x-normal. Consider the following Poisson equation for a real function f_2 on $V \cap G$:

$$\begin{aligned} \Delta f_2(y,z) = f_{2,yy}(y,z) + f_{2,zz}(y,z) &= -q_{2,x}(a,y,z) \\ &= p_z(a,y,z) - r_y(a,y,z) \end{aligned}$$

where one of the generalized Cauchy-Riemann equations has been used. It can be proved ([20]) that as $-q_{2,x} \in C^\infty(V \cap G)$, a smooth solution f_2 of this equation exists. Next, consider the following set of coupled partial differential equations for a real, differentiable function f_1 on $V \cap G$:

$$\begin{aligned} f_{1,y}(y,z) &= p(a,y,z) - f_{2,z}(y,z) \\ f_{1,z}(y,z) &= r(a,y,z) + f_{2,y}(y,z) \end{aligned}$$

One proves by elementary methods that a solution f_1 exists because $V \cap G$ is simply connected and because the condition of compatibility is satisfied:

$$(p - f_{2,z})_z = p_z - f_{2,zz} = p_z + f_{2,yy} - p_z + r_y = (r + f_{2,y})_y$$

As G is x-normal, we can define functions Q_1 and Q_2 on G by

$$Q_i(x,y,z) = f_i(y,z) + \int_a^x q_i(\xi,y,z)d\xi \qquad i = 1,2$$

Using the generalized Cauchy-Riemann equation again, one checks:

$$\begin{aligned} Q_{1,x} &= q_1 & \qquad Q_{1,y} + Q_{2,z} &= p \\ Q_{2,x} &= q_2 & \qquad -Q_{2,y} + Q_{1,z} &= r \end{aligned} \tag{2.46}$$

In the same way, one constructs functions P and R satisfying

$$\begin{array}{ll} P_x = p & P_y + R_z = -q_1 \\ R_x = r & -R_y + P_z = -q_2 \end{array} \qquad (2.47)$$

The relations (2.45) follow after substitution of the relations (2.46) and (2.47) into identity (2.43). □

Remark 2.20 As the grade involution of a left monogenic function is anti-left monogenic ($Df = 0 \Rightarrow \overline{Df} = \overline{D}\,\overline{f} = 0$), it follows immediately from Proposition 2.19 that for every anti-left monogenic function f on an open x-normal domain G, an anti-left monogenic function F exists on G such that $DF = f$.

Remark 2.21 It has already been remarked that Proposition 2.19 is a generalization of the theorem which states that every analytic function on a simply connected domain $G \subset \mathbb{C}$ has a primitive. The reverse of this statement is also true: a connected domain $G \subset \mathbb{C}$ which possesses the property that every analytical function on G has a (single-valued, analytic) primitive function on G is simply connected. One therefore may ask whether domains $G \subset \mathbb{R}^3$ exist which are not x-normal but do possess the property that every left monogenic function on it has a primitive. We give a partial answer to this question in Appendix A.

Proposition 2.22 *Let n and m be two integers not both equal to zero, let $G \subset \mathbb{R}^3$ be an open x-normal domain and let f be a smooth $Cl_{0,2}$-valued function on G satisfying*

$$D^n \overline{D}^m f = 0 \qquad (2.48)$$

Then left monogenic functions g_k, $k = 1, ..., n$ and h_k, $k = 1, ..., m$ on G exist such that

$$f = \sum_{k=1}^{n} x^{k-1} g_k + \sum_{k=1}^{m} x^{k-1} \overline{h_k} \qquad (2.49)$$

Proof

The assertions for $n = 1$, $m = 0$ and $n = 0$, $m = 1$ are trivial. The assertion for an arbitrary pair n, m can then be proved by two inductions. Only the following step is proven: we assume that the assertion in the proposition holds for a certain pair n, m and show that it holds also for the pair $n, m+1$; the other induction step can be proved in a similar way.

Consider the equation

$$D^n \overline{D}^{m+1} f = (D^n \overline{D}^m)(\overline{D} f) = 0$$

We assume that left monogenic functions g_k, $k = 1, ..., n$ and h_k, $k = 1, ..., m$ exist such that

$$\overline{D} f = \sum_{k=1}^{n} x^{k-1} g_k + \sum_{k=1}^{m} x^{k-1} \overline{h}_k \qquad (2.50)$$

With the aid of Proposition 2.19, one argues that a set of left monogenic functions $G_k, k = 1, ..., n$ exists satisfying the following n coupled partial differential equations:

$$\begin{aligned} \overline{D}G_n &= g_n \\ \overline{D}G_k + kG_{k+1} &= g_k \qquad\qquad k = 1, ..., n-1 \end{aligned}$$

We check that the function F defined by

$$F = \sum_{k=1}^{n} x^{k-1} G_k + \sum_{k=2}^{m+1} \frac{x^{k-1}}{k-1} \overline{h_{k-1}} \tag{2.51}$$

is a particular solution of equation (2.50):

$$\begin{aligned} \overline{D}F &= \sum_{k=1}^{n} \left((\overline{D}x^{k-1})G_k + x^{k-1}\,\overline{D}G_k \right) \\ &\quad + \sum_{k=2}^{m+1} \left(\left(\overline{D}\,\frac{x^{k-1}}{k-1}\right) \overline{h_{k-1}} + \frac{x^{k-1}}{k-1}\,\overline{D}\,\overline{h_{k-1}} \right) \\ &= \sum_{k=1}^{n} \left((k-1)x^{k-2}\,G_k + x^{k-1}\,\overline{D}G_k \right) + \sum_{k=2}^{m+1} x^{k-2}\,\overline{h_{k-1}} \\ &= \sum_{k=1}^{n-1} \left(kx^{k-1}\,G_{k+1} + x^{k-1}\,\overline{D}G_k \right) + x^{n-1}\,\overline{D}G_n + \sum_{k=2}^{m+1} x^{k-2}\,\overline{h_{k-1}} \\ &= \sum_{k=1}^{n} x^{k-1}\,g_k + \sum_{k=1}^{m} x^{k-1}\,\overline{h_k} \end{aligned}$$

Hence, a left monogenic function h_0 exists on G such that: $f = F + \overline{h_0}$. One checks that the substitution of this relation into identity (2.51) corresponds to relation (2.49) with m replaced by $m+1$. □

Remark 2.23 The reverse of this proposition also holds: a function f that can be written in the form of expression (2.49) satisfies equation (2.48). This can easily be verified.

Remark 2.24 It has been remarked in Section 2.1 that every smooth complex valued function f on an open simply connected domain $G \subset \mathbb{R}^2$ satisfying equation (2.7) can be written as in expression (2.8). It then follows that such a function f also can be written as

$$f = \sum_{k=1}^{n} x^{k-1}\tilde{g}_k + \sum_{k=1}^{m} x^{k-1}\overline{\tilde{h}_k}$$

where $\tilde{g}_k, k = 1, \ldots, n$ and $\tilde{h}_k, k = 1, \ldots, m$ are analytic functions. This expression corresponds to expression (2.49).

Remark 2.25 Special cases of Proposition 2.22 are:

- $n = m = 1$: a $Cl_{0,2}$-valued harmonic function f can be written as the sum of a left monogenic function g and an anti-left monogenic function h. If f is a real harmonic function, it is the scalar part of a left monogenic function. Compare this result to Proposition 8.6 in [13], where it is shown that every real harmonic function on a star-shaped domain can be written as the scalar part of a left monogenic function.

- $n = 2$, $m = 1$: the general solution of equation (2.40) in the previous subsection is given by

$$F = xg_1 + g_2 + \overline{h} \tag{2.52}$$

  where g_1, g_2 and h are left monogenic functions.

- $n = m = 2$: one easily obtains a representation of biharmonic, real or $Cl_{0,2}$-valued functions in terms of left monogenic functions.

Remark 2.26 The functions g_k, $k = 1, ..., n$ and h_k, $k = 1, ..., m$ in relation (2.49) are not uniquely determined. In order to answer the question to which extent these functions are determined, we need the general solution of the following equations for a smooth function f:

$$\overline{D}^n f = Df = 0 \qquad n \in I\!N \tag{2.53}$$

For example, let $g_1, g_2, h, \tilde{g}_1, \tilde{g}_2$ and $\tilde{h}$ be left monogenic functions on an open (x-normal) domain such that:

$$xg_1 + g_2 + \overline{h} = x\tilde{g}_1 + \tilde{g}_2 + \overline{\tilde{h}}$$

(see also relation (2.52)). Applying the operators $D\overline{D}$, $\overline{D}$ and D^2 to both sides of this relation, we get:

$$\overline{D}(g_1 - \tilde{g}_1) = 0 \qquad \overline{D}^2(g_2 - \tilde{g}_2) = 0 \qquad \overline{D}^2(h - \tilde{h}) = 0$$

The general solution of equation (2.53) is given in [45] but we will not elaborate on this result.

2.3.3 Representations of Stokes flows

We show in this subsection how the results of the previous subsections can be used to represent solutions of Stokes' equations three dimensions by a pair of left monogenic functions (see also [66, 48]).
Let p and v satisfy Stokes' equations on an open x-normal domain $G \subset I\!R^3$ and

let σ, ψ and F be as in relations (2.38) and (2.39). It follows from equation (2.40) and Proposition 2.22 that F can be written as

$$F = x\varphi + \chi \tag{2.54}$$

where φ is a left monogenic function and where χ is a $Cl_{0,2}$-valued harmonic function. We already remarked in Subsection 2.3.1 that if f and g are real harmonic functions on G, then the functions $\tilde{\psi}$ and $\tilde{\sigma}$ defined by:

$$\tilde{\psi} = \psi + Df \qquad\qquad \tilde{\sigma} = \sigma + g \tag{2.55}$$

play the same role as ψ and σ. It is clear that the function $\tilde{F}$ defined by

$$\tilde{F} = \tilde{\sigma} + \iota\tilde{\psi}$$

can be written as

$$\tilde{F} = x\varphi + \chi + g + \iota Df \tag{2.56}$$

Now we want to follow the strategy explained in Subsection 2.1; that is: we want to choose f and g such that the function $\tilde{\chi}$ defined by

$$\tilde{\chi} = \chi + g + \iota Df \tag{2.57}$$

corresponds to a left monogenic function. It turns out that it is impossible to gauge f and g such that $\tilde{\chi}$ is a left monogenic function but we will show that these functions can be chosen such that $\tilde{\chi}$ is anti-left monogenic. We then omit the tildes and find that F can be written as in relation (2.54) where φ is a left monogenic and χ is an anti-left monogenic function. So, the next lemma remains to be proved.

Lemma 2.27 *Let χ be a $Cl_{0,2}$-valued harmonic function on an open x-normal domain $G \subset \mathbb{R}^3$. Then real harmonic functions f and g on G exist such that*

$$\overline{D}(\chi + g + \iota D f) = 0 \tag{2.58}$$

Proof

We decompose χ in the standard way:

$$\chi = \chi_0 + \chi_1 e_1 + \chi_2 e_2 + \chi_{12}\iota$$

Let $V : x = a$ be a plane to which respect G is x-normal. It is possible to construct functions g_1 and F_1 on $G \cap V$ satisfying the following Poisson equations:

$$\Delta g_1(y,z) = (\chi_{0,xx} + \chi_{1,xy} + \chi_{2,xz})|_{(a,y,z)}$$
$$\Delta F_1(y,z) = (\chi_{12,xx} - \chi_{2,xy} + \chi_{1,xz})|_{(a,y,z)}$$

We define a function σ on $G \cap V$ by

$$\begin{aligned} \sigma_1(y,z) &= (\chi_{1,x} - \chi_{0,y} - \chi_{12,z})|_{(a,y,z)} - g_{1,y}(y,z) - F_{1,z}(y,z) \\ \sigma_2(y,z) &= (\chi_{2,x} + \chi_{12,y} - \chi_{0,z})|_{(a,y,z)} - g_{1,z}(y,z) + F_{1,y}(y,z) \\ \sigma(y,z) &= \sigma_1(y,z) - \iota\sigma_2(y,z) \end{aligned}$$

One checks that σ is an analytic function of the variable $w = y + \iota z$. As $G \cap V$ is simply connected, functions g_2 and F_2 on $G \cap V$ exist such that

$$g_{2,y} = F_{2,z} = \sigma_1/2 \qquad g_{2,z} = -F_{2,y} = \sigma_2/2$$

As G is x-normal, we can define functions g and F on G by

$$\begin{aligned} g(x,y,z) &= -\int_a^x (\chi_{0,\xi} + \chi_{1,y} + \chi_{2,z})|_{(\xi,y,z)}\, d\xi + g_1(y,z) + g_2(y,z) \\ F(x,y,z) &= -\int_a^x (\chi_{12,\xi} - \chi_{2,y} + \chi_{1,z})|_{(\xi,y,z)}\, d\xi + F_1(y,z) + F_2(y,z) \end{aligned}$$

The following identities can then be checked by differentiation:

$$\begin{aligned} &\Delta g = \Delta F = 0 \\ &g_x = -(\chi_{0,x} + \chi_{1,y} + \chi_{2,x}) \qquad F_x = -\chi_{12,x} + \chi_{2,y} - \chi_{1,z} \\ &g_y + F_z = \chi_{1,x} - \chi_{0,y} - \chi_{12,z} \quad g_z - F_y = \chi_{2,x} + \chi_{12,y} - \chi_{0,z} \end{aligned} \tag{2.59}$$

Finally, one shows that it is possible to construct a harmonic function f on G such that $f_x = F/2$. The identity (2.58) follows from this relation and the relations (2.59). □

We are now able to formulate our final result, which can be compared with the results obtained in Sections 2.1 and 2.2.

Theorem 2.28 *Let p and v be smooth functions satisfying Stokes'equations on an open x-normal domain $G \subset \mathbb{R}^3$. Then left monogenic functions φ and χ exist on G such that*

$$\begin{aligned} p &= 2\varphi_{0,x} \\ v_1 &= x\varphi_{0,x} - \chi_{0,x} \\ v_2 &= x\varphi_{0,y} - \varphi_1 - \chi_{1,x} - \chi_{12,z} \\ v_3 &= x\varphi_{0,z} - \varphi_2 - \chi_{2,x} + \chi_{12,y} \end{aligned} \tag{2.60}$$

Conversely, if φ and χ are left monogenic functions on an open domain $G \subset \mathbb{R}^3$, then the functions p, v_1, v_2 and v_3 given by the expressions (2.60) *satisfy Stokes' equations.*

Proof

The proof of the first assertion runs as follows. Let p and v satisfy Stokes' equations on G. We showed in the beginning of this subsection that left monogenic functions φ and χ on G exist such that:

$$F = \sigma + \iota\psi = x\varphi + \overline{\chi}$$
$$p = \Delta\sigma \qquad v = -D\iota\psi$$

The expressions (2.60) follow from these relations. The second assertion can be proved by a straightforward differentiation of the relations (2.60) in combination with the generalized Cauchy-Riemann equations. □

We end this subsection by remarking that the relation between solutions of Stokes' equations and pairs of left monogenic functions is not one-to-one; there are a lot of pairs of left monogenic functions which correspond to the trivial solution $p = v = 0$ ([45]). The problem which pairs of left monogenic functions correspond to the solution where p is constant and v is a rigid-body motion is unsolved.

2.4 Traction formulae

In this section we rewrite the boundary value problem (1.8) in the axially symmetric case with the use of the representations obtained in Section 2.2. We also state the analogous, known result in the two-dimensional case.

Let $G \subset I\!R^3$ be an axially symmetric domain and let V denote the closed half plane $\phi = 0$ including the z-axis (see Section 2.2). We assume that G contains the origin $\underline{0}$, that G is 2-connected and that ∂G is a compact C^2-surface. This implies that the intersection $\tilde{G}$ of G and V is connected and simply connected and that the intersection γ of ∂G and V is a compact C^2-curve. (This curve has of course nothing to do with the surface tension coefficient which was also denoted by γ in Chapter 1).
Let $p, \underline{v} \in C^2(\overline{G})$ satisfy Stokes' equations on G and satisfy the boundary condition

$$\underline{\underline{T}}(\underline{x})\, \underline{n}(\underline{x}) = -\kappa(\underline{x})\, \underline{n}(\underline{x}) \qquad \underline{x} \in \partial G \qquad (2.61)$$

(see also Section 1.2). It follows from symmetry considerations that p and $\underline{v}$ do not depend on the variable ϕ and it therefore suffices to solve equation (2.61) for points $\underline{x}$ on γ.
First we give an expression for the right-hand side of relation (2.61) for points $\underline{x}$ on γ. Let γ be parameterized by its arclength s such that $r(0) = 0$ and $z(0) < 0$. Let $\underline{x} = (r(s), 0, z(s))$ denote a point on γ. As the basis $\alpha = \{\partial_r, \frac{1}{r}\partial_\phi, \partial_z\}$ is orthonormal, we get the following expression for the outward pointing normal vector $\underline{n}(\underline{x})$ at $\underline{x}$ with respect to this basis α:

$$\underline{n}(\underline{x}) = (\dot{z}(s) \;\; 0 \;\; -\dot{r}(s))^T \qquad (2.62)$$

where the dot denotes differentiation with respect to s. Let $W : z = z(s)$ be the plane through the point $\underline{x}$ orthogonal to the z-axis. It is clear that the radius of the circle in this plane with centre $(0, 0, z(s))$ through the point $\underline{x}$ is $r(s)$. It is also clear that the inner product of the normal vector of W and the normalized vector tangent to γ at $\underline{x}$ is $\dot{z}(s)$. Together with some standard differential geometry ([83]), this leads to the following expression for the right-hand side of relation (2.61):

$$-\kappa(\underline{x})\ \underline{n}(\underline{x}) = \begin{pmatrix} \ddot{r}(s) - \frac{\dot{z}(s)^2}{r(s)} \\ 0 \\ \ddot{z}(s) + \frac{\dot{z}(s)\dot{r}(s)}{r(s)} \end{pmatrix} \qquad \underline{x} = (r(s), 0, z(s)) \in \gamma \tag{2.63}$$

Next we consider the left-hand side of relation (2.61). The components of the matrix T corresponding to the stress tensor $\underline{\underline{T}}$ with respect to the basis α and its dual are given by expression (1.3):

$$T = \begin{pmatrix} -p + 2v_{r,r} & r(\frac{v_\phi}{r})_{,r} & v_{r,z} + v_{z,r} \\ r(\frac{v_\phi}{r})_{,r} & -p + \frac{2v_r}{r} & v_{\phi,z} \\ v_{r,z} + v_{z,r} & v_{\phi,r} & -p + 2v_{z,z} \end{pmatrix} \tag{2.64}$$

Substituting the representations found in Section 2.2 and using identity (2.62), we find the following expression for the left-hand side of relation (2.61):

$$\underline{\underline{T}}(\underline{x})\,\underline{n}(\underline{x}) = \begin{pmatrix} -2\left\{\frac{d}{ds}\Big(v_z(\underline{x}(s)) - 2\varphi_1(\underline{x}(s))\Big) + v_r(\underline{x}(s))\frac{\dot{z}(s)}{r(s)}\right\} \\ -r(s)\frac{\partial}{\partial_n}(\frac{v_\phi}{r})|_{(\underline{x}(s))} \\ 2\left\{\frac{d}{ds}\Big(v_r(\underline{x}(s)) + 2\varphi_2(\underline{x}(s))\Big) + \Big(v_r(\underline{x}(s)) + 2\varphi_2(\underline{x}(s))\Big)\frac{\dot{r}(s)}{r(s)}\right\} \end{pmatrix} \tag{2.65}$$

where $\underline{x}(s) = (r(s), 0, z(s)) \in \gamma$ and where $\frac{\partial}{\partial_n}$ denotes differentiation in the direction of the normal vector $\underline{n}$.
Now we substitute identities (2.63) and (2.65) into equation (2.61), integrate and find the following relations:

$$v_r(\underline{x}(s)) + 2\varphi_2(\underline{x}(s)) = \dot{z}(s)/2 \tag{2.66a}$$

$$v_\phi(\underline{x}(s)) = \alpha r(s) \tag{2.66b}$$

$$v_z(\underline{x}(s)) - 2\varphi_1(\underline{x}(s)) = -\dot{r}(s)/2 + 2\int_0^s \frac{\varphi_2(\underline{x}(\sigma))\dot{z}(\sigma)}{r(\sigma)}\, d\sigma + \beta \tag{2.66c}$$

where α and β are arbitrary real constants. We call equations (2.66) traction formulae. It follows from Remark 2.17 that we may take $\alpha = \beta = 0$ without loss of generality.

Remark 2.29 Let l denote the length of γ. For each $s \in [0, l]$ we can define

$$\underline{x}(-s) = (r(-s), 0, z(-s)) = (-r(s), 0, z(s)) \in \partial G$$

The integrand at the right-hand side of formula (2.66c) can then be considered as a function on $[-l, l]$. One checks that this integrand is an odd function in accordance with the symmetry properties of the other terms in relation (2.66c).

Example 2.30 We consider a ball of viscous matter. A solution of Stokes' equations with boundary condition (2.61) is given by

$$p(r, \phi, z) = p_0 \qquad \underline{v}(r, \phi, z) = \underline{0}$$

A pair of deformed analytic functions φ and χ representing this flow is

$$\varphi(r, z) = -(p_0/4)T_1(r, z) \qquad \chi(r, z) = 0$$

where T_1 is defined in Definition 2.11. One checks that equations (2.66a) and (2.66c) are satisfied.

We end this section by formulating the analogous, well-known results in the two-dimensional case. These results can be obtained in the same way as above ([35, 30, 78]).
Let $G \in I\!R^2$ be an open simply connected domain with a boundary ∂G which is a compact C^2-curve. Let $\underline{v}, p \in C^2(\overline{G})$ satisfy Stokes' equations on G and boundary condition (2.61). Let (φ, χ) be a pair of analytic functions on G representing this Stokes flow (see Section 2.1). The analogons of formulae (2.66) are

$$v_1(\underline{x}(s)) + 2\varphi_1(\underline{x}(s)) = \dot{y}(s)/2$$
$$v_2(\underline{x}(s)) + 2\varphi_2(\underline{x}(s)) = -\dot{x}(s)/2$$

where $\underline{x}(s) = (x(s), y(s)) \in \partial G$. We rewrite these relations in complex notation, substitute relation (2.12) and get the following traction formula (see also [75]):

$$v(z) + 2\varphi(z) = \varphi(z) + z\overline{\varphi'(z)} + \overline{\chi'(z)} = n(z)/2 \qquad z \in \partial G \qquad (2.68)$$

We notice that this implies the following relation on ∂G:

$$v_n = \frac{1}{2} - 2\mathrm{Re}(\varphi\overline{n}) \qquad (2.69)$$

where v_n denotes the normal component of $\underline{v}$.

Remark 2.31 We stress that we did not solve boundary value problem (1.8) by these relations. However, the identities above turn out to be useful when we consider the moving boundary problem for Stokes flow driven by surface tension

and multi-poles in the next chapter. Actually, we will use the following, reversed result. Let $G \subset \mathbb{R}^2$ be an open domain which boundary ∂G is a smooth curve (e.g. a Liapunov-curve ([81])). Let (φ, χ) be a pair of analytic functions on G with derivatives which can be extended continuously to $\overline{G}$ and which satisfy boundary condition (2.68). The corresponding Stokes flow, given by the relations (2.12), then satisfies boundary condition (1.8).
We finally note that the traction formula for boundary value problem (1.8) with $\gamma = 0$ is

$$v(z) + 2\varphi(z) = \varphi(z) + z\overline{\varphi'(z)} + \overline{\chi'(z)} = 0 \qquad z \in \partial G \tag{2.70}$$

Chapter 3

Moving Boundary Problems

This chapter concerns quasi-static moving boundary problems. In Section 3.1 we present a Lagrangian and a geometric formulation of such problems in a general setting. We consider two-dimensional quasi-static moving boundary problems in more detail in Section 3.2 where we show a relationship between these problems and what we will call quasi-linear Löwner-Kufareev equations. We use these considerations in Sections 3.3 and 3.4 where we present equations for the solutions of the moving boundary problems for Stokes flow driven by surface tension and multi-poles and for Hele-Shaw flow.

3.1 Quasi-static moving boundary problems

In this section we will give two non-equivalent definitions of quasi-static moving boundary problems. We will also return to the moving boundary problems mentioned in Chapter 1.

Let S denote a class of domains in $\mathbb{R}^n$ and let $\mathcal{F}$ denote a mapping from S to vector-fields on $\mathbb{R}^n$:

$$\mathcal{F} : G \in S \quad \mapsto \quad \mathcal{F}(G) = \underline{v}_{[G]}$$

such that the domain of the vector-field $\underline{v}_{[G]}$ contains G:

$$\underline{v}_{[G]} : \underline{x} \in G \quad \mapsto \quad \underline{v}_{[G]}(\underline{x}) \in \mathbb{R}^n$$

Let $G_0 \in S$ and let I be an interval containing 0.

Definition 3.1 A mapping

$$\underline{\tilde{x}} : (\underline{x}_0, t) \in G_0 \times I \quad \mapsto \underline{\tilde{x}}(\underline{x}_0, t) \in \mathbb{R}^n \tag{3.1}$$

is said to be a solution of the quasi-static moving boundary problem for $\mathcal{F}$ with initial data G_0 if it satisfies the following conditions:

i). $\tilde{\underline{x}}(\underline{x}_0, 0) = \underline{x}_0$ for all $\underline{x}_0 \in G_0$

ii). $\tilde{\underline{x}}$ is differentiable with respect to the variable t

iii). the mapping $\tilde{\underline{x}}$ is a homeomorphism from G_0 to its image G_t in S for all fixed $t \in I$

iv). $\dot{\tilde{\underline{x}}}(\underline{x}_0, t) = \underline{v}_{[G_t]}(\tilde{\underline{x}}(\underline{x}_0, t))$ for all $(\underline{x}_0, t) \in G_0 \times I$

We note that this definition excludes the possibility of a change of connectivity properties of the evoluting domain G_t as all the domains $G_t, t \in I$ are homeomorphic, i.e. topologically equivalent.

Consider the moving boundary problem (1.9) for Stokes flow driven by surface tension. Let S be the class of closed C^2-domains in $I\!R^n, n \geq 2$. It has already been remarked in Chapter 1 that a solution of the corresponding boundary value problem (1.8) is not unique; in particular, the velocity $\underline{v}$ is determined up to a rigid-body motion. This implies that the mapping $\mathcal{F}$ is not completely specified in this case. We can circumvent this problem in the following ways:
• It is possible to suppress the rigid-body motions. For example, this can be done as follows. We may assume without loss of generality that $\underline{0} \in G_0$. We define the subclass $\hat{S} \subset S$ as the set of all domains in S which contain $\underline{0}$. The mapping $\mathcal{F}$ on $\hat{S}$ can then be defined as the mapping from $G \in \hat{S}$ to the unique vector-field $\underline{v}_{[G]}$ that satisfies:

i). a function $p_{[G]}$ on G exists such that $p_{[G]}$ and $\underline{v}_{[G]}$ satisfy Stokes' equations on the interior of G and the boundary value problem (1.8) on ∂G

ii). $\underline{v}_{[G]}(\underline{0}) = \underline{0}$ $\nabla \times \underline{v}_{[G]}|_{\underline{0}} = \underline{0}$
where $\nabla \times \underline{v}$ denotes the (generalized) curl of $\underline{v}$

It is important to realize that although $\mathcal{F}$ is completely specified in this way –i.e. is a single-valued mapping– this does not necessarily imply that the solution of the moving boundary problem is unique.
• It is possible to change Definition 3.1 in such a way that $\mathcal{F}$ is a mapping from S to *classes* of vector-fields. In the above mentioned case, the vector-fields in such a class differ by a rigid-body motion. It is then understood in condition iv) of Definition 3.1 that the vector-field $\underline{v}_{[G_t]}$ in the right-hand side is in the class $\mathcal{F}(G_t)$ for all $t \in I$.
The moving boundary problem for Hele-Shaw flow can also be formulated in this way.

In several cases one is not interested in all properties of $\tilde{\underline{x}}$ but mainly in the shape evolution $t \mapsto G_t \in S$ determined by it. In the following we will consider

this shape evolution in more detail.
Assume that the class S contains only closed C^1-domains and let $\underline{\tilde{x}}$ be a solution of the quasi-static moving boundary problem for $\mathcal{F}$ with initial data G_0 such that $\underline{\tilde{x}}$ is a diffeomorphism from G_0 to G_t for all fixed $t \in I$. Let $\mathcal{E} \subset I\!R^n$ be a fixed, closed domain of reference with a smooth boundary (such as the unit ball, the half space $x_1 \leq 0$, etc.). We look for mappings

$$\underline{\Omega} : (\underline{\xi}, t) \in \mathcal{E} \times I \quad \mapsto \quad \underline{\Omega}(\underline{\xi}, t) \in I\!R^n \tag{3.2}$$

such that:

i). $\underline{\Omega}$ is differentiable with respect to t

ii). $\underline{\Omega}$ is a diffeomorphism from $\mathcal{E}$ to G_t for all fixed $t \in I$

Such mappings describe the shape evolution as well the mapping $\underline{\tilde{x}}$ in Definition 3.1 but they do not provide a Lagrangian description of the fluid under consideration in the classical sense. To put it otherwise, the vector-field $\underline{v} = \mathcal{F}(G_t)$ on $G_t, t \in I$ –to be interpretated as the hydrodynamic velocity– is only partially determined by such a mapping $\underline{\Omega}$.

Lemma 3.2 *Let the mapping $\underline{\tilde{x}}$ be as above. There exists a class of mappings $\underline{\Omega}$ satisfying the conditions i) and ii) above. Each mapping $\underline{\Omega}$ of this type satisfies*

$$\dot{\underline{\Omega}}(\underline{\Omega}^{\leftarrow}(\underline{x}, t), t) \cdot \underline{n}(\underline{x}, t) = v_{n,[G_t]}(\underline{x}) \qquad \text{for all } (\underline{x}, t) \in \partial G_t \times I \tag{3.3}$$

where $v_{n,[G_t]}(\underline{x})$ denotes the normal component of the vector $\underline{v}_{[G_t]}(\underline{x})$ at a point $\underline{x} \in \partial G_t$.

Proof
Consider the class of mappings

$$\underline{P} : (\underline{\xi}, t) \in \mathcal{E} \times I \quad \mapsto \quad \underline{P}(\underline{\xi}, t) \in \mathcal{E}$$

such that $\underline{P}$ is differentiable with respect to t and such that $\underline{P}$ is a diffeomorphism from $\mathcal{E}$ to itself for all fixed $t \in I$. It is clear that this class is not empty as the identity mapping is an element in it (in general, this class is very large). Let $\underline{F}$ be a diffeomorphism from $\mathcal{E}$ to G_0. One checks that a mapping $\underline{\Omega}$ defined by

$$\underline{\Omega} : (\underline{\xi}, t) \in \mathcal{E} \times I \quad \mapsto \quad \underline{\Omega}(\underline{\xi}, t) = \underline{\tilde{x}}(\underline{F}(\underline{P}(\underline{\xi}, t)), t)$$

with an arbitrary $\underline{P}$ as above satisfies the conditions i) and ii).
The proof of the second assertion runs as follows. Let $\underline{\Omega}$ satisfy conditions i) and ii) above. The trajectory in $\mathcal{E}$ of a point labelled by $\underline{x}_0 \in G_0$ is given by

$$\underline{\xi}(\underline{x}_0, t) = \underline{\Omega}^{\leftarrow}(\underline{\tilde{x}}(\underline{x}_0, t), t) \tag{3.4}$$

We remark that since $\underline{\tilde{x}}$ and $\underline{\Omega}^{\leftarrow}$ are homeomorphisms for all fixed $t \in I$, $\underline{x}_0 \in \partial G_0$ implies $\underline{\xi}(\underline{x}_0, t) \in \partial \mathcal{E}$ for all $t \in I$. One checks that $\underline{\Omega}^{\leftarrow}$ is differentiable with respect to t and we have

$$\dot{\underline{\Omega}}^{\leftarrow}(\underline{x}, t) = -D\underline{\Omega}^{\leftarrow}(\underline{x}, t)\, \dot{\underline{\Omega}}(\underline{\Omega}^{\leftarrow}(\underline{x}, t), t) \qquad (t, \underline{x}) \in I \times G_t \qquad (3.5)$$

We differentiate relation (3.4) with respect to the variable t, substitute relation (3.5) and identity iv) of Definition 3.1, multiply the result from the left with $D\underline{\Omega}(\underline{\Omega}^{\leftarrow}(\underline{\tilde{x}}(\underline{x}_0, t), t), t)$ and find

$$D\underline{\Omega}(\underline{\Omega}^{\leftarrow}(\underline{\tilde{x}}(\underline{x}_0, t), t), t)\, \dot{\underline{\xi}}(\underline{x}_0, t) = \underline{v}_{[G_t]}(\underline{\tilde{x}}(\underline{x}_0, t)) - \dot{\underline{\Omega}}(\underline{\Omega}^{\leftarrow}(\underline{\tilde{x}}(\underline{x}_0, t), t), t) \qquad (3.6)$$

This relation holds for all $t \in I$ and all $\underline{x}_0 \in G_0$, in particular for all $\underline{x}_0 \in \partial G_0$. As $\underline{\xi}(\underline{x}_0, t) \in \partial \mathcal{E}$ for all $(\underline{x}_0, t) \in \partial G_0 \times I$, $\dot{\underline{\xi}}(\underline{x}_0, t)$ is tangent to $\partial \mathcal{E}$. This implies that the inner product of the normal vector $\underline{n}$ at a point $\underline{x} = \underline{\tilde{x}}(\underline{x}_0, t) \in \partial G_t$ (so: $\underline{x}_0 \in G_0, t \in I$) and the vector at the left-hand side of relation (3.6) equals zero. Identity (3.3) follows. □

So, relation (3.3) is a necessary condition for a mapping $\underline{\Omega}$ on $\mathcal{E} \times I$ to describe the shape evolution of a solution of a moving boundary problem in the sense of Definition 3.1. The next example illustrates that this condition is also almost sufficient.

Example 3.3 Let S be the class of closed half planes $V_c : x_1 \leq c, c \in \mathbb{R}$ in $\mathbb{R}^2$. Let f denote a real function on $\mathbb{R}$ and let $\mathcal{F}$ on S be defined by

$$\begin{aligned} \mathcal{F} \;&:\; V_c \in S \;\mapsto\; \mathcal{F}(V_c) = \underline{v}_{[V_c]} \\ \underline{v}_{[V_c]} \;&:\; (x_1, x_2) \in V_c \mapsto \underline{v}_{[V_c]}(x_1, x_2) = (1 \;\; f(x_2))^T \end{aligned}$$

Consider the quasi-static moving boundary problem for $\mathcal{F}$ with initial data V_0. Let the reference domain $\mathcal{E}$ be V_0. We define $\underline{\Omega}$ by

$$\underline{\Omega} \;:\; (\underline{x}, t) \in \mathcal{E} \times \mathbb{R} \;\mapsto\; \underline{\Omega}(\underline{x}, t) = (x_1 + t, x_2)$$

It is clear that the image of $\mathcal{E}$ under $\underline{\Omega}$ is V_t for all $t \in \mathbb{R}$. It is easily checked that relation (3.3) holds. However, the moving boundary problem for $\mathcal{F}$ only has a solution for all $t \in \mathbb{R}$ if the initial value problem

$$\begin{aligned} \dot{y}(x_2, t) &= f(y(x_2, t)) \\ y(x_2, 0) &= x_2 \end{aligned} \qquad (3.8)$$

has a solution y on $\mathbb{R}^2$. It is not difficult to construct functions f such that this initial value problem has a global solution, nor is it difficult to construct functions f such that this initial value problem does not have a global solution due to the non-smoothness of f or due to the non-compactness of the domain of f.

It can be shown that relation (3.3) is a sufficient condition for a mapping $\underline{\Omega}$ on $\mathcal{E} \times I$ to describe the shape evolution of a solution of a moving boundary problem in the sense of Definition 3.1 if the domain $\mathcal{E}$ is compact and if the mapping $\mathcal{F}$ is smooth. We will not go into the proof of this assertion (see for instance the theory of groups of homeomorphisms explained in [17]). However, we formulate a slightly more general definition of a quasi-static moving boundary problem which qualifies the aforementioned condition to be sufficient.
Let S denote a class of closed C^1-domains in $I\!R^n$ and let $\mathcal{F}$ denote a mapping from S to functions on $I\!R^n$:

$$\mathcal{F} \ : \ G \in S \ \mapsto \ \mathcal{F}(G) = v_{n,[G]}$$

such that the domain of $v_{n,[G]}$ contains the boundary ∂G:

$$v_{n,[G]} \ : \ \underline{x} \in \partial G \ \mapsto \ v_{n,[G]}(\underline{x}) \in I\!R$$

Remark 3.4 This mapping $\mathcal{F}$ is another type of mapping as the one in the very beginning of this section. We will use the same symbol $\mathcal{F}$ throughout this and the following chapters to denote several types of mappings that are related to quasi-static moving boundary problems. It will always be clear from the context which type of mapping is meant.

Definition 3.5 A mapping

$$\underline{\Omega} \ : \ (\underline{\xi}, t) \in \mathcal{E} \times I \ \mapsto \ \underline{\Omega}(\underline{\xi}, t) \in I\!R^n$$

is said to be a solution (in the geometric sense) of the quasi-static moving boundary problem for $\mathcal{F}$ with initial data $G_0 \in S$ if it satisfies the following conditions:

i). $\underline{\Omega}$ is differentiable with respect to the variable t

ii). the mapping $\underline{\Omega}$ is a diffeomorphism from $\mathcal{E}$ to its image G_t in S for all fixed $t \in I$

iii). $\dot{\underline{\Omega}}(\underline{\Omega}^{\leftarrow}(\underline{x}, t), t) \cdot \underline{n}(\underline{x}, t) = v_{n,[G_t]}(\underline{x})$ for all $(\underline{x}, t) \in \partial G_t \times I$

Remark 3.6 We return to the moving boundary problems discussed in the Subsections 1.2.2 and 1.2.3. A precise formulation of these problems consists of two parts: a mapping $\mathcal{F}$ which maps a domain G to the solution of the corresponding boundary value problem (see also page 42) and a mapping $\underline{\Omega}$ as in Definition 3.5. We will not present these precise formulations although it is straightforward. We only note that in these formulations, the kinematic boundary condition (1.9c) or (1.11c) is replaced by condition iii) in Definition 3.5 above; this condition can roughly be explained by stating that the normal component of the velocity of the boundary equals the normal component of the velocity determined by the mapping $\mathcal{F}$, that is: determined by the solution of the corresponding boundary value problem at that time.

3.2 The quasi-linear Löwner-Kufareev equation

In this section we introduce the quasi-linear Löwner-Kufareev equation and explain in which way it is related to quasi-static moving boundary problems in two dimensions.
In the previous section we gave two definitions of solutions of a quasi-static moving boundary problem. Lemma 3.2 states that in general there are many equivalent solutions in the sense of Definition 3.5 corresponding to one solution in the sense of Definition 3.1. We want to reduce this arbitrariness by requiring the mapping $\underline{\Omega}$ in Definition 3.5 to satisfy some additional conditions. Stated otherwise, we look for a class $\mathcal{H}$ of diffeomorphisms from $\mathcal{E}$ to domains G in S such that:

i). for every domain G in S, there is a diffeomorphism in $\mathcal{H}$ mapping $\mathcal{E}$ to G

ii). the space of diffeomorphisms in $\mathcal{H}$ mapping $\mathcal{E}$ to a domain G is relatively small

In the case that S contains only compact, simply connected domains in $\mathbb{R}^2$ of class $C^{1,\alpha}, \alpha > 0$, we can make the following, more or less natural choices for $\mathcal{E}$ and $\mathcal{H}$. The reference domain $\mathcal{E}$ is the closed unit disc $\overline{D}$ in $\mathbb{R}^2$ and $\mathcal{H}$ is the class of conformal mappings from $\overline{D}$ to domains G in S. The Riemann mapping theorem ([71, 3]) and the Kellog-Warschawski theorem ([71]) state that both conditions i) and ii) are fulfilled; in particular we notice that the space of conformal mappings from $\overline{D}$ to a domain G in S is a real three-dimensional manifold.

Remark 3.7 If S contains closed and simply connected domains in $\mathbb{R}^2$ of class $C^{1,\alpha}, \alpha > 0$ we can make the same choices as above for $\mathcal{E}$ and $\mathcal{H}$ by identifying $\mathbb{R}^2$ with the complex plane and compactify the complex plane $\mathbb{C}$ by extending it to the Riemann sphere $\overline{\mathbb{C}}$. However, we will only consider compact domains although many results in the following can be generalized in this sense.
In the case that S is another class of domains (e.g. doubly connected domains in $\mathbb{R}^2$ or domains in $\mathbb{R}^3$ which are diffeomorphic to the unit ball), we cannot choose $\mathcal{H}$ to be the class of conformal mappings from some reference domain (e.g. an annulus or the unit ball in $\mathbb{R}^3$) to domains in S in general. The reason is that condition i) above is not fulfilled unless all the domains in S happen to be of the same conformal type (see e.g.[65]).

In the remaining part of this section we consider the above mentioned case in more detail; that is: S contains only compact, simply connected domains in the two-dimensional plane of type $C^{1,\alpha}, \alpha > 0$, $\mathcal{E} = \overline{D}$ and $\underline{\Omega}$ on $\overline{D} \times I$ is a solution of a quasi-static moving boundary problem in the geometric sense such that $\underline{\Omega}$ is a conformal mapping for all fixed $t \in I$. We will sometimes call such mappings conformal solutions. From now on we will use complex notation.

Lemma 3.8 *The mapping*

$$\Omega \ : \ (\zeta, t) \in \overline{D} \times I \ \mapsto \ \Omega(\zeta, t) \in \mathbb{C}$$

is a solution of the quasi-static moving boundary problem for $\mathcal{F} : G \mapsto v_{n,[G]}$ *with initial data given by the image of* $\overline{D}$ *under* $\Omega(\cdot, 0)$ *if it satisfies the following conditions:*

i). Ω *is differentiable with respect to the variable* t

ii). Ω *is a conformal mapping from* $\overline{D}$ *to its image* G_t *in* S *for all fixed* $t \in I$

iii).

$$\text{Re}\, \frac{\dot{\Omega}(\zeta, t)}{\Omega'(\zeta, t)\zeta} = \frac{v_{n,[G_t]}(\Omega(\zeta, t))}{|\Omega'(\zeta, t)|} \qquad \text{for all } (\zeta, t) \in \partial D \times I \tag{3.9}$$

Conversely, if Ω *is a solution of a quasi-static moving boundary problem in the sense of Definition 3.5 and* Ω *satisfies condition ii) above, then it also satisfies condition iii).*

Proof
We only prove the first assertion; the second one can easily be checked. Let Ω on $\overline{D} \times I$ satisfy the conditions i)–iii) above. The conditions i) and ii) of Definition 3.5 are satisfied; it only remains to be checked that condition iii) is satisfied. The normal vector n at a point $\Omega(\zeta, t)$ with $(\zeta, t) \in \partial D \times I$ is given in complex notation by

$$n(\Omega(\zeta, t)) = \frac{\Omega'(\zeta, t)\zeta}{|\Omega'(\zeta, t)|}$$

So, we get for all $(\zeta, t) \in \partial D \times I$

$$\text{Re}\, \dot{\Omega}(\zeta, t)\overline{n(\Omega(\zeta, t))} = \text{Re}\, \Omega'(\zeta, t)\zeta \frac{v_{n,[G_t]}(\Omega(\zeta, t))}{|\Omega'(\zeta, t)|} \frac{\overline{\Omega'(\zeta, t)\,\zeta}}{|\Omega'(\zeta, t)|} = v_{n,[G_t]}(\Omega(\zeta, t))$$

The relation iii) in Definition 3.5 then follows from this relation after changing the notation and the names of the variables. □

Remark 3.9 Let Ω on $\overline{D} \times I$ satisfy the conditions i)–iii) of Lemma 3.8 and let ω and α be differentiable functions from I to $\mathbb{R}$ and D respectively. It is easily verified that the mapping $\tilde{\Omega}$ on $\overline{D} \times I$ defined by

$$\tilde{\Omega}(\zeta, t) = \Omega\left(e^{i\omega(t)} \frac{\zeta - \alpha(t)}{1 - \overline{\alpha}(t)\zeta}, t\right) \tag{3.10}$$

also satisfies the conditions i)–iii) in the lemma. In order to get rid of this arbitrariness, the solution should be normalized (see condition ii) on page 46).

To illustrate how this can be done, we assume that for all $t \in I$ there is a point $\zeta(t) \in D$ such that $\Omega(\zeta(t), t) = 0$ (so, 0 is an inner point of G_t for all $t \in I$). It follows from the Riemann mapping theorem that there is exactly one mapping $\tilde{\Omega}$ on $\overline{D} \times I$ such that:

$$\tilde{\Omega}(0,t) = 0 \qquad \tilde{\Omega}'(0,t) > 0 \tag{3.11}$$

and such that the image of $\overline{D}$ under $\tilde{\Omega}(\cdot, t)$ is G_t. One can construct such normalized solution $\tilde{\Omega}$ –given the mapping Ω– by solving the equations for α and ω obtained by substituting the normalization (3.11) into identity (3.10).

Consider a quasi-static moving boundary problem for a mapping $\mathcal{F} : G \in S \mapsto v_{n,[G]}$ where the class S contains only simply connected and compact domains G in $\mathbb{C}$ of class $C^{1,\alpha}, \alpha > 0$ such that $0 \in G \backslash \partial G$. The latter condition on the domains enables us to normalize solutions in the sense of Remark 3.9. We show that such normalized conformal solutions can be described by a partial differential equation by rewriting condition iii) in Lemma 3.8.

Definition 3.10 Let $\mathcal{H}$ denote the space of conformal mappings Ω on $\overline{D}$ such that the image $\Omega(\overline{D})$ of $\overline{D}$ under Ω is in S and such that

$$\Omega(0) = 0 \qquad \Omega'(0) > 0$$

For each $\Omega \in \mathcal{H}$, we define an analytic function $f_{[\Omega]}$ on D with a continuous real part on $\overline{D}$ by

$$\text{Re } f_{[\Omega]}(\zeta) = \frac{v_{n,[\Omega(\overline{D})]}(\Omega(\zeta))}{|\Omega'(\zeta)|} \qquad \zeta \in \partial D \tag{3.12a}$$

$$\text{Im } f_{[\Omega]}(0) = 0 \tag{3.12b}$$

Remark 3.11 We say that the function f depends on Ω in a functional way. We will use this terminology ("depending in a functional way", "functional dependency", etc.) in this and the following chapters to indicate mappings from a function space to another function space.

Lemma 3.12 *Let $\Omega_0 \in \mathcal{H}$. A mapping*

$$\Omega \;:\; (\zeta, t) \in \overline{D} \times I \;\mapsto\; \Omega(\zeta, t) \in \mathbb{C}$$

is a solution of the quasi-static moving boundary problem for $\mathcal{F} : G \mapsto v_{n,[G]}$ with initial data given by the image of $\overline{D}$ under Ω_0 if it satisfies the following conditions:

i). Ω is differentiable with respect to the variable t

ii). Ω is in $\mathcal{H}$ for all fixed $t \in I$

iii).

$$\dot{\Omega}(\zeta,t) = \Omega'(\zeta,t) f_{[\Omega(\cdot,t)]}(\zeta)\zeta \qquad \text{for all } (\zeta,t) \in D \times I \tag{3.13}$$

Conversely, if Ω satisfies the conditions in Lemma 3.8 and is normalized, then Ω satisfies the conditions above.

Proof
Straightforward from Lemma 3.8. □

Example 3.13 We illustrate the notations and definitions by a trivial example. Let S be the class of closed discs $\overline{D_r} \subset \mathbb{C}, r > 0$. Moving boundary problem (1.11) with Q_1 identical to 1 give rise to a mapping $\mathcal{F} : G \in S \mapsto v_{n,[G]}$ given by

$$v_{n,[\overline{D_r}]}(\zeta) = \frac{1}{2\pi r} \qquad \zeta \in \partial D_r$$

One checks that Definition 3.10 in this case corresponds to

$$f_{[\Omega_r]}(\zeta) = \frac{1}{2\pi r^2}$$

where $\Omega_r(\zeta) = r\zeta$. The mentioned moving boundary problem with initial data given by $\overline{D}$ is then equivalent to the initial value problem given by equation (3.13) and initial data $\Omega(\zeta, 0) = \zeta$. The solution of this problem is given by

$$\Omega(\zeta,t) = \sqrt{\frac{t}{\pi} + 1}\,\zeta$$

An equation for a function Ω of the form

$$\dot{\Omega}(\zeta,t) = \Omega'(\zeta,t) f(\zeta,t)\zeta$$

where f is a given function, is called a linear Löwner-Kufareev equation. We consider such equations in Section 5.1. If the function f depends on Ω in a functional way, we call such equation a quasi-linear Löwner-Kufareev equation. (The adjective quasi-linear should not be interpretated in the sense used in the theory of partial differential equations but in the sense used in the theory of semi-groups; see also Appendix C). We consider such equations in Section 5.4. We remark that Lemma 3.12 actually states that many two-dimensional moving boundary problems with initial data can be considered to be initial value problems for quasi-linear Löwner-Kufareev equations. We loosely discuss some advantages of this approach to quasi-static moving boundary problems.
First, we note that –in the formulation given above, see e.g (3.13)– the function Ω does not have to be continuously extendable to $\partial D \times I$. This enables us

to study also quasi-static moving boundary problems where the shape evolution $t \in I \mapsto G_t \in S$ is such that not every domain G_t has a smooth boundary (cf. [34]). We remark that it is possible to solve the problem: i) f analytic in D and ii) Re $f(\zeta) = h(\zeta), \zeta \in \partial D$, for non-smooth functions h on ∂D (e.g. hyper-functions [56]). In other words: this approach enables us to formulate a quasi-static moving boundary problem as soon as the normal component of the velocity $\underline{v}$ at the boundary makes sense in some generalized way.
Secondly, we note that the formulation above enables us to generalize the concept of a solution of a moving boundary problem by omitting the condition that such a solution is injective for all fixed $t \in I$; we note that this injectivity of the solution was implicitly understood by requiring the solution to be diffeomorphic (Definition 3.5) or conformal (Lemmas 3.8 and 3.12). In the following we will consider also solutions of moving boundary problems which are only locally injective.
Thirdly, we remark that the solvability of quasi-linear Löwner-Kufareev equations is relatively easy to study; we will do so in Section 5.4.
Finally, we remark that the formulation of a quasi-static moving boundary as an initial value problem for a quasi-linear Löwner-Kufareev equation may also be useful in numerical studies. To understand this, we write:

$$\Omega_0(\zeta) = \sum_{n=1}^{\infty} a_{n,0}\zeta^n \qquad \Omega(\zeta, t) = \sum_{n=1}^{\infty} a_n(t)\zeta^n \qquad f_{[\Omega]}(\zeta) = \sum_{k=0}^{\infty} c_{k,[\underline{a}]}(t)\zeta^k \tag{3.14}$$

where we defined

$$c_{k,[\underline{a}]} = \frac{1}{2\pi}\int_0^{2\pi} \frac{v_{n,[\Omega]}(\Omega(e^{i\phi}))}{|\Omega'(e^{i\phi})|} e^{-ik\phi}\, d\phi \qquad\qquad k \in \mathbb{N}_0$$

(It is understood that the Taylor coefficients of Ω in this relation are given as a vector $\underline{a} = (a_1, a_2, ...)$.) The initial value problem then gets the form

$$\begin{aligned} \dot{a}_n(t) &= \sum_{l+k=n} l\, a_l(t) c_{k,[\underline{a}(t)]} \\ a_n(0) &= a_{n,0} \end{aligned}$$

An approximative solution of this initial value problem can be obtained by truncating the Taylor-series. Such an approximation of the solution may be more appropriate than one where the domain G_t is described by a mesh of points at the boundary ([91, 92, 89, 7]).

The following two sections are concerned with the moving boundary problems introduced in Section 1.2. The Löwner-Kufareev equation will play an important role. The solutions will be required to be smooth time-dependent locally conformal mappings in the following sense:

Definition 3.14 A mapping Ω on $\overline{D} \times I$ is said to be a smooth time-dependent locally conformal mapping if:

i). Ω is analytic on D for all fixed $t \in I$

ii). $\Omega|_{\partial D} \in C^3(\partial D)$ for all fixed $t \in I$

iii). Ω' is a non-vanishing function on $\overline{D} \times I$ that is continuously differentiable with respect to the variable t

The condition ii) may sometimes be weakened but we will not go into that matter.

3.3 Hopper's equation

In this section we consider Hopper's equation. We show that if a mapping Ω satisfies this equation, then it is a solution of the moving boundary problem for Stokes flow driven by surface tension. It is shown in the literature ([35, 30, 78]) that if Ω is a solution of this problem, then it satisfies Hopper's equation.

Definition 3.15 Let Ω be an analytic function on D such that its derivative has a continuous, non-vanishing extension to $\overline{D}$. We define $F_{[\Omega]}$ as the unique analytic function on D with a continuous real part on $\overline{D}$ such that

$$\begin{aligned} \operatorname{Re} F_{[\Omega]}(\zeta) &= \frac{1}{2|\Omega'(\zeta)|} \qquad \zeta \in \partial D \\ \operatorname{Im} F_{[\Omega]}(0) &= 0 \end{aligned}$$

We remark that if $\Omega \in C^3(\partial D)$, then the derivative $F'_{[\Omega]}$ of $F_{[\Omega]}$ can be extended continuously to $\overline{D}$.

Definition 3.16 Let f be a complex valued function on a domain $G \subset \overline{\mathbb{C}}$. Let $R(G)$ denote the domain obtained by reflecting G with respect to the unit circle:

$$R(G) = \{\zeta \in \overline{\mathbb{C}} \mid 1/\overline{\zeta} \in G\}$$

The function $\overline{f}$ on $R(G)$ is defined by

$$\overline{f}(\zeta) = \overline{f(1/\overline{\zeta})}$$

Remark 3.17 We used the bar in the previous chapter to denote the complex conjugated function; in formula (2.12) for example, $\overline{\chi'}$ denotes an anti-analytic function with the same domain as χ'. From now on, we will always explicitly state when $\overline{f}$ denotes the function obtained by complex conjugating f and not the function defined above. We note that no confusion can arise on the unit disc as, if $\partial D \subset G$, then $\partial D \subset R(G)$ and we get $\overline{f}(\zeta) = \overline{f(\zeta)}$ on ∂D.

It is not difficult to show that if f is analytic on an open domain G, then $\overline{f}$ is analytic on $R(G)$ and we get

$$\overline{f}'(\zeta) = -\overline{f'}(\zeta)/\zeta^2 \qquad \text{for all } \zeta \in R(G) \tag{3.16}$$

Definition 3.18 Let f be a differentiable complex valued function on ∂D. The function f' on ∂D is defined by

$$f'(e^{i\theta}) = \frac{df(e^{i\theta})}{de^{i\theta}} = -ie^{-i\theta}\frac{df(e^{i\theta})}{d\theta}$$

Remark 3.19 We remark that the relation (3.16) also holds in this sense. We also make the important remark that if f is analytic on a neighbourhood of ∂D, then the definition of f' is consistent with $f'(\zeta) = \frac{df(\zeta)}{d\zeta}$.

Proposition 3.20 *Let Ω be a smooth time-dependent conformal mapping on $\overline{D} \times I$. If a continuous function θ on $\overline{D} \times I$ exists such that θ is analytic on D for all $t \in I$ and such that the following equation is satisfied for all $(\zeta, t) \in \partial D \times I$:*

$$\theta(\zeta,t) = \left(\Omega'(\zeta,t)\overline{\Omega}(\zeta,t)\right)^{\cdot} - \left(\Omega'(\zeta,t)\overline{\Omega}(\zeta,t)F_{[\Omega(\cdot,t)]}(\zeta)\zeta\right)' \tag{3.17}$$

then Ω is a solution of the moving boundary for Stokes flow driven by surface tension.

We call equation (3.17) Hopper's equation.

Proof

Step 1. We define functions $F, \tilde{\chi}$ and $\tilde{\varphi}$ on $\overline{D} \times I$ by

$$F(\zeta,t) = F_{[\Omega(\cdot,t)]}(\zeta) \tag{3.18a}$$

$$\tilde{\chi}(\zeta,t) = \frac{1}{2}\int_0^\zeta \theta(z,t)\,dz \tag{3.18b}$$

$$\tilde{\varphi}(\zeta,t) = \frac{1}{2}\left(\Omega'(\zeta,t)F(\zeta,t)\zeta - \dot{\Omega}(\zeta,t)\right) \tag{3.18c}$$

Let G_t denote the image of D under $\Omega(\cdot,t)$. We define functions χ and φ on $\{(z,t) \in \mathbb{C} \times I |\ z \in \overline{G_t}\}$ by

$$\chi(z,t) = \tilde{\chi}(\Omega^{\leftarrow}(z,t),t) \qquad \varphi(z,t) = \tilde{\varphi}(\Omega^{\leftarrow}(z,t),t)$$

We note that these functions φ and χ have derivatives φ' and χ' which can be extended continuously to $\overline{G_t}$ for all $t \in I$. We finally define functions p and v on $\{(z,t)| \in \mathbb{C} \times I|\ z \in \overline{G_t}\}$ according to formulae (2.12). It follows from Remark 2.1 that p and v satisfy Stokes' equations G_t for all $t \in I$.

Step 2. We show that boundary condition (1.9b) with $\gamma = 1$ is fulfilled. It follows from relation (3.16) that

$$\left(\Omega'(\zeta,t)\overline{\Omega}(\zeta,t)F(\zeta,t)\zeta\right)' = \overline{\Omega}(\zeta,t)\left(\Omega'(\zeta,t)F(\zeta,t)\zeta\right)' - \Omega'(\zeta,t)\overline{\Omega'}(\zeta,t)F(\zeta,t)\overline{\zeta} \tag{3.19}$$

for all $(\zeta, t) \in \partial D \times I$. It follows immediately from the definition of the function F that

$$F(\zeta,t) = -\overline{F}(\zeta,t) + |\Omega'(\zeta,t)|^{-1} \qquad \text{for all } (\zeta,t) \in \partial D \times I \tag{3.20}$$

Now we substitute successively relations (3.18b),(3.19),(3.20) and (3.18c) into Hopper's equation. We suppress the variables in the notation and note that the following identities hold on $\partial D \times I$:

$$\begin{aligned} 2\tilde{\chi}' = \theta &= (\Omega'\dot{\overline{\Omega}}) - (\Omega'\overline{\Omega}F\zeta)' = (\Omega'\dot{\overline{\Omega}}) - \overline{\Omega}(\Omega'F\zeta)' + \Omega'\overline{\Omega'}F\overline{\zeta} \\ &= (\Omega'\dot{\overline{\Omega}}) - \overline{\Omega}(\Omega'F\zeta)' - \Omega'\overline{\Omega'}\,\overline{F}\,\overline{\zeta} + \overline{\zeta}\Omega'\overline{\Omega'}|\Omega'|^{-1} \\ &= \Omega'\dot{\overline{\Omega}} + \overline{\Omega}\dot{\Omega}' - \overline{\Omega}(\Omega'F\zeta)' - \Omega'\overline{(\Omega'F\zeta)} + \overline{\zeta}|\Omega'| \\ &= \overline{\Omega}(\dot{\Omega} - \Omega'F\zeta)' + \Omega'\overline{(\dot{\Omega} - \Omega'F\zeta)} + \overline{\zeta}|\Omega'| \\ &= \overline{\Omega}(-2\tilde{\varphi})' + \Omega'\overline{(-2\tilde{\varphi})} + \overline{\zeta}|\Omega'| \end{aligned}$$

We divide both sides of this relation by Ω', rearrange some terms and find

$$2\left(\overline{\tilde{\varphi}} + \overline{\Omega}\frac{\tilde{\varphi}'}{\Omega'} + \frac{\tilde{\chi}'}{\Omega'}\right) = \frac{\overline{\Omega'}\,\overline{\zeta}}{|\Omega'|}$$

Thus, we get

$$2\left(\overline{\varphi(z,t)} + \overline{z}\varphi'(z,t) + \chi'(z,t)\right) = \overline{n(z,t)} \qquad \text{for all } (z,t) \in \partial G_t \times I$$

where $n(z,t)$ denotes the normal vector at $z \in \partial G_t$. It follows from Remark 2.31 that boundary condition (1.9b) is satisfied indeed for all $t \in I$.

Step 3. It remains to be checked that Ω satisfies condition iii) of Lemma 3.8. The relation (3.18c) can be written as

$$\frac{\dot{\Omega}(\zeta,t)}{\Omega'(\zeta,t)\zeta} = F(\zeta,t) - \frac{2\tilde{\varphi}(\zeta,t)}{\Omega'(\zeta,t)\zeta}$$

By means of this relation and relation (2.69), we find

$$\begin{aligned} &\operatorname{Re}\left\{\frac{\dot{\Omega}(\zeta,t)}{\Omega'(\zeta,t)\zeta}\right\} = F(\zeta,t) - \frac{2\tilde{\varphi}(\zeta,t)}{\Omega'(\zeta,t)\zeta} = \\ &|\Omega'(\zeta,t)|^{-1}\operatorname{Re}\left(\frac{1}{2} - 2\tilde{\varphi}(\zeta,t)\overline{\Omega'(\zeta,t)\zeta}|\Omega'(\zeta,t)|^{-1}\right) = \\ &|\Omega'(\zeta,t)|^{-1}\operatorname{Re}\left(\frac{1}{2} - 2\varphi(\Omega(\zeta,t),t)\overline{n(\Omega(\zeta,t),t)}\right) = \\ &|\Omega'(\zeta,t)|^{-1}v_n(\Omega(\zeta,t),t) \qquad (\zeta,t) \in \partial D \times I \end{aligned}$$

It follows from Lemma 3.8 that Ω is a solution of the problem. □

Remarks 3.21 Let Ω on $\overline{D} \times I$ satisfy the conditions i)–iii) of the proposition above and let ω and f be differentiable real and complex valued functions on I. A mapping $\tilde{\Omega}$ defined by

$$\tilde{\Omega}(\zeta,t) = e^{i\omega(t)}\Omega(\zeta,t) + f(t) \tag{3.21}$$

also satisfies the conditions i)–iii) in Proposition 3.20. This freedom in the solution has its origin in the fact that the solution of boundary value problem (1.8) is determined up to a rigid-body motion.
The mapping $\tilde{\Omega}$ defined by relation (3.10) generally does not satisfy Hopper's equation (although it produces the same shape evolution as Ω). However, if Definition 3.15 is changed in such a way that $F_{[\Omega]}$ is admitted to have a different behaviour at $\zeta = 0$ (in particular: is admitted to have a pole of order one in $\zeta = 0$), then this this mapping $\tilde{\Omega}$ is also a solution of Hopper's equation.

Remark 3.22 We roughly explain the role of the function θ in Hopper's equation. We stress that θ should not be considered a given function. Actually, Hopper's equation can equivalently be formulated without reference to this function by stating that the imaginary part of the expression at the right-hand side of equation (3.17) is the Hilbert transform of the real part of this same expression. However, it is clear that, once it is known that Ω satisfies Hopper's equation or this equivalent formulation, then the function θ can be determined. We therefore stated in the introduction that this function can be determinded only a posteriori.

Proposition 3.23 *Let Q_1 be a real valued function on I and let $Q_n, n = 2, ..., N$ be complex valued functions on I. Let Ω be a smooth time-dependent locally conformal mapping on $\overline{D} \times I$ such that $\Omega(0,t) = 0$ for all $t \in I$. If a continuous function θ on $\overline{D}\backslash\{0\} \times I$ exists such that:*

i). θ is analytic on $D\backslash\{0\}$ for all fixed $t \in I$

ii). $\theta(\zeta,t) - \sum_{n=1}^{N} \frac{Q_n(t)\Omega'(\zeta,t)}{\pi\Omega(\zeta,t)^n} \to \text{const.}$ if $\zeta \to 0$ *for all $t \in I$*

iii). the following equation is satisfied on $\partial D \times I$:

$$\theta(\zeta,t) = \left(\Omega'(\zeta,t)\overline{\Omega}(\zeta,t)\right)^{\cdot} - \gamma\left(\Omega'(\zeta,t)\overline{\Omega}(\zeta,t)F_{[\Omega(\cdot,t)]}(\zeta)\zeta\right)' \tag{3.22}$$

then Ω is a solution of the moving boundary problem for Stokes flow driven by surface tension and/or ($\gamma = 1, 0$) multi-poles $Q_n, n = 1, .., N$ in $z = 0$.

Proof
We define the function $\tilde{\chi}'$ on $\overline{D}\backslash\{0\} \times I$ by

$$\tilde{\chi}'(\zeta,t) = \frac{1}{2}\theta(\zeta,t)$$

(the function $\tilde{\chi}$ itself is multiple-valued if Q_1 is not identically zero). We define the function χ' on $\{(z,t) \in \mathbb{C} \times I \mid z \in \overline{G_t}, z \neq 0\}$ by

$$\chi'(z,t) = \frac{\tilde{\chi}'(\Omega^{\leftarrow}(z,t),t)}{\Omega'(\Omega^{\leftarrow}(z,t),t)}$$

The function χ' is analytic on $G_t \backslash \{0\}$ for all fixed $t \in I$. Moreover, it follows from condition ii) that

$$\chi'(z,t) - \sum_{n=1}^{N} \frac{Q_n(t)}{2\pi z^n} \to \text{const.} \qquad \text{if } z \to 0$$

One then shows that Ω is a solution of the problem by reasoning in the same way as in the proof of Proposition 3.20 (see also (1.7)). □

We end this section by a proposition that can be considered as a kind of physical interpretation of the function φ that appears in the representation of the solution of the moving boundary problem for Stokes flow driven by surface tension and multi-poles (see also [76] and [4]). We define the nth moment $C_n(G;\varphi)$ of a complex valued function φ on a domain $G \subset \mathbb{R}^2$ by

$$C_n(G;\varphi) = \iint_G (x+iy)^n \varphi(x,y)\, dx\, dy$$

If the function φ is identical to one, we simply write $C_n(G)$ in stead of $C_n(G;1)$.

Proposition 3.24 *Let Ω be a smooth time-dependent conformal mapping satisfying the conditions of Proposition 3.23. Let $G_t, t \in I$ denote the image of D under $\Omega(\cdot,t)$ as before and let φ_t denote the representing function for the corresponding Stokes flow on the domain G_t that appears in relation* (2.12) *and which is normalized by*

$$\varphi_t(0) = 0 \qquad \varphi_t'(0) \in \mathbb{R}$$

(see Remark 2.3). Then:

$$\frac{d}{dt} C_n(G_t) = Q_{n+1}(t) - 2nC_{n-1}(G_t;\varphi_t) \qquad n \geq 1 \qquad (3.23a)$$

$$\frac{d}{dt} C_0(G_t) = Q_1(t) \qquad (3.23b)$$

Proof

We only prove identity (3.23a); the proof of identity (3.23b) runs similarly. Applying Gauss' theorem and using complex notation, we find:

$$\begin{aligned}\frac{d}{dt} C_n(G_t) &= \frac{d}{dt} \iint_{G_t} z^n\, dx\, dy = (2i)^{-1} \frac{d}{dt} \oint_{\partial G_t} z^n \bar{z}\, dz \\ &= (2i)^{-1} \frac{d}{dt} \oint_{\partial D} \Omega(\zeta,t)^n \overline{\Omega}(\zeta,t) \Omega'(\zeta,t)\, d\zeta\end{aligned}$$

We suppress the variables in the notation and substitute the relations (3.18c) and (3.22):

$$\begin{aligned}
\frac{d}{dt}C_n(G_t) &= (2i)^{-1}\frac{d}{dt}\oint_{\partial D} \Omega^n\overline{\Omega}\Omega'\,d\zeta \\
&= (2i)^{-1}\oint_{\partial D} n\Omega^{n-1}\dot{\Omega}\overline{\Omega}\Omega' + \Omega^n(\Omega'\dot{\overline{\Omega}})\,d\zeta \\
&= (2i)^{-1}\oint_{\partial D} n\Omega^{n-1}(\Omega'\gamma F_{[\Omega]}\zeta - 2\tilde{\varphi})\Omega'\overline{\Omega} + \Omega^n((\Omega'\overline{\Omega}\gamma F_{[\Omega]}\zeta)' + \theta)\,d\zeta \\
&= (2i)^{-1}\oint_{\partial D} (\Omega^n\Omega'\overline{\Omega}\gamma F_{[\Omega]}\zeta)' + \theta\Omega^n - 2n\tilde{\varphi}\Omega^{n-1}\Omega'\overline{\Omega}\,d\zeta \\
&= \sum_{m=1}^{N}\frac{Q_m}{2\pi i}\oint_{\partial D}\Omega^{n-m}\Omega'\,d\zeta - \frac{2n}{2i}\oint_{\partial G_t}\varphi z^{n-1}\overline{z}\,dz \\
&= Q_{n+1} - 2nC_{n-1}(G_t;\varphi_t)
\end{aligned}$$

□

Remark 3.25 Formula (3.23b) can be considered as the conservation of mass. The other identities state in which way the moments $C_n(G_t), n \in \mathbb{N}$ change in time. We stress that the relations (3.23) are independent of the value of γ; these relations hold whether surface tension is present or is not.

3.4 The Hopper equation for Hele-Shaw flow

In this section we present several equations with the property that a time dependent conformal mapping satisfies this equation if and only if this mapping is a solution of the moving boundary problem for Hele-Shaw flow. We treat three well-known equations of this type ([75, 39, 53, 85]) and add one which resembles Hopper's equation.

Definition 3.26 Let $Q_1 \in \mathbb{R}$ and let Ω be an analytic function on D such that its derivative has a continuous, non-vanishing extension to $\overline{D}$. We define the function $F_{[\Omega]}^{Q_1}$ as the unique analytic function on D with a continuous real part on $\overline{D}$ such that:

$$\begin{aligned}
\text{Re } F_{[\Omega]}^{Q_1}(\zeta) &= \frac{Q_1}{2\pi|\Omega'(\zeta)|^2} \qquad \zeta \in \partial D \\
\text{Im } F_{[\Omega]}^{Q_1}(0) &= 0
\end{aligned}$$

The superscript Q_1 prevents confusion between this definition and Definition 3.15.

Proposition 3.27 *Let Q_1 be a real continuous function on an interval $I \subset \mathbb{R}$. Let Ω be a smooth time-dependent locally conformal mapping on $\overline{D} \times I$ such that*

$$\Omega(0,t) = 0 \qquad \Omega'(0,t) > 0 \qquad \text{for all } t \in I$$

The following assertions are equivalent:

i). $\dot{\Omega}(\zeta,t) = \Omega'(\zeta,t) F^{Q_1(t)}_{[\Omega(\cdot,t)]}(\zeta)\zeta \qquad \text{for all } (\zeta,t) \in \overline{D} \times I$

ii). $\operatorname{Re} \dot{\Omega}(\zeta,t)\overline{\Omega'}(\zeta,t)\overline{\zeta} = Q_1(t)/2\pi \qquad \text{for all } (\zeta,t) \in \partial D \times I$

iii). a continuous function θ on $\overline{D}\backslash\{0\} \times I$ exists such that θ is analytic on $D\backslash\{0\}$ with a first order pole at $\zeta = 0$ with residue $Q_1(t)/\pi$ for all fixed $t \in I$ and such that the following equation is satisfied for all $(\zeta,t) \in \partial D \times I$:

$$\theta(\zeta,t) = \dot{\overline{\Omega}}(\zeta,t)\Omega'(\zeta,t) - \dot{\Omega}(\zeta,t)\overline{\Omega}'(\zeta,t)$$

iv). Ω is a solution of the moving boundary problem for Hele-Shaw flow.

Equation ii) is called the Polubarinova-Galin equation.

Proof

We do not go into all details of the proof as most of the results are already established in the literature. First we show i)$\rightarrow$iv) in three steps. We will denote the image of D under $\Omega(\cdot,t)$ by G_t.

Step 1. We define a multiple-valued function $\tilde{\chi}$ on $\overline{D}\backslash\{0\} \times I$ by

$$\tilde{\chi}(\zeta,t) = (Q_1(t)/2\pi)\ln\zeta$$

We define a function χ on $\{(z,t) \in \mathbb{C} \times I \mid z \in \overline{G_t}, z \neq 0\}$ by

$$\chi(z,t) = \tilde{\chi}(\Omega^{\leftarrow}(z,t),t)$$

We finally define functions p and v on $\{(z,t) \in \mathbb{C} \times I \mid z \in \overline{G_t}, z \neq 0\}$ according to the formulae (1.5). One checks that p and v are single-valued and satisfy Darcy's equations.

Step 2. As

$$\chi'(z,t) = \frac{Q_1(t)\Omega^{\leftarrow\prime}(z,t)}{2\pi\Omega^{\leftarrow}(z,t)} \qquad z \in \overline{G_t}\backslash\{0\}, t \in I$$

and as $\Omega^{\leftarrow}$ has a first order zero at $z = 0$, the function χ' has a first order pole at $z = 0$ with a residue $Q_1(t)/2\pi$ for all fixed $t \in I$. So, condition (1.11a) is satisfied for all $t \in I$ (see also (1.6)). The boundary condition (1.11b) is also satisfied for all $t \in I$ as

$$p(z,t) = -(Q_1(t)/2\pi)(\operatorname{Re}\ \ln\Omega^{\leftarrow}(z,t)) = 0 \qquad \text{for all } z \in \partial G_t \text{ and all } t \in I$$

Step 3. It remains to be checked that Ω satisfies condition iii) of Lemma 3.12. We denote differentiation in the direction of the normal vector at ∂D by $\frac{\partial}{\partial r}$ and we denote differentiation in the direction of the normal vector at ∂G_t by $\frac{\partial}{\partial n}$. We get for all $(\zeta, t) \in \partial D \times I$:

$$\begin{aligned} \text{Re } F^{Q_1(t)}_{[\Omega(\cdot,t)]}(\zeta) &= (Q_1(t)/2\pi)|\Omega'(\zeta,t)|^{-2} = |\Omega'(\zeta,t)|^{-2}\frac{\partial}{\partial r}(\text{Re } \tilde{\chi}(\zeta,t)) \\ &= |\Omega'(\zeta,t)|^{-1}\frac{\partial}{\partial n}(\text{Re } \chi(\Omega(\zeta,t),t)) = |\Omega'(\zeta,t)|^{-1}v_n(\Omega(\zeta,t)) \end{aligned} \quad (3.24)$$

This relation and Lemma 3.12 then imply that Ω is a solution of the moving boundary problem for Hele-Shaw flow.
The proof of the whole proposition follows from the following implications.
ii)$\rightarrow$i). We define the function f on $\overline{D} \times I$ by

$$f(\zeta,t) = \frac{\dot{\Omega}(\zeta,t)}{\Omega'(\zeta,t)\zeta} \quad (3.25)$$

We note that f is well-defined in $\zeta = 0$ and we have Im $f(0,t) = 0$ for all $t \in I$. It is clear that f is analytic on D and is continuous on $\overline{D}$ for all $t \in I$. It follows from the Polubarinova-Galin equation that

$$\begin{aligned} \text{Re } f(\zeta,t) = \text{Re } \frac{\dot{\Omega}(\zeta,t)}{\Omega'(\zeta,t)\zeta} &= |\Omega'(\zeta,t)|^{-2}\text{Re } \overline{\Omega'}(\zeta,t)\overline{\zeta}\dot{\Omega}(\zeta,t) \\ &= (Q_1(t)/2\pi)|\Omega(\zeta,t)|^{-2} \qquad \text{for all } (\zeta,t) \in \partial D \times I \end{aligned}$$

This implies $f(\zeta,t) = F^{Q_1(t)}_{[\Omega(\cdot,t)]}(\zeta)$ for all $(\zeta,t) \in \overline{D} \times I$ and relation i) follows from relation (3.25).
iii)$\rightarrow$ii). We define the function ϕ on $\overline{D} \times I$ by

$$\phi(\zeta,t) = \theta(\zeta,t) - \frac{Q_1(t)}{\pi\zeta} \quad (3.26)$$

One checks that ϕ is continuous on its domain and analytic on D for all $t \in I$. We suppress the variables in the notation and note that the following identity holds on $\partial D \times I$:

$$\text{Re } \dot{\Omega}\overline{\Omega'}\,\overline{\zeta} = \frac{1}{2}(\dot{\Omega}\overline{\Omega'}\,\overline{\zeta} + \dot{\overline{\Omega}}\Omega'\zeta) = \frac{1}{2}\zeta(\dot{\overline{\Omega}}\Omega' - \dot{\Omega}\overline{\Omega}') = \frac{1}{2}\zeta\theta = \frac{Q_1}{2\pi} + \zeta\phi/2 \quad (3.27)$$

This relation implies that the function Im $\zeta\phi$ vanishes on the boundary $\partial D \times I$. As Im $\zeta\phi$ is a harmonic function on D for all fixed $t \in I$, ϕ is identically zero and (3.27) is just the Polubarinova-Galin equation.
iv)$\rightarrow$iii). We first note that it follows from the relations (1.11a) and (1.11b) that the continuous function $\hat{p}$ on $\{(z,t) \in \mathbb{C} \times I \mid z \in \overline{G_t}\}$ defined by

$$\hat{p}(z,t) = \frac{Q_1(t)}{2\pi}\ln|z| + p(z,t)$$

satisfies the following Dirichlet problem:

$$\Delta\hat{p}(z,t) = 0 \qquad z \in G_t, t \in I$$
$$\hat{p}(z,t) = \frac{Q_1(t)}{2\pi}\ln|z| \qquad z \in \partial G_t, t \in I$$

This Dirichlet problem has a unique solution given by

$$\hat{p}(z,t) = \frac{Q_1(t)}{2\pi}\mathrm{Re}\left(\ln\frac{z}{\Omega^{\leftarrow}(z,t)}\right)$$

Hence,

$$p(z,t) = \frac{Q_1(t)}{2\pi}\mathrm{Re}\left(\ln\Omega^{\leftarrow}(z,t)\right)$$

and we get

$$v_n(\Omega(\zeta,t),t)|_{\zeta\in\partial D} = \frac{Q_1(t)}{2\pi|\Omega(\zeta,t)|} \tag{3.28}$$

(see also relation (3.24)). It follows from Lemma 3.12 that Ω satisfies

$$\dot{\Omega}(\zeta,t) = \Omega'(\zeta,t)F^{Q_1(t)}_{[\Omega(\cdot,t)]}(\zeta)\zeta \qquad \text{for all } \zeta\in\overline{D}, t\in I$$

One then straightforwardly checks that condition iii) is also fulfilled: just put $\theta(\zeta,t) = Q_1(t)/\pi\zeta$. □

Proposition 3.28 *Let Q_1 and Ω be as in Proposition 3.27. The assertions in Proposition 3.27 are equivalent to the following assertion:*

v). a continuous function θ on $\overline{D}\backslash\{0\}\times I$ exists such that θ is analytic on $D\backslash\{0\}$ with a first order pole in $\zeta = 0$ with residue $Q_1(t)/\pi$ for all fixed $t\in I$ and such that the following equation is satisfied for all $(\zeta,t)\in\partial D\times I$:

$$\theta(\zeta,t) = \left(\Omega'(\zeta,t)\overline{\Omega}(\zeta,t)\right)^{\cdot} - \left(\Omega'(\zeta,t)\overline{\Omega}(\zeta,t)F^{Q_1(t)}_{[\Omega(\cdot,t)]}\zeta\right)' \tag{3.29}$$

As equation (3.29) resembles Hopper's equation, we call it a Hopper equation.

Proof

i),iii)→v). Let θ be the function as in condition iii). We suppress the variables in the notation, substitute relation i) and find on $\partial D\times I$:

$$\theta = \dot{\overline{\Omega}}\Omega' - \dot{\Omega}\overline{\Omega}' = (\overline{\Omega}\Omega')^{\cdot} - (\overline{\Omega}\dot{\Omega})' = (\Omega'\overline{\Omega})^{\cdot} - (\Omega'\overline{\Omega}F^{Q_1}_{[\Omega]}\zeta)'$$

v)→i). We define the function g on $\overline{D}\times I$ by

$$g(\zeta,t) = \dot{\Omega}(\zeta,t) - \Omega'(\zeta,t)F^{Q_1(t)}_{[\Omega(\cdot,t)]}(\zeta)\zeta \tag{3.30}$$

It is clear that g is analytic on D for all $t \in I$. One checks that as Ω is a smooth time-dependent conformal mapping, the derivative g' has a continuous extension to $\overline{D}$ for all $t \in I$. We show that g is identical to zero.
We suppress the variables in the notation, substitute relation (3.30) into equation (3.29) and find on $\partial D \times I$:

$$\begin{aligned}\theta &= \dot{(\Omega'\overline{\Omega})} - (\Omega'\overline{\Omega} f^{Q_1}_{[\Omega]}\zeta)' = \dot{(\Omega'\overline{\Omega})} - (\overline{\Omega}(\dot{\Omega} - g))' = \dot{\overline{\Omega}}\Omega' - \overline{\Omega}'\dot{\Omega} + (\overline{\Omega}g)' \\ &= (\overline{\Omega'}\overline{F^{Q_1}_{[\Omega]}}\overline{\zeta} + \overline{g})\Omega' - \overline{\Omega}'(\Omega' F^{Q1}_{[\Omega]}\zeta + g) + (\overline{\Omega}g)' \\ &= |\Omega'|^2\overline{\zeta}2\text{Re } F^{Q1}_{[\Omega]} + \overline{g}\Omega' + \overline{\Omega}g' = \overline{g}\Omega' + \overline{\Omega}g' + \frac{Q_1}{\pi\zeta}\end{aligned}$$

So, the function ϕ defined as in relation (3.26) is analytic on D for all $t \in I$ while its boundary values are given by the following relation on $\partial D \times I$:

$$\phi = \overline{g}\Omega' + \overline{\Omega}g'$$

Now we apply Lemma 4.12; this lemma will proved in the next chapter. It follows that there are functions C_1 and C_2 on I with values in $\mathbb{R}$ and $\mathbb{C}$ respectively such that

$$g = iC_1\Omega + C_2$$

Substitution of this result together with relation (3.30) into the normalization

$$\Omega(0,t) = 0 \qquad \Omega'(0,t) \in \mathbb{R} \qquad\qquad \text{for all } t \in I$$

leads to the conclusion that both C_1 and C_2, and hence g, are identically zero. □

Remark 3.29 The analogons of formulae (3.23) for Hele-Shaw flow are

$$\frac{d}{dt}C_n(g_t) = \delta_{n0}Q_1(t) \qquad\qquad n \in \mathbb{N}_0$$

(see also [76, 4]). These identities –which imply the conservation of the moments of the domain except for the area– can easily be obtained by putting φ identical zero in the proof of Proposition 3.24: compare relation (3.18c) to assertion i) in Proposition 3.27 and note that the reasoning in the proof of Proposition 3.24 does not depend on the values of the fuction F (see also Remark 3.25).

Remark 3.30 The similarity of the Hopper equation (3.29) for Hele-Shaw flow and Hopper's equation (in particular equation (3.22) with $\gamma = 1, N = 1$) is remarkable. It is therefore interesting to study Hopper equations such as equation (3.29) with $F^{Q_1}_{[\Omega]}$ replaced by another function that depends on Ω' in a functional way; compare Definitions 3.15 and 3.26. This may lead to regularized models of Hele-Shaw flow (see also [88, 33]) which are some kind of intermediate between models of Stokes flow driven by surface tension and a sink and of Hele-Shaw flow. We do not go into this matter because it is difficult to deduce what kind of moving boundary problems such Hopper equations describe. We only state that such ad hoc methods turn out to work quite well.

Chapter 4

Hopper Equations

It will be shown in Section 4.1 that a Hopper equation for a rational function is equivalent to a finite set of differential equations. We prove in Section 4.2 that this set of equations has a local solution under general conditions. We show in Section 4.3 that this set of equations has a global solution in a particular case. A comprehensive introduction to this chapter can be found in Section 1.3.

4.1 Rational solutions of Hopper equations

This section concerns Hopper equations. We show how such an equation reduces to a finite set of differential equations if we make the Ansatz that the mapping Ω is a time-dependent rational function.

In Propositions 3.20, 3.23 and 3.28 we met equations for time-dependent analytic functions Ω on $\overline{D} \times I$ of the following form:

$$\theta(\zeta,t) = \left(\overline{\Omega}(\zeta,t)\Omega'(\zeta,t)\right)^{\cdot} - \left(\overline{\Omega}(\zeta,t)\Omega'(\zeta,t)f(\zeta,t)\zeta\right)' \qquad (\zeta,t) \in \partial D \times I \quad (4.1)$$

where:

i). f is a function on $\overline{D} \times I$ which is analytic on D for all fixed $t \in I$; this function may depend on Ω in a functional way although this is not expressed in the notation.

ii). θ is some function on $\overline{D} \times I$ with a prescribed pole at $\zeta = 0$; i.e. θ is analytic on $D\backslash\{0\}$ with a pole of order N at $\zeta = 0$ for all $t \in I$–where it is understood that θ is analytic on D if $N = 0$– and such that

$$\frac{1}{2\pi i}\oint_{\gamma_0} \zeta^{n-1}\theta(\zeta,t)\, d\zeta = a_n(t) \qquad n = 1,..,N; t \in I$$

In this expression, γ_0 denotes a closed Jordan curve around $\zeta = 0$ and $a_n, n = 1,..,N$ are given complex valued functions on I which may depend on Ω. We stress again that θ should not be considered a given function; only the principal part of its Laurent series is prescribed.

We call equation (4.1) a Hopper equation. It turns out that solutions of such an equation only exist if a_1 is real valued. We note that a Hopper equation is an equation on $\partial D \times I$ although the domains of the functions which appear in it are larger.
A straightforward construction of solutions Ω of a Hopper equation does not seem to exist if f depends on Ω in a functional way (we return to this point in Chapter 5 and in Appendix C). We therefore follow Hopper ([35]) and make an Ansatz for the mapping Ω:

$$\Omega(\zeta, t) = \sum_{m=1}^{M} \sum_{k=1}^{K(m)} c_{mk}(t) \frac{\zeta^k}{(1 - \zeta_m(t)\zeta)^k} \tag{4.2}$$

where $M, K(1),.., K(M)$ are positive integers and where:

i). $\zeta_m, m = 1,..,M$ and $c_{mk}, m = 1,..,M, k = 1,..,K(m)$ are continuously differentiable functions from I to D and $\mathbb{C}$ respectively such that the derivative Ω' of the function in expression (4.2) does not vanish on $\overline{D} \times I$

ii). $\text{Im} \sum_{m=1}^{M} c_{m1}(t) = 0$ for all $t \in I$

iii). (a) $m_1 \neq m_2$ implies $\zeta_{m_1}(t) \neq \zeta_{m_2}(t)$ for all $t \in I$
(b) for all $m \in \{1,..,M\}$ and all $t \in I$, there is a $k \in \{1,..,K(m)\}$ such that $c_{mk}(t) \neq 0$

iv). (a) ζ_1 is identical to zero unless $N = 0$
(b) $K(1) \geq N$

Remark 4.1 We discuss the conditions i)–iv) on the parameterization of the rational Ansatz.
The condition i) assures that the function Ω is a smooth time-dependent locally conformal mapping.
One easily verifies that if Ω on $\overline{D} \times I$ is a solution of a Hopper equation, then the function $\tilde{\Omega}$ defined by relation (3.21) is also a solution. Therefore, we do not lose any generality by normalizing Ω as follows:

$$\Omega(0, t) = 0 \qquad \text{Im } \Omega'(0, t) = 0 \qquad \text{for all } t \in I$$

The first normalization is already contained in the parametrization of the function Ω in expression (4.2). The second normalization is equivalent to condition ii).

The mapping Ω is thus determined by $L = 2(M+P)-1$ real time-dependent parameters, where $P = \sum_{m=1}^{M} K(m)$.
Consider for example a function Ω in expression (4.2) with $M = 4, K(m) = 1$, $m = 1,..,4$, where ζ_2, ζ_4 and c_{21} are arbitrary functions and where

$$\zeta_1 = 0 \quad \zeta_2 = \zeta_3 \quad c_{11} = 1 \quad c_{31} = -c_{21} \quad c_{41} = 0$$

One checks that this function is actually the identity: $\Omega(\zeta, t) = \zeta$. In order to avoid that a function can be parameterised in several ways, we require that condition iii) is satisfied: the conditions i)–iii) imply that every normalized, rational, smooth time-dependent locally conformal mapping with exactly M poles outside $\overline{D}$ can uniquely be parameterized in the form of expression (4.2). We note that we pay the following price for this: we cannot consider rational solutions of Hopper equations where poles appear or disappear. We will show in Chapter 5 that this is not a loss of generality.
It can be checked that if the function θ in equation (4.1) is allowed to have poles in $\zeta = 0$ (i.e. $N \neq 0$), then a parametrisation as in expression (4.2) only makes sense if we put $\zeta_m(t) = 0$ for all $t \in I$ and $K(m) \geq N$ for some $m \in \{1,..,M\}$.

We substitute expression (4.2) into equation (4.1) and show that this leads to a finite set of differential equations for the parameters $\zeta_.$ and $c_{..}$. Before we do so, we remark that the function $\overline{\Omega}$ is meromorphic on D for all $t \in I$ as

$$\overline{\Omega}(\zeta, t) = \sum_{m=1}^{M} \sum_{k=1}^{K(m)} \overline{c_{mk}(t)}\, (\zeta - \overline{\zeta_m(t)})^{-k}$$

To be more precise, $\overline{\Omega}$ has poles in $\zeta = \overline{\zeta_m(t)} \in D, m \in \{1,..,M\}$ of orders not larger than $K(m)$ for all $t \in I$. This implies that the function θ on $\overline{D} \times I$ defined by relation (4.1) is analytic on $D \backslash \{\overline{\zeta_1(t)},..,\overline{\zeta_M(t)}\}$ for all fixed $t \in I$. Moreover, if θ has a singularity in $\zeta = \overline{\zeta_m(t)}, m = 1,..,M$ for some $t \in I$, then this singularity is a pole and the order of this pole does not exceed $K(m)+1$. This implies that the function Ω in expression (4.2) is a solution of equation (4.1) if and only if it satisfies the following equations for all $t \in I$, all $m \in \{1,..,M\}$ and all $n \in \{1,..,K(m)+1\}$:

$$\frac{1}{2\pi i} \oint_{\gamma_m(t)} (\zeta - \overline{\zeta_m(t)})^{n-1} \frac{\partial}{\partial t} \left(\overline{\Omega}(\zeta,t)\Omega'(\zeta,t)\right) d\zeta - \\ \frac{1}{2\pi i} \oint_{\gamma_m(t)} (\zeta - \overline{\zeta_m(t)})^{n-1} \frac{\partial}{\partial \zeta} \left(\overline{\Omega}(\zeta,t)\Omega'(\zeta,t)f(\zeta,t)\zeta\right) d\zeta = \delta_{m1} a_n(t) \quad (4.3)$$

where $\gamma_m(t), m = 1,..,M$ is a closed Jordan curve in D such that $\overline{\zeta_m(t)}$ is inside $\gamma_m(t)$ and $\overline{\zeta_l(t)}$ is outside $\gamma_m(t)$ if $l \neq m$. First we calculate the first term in the left-hand side:

$$\frac{1}{2\pi i}\oint\limits_{\gamma_m(t)} (\zeta-\overline{\zeta_m(t)})^{n-1}\frac{\partial}{\partial t}\left(\overline{\Omega}(\zeta,t)\Omega'(\zeta,t)\right)\,d\zeta =$$

$$\frac{1}{2\pi i}\oint\limits_{\gamma_m(t)} \frac{\partial}{\partial t}\left((\zeta-\overline{\zeta_m(t)})^{n-1}\overline{\Omega}(\zeta,t)\Omega'(\zeta,t)\right)\,d\zeta +$$

$$\frac{1}{2\pi i}\oint\limits_{\gamma_m(t)} (n-1)(\zeta-\overline{\zeta_m(t)})^{n-2}\overline{\dot{\zeta}_m(t)}\,\overline{\Omega}(\zeta,t)\Omega'(\zeta,t)\,d\zeta =$$

$$\frac{\partial}{\partial t}\left(\frac{1}{2\pi i}\oint\limits_{\gamma_m(t)}\sum_{k=1}^{K(m)}\sum_{l=0}^{\infty}\frac{\overline{c_{mk}(t)}}{l!}\Omega^{(l+1)}(\overline{\zeta_m(t)},t)(\zeta-\overline{\zeta_m(t)})^{n+l-k-1}\,d\zeta\right)+$$

$$\overline{\dot{\zeta}_m(t)}\,\frac{n-1}{2\pi i}\oint\limits_{\gamma_m(t)}\sum_{k=1}^{K(m)}\sum_{l=0}^{\infty}\frac{\overline{c_{mk}(t)}}{l!}\Omega^{(l+1)}(\overline{\zeta_m(t)},t)(\zeta-\overline{\zeta_m(t)})^{n+l-k-2}\,d\zeta =$$

$$(1-\delta_{n\,K(m)+1})\frac{\partial}{\partial t}\left(\sum_{k=0}^{K(m)-n}\frac{\overline{c_{m\,k+n}(t)}}{k!}\Omega^{(k+1)}(\overline{\zeta_m(t)},t)\right)+$$

$$(n-1)\overline{\dot{\zeta}_m(t)}\left(\sum_{k=0}^{K(m)-n+1}\frac{\overline{c_{m\,k+n-1}(t)}}{k!}\Omega^{(k+1)}(\overline{\zeta_m(t)},t)\right) \tag{4.4}$$

Next we calculate the second term in the left-hand side:

$$\frac{1}{2\pi i}\oint\limits_{\gamma_m(t)} (\zeta-\overline{\zeta_m(t)})^{n-1}\frac{\partial}{\partial \zeta}\left(\overline{\Omega}(\zeta,t)\Omega'(\zeta,t)f(\zeta,t)\zeta\right)\,d\zeta =$$

$$-\frac{1}{2\pi i}\oint\limits_{\gamma_m(t)} (n-1)(\zeta-\overline{\zeta_m(t)})^{n-2}\,\overline{\Omega}(\zeta,t)\Omega'(\zeta,t)f(\zeta,t)\zeta\,d\zeta =$$

$$-\frac{n-1}{2\pi i}\oint\limits_{\gamma_m(t)}\left(\sum_{k=1}^{K(m)}\overline{c_{mk}(t)}(\zeta-\overline{\zeta_m(t)})^{n-2-k}\right)(\overline{\zeta_m(t)}+\zeta-\overline{\zeta_m(t)})\,\cdot$$

$$\cdot\left(\sum_{l=0}^{\infty}\sum_{j=0}^{l}\frac{1}{(l-j)!j!}\Omega^{(l+1-j)}(\overline{\zeta_m(t)},t)f^{(j)}(\overline{\zeta_m(t)},t)(\zeta-\overline{\zeta_m(t)})^{l}\right)\,d\zeta =$$

$$-(n-1)\left\{\overline{\zeta_m(t)}\sum_{k=0}^{K(m)-n+1}\sum_{l=0}^{k}\frac{\overline{c_{m\,k+n-1}(t)}}{(k-l)!l!}\Omega^{(k+1-l)}(\overline{\zeta_m(t)},t)f^{(l)}(\overline{\zeta_m(t)},t)+\right.$$

$$\left.(1-\delta_{n\,K(m)+1})\sum_{k=0}^{K(m)-n}\sum_{l=0}^{k}\frac{\overline{c_{m\,k+n}(t)}}{(k-l)!l!}\Omega^{(k+1-l)}(\overline{\zeta_m(t)},t)f^{(l)}(\overline{\zeta_m(t)},t)\right\} \tag{4.5}$$

Substitution of these results for $n = K(m) + 1$ into equation (4.3) leads to

$$K(m)\overline{\dot{\zeta}_m(t)}\ \overline{c_{m\ K(m)}(t)}\Omega'(\overline{\zeta_m(t)}, t) +$$

$$K(m)\overline{\zeta_m(t)}\ \overline{c_{m\ K(m)}(t)}\Omega'(\overline{\zeta_m(t)}, t) f(\overline{\zeta_m(t)}, t) = 0 \tag{4.6}$$

for all $m \in \{1, .., M\}$; this relation holds for $m = 1$ as $K(1) + 1 > N$. After complex conjugating this equation and rearranging some terms –see also Remark 4.2 below– we obtain

$$\dot{\zeta}_m(t) = -\zeta_m(t)\overline{f(\overline{\zeta_m(t)}, t)} \qquad m = 1, .., M \tag{4.7}$$

Substituting the relations (4.4),(4.5) and (4.7) into equation (4.3), we find for all $m \in \{1, .., M\}$ and all $n \in \{1, .., K(m)\}$

$$\frac{\partial}{\partial t}\left(\sum_{k=0}^{K(m)-n} \frac{\overline{c_{m\ k+n}(t)}}{k!}\Omega^{(k+1)}(\overline{\zeta_m(t)}, t)\right) +$$

$$(n-1)\sum_{k=0}^{K(m)-n}\sum_{l=0}^{k}\left(\frac{\overline{c_{m\ k+n}(t)}}{l!(k-l)!}\Omega^{(k+1-l)}(\zeta, t)\left(f^{(l)}(\zeta, t) + \frac{\zeta}{l+1}f^{(l+1)}(\zeta, t)\right)\right)\Bigg|_{\zeta=\overline{\zeta_m(t)}}$$

$$= \delta_{m1}a_n(t) \tag{4.8}$$

Remark 4.2 In order to obtain relation (4.7) from relation (4.6), we divided this relation by $c_{m\,K(m)}(t)$. This is not allowed if $c_{m\,K(m)}(t) = 0$ for some $t \in I$. However, one carefully checks that –because of condition iiib) on page 64– relation (4.7) follows anyway.

Corollary 4.3 *A function Ω as in expression* (4.2) *is a solution of equation* (4.1) *if and only if equations* (4.7) *and* (4.8) *are satisfied.*

Example 4.4 Let $M = 1$ and let ζ_1 be identical to zero, i.e. we make the Ansatz that Ω is a polynomial mapping (see also [30]). We change the notation and write

$$\Omega(\zeta, t) = \sum_{k=1}^{K} c_k(t)\zeta^k$$

with $K \geq N$. Equation (4.7) is trivially satisfied while the equations (4.8) reduce to the following set of differential equations:

$$\frac{\partial}{\partial t}\left(\sum_{k=0}^{K-n}(k+1)c_{k+1}(t)\overline{c_{k+n}(t)}\right) +$$

$$(n-1)\sum_{k=0}^{K-n}\sum_{l=0}^{k}(k+1-l)\overline{c_{k+n}(t)}c_{k-l+1}(t)\frac{f^{(l)}(0,t)}{l!}$$

$$= a_n(t) \tag{4.9}$$

where $n = 1,..,K$. We define

$$b_k(t) = \sum_{l=1}^{K-k+1} l c_l(t)\overline{c_{l+k-1}(t)} \qquad k = 1,..,K$$
$$d_l(t) = f^{(l)}(0,t)/l! \qquad l = 0,..,K$$

and find that this set of equations can be written as

$$\dot{b}_n(t) + (n-1)\sum_{k=0}^{K-n} b_{n+k}(t)d_k(t) = a_n(t) \qquad n = 1,..,K \tag{4.10}$$

Example 4.5 Let $N = 0$ and $K(m) = 1$ for all $m \in \{1,..,M\}$, i.e. we consider a particular type of Hopper equation and make the Ansatz that Ω is a so-called partial fraction mapping ([35]). We change the notation and write

$$\Omega(\zeta,t) = \zeta\sum_{m=1}^{M}\frac{A_m(t)}{1-\zeta_m(t)\zeta} \tag{4.11}$$

The equations (4.7) remain while the equations (4.8) reduce to a set of M algebraic relations:

$$q_m = \overline{A_m(t)}\sum_{k=1}^{M}\frac{A_k(t)}{(1-\zeta_k(t)\overline{\zeta_m(t)})^2} \qquad m = 1,..,M \tag{4.12}$$

where the quantities $q_m, m = 1,..,M$ do not depend on the variable t. We consider these equations in more detail in Section 4.3. We present a result from a numerical study of these equations in Appendix C.

We end this section by counting the total number of equations. As the terms $\Omega^{(l)}(\overline{\zeta_m(t)},t), m = 1,..,M, l = 1,..,K(m)$ in equations (4.8) can straightforwardly be expressed in terms of the parameters $c_{..}$ and $\zeta_.$, the equations (4.7) and (4.8) can be considered as a set of $2(M+P) = L+1$ coupled real differential equations for the L parameters $c_{..}$ and $\zeta_.$ (see also Remark 4.1). We show below that the imaginary parts of the equations (4.8) with $n = 1$ are linearly dependent. We thus end up with not more than L differential equations for L parameters.
The equations (4.8) with $n = 1$ lead to a set of $M-1$ complex valued conserved quantities

$$q_m = \sum_{k=0}^{K(m)-1}\frac{\overline{c_{m\,k+1}(t)}}{k!}\Omega^{(k+1)}(\overline{\zeta_m(t)},t) \qquad m = 2,..,M$$

while the equation with $m = 1$ remains:

$$\frac{\partial}{\partial t}\sum_{k=0}^{K(1)-1}\frac{\overline{c_{1\,k+1}(t)}}{k!}\Omega^{(k+1)}(\overline{\zeta_1(t)},t) = a_1(t)$$

(If a_1 is identical to zero (e.g. if $N = 0$), this equation also leads to a conserved quantity). The sum of these equations leads to a differential equation for a real function A on I defined by

$$A(t) = (2i)^{-1} \oint_{|\zeta|=1} \Omega'(\zeta,t)\overline{\Omega}(\zeta,t)d\zeta$$

as

$$\begin{aligned}\frac{d}{dt}A(t) &= (2i)^{-1}\frac{d}{dt} \oint_{|\zeta|=1} \Omega'(\zeta,t)\overline{\Omega}(\zeta,t)d\zeta = \pi\frac{d}{dt}\sum_{m=1}^{M}\frac{1}{2\pi i}\oint_{\gamma_m(t)} \Omega'(\zeta,t)\overline{\Omega}(\zeta,t)d\zeta \\ &= \pi\frac{d}{dt}\sum_{m=1}^{M}\sum_{k=0}^{K(m)-1}\frac{\overline{c_{m\,k+1}(t)}}{k!}\overline{\Omega^{(k+1)}(\zeta_m(t),t,t)} = \pi a_1(t) \qquad (4.13)\end{aligned}$$

We note that if the mapping Ω is conformal –and not only locally conformal– the quantity $A(t)$ is the area of the image of D under $\Omega(\cdot,t)$. The equation (4.13) is real as both A and a_1 are real valued. So, the imaginary parts of the equations (4.8) with $n = 1$ are linearly dependent indeed. We repeat that this leads to the conclusion that the set of equations for the parameters is not overdetermined.

4.2 Proof of local existence of solutions

In this section we reconsider the set of differential equations obtained in the previous section by substitution of a rational time dependent mapping Ω into a Hopper equation. We show in Subsection 4.2.1 that these equations can be solved if the given functions f and $a_n, n = 1,..,N$ are continuous with respect to the variable t, by revealing the structure of equations (4.7) and (4.8). This enables us to study a Hopper equation where f and $a_n, n = 1,..,N$ depend on Ω in a functional way. In Subsection 4.2.2 we prove a result on the solvability of this equation under certain restrictions on the functions f and $a_n, n = 1,..,N$.

4.2.1 The structure of the equations

We discuss equations (4.7) and (4.8) in the case where f and $a_n, n = 1,..,N$ are given functions on $\overline{D} \times I$ and I respectively. We assume that these functions are continuous with respect to the variable t and prove that the mentioned equations can be solved. First we introduce some notation.
In stead of $\zeta_.$ or $\zeta_m, m = 1,..,M$, we write

$$\underline{\zeta} = (\text{Re}\,\zeta_1, \text{Im}\,\zeta_1, ..., \text{Im}\,\zeta_M) \in \mathbb{R}^{2M}$$

In stead of $c_{..}$ or $c_{mk}, m = 1,..,M, k = 1,..,K(M)$ we write

$$\underline{c} = (\text{Re}\,c_{11}, \text{Im}\,c_{11}, .., \text{Im}\,c_{1K(1)}, \text{Re}\,c_{21}, .., .., \text{Im}\,c_{MK(M)}) \in \mathbb{R}^{2P}$$

where $P = \sum_{m=1}^{M} K(m)$. Because we only consider functions Ω normalized by $\Omega'(0) \in I\!R$, we take $\underline{c}$ in the $(2P-1)$-dimensional subspace corresponding to $\mathrm{Im} \sum_{m=1}^{M} c_{1m} = 0$ and we will simply write $\underline{c} \in I\!R^{2P-1}$ –see also condition ii) on page 64. Each $\underline{q} = (\underline{\zeta}, \underline{c}) \in I\!R^{2M} \times I\!R^{2P-1} = I\!R^{L}$ corresponds in this way to a rational function $\Omega_{\underline{q}}$ given by:

$$\Omega_{\underline{q}}(\zeta) = \sum_{m=1}^{M} \sum_{k=1}^{K(m)} c_{mk} \frac{\zeta^k}{(1-\zeta_m\zeta)^k} \tag{4.14}$$

The points in $I\!R^L$ which correspond to functions which are analytic on some neighbourhood of $\overline{D}$ and which have a derivative which does not vanish on $\overline{D}$ deserve special attention.

Definition 4.6 We define the poly-disc $\mathcal{D}^M$ as the set of points $\underline{\zeta} \in I\!R^{2M}$ such that:

$$\zeta_{2m-1}^2 + \zeta_{2m}^2 < 1 \qquad\qquad m = 1,..,M$$

We define $\mathcal{R}^{M;K(1),..,K(M)}$ as the set of points $\underline{q} = (\underline{\zeta}, \underline{c}) \in I\!R^L$ such that:

i). $\underline{\zeta} \in \mathcal{D}^M$

ii). the function $\Omega_{\underline{q}}$ corresponding to $\underline{q}$ by relation (4.14) has a derivative which does not vanish on $\overline{D}$

iii). (a) $m_1 \neq m_2$ implies $\zeta_{2m_1-1} \neq \zeta_{2m_2-1}$ or $\zeta_{2m_1} \neq \zeta_{2m_2}$

(b) for every $m \in \{1,..,M\}$ there is a $k \in \{1,..,K(m)\}$ such that $c_{mk} \neq 0$

The space of functions $\Omega_{\underline{q}}$ with $\underline{q} \in \mathcal{R}^{M;K(1),..,K(M)}$ will be denoted by $\mathcal{T}^{M;K(1),..,K(M)}$.

Remarks 4.7 We make a number of remarks which are helpful later on.
We note that the spaces $\mathcal{R}^{M;K(1),..,K(M)}$ and $\mathcal{T}^{M;K(1),..,K(M)}$ differ only by the nature of their elements; this can be checked on the basis of the assertion that a rational function is completely determined by its poles and a value at a regular point. We can characterize the space $\mathcal{T}^{M;K(1),..,K(M)}$ also as follows: a function Ω is an element of $\mathcal{T}^{M;K(1),..,K(M)}$ if and only if:

i). Ω is a meromorphic function on $\overline{\mathbb{C}}$ with M poles which all lay outside $\overline{D}$

ii). the order of the mth pole does not exceed $K(m)$

iii). Ω' does not vanish on $\overline{D}$

iv). $\Omega(0) = 0 \qquad \mathrm{Im}\ \Omega'(0) = 0$

In the following we will use a shorter notation and write simply $\mathcal{D}$ in stead of $\mathcal{D}^M$, etc.; we implicitly understand that the numbers M and $K(1),..,K(M)$ are fixed.
We make the important remark that $\mathcal{R}$ is an open set in $I\!R^L$. This can straightforwardly be checked by means of the theorem that states that zeros of polynomials in a complex variable depend on the coefficients in a continuous way ([58]). We finally notice that the set $\mathcal{R}$ is not connected: there is no continuous path from points $\underline{q} \in \mathcal{R}$ with $\sum_{m=1}^{M} c_{1m} > 0$ to points $\underline{q} \in \mathcal{R}$ with $\sum_{m=1}^{M} c_{1m} < 0$.

Before we rewrite equations (4.7) and (4.8), we introduce some more notation. Let K denote the maximum of the numbers $K(1),..,K(M)$. We define the mappings $\underline{f}^k, k = 0,..,K+1$ by

$$\begin{aligned} \underline{f}^k \; &: \; (\underline{\zeta}, t) \in \mathcal{D} \times I \;\mapsto\; \underline{f}^k(\underline{\zeta}, t) \in \mathbb{C}^M \\ f^k_m(\underline{\zeta}, t) &= f^{(k)}(\zeta_{2m-1} - i\zeta_{2m}, t) \qquad\qquad m = 1,..,M \end{aligned}$$

Finally, we write $\underline{a} \in I\!R^{2N-1}$in stead of $a_k, k = 1,..,N$.
The equations (4.7) can now be written as

$$\dot{\underline{\zeta}}(t) = \underline{b}\left(\underline{\zeta}(t), \underline{f}^0(\underline{\zeta}(t), t)\right) \tag{4.16}$$

where $\underline{b}$ is a mapping from $I\!R^{2M} \times \mathbb{C}^M$ to $I\!R^{2M}$. The equations (4.8) can be rewritten as follows. We express $\Omega^{(l)}(\overline{\zeta_m(t)}, t)$, $m = 1,..,M, l = 1,..,K(m)$ in terms of the coefficients $\zeta_.$ and $c_{..}$, differentiate the first term in equation (4.8) with respect to the variable t, use relation (4.7) whenever a time derivative of ζ_m appears, bring every term without time derivatives to the right-hand side and skip one of the linear dependent real equations discussed in the end of the previous section. This leads to an equation of the following form:

$$\underline{\underline{A}}\left(\underline{\zeta}(t), \underline{c}(t)\right)\; \dot{\underline{c}}(t) = \underline{d}\left(\underline{\zeta}(t), \underline{c}(t), \underline{f}^0(\underline{\zeta}(t), t), .., \underline{f}^{K+1}(\underline{\zeta}(t), t), \underline{a}(t)\right) \tag{4.17}$$

where:

i). $\underline{\underline{A}}$ is a mapping from $I\!R^L$ to $\mathcal{M}^{(2P-1)\times(2P-1)}$, the space of real $(2P-1) \times (2P-1)$-matrices

ii). $\underline{d}$ is a mapping from $I\!R^{2M} \times I\!R^{2P-1} \times \overbrace{\mathbb{C}^M \times \cdots \times \mathbb{C}^M}^{K+2} \times I\!R^{2N-1}$ to $I\!R^{2P-1}$

Next we apply a proposition that will be proved at the end of this subsection.

Proposition 4.8 *The matrix $\underline{\underline{A}}(\underline{q})$ is regular for all $\underline{q} \in \mathcal{R}$.*

The Hopper equation (4.1) for a mapping Ω as in expression (4.2) is therefore equivalent to the following set of equations:

$$\dot{\underline{\zeta}}(t) = \underline{b}\left(\underline{\zeta}(t), \underline{f}^0(\underline{\zeta}(t), t)\right)$$
$$\dot{\underline{c}}(t) = \underline{\underline{A}}^{\leftarrow}\left(\underline{\zeta}(t), \underline{c}(t)\right)\ \underline{d}\left(\underline{\zeta}(t), \underline{c}(t), \underline{f}^0(\underline{\zeta}(t), t), .., \underline{f}^{K+1}(\underline{\zeta}(t), t), \underline{a}(t)\right)$$

Let these equations be complemented by initial data

$$(\underline{\zeta}(0), \underline{c}(0)) = (\underline{\zeta}_0, \underline{c}_0) \in \mathcal{R}$$

We repeat that we assumed that the functions f and $a_n, n = 1, .., N$ are continuous with respect to the variable t. One then shows that this initial value problem has a unique solution by using the following arguments:

i). the functions $\underline{f}^k, k = 0, .., K+1$ are continuously differentiable with respect to $\underline{\zeta}$ and continuous with respect to t

ii). $\underline{a}$ is a continuous function on I

iii). $\underline{b}, \underline{d}$ and $\underline{\underline{A}}^{\leftarrow}$ are continuously differentiable functions: they depend on their arguments in a rational way

iv). an initial value problem of this type has a unique maximal solution ([19])

Corollary 4.9 *Let f be a continuous function on $\overline{D} \times I$ that is analytic on D for all fixed $t \in I$. Let $a_n, n = 1, .., N$ be continuous functions on I (a_1 real valued). Let Ω_0 be a rational function such that all singularities of Ω_0 and all zeros of Ω_0' lay outside $\overline{D}$. Then the Hopper equation* (4.1) *with initial data Ω_0 has a rational solution. This solution is unique in the class of normalized, smooth time-dependent locally conformal and rational functions with a time-independent numbers of poles.*

The last part of this subsection is devoted to the proof of Proposition 4.8. We first prove two lemmas.

Lemma 4.10 *Let $\underline{q} = (\underline{\zeta}, \underline{c}) \in \mathcal{R}$ and let $\underline{x} \in I\!R^{2P-1}$ satisfy $\underline{\underline{A}}(\underline{q})\ \underline{x} = \underline{0}$. Let Ω denote the function corresponding to $\underline{q}$ (so: $\Omega = \Omega_{\underline{q}}$) and let g denote the function corresponding to $(\underline{\zeta}, \underline{x})$ (so: $g = \Omega_{(\underline{\zeta}, \underline{x})}$). The function ϕ defined by*

$$\phi = \overline{\Omega} g' + \Omega' \overline{g} \tag{4.18}$$

is analytic on an open domain that contains $\overline{D}$.

Proof

Part 1. The functions Ω and g are both analytic on an open domain that contains $\overline{D}$. More precisely, the rational functions Ω and g are analytic on $\overline{\mathbb{C}}\backslash\{1/\zeta_1,..,1/\zeta_M\}$. So, the function ϕ is analytic on $\overline{\mathbb{C}}\backslash\{\overline{\zeta_1},..,\overline{\zeta_M},1/\zeta_1,..,1/\zeta_M\}$. As $\underline{q} = (\underline{\zeta},\underline{c}) \in \mathcal{R}$ implies $\underline{\zeta} \in \mathcal{D}$ and hence, $|\zeta_m| < 1, m = 1,..,M$, it is sufficient to prove that ϕ is analytic on D.

Part 2. We consider the homogeneous Hopper equation, i.e. equation (4.1) with f identically zero:

$$\theta(\zeta,t) = \overline{\Omega}(\zeta,t)\dot{\Omega}'(\zeta,t) + \Omega'(\zeta,t)\dot{\overline{\Omega}}(\zeta,t) \tag{4.19}$$

where it is understood that θ is an analytic function on D for all $t \in I$. We substitute the Ansatz (4.2) into this equation and get equations (4.7) and (4.8) with $f^{(l)}(\overline{\zeta_m(t)},t) = 0$ for all $m = 1,..,M$ and all $l = 0,..,K+1$. So, the equivalent equations (4.16) and (4.17) in this case reduce to

$$\dot{\underline{\zeta}}(t) = \underline{b}\left(\underline{\zeta}(t),\underline{0}\right) = \underline{0}$$
$$\underline{\underline{A}}\left(\underline{\zeta}(t),\underline{c}(t)\right)\ \dot{\underline{c}}(t) = \underline{d}\left(\underline{\zeta}(t),\underline{c}(t),\underline{0},..,\underline{0},\underline{0}\right) = \underline{0}$$

Conversely, if $\underline{q} : t \in I \mapsto \underline{q}(t) = (\underline{\zeta}(t),\underline{c}(t)) \in \mathcal{R}$ satisfies these equations, then $\Omega_{\underline{q}(t)}$ is a solution of the homogeneous Hopper equation (see Corollary 4.3). One checks that –because only first order derivatives with respect to the variable t appear in the calculations– the following assertion is also true:
if $\underline{q} : t \in I \mapsto \underline{q}(t) = (\underline{\zeta}(t),\underline{c}(t)) \in \mathcal{R}$ is such that

$$\dot{\underline{\zeta}}(0) = \underline{0} \qquad \underline{\underline{A}}(\underline{\zeta}(0),\underline{c}(0))\,\dot{\underline{c}}(0) = \underline{0} \tag{4.20}$$

then the function $\Omega_{\underline{q}(t)}$ satisfies the homogeneous Hopper equation at $t = 0$; i.e. the function $\theta(\zeta,t)$ defined by (4.19) –with $\Omega(\zeta,t) = \Omega_{\underline{q}(t)}(\zeta)$– is analytic on D for $t = 0$.

Part 3. Let $\underline{q} = (\underline{\zeta},\underline{c}) \in \mathcal{R}$ and let $\underline{x} \in \mathbb{R}^{2P-1}$ satisfy $\underline{\underline{A}}(\underline{q})\ \underline{x} = \underline{0}$. As $\mathcal{R}$ is an open set of $\mathbb{R}^L$ (see Remarks 4.7), there is an ε such that the mapping

$$\underline{q} : t \in I \mapsto \underline{q}(t) = (\underline{\zeta}(t),\underline{c}(t)) = (\underline{\zeta},\underline{c} + \underline{x}t) \in \mathbb{R}^L$$

is such that $\underline{q}(t) \in \mathcal{R}$ for all $t \in (-\varepsilon,\varepsilon)$. We note that this mapping is such that relations (4.20) hold. It then follows from Part 2 that the function ϕ defined by

$$\phi(\zeta) = \overline{\Omega_{(\underline{\zeta},\underline{c})}}(\zeta)\Omega'_{(\underline{\zeta},\underline{x})}(\zeta) + \Omega'_{(\underline{\zeta},\underline{c})}(\zeta)\overline{\Omega_{(\underline{\zeta},\underline{x})}}(\zeta)$$

is analytic on D, where it has been used that

$$\frac{\partial}{\partial t}\left(\Omega_{\underline{q}(t)}(\zeta)\right) = \Omega_{(\underline{\zeta},\dot{\underline{c}}(t))}(\zeta) = \Omega_{(\underline{\zeta},\underline{x})}(\zeta)$$

The assertion in the lemma follows. □

Remark 4.11 We sketch an alternative proof of Parts 2 and 3. It has been stated in Part 1 that the function ϕ is analytic on $\overline{\mathbb{C}}\backslash\{\overline{\zeta_1},..,\overline{\zeta_M},1/\zeta_1,..,1/\zeta_M\}$. Moreover, if ϕ has a singularity in $\zeta=\overline{\zeta_m}\in D, m=1,..,M$, then this singularity is a pole and the order of this pole does exceed $K(m)+1$. This implies that ϕ is analytic on an open domain that contains $\overline{D}$ if the following relations hold for all $m\in\{1,..,M\}$ and all $n\in\{1,..,K(m)+1\}$:

$$\frac{1}{2\pi i}\oint_{\gamma_m}(\zeta-\overline{\zeta_m})^{n-1}\phi(\zeta)\,d\zeta=0 \tag{4.21}$$

where $\gamma_m, m=1,..,M$ is a closed Jordan curve in D such that $\overline{\zeta_m}$ is inside γ_m and $\overline{\zeta_l}$ is outside γ_m if $l\neq m$. We substitute relation (4.18), calculate the integrals as we did in the previous section and find:

$$\frac{1}{2\pi i}\oint_{\gamma_m}(\zeta-\overline{\zeta_m})^{n-1}\phi(\zeta)\,d\zeta=$$
$$\sum_{k=0}^{K(m)-n}\sum_{l=1}^{M}\sum_{j=1}^{K(l)}\frac{\overline{x_{m\;k+n}c_{lj}+c_{m\;k+n}x_{lj}}}{k!}\left(\left(\frac{\partial}{\partial\zeta}\right)^{k+1}\left(\frac{\zeta}{1-\zeta_l\zeta}\right)^{j}\right)\Bigg|_{\zeta=\overline{\zeta_m}} \tag{4.22}$$

for $m=1,..,M$ and $n=1,..,K(m)$ while the integrals in the left-hand sides of (4.21) with $n=K(m)+1$ and $m=1,..,M$ vanish. One shows by straightforward calculations that the relation $\underline{\underline{A}}(\underline{q})\;\underline{x}=\underline{0}$ implies that the right-hand sides of identities (4.22) vanish for all $m=1,..,M$ and all $n=1,..,K(m)$.

Next we prove an important lemma which has already been used in Section 3.4.

Lemma 4.12 *Let ϕ,Ω and g be analytic functions on D and let ϕ,Ω' and g' be continuously extendable to $\overline{D}$. If Ω' does not vanish on $\overline{D}$ and if*

$$\phi(\zeta)=\overline{\Omega(\zeta)}g'(\zeta)+\Omega'(\zeta)\overline{g(\zeta)} \tag{4.23}$$

for all $\zeta\in\partial D$, then constants $C_1\in\mathbb{R}$ and $C_2\in\mathbb{C}$ exist such that

$$g=iC_1\Omega+C_2 \tag{4.24}$$

Proof
We define functions h,f and p on D by

$$h(\zeta)=\frac{\phi(\zeta)}{\Omega'(\zeta)} \tag{4.25a}$$
$$f(\zeta,\overline{\zeta})=\left(\Omega(\zeta)\overline{\left(\frac{g'(\zeta)}{\Omega'(\zeta)}\right)'}-\overline{h'(\zeta)}\right)\left(-\overline{g(\zeta)}+\overline{\Omega(\zeta)}\frac{g'(\zeta)}{\Omega'(\zeta)}-h(\zeta)\right)$$

$$- \left(\overline{\Omega'(\zeta)} \frac{g'(\zeta)}{\Omega'(\zeta)} + \overline{g'(\zeta)} \right) \left(-g(\zeta) + \Omega(\zeta) \overline{\left(\frac{g'(\zeta)}{\Omega'(\zeta)} \right)} - \overline{h(\zeta)} \right) \quad (4.25b)$$

$$p(\zeta, \overline{\zeta}) = \frac{\phi(\zeta)}{\Omega'(\zeta)} - \overline{g(\zeta)} - \overline{\Omega(\zeta)} \frac{g'(\zeta)}{\Omega'(\zeta)} \quad (4.25c)$$

We note that the functions h and p are continuously extendable to $\overline{D}$. We also note that h is analytic on D while f and p are real analytic on D as they are sums of products of analytic and anti-analytic functions. In particular, f is differentiable and we can apply Gauss' theorem on each disc $\overline{D_r}$ with $0 < r < 1$:

$$\iint_{B_r} \mathrm{Re}\ (\partial_\zeta f(\zeta, \overline{\zeta}))\ dV = \frac{r}{2} \int_0^{2\pi} \mathrm{Re}\ \left(f(re^{i\theta}, re^{-i\theta}) e^{-i\theta} \right)\ d\theta \quad (4.26)$$

We rewrite the integrands in this relation. First we calculate $\partial_\zeta f$. We omit the variables and stress that $\overline{\Omega}, \overline{g}$, etc. denote the anti-analytic functions obtained by complex conjugating Ω, g, etc.; so $\overline{\Omega}$ is *not* the function defined in Definition 3.16. We get

$$\partial_\zeta f = \Omega' \overline{\left(\frac{g'}{\Omega'} \right)'} \left(-\overline{g} + \overline{\Omega} \frac{g'}{\Omega'} - h \right) + \left(\Omega \overline{\left(\frac{g'}{\Omega'} \right)'} - \overline{h'} \right) \left(\overline{\Omega} \left(\frac{g'}{\Omega'} \right)' - h' \right)$$
$$- \overline{\Omega'} \left(\frac{g'}{\Omega'} \right)' \left(-g + \Omega \overline{\left(\frac{g'}{\Omega'} \right)} - \overline{h} \right) - \left(\overline{\Omega'} \frac{g'}{\Omega'} + \overline{g'} \right) \left(-g' + \Omega' \overline{\left(\frac{g'}{\Omega'} \right)} \right)$$

and this leads to

$$\mathrm{Re}\ \partial_\zeta f = \left| \overline{\Omega} \left(\frac{g'}{\Omega'} \right)' - h' \right|^2 \quad (4.27)$$

Next we rewrite the integrand $\mathrm{Re}\ f e^{-i\theta}|_{\partial D_r}$ in the right-hand side of relation (4.26). We again omit the variables, we do not indicate that we restrict the functions to ∂D_r and find

$$\mathrm{Re}\ f e^{-i\theta} =$$
$$\mathrm{Re}\ \left(\left(-g + \Omega \overline{\left(\frac{g'}{\Omega'} \right)} - \overline{h} \right) \left(e^{i\theta} \left(\overline{\Omega} \left(\frac{g'}{\Omega'} \right)' - h' \right) - e^{-i\theta} \left(\overline{\Omega'} \frac{g'}{\Omega'} + \overline{g'} \right) \right) \right) =$$
$$\frac{1}{r}\ \mathrm{Re}\ \left((-\overline{p} - 2g) \left(-i\overline{\Omega} \frac{d}{d\theta} \left(\frac{g'}{\Omega'} \right) + i \frac{d}{d\theta} h - \frac{g'}{\Omega'} i \frac{d}{d\theta} \overline{\Omega} - i \frac{d}{d\theta} \overline{g} \right) \right) =$$
$$-\frac{1}{r}\ \mathrm{Re}\ \left((\overline{p} + 2g) i \frac{d}{d\theta} p \right) = \frac{2}{r}\ \mathrm{Im}\ \left(g \frac{d}{d\theta} p \right) + \frac{1}{r} \frac{d}{d\theta} (\mathrm{Re}\ p\ \ \mathrm{Im}\ p) \quad (4.28)$$

We substitute the relations (4.27) and (4.28) into relation (4.26), integrate by parts and obtain for all positive $r < 1$:

$$\iint_{B_r} \left| \overline{\Omega} \left(\frac{g'}{\Omega'} \right)' - h' \right|^2 dV = \mathrm{Re} \int_0^{2\pi} e^{i\theta} (pg') \Big|_{\partial D_r} d\theta$$

We let r approach to 1. The right-hand side vanishes as both g' and p are continuous on $\overline{D}$ and as p restricted to ∂D equals zero as follows from relations (4.23) and (4.25c). This leads to the following identity on D:

$$\overline{\Omega}\left(\frac{g'}{\Omega'}\right)' = h'$$

One then argues that this implies the existence of a constant $C \in \mathbb{C}$ such that

$$g' = C\Omega'$$

It follows from this relation that constants C and C_2 exist such that $g = C\Omega + C_2$. Substitution of this relation into relation (4.23) leads to Re $C = 0$. □

Remark 4.13 If Ω is an univalent function on D such that Ω'' can be extended continuously to $\overline{D}$ and such that Ω' does not vanish on $\overline{D}$, then this result can be proved alternatively as follows.
We define functions $\tilde{\chi}$ and $\tilde{\varphi}$ on $\overline{D}$ by

$$\tilde{\chi}(\zeta) = -\int_0^\zeta \phi(z)\,dz \qquad \tilde{\varphi}(\zeta) = g(\zeta)$$

Let G denote the image of D under Ω. We define functions χ and φ on $\overline{G}$ by

$$\chi(z) = \tilde{\chi}(\Omega^{\leftarrow}(z)) \qquad \varphi(z) = \tilde{\varphi}(\Omega^{\leftarrow}(z))$$

Finally, we define the functions p and v according to formulae (2.12). It follows from Remark 2.1 that p and v satisfy Stokes' equations. Moreover, one checks that the homogeneous boundary condition is fulfilled; i.e. $\underline{\underline{T}}\,\underline{n} = \underline{0}$ on ∂G as follows from the easily established relation $\overline{z}\varphi' + \overline{\varphi} + \chi' = 0$ on ∂G and relation (2.70). This implies ([72]) that p is constant and v is a rigid body motion. Relation (4.24) can then be established by considering the homogeneous Hopper equation (4.19) for a fixed t.

Remark 4.14 The condition in Lemma 4.12 that Ω has a non-vanishing derivative cannot regardless be omitted as the following counterexample shows. Let the functions Ω, g and ϕ be defined by

$$\Omega(\zeta) = \sqrt{2}\zeta + \zeta^2 \qquad g(\zeta) = \sqrt{2}\zeta - \zeta^2 \qquad \phi(\zeta) = 0$$

One easily checks that relation (4.23) holds but that there are not any constants $C_1 \in \mathbb{R}, C_2 \in \mathbb{C}$ such that relation (4.24) holds.

Remark 4.15 If we substitute relation (4.24) into relation (4.23), we find $\phi = C_2\Omega'$ on ∂D. As ϕ and Ω' are both analytic on D, this implies $\phi = C_2\Omega'$ on $\overline{D}$.

One checks that this in turn implies that relation (4.24) holds on $\overline{D}$ and not only on ∂D (where it is understood that $\overline{\Omega}$ denotes the complex conjugate of Ω). We did not find a way to prove this result directly and we believe that this is not easy; the proof of this assertion may be connected with the following exercise: let the real functions u and v on an open simply connected domain in $G \subset I\!R^2$ and their product uv be harmonic on G, show that if u is not identically zero, then constants $c_1, c_2 \in I\!R$ exist such that $v = c_1 u + c_2$ ([15]). We finally note that for example: ϕ, Ω and g analytic on D and continuous on $\overline{D}$ and $\phi = \overline{\Omega} g + \Omega \overline{g}$ on ∂D does not imply $\phi = \overline{\Omega} g + \Omega \overline{g}$ on D (take e.g. $\Omega(\zeta) = g(\zeta) = \zeta$).

The previous lemmas enable us to prove Proposition 4.8.

Proof of Proposition 4.8
Let $\underline{q} = (\underline{\zeta}, \underline{c})$ be an arbitrary point in $\mathcal{R}$ and let $\underline{x} \in I\!R^{2P-1}$ satisfy the equation $\underline{\underline{A}}(\underline{q})\ \underline{x} = \underline{0}$. Let Ω and g be the rational functions corresponding to $\underline{q} \in \mathcal{R}$ and $(\underline{\zeta}, \underline{x}) \in \mathcal{D} \times I\!R^{2P-1}$ respectively. It follows from Lemma 4.10 that the functions Ω, g and the function ϕ defined by (4.18) satisfy the conditions of Lemma 4.12. So, constants $C_1 \in I\!R, C_2 \in \mathbb{C}$ exist such that

$$g = iC_1 \Omega + C_2$$

As the functions Ω and g are normalized by

$$\Omega(0) = g(0) = 0 \qquad \text{Im } \Omega'(0) = \text{Im } g'(0) = 0$$

we conclude that the constants C_1 and C_2 and hence g are identical to zero. As $\zeta_{m_1} \neq \zeta_{m_2}$ if $m_1 \neq m_2$ (see condition iiia) of Definition 4.6), this implies $x_{mk} = 0$ for all $m = 1, .., M$ and all $k = 1, .., K(m)$ and hence $\underline{x} = \underline{0}$. So, $\underline{\underline{A}}(\underline{q})\ \underline{x} = \underline{0}$ implies $\underline{x} = \underline{0}$ and it follows that $\underline{\underline{A}}(\underline{q})$ is regular for all $\underline{q} \in \mathcal{R}$. □

Remark 4.16 It has already been stated that the proof of Proposition 4.8 is based on the Lemmas 4.10 and 4.12 which in turn are based on the fact that the only solutions of boundary value problem (1.8) with $\gamma = 0$ are the rigid-body motions. So, we can state in retrospect that the set of differential equations obtained by substitution of a rational function into a Hopper equation is uniquely solvable because of the linear structure of these equations and because the solutions of the homogeneous Hopper equation (4.19) correspond to rigid-body motions which can be suppressed by normalization.

4.2.2 Functional dependency

We consider Hopper equations where the functions f and $a_n, n = 1, .., N$ depend on Ω in a functional way. As far as the solvability of equations (4.7) and (4.8) is concerned, we can repeat the same arguments as in the previous subsection

with the exception of the arguments i) and ii) on page 72. In this subsection we present sufficient conditions on the functions f and $a_n, n = 1, .., N$ for the equations (4.7) and (4.8) to be solvable.
The standard norm on $\mathcal{R} \subset I\!R^L$ can be carried over to the space $\mathcal{T}$ (see Definition 4.6 and Remark 4.7):

$$\|\Omega_{\underline{q}_1} - \Omega_{\underline{q}_2}\| = |\underline{q}_1 - \underline{q}_2|$$

Let $\mathcal{A}$ denote the space of analytic functions on D.

Definition 4.17 A mapping

$$\mathcal{F} : \Omega \in \mathcal{T} \quad \mapsto \quad f_{[\Omega]} \in \mathcal{A}$$

is called Lipschitz continuous if a constant C exists such that for all $\Omega_1, \Omega_2 \in \mathcal{T}$, all $m \in \{1, .., M\}$ and all $k \in \{1, .., K(M) + 1\}$

$$\left| f^{(k)}_{[\Omega_1]}\left(\overline{\zeta^{(1)}_m}\right) - f^{(k)}_{[\Omega_2]}\left(\overline{\zeta^{(2)}_m}\right)\right| < C \, \|\Omega_1 - \Omega_2\|$$

where it is understood that:

$$\Omega_j(\zeta) = \sum_{m=1}^{M} \sum_{k=1}^{K(m)} c^{(j)}_{mk} \frac{\zeta^k}{(1 - \zeta^{(j)}_m \zeta)^k} \qquad j = 1, 2$$

A mapping

$$a : (\Omega, t) \in \mathcal{T} \times I \quad \mapsto \quad a_{[\Omega]}(t) \in \mathbb{C}$$

is said to be Lipschitz continuous if it is continuous with respect to the variable t and if a constant C exists such that for all $\Omega_1, \Omega_2 \in \mathcal{T}$ and all $t \in I$

$$\left| a_{[\Omega_1]}(t) - a_{[\Omega_2]}(t)\right| < C \, \|\Omega_1 - \Omega_2\| \tag{4.29}$$

Theorem 4.18 *Let $\mathcal{F}$ be a locally Lipschitz continuous mapping from $\mathcal{T}$ to $\mathcal{A}$ and let $a_N, n = 1, .., N$ be locally Lipschitz continuous mappings from $\mathcal{T} \times I$ to $\mathbb{C}$ (a_1 from $\mathcal{T} \times I$ to $I\!R$). Let Ω_0 be rational function such that all singularities of Ω_0 and all zeros of Ω_0' lay outside $\overline{D}$. Then the Hopper equation (4.1) with initial data Ω_0 corresponding to $\mathcal{F}$ and $a_n, n = 1, .., N$ has a rational solution. This solution is unique in the class of normalized, smooth time-dependent locally conformal rational functions with a time-independent number of poles.*

Proof
We define mappings $\underline{f}^k, k = 0, .., K + 1$ by

$$\underline{f}^k : \underline{q} = (\underline{\zeta}, \underline{c}) \in \mathcal{R} \mapsto \underline{f}^k(\underline{q}) \in \mathbb{C}^M$$

$$f^k_m(\underline{q}) = f^{(k)}_{[\Omega_{\underline{q}}]}(\zeta_{2m-1} - i\zeta_{2m}) \qquad m = 1, .., M$$

It follows immediately from the Lipschitz continuity of $\mathcal{F}$ and the property that $f_{[\Omega]}$ is analytic on D for all $\Omega \in \mathcal{T}$ that the mappings $\underline{f}^k, k = 0, .., K+1$ are Lipschitz continuous functions on $\mathcal{R}$. We define the mapping $\underline{a}$ by

$$\underline{a} : (\underline{q}, t) \in \mathcal{R} \times I \mapsto$$
$$\underline{a}(\underline{q}, t) = (a_{1,[\Omega_{\underline{q}}]}(t), \text{Re } a_{2,[\Omega_{\underline{q}}]}(t), .., \text{Im } a_{N,[\Omega_{\underline{q}}]}(t)) \in I\!R^{2N-1}$$

The initial value problem given by the Hopper equation (4.1) corresponding to $\mathcal{F}$ and $a_n, n = 1, .., N$ with initial data Ω_0, in the notation explained in Subsection 4.2.1, can now be written as

$$\dot{\underline{\zeta}}(t) = \underline{b}\left(\underline{\zeta}(t), \underline{f}^0(\underline{q}(t))\right)$$
$$\dot{\underline{c}}(t) = \underline{\underline{A}}^{-1}(\underline{q}(t))\, \underline{d}\left(\underline{q}(t), \underline{f}^0(\underline{q}(t)), .., \underline{f}^{K+1}(\underline{q}(t)), \underline{a}(\underline{q}(t), t)\right)$$
$$\underline{q}(0) = \underline{q}_0$$

where $\underline{q}(t) = (\underline{\zeta}(t), \underline{c}(t))$ and where $\underline{q}_0$ corresponds to Ω_0. One shows that this set of differential equations has a unique solution by using arguments similar to those on page 72. □

We consider the moving boundary problem for Stokes flow driven by surface tension with initial data $\Omega_0 \in \mathcal{R}$. It follows from Proposition 3.20 and Theorem 4.18 that this problem has a local rational solution if the mapping $\mathcal{F} : \Omega \in \mathcal{T} \mapsto F_{[\Omega]}$ defined by Definition 3.15 is locally Lipschitz continuous. We show that this mapping $\mathcal{F}$ has this property.
An arbitrary function $\Omega \in \mathcal{T}$ can uniquely be written as

$$\Omega(\zeta) = \sum_{m=1}^{M} \sum_{k=1}^{K(m)} c_{mk} \frac{\zeta^k}{(1 - \zeta_m \zeta)^k}$$

The values of $\mathcal{F}(\Omega) = F_{[\Omega]}$ in the points $\overline{\zeta_m} \in D, m = 1, .., M$ can be found by applying Schwarz' integral relation ([63, 35]):

$$F_{[\Omega]}(\overline{\zeta_m}) = \frac{1}{2\pi i} \oint_{|z|=1} \frac{1}{|\Omega'(z)|} \left(\frac{1}{z - \overline{\zeta_m}} - \frac{1}{2z} \right) dz$$
$$= \frac{1}{2\pi i} \oint_{|z|=1} \left| \sum_{l=1}^{M} \sum_{j=1}^{K(l)} j c_{lj} \frac{z^{j-1}}{(1 - \zeta_l z)^{j+1}} \right|^{-1} \left(\frac{1}{z - \overline{\zeta_m}} - \frac{1}{2z} \right) dz \quad (4.30)$$

The values of the kth derivative of $F_{[\Omega]}$ in the points $\overline{\zeta_m} \in D, m = 1, .., M$ can be found by differentiating this relation:

$$F_{[\Omega]}^{(k)}(\overline{\zeta_m}) = \frac{k!}{2\pi i} \oint_{|z|=1} \left| \sum_{l=1}^{M} \sum_{j=1}^{K(l)} j c_{lj} \frac{z^{j-1}}{(1 - \zeta_l z)^{j+1}} \right|^{-1} (z - \overline{\zeta_m})^{-(k+1)} dz \quad (4.31)$$

Roughly speaking, the integrands of these expressions are smooth functions of the coefficients $\zeta.$ and $c..$ as long as the derivative of Ω does not vanish on ∂D. To be more precise, it follows from these expressions that for every $\Omega_0 \in \mathcal{T}$ an open neighbourhood $\mathcal{N} \subset \mathcal{T}$ and a constant C exist such that for all $\Omega \in \mathcal{N}$, for all $m \in \{1,..,M\}$ and all $k \in \{1,..,K(M)+1\}$

$$\left| F^{(k)}_{[\Omega_0]}\left(\overline{\zeta^{(0)}_m}\right) - F^{(k)}_{[\Omega]}\left(\overline{\zeta_m}\right)\right| < C \left\|\Omega_0 - \Omega\right\|$$

The other moving boundary problems mentioned in the Sections 3.3 and 3.4 can be treated analogously.

Theorem 4.19 *Let Ω_0 be a rational function such that all singularities of Ω_0 and all zeros of Ω_0' lay outside $\overline{D}$. Then the moving boundary problems for Stokes flow driven by surface tension and multi-poles and for Hele-Shaw flow with initial data Ω_0 have a local rational solution. This solution is unique in the class of normalized, smooth time-dependent locally conformal rational functions with a time-independent number of poles.*

We end this section by a discussion why the results obtained so far are local results. Let

$$\gamma \ : \ t \in I \ \mapsto \ \gamma(t) = \Omega_{\underline{q}(t)} \in \mathcal{T}$$

denote the unique maximal solution of a problem mentioned in Theorem 4.19. In general, the maximal interval I is a strict subset of $\mathbb{R}$ and we ask ourselves what can happen at the end point T_+ of the interval $I = (T_-, T_+)$ in these cases. It follows from the theory of ordinary differential equations ([31]) that in some sense

$$\Omega_{\underline{q}(t)} \ \rightarrow \ \partial\mathcal{T} \qquad \text{if } t \uparrow T_+$$

Roughly speaking, the boundary $\partial\mathcal{T}$ has three parts (see Definition 4.6) and we have the following possibilities in which way γ approaches this boundary:

i). One of the singularities of $\Omega_{\underline{q}(t)}$ moves to ∂D as $t \uparrow T_+$. We remark that the singularities of $\Omega_{\underline{q}(t)}$ move in the extended complex plane according to the relations (4.7). It follows from these relations that the singularities move away from $\overline{D}$ if the real part of the function f is non-negative on D; we will discuss this property in more detail in Chapter 5. So, for the moving boundary problems for Stokes flow driven by surface tension and multi-poles and for Hele-Shaw flow with a source, this possibility does not occur; i.e. solutions of these problems in $\mathcal{T}$ cannot approach that part of the boundary $\partial\mathcal{T}$ which consists of functions with singularities on ∂D.

ii). The function $\Omega_{\underline{q}(t)}$ approaches a function which has a derivative which vanishes somewhere on $\overline{D}$. Assume that the following limit exists:

$$\lim_{t\uparrow T_+} \Omega_{\underline{q}(t)} = \Omega_{\underline{q}(T_+)} \in \partial\mathcal{T}$$

We distinguish two cases (see e.g. [27]):

(a) The function $\Omega_{\underline{q}(t)}$ approaches the function which is identical to zero; that is: $\Omega_{\underline{q}(T_+)}(\zeta) = 0$. This cannot happen if a_1 is non-negative as follows from relation (4.13).

(b) One of the zeros of $\Omega'_{\underline{q}(t)}$ moves to ∂D as $t \uparrow T_+$. There are two reasons why the solution cannot be extended regardless beyond this point. First, the matrix $\underline{\underline{A}}(\underline{q}(T_+))$ may be singular and the equations (4.17), and hence the equations (4.8) and (4.1), may be incompatible. Secondly, the function $F_{[\Omega_{\underline{q}(T_+)}]}$ may not exist.

iii). Two singularities of $\Omega_{\underline{q}(t)}$ move to each other or a singularity vanishes as $t \uparrow T_+$. We may argue that this cannot occur in general by considering the time inversed problem. We will give a strict proof of this result in the next chapter.

Roughly speaking, the main reason why a non-global solution of a Hopper equation cannot be extended is the formation of cusps (see [38, 40]).

4.3 A class of global solutions

We showed in the previous section that the moving boundary problem for Stokes flow driven by surface tension has a local solution if the initial data is a rational function. In this section we show that this problem has a global solution if the initial data is a rational function of a particular type.

We reconsider the equations obtained by substitution of a partial fraction mapping in Hopper's equation. We showed in Example 4.5 that Hopper's equation for a mapping Ω parameterized as in expression (4.11), reduces to a set of M differential and M algebraic equations. Before we show that these equations have a global solution under certain restrictions on the initial data, we study the algebraic equations (4.12).

We define $\overline{\underline{x}}$ and $\underline{\underline{D}}_{\underline{x}}$ for an element $\underline{x} = (x_1, .., x_M) \in \mathbb{C}^M$ by

$$\overline{\underline{x}} = (\overline{x_1}, .., \overline{x_M}) \qquad \underline{\underline{D}}_{\underline{x}} = \mathrm{diag}(x_1, .., x_M)$$

We define the hermitian $M \times M$ matrix $\underline{\underline{M}}(\underline{\zeta})$ for an element $\underline{\zeta} \in \mathcal{D}^M$ by

$$M_{kl}(\underline{\zeta}) = (1 - \zeta_k \zeta_l)^{-2} \qquad k, l = 1, .., M$$

The M algebraic relations mentioned above can now be written as

$$\underline{\underline{D}}_{\overline{\underline{A}(t)}} \underline{\underline{M}}(\underline{\zeta}(t))\, \underline{A}(t) = \underline{q} = \underline{\underline{D}}_{\overline{\underline{A}(0)}} \underline{\underline{M}}(\underline{\zeta}(0))\, \underline{A}(0) \qquad (4.32)$$

Before we proceed, we note that the matrix $\underline{\underline{M}}(\underline{\zeta})$ is positive definite for all $\underline{\zeta} \in \mathcal{D}^M$ such that $m_1 \neq m_2$ if $\zeta_{m_1} \neq \zeta_{m_2}$. The proof of this assertion runs as follows. First one proves the following inequality for all $\underline{\zeta} \in \mathcal{D}^M$ and all $\underline{x} \in \mathbb{C}^M$:

$$< \underline{x}, \underline{\underline{M}}(\underline{\zeta})\, \underline{x} > \; \geq \; \left| \underline{\underline{B}}\, \underline{x} \right|^2$$

where $\underline{\underline{B}}$ is the matrix defined by

$$B_{kl} = (\zeta_l)^{k-1} \qquad\qquad k, l = 1, .., M$$

Secondly, one shows by induction that

$$\left| \det \underline{\underline{B}} \right| = \prod_{\substack{k,l=1 \\ k<l}}^{M} |\zeta_k - \zeta_l|$$

We show that the equations (4.32) with

$$A_m(0) \in I\!R_+ \qquad\qquad \zeta_m(0) \in (-1, 1) \qquad\qquad m = 1, .., M$$

have exactly one proper solution. To be more precise, we prove the following lemma.

Lemma 4.20 *Let $\underline{q} \in I\!R_+^M$ and let $\underline{\underline{M}}$ be a positive definite, symmetric $M \times M$ matrix with positive matrix elements. The following set of M quadratic equations:*

$$\underline{\underline{D}}_{\underline{x}} \left(\underline{\underline{M}}\, \underline{x} \right) = \underline{q} \tag{4.33}$$

has exactly one solution in $I\!R_+^M$. This solution depends on the matrix elements of $\underline{\underline{M}}$ in a continuously differentiable way.

The proof of this lemma is based on the following lemma.

Lemma 4.21 *Let $\underline{a}, \underline{q} \in I\!R_+^M$ and let $\underline{\underline{M}}$ be a positive definite, symmetric $M \times M$ matrix with positive matrix elements. The following set of M quadratic equations:*

$$\underline{\underline{D}}_{\underline{x}} \left(\underline{\underline{M}}\, \underline{x} + \underline{a} \right) = \underline{q} \tag{4.34}$$

has exactly one solution in $I\!R_+^M$. This solution depends on $\underline{a}$ in a continuously differentiable way.

Proof

Let $\underline{\underline{P}}$ be a permutation matrix and define

$$\underline{\tilde{x}} = \underline{\underline{P}}\, \underline{x} \qquad \underline{\tilde{a}} = \underline{\underline{P}}\, \underline{a} \qquad \underline{\tilde{q}} = \underline{\underline{P}}\, \underline{q} \qquad \underline{\underline{\tilde{M}}} = \underline{\underline{P}}\, \underline{\underline{M}}\, \underline{\underline{P}}^{\leftarrow}$$

One shows that

$$\underline{\underline{P}}\,\underline{\underline{D}}_{\underline{x}}\,\underline{\underline{P}}^{\leftarrow} = \underline{\underline{D}}_{(\underline{\underline{P}}\,\underline{x})}$$

and this leads to the following equivalency:

$$\underline{\underline{D}}_{\underline{x}}\left(\underline{\underline{M}}\,\underline{x} + \underline{a}\right) = \underline{q} \quad \text{if and only if} \quad \underline{\underline{D}}_{\tilde{\underline{x}}}\left(\underline{\underline{\tilde{M}}}\,\tilde{\underline{x}} + \tilde{\underline{a}}\right) = \tilde{\underline{q}}$$

It therefore suffices to prove the assertion in the lemma under the assumption

$$q_k \leq q_l \qquad \text{if} \qquad k \geq l \tag{4.35}$$

We prove this assertion by induction. The case $M = 1$ is trivial. Assume that the assertion holds for a certain $N \in \mathbb{N}$. We show in four steps that the assertion then also holds for $M = N + 1$.
Step 1. Consider the following equations:

$$x_k\left(\sum_{l=1}^{N} M_{kl}x_l + M_{k\,N+1}t + a_k\right) = q_k \qquad k = 1,..,N; t \geq 0 \tag{4.36}$$

As the assertion is assumed to hold for $M = N$, there is a unique differentiable curve

$$\gamma \;:\; t \in \mathbb{R}_{0,+} \;\mapsto\; \gamma(t) = \underline{x}(t) \in \mathbb{R}_+^N$$

such that $\underline{x}(t)$ satisfies equations (4.36).
Next consider the following quadratic equation for x_{N+1}:

$$x_{N+1}\left(\sum_{l=1}^{N+1} M_{N+1\,l}x_l + a_{N+1}\right) = q_{N+1} \tag{4.37}$$

with $x_l \geq 0, l = 1,..,N$. This equation has exactly one positive solution given by

$$x_{N+1} = f(\underline{x}) = 2q_{N+1}\left(h(\underline{x}) + (h(\underline{x})^2 + C)^{1/2}\right)^{-1} \tag{4.38}$$

where

$$h(\underline{x}) = \sum_{k=1}^{N} M_{N+1\,k}x_k + a_{N+1} \qquad\qquad C = 4q_{N+1}M_{N+1\,N+1}$$

We define a function g by

$$g \;:\; t \in \mathbb{R}_{0,+} \;\mapsto g(t) = f(\underline{x}(t)) - t \in \mathbb{R}$$

This function g is continuously differentiable. One checks that $g(t_0) = 0$ if and only if $\hat{\underline{x}}(t_0) = (\underline{x}(t_0), t_0) \in \mathbb{R}_+^{N+1}$ is a solution of equations (4.36) and (4.37). So, the number of zeros of g equals the number of solutions in $\mathbb{R}_+^M$ of equation (4.34) with $M = N + 1$.

Step 2. We show that g has at least one zero. We differentiate the function f with respect to $x_k, k = 1,..,N$:

$$\frac{\partial f(\underline{x})}{\partial x_k} = -2q_{N+1}M_{N+1\,k}\left(h(\underline{x})^2 + C\right)^{-1/2}\left(h(\underline{x}) + (h(\underline{x})^2 + C)^{1/2}\right)^{-1}$$

All partial derivatives are negative and this implies that f on $I\!R^N_{0,+}$ is bounded by $f(\underline{0})$. We conclude that

$$g(f(\underline{0})) = f(\underline{x}(f(\underline{0}))) - f(\underline{0}) \leq 0$$

As $g(0) > 0$ and as g is continuous, it follows that g has at least one zero.
Step 3. We show that g is monotonously decreasing and therefore cannot have more than one zero. We differentiate the relations (4.36) with respect to t:

$$\dot{x}_k(t)\left(\frac{q_k}{x_k(t)}\right) + x_k(t)\left(\sum_{l=1}^{N} M_{kl}\dot{x}_l(t) + M_{k\,N+1}\right) = 0$$

and find for all $k = 1,..,N$

$$\begin{aligned}\dot{x}_k(t) &= -\frac{x_k(t)^2}{q_k}\left(\sum_{l=1}^{N} M_{kl}\dot{x}_l(t) + M_{k\,N+1}\right)\\ &= \frac{\dot{x}_k(t)}{M_{kN+1}}\left(\sum_{l=1}^{N} M_{kl}\dot{x}_l(t)\right) + \frac{x_k(t)^2}{q_k M_{k\,N+1}}\left(\sum_{l=1}^{N} M_{kl}\dot{x}_l(t)\right)^2 - \frac{x_k(t)^2}{q_k}M_{k\,N+1}\end{aligned}$$

Hence,

$$\begin{aligned}g'(t) &= \frac{d}{dt}\left(f(\underline{x}(t)) - t\right) = \sum_{k=1}^{N}\frac{\partial f(\underline{x})}{\partial x_k}\dot{x}_k(t) \;-1\\ &= \sum_{k,l=1}^{N}\frac{\partial f(\underline{x})}{\partial x_k}\frac{M_{kl}}{M_{kN+1}}\dot{x}_k(t)\dot{x}_l(t) + \sum_{k=1}^{N}\frac{\partial f(\underline{x})}{\partial x_k}\frac{x_k(t)^2}{q_k M_{kN+1}}\left(\sum_{l=1}^{N} M_{kl}\dot{x}_l(t)\right)^2\\ &\quad -\sum_{k=1}^{N}\frac{\partial f(\underline{x})}{\partial x_k}\frac{x_k(t)^2 M_{kN+1}}{q_k} \;-1\end{aligned} \tag{4.39}$$

The first two terms in the right-hand side of this identity can be shown to be negative by means of the positive definiteness of and the symmetry of the matrix $\underline{\underline{M}}$. We omit the variables and show that the third term is bounded by 1 by using the symmetry of $\underline{\underline{M}}$ and inequality (4.35):

$$-\sum_{k=1}^{N}\frac{\partial f}{\partial x_k}\frac{x_k^2 M_{k\,N+1}}{q_k}$$

$$= 2q_{N+1}(h^2+C)^{-1/2}(h+(h^2+C)^{1/2})^{-1}\sum_{k=1}^{N}\frac{x_k^2 M_{kN+1}M_{N+1k}}{q_k}$$

$$< h^{-2}\sum_{k=1}^{N}\frac{q_{N+1}}{q_k}x_k^2 M_{N+1k}^2 < \left(\sum_{k=1}^{N} M_{N+1k}x_k\right)^{-2}\left(\sum_{k=1}^{N} M_{N+1k}^2 x_k^2\right)$$

$$\leq 1$$

We conclude that $g'(t)$ is negative for all $t \geq 0$.
Step 4. Now that we have shown that equation (4.33) with $M = N+1$ has exactly one solution in $I\!R_+^{N+1}$ for all $\underline{a}, \underline{q} \in I\!R_+^{N+1}$, it remains to be shown that this solution depends on $\underline{a}$ in a continuously differentiable way. We define the function $\underline{F}$ by

$$\underline{F} \,:\, (\underline{x},\underline{a}) \in I\!R_+^{N+1}\times I\!R_+^{N+1} \;\mapsto\; \underline{F}(\underline{x},\underline{a}) = \underline{\underline{D}}_{\underline{x}}\left(\underline{\underline{M}}\;\underline{x}+\underline{a}\right) \in I\!R^{N+1}$$

This function $\underline{F}$ is continuously differentiable. Its Jacobian $\underline{\underline{J}}$ has matrix elements

$$J_{kl}(\underline{x},\underline{a}) = \frac{\partial F_k(\underline{x},\underline{a})}{\partial x_l} = \delta_{kl}\left(\sum_{n=1}^{N+1} M_{ln}x_n + a_l\right) + x_k M_{kl} \qquad k,l = 1,..,N+1$$

and can therefore be written as

$$\underline{\underline{J}}(\underline{x},\underline{a}) = \underline{\underline{D}}_{(\underline{\underline{M}}\,\underline{x}+\underline{a})} + \underline{\underline{D}}_{\underline{x}}\,\underline{\underline{M}} = \underline{\underline{D}}_{\underline{x}}\left(\underline{\underline{D}}_{\underline{x}}^{\leftarrow}\,\underline{\underline{D}}_{(\underline{\underline{M}}\,\underline{x}+\underline{a})} + \underline{\underline{M}}\right)$$

We note that $\underline{\underline{D}}_{\underline{x}}$ is regular for all $\underline{x} \in I\!R_+^{N+1}$ while the diagonal matrices $\underline{\underline{D}}_{\underline{x}}^{\leftarrow}$ and $\underline{\underline{D}}_{(\underline{\underline{M}}\,\underline{x}+\underline{a})}$ are positive definite for all $(\underline{x},\underline{a}) \in I\!R_+^{N+1}\times I\!R_+^{N+1}$. The previous results and the inverse function theorem then imply that the solution of equation (4.33) with $M = N+1$ depends on $\underline{a}$ in a continuously differentiable way. □

This lemma enables us to prove Lemma 4.20 in a way which resembles the proof above but differs from it in some subtle aspects.

Proof of lemma 4.20
Consider the following equations:

$$x_k\left(\sum_{l=1}^{M-1} M_{kl}x_l + M_{kM}t\right) = q_k \qquad k = 1,..,M-1; t > 0 \qquad (4.40)$$

It follows from Lemma 4.20 that there is a unique differentiable curve

$$\gamma \,:\, t \in I\!R_+ \;\mapsto\; \gamma(t) = \underline{x}(t) \in I\!R_+^{M-1}$$

such that $\underline{x}(t)$ satisfies the equations (4.40).
Next consider the following quadratic equation for x_M:

$$x_M\left(\sum_{k=1}^{M} M_{Mk}x_k\right) = q_M$$

with $x_k > 0, k = 1,..,M-1$. This equation has exactly one positive solution given by

$$x_M = f(\underline{x}) = 2q_M \left(h(\underline{x}) + (h(\underline{x})^2 + C)^{1/2}\right)^{-1} \tag{4.41}$$

where

$$h(\underline{x}) = \sum_{k=1}^{M-1} M_{Mk} x_k \qquad C = 4q_M M_{MM}$$

One checks that the number of solutions in $\mathbb{R}_+^M$ of equation (4.33) equals the number of zeros of the function g defined by

$$g \,:\, t \in \mathbb{R}_+ \mapsto g(t) = f(\underline{x}(t)) - t \in \mathbb{R}$$

One shows that g has exactly one zero as in Step 2 and 3 of the proof of Lemma 4.21. One then shows that the solution $\underline{x} \in \mathbb{R}_+^M$ of equation (4.33) depends on the matrix elements of $\underline{\underline{M}}$ in a continuously differentiable way as in Step 4 with some small modifications. □

We use Lemma 4.20 to prove the following proposition.

Proposition 4.22 *Let $A_{m,0} \in \mathbb{R}_+, m = 1,..,M$, let $\zeta_{m,0} \in (-1,1), m = 1,..,M$ and let Ω_0 be defined by*

$$\Omega_0 = \zeta \sum_{m=1}^{M} \frac{A_{m,0}}{(1 - \zeta_{m,0}\zeta)}$$

The moving boundary problem for Stokes flow driven by surface tension with initial data Ω_0 has a global solution. This solution satisfies

$$\lim_{t\to\infty} \Omega(\zeta, t) = \sqrt{A/\pi}\,\zeta \qquad \text{for all } \zeta \in \overline{D} \tag{4.42}$$

where A denotes the area of the image of D under Ω_0.

Proof

Before we start with the actual proof, we present some definitions and estimates. We may assume $\zeta_{m_{1,0}} \neq \zeta_{m_{2,0}}$ if $m_{1,0} \neq m_{2,0}$. We define

$$q_m = A_{m,0} \sum_{n=1}^{M} \frac{A_{n,0}}{(1 - \zeta_{m,0}\zeta_{n,0})^2}$$

$$\varepsilon = \frac{1}{2}\left(\min_{m\in\{1,..,M\}} 1 - |\zeta_{m,0}|\right)$$

$$\mathcal{D}_\varepsilon^M = \{\, \underline{\zeta} \in [-1+\varepsilon, 1-\varepsilon]^M \mid \zeta_{m_1} \neq \zeta_{m_2} \text{ if } m_1 \neq m_2\}$$

Lemma 4.20 enables us to define $\underline{A}(\underline{\zeta})$ for each $\underline{\zeta} \in \mathcal{D}_\varepsilon^M$ as the unique solution in $\mathbb{R}_+^M$ of the M quadratic equations

$$\underline{\underline{D}}_{\underline{A}(\underline{\zeta})} \left(\underline{\underline{M}}(\underline{\zeta})\, \underline{A}(\underline{\zeta})\right) = \underline{q}$$

We finally define the function $\Omega_{\underline{\zeta}}, \underline{\zeta} \in \mathcal{D}_\varepsilon^M$ by

$$\Omega_{\underline{\zeta}}(\zeta) = \zeta \sum_{m=1}^{M} \frac{A_m(\underline{\zeta})}{1 - \zeta_m \zeta}$$

Next we derive some estimates. We easily obtain the following inequality for all $\zeta \in \partial D$ and all $\underline{\zeta} \in \mathcal{D}_\varepsilon^M$:

$$\begin{aligned} \left|\Omega'_{\underline{\zeta}}(\zeta)\right| &= \left|\sum_{m=1}^{M} \frac{A_m(\underline{\zeta})}{(1-\zeta_m\zeta)^2}\right| \geq \frac{1}{4}\sum_{m=1}^{M} A_m(\underline{\zeta}) \\ &\geq \frac{\varepsilon}{4}\left(\sum_{m,n=1}^{M} \frac{A_m(\underline{\zeta})A_n(\underline{\zeta})}{(1-\zeta_m\zeta_n)^2}\right)^{1/2} \geq \frac{\varepsilon}{4}\left(\sum_{m=1}^{M} q_m\right)^{1/2} = \frac{\varepsilon}{4}\sqrt{A/\pi} \end{aligned}$$

In the same way we get

$$\left|\Omega'_{\underline{\zeta}}(\zeta)\right| \leq \frac{2}{\varepsilon^2}\sqrt{A/\pi}$$

for all $\zeta \in \partial D$ and all $\underline{\zeta} \in \mathcal{D}_\varepsilon^M$. The first inequality implies that the function $F_{[\Omega_{\underline{\zeta}}]}$ is properly defined for all $\underline{\zeta} \in \mathcal{D}_\varepsilon^M$. It follows from these inequalities and the maximum principle for harmonic functions that

$$\varepsilon^2\sqrt{\pi/A} \leq \text{Re } F_{[\Omega_{\underline{\zeta}}]}(\zeta) \leq 2\varepsilon^{-1}\sqrt{\pi/A} \tag{4.43}$$

for all $\zeta \in \overline{D}$ and all $\underline{\zeta} \in \mathcal{D}_\varepsilon^M$. Finally, it follows from Schwarz' integral relation (see also expression (4.31) with $k = 1$) that:

$$\left|F'_{[\Omega_{\underline{\zeta}}]}(\zeta_m)\right| = \left|\frac{1}{2\pi i}\oint_{|z|=1} |\Omega'_{\underline{\zeta}}(z)|^{-1}(z-\zeta_m)^{-2}\,dz\right| < \frac{4}{\varepsilon^3}\sqrt{\pi/A} \tag{4.44}$$

for all $m \in \{1, .., M\}$ and all $\underline{\zeta} \in \mathcal{D}_\varepsilon^M$.

We now come to the actual proof of the proposition. It follows from Proposition 3.20, the previous sections and the definitions given above that the moving boundary problem for Stokes flow driven by surface tension with initial data Ω_0 has a global solution if the following initial value problem has a global solution:

$$\dot{\zeta}_m(t) = -\zeta_m(t)F_{[\Omega_{\underline{\zeta}(t)}]}(\zeta_m(t)) \tag{4.45a}$$

$$\zeta_m(0) = \zeta_{m,0} \tag{4.45b}$$

The right-hand sides of the differential equations (4.45a) depend on $\underline{\zeta}$ in a Lipschitz continuous way as follows from the considerations in the previous section

-especially expression (4.30)- and Lemma 4.20 which states that $\underline{A}(\underline{\zeta})$ depends on $\underline{\zeta}$ in a continuously differentiable way. It then follows from the theory of ordinary differential equations that the initial value problem (4.45) has a local solution; see also Theorem 4.18. Now assume that the maximal solution $\underline{\zeta}$ of initial value problem (4.45) is not global. That is (see [31]): assume that a $T \in I\!R_+$ exists such that for each compactum $K \subset \mathcal{D}_\varepsilon^M$ there is a $t \in (0,T)$ such that $\underline{\zeta}(t) \notin K$. We show below that for every $T \in I\!R_+$ a compactum $K \subset \mathcal{D}_\varepsilon^M$ exists such that $\underline{\zeta}(t) \in K$ for all $t \in [0,T)$ and this contradiction leads to the conclusion that the solution of the problem is global.
Let T be an arbitrary positive number and assume that the solution $\underline{\zeta}$ of initial value problem (4.45) exists for all $t \in [0,T)$. We first note that inequality (4.43) implies

$$\frac{d}{dt}|\zeta_m(t)| = -|\zeta_m(t)|\; F_{[\Omega_{\underline{\zeta}(t)}]}(\zeta_m(t)) < -\varepsilon^2\sqrt{\pi/A} < 0 \tag{4.46}$$

for all $m \in \{1,..,M\}$ and all $t \in [0,T)$. As $\underline{\zeta}(0) \in \mathcal{D}_\varepsilon^M$, this leads to

$$\underline{\zeta}(t) \in \mathcal{D}_\varepsilon^M \qquad \text{for all } t \in [0,T)$$

Secondly, we show that a constant $C > 0$ exists such that

$$|\zeta_{m_1}(t) - \zeta_{m_2}(t)| \geq C \tag{4.47}$$

for all $t \in [0,T)$ and all $m_1, m_2 \in \{1,..,M\}$ with $m_1 \neq m_2$. Therefore we consider the following relation:

$$\dot{\zeta}_{m_1}(t) - \dot{\zeta}_{m_2}(t) = -\zeta_{m_1}(t)F_{[\Omega_{\underline{\zeta}(t)}]}(\zeta_{m_1}(t)) + \zeta_{m_2}(t)F_{[\Omega_{\underline{\zeta}(t)}]}(\zeta_{m_2}(t))$$
$$= -(\zeta_{m_1}(t) - \zeta_{m_2}(t))\left(F_{[\Omega_{\underline{\zeta}(t)}]}(\zeta_{m_1}(t)) + \zeta_{m_2}(t)\frac{F_{[\Omega_{\underline{\zeta}(t)}]}(\zeta_{m_1}(t)) - F_{[\Omega_{\underline{\zeta}(t)}]}(\zeta_{m_2}(t))}{\zeta_{m_1}(t) - \zeta_{m_2}(t)}\right)$$

It follows straightforwardly from the inequalities given above that:

$$\left|F_{[\Omega_{\underline{\zeta}(t)}]}(\zeta_{m_1}(t)) + \zeta_{m_2}(t)\frac{F_{[\Omega_{\underline{\zeta}(t)}]}(\zeta_{m_1}(t)) - F_{[\Omega_{\underline{\zeta}(t)}]}(\zeta_{m_1}(t))}{\zeta_{m_1}(t) - \zeta_{m_2}(t)}\right| \leq (\frac{2}{\varepsilon} + \frac{4}{\varepsilon^3})\sqrt{\pi/A}$$

for all $m_1, m_2 \in \{1,..,M\}$ and all $t \in [0,T)$. One checks that it follows from this inequality that a constant $C > 0$ as in inequality (4.47) exists indeed. We conclude that there is a compactum K such that $\underline{\zeta}(t) \in K$ for all $t \in [0,T)$.
We repeat that this leads to the conclusion that the solution of the problem exists for all $t \geq 0$. It follows from inequality (4.46) that $|\zeta_m(t)| \downarrow 0$ if $t \uparrow \infty$ for all $m \in \{1,..,M\}$ and this implies (4.42). □

Chapter 5

Löwner-Kufareev equations

In this chapter we study several types of Löwner-Kufareev equations. We show in Section 5.1 the existence and uniqueness of solutions of initial value problems corresponding to linear Löwner-Kufareev equations; some of these results are known in literature ([69, 23]). In Section 5.2 we show the existence and uniqueness of solutions of initial value problems corresponding to what we call extended linear Löwner-Kufareev equations. We also show how isolated singularities in the initial data propagate. In Section 5.3 we reconsider Hopper equations. We first remark that a Hopper equation can be considered as a kind of extended Löwner-Kufareev equation for the function $\Xi = \Omega'\overline{\Omega}$. The relation between the functions Ξ and Ω is treated in Subsection 5.3.1. We then prove in Subsections 5.3.2 and 5.3.3 some properties of solutions of Hopper equations on the basis of the results obtained in Section 5.2. We finally return in Section 5.4 to quasi-linear Löwner-Kufareev equations –already introduced in Section 3.2– and prove a result on the solvability.

5.1 Linear Löwner-Kufareev equations

The existence and uniqueness of solutions of initial value problems corresponding to linear Löwner-Kufareev equations are based on the following result (see also [49, 50]).

Lemma 5.1 *Let f be a continuous function on $D \times I$ such that f is analytic on D for all fixed $t \in I$. The initial value problem*

$$\dot{\varphi}(\zeta, t) = -\varphi(\zeta, t) f(\varphi(\zeta, t), t) \tag{5.1a}$$

$$\varphi(\zeta, 0) = \zeta \tag{5.1b}$$

has a unique maximal solution. This solution is univalent for all fixed $t \in I$.

Proof
One shows that f' is a continuous function on $D \times I$ by using Cauchy's integral

formula. This implies that f is locally Lipschitz continuous on its domain $D \times I$ and it follows from Picard's theorem ([18]) that initial value problem (5.1) has a unique maximal solution for all fixed $\zeta \in D$.
Next we show that for all $\zeta \in D$, there is an $\varepsilon > 0$ such that φ is analytic at ζ for all fixed $t \in (-\varepsilon, \varepsilon)$. It follows from the theory of ordinary differential equations that for all $\zeta \in D$, there is an $\varepsilon > 0$ and an open neighbourhood $\mathcal{N}$ of ζ such that the sequence of functions $\varphi_n, n \in \mathbb{N}_0$ defined on $\mathcal{N} \times (-\varepsilon, \varepsilon)$ by

$$\varphi_0(\zeta, t) = \zeta$$
$$\varphi_{n+1}(\zeta, t) = \zeta - \int_0^t \varphi_n(\zeta, \tau) f(\varphi_n(\zeta, \tau), \tau)\, d\tau$$

converges uniformly on compacta in $\mathcal{N}$ to φ for all fixed $t \in (-\varepsilon, \varepsilon)$. One shows by induction that φ_n is analytic on $\mathcal{N}$ for all fixed $t \in (-\varepsilon, \varepsilon)$ and all $n \in \mathbb{N}$. This implies that φ is analytic in ζ for all fixed $t \in (-\varepsilon, \varepsilon)$ ([27]). It then follows from set theoretical arguments that the maximal solution of initial value problem (5.1) is analytic on its domain for all $t \in I$.
In order to prove that the function φ is injective on its domain for all fixed $t \in I$, we consider the time-inversed problem. It follows from the arguments given above that for all fixed $T \in I$ and all fixed $\zeta \in D$, the initial value problem

$$\dot{\psi}(z, t, T) = \psi(z, t, T) f(\psi(z, t, T), T - t) \tag{5.2a}$$
$$\psi(z, 0, T) = z \tag{5.2b}$$

has a unique maximal solution. Now, let $\zeta_1, \zeta_2 \in D$ and $T \in I$ be such that $\varphi(\zeta_1, T) = \varphi(\zeta_2, T)$. One straightforwardly checks that the functions ψ_1 and ψ_2 defined by

$$\psi_1(t) = \varphi(\zeta_1, T - t) \qquad \psi_2(t) = \varphi(\zeta_2, T - t)$$

both satisfy initial value problem (5.2) with $z = \varphi(\zeta_1, T) = \varphi(\zeta_2, T)$ on an interval that contains 0 and T. The uniqueness of the solution of initial value problem (5.2) implies

$$\zeta_1 = \varphi(\zeta_1, 0) = \psi_1(T) = \psi_2(T) = \varphi(\zeta_2, 0) = \zeta_2$$

We conclude that φ is injective on its domain. □

Lemma 5.2 *Let f be a continuous function on $D \times I$ such that f is analytic on D for all fixed $t \in I$. Let φ denote the maximal solution of initial value problem (5.1). The function σ defined by*

$$\sigma(\zeta, t) = \frac{1}{\overline{\varphi(\zeta, t)}}$$

is meromorphic with only a first order singularity at ∞ for all fixed $t \in I$ and is the unique maximal solution of the following initial value problem:

$$\dot{\sigma}(\zeta,t) = \sigma(\zeta,t)\overline{f}(\sigma(\zeta,t),t) \tag{5.3a}$$

$$\sigma(\zeta,0) = \zeta \tag{5.3b}$$

Proof

We note that differential equation (5.3a) is satisfied as

$$\dot{\sigma}(\zeta,t) = \frac{-\overline{\dot{\varphi}(1/\bar{\zeta},t)}}{(\overline{\varphi(1/\bar{\zeta},t)})^2} = \frac{\overline{f(\overline{\varphi}(\zeta,t),t)}}{\overline{\varphi(1/\bar{\zeta},t)}} = \sigma(\zeta,t)\overline{f}(\sigma(\zeta,t),t)$$

The assertions in the lemma then follow immediately from Lemma 5.1. □

Initial value problem (5.1) will turn out to determine the characteristics of Löwner-Kufareev equations; this will become clear in the proof of Propositions 5.6 and 5.9. The initial value problem (5.3) will turn out to play an important role in the study of Hopper equations; this will become clear in Section 5.3. It is therefore important to know the image of the functions φ and σ.

Definition 5.3 Let f be a continuous function on $D \times I$ such that f is analytic on D for all fixed $t \in I$. Let φ be the maximal solution of initial value problem (5.1) and let σ be the maximal solution of initial value problem (5.3). We define I_ζ as the interval on which the maximal solution φ exists for a fixed $\zeta \in D$. Let B be some subset of $\overline{\mathbb{C}}$. We define the sets $\varphi_t(B), \varphi(B), \sigma_t(B)$ and $\sigma(B)$ by

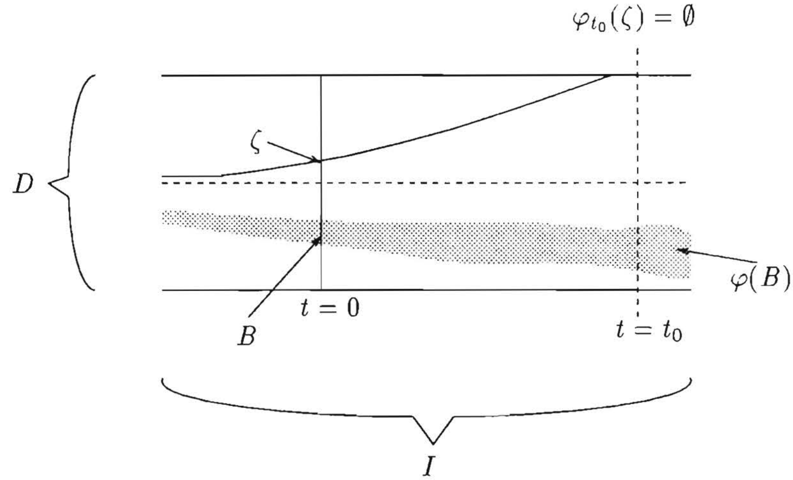
$$\varphi_t(B) = \{z \in D \mid \exists\, \zeta \in B \cap D: \;\; t \in I_\zeta,\; \varphi(\zeta,t) = z\}$$
$$\varphi(B) = \{(z,t) \in D \times I \mid \exists\, \zeta \in B: \;\; t \in I_\zeta,\; \varphi(\zeta,t) = z\}$$
$$\sigma_t(B) = \{z \in \overline{\mathbb{C}}\backslash\overline{D} \mid \exists\, \zeta \in B \cap (\overline{\mathbb{C}}\backslash\overline{D}): \;\; t \in I_{1/\bar{\zeta}},\; \sigma(\zeta,t) = z\}$$
$$\sigma(B) = \{(z,t) \in (\overline{\mathbb{C}}\backslash\overline{D}) \times I \mid \exists\, \zeta \in B \cap (\overline{\mathbb{C}}\backslash\overline{D}): \;\; t \in I_{1/\bar{\zeta}},\; \sigma(\zeta,t) = z\}$$

The set $\sigma_t(B)$ can alternatively be defined by

$$\sigma_t(B) = R(\varphi_t(R(B)))$$

where R denotes reflection with respect to the unit circle (see Definition 3.16)

The figure on the next page illustrates this definition.



Lemma 5.4 *Let f be a continuous function on $D \times I$ such that f is analytic on D for all fixed $t \in I$. If $M \in \mathbb{R}_{0,+}, T_0 \in I_+ = I \cap \mathbb{R}_{0,+}$ and $r \in (0,1]$ are such that*

$$\max_{t \in [0,T_0]} \sup_{\zeta \in D_r} \operatorname{Re} f(\zeta, t) \leq M \tag{5.4}$$

then

$$\{(z,t) \in D \times [0,T_0] \,\big|\, |z| < re^{-Mt}\} \subset \varphi(D_r) \tag{5.5}$$

Proof

Consider initial value problem (5.2) with f restricted to $D_r \times I$ for a $T \in [0, T_0]$ and a fixed z with $|z| < re^{-MT}$. We show that the maximal solution of this problem exists on an interval that contains $[0,T]$. Assume that this is not true, i.e. assume that the maximal interval is $[0,T_1)$ with $T_1 < T$. It follows from the theory of ordinary differential equations ([31]) that for every compactum $K \subset D_r$, there is a $t \in [0,T_1)$ such that $\psi(z,t,T) \notin K$. However, as the solution of initial value problem (5.2) satisfies

$$\psi(z,t,T) = ze^{\int_0^t f(\psi(z,\tau,T),T-t)\,d\tau}$$

for all $t \in [0,T_1)$, we find from inequality (5.4) that

$$|\psi(z,t,T)| \leq |z|e^{MT} < r$$

and this implies that there is a compactum $\tilde{K} \subset D_r$ such that $\psi(\zeta,t,T) \in \tilde{K}$ for all $t \in [0,T_1)$. This contradiction leads to the conclusion that initial value problem (5.2) for all z with $|z| < re^{-MT}$ can be solved on an interval that contains

$[0, T]$. Moreover, we have $|\psi(z, T, T)| < r$.
Let z with $|z| < re^{-MT}, T \in [0, T_0]$ be arbitrary. One verifies that the function φ on $[0, T]$ defined by

$$\varphi(t) = \psi(z, T - t, T)$$

satisfies initial value problem (5.1) with $\zeta = \psi(z, T, T) \in D_r$. So: for all z with $|z| < re^{-MT}$, there is a $\zeta \in D_r$ such that $\varphi(\zeta, T) = z$. This is equivalent with inclusion (5.5). □

Remark 5.5 It is clear that for all $T_0 \in I_+$ and all $r \in (0, 1)$, there is a $M \in I\!R_{0,+}$ such that inequality (5.4) holds. This implies $\{0\} \times I \subset \varphi(D_r)$ for all $r \in (0, 1)$. We state without proof, although it is straightforward, that $\varphi_t(D)$ is a non-empty, open, connected and simply connected subdomain of D for all $t \in I$.
We notice that if the real part of f on I_+ is non-positive (i.e. $r = 1, M = 0$ and T arbitrary in inequality (5.4)), we get: $D \times I_+ \subset \varphi(D)$. We finally notice that it can be shown in more or less the same way that if the real part of f is non-negative, then $I_+ \subset I_\zeta$ for all $\zeta \in D$.

Proposition 5.6 *Let f be a continuous function on $D \times I$ such that f is analytic on D for all fixed $t \in I$. Let Ω_0 be an analytic function on D. The initial value problem*

$$\dot{\Omega}(\zeta, t) = \Omega'(\zeta, t) f(\zeta, t)\zeta \tag{5.6a}$$
$$\Omega(\zeta, 0) = \Omega_0(\zeta) \tag{5.6b}$$

has an unique solution on $\varphi(D)$. This solution is analytic on its domain $\varphi_t(D)$ for all fixed $t \in I$. If Ω_0 is locally or globally univalent, then so is this solution for all fixed $t \in I$.

Equation (5.6a) is called the linear Löwner-Kufareev equation. This result is proved in a slightly different way in e.g. [69].
Proof
Lemma 5.1 states that the maximal solution φ of initial value problem (5.1) is univalent on its domain for all fixed $t \in I$. This implies that we can define a function $\varphi^{\leftarrow}$ on $\varphi(D)$ by the relation

$$\varphi^{\leftarrow}(\varphi(\zeta, t), t) = \zeta$$

(One can alternatively define this function by $\varphi^{\leftarrow}(\zeta, t) = \psi(\zeta, t, t)$ for all $(\zeta, t) \in D \times I$ for which the right-hand side makes sense.) We differentiate the relation with respect to t and get:

$$\dot{\varphi}^{\leftarrow}(\varphi(\zeta, t), t) + \varphi^{\leftarrow\prime}(\varphi(\zeta, t), t)\dot{\varphi}(\zeta, t) =$$
$$\dot{\varphi}^{\leftarrow}(\varphi(\zeta, t), t) - \varphi^{\leftarrow\prime}(\varphi(\zeta, t), t) f(\varphi(\zeta, t), t)\zeta = 0$$

One checks that this implies that $\varphi^{\leftarrow}$ satisfies the following partial differential equation on its domain $\varphi(D)$:

$$\dot{\varphi^{\leftarrow}}(\zeta,t) = \varphi^{\leftarrow\prime}(\zeta,t)f(\zeta,t)\zeta \tag{5.7}$$

(It would be more natural to use the variable z in this formula in stead of ζ but there are reasons to do not.) As $|\varphi^{\leftarrow}(\zeta,t)| < 1$ for all $(\zeta,t) \in \varphi(D)$, we can define a function Ω on $\varphi(D)$ by

$$\Omega(\zeta,t) = \Omega_0(\varphi^{\leftarrow}(\zeta,t)) \tag{5.8}$$

It is easily checked that this function Ω satisfies initial value problem (5.6). It is clear that Ω is analytic on its domain $\varphi_t(D)$ for all fixed $t \in I$.
The uniqueness of the solution of the problem is a consequence of the fact that $\varphi(D)$ is the so-called domain of determinacy ([32]). One can prove this uniqueness directly as follows. Let $\tilde{\Omega}$ on $\varphi(D)$ be a solution of initial value problem (5.6). One easily checks that

$$\frac{d}{dt}\tilde{\Omega}(\varphi(\zeta,t),t) = 0$$

and this implies that $\tilde{\Omega}(\zeta,t) = \Omega_0(\varphi^{\leftarrow}(\zeta,t))$; i.e. the solution $\tilde{\Omega}$ is identical to the solution constructed above.
Let Ω_0 be locally univalent on D; i.e. Ω_0' does not vanish on D. As $\varphi^{\leftarrow}$ is injective for fixed $t \in I$, $\varphi^{\leftarrow\prime}$ does not vanish on $\varphi(D)$ and it follows immediately from relation (5.8) that Ω' does not vanish on $\varphi(D)$. In other words: Ω is locally univalent on its domain $\varphi_t(D)$ for all fixed $t \in I$. In more or less the same way, one proves that Ω is globally univalent for all fixed $t \in I$ if Ω_0 is globally univalent on D. □

We will construct solutions of a particular type of linear Löwner-Kufareev equations in Appendix B by the method of separation of variables. These solutions of initial value problem (5.6) may exist on a domain that properly includes $\varphi(D)$; the following example shows that such extensions of the unique solution on $\varphi(D)$ are not unique themselves.

Example 5.7 Let f on $D \times I\!R$ be identical to 1. One easily checks that the domain $\varphi(D)$ in this case is given by

$$\varphi(D) = D \times I\!R_- \cup \{(z,t) \in D \times I\!R_{0,+} \mid |z| < e^{-t}\}$$

Consider initial value problem (5.6) with initial data given by

$$\Omega_0(\zeta) = \sqrt{\zeta - 1} \qquad \zeta \in D$$

The unique solution Ω on $\varphi(D)$ of this problem is given by

$$\Omega(\zeta,t) = \sqrt{e^t\zeta - 1}$$

This solution can be extended outside $\varphi(D)$ in several ways by making branch cuts for every $t > 0$ from the points $(e^{-t}, t) \in \partial\varphi(D)$ to $\partial D \times R_+$.

The following lemma can be useful if the uniqueness of a solution outside $\varphi(D)$ is to be proved.

Lemma 5.8 Let f be a continuous function on $D \times I$ such that f is analytic on D for all fixed $t \in I$. Let $r > 0$ and let Ω_1 and Ω_2 be functions on $D_r \times I$ such that Ω_1 and Ω_2 are analytic on D_r for all fixed $t \in I$. If Ω_1 and Ω_2 both satisfy initial value problem (5.6), then Ω_1 and Ω_2 are identical.

Proof
It is clear that the function $\Omega = \Omega_1 - \Omega_2$ on $D_r \times I$ solves initial value problem (5.6) with Ω_0 identical to zero. It follows from Cauchy's integral formula that there are continuously differentiable functions $a_n, n \in I\!N_0$ and continuous functions $c_n, n \in I\!N_0$ on I such that:

$$\Omega(\zeta, t) = \sum_{n=0}^{\infty} a_n(t)\zeta^n \qquad f(\zeta, t) = \sum_{n=0}^{\infty} c_n(t)\zeta^n$$

Substituting these relations into initial value problem (5.6) with Ω_0 identical to zero, we get

$$\begin{aligned} \dot{a}_n(t) &= (1 - \delta_{n0}) \sum_{k=1}^{n} k a_k(t) c_{n-k}(t) \\ a_n(0) &= 0 \end{aligned}$$

for all $n \in I\!N_0$. One shows by induction, using some standard results on ordinary linear differential equations, that this implies $a_n(t) = 0$ for all $n \in I\!N_0$ and all $t \in I$. So, Ω is identically zero and this implies that Ω_1 and Ω_2 are identical. □

We will study the quasi-linear Löwner-Kufareev in Section 5.4.

5.2 Extended linear Löwner-Kufareev equations

We call equation (5.9a) below an extended linear Löwner-Kufareev equation. The existence and uniqueness of solutions of initial value problems corresponding to such equations can be proved in more or less the same way as we did in the previous subsection for the (non-extended) linear Löwner-Kufareev equation. We consider in Subsection 5.2.1 the case where the initial data is given by an analytic function Ξ_0 on an open subset of D. In the Subsections 5.2.2 and 5.2.3, we consider the same initial value problem with initial data given by a function Ξ_0 which is not analytic but is meromorphic or multiple-valued. We summarize the results in Subsection 5.2.4.

5.2.1 Regular Points

The following proposition on solutions of extended linear Löwner-Kufareev can be considered as the analogon of Proposition 5.6.

Proposition 5.9 *Let f and θ be continuous functions on $D \times I$ such that f and θ are analytic on D for all fixed $t \in I$. Let B be an open subdomain of D and let Ξ_0 be an analytic function on B. The initial value problem*

$$\dot{\Xi}(\zeta,t) = (\Xi(\zeta,t)f(\zeta,t)\zeta)' + \theta(\zeta,t) \quad (5.9a)$$

$$\Xi(\zeta,0) = \Xi_0(\zeta) \quad (5.9b)$$

has a unique solution on $\varphi(B)$. This solution is analytic on its domain $\varphi_t(B)$ for all fixed $t \in I$.

Proof
We define functions $\tilde{f}$ and $\tilde{\theta}$ on $\{(\zeta,t) \in D \times I \mid t \in I_\zeta\}$ as follows:

$$\tilde{f}(\zeta,t) = f'(\varphi(\zeta,t),t)\varphi(\zeta,t) + f(\varphi(\zeta,t),t) \qquad \tilde{\theta}(\zeta,t) = \theta(\varphi(\zeta,t),t)$$

Consider the following initial value problem for a fixed $\zeta \in B$:

$$\dot{\tilde{\Xi}}(\zeta,t) = \tilde{f}(\zeta,t)\tilde{\Xi}(\zeta,t) + \tilde{\theta}(\zeta,t) \quad (5.10a)$$

$$\tilde{\Xi}(\zeta,0) = \Xi_0(\zeta) \quad (5.10b)$$

The differential relation (5.10a) is an ordinary linear differential equation and the initial value problem has therefore a unique solution on $I_\zeta \subset I$. Solving this initial value problem for all $\zeta \in B$ and reasoning as in the proof of Lemma 5.1, one shows that $\tilde{\Xi}$ is analytic on its domain for all fixed $t \in I$.
Next we define the function Ξ on $\varphi(B)$ by

$$\Xi(\zeta,t) = \tilde{\Xi}(\varphi^{\leftarrow}(\zeta,t),t) \quad (5.11)$$

We differentiate this function with respect to t, substitute the relations (5.10a) and (5.7), and find:

$$\begin{aligned}
\dot{\Xi}(\zeta,t) &= \dot{\tilde{\Xi}}(\varphi^{\leftarrow}(\zeta,t),t) + \tilde{\Xi}'(\varphi^{\leftarrow}(\zeta,t),t)\dot{\varphi}^{\leftarrow}(\zeta,t) \\
&= \tilde{f}(\varphi^{\leftarrow}(\zeta,t),t)\tilde{\Xi}(\varphi^{\leftarrow}(\zeta,t),t) + \tilde{\theta}(\varphi^{\leftarrow}(\zeta,t),t) \\
&\qquad + \tilde{\Xi}'(\varphi^{\leftarrow}(\zeta,t),t)\varphi^{\leftarrow\prime}(\zeta,t)f(\zeta,t)\zeta \\
&= (f(\zeta,t) + f'(\zeta,t)\zeta)\Xi(\zeta,t) + \theta(\zeta,t) + \Xi'(\zeta,t)f(\zeta,t)\zeta \\
&= (\Xi(\zeta,t)f(\zeta,t)\zeta)' + \theta(\zeta,t)
\end{aligned}$$

We conclude that the function Ξ solves initial value problem (5.9). It follows from relation (5.11) and the analyticity of the functions $\varphi^{\leftarrow}$ and $\tilde{\Xi}$ for fixed $t \in I$ that the function Ξ is analytic for fixed $t \in I$. The uniqueness of the solution can be shown by means of the arguments given in the proof of Proposition 5.6. □

5.2.2 Poles and essential singularities

The function Ξ_0 in the previous proposition is analytic on its domain. The next proposition deals with a function Ξ_0 which has a pole or an essential singularity. The proposition can be formulated most easily if we agree that an essential singularity is a pole of order ∞.

Proposition 5.10 *Let f and θ be continuous functions on $D \times I$ such that f and θ are analytic on D for all fixed $t \in I$. Let $\zeta_0 \in D$ and let $\mathcal{N} \subset D$ be an open neighbourhood of ζ_0. If Ξ_0 is an analytic function on $\mathcal{N}\backslash\{\zeta_0\}$ with a pole of order $N \in \mathbb{N}_\infty$ at $\zeta = \zeta_0$, then the solution of initial value problem (5.9) has a pole of order N at $\varphi(\zeta_0, t)$ for all fixed $t \in I_{\zeta_0}$.*

Proof

It follows from Proposition 5.9 that the solution Ξ of initial value problem (5.9) is analytic on $\varphi_t(\mathcal{N}\backslash\{\zeta_0\}) = \varphi_t(\mathcal{N})\backslash\varphi_t(\{\zeta_0\})$ for all $t \in I$. This implies that for all $t \in I_{\zeta_0}$, there is a $r(t) \in (0,1)$ such that Ξ is analytic on the punctured disc $\{\zeta \in D \mid 0 < |\zeta - \varphi(\zeta_0, t)| < r(t)\} \subset D$. It follows from Cauchy's integral formula that there are continuously differentiable functions $d_n, n \in \mathbb{Z}$ on I_{ζ_0} such that

$$\Xi(\zeta, t) = \sum_{n=-\infty}^{\infty} d_n(t)(\zeta - \varphi(\zeta_0, t))^n$$

for all $(\zeta, t) \in D \times I$ such that $0 < |\zeta - \varphi(\zeta_0, t)| < r(t)$.

We derive a set of differential relations for the coefficients $d_{-n}, n \in \mathbb{N}$. Before we do so, we note that there are continuous functions $c_n, n \in \mathbb{N}_0$ on I_{ζ_0} such that

$$f(\zeta, t) = \sum_{n=0}^{\infty} c_n(t)(\zeta - \varphi(\zeta_0, t))^n$$

Let $\gamma(t)$ for all $t \in I_{\zeta_0}$ denote a simple closed Jordan curve in $\varphi_t(\mathcal{N})$ enclosing $\varphi(\zeta_0, t)$. Using relations (5.9a) and (5.1a), we get for all $n \in \mathbb{N}$:

$$\begin{aligned}
\dot{d}_{-n}(t) &= \frac{d}{dt}\left(\frac{1}{2\pi i}\oint_{\gamma(t)} \Xi(\zeta, t)(\zeta - \varphi(\zeta_0, t))^{n-1}\, d\zeta\right) \\
&= \frac{1}{2\pi i}\oint_{\gamma(t)} \dot{\Xi}(\zeta, t)(\zeta - \varphi(\zeta_0, t))^{n-1}\, d\zeta \\
&\quad - \frac{n-1}{2\pi i}\oint_{\gamma(t)} \Xi(\zeta, t)(\zeta - \varphi(\zeta_0, t))^{n-2}\dot{\varphi}(\zeta_0, t)\, d\zeta \\
&= \frac{1}{2\pi i}\oint_{\gamma(t)} \Big((\Xi(\zeta, t) f(\zeta, t)\zeta)' + \theta(\zeta, t)\Big)(\zeta - \varphi(\zeta_0, t))^{n-1}\, d\zeta
\end{aligned}$$

$$+\frac{n-1}{2\pi i}\oint_{\gamma(t)} \Xi(\zeta,t)(\zeta-\varphi(\zeta_0,t))^{n-2}\varphi(\zeta_0,t)f(\varphi(\zeta_0,t),t)\,d\zeta$$

$$=-\frac{n-1}{2\pi i}\oint_{\gamma(t)} \Xi(\zeta,t)f(\zeta,t)\zeta(\zeta-\varphi(\zeta_0,t))^{n-2}\,d\zeta$$

$$+\frac{n-1}{2\pi i}\oint_{\gamma(t)} \Xi(\zeta,t)f(\varphi(\zeta_0,t),t)\varphi(\zeta_0,t)(\zeta-\varphi(\zeta_0,t))^{n-2}\,d\zeta$$

$$=\frac{1-n}{2\pi i}\oint_{\gamma(t)} \Xi(\zeta,t)f(\zeta,t)(\zeta-\varphi(\zeta_0,t))^{n-1}\,d\zeta$$

$$+\varphi(\zeta_0,t)\frac{1-n}{2\pi i}\oint_{\gamma(t)} \Xi(\zeta,t)(f(\zeta,t)-f(\varphi(\zeta_0,t),t))(\zeta-\varphi(\zeta_0,t))^{n-2}\,d\zeta$$

$$=\frac{1-n}{2\pi i}\oint_{\gamma(t)} \sum_{k=-\infty}^{\infty}\sum_{l=0}^{\infty} d_k(t)c_l(t)(\zeta-\varphi(\zeta_0,t))^{k+l+n-1}\,d\zeta$$

$$+\varphi(\zeta_0,t)\frac{1-n}{2\pi i}\oint_{\gamma(t)} \sum_{k=-\infty}^{\infty}\sum_{l=1}^{\infty} d_k(t)c_l(t)(\zeta-\varphi(\zeta_0,t))^{k+l+n-2}\,d\zeta$$

$$=(1-n)\sum_{l=0}^{\infty} d_{-(n+l)}(t)\Big(c_l(t)+\varphi(\zeta_0,t)c_{l+1}(t)\Big) \tag{5.12}$$

We first consider the case where the order of the pole of Ξ_0 is finite, i.e. $N\neq\infty$. We conclude from the calculations above that the set of functions $d_{-n}, n\in\mathbb{N}$ on I_{ζ_0} solve the following initial value problem:

$$\dot{d}_{-n}(t)=(1-n)\sum_{l=0}^{\infty} d_{-(n+l)}(t)\Big(c_l(t)+\varphi(\zeta_0,t)c_{l+1}(t)\Big) \tag{5.13a}$$

$$d_{-n}(0)=\begin{cases}\frac{1}{2\pi i}\oint_{\gamma(0)}\Xi_0(\zeta)(\zeta-\zeta_0)^{n-1}\,d\zeta & n=1,..,N\\ 0 & n>N\end{cases} \tag{5.13b}$$

As initial value problem (5.9) has a unique solution, there is exactly one set of continuously differentiable functions $d_{-n}, n\in\mathbb{N}$ on I_{ζ_0} solving initial value problem (5.13) and such that $\sum_{n=1}^{\infty} d_{-n}(t)\zeta^{-n}$ constitutes an analytic function on $\mathbb{C}\backslash\{0\}$ for all fixed $t\in I_{\zeta_0}$. Now we remark that it follows from the theory of ordinary linear differential equations, and the continuity of the functions $c_l, l\in\mathbb{N}_0$ and φ with respect to t, that there are functions $\tilde{d}_{-n}, n\in\{1,..,N\}$ on I_{ζ_0} such that

$$\dot{\tilde{d}}_{-n}(t)=(1-n)\sum_{l=0}^{N-n} \tilde{d}_{-(n+l)}(t)\Big(c_l(t)+\varphi(\zeta_0,t)c_{l+1}(t)\Big) \tag{5.14a}$$

$$\tilde{d}_{-n}(0)=d_{-n}(0) \tag{5.14b}$$

for all $n \in \{1, .., N\}$. Hence, d_{-n} equals $\tilde{d}_{-n}$ for all $n \in \{1, .., N\}$ and d_{-n} vanishes on I_{ζ_0} for all $n > N$. We finally remark that it follows from relation (5.14a) for $n = N$, that, as $d_{-N}(0) \neq 0$, d_{-N} does not vanish on I_{ζ_0}. The assertion in the proposition for $N \in I\!N$ follows.
The assertion in the proposition for $N = \infty$ can then be shown as follows. Let Ξ_0 have an essential singularity at $\zeta = \zeta_0$ and assume that there is a $t_0 \in I_{\zeta_0} \backslash \{0\}$ such that Ξ at $t = t_0$ does not have an essential singularity at $\varphi(\zeta_0, t_0)$. As Ξ at $t = t_0$ is analytic on some punctured neighborhood of $\varphi(\zeta_0, t_0)$, this implies that Ξ at $t = t_0$ has a finite order pole in $\zeta = \varphi(\zeta_0, t_0)$ (or can be extended analytically in $\zeta = \varphi(\zeta_0, t_0)$). Next one considers the time-inversed problem and shows by the assertion proved above (or by Proposition 5.9) that Ξ at $t = t_0$ has a finite order pole at $\zeta = \zeta_0$ (or can be extended analytically at $\zeta = \zeta_0$). This contradiction leads to the conclusion that Ξ has an essential singularity in $\varphi(\zeta_0, t)$ for all $t \in I_{\zeta_0}$. □

Because we will need the result later on, we also consider what happens with a pole of Ξ_0 at $\zeta = 0$ if the function θ is analytic on $D \backslash \{0\}$ with a pole at $\zeta = 0$.

Proposition 5.11 *Let f and be a continuous function on $D \times I$ such that f is analytic on D for all fixed $t \in I$ and let θ be a continuous function on $D \backslash \{0\} \times I$ such that θ is analytic on $D \backslash \{0\}$ with a pole of an order not larger than $N \in I\!N$ at $\zeta = 0$ for all fixed $t \in I$. Let $\mathcal{N} \subset D$ denote an open neighbourhood of $\zeta = 0$. If Ξ_0 is an analytic function on $\mathcal{N} \backslash \{0\}$ with a pole of an order not larger than N in $\zeta = 0$, then the solution of initial value problem (5.9) has a pole of an order not larger than N in $\zeta = 0$ for all $t \in I$.*

Proof
It follows from Proposition 5.9 that the solution Ξ of initial value problem (5.9) is analytic on $\varphi_t(\mathcal{N} \backslash \{0\})$ for all $t \in I$. It follows from Remark 5.5 that there is no $t \in I$ such that $\varphi_t(\mathcal{N} \backslash \{0\}) = \emptyset$. We define the functions $\tilde{f}$ and $\tilde{\theta}$ on $\{(\zeta, t) \in D \times I \mid t \in I_\zeta\}$ as in the beginning of the proof of Proposition 5.9. As the function φ is univalent on some open neighbourhood of $\zeta = 0$ for all $t \in I$ –see Lemma 5.1–, the function $\tilde{\theta}$ has a pole of an order not larger than N at $\zeta = 0$ for all fixed $t \in I$.
Consider the following initial value problem for a fixed $\zeta \in \mathcal{N}$:

$$\dot{\hat{\Xi}}(\zeta, t) = \tilde{f}(\zeta, t)\hat{\Xi}(\zeta, t) + \tilde{\theta}(\zeta, t)\zeta^N \qquad (5.15a)$$
$$\hat{\Xi}(\zeta, 0) = \Xi_0(\zeta)\zeta^N \qquad (5.15b)$$

One shows as in the proof of Proposition 5.9 that the unique solution $\hat{\Xi}$ is analytic on its domain for all fixed $t \in I$. In particular, as $\varphi(0, t) = 0$ for all $t \in I$, this solution is analytic at $\zeta = 0$ for all $t \in I$. One checks that the function $\bar{\Xi}$ defined by

$$\bar{\Xi}(\zeta, t) = \hat{\Xi}(\zeta, t)/\zeta^N$$

has a pole of an order not larger than N in $\zeta = 0$ and solves initial value problem (5.15). It is then easily shown that the function Ξ defined as in relation (5.11) has a pole of an order not larger than N at $\zeta = 0$ and solves initial value problem (5.9). □

Remark 5.12 The proof of this proposition can alternatively be based on the relations (5.12) with slight modifications.

5.2.3 Branch points

The function Ξ_0 in Propositions 5.9 and 5.10 is single-valued on its domain. In this subsection, we consider functions Ξ_0 which are multiple-valued. A comprehensive account of multiple-valued analytic functions is given in e.g. [10] and [3]. We will mainly consider multiple-valued functions on an open punctered neighbourhood of a point $\zeta_0 \in \mathbb{C}$. We implicitly understand that such a function is not single-valued (i.e. such a function cannot be expanded in a Laurent series). Conversely, an analytic function on an open domain is understood to be single-valued unless it is explicitly stated that it is multiple-valued.
We recapitulate (cf. [3]) that a multiple-valued analytic function Ξ on an open punctured neighbourhood of a point $\zeta_0 \in \mathbb{C}$ has a branch point of order $n \in \mathbb{N}$ if and only if the function Ξ_n defined on an open punctured neighbourhood of $\zeta = 0$ by

$$\Xi_n(\zeta) = \Xi(\zeta_0 + \zeta^{n+1}) \tag{5.16}$$

is analytic while there is no $k \in \mathbb{N}_0$ smaller than n such that the function Ξ_k defined in the same way is analytic. The point ζ_0 is then called an (ordinary) algebraic branch point if the principal part of the Laurent series of Ξ_n contains only a finite number of terms (vanishes). Otherwise the branch point is called transcendental. If there is no $n \in \mathbb{N}$ such that the function Ξ_n defined above is analytic on its domain, then the branch point is called logarithmic. The results proved below are most easily formulated if we agree that a logarithmic branch point is of order ∞. The proof of the analogon of Propositions 5.9 and 5.10 where the initial data is given by a multiple-valued function is based on the following lemma.

Lemma 5.13 *Let g, h and ψ be analytic functions on an open neighbourhood of a point $\hat{\zeta}_0 \in \mathbb{C}$ where g does not vanish and where ψ is univalent. Let Ξ be a multiple-valued analytic function on an open punctured neighbourhood of $\zeta_0 = \psi(\hat{\zeta}_0)$. This function Ξ has a ((ordinary) algebraic) branch point of order $n \in \mathbb{N}$ if and only if the function $\hat{\Xi}$ defined on an open punctured neighbourhood of $\hat{\zeta}_0$ by*

$$\hat{\Xi}(\zeta) = \Xi(\psi(\zeta))g(\zeta) + h(\zeta) \tag{5.17}$$

has a ((ordinary) algebraic) branch point of order n at $\hat{\zeta}_0$.

Proof

Before we start with the actual proof of the lemma, we note that the relation (5.17) implies the relation

$$\Xi(\zeta) = \frac{\hat{\Xi}(\psi^{\leftarrow}(\zeta)) - h(\psi^{\leftarrow}(\zeta))}{g(\psi^{\leftarrow}(\zeta))}$$

(with ζ in some open punctured neighbourhood of ζ_0). This relation implies that the function Ξ can be written as

$$\Xi(\zeta) = \hat{\Xi}(\hat{\psi}(\zeta))\hat{g}(\zeta) + \hat{h}(\zeta) \tag{5.18}$$

where $\hat{g}, \hat{h}$ and $\hat{\psi}$ are analytic functions on an open neighbourhood of ζ_0 where $\hat{g}$ does not vanish and where $\hat{\psi}$ is univalent.

Part 1. Let the multiple-valued analytic function Ξ have a branch point of order $n \in \mathbb{N}$ at ζ_0. We show that the function $\hat{\Xi}$ defined by relation (5.17) has a branch point of order n at $\hat{\zeta}_0$. We first prove that the function $\hat{\Xi}_n$ defined on an open punctured neighbourhood of $\zeta = 0$ by

$$\hat{\Xi}_n(\zeta) = \hat{\Xi}(\hat{\zeta}_0 + \zeta^{n+1}) \tag{5.19}$$

is analytic. It can straightforwardly be shown (cf. [23]) that, as ψ is univalent on its domain, a univalent function σ on an open neighbourhood of $\zeta = 0$ exists such that

$$\sigma(\zeta)^{n+1} = \psi(\hat{\zeta}_0 + \zeta^{n+1}) - \psi(\hat{\zeta}_0)$$

This implies that the function $\hat{\Xi}_n$ can be written as

$$\begin{aligned}
\hat{\Xi}_n(\zeta) &= \hat{\Xi}(\hat{\zeta}_0 + \zeta^{n+1}) \\
&= \Xi(\psi(\hat{\zeta}_0 + \zeta^{n+1}))g(\hat{\zeta}_0 + \zeta^{n+1}) + h(\hat{\zeta}_0 + \zeta^{n+1}) \\
&= \Xi(\zeta_0 + \sigma(\zeta)^{n+1})g(\hat{\zeta}_0 + \zeta^{n+1}) + h(\hat{\zeta}_0 + \zeta^{n+1}) \\
&= \Xi_n(\sigma(\zeta))g(\hat{\zeta}_0 + \zeta^{n+1}) + h(\hat{\zeta}_0 + \zeta^{n+1})
\end{aligned} \tag{5.20}$$

This relation, together with the analytic properties of the functions in the right-hand side, implies that the function $\hat{\Xi}_n$ is indeed an analytic function on some open punctured neighbourhood of $\zeta = 0$.

We next show that there is no $k \in \mathbb{N}_0$ smaller than n such that the function $\hat{\Xi}_k$ defined as $\hat{\Xi}_n$ (see relation (5.19)) is analytic on an open punctured neighbourhood of $\zeta = 0$. Assume that this is not true, i.e. let $k \in \{0, 1, .., n-1\}$ be such that $\hat{\Xi}_k$ is analytic on an open punctured neighbourhood of $\zeta = 0$. It follows from relation (5.18) and the reasoning above (replacing functions with a tilde by functions without a tilde and contariwise) that the function Ξ_k defined as Ξ_n in relation (5.16) is analytic on an open punctured neighbourhood of $\zeta = 0$. However, this contradicts the characterization of a branch point of Ξ of order n

at ζ_0 as we gave it in the beginning of this subsection. Together with the result above, this leads to the conclusion that $\tilde{\Xi}$ has a branch point of order n at $\hat{\zeta}_0$ if Ξ has a branch point of order n at ζ_0.

Part 2. We next show that if Ξ has an (ordinary) algebraic branch point of order n at ζ_0, then the function $\hat{\Xi}$ has an (ordinary) algebraic branch point of order n at $\hat{\zeta}_0$. If Ξ has an ordinary algebraic branch point of order n at ζ_0, then the function Ξ_n defined by relation (5.16) is not only analytic on some open punctured neighbourhood of $\zeta = 0$ but is analytic on a whole neighbourhood of $\zeta = 0$. It then immediately follows from relation (5.20) that the function $\hat{\Xi}_n$ defined by relation (5.19) is analytic at $\zeta = 0$. This in turn implies that the branch point of $\hat{\Xi}$ at $\hat{\zeta}_0$ is an ordinary algebraic branch point. It can be shown in more or less the same way that if Ξ has an algebraic branch point of order n at ζ_0, then the function $\hat{\Xi}$ has an algebraic branch point of order n at $\hat{\zeta}_0$.

Part 3. We have shown in Parts 1 and 2 that $\hat{\Xi}$ has a ((ordinary) algebraic) branch point of order n at $\hat{\zeta}_0$ if Ξ has a ((ordinary) algebraic) branch point of order n at ζ_0. It follows from relation (5.18) and this result (replacing the functions without a tilde by the functions with a tilde and contariwise) that Ξ has a ((ordinary) algebraic) branch point of order n at ζ_0 if $\hat{\Xi}$ has a ((ordinary) algebraic) branch point of order n at $\hat{\zeta}_0$. The statement in the lemma follows from these two assertions. □

Remark 5.14 As a branch point which is not of finite order is of order ∞, it follows immediately from this lemma that a multiple-valued analytic function Ξ on an open punctured neighbourhood of a point ζ_0 has a branch point of order ∞ at ζ_0 if and only if the function $\hat{\Xi}$ defined by relation (5.17) – with g, h and ψ as in Lemma 5.13– has a branch point of order ∞ at $\hat{\zeta}_0$.

Proposition 5.15 *Let f and θ be continuous functions on $D \times I$ such that f and θ are analytic on D for all fixed $t \in I$. Let $\zeta_0 \in D$ and let $\mathcal{N} \subset D$ be an open neighbourhood of ζ_0. If Ξ_0 is a multiple-valued analytic function on $\mathcal{N}\backslash\{\zeta_0\}$ with a branch point of order $N \in I\!N_\infty$ at $\zeta = \zeta_0$, then the solution of initial value problem* (5.9) *has a branch point of order N at $\varphi(\zeta_0, t)$ for all fixed $t \in I_{\zeta_0}$. Moreover, if the branch point of Ξ_0 is (ordinary) algebraic, then the branch point of Ξ is (ordinary) algebraic for all fixed $t \in I_{\zeta_0}$.*

Proof

We can define a function Ξ on $\varphi(\mathcal{N})\backslash\{\zeta_0\}$ exactly as in the proof of Proposition 5.9. It is clear that as Ξ_0 is now multiple-valued, this function Ξ may also be multiple-valued (see in particular relation (5.11) where $\tilde{\Xi}$ is the solution of initial value problem (5.10)). Proposition 5.9 states that the thus defined function Ξ solves initial value problem (5.9) in the sense that if Ξ_0 is made single-valued by making a cut K in $\mathcal{N}$ from ζ_0 to $\partial\mathcal{N}$ and choosing a branch of Ξ_0, this determines a branch cut $\varphi_t(K)$ in the domain of Ξ for all fixed $t \in I$ –as the mapping φ is

univalent on its domain for all fixed $t \in I-$ and a branch of the function Ξ which uniquely solves the corresponding initial value problem.
In order to reveal the character of the function Ξ on its domain $\varphi_t(\mathcal{N}\backslash\{\zeta_0\})$ for a fixed $t \in I_{\zeta_0}$, we reconsider its construction given in the proof of Proposition 5.9. More particularly, we determine in which way this function Ξ depends on the initial data given by the function Ξ_0 on $\mathcal{N}\backslash\{\zeta_0\}$. The proof of Proposition 5.9 starts with the definition of the functions $\tilde{f}$ and $\tilde{\theta}$ on $\{(\zeta,t) \in D \times I \mid t \in I_\zeta\}$. The precise form of these functions is not important for the following discussion; what is essential is that these functions are both analytic on their domains for all fixed $t \in I$. The next step in the proof is the observation that initial value problem (5.10) has a unique solution. This solution $\tilde{\Xi}$ is given by

$$\tilde{\Xi}(\zeta,t) = \Xi_0(\zeta)e^{\int_0^t \tilde{f}(\zeta,\tau)d\tau} + \int_0^t \tilde{\theta}(\zeta,\tau)e^{\int_\tau^t \tilde{f}(\zeta,\tau_1)d\tau_1}d\tau$$

as can be verified directly. Hence, this function $\tilde{\Xi}$ can be written as

$$\tilde{\Xi}(\zeta,t) = \Xi_0(\zeta)\tilde{g}(\zeta,t) + \tilde{h}(\zeta,t)$$

where $\tilde{g}$ and $\tilde{h}$ are functions on $\{(\zeta,t) \in D \times I \mid t \in I_\zeta\}$ which are analytic for all fixed $t \in I$. This implies that the function Ξ on $\varphi(\mathcal{N}\backslash\{\zeta_0\})$ defined as in relation (5.11) can be written as

$$\Xi(\zeta,t) = \Xi_0(\varphi^{\leftarrow}(\zeta,t),t)g(\zeta,t) + h(\zeta,t) \tag{5.21}$$

where g and h are functions on $\varphi(\mathcal{N})$ which are analytic on their domains $\varphi_t(\mathcal{N})$ for all fixed $t \in I$. We note that the function $\varphi^{\leftarrow}$ is univalent on its domain for all fixed $t \in I$ and that the function g does not vanish. So, for a fixed $t \in I_{\zeta_0}$, the function Ξ_t defined by $\Xi_t(\zeta) = \Xi(\zeta,t)$ can be written as

$$\Xi_t(\zeta) = \Xi_0(\psi_t(\zeta))g(\zeta) + h(\zeta)$$

where ψ_t, g and h are analytic function on an open neighbourhood of $\varphi(\zeta_0,t)$ where ψ_t is univalent and where g does not vanish. The assertion in the proposition then follows from Lemma 5.13. □

5.2.4 Propagation of isolated singularities

The Propositions 5.10 and 5.15 can be considered as assertions on the propagation of isolated singularities of solutions of extended linear Löwner-Kufareev equations. The isolated singularities of a function Ξ satisfying an extended linear Löwner-Kufareev equation move along the characteristics determined by equation (5.1a). A pole or a branch point cannot appear and can only disappear if the corresponding characteristic reaches the boundary of the unit disc. We note that if the real part of the function f that appears in the extended linear Löwner-Kufareev equation is non-negative, then the characteristics cannot reach the boundary of the unit disc -see Remark 5.5- and the number of poles and branch points is conserved. It is needless to state that these assertions are in complete accordance with Proposition 5.9. We finally note that the order of a pole or branch point cannot change in the sense that the order of the pole or branch point at $\varphi(\zeta_0, t)$ is the same for all $t \in I_{\zeta_0}$. We stress that these assertions are only true for solutions of extended linear Löwner-Kufareev equations where the function θ is analytic on $D \times I$; if the function θ is admitted to have poles, then singularities may appear or disappear as the proof of Proposition 5.11 shows.

5.3 Hopper equations reconsidered

In the previous chapters we met the Hopper equation which can be written as:

$$\left(\Omega'(\zeta,t)\overline{\Omega}(\zeta,t)\right)^{\cdot} = \left(\Omega'(\zeta,t)\overline{\Omega}(\zeta,t)f(\zeta,t)\zeta\right)' + \theta(\zeta,t) \tag{5.22}$$

(For the details on how to read this equation we refer to Section 4.1.) This Hopper equation can be regarded as a kind of extended Löwner-Kufareev equation for the function $\Omega'\overline{\Omega}$. There are nevertheless three important differences between the Hopper equations introduced in Section 4.1 and the extended linear Löwner-Kufareev equations for $\Xi = \Omega'\overline{\Omega}$ considered in Section 5.2. First, a function Ω is said to satisfy a Hopper equation if the relation (5.22) holds on $\partial D \times I$ while a function Ξ is said to satisfy an extended linear Löwner-Kufareev equation if this same relation -with a different interpretation of the prime, see Definition 3.18- holds on some open domain contained in $D \times I$. Secondly, the function θ that appears in a Hopper equation should not be considered to be given while the function θ in the right-hand side of an extended Löwner-Kufareev *is* a given function. Thirdly, the function f in a Hopper equation may depend on Ω in a functional way while the function f in an extended linear Löwner-Kufareev equation does not. We can summarize these differences by stating that a Hopper equation is an extended quasi-linear Löwner-Kufareev equation restricted to $\partial D \times I$ where the function θ that appears remains undetermined.
This view on Hopper equations turns out to be fruitful. In the Subsections 5.3.2 and 5.3.3 we show how the results obtained in the previous section can be used to prove properties of solutions of Hopper equations. In order to make the above explained point of view on Hopper equations more convincing, we reveal the relation between the functions $\Xi = \Omega'\overline{\Omega}|_{\partial D}$ and Ω in Subsection 5.3.1.

5.3.1 The inverse problem

We explained above that a Hopper equation can be considered to be a kind of Löwner-Kufareev equation for the function $\Xi = \Omega'\overline{\Omega}|_{\partial D}$. Theorem 5.20 below states for which functions Ξ on ∂D there is a locally conformal mapping Ω such that $\Xi = \Omega'\overline{\Omega}|_{\partial D}$. We first show that if such a function Ω exist, then it is almost unique.

Proposition 5.16 *Let $\Omega_k, k = 1,2$ be continuous functions on $\overline{D}$ such that:*

i). $\Omega_k(0) = 0$

ii). Ω_k is analytic on D and does not vanish on ∂D

iii). the derivative Ω_k' on D can be extended continuously to $\overline{D}$ and Ω_k' does not vanish on $\overline{D}$

If

$$\Omega_1'\overline{\Omega_1}|_{\partial D} = \Omega_2'\overline{\Omega_2}|_{\partial D}$$

then there is a real constant α such that $\Omega_1 = e^{i\alpha}\Omega_2$.

Proof

We first show that Ω_1 has only a finite number of zeros. Assume that this is not true, i.e. let Ω_1 have an infinite number of zeros. As $\overline{D}$ is compact, there is an accumulation point $\zeta_0 \in \overline{D}$ of zeros. As Ω_1 is continuous and does not have any zeros on ∂D, this point ζ_0 lies in D. Since Ω_1 is analytic on D, this implies that Ω_1 is identically zero on D and hence on $\overline{D}$. This contradicts assumption ii) in the proposition and we are led to the conclusion that Ω_1 has only a finite number of zeros. We denote these zeros outside 0 by $\zeta_n, n = 1,..,N$. It follows from the other assumptions in the proposition that all these zeros lay in D and are of first order.

The function h on $\overline{\mathbb{C}}\backslash\overline{D}$ defined by

$$h(\zeta) = \frac{\overline{\Omega_2}(\zeta)}{\overline{\Omega_1}(\zeta)}$$

is therefore meromorphic and has only a finite number of first order poles. It follows from the assumptions in the proposition that this function can be extended continuously to ∂D:

$$h(\zeta) = \frac{\overline{\Omega_2}(\zeta)}{\overline{\Omega_1}(\zeta)} = \frac{\Omega_1'(\zeta)}{\Omega_2'(\zeta)} \qquad \zeta \in \partial D \tag{5.23}$$

The function in the right-hand side can in turn be extended analytically on D. It then follows from the principle of analytic continuation ([26]) that h is a meromorphic function on $\overline{\mathbb{C}}$ with a finite number of poles which are all of first order.

It is then straightforward to show that there are numbers $c_n \in \mathbb{C}, n = 0, .., N$ such that

$$h(\zeta) = c_0 + \sum_{n=1}^{N} \frac{c_n}{\zeta - 1/\overline{\zeta_n}} \tag{5.24}$$

We note that

$$\overline{h}(\zeta) = \overline{c_0} + \sum_{n=1}^{N} \frac{\overline{c_n}\zeta_n\zeta}{\zeta_n - \zeta} \qquad \overline{h}'(\zeta) = \sum_{n=1}^{N} \frac{\overline{c_n}\zeta_n^2}{(\zeta_n - \zeta)^2} \tag{5.25}$$

It follows from the relations (5.23) that the following relation holds on ∂D:

$$\Omega_1' = h(\overline{h}'\Omega_1 + \overline{h}\Omega_1')$$

We multiply both sides of this identity by $\prod_{n=1}^{N}(\zeta_n - \zeta)$ and get

$$\Omega_1'(\zeta)\prod_{n=1}^{N}(\zeta_n - \zeta) = h(\zeta)\prod_{\substack{n=1\\n\neq k}}^{N}(\zeta_n - \zeta)\Big((\zeta_k - \zeta)^2\overline{h}'(\zeta)\frac{\Omega_1(\zeta)}{\zeta_k - \zeta} + (\zeta_k - \zeta)\overline{h}\Omega_1'(\zeta)\Big)$$

for all $\zeta \in \partial D$ and for an arbitrary $k \in \{1, .., N\}$. After substitution of the identities (5.24) and (5.25), it can be checked that all terms that appear in this relation are continuous functions on ∂D which can be extended analytically on D (observing that Ω_1 has first order zeros in $\zeta = \zeta_n, n = 1, ..N$). This implies that the relation above holds for all $\zeta \in D$. In particular, it holds for $\zeta = \zeta_k, k = 1, .., N$ and we find for all $k = 1, .., N$:

$$0 = h(\zeta_k)\prod_{\substack{n=1\\n\neq k}}^{N}(\zeta_n - \zeta_k)\left(2\zeta_k^2\overline{c_k}\Omega_1'(\zeta_k)\right) = 2\overline{c_k}\frac{(\zeta_k\Omega_1'(\zeta_k))^2}{\Omega_2'(\zeta_k)}\prod_{\substack{n=1\\n\neq k}}^{N}(\zeta_n - \zeta_k)$$

This implies $c_k = 0$ for $k = 1, .., N$ and we find that h is identically a constant c_0. Substition of this result into relation (5.23) leads to $\Omega_2 = \overline{c_0}\Omega_1$ on ∂D with $|c_0| = 1$. The assertion in the proposition follows. □

Before we formulate a theorem that states under which conditions a function Ξ on ∂D is such that there is a locally conformal mapping such that $\Xi = \Omega'\overline{\Omega}|_{\partial D}$, we introduce some notations and definitions.
In the following we denote the Hilbert transform of a function f on ∂D by $\mathcal{H}(f)$. We will write $\mathcal{H}(f(\zeta))$ in stead of $\mathcal{H}(f)(\zeta)$ as this will make formulae easier to write down. It is known that the Hilbert transform of a Hölder continuous function is Hölder continuous ([86]).

Definition 5.17 We define $\mathcal{K}_1$ as the space of functions Ξ on ∂D such that:

i). Ξ is Hölder continuous

ii). Ξ does not vanish on ∂D

iii). the function P on ∂D defined by

$$P(\zeta) = \frac{1}{2}\Big(\mathcal{H}(\ln|\Xi(\zeta)|) - \text{Im } \ln(\Xi(\zeta)\zeta)\Big) \tag{5.26}$$

is single-valued and has a Hölder continuous derivative.

Remark 5.18 It follows from the conditions i) and ii) that $\mathcal{H}(\ln|\Xi|)$ is a properly defined function on ∂D. It then follows from the Argument Principle (cf. [15]) that P is single-valued if and only if the increase of the argument of $\Xi(\zeta)$ as ζ tranverses ∂D in the positive direction is -2π. This implies for example that among the functions $\Xi_n(\zeta) = \zeta^n, n \in \mathbb{Z}$, the function Ξ_{-1} is the only one in $\mathcal{K}_1$.

Definition 5.19 We define $\mathcal{K}_2$ as the space of functions Ω on D such that:

i). Ω is analytic on D

ii). Ω' can be extended continuously to a function on $\overline{D}$ such that $\Omega'|_{\partial D}$ is Hölder continuous

iii). $\Omega(\zeta) = 0$ if and only if $\zeta = 0$

iv). the function Ω' does not vanish on $\overline{D}$

Theorem 5.20 *If $\Omega \in \mathcal{K}_2$, then the function Ξ on ∂D defined by*

$$\Xi = \Omega'\overline{\Omega}|_{\partial D} \tag{5.27}$$

is in $\mathcal{K}_1$ and satisfies

$$\frac{\Xi(\zeta)\zeta}{|\Xi(\zeta)|}e^{-\mathcal{H}(\text{Im } \ln \Xi(\zeta)\zeta)} = 1 + i\zeta\Big(P(\zeta) + i\mathcal{H}(P)(\zeta)\Big)' \quad \zeta \in \partial D \tag{5.28}$$

where the function P is defined by relation (5.26). *Conversely, if a function $\Xi \in \mathcal{K}_1$ satisfies equation* (5.28), *a function $\Omega \in \mathcal{K}_2$ exists such that relation* (5.27) *holds.*

Proof

Part 1. Let $\Omega \in \mathcal{K}_2$ and let Ξ be defined by relation (5.27). We prove that $\Xi \in \mathcal{K}_1$ and that Ξ satisfies equation (5.28).

It is easily checked that Ξ satisfies condition i) and ii) of Definition 5.17 as Ω satisfies condition ii), iii) and iv) of Definition 5.19. In order to show that Ξ satisfies condition iii) of Definition 5.17, we calculate P. We define $r, \alpha \in \mathbb{R}$ by

$\Omega'(0) = re^{i\alpha}$, omit the variables and note that the following identities hold on ∂D:

$$\begin{aligned} P &= \frac{1}{2}\Big(\mathcal{H}(\ln|\Xi|) - \text{Im } \ln(\Xi\zeta)\Big) = \frac{1}{2}\mathcal{H}(\ln|\Omega'\overline{\Omega}|) - \frac{1}{2}\text{Im } \ln(\Omega'\overline{\Omega}\zeta) \\ &= \frac{1}{2}\mathcal{H}\left(\text{Re } \ln\left(\Omega'\frac{\Omega}{\zeta}\right)\right) + \frac{1}{2}\text{Im } \ln\left(\overline{\Omega'}\frac{\Omega}{\zeta}\right) \\ &= \frac{1}{2}\text{Im } \ln\left(\Omega'\frac{\Omega}{\zeta}\right) - \alpha + \frac{1}{2}\text{Im } \ln\left(\overline{\Omega'}\frac{\Omega}{\zeta}\right) \\ &= \frac{1}{2}\text{Im } \ln\left(|\Omega'|\frac{\Omega}{\zeta}\right)^2 - \alpha = \text{Im } \ln\left(\frac{\Omega}{\zeta}\right) - \alpha \end{aligned} \tag{5.29}$$

where we used the observation that $\Omega'\Omega/\zeta$ is an analytic function on D with a non-vanishing continuous extension to ∂D. It follows from this relation that P is single-valued and has a Hölder continuous derivative on ∂D. We conclude that $\Xi \in \mathcal{K}_1$.

In order to show that Ξ satisfies equation (5.28), we calculate both sides of this equation. We again omit the variables and find for the left-hand side:

$$\frac{\Xi\zeta}{|\Xi|}e^{-\mathcal{H}(\text{Im } \ln(\Xi\zeta))} = \frac{\Omega'\overline{\Omega}\zeta}{|\Omega'\overline{\Omega}|}e^{-\mathcal{H}(\text{Im } \ln(\Omega'\overline{\Omega}\zeta))} =$$

$$\frac{\Omega'\overline{\Omega}\zeta}{|\Omega'\Omega|}e^{\mathcal{H}(\text{Im } \ln(\frac{\Omega}{\zeta})}e^{-\mathcal{H}(\text{Im } \ln(\Omega'))} = \frac{\Omega'\overline{\Omega}\zeta}{|\Omega'\Omega|}re^{-\ln|\Omega|}r^{-1}e^{\ln|\Omega'|} = \frac{\Omega'\zeta}{\Omega}$$

Next we calculate the right-hand side of equation (5.28):

$$1 + i\zeta(P + i\mathcal{H}(P))' = 1 + i\zeta\left(\text{Im } \ln\left(\frac{\Omega}{\zeta}\right) - \alpha + i\mathcal{H}\left(\text{Im } \ln\left(\frac{\Omega}{\zeta}\right) - \alpha\right)\right)' =$$

$$1 + i\zeta\left(\text{Im } \ln\left(\frac{\Omega}{\zeta}\right) - i\left(\text{Re } \ln\left(\frac{\Omega}{\zeta}\right) - r\right)\right)' = 1 + \zeta\left(\ln\left(\frac{\Omega}{\zeta}\right)\right)' = \frac{\Omega'\zeta}{\Omega}$$

It follows immediately from these identities that the function Ξ satisfies equation (5.28).

Part 2. Let $\Xi \in \mathcal{K}_1$ satisfy equation (5.28). We define the function Ω on ∂D by

$$\Omega(\zeta) = \zeta re^{i(P(\zeta)+i\mathcal{H}(P(\zeta)))} \tag{5.30}$$

where

$$r = e^{\frac{1}{4\pi}\int_0^{2\pi}\ln|\Xi(e^{i\theta})|d\theta} \tag{5.31}$$

It is easily checked that as Ξ satisfies condition iii) of Definition 5.17, this function Ω is differentiable on ∂D. It follows from relation (5.28) that equation (5.27) is satisfied as

$$\begin{aligned} \Omega'\overline{\Omega} &= re^{i(P+i\mathcal{H}(P))}\left(1 + i\zeta(P + i\mathcal{H}(P))'\right)\overline{\zeta}re^{-i(P-i\mathcal{H}(P))} \\ &= \frac{r^2\Xi}{|\Xi|}e^{-2\mathcal{H}(P)}e^{-\mathcal{H}(\text{Im } \ln(\Xi\zeta))} = \frac{\Xi}{|\Xi|}e^{\ln|\Xi|-\mathcal{H}(\text{Im } \ln(\Xi\zeta))}e^{\mathcal{H}(\text{Im}\ln(\Xi\zeta))} = \Xi \end{aligned}$$

It remains to be shown that Ω on ∂D can be extended to a function on $\overline{D}$ in $\mathcal{K}_2$. It immediately follows from relation (5.30) that the function Ω on ∂D can be extended to an analytic function on D. It also follows from this relation that $\Omega(\zeta) = 0$ if and only if $\zeta = 0$. In order to show that the conditions ii) and iv) in Definition 5.19 are also satisfied, we note that the following identities hold on ∂D:

$$\begin{aligned}\mathcal{H}(\ln|\Xi|) - P &= \frac{1}{2}\Big(\mathcal{H}(\ln|\Xi|) + \mathrm{Im}\ln(\Xi\zeta)\Big)\\ &= \frac{1}{2}\left(\mathcal{H}\Big(\mathrm{Re}\ \ln\left(\frac{\Omega'\Omega}{\zeta}\right)\Big) + \mathrm{Im}\ \ln\left(\Omega'\overline{\Omega}\zeta\right)\right)\\ &= \frac{1}{2}\mathrm{Im}\ \ln\left(\frac{\Omega\Omega'}{\zeta}\right) - \alpha + \frac{1}{2}\mathrm{Im}\ \ln\left(\frac{\zeta\Omega'}{\Omega}\right)\\ &= \mathrm{Im}\ \ln\Omega' - \alpha \end{aligned} \tag{5.32}$$

where $\alpha \in I\!\!R$ is such that $\Omega'(0) = re^{i\alpha}$. The left-hand side of this identity is single-valued as Ξ satisfies condition iii) of Definition 5.17 (see also Remark 5.18). This implies that the function Im $\ln\Omega'$ is single-valued on ∂D and it follows from the Argument Principle that Ω' does not vanish on D. We finally note the function Ω satisfies condition ii) of Definition 5.19 because of relation (5.32) and because Ξ satisfies conditions i) and iii) of Definition 5.17. □

Remark 5.21 It is clear that the right-hand side of equation (5.28) can be extended analytically on D. The left-hand side of this equation can also be extended analytically on D, whatever the function $\Xi \in \mathcal{K}_1$ is, as it can be written as

$$\frac{\Xi\zeta}{|\Xi|}e^{-\mathcal{H}(\mathrm{Im}\ \ln(\Xi\zeta))} = e^{i(\mathrm{Im}\ \ln(\Xi\zeta) + i\mathcal{H}(\mathrm{Im}\ \ln(\Xi\zeta))}$$

This implies that Theorem 5.20 also holds if equation (5.28) is replaced by the equation

$$\mathrm{Re}\ \Xi(\zeta)\zeta = |\Xi(\zeta)|\,(1 - \mathrm{Im}\ P'(\zeta)\zeta)\ e^{\mathcal{H}(\mathrm{Im}\ \ln(\Xi(\zeta)\zeta))} \tag{5.33}$$

which is obtained from equation (5.28) by taking the real part of both sides and multiplying by $|\Xi|e^{\mathcal{H}(\mathrm{Im}\ \ln(\Xi\zeta))}$.

Remark 5.22 We note that the proof of Theorem 5.20 is constructive in the sense that if a function $\Xi \in \mathcal{K}_1$ satisfies equation (5.28), then a function Ω such that $\Omega'\overline{\Omega}|_{\partial D} = \Xi$ is easily obtained from relation (5.30) and Schwarz' integral formula:

$$\Omega(\zeta) = \zeta r e^{\frac{1}{4\pi}\oint P(z)\frac{\zeta+z}{z(z-\zeta)}dz} \qquad \zeta \in D \tag{5.34}$$

where the integral is over ∂D. It is not difficult to prove that the function $\Omega_\alpha, \alpha \in I\!R$ defined by:

$$\Omega_\alpha(\zeta) = e^{i\alpha}\left(2\int_0^\zeta z e^{\frac{1}{2\pi i}\oint \ln|\Xi(w)|\frac{w+\zeta}{w(w-\zeta)}dw}dz\right)^{\frac{1}{2}} \qquad \zeta \in D \tag{5.35}$$

can be extended continuously to a function on $\overline{D}$ in $\mathcal{K}_2$ that satisfies $\Omega'_\alpha\overline{\Omega_\alpha}|_{\partial D} = \Xi$. The advantage of the last expression over the first one is that no singular integrals appear in the expression in the right-hand side of relation (5.35) while the function P – defined in Definition 5.17– in the right-hand side of relation (5.34) is related to the function Ξ via a Hilbert transform (which can be obtained only directly by calculating an integral with the singular Hilbert-Cauchy kernel, see e.g. [86]).

It turns out that some of the geometric properties of the image of D under Ω can relatively easy be revealed from the function $\Omega'\overline{\Omega}|_{\partial D}$. Before we show how this can be done, we introduce some definitions. We adopt the convention (cf. [23]) that a function Ω on D is called starlike (convex) if:

i). Ω is univalent on D

ii). $\Omega(0) = 0$

iii). the image of D under Ω is starlike with respect to the origin (is convex)

Definition 5.23 We define $\mathcal{K}_1^{(\text{star})}$ as the space of functions $\Xi \in \mathcal{K}_1$ such that the function P, defined in Definition 5.17, satisfies

$$\partial_\theta P(e^{i\theta}) \geq -1 \qquad \text{for all } \theta \in I\!R \tag{5.36}$$

We define $\mathcal{K}_1^{(\text{convex})}$ as the space of functions $\Xi \in \mathcal{K}_1$ such that the function Q defined on ∂D by

$$Q(\zeta) = \frac{1}{2}\Big(\mathcal{H}(\ln|\Xi(\zeta)|) + \text{Im}\ \ln(\Xi(\zeta)\zeta)\Big)$$

has a Hölder continuous derivative and satisfies

$$\partial_\theta Q(e^{i\theta}) \geq -1 \qquad \text{for all } \theta \in I\!R \tag{5.37}$$

Proposition 5.24 *If a function* $\Xi \in \mathcal{K}_1^{(\text{star})}$ ($\Xi \in \mathcal{K}_1^{(\text{convex})}$) *satisfies equation* (5.28), *then every function* $\Omega \in \mathcal{K}_2$ *such that* $\Omega'\overline{\Omega}|_{\partial D} = \Xi$ *is starlike (convex).*

Proof

Part 1. Let $\Xi \in \mathcal{K}_1^{(\text{star})}$. As the function P satisfies inequality (5.36), we have

$$(1 - \text{Im } P'(\zeta)\zeta) \geq 0 \qquad \text{for all } \zeta \in \partial D$$

If the function Ξ satisfies equation (5.28), it also satisfies equation (5.33) and we get

$$\text{Re } \Xi(\zeta)\zeta \geq 0 \qquad \text{for all } \zeta \in \partial D$$

Let $\Omega \in \mathcal{K}_2$ be a function such that $\Omega'\overline{\Omega}|_{\partial D} = \Xi$. We find

$$\text{Re } \frac{\zeta\Omega'(\zeta)}{\Omega(\zeta)} = |\Omega(\zeta)|^{-2}\text{Re } \Omega'(\zeta)\overline{\Omega}(\zeta)\zeta \geq 0 \qquad \text{for all } \zeta \in \partial D$$

It then follows immediately from the maximum principle for harmonic function that this inequality also holds for all $\zeta \in D$. The assertion on star-like functions in the proposition then follows immediately from a theorem –formulated in e.g. [23]– which states that an analytic function Ω with $\Omega(0) = 0$ such that

$$\text{Re } \frac{\zeta\Omega'(\zeta)}{\Omega(\zeta)} > 0 \qquad \text{for all } \zeta \in D$$

is starlike.

Part 2. It follows from Proposition 5.16 that it is sufficient to prove that if a function $\Xi \in \mathcal{K}_1^{(\text{convex})}$ satisfies equation (5.28), then an arbitrary function $\Omega \in \mathcal{K}_2$ such that $\Omega'\overline{\Omega}|_{\partial D} = \Xi$ is convex. One checks that there is function $\Omega \in \mathcal{K}_1$ such that

$$\Omega'|_{\partial D} = re^{i(Q+i\mathcal{H}(Q))}$$

where r is defined as in relation (5.31). As $\Xi \in \mathcal{K}_1^{(\text{convex})}$, the function Q is continuously differentiable and satisfies inequality (5.37). This implies

$$\text{Re } \frac{\zeta\Omega''(\zeta)}{\Omega'(\zeta)} = \text{Re } i\zeta\Big(Q(\zeta) + i\mathcal{H}(Q(\zeta))\Big)' \geq -1 \qquad \text{for all } \zeta \in \partial D$$

It then follows from the maximum principle for harmonic functions that

$$\text{Re } \left(1 + \frac{\zeta\Omega''(\zeta)}{\Omega'(\zeta)}\right) > 0 \qquad \text{for all } \zeta \in D$$

The assertion on convex functions in the proposition then follows immediately from a theorem –formulated in e.g. [23]– which states that an analytic function Ω with $\Omega(0) = 0$ that satisfies this inequality, is convex. □

Example 5.25 We consider the functions $\Xi_{a,b}$ with $a, b \in I\!R$ on ∂D defined by

$$\Xi_{a,b}(e^{i\theta}) = e^{2a\cos\theta}(b + e^{-i\theta})$$

It follows from Definition 5.17 and Remark 5.18 that $\Xi_{a,b} \in \mathcal{K}_1$ if $|b| \neq 1$ and if the increase of the argument of $1 + b\zeta$ as ζ transverses ∂D is equal to zero. This implies that $\Xi_{a,b} \in \mathcal{K}_1$ if and only if $|b| < 1$. It follows from Theorem 5.20 and the straightforward calculations of the expressions in equation (5.28) that there is a function $\Omega \in \mathcal{K}_2$ such that relation (5.27) holds with Ξ replaced by $\Xi_{a,b}$ if and only if $a = b$ and $|b| < 1$. This function Ω, which we will now denote by Ω_a, can be constructed by one of the relations given in Remark 5.22. We get:

$$\Omega_a(\zeta) = \zeta e^{a\zeta}$$

It follows from Proposition 5.24 that all these functions Ω_a with $|a| < 1$ are star-like, and hence univalent, as

$$\partial_\theta P(e^{i\theta}) = \partial_\theta(a\sin\theta) = a\cos\theta \geq -1 \qquad \text{for all } \theta \in I\!R$$

The function Ω_a with $|a| < 1$ is then convex if and only if

$$\partial_\theta Q(e^{i\theta}) = a\cos\theta + \operatorname{Im} \frac{aie^{i\theta}}{2|1 + ae^{i\theta}|} \geq -1 \qquad \text{for all } \theta \in I\!R$$

This inequality holds if and only if $|a| < (3 - \sqrt{5})/2$.

5.3.2 Analytically extendable mappings

We show in this subsection how the results on extended linear Löwner-Kufareev equations can be used to prove properties of solutions of Hopper equations. Before we do so, we first sketch the line of reasoning.

Let the mapping Ω on $\overline{D} \times I$ be a solution of a Hopper equation (5.22). We assume for the sake of simplicity that the function θ is analytic on D for all $t \in I$; the case where θ has a prescribed pole in $\zeta = 0$ is treated later on. The functions f and θ can –so to speak *a posteriori*– be considered as given functions. We then consider initial value problem (5.9) with initial data given by relation (5.38) below. Proposition 5.9 states that this problem has a unique solution Ξ on an open subdomain of $D \times I$. The function $\Omega'\overline{\Omega}$ can be considered as a solution of this problem on $\partial D \times I$. If the domains of the functions Ξ and $\Omega'\overline{\Omega}$ have some overlap, we may be able to show that these functions are identical: the solution Ξ is unique and analytic functions which are locally the same are globally the same. This in turn may enable us to deduce properties of the solution Ω of the original problem. However, it is in general not possible to show without further assumptions that the domains of the functions Ξ and $\Omega'\overline{\Omega}$ have some overlap. Therefore we assume in the next proposition that the function Ω is analytically extendable.

Proposition 5.26 *Let Ω be a smooth time dependent locally conformal mapping on $\overline{D} \times I$ satisfying a Hopper equation where the function θ is analytic on D for all $t \in I$. Let Ω be analytically extendable to $D_{1+\varepsilon}, \varepsilon > 0$ for all fixed $t \in I$. If Ω at $t = 0$ is analytically extendable on an open domain $B = \overline{D} \cup B_0$, then Ω is analytically extendable to $\overline{D} \cup (\sigma_t(B_0))$ for all fixed $t \in I$.*

The mapping σ_t was defined in Definition 5.3.
Proof
Part 1. As Ω is analytic on $D_{1+\varepsilon}$ for all fixed $t \in I$, the function $\Omega'\overline{\Omega}$ is analytic on the annulus $A_1 = \{\zeta \in \mathbb{C} \mid (1+\varepsilon)^{-1} < |\zeta| < 1+\varepsilon\}$ for all fixed $t \in I$. One carefully checks that this implies that relation (5.22) holds for all $(\zeta, t) \in A_2 \times I$ where $A_2 = \{\zeta \in \mathbb{C} \mid (1+\varepsilon)^{-1} < |\zeta| < 1\}$.
Part 2. Consider the initial value problem given by equation (5.9) where f and θ are the functions which appear in the Hopper equation (5.22) and initial data given by

$$\Xi(\zeta, 0) = \Omega'(\zeta, 0)\overline{\Omega}(\zeta, 0) \tag{5.38}$$

Proposition 5.9 states that this initial value problem has a solution Ξ on $\varphi(R(B) \cap D) = \varphi(R(B_0))$ such that Ξ is analytic on $\varphi_t(R(B_0))$ for all $t \in I$. One carefully checks that because $A_2 \subset R(B_0)$, there is an annulus $A_3 = \{\zeta \in \mathbb{C} \mid (1+\varepsilon)^{-1} \leq r_1 < |\zeta| < r_2 \leq 1\}$ and an open interval $\tilde{I} \subset I$ containing 0 such that $A_3 \times \tilde{I} \subset \varphi(R(B_0))$. Proposition 5.9 also states that the solution of the initial value problem is unique on the domain of determinacy $\varphi(R(B_0))$. As $A_3 \times \tilde{I} \subset A_2 \times I$, this implies that the functions Ξ and $\Omega'\overline{\Omega}$ are identical on this subdomain $A_3 \times \tilde{I}$. We draw two conclusions from this observation:

i). the function Ξ on $\varphi_t(R(B_0) \cap D)$ can be extended as an analytic function on $\varphi_t(R(B_0)) \cup A_2$ for all fixed $t \in \tilde{I}$

ii). the function $\Omega'\overline{\Omega}$ on A_2 can be extended analytically on $A_2 \cup \varphi_t(R(B_0))$ for all $t \in \tilde{I}$. As Ω' is a non-vanishing function on D for all fixed $t \in I \supset \tilde{I}$, it follows that the function $\overline{\Omega}$ on $R(D_{1+\varepsilon})$ can be extended analytically on $\varphi_t(R(B_0)) = R(\sigma_t(B_0))$ for all fixed $t \in \tilde{I}$. This is equivalent with stating that Ω on $D_{1+\varepsilon}$ can be extended analytically on $\sigma_t(B_0)$ for all fixed $t \in \tilde{I}$.

Part 3. One shows that this statement –Ω is analytically extendable on $\sigma_t(B_0)$– does not only hold for all $t \in \tilde{I}$ but for all $t \in I$ by using some set-theoretical arguments and the conclusion i) above. □

Remark 5.27 It is assumed in Proposition 5.26 that the function Ω is analytically extendable to $D_{1+\varepsilon}$ for all $t \in I$. The necessity of requiring Ω to be analytically extendable outside D has its origin in Part 2 of the proof and is already explained in the beginning of this subsection. We may have formulated the proposition under the less restrictive condition that Ω is analytically extendable on some open domain B_t for all $t \in I$ such that there is a continuous mapping $\gamma : t \in I \mapsto \gamma(t) \in \partial D \cap B_t$.

In order to illustrate the significance of Proposition 5.26 we consider some special cases. We use the same notation and make the same assumptions as in the proposition; i.e. in the following the function Ω on $\overline{D} \times I$ denotes a smooth time-dependent locally conformal mapping that is analytically extendable outside D and which satisfies a Hopper equation.
First we consider the case where $B = \overline{\mathbb{C}} \backslash \{\zeta_0\}$ with $\zeta_0 \in \overline{\mathbb{C}} \backslash \overline{D}$; i.e. ζ_0 is the only singularity of the function Ω at $t = 0$. It follows from Proposition 5.26 that the function Ω is analytically extendable on $\overline{D} \cup \sigma_t(\overline{\mathbb{C}} \backslash \{\zeta_0\}) = \overline{\mathbb{C}} \backslash \sigma_t(\{\zeta_0\})$ for all fixed $t \in I$. It follows from Lemma 5.2 and Definition 5.3 that the singularity $\zeta(t)$ of the function Ω uniquely solves initial value problem (5.3):

$$\dot{\zeta}(t) = \zeta(t)\overline{f}(\zeta(t), t)$$
$$\zeta(0) = \zeta_0$$

If the real part of the function f is positive definite on $D \times I$, the singularity tends to move to infinity as

$$\frac{d}{dt}|\zeta(t)|^2 = |\zeta(t)|^2 \text{Re } \overline{f}(\zeta(t), t) > 0$$

(We remark that also if the real part of the function f is not positive definite, the singularity cannot reach the boundary of the unit disc as this would contradict the assumption that Ω is analytically extendable; so: $\sigma_t(\{\zeta_0\}) \neq \emptyset$).) So to speak, the singularity of Ω can be obtained by reflecting the trajectory determined by equation (5.1a) which passes through $(\zeta_0, 0)$ with respect to the unit circle.
Secondly, we consider the case where Ω at $t = 0$ is a rational function with M poles at $\zeta_{m,0}, m = 1, .., M$ of order $K(m), m = 1, .., M$. It follows from Proposition 5.26 that the function Ω is analytically extendable on $\overline{\mathbb{C}}$ minus a set of M points outside $\overline{D}$ for all $t \in I$. One shows that the function $\Omega' \overline{\Omega}$ satisfies equation (5.22) on $(D \times I) \backslash \{\sigma(\zeta_{m,0}, t)\}_{m=1}^{M}$. It then follows from Proposition 5.10 that the function $\Omega' \overline{\Omega}$ is meromorphic on D with M poles of order $K(m), m = 1, .., M$ for all fixed $t \in I$. As the function Ω' does not vanish on its domain, this implies that the function $\overline{\Omega}$ is meromorphic on D with M poles of order $K(m), m = 1, .., M$ for all fixed $t \in I$. We conclude that the function Ω is a rational function with M poles of order $K(m), m = 1, .., M$ for all fixed $t \in I$. We note that the poles $\zeta_m(t), m = 1, .., M$ satisfy equation (5.3a):

$$\dot{\zeta}_m(t) = \zeta_m(t)\overline{f}(\zeta_m(t), t)$$

It is easily checked that these equations are equivalent to the equations (4.7). We note that if $\zeta_{m,0} = \infty$, then $\zeta_m(t) = \infty$ for all $t \in I$ and this implies that if Ω at $t = 0$ is a polynomial of order N (i.e. $M = 1, K(1) = N, \zeta_{1,0} = \infty$), then Ω is a polynomial of order N for all $t \in I$.
The case where the function Ω on $D_{1+\varepsilon}$ at $t = 0$ is a branch of an algebraic function can be treated almost analogously apart from some technical difficulties. We note that an algebraic function can be characterized by the property that such function has only a finite number of algebraic branch points ([3]). It can be shown –by reasoning as above and applying Proposition 5.15– that if Ω at $t = 0$ is a M-valued algebraic function at $t = 0$, then Ω is a M-valued algebraic function for all $t \in I$.

Remark 5.28 The function Ω in Proposition 5.26 is required to satisfy a Hopper equation where the function θ is analytic on D for all fixed $t \in I$. If the function θ is analytic on $D\backslash\{0\}$ with a pole of an order not larger than $N \in \mathbb{N}$ for all fixed $t \in I$, then the statements made above hold only with $\overline{\mathbb{C}}$ replaced by $\mathbb{C}$. For example, if Ω at $t = 0$ is a rational function with M poles outside ∞ of order $K(m), m = 1,..,M$ and a pole at ∞ of an order not larger than N, then Ω is a rational function with M poles outside ∞ of order $K(m), m = 1,..,M$ and a pole at ∞ with an order not larger than N for all $t \in I$. This assertion can be proved in more or less the same way as we did above –with special attention to the point $R(\infty) = 0$– with use of Proposition 5.11.

Corollary 5.29 *Let Ω be a smooth time-dependent locally conformal mapping on $\overline{D} \times I$ satisfying a Hopper equation. Let Ω be analytically extendable outside D for all fixed $t \in I$. If Ω at $t = 0$ is a polynomial, rational or algebraic function then Ω is polynomial, rational or algebraic function for all $t \in I$.*

We end this subsection with a discussion of what these results imply for solutions of the moving boundary problems discussed in Chapter 3. Consider a smooth-time dependent locally conformal mapping Ω on $\overline{D} \times I$ solving the moving boundary problem for Stokes flow driven by surface tension and multi-poles or for Hele-Shaw-flow with a source and assume that Ω is analytically extendable. It has been remarked in Chapter 3 that this mapping Ω must satisfy the corresponding Hopper equation. Corollary 5.29 and the remark that the real part of the function f that appears in the corresponding Hopper equations is positive, enables us to draw the following conclusions:

i). Singularities of the mapping Ω cannot appear or disappear; the only exception is a pole at ∞ if a_1 (or Q_1) is not identically zero. An isolated singularity cannot change its character.

ii). If the mapping Ω is a rational function at $t = 0$, then Ω is a rational function for all $t \in I$ with the same number and order of poles. We recapitulate that

Theorem 4.19 states that the solution of the corresponding initial value problem is unique in the class of rational functions with a fixed number of poles. It follows that the solution is even unique in the class of analytically extendable functions.

iii). The singularities of the mapping Ω tend to move to infinity. The radius of the largest disc on which the function Ω is analytic for some fixed $t \in I$ is monotonously increasing with time.

We stress that these conclusions can only be made under the assumption that Ω is analytically extendable outside D for all $t \in I$. Without this assumption, we cannot exclude the possibility that Ω at $t = 0$ is analytic on some open domain containing D while Ω has a set of singularities lying dense in ∂D for all $t \in I \backslash \{0\}$. It may be possible to show that this cannot happen by means of the methods explained in [6, 5] and [73] in which a slightly different model is studied.

5.3.3 Mappings in the neighbourhood of the identity

We showed in the previous subsection that a solution Ω of a Hopper equation such that Ω at $t = 0$ is rational remains rational if it is a priori known that the function Ω is analytically extendable. In this subsection we prove a comparable result; if stead of requiring Ω to be analytically extendable, we lay certain restrictions on the function f that appears in the Hopper equation.

Definition 5.30 We define $\mathcal{S}$ as the space of analytic functions f on D such that the derivative f' is continuously extendable to $\overline{D}$ and such that the Taylor coefficients $c_n, n \in I\!N_0$ of f satisfy

$$\text{Re } c_0 \geq \sum_{n=1}^{\infty} |c_n| \tag{5.39}$$

Proposition 5.31 *Let Ω on $\overline{D} \times I$ be a smooth time-dependent locally conformal mapping satisfying a Hopper equation where θ is analytic on D for all $t \in I$ and where the function f is in $\mathcal{S}$ for all fixed $t \in I_+ = I \cap I\!R_{0,+}$. If Ω at $t = 0$ is a rational function with M poles of order $K(m), m = 1, .., M$, then Ω is a rational function with M poles of order $K(m), m = 1, .., M$ for all $t \in I_+$.*

Proof

Part 1. We first show that there is a $T_0 \in I \cap I\!R_+$ such that Ω is a rational function with M poles of order $K(m), m = 1, .., M$ for all $t \in [0, T_0)$.
It follows from Corollary 4.9 that there is a $T_0 > 0$ and a smooth time-dependent locally conformal mapping $\tilde{\Omega}$ on $\overline{D} \times [0, T_0)$ such that:

i). $\tilde{\Omega}|_{t=0} = \Omega|_{t=0}$

ii). $\tilde{\Omega}$ is a rational function for all $t \in [0, T_0)$ with M poles of order $K(m)$, $m = 1, .., M$

iii). there is a continuous function $\tilde{\theta}$ on $\overline{D} \times [0, T_0)$, such that $\tilde{\theta}$ is an analytic function on D for all $t \in [0, T_0)$ and such that for all $(\zeta, t) \in \partial D \times [0, T_0)$:

$$\tilde{\theta}(\zeta, t) = \left(\tilde{\Omega}'(\zeta, t)\overline{\tilde{\Omega}}(\zeta, t)\right)^{\cdot} - \left(\tilde{\Omega}'(\zeta, t)\overline{\tilde{\Omega}}(\zeta, t)f(\zeta, t)\zeta\right)'$$

It can easily be checked that the function Ξ on $\partial D \times [0, T_0)$ defined by

$$\Xi(\zeta, t) = \Omega'(\zeta, t)\overline{\Omega}(\zeta, t) - \tilde{\Omega}'(\zeta, t)\overline{\tilde{\Omega}}(\zeta, t) \tag{5.40}$$

solves the initial value problem

$$\begin{aligned} &\dot{\Xi}(\zeta, t) = (\Xi(\zeta, t)f(\zeta, t)\zeta)' + \theta(\zeta, t) - \tilde{\theta}(\zeta, t) && (\zeta, t) \in \partial D \times [0, T_0) \\ &\Xi(\zeta, 0) = 0 && \zeta \in \partial D \end{aligned}$$

We decompose the functions Ξ and $\theta - \tilde{\theta}$ by

$$\Xi(\zeta, t) = \sum_{n=-\infty}^{\infty} d_n(t)\zeta^n \qquad \theta(\zeta, t) - \tilde{\theta}(\zeta, t) = \sum_{n=0}^{\infty} a_n(t)\zeta^n$$

and find that the initial value problem can be written as

$$\begin{aligned} &\dot{d}_n(t) = (n+1)\sum_{k=0}^{\infty} c_k(t)d_{n-k}(t) + a_n(t) && n \in \mathbb{Z} \\ &d_n(0) = 0 && n \in \mathbb{Z} \end{aligned}$$

where $a_{-n}(t) = 0$ for all $n \in \mathbb{N}$ and all $t \in [0, T_0)$. It will be shown in Appendix C, see Remark C.4, that the condition $f \in S$ for all $t \in I \supset [0, T_0)$ then implies $d_{-n}(t) = 0$ for all $n \in \mathbb{N}$ and all $t \in [0, T_0)$. We conclude that the function Ξ on ∂D can be analytically extended on D for all $t \in [0, T_0)$.
Now we rewrite relation (5.40) on $\partial D \times [0, T_0)$ as

$$\overline{\Omega} = \left(\overline{\tilde{\Omega}}\tilde{\Omega}' + \Xi\right) / \Omega'$$

One checks that it follows from the properties of the functions appearing in the right-hand side of this relation that the function $\overline{\Omega}$ on ∂D can be extended as a meromorphic function on D with M poles of order $k(m), m = 1, .., M$ for all $t \in [0, T_0)$. This implies that the function Ω is a rational function with M poles of order $K(m), m = 1, .., M$ for all $t \in [0, T_0)$.
Part 2. It remains to be shown that Ω is not only a rational function with M

poles of order $K(m), m = 1, .., M$ for all $t \in [0, T_0)$ for some $T_0 \in I \cap \mathbb{R}_+$ but for all $t \in I_+$. Let T denote the supremum of all $T_0 \in I$ such that Ω is a rational function with M poles of order $K(m), m = 1, .., M$ for all $t \in [0, T_0)$. We assume that T is not the right boundary of I and show that this leads to a contradiction. As Ω is a continuous function, there is a constant $C \in \mathbb{R}$ such that

$$\max_{\zeta \in \overline{D}} \max_{t \in [0,T]} |\Omega(\zeta, t)| \leq C$$

Let $\mathcal{U}$ denote the space of rational functions with at most M poles outside $\overline{D}$ of orders not larger than $K(m), m = 1, .., M$ equipped with the sup-norm on $\overline{D}$. Let $\mathcal{U}_C$ denote the ball of radius C in $\mathcal{U}$. This space $\mathcal{U}_C$ is compact and this implies that the sequence of functions $\Omega_n \in \mathcal{U}_C, n \in \mathbb{N}$ defined by

$$\Omega_n(\zeta) = \Omega\left(\zeta, (1 - \frac{1}{n})T\right)$$

has a subsequence converging in $\mathcal{U}_C$. This implies that Ω at $t = T$, that is: the limit of Ω_n as $n \to \infty$, is a rational function (with at most M poles outside $\overline{D}$ of orders not larger than $K(m), m = 1, .., M$). Reasoning as in Part 1, one then shows that there is an $\varepsilon > 0$ such that Ω is a rational function for all $t \in [T, T+\varepsilon)$ and hence for all $t \in [0, T + \varepsilon)$. It then follows from Proposition 5.10 that Ω is a rational function with M poles of order $K(m), m = 1, .., M$ for all $t \in [0, T + \varepsilon)$ as:

i). Ω at $t = 0$ is a rational function with M poles of order $K(m), m = 1, .., M$

ii). Re $f(\zeta, t) \geq 0$ for all $(\zeta, t) \in D \times I$

In other words, T is not the supremum of all $T_0 \in I$ such that Ω is a rational function with M poles of order $K(m), m = 1, .., M$ for all $t \in [0, T_0)$. □

Remark 5.32 The function Ω in Proposition 5.31 is required to satisfy a Hopper equation where θ is analytic on D for all fixed $t \in I$. If the function θ is analytic on $D \backslash \{0\}$ with a prescribed pole in $\zeta = 0$ for all $t \in I$, then a similar assertion can be formulated; see also Remark 5.28.

Consider a smooth time dependent locally conformal mapping Ω on $\overline{D} \times I$ that satisfies Hopper's equation and which is rational at $t = 0$. It follows from Proposition 5.31 that Ω is rational for all fixed $t \in I_+$ if it is known that $F_{[\Omega(\cdot,t)]} \in \mathcal{S}$ for all $t \in I_+$. This leads to the following problem: how can we characterize the space $\mathcal{F}^{\leftarrow}(\mathcal{S})$ of functions Ω on D such that:

i). Ω is analytic on D, $\Omega(0) = 0$

ii). Ω' can be extended continuously to a non-vanishing function on $\overline{D}$

iii). the function $F_{[\Omega]}$ has Taylor coefficients $c_n, n \in \mathbb{N}_0$ such that inequality (5.39) holds

We will not elaborate on this problem but only make some remarks. First we note that the expressions (4.30) and (4.31) can be used to characterize the functions in $\mathcal{F}^{\leftarrow}(\mathcal{S})$ by their Taylor coefficients. Secondly, we remark that the condition iii) implies that a function $\Omega \in \mathcal{F}^{\leftarrow}(\mathcal{S})$ is in some sense in the neighbourhood of the function $\Omega'(0)\zeta$; this explains the title of this subsection. Finally we remark that the methods explained in e.g. [6],[73] may lead to an affirmative answer to the following question: can it be shown that for each solution Ω of Hopper's equation such that Ω at $t = 0$ is in $\mathcal{F}^{\leftarrow}(\mathcal{S})$, there is a $T \in \mathbb{R}_+$ such that $F_{[\Omega(\cdot,t)]} \in \mathcal{S}$ for all $t \in [0, T)$?

5.4 Quasi-linear Löwner-Kufareev equations

In this section we prove a theorem on the local solvability of quasi-linear Löwner-Kufareev equations. The proof of this theorem is based on estimates on solutions of linear Löwner-Kufareev equations and an iteration technique. Most of these estimates, given in Subsection 5.4.1, can be found in [43] in a somewhat sharper formulation.

5.4.1 Preliminary Results

We start by making some estimates on the solutions of initial value problem (5.6) for a linear Löwner-Kufareev equation. In this subsection, f denotes a continuous function on $D \times I$, where I is an open interval that contains $\mathbb{R}_{0,+}$, such that f is analytic for all fixed $t \in I$ and such that:

$$\text{Re } f(\zeta, t) \leq 0 \qquad \text{for all } (\zeta, t) \in D \times \mathbb{R}_{0,+} \tag{5.42}$$

We note that a Löwner-Kufareev equation where the function f in the right-hand side has a non-positive definite real part corresponds to a shrinking domain (see relation (3.12a)).

Lemma 5.33 *Let f be bounded on $D \times [0, T]$ for all $T \geq 0$; i.e. a continuous function M_0 on $\mathbb{R}_{0,+}$ exists such that*

$$\max_{\tau \in [0,t]} \sup_{\zeta \in D} |f(\zeta, \tau)| \leq M_0(t)$$

If the derivative of the function Ω_0 is bounded, then the solution of initial value problem (5.6) satisfies for all $t \in \mathbb{R}_{0,+}$

$$\sup_{\zeta \in D} |\Omega(\zeta, t) - \Omega_0(\zeta)| \leq t M_0(t) \sup_{\zeta \in D} |\Omega_0'(\zeta)|$$

Proof
It has already been stated in Remark 5.5 that inequality (5.42) implies that $\varphi(D)$ contains $D \times I_+$. It is easily checked that the solution of initial value problem (5.2) for arbitrary $T \geq 0$ therefore has a domain that contains $D \times [0,T]$. It is not difficult to show that this solution ψ satisfies the following integral equation:

$$\psi(\zeta,t,T) = \zeta + \int_0^t \psi(\zeta,\tau,T)f(\psi(\zeta,\tau,T),T-\tau)\,d\tau$$

This implies

$$|\psi(\zeta,t,T) - \zeta| \leq t\ M_0(T) \tag{5.43}$$

for all $\zeta \in D$, all $T \geq 0$ and all $t \in [0,T]$. The solution of initial value problem (5.6) is given by relation (5.8):

$$\Omega(\zeta,t) = \Omega_0(\varphi^{\leftarrow}(\zeta,t)) = \Omega_0(\psi(\zeta,t,t)) \tag{5.44}$$

So:

$$\Omega(\zeta,t) - \Omega_0(\zeta) = \int_{\zeta}^{\psi(\zeta,t,t)} \Omega_0'(z)\,dz$$

and the inequality in the lemma follows straightforwardly from inequality (5.43).□

Lemma 5.34 *Let the function f be such that:*

i). a continuous function M_0 on $\mathbb{R}_{0,+}$ exists such that:

$$\max_{\tau\in[0,t]}\ \sup_{\zeta\in D} |f(\zeta,\tau)| \leq M_0(t)$$

ii). a continuous function M_1 on $\mathbb{R}_{0,+}$ exists such that:

$$\max_{\tau\in[0,t]}\ \sup_{\zeta\in D} \mathrm{Re}(f(\zeta,t) + f'(\zeta,t)\zeta) \leq M_1(t)$$

If the derivative of the function Ω_0 is bounded, then the solution of initial value problem (5.6) satisfies for all $t_1 \geq t_2 \geq 0$:

$$\sup_{\zeta\in D} |\Omega(\zeta,t_1) - \Omega(\zeta,t_2)| \leq (t_1 - t_2)M_0(t_1)e^{t_1 M_1(t_1)} \sup_{\zeta\in D} |\Omega_0'(\zeta)|$$

Proof

Part 1. It follows straightforwardly from relations (5.2) that the derivative of ψ can be written as

$$\psi'(\zeta,t,T)=e^{\int_0^t (f(\psi(\zeta,\tau,T),T-\tau)+f'(\psi(\zeta,t,T),T-\tau)\psi(\zeta,\tau,T))\,d\tau}$$

The condition ii) in the lemma then implies for all $T\geq 0$ and all $t\in[0,T]$:

$$\sup_{\zeta\in D}|\psi'(\zeta,t,T)|\leq e^{tM_1(T)}$$

Part 2. It follows from the theory of ordinary differential equations that

$$\psi(\psi(\zeta,t,T),\tau,T-t)=\psi(\zeta,t+\tau,T)$$

for all $\zeta\in D$ and all $T,t,\tau\geq 0$ such that $t+\tau\leq T$. With the use of this relation for $T=t_1$, $t=t_1-t_2$ and $\tau=t_2$, we get from the result obtained in Part 1 for all $t_1\geq t_2\geq 0$:

$$\sup_{\zeta\in D}|\psi(\zeta,t_1,t_1)-\psi(\zeta,t_2,t_2)|=\sup_{\zeta\in D}|\psi(\psi(\zeta,t_1-t_2,t_1),t_2,t_2)-\psi(\zeta,t_2,t_2)|$$

$$=\sup_{\zeta\in D}\left|\int_{\zeta}^{\psi(\zeta,t_1-t_2,t_1)}\psi'(z,t_2,t_2)\,dz\right|\leq e^{t_1M_1(t_1)}\sup_{\zeta\in D}|\psi(\zeta,t_1-t_2,t_1)-\zeta|$$

The assertion in the lemma then follows from relation (5.44), this inequality and inequality (5.43). □

We end this section with a lemma on the solutions Ω_1 and Ω_2 of initial value problems given by:

$$\dot{\Omega}_k(\zeta,t)=\Omega_k'(\zeta,t)f_k(\zeta,t)\zeta \qquad k=1,2 \tag{5.45a}$$

$$\Omega_k(\zeta,0)=\Omega_0(\zeta) \qquad k=1,2 \tag{5.45b}$$

where f_1 and f_2 both possess the properties mentioned in the beginning of this subsection. The corresponding functions ψ (see relations (5.2)) are denoted by $\psi_k, k=1,2$.

Lemma 5.35 *Let the functions f_1 and f_2 be such that:*

i). a continuous function C_1 on $\mathbb{R}_{0,+}$ exists such that

$$\max_{\tau\in[0,t]}\sup_{\zeta\in D}|f_1'(\zeta,\tau)|\leq C_1(t)$$

ii). a continuous function C_2 on $\mathbb{R}_{0,+}$ exists such that:

$$\max_{\tau\in[0,t]} \sup_{\zeta\in D} |f_2(\zeta,\tau)| \leq C_2(t)$$

If the derivative of the function Ω_0 is bounded, then the solutions Ω_1 and Ω_2 of initial value problems (5.45) *satisfy for all $t \in \mathbb{R}_{0,+}$:*

$$\sup_{\zeta\in D} |\Omega_1(\zeta,t) - \Omega_2(\zeta,t)| \leq$$
$$te^{t(C_1(t)+C_2(t))} \sup_{\zeta\in D} |\Omega_0'(\zeta)| \max_{\tau\in[0,t]} \sup_{\zeta\in D} |f_1(\zeta,t) - f_2(\zeta,t)| \tag{5.46}$$

Proof

Reasoning as before, we get for all $T \geq 0$ and all $t \in [0,T]$:

$$\sup_{\zeta\in D} |\psi_1(\zeta,t,T) - \psi_2(\zeta,t,T)| =$$
$$\sup_{\zeta\in D} \left| \int_0^t (\psi_1(\zeta,\tau,T) f_1(\psi_1(\zeta,\tau,T),T-\tau) - \psi_2(\zeta,\tau,T) f_2(\psi_2(\zeta,\tau,T),T-\tau))\, d\tau \right| \leq$$
$$\int_0^t \sup_{\zeta\in D} |\psi_1(\zeta,\tau,T)|\, |f_1(\psi_1(\zeta,\tau,T),T-\tau) - f_1(\psi_2(\zeta,\tau,T),T-\tau)|\, d\tau \;+$$
$$\int_0^t \sup_{\zeta\in D} |\psi_1(\zeta,\tau,T)|\, |f_1(\psi_2(\zeta,\tau,T),T-\tau) - f_2(\psi_2(\zeta,\tau,T),T-\tau)|\, d\tau \;+$$
$$\int_0^t \sup_{\zeta\in D} |\psi_1(\zeta,\tau,T) - \psi_2(\zeta,\tau,T)|\, |f_2(\psi_2(\zeta,\tau,T),T-\tau)|\, d\tau \leq$$
$$\int_0^t \sup_{\zeta\in D} |\psi_1(\zeta,\tau,T) - \psi_2(\zeta,\tau,T)| \cdot (|f_1'(\zeta,T-\tau)| + |f_2(\zeta,T-\tau)|)\; d\tau \;+$$
$$\int_0^t \sup_{\zeta\in D} |f_1(\zeta,T-\tau) - f_2(\zeta,T-\tau)|\, d\tau \tag{5.47}$$

As the last term can be estimated by

$$\int_0^t \sup_{\zeta\in D} |f_1(\zeta,T-\tau) - f_2(\zeta,T-\tau)|\, d\tau \leq t \max_{\tau\in[0,T]} \sup_{\zeta\in D} |f_1(\zeta,\tau) - f_2(\zeta,\tau)|$$

we get from inequality (5.47) and the lemma of Grönwall ([31]):

$$\sup_{\zeta\in D} |\psi_1(\zeta,t,T) - \psi_2(\zeta,t,T)| \leq$$
$$t\, e^{T(C_1(T)+C_2(T))} \max_{\tau\in[0,T]} \sup_{\zeta\in D} |f_1(\zeta,\tau) - f_2(\zeta,\tau)| \tag{5.48}$$

Substitution of this result into the relation

$$\Omega_1(\zeta,t) - \Omega_2(\zeta,t) = \int_{\psi_2(\zeta,t,t)}^{\psi_1(\zeta,t,t)} \Omega_0'(z)\,dz$$

leads to the assertion in the lemma. □

5.4.2 Existence of solutions

We return to initial value problems for quasi-linear Löwner-Kufareev equations; see Section 3.2. We show that these problems have a local solution if the mapping $\mathcal{F}$ is smooth in the following sense:

Definition 5.36 Let $\mathcal{H}$ denote the space of all bounded univalent functions on D equiped with the sup-norm. Let $\mathcal{A}$ denote the space of analytic functions on D with a bounded derivative and a non-positive definite real part. A mapping $\mathcal{F} : \mathcal{H} \mapsto \mathcal{A}$ is called Lipschitz continuous if a constant K exist such that

$$\sup_{\zeta\in D} |f_{[\Omega_1]}(\zeta) - f_{[\Omega_2]}(\zeta)| \leq K\|\Omega_1 - \Omega_2\|$$

for all $\Omega_1, \Omega_2 \in \mathcal{H}$.
A mapping $\mathcal{F} : \mathcal{H} \mapsto \mathcal{A}$ is called continuous with respect to the derivative if the functional

$$\mathcal{F}' \;:\; \Omega \in \mathcal{H} \;\mapsto\; \sup_{\zeta\in D} |f'_{[\Omega]}|$$

is continuous.

Theorem 5.37 *Let the mapping $\mathcal{F} : \mathcal{H} \mapsto \mathcal{A}$ be Lipschitz continuous and continuous with respect to the derivative on an open neighbourhood of a function $\Omega_0 \in \mathcal{H}$ which has a bounded derivative. Then the initial value problem given by*

$$\dot\Omega(\zeta,t) = \Omega'(\zeta,t) f_{[\Omega(\cdot,t)]}\zeta \tag{5.49a}$$
$$\Omega(\zeta,0) = \Omega_0(\zeta) \tag{5.49b}$$

has a local solution. This solution is univalent for all fixed t in its domain.

Proof
Part 1. At this stage we do not bother whether the functions Ω_n to be defined are in a neighbourhood of Ω_0. Let Ω_n for each $n \in \mathbb{N}_0$ be defined by

$$\Omega_0(\zeta,t) = \Omega_0(\zeta)$$
$$\begin{cases} \dot\Omega_{n+1}(\zeta,t) = \Omega'_{n+1}(\zeta,t) f_{[\Omega_n(\cdot,t)]}(\zeta)\zeta \\ \Omega_{n+1}(\zeta,0) = \Omega_0(\zeta) \end{cases}$$

It follows from Proposition 5.6 that the functions Ω_n are properly defined on $D \times I\!R_{0,+}$ if the functions $f_n, n \in I\!N_0$ defined by

$$f_n(\zeta, t) := f_{[\Omega_n(\cdot,t)]}(\zeta)$$

are continuous. We prove by induction that these functions possess the following properties:

i). f_n is continuous

ii). $\sup_{\zeta \in D} |f_n'(\zeta, t)|$ is a continuous function of the variable t

As Ω_0 does not depend on t, neither does f_0 and it is clear that f_0 possesses these properties. Now let f_n for an arbitrary $n \in I\!N_0$ have the mentioned properties. It follows from Proposition 5.6 that Ω_{n+1} is properly defined as the solution of the initial value problem given above (i.e. the function Ω_{n+1} on $D \times R_{0,+}$ exists and is unique). It follows from the same proposition that Ω_{n+1} is univalent for all fixed $t \in I\!R_{0,+}$ and the function f_{n+1} is therefore well defined. Moreover, it follows from Lemma 5.34 that for all $t_1, t_2 \geq 0$ there is a constant k such that

$$\sup_{\zeta \in D} |\Omega_{n+1}(\zeta, t_1) - \Omega_{n+1}(\zeta, t_2)| < k|t_1 - t_2|$$

One checks that this inequality, the Lipschitz continuity of the mapping $\mathcal{F}$ and the continuity of f_{n+1} with respect to the first variable, imply the continuity of f_{n+1}. The continuity of the function $\sup_{\zeta \in D} |f_{n+1}'(\zeta, t)|$ then follows from the continuity of the mapping $\mathcal{F}$ with respect to the derivative.
Part 2. We show that for every $d > 0$, there is a $T > 0$ such that for all non-negative $t < T$ and all $n \in I\!N_0$:

$$\sup_{\zeta \in D} |\Omega_n(\zeta, t) - \Omega_0(\zeta)| < d \tag{5.50}$$

We define constants M, C and T by

$$M = \sup_{\zeta \in D} |f_0(\zeta)| \quad C = \sup_{\zeta \in D} |\Omega_0'(\zeta)| \quad T = \frac{d}{C(M + Kd)}$$

and prove by induction that for all non-negative $t \leq T$ and all $n \in I\!N$:

$$\sup_{\zeta \in D} |\Omega_n(\zeta, t) - \Omega_0(\zeta)| \leq \frac{M}{K} \sum_{k=1}^{n} (CKt)^k \tag{5.51}$$

The inequality for $n = 1$ can easily be shown. We then assume that this inequality holds for a certain $n \in I\!N$, apply Lemma 5.33 and we find that for all $t \in [0, T]$:

$$\sup_{\zeta \in D} |\Omega_{n+1}(\zeta, t) - \Omega_0(\zeta)| \leq Ct \max_{\tau \in [0,t]} \sup_{\zeta \in D} |f_n(\zeta, \tau)| \leq$$

$$Ct \max_{\tau\in[0,t]} \sup_{\zeta\in D} \left(|f_n(\zeta,\tau) - f_0(\zeta)| + |f_0(\zeta)|\right) \leq$$

$$Ct \left(M + K \max_{\tau\in[0,t]} \sup_{\zeta\in D} |\Omega_n(\zeta,\tau) - \Omega_0(\zeta)|\right) \leq$$

$$MCt \max_{\tau\in[0,t]} \sum_{k=0}^{n} (CKt)^k \leq \frac{M}{K} \sum_{k=1}^{n+1} (CKt)^k$$

Inequality (5.51) for all non-negative $t \leq T$ and all $n \in I\!N$ follows. The inequality (5.50) for arbitray $n \in I\!N_0$ and all $t \in [0,T)$ is a direct consequence.
Part 3. It follows from the result deduced in Part 2 and from the continuity of $\mathcal{F}$ with respect to the derivative that there exists a $T > 0$ and numbers K_2, K_3 such that for all $n \in I\!N_0$ and all non-negative $t < T$

$$\sup_{\zeta\in D} |f_n'(\zeta,t)| < K_1 \qquad \sup_{\zeta\in D} |f_n(\zeta,t)| < K_2$$

We define $L = Ce^{T(K_1+K_2)}$, apply Lemma 5.35 and find for arbitrary $n \in I\!N$ and all non-negative $t < T$:

$$\sup_{\zeta\in D} |\Omega_{n+1}(\zeta,t) - \Omega_n(\zeta,t)| \leq$$
$$Lt \max_{t\in[0,t]} \sup_{\zeta\in D} |f_n(\zeta,\tau) - f_{n+1}(\zeta,\tau)| \leq$$
$$LKt \max_{t\in[0,t]} \sup_{\zeta\in D} |\Omega_n(\zeta,\tau) - \Omega_{n-1}(\zeta,\tau)|$$

One then shows in the standard way that $\Omega_n, n \in I\!N_0$ is a Cauchy–sequence in $\mathcal{H}$ for all fixed $t < \tilde{T} = \min\{T, (LK)^{-1}\}$. So, Ω_n is a sequence of univalent functions converging uniformly to an analytic function $\Omega_n, n \in I\!N_0$ on D for all non-negative $t < \tilde{T}$. One shows by standard techniques (cf. [27]) that the function Ω on $D \times [0,\tilde{T})$ thus defined satisfies initial value problem (5.49) and is univalent for all $t \in [0,\tilde{T})$. □

Remark 5.38 Direct applications of this theorem for standard moving boundary problems are restricted for two reasons. The first reason is that $\mathcal{F}$ maps into the space of functions which have a non-positive real part. We already remarked that this corresponds to moving boundary problems where the domain is shrinking. If one wants to generalize the theorem is such a way that $\mathcal{F}$ maps into a space that contains also functions which have real parts which are not purely negative, the same methods only apply if these functions can be extended analytically outside D. The second reason is that the conditions on the mapping $\mathcal{F}$ in the theorem can be formulated as conditions on how smooth the normal component of the velocity depends on the shape of the boundary (see Section 3.2). For problems such as the ones discussed in Sections 3.3 and 3.4, it is non-trivial –and it may even be impossible– to show that these conditions are satisfied indeed. We present in Appendix C another point of view on this problem.

Appendix A

Domains of left monogenic functions without primitives

In Section 2.3 we met the following problem: do domains $G \subset \mathbb{R}^3$ exist which are not x-normal but do possess the property that every left monogenic function on G has a primitive? Unfortunately, we cannot answer this question but the proposition below actually shows that a domain G which is such that every left monogenic function on G has a primitive must have geometric properties which resemble x-normality.

Proposition A.1 *Let $G \subset \mathbb{R}^3$ be a domain such that a curve $\gamma \subset G$ exists with the following properties:*

i). γ connects two points P_1 and P_2 with the same y- and z-coordinates such that the line segment connecting P_1 and P_2 is not entirely in G

ii). the orthogonal projection $\tilde{\gamma}$ of γ on a plane $V : x = a$ is such that $G/\tilde{\gamma}$ is connected.

Then a left monogenic function on G exists which does not have a primitive.

Proof
We may assume without loss of generality that $P_1 = (x_1, 0, 0)$ and $P_2 = (x_2, 0, 0)$ with $x_1 > 0, x_2 < 0$ while the origin O is not in G. Consider the function

$$e(x, y, z) = -2\,\frac{x - ye_1 - ze_2}{r^3} \qquad (x, y, z) \in G \qquad \text{(A.1)}$$

where $r = (x^2+y^2+z^2)^{\frac{1}{2}}$. One checks that this a properly defined, left monogenic function on G. We assume that a left monogenic primitive function

$$E = P + Q_1e_2 + Q_2e_2 + R\iota$$

on G exists and show that this leads to a contradiction. This proves the proposition.
The relations $\overline{D}E = e$, $DE = 0$ imply that the function $Q = Q_1$ satisfies

$$Q_x = \frac{y}{r^3} \qquad \Delta Q = 0 \tag{A.2}$$

On G minus the x-axis, Q can be written as

$$Q(x,y,z) = \frac{xy}{(y^2+z^2)r} - \frac{y}{y^2+z^2} + f(x,y,z) \tag{A.3}$$

Substituting this relation into relations (A.2), we find that the function f on G minus the x-axis satisfies

$$f_x = 0 \qquad \Delta f = 0 \tag{A.4}$$

Now consider the restriction $\tilde{f}$ of f to the intersection $\tilde{G}$ of G and the plane $V_1 : x = x_1$. It follows from the relations (A.4) that $\tilde{f}$ on $\tilde{G}\backslash\{(0,0)\}$ is harmonic. It follows from relation (A.3) that $\tilde{f}$ can be extended continuously to $(0,0)$ as

$$\lim_{(y,z)\to(0,0)} \tilde{f}(y,z) = \lim_{(y,z)\to(0,0)} Q(x_1,y,z) = Q(x_1,0,0)$$

and this implies that $\tilde{f}$ is harmonic on $\tilde{G}$.
Let γ be a curve with the properties as mentioned in the proposition and let $\mathcal{N}$ be a simply connected neighbourhood of its projection $\tilde{\gamma}$ –condition ii) implies that such a neighbourhood of exists. It follows from the relations (A.4) and the monodromy theorem ([3]) that f on some neighbourhood of γ does not depend on the variable x. In particular this implies:

$$\lim_{(y,z)\to(0,0)} f(x_2,y,z) = \lim_{(y,z)\to(0,0)} f(x_1,y,z) = Q(x_1,0,0)$$

On the other hand, it follows from relation (A.3) that this limit does not exist because:

$$\lim_{y\to 0} f(x_2,y,0) = \lim_{y\to 0} \frac{-x_2 y}{y^2(x_2^2+y^2)^{\frac{1}{2}}} + \frac{1}{y} + Q(x_2,y,0) = Q(x_2,0,0) + 2\lim_{y\to 0} \frac{1}{y}$$

We conclude that there is no primitive E of e. □

We discuss why the conditions i) and ii) in Proposition A.1 can not regardlessly be replaced by the condition that G is not x-normal. A glance at the proof shows that the domain G in the proposition must be such that every real function f on G satisfying equation (A.4) can be considered as a function of the variables y and z only. We say that a function f on a domain $G \subset I\!R^3$ can be considered as a function of the variables x and y if:

$$(x_1,y,z),(x_2,y,z) \in G \quad \text{implies} \quad f(x_1,y,z) = f(x_2,y,z)$$

First we treat an example which shows that not every domain $G \subset \mathbb{R}^3$ has this property. Consider the domain

$$\{(x,y,z) \in \mathbb{R}^3 \;\big|\; x \in (-\pi,\pi),\ (y-\cos x)^2 + (z - \sin x)^2 < 1\}$$

We define the function f on G by

$$f(x,y,z) = \operatorname{Im} \int_{\tilde{\gamma}} \frac{1}{w}\, dw$$

where $\tilde{\gamma}$ is the orthogonal projection of a path γ in G from $(0,1,0)$ to (x,y,z) on the plane $V : x = 0$ which is identified with the complex w-plane. One checks that f is well-defined, satisfies differential equations (A.4) but cannot be considered to be a function of the variables x and y as

$$f(-\tfrac{5\pi}{6},-1,0) = -\pi \qquad\qquad f(\tfrac{5\pi}{6},-1,0) = \pi$$

Next we show that even if the domain G is such that its projection $\tilde{G}$ on the plane $V : x = 0$ is simply connected, then a function f on G may exist that satisfies equations (A.4) but which cannot be considered as a function of the variables y and z only. It is possible to construct a non-compact Riemannian surface ([3, 25]) such that:

i). the projection of G on the plane $V : x = 0$ is simply connected

ii). the points $P_1 = (1,0,0)$ and $P_2 = (-1,0,0)$ are in G

iii). for every point $(x,y,z) \in G$ and for all $t \in (-\frac{1}{2},\frac{1}{2})$, $(x+t,y,z) \in G$ implies $t = 0$

It follows from Weierstrass' product theorem (also called: Weierstrass' theorem for the construction of a meromorphic function with prescribed poles and zeros, [25]) for non-compact Riemannian surfaces that an analytic function g on H exists with precisely one zero in $(1,0,0) \in H$. We may assume Re $g(-1,0,0) \neq 0$ (if Re $g(-1,0,0) = 0$ then Re $ig(-1,0,0) \neq 0$).
Now we define the open domain $G \subset \mathbb{R}^3$ by

$$G = \{(x,y,z) \in \mathbb{R}^3 \mid \exists\, t \in (-\frac{1}{2},\frac{1}{2}) \quad (x+t,y,z) \in H\}$$

and we define the function f on G by

$$f(x,y,z) = \operatorname{Re}\, g(x+t,y,z)$$

where t is the unique number in $(-\frac{1}{2},\frac{1}{2})$ such that $(x+t,y,z) \in H$. We notice that the projection $\tilde{G}$ of G on V is the open unit disc, that the above constructed function f satisfies equations (A.4) but that this function cannot be considered to be a function of the variables y and z.

Remark A.2 It follows immediately from the construction given above that an analytic function g on D (in the sense of Weierstrass, [11]) exists such that:

i). for every $z \in D$ there is a path from $z_0 = 0$ to z along which g can be extended analytically

ii). the so obtained analytical function g is not single-valued.

We stress that this does not contradict the monodromy theorem.

We end this appendix with a lemma which states under which conditions a domain $G \subset \mathbb{R}^3$ is such that every function f on G satisfying equations (A.4) can be considered to be a function of two variables.

Lemma A.3 *Let $G \subset \mathbb{R}^3$ be an open domain such that:*

i). the orthogonal projection $\tilde{G}$ of G on the plane $V : x = 0$ is simply connected

ii). there is a point $\underline{x}_0 \in G$ such that every path $\tilde{\gamma}$ in $\tilde{G}$ starting at the projection $\tilde{\underline{x}}_0$ of $\underline{x}_0$ is the projection of a path γ starting at $\underline{x}_0$

Let the function f on G satisfy differential equations (A.4). *Then f can be considered to be a function of the variables y and z only.*

Proof
First we make the following remark. Let $\underline{x} = (x, y, z)$ be an arbitrary point in G. There is an $\varepsilon > 0$ such that the ball $B_\varepsilon(\underline{x})$ with radius ε and centre $\underline{x}$ is a subdomain of G. As $\frac{\partial f}{\partial x} = 0$ on $B_\varepsilon(\underline{x})$, the function f restricted to $B_\varepsilon(\underline{x})$ can be considered a function of the variables y and z only and therefore it makes sense to consider the projected function $\tilde{f}$ on the projection $\tilde{B}_\varepsilon(\underline{x})$ of $B_\varepsilon(\underline{x})$ on V. This function $\tilde{f}$ is harmonic on $\tilde{B}_\varepsilon(\underline{x})$.
Let $\underline{x}_0$ be a point as in the lemma. We will show that it is possible to extend the function (or function element) $\tilde{f}$ on $\tilde{B}_{\varepsilon_0}(\underline{x}_0)$ harmonically along every path $\tilde{\gamma}$ in $\tilde{G}$ starting at $\tilde{\underline{x}}_0$. Let $\tilde{\gamma}$ be an arbitrary path in $\tilde{G}$ starting at $\tilde{\underline{x}}_0$ and let γ be a path in G starting in $\underline{x}_0$ such that its projection on V is $\tilde{\gamma}$ –such a γ exists because of condition ii). As γ is compact (it is the image of $[0,1]$ under a continuous mapping) and as the distance between γ and ∂G (if it exists) is non-zero, it is possible to cover γ with a finite number of balls $B_n, n = 0, .., N$ in G. It follows from the remark made above that the projected functions $\tilde{f}_n, n = 0, .., N$ on $\tilde{B}_n$ are harmonic. As the path $\tilde{\gamma}$ is covered by $\{\tilde{B}_n\}_{n=0}^N$, this construction leads to a harmonic extension of $\tilde{f}$ on $\tilde{B}_{\varepsilon_0}(\underline{x}_0)$ along $\tilde{\gamma}$.
As the function $\tilde{f}_0$ thus can be extended harmonically along every path in the simply connected domain $\tilde{G}$, it follows from the monodromy theorem that all these extensions lead to the same result. To be more precise: given two paths $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ at $\tilde{G}$ starting at $\tilde{\underline{x}}_0$ and ending at a point $\tilde{\underline{x}} \in G$, we find $\tilde{f}_1(\tilde{\underline{x}}) = \tilde{f}_2(\tilde{\underline{x}})$

where $\tilde{f}_i, i = 1, 2$ denotes the function (or function element) in a neighbourhood of $\tilde{\underline{x}}_i$ obtained by extending $\tilde{f}$ on $\tilde{B}_\varepsilon(\underline{x_0})$ along the path $\tilde{\gamma}_i$.
Now, let $\underline{x}_1 = (x_1, y, z)$ and $\underline{x}_2 = (x_2, y, z)$ be arbitrary points in G with the same $y-$ and $z-$coordinates. As G is connected, there are paths γ_1 and γ_2 from $\underline{x}_0$ to $\underline{x}_1$ and $\underline{x}_2$ respectively. We can extend the function $\tilde{f}$ on $\tilde{B}_\varepsilon(\underline{x_0})$ harmonically along the projections of the paths $\tilde{\gamma}_i$ of the paths $\gamma_i, i = 1, 2$ on V as we showed above. As the end points of the paths $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ are the same, namely $(y, z) \in \tilde{G}$, we get $\tilde{f}_1(y, z) = \tilde{f}_2(y, z)$ and this implies $f(x_1, y, z) = f(x_2, y, z)$. □

Appendix B

Separation of variables in linear Löwner-Kufareev equations

We show in this appendix that the initial value problem for a linear Löwner-Kufareev equation can be solved by the method of separation of variables if the function f that appears in this equation does not depend on the variable t.
Let f be an analytic function on an open, simply connected domain $B \subset \mathbb{C}$ that contains 0 and consider the initial value problem

$$\dot{\Omega}(\zeta,t) = \Omega'(\zeta,t)f(\zeta)\zeta \tag{B.1a}$$
$$\Omega(\zeta,0) = \Omega_0(\zeta) \tag{B.1b}$$

where Ω_0 is an analytic function on B. We assume that there is a solution Ω of equation (B.1a) of the following form:

$$\Omega(\zeta,t) = u(g(t)\psi(\zeta))$$

where u is an arbitrary analytic function and where g and ψ are functions to be determined. A solution of this form exists indeed if there is a constant $C \in \mathbb{C}$ such that the following differential equations can be solved:

$$\dot{g}(t) = Cg(t) \tag{B.2a}$$
$$\psi'(\zeta)f(\zeta)\zeta = C\psi(\zeta) \tag{B.2b}$$

First we consider equation (B.2b) on an open neighbourhood of $\zeta = 0$. One verifies that a non-trivial solution of this equation exists if and only if $f(0) \neq 0$ and $C = nf(0)$ with $n \in \mathbb{N}$. This solution, which is unique up to a multiplicative constant, has a zero of order n in $\zeta = 0$. As we will require the function ψ to be univalent later on (see relation (B.4)), we put $n = 1$ and solve the equations (B.2):

$$g(t) = K_1 e^{f(0)t} \qquad \psi(\zeta) = K_2 \zeta e^{\int_0^{\zeta} \left(\frac{f(0)}{f(z)}-1\right)\frac{1}{z}\,dz}$$

where K_1 and K_2 are arbitrary constants. This function ψ is univalent in some open neighbourhood of $\zeta = 0$. We remark that ψ is injective on $\partial D_r \subset B, r > 0$, and hence univalent on D_r ([23]), if the relation

$$\int_{\theta_1}^{\theta_2} \frac{f(0)}{f(re^{i\theta})}\,d\theta = 2\pi k \tag{B.3}$$

implies $\theta_1 = \theta_2$ modulus 2π for all $k \in \mathbb{Z}$. Furthermore, the function ψ is well-defined on the whole domain B of f if the function f does not have any zeros. One verifies that if f does have a zero at a point ζ_0, then the behaviour of the function ψ at ζ_0 depends on the value of $f'(\zeta_0)$:

i). if there is a $n \in \mathbb{N}$ such that $n\zeta_0 f'(\zeta_0) = f(0)$, then ψ is analytic at $\zeta = \zeta_0$ with a zero of order n

ii). if there is a $n \in \mathbb{N}$ such that $n\zeta_0 f'(\zeta_0) = -f(0)$, then ψ has a pole of order n in $\zeta = \zeta_0$

iii). otherwise there is no solution of equation (B.2b) in the class of meromorphic functions on B; in particular, if $f'(\zeta_0) = 0$ then ψ has an essential singularity in $\zeta = \zeta_0$, and if $\zeta_0 f'(\zeta_0)/f(0) \in \mathbb{R}\backslash\mathbb{Z}$ then ψ is multiple-valued

So, differential equation (B.2b) has a non-trivial analytic solution on B such that $\psi'(0) \neq 0$ if and only if $f(0) \neq 0$ and for every zero $\zeta_i \in B, i = 1,..,N$ of f there is a $n_i \in \mathbb{N}$ such that $n_i f'(\zeta_i)\zeta_i = f(0)$. The general solution of this equation is then given by

$$\psi(\zeta) = K_2 \zeta \prod_{i=1}^{N} (\zeta - \zeta_i)^{n_i} e^{\int_0^{\zeta} \left(\frac{1}{\zeta}\left(\frac{f(0)}{f(\zeta)}-1\right) - \sum_{i=1}^{N} \frac{n_i}{\zeta-\zeta_i}\right) d\zeta}$$

It follows that the function Ω on an open domain $\mathcal{N} \subset \mathbb{C} \times \mathbb{R}$ defined by

$$\Omega(\zeta, t) = \Omega_0(\psi^{\leftarrow}(e^{f(0)t}\psi(\zeta))) \tag{B.4}$$

is a solution of initial value problem (B.1). We note that it demands some analysis to determine the domain $\mathcal{N}$ of Ω; in particular, this domain depends on the subdomain on which the function ψ is univalent. However, it follows from the construction above that $\{0\} \times \mathbb{R} \subset \mathcal{N}$.

Example B.1 Let the function f on D be given by:

$$f(\zeta) = \frac{\zeta+1}{\zeta-1}$$

and let Ω_0 be the identity. We solve the corresponding initial value problem (B.1) in the way explained above. We put $K_2 = 1$ and find

$$\psi(\zeta) = \frac{\zeta}{(\zeta+1)^2}$$

(We note that ψ is analytic on D in accordance with the observation that f does not have any zeros on D; the pole in $\zeta = -1$ has order two according to the analysis presented above. We also note that ψ is univalent on D according to the remark made above; see relation (B.3)). This function, that is called the Koebe function ([70]), maps the unit disc D univalently on $\mathbb{C}$ minus a slit S on the real axis from 1/4 to infinity. The inverse function $\psi^{\leftarrow}$ on $\mathbb{C} \setminus S$ is given by

$$\psi^{\leftarrow}(\zeta) = -1 + \frac{1}{2\zeta}\left(1 - \sqrt{1-4\zeta}\right)$$

We get the solution of the initial value problem from relation (B.4):

$$\Omega(\zeta,t) = -1 + \frac{e^t(\zeta+1)}{2\zeta}\left(\zeta + 1 - \sqrt{(\zeta+1)^2 - 4e^{-t}\zeta}\right)$$

One verifies that the domain of this function is given by

$$\{(\zeta,t) \in D \times \mathbb{R}_- \mid \zeta \notin [-1 + 2e^{-t}(1-\sqrt{1-e^t}),\, 1) \subset \mathbb{R}\} \cup (D \times \mathbb{R}_{0,+})$$

We note that Ω is well-defined on D for all $t \in \mathbb{R}_{0,+}$, according to Remark 5.5 and the fact that the real part of f on D is negative. It turns out that the function $\Omega(\cdot,-t), t \in \mathbb{R}_{0,+}$ is just the inverse of the function $\Omega(\cdot,t)$ and this implies that the image of D under $\Omega(\cdot,t), t \in \mathbb{R}_{0,+}$ is the unit disc minus a slit on the real axis from $-1 + 2e^t(1-\sqrt{1-e^{-t}})$ to 1. We finally remark that this solution can also be obtained by solving initial value problem (5.1); one gets the following identity:

$$\varphi(\zeta,t) = \Omega(\zeta,-t)$$

We stress that this identity does not hold in general.

Appendix C

Existence of solutions of Hopper equations

This appendix is concerned with the solvability of Hopper equations under general conditions. We certainly do not establish a complete proof; we only show some ingredients in the hope that this may contribute to the understanding of Hopper equations.

It is clear from the considerations in Chapter 5 that the solvability of a Hopper equation is related with the existence of solutions of the following initial value problem for a function Ξ on $\partial D \times I$:

$$\dot{\Xi}(\zeta,t) = (\Xi(\zeta,t)f(\zeta,t)\zeta)' + \theta(\zeta,t) \tag{C.1a}$$

$$\Xi(\zeta,0) = \Xi_0 \tag{C.1b}$$

where Ξ_0 is a given function on ∂D and where the functions f and θ are assumed to be decomposable in the following way:

$$f(\zeta,t) = \sum_{n=0}^{\infty} c_n(t)\zeta^n \qquad \theta(\zeta,t) = \sum_{n=0}^{\infty} a_n(t)\zeta^n$$

We will explain the precise role of these functions at the end of this appendix. We decompose Ξ and Ξ_0 as:

$$\Xi(\zeta,t) = \sum_{n=-\infty}^{\infty} d_n(t)\zeta^n \qquad \Xi_0(\zeta,t) = \sum_{n=-\infty}^{\infty} d_{n,0}(t)\zeta^n$$

and find that initial value problem (C.1) can be written as

$$\dot{d}_n(t) = (n+1)\sum_{k=0}^{\infty} c_k(t)d_{n-k}(t) + a_n(t) \qquad n \in \mathbb{N}_0 \tag{C.2a}$$

$$\dot{d}_n(t) = (n+1)\sum_{k=0}^{\infty} c_k(t)d_{n-k}(t) \qquad n \in \mathbb{Z}\backslash\mathbb{N}_0 \tag{C.2b}$$

$$d_n(0) = d_{n,0} \qquad n \in \mathbb{Z} \tag{C.2c}$$

(A comparable set of equations already appeared in the proof of Proposition 5.31). In the Chapters 4 and 5, the time interval I –in which the variable t took its values– was open and the functions Ξ, f and θ were typically continuous on their domains. It turns out that if we want to discuss the existence of solutions, it is more appropriate to consider time intervals $[0,T], T \in \mathbb{R}_+$ and to consider time dependent functions in $L_2(\partial D)$. That is, we assume that $c(t) = (c_0(t), c_1(t), ..) \in l_2$ and $a(t) = (a_0(t), a_1(t), ..) \in l_2$ for all $t \in [0,T]$, take $\sum_{-\infty}^{\infty} |d_{n,0}|^2 < \infty$ and look for solutions Ξ such that $\sum_{-\infty}^{\infty} |d_n(t)|^2 < \infty$ for all $t \in [0,T]$.
We first consider differential equations (C.2b) with $n \in \{-2,-3,..\}$ in this context; the equation with $n = -1$ is trivial while the equations (C.2a) will be considered later on. We can rewrite the mentioned set of equations as follows. We define $b(t), t \in [0,T]$ and b_0 by

$$b(t) = (d_{-2}(t), d_{-3}(t), ...) \qquad b_0 = (d_{-2,0}, d_{-3,0}, ...)$$

For each $c = (c_0, c_1, ..) \in l_2$, we define the operator $A(c)$ by

$$A(c) = -\sum_{n=0}^{\infty} c_n N J_n$$

where N and $J_n, n \in \mathbb{N}_0$ are operators corresponding to the matrices with elements given by

$$N_{kl} = k\delta_{kl} \qquad (J_n)_{kl} = \delta_{n+k\,l} \qquad k,l \in \mathbb{N}$$

The equations (C.2b) and (C.2c) can then be written as

$$\dot{b}(t) = A(c(t))b(t) \tag{C.3a}$$

$$b(0) = b_0 \tag{C.3b}$$

Proposition C.1 *If $c : t \in I \mapsto c(t) \in l_2$ is such that:*

i). Re $c_0(t) \geq \sum_{n=1}^{\infty} |c_n(t)|$ for all $t \in [0,T]$

ii). the function M on $[0,T]$ defined by: $M(t) = \sum_{n=0}^{\infty} n|c_n(t)|$ is continuous *then the initial value problem* (C.3) *has a unique l_2-valued solution.*

Proof

It follows from a theorem formulated in [42] (see also [41]) that it is sufficient to check that the family of operators $\{A(t)\}_{t\in[0,T]}$ where $A(t) = A(c(t))$, satisfies the following three conditions:

i). $A(t)$ is an infinitesimal generator of a C_0-semigroup on l_2 for all fixed $t \in [0,T]$

ii). there is a Banach-space h, continuously and densely embedded in l_2, and an isomorphism S from h to l_2 such that the operator $B(t), t \in [0,T]$ on l_2 defined by

$$B(t) = SA(t)S^{\leftarrow} - A(t) \tag{C.4}$$

is bounded for all $t \in [0,T]$; moreover, $t \mapsto B(t)$ is continuous with respect to the operator norm

iii). $A(t)$ is a bounded operator from h to l_2 for all $t \in [0,t]$; moreover, $t \mapsto A(t)$ is continuous with respect to the operator norm

Before we check that these conditions are indeed satisfied, we define the Hilbert space h by

$$h = \{(u_1, u_2, ...) \in l_2 \mid \sum_{n=1}^{\infty} n^2 |u_n|^2 < \infty\}$$

$$((u_1, u_2, ...), (v_1, v_2, ...))_h = \sum_{n=1}^{\infty} n^2 \overline{u_n} v_n$$

It is not difficult to show that h is continuously and densely embedded in l_2. It is clear that $\mathcal{D}(N) = h$, that N is a self-adjoint, positive definite operator and that N is an isomorphism from h to l_2 such that

$$\|Nu\|_{l_2} = \|u\|_h \qquad \text{for all } u \in h \tag{C.5}$$

Now we show that the conditions i)-iii) above are indeed satisfied.
Condition i). We write:

$$A(t) = -Q(t) + A_1(t)$$

$$Q(t) = (\text{Re } c_0(t))N \qquad A_1(t) = -(\text{Im } c_0(t))N - \sum_{n=1}^{\infty} c_n(t) N J_n$$

It follows directly from results proved in [29] that $A(t), t \in [0,T]$ is a generator of a C_0-semigroup of contractions if

$$\text{Re } (u, A(t)u) \leq 0 \qquad \text{for all } u \in h \tag{C.6a}$$

$$\text{Re } (Q(t)u, A(t)u) \leq 0 \qquad \text{for all } u \in h \tag{C.6b}$$

We only prove the first inequality; the second one can be proved in a similar way. We first note that for all $n \in I\!N$ and all $k, l \in I\!N$:

$$\left(\sqrt{N} J_n \sqrt{N}^{\leftarrow}\right)_{kl} = \sum_{m,j=1}^{\infty} \sqrt{k} \delta_{km} \delta_{m+n\,j} \sqrt{l}^{-1} \delta_{jl} = \sqrt{k/l}\, \delta_{k+n\,l}$$

and this implies

$$\|\sqrt{N} J_n \sqrt{N}^{\leftarrow}\| \leq 1 \qquad n \in I\!N_0 \tag{C.7}$$

The inequality (C.6a) then follows from condition i) in the proposition as

$$\text{Re } (u, A(t)u) = -\text{Re } (u, Q(t)u) + \text{Re } (u, A_1(t)u)$$

and as

$$|\text{Re } (u, A_1(t)u)| \leq \left|(u, \sum_{n=1}^{\infty} c_n(t) N J_n u)\right| \leq \sum_{n=1}^{\infty} \left|c_n(t) \left(\sqrt{N} u, \sqrt{N} J_n \sqrt{N}^{\leftarrow} \sqrt{N} u\right)\right|$$

$$\leq \sum_{n=1}^{\infty} |c_n(t)|\, \|\sqrt{N} u\|^2 \leq (\text{Re } c_0(t)) |(u, Nu)| = (u, Q(t)u) \tag{C.8}$$

for all $u \in h$ and all $t \in [0, T]$.
Condition ii). We put $S = N$ and find

$$B(t) = \sum_{n=1}^{\infty} c_n(t) N \left(J_n - N J_n N^{\leftarrow}\right) \tag{C.9}$$

In order to show that $B(t)$ is a bounded operator on l_2 for all $t \in [0, T]$, we first note that

$$\left(N \left(J_n - N J_n N^{\leftarrow}\right)\right)_{kl} = \left(\frac{k(l-k)}{l}\right) \delta_{k+n\,l} \qquad k, l \in I\!N$$

for all $n \in I\!N_0$, and this implies

$$\|N \left(J_n - N J_n N^{\leftarrow}\right)\| \leq \sup_{k \in I\!N} \frac{kn}{k+n} = n \qquad n \in I\!N_0$$

It then follows from condition ii) in the proposition that the mapping $t \mapsto B(t)$ is continuous for all $t \in [0, T]$ as

$$\|B(t)\| \leq \sum_{n=1}^{\infty} n |c_n(t)| = M(t)$$

Condition iii). We have for all $u \in h$ and all $t \in [0, T]$:

$$\|A(t)u\|_{l_2} = \|\sum_{n=0}^{\infty} c_n(t) N J_n u\|_{l_2} \leq \sum_{n=0}^{\infty} |c_n(t)|\, \|N J_n u\|_{l_2}$$

$$= \sum_{n=0}^{\infty} |c_n(t)|\, \|J_n u\|_h \leq M(t) \|u\|_h$$

The continuity of the mapping $t \mapsto A(t)$ follows. □

Remark C.2 Let $L_2^-(\partial D)$ denote the subspace of $L_2(\partial D)$ containing those functions $u \in L_2(\partial D)$ such that

$$\int_0^{2\pi} e^{i(1-n)\theta} u(e^{i\theta})\, d\theta = 0 \qquad \text{for all } n \in \mathbb{N}_0$$

(This space $L_2^-(\partial D)$ can be identified with a particular Hardy-space, see e.g. [95]). Let P^- denote the projection operator from $L_2(\partial D)$ onto $L_2^-(\partial D)$. One checks that the Hilbert space h introduced in the proof is isomorphic to $P^-(H^1(\partial D))$, where $H^1(\partial D)$ denotes the Sobolev space of functions $u \in L_2(\partial D)$ with a generalized derivative u' in $L_2(\partial D)$. One also checks that the operator $B(t), t \in [0,T]$ on l_2 corresponds to the operator $\tilde{B}(t)$ on $L_2^-(\partial D)$ defined by:

$$\tilde{B}(t) \;:\; u \in L_2^-(\partial D) \;\mapsto\; P^-(\hat{B}(t)u) \in L_2^-(\partial D)$$

$$\left(\hat{B}(t)u\right)(e^{i\theta}) = -ie^{-i\theta}\frac{\partial}{\partial\theta}\left(\frac{\partial f(e^{i\theta},t)}{\partial\theta}\int_0^{\theta} e^{i\phi}u(e^{i\phi})\, d\phi\right)$$

It follows from this expression that the condition ii) in the proposition cannot be weakened very much; the estimates in the proof are quite sharp.

Remark C.3 We note that a formal substitution

$$\begin{aligned} \tilde{d}_n(t) &= e^{(n+1)Ct} d_n(t) && n \in \mathbb{Z} \\ \tilde{c}_n(t) &= e^{nCt} c_n(t) && n \in \mathbb{N} \\ \tilde{c}_0(t) &= c_0(t) + C \end{aligned}$$

for some $C \in \mathbb{C}$ in the equations (C.2b) (and (C.2a)) leads to a same set of equations with c_n and d_n replaced by $\tilde{c}_n$ and $\tilde{d}_n$. This observation can be used to recover some of the results obtained in Subsection 5.3.3: if it is assumed that the function f or Ξ can be extended analytically, then it is possible to choose $C \in \mathbb{R}_+$ such that the conditions in the proposition reduce to much weaker conditions. We do not go into the details of this consideration as the results will not be stronger than the results obtained before.

Remark C.4 The proof of inequality (C.6a) was completely based on the inequality Re $c_0(t) \geq \sum_{n=1}^{\infty} |c_n(t)|$. This implies that the uniqueness of the solution of initial value problem (C.3) can be proved without reference to condition ii) in the proposition. We show this result explicitly because we used it in the proof of Proposition 5.31.
Let $b : t \mapsto b(t) \in l_2$ be a solution of initial value problem (C.3) with $b_0 = 0$ and with $c : t \mapsto c(t) \in l_2$ such that it satisfies condition i) in the proposition. We get for all $t \in [0,T]$:

$$\frac{d}{dt}|b(t)|^2 = 2\mathrm{Re}\;(b(t), A(c(t))b(t)) \leq 0$$

This inequality, together with $|b(0)| = |b_0| = 0$, implies $|b(t)| = 0$, and hence $b(t) = 0$, for all $t \in [0,T]$.

We return to initial value problem (C.2). Proposition C.1 states that the initial value problem given by (C.2b) and (C.2c) has a unique solution under general conditions. The remaining part of the initial value problem, given by (C.2a) and (C.2c), can then easily be solved by iteration. Roughly speaking, the solutions $d_n, n \in I\!N_0$ exhibit in general a typically exponential behaviour

$$d_n \approx d_{n,0} e^{(n+1)\int_0^t c_0(\tau)d\tau}$$

and do not constitute a function Ξ which is in $L_2(\partial D)$ for a fixed $t > 0$. However, this does not mean that a Hopper equation does not have a solution in general: it is important to realize that the function θ that figures in a Hopper equation is undetermined and that the function Ξ is actually an abbrevation of the function $\Omega'\overline{\Omega}$ (we are not looking for a solution Ξ of initial value problem (C.1) but for a function Ω such that $\Xi = \Omega'\overline{\Omega}$ solves this initial value problem). So, the question is not whether initial value problem (C.2) can be solved for a given set of functions $a_n, n \in I\!N_0$, but the question is whether functions $a_n, n \in I\!N_0$ exist such that initial value problem (C.2) has a solution with the property that a time dependent (locally) conformal mapping Ω on $\overline{D} \times [0,T]$ exist such that for all $(\zeta,t) \in \partial D \times [0,T]$:

$$\Omega'(\zeta,t)\overline{\Omega}(\zeta,t) = \sum_{n=-\infty}^{\infty} d_n(t)\zeta^n$$

This question remains unanswered; we only make some comments on how this problem may be solved.

First one must try to show that for each function Ξ_- –in some proper space of functions on ∂D such as $H^1(\partial D)$, see remark C.2– with the property

$$\int_0^{2\pi} e^{-in\theta}\Xi_-(e^{i\theta})\, d\theta = 0 \qquad \text{for all } n \in I\!N_0$$

there exists a unique function Ξ_+ –in the same space of functions– with the property

$$\int_0^{2\pi} e^{in\theta}\Xi_+(e^{i\theta})\, d\theta = 0 \qquad \text{for all } n \in I\!N$$

such that the function $\Xi = \Xi_- + \Xi_+$ satisfies equation (5.28) (in some generalized sense).

Secondly, one shows that for each function d from $[0,T]$ to some subspace of l_2 –such as h, see again remark C.2– there is a function a from $[0,T]$ to l_2 such that the equations (C.2a)- are satisfied. This problem is probably not hard to solve because of the structure of the equations (C.2a).

In order to establish a proof of the existence of solutions of Hopper equations where the function f depends on Ω in a functional way, one should generalize Proposition C.1 in such a way that the function c may depend on the function b in a functional way. This is possible as the proof of this proposition is based on a theorem which has been generalized in this sense (see e.g. [42]).

Appendix D

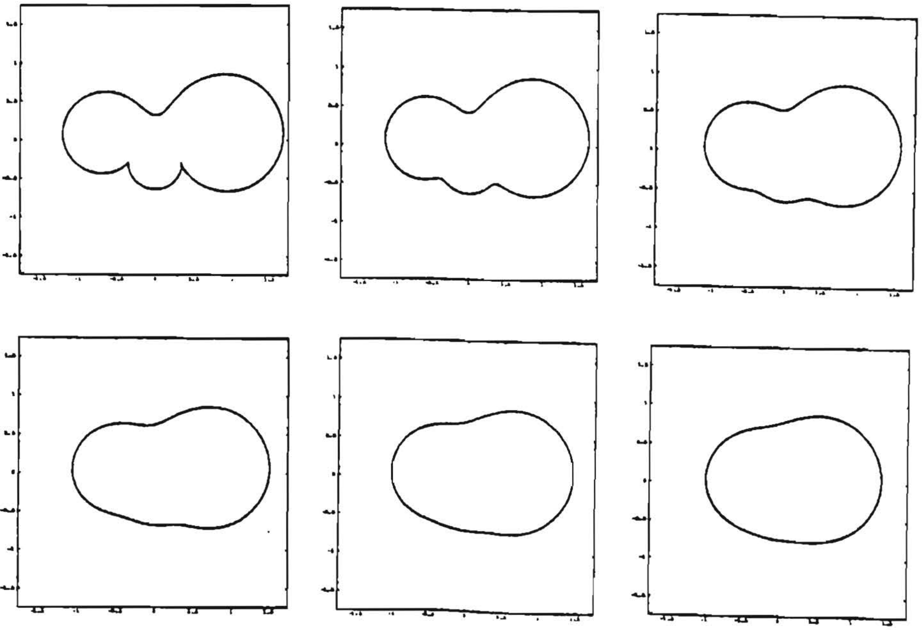
Some numerical results

It has been shown in Section 4.1 that a Hopper equation for a rational time-dependent conformal mapping is equivalent to a set of differential equations. In order to illustrate the usefulness of this equivalency, we present a numerical result from [7]. The figures below are the output of a Mathematica program which solves the mentioned equations in the following case (see also Example 4.5):

$$\Omega(\zeta, t) = \zeta \sum_{m=1}^{3} \frac{A_m(t)}{1 - \zeta_m(t)\zeta}$$

with initial given by

$$A_1(0) = 0.140 \quad A_2(0) = 0.240 \quad A_3(0) = 0.141$$
$$\zeta_1(0) = -0.850 \quad \zeta_2(0) = -0.500i \quad \zeta_3(0) = 0.900$$



Glossary and Index

$\mathbb{N}_0, \mathbb{N}_\infty$	$\mathbb{N} \cup \{0\}, \mathbb{N} \cup \{\infty\}$
$\mathbb{R}_+, \mathbb{R}_{0,+}$	$\{x \in \mathbb{R} \mid x > 0\}, \{x \in \mathbb{R} \mid x \geq 0\}$
D_r	$\{z \in \mathbb{C} \mid \lvert z\rvert < r\}$
$\overline{\mathbb{C}}$	$\mathbb{C} \cup \{\infty\}$, Riemann sphere
I	open interval of $\mathbb{R}$ containing 0
$\underline{v}$	vector (in a finite dimensional vector space)
$\underline{\underline{T}}$	tensor of order two
$(v_1 \ \ v_2)^T$	$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$
$S^{\leftarrow}$	inverse of operator or mapping S
$[a]$	largest integer smaller than or equal to number a
$\overline{f}$	i) complex conjugate or grade involution (see page 27) of function or number f ii) function obtained by reflecting function f with respect to the unit circle (see Definition 3.16)
f'	i) derivative of function f with respect to variable in complex plane or on unit circle (see Definition 3.18) ii) derivative of deformed analytic function (see page 17)
$\dot{f} = f^{\cdot}$	derivative of function f with respect to variable t

analytic The adjective analytic is used in the sense of holomorphic: a complex valued function on an open domain $G \subset \overline{\mathbb{C}}$ is analytic on G if it is single-valued and satisfies the Cauchy-Riemann equations. If a function is multiple-valued or meromorphic (and analytic in another, more general sense), it is always explicitly stated.

complex notation After identifying $\mathbb{R}^2$ with $\mathbb{C}$, we rewrite $f(x, y)$ as $f(z, \overline{z})$, $\underline{v} = (v_1 \ \ v_2)^T$ as $v = v_1 + iv_2$, etc..

conformal A (locally) conformal mapping on an open domain is nothing more or less than a (locally) univalent function. A conformal mapping on a closed domain $G \subset \mathbb{C}$ can be defined as an injective orientation preserving diffeomorphism with the property of preserving angles. The following characterization may be more convenient: Ω is (locally) injective on G, the restriction of Ω to the interior of G

is an analytic function and the derivative Ω' on the interior of G can be extended continuously to a non-vanishing function on G. A smooth time-dependent locally conformal mapping is defined in Definition 3.14.
domain Throughout the text, with the exception of Sections 3.1 and 3.2, we only consider *open domains*: open connected subsets of $\mathbb{R}^n$. A *closed domain* in the sense used in Sections 3.1 and 3.2 is: a subset of $\mathbb{R}^n$ which is the closure of an open domain. A *domain* is then an open or a closed domain.
univalent A function is said to be (locally) univalent if it is analytic and (locally) injective.

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Summary

The starting point of this study is Hopper's equation. This is an evolution equation for a time-dependent conformal mapping from the unit disc to a two-dimensional domain occupied by a viscous fluid satisfying Stokes' equations and driven by surface tension. Generalizations of this equation we call Hopper equations. We show that some other moving boundary problems from fluid mechanics can be treated by means of these generalizations. An example is the Hele-Shaw flow. As a consequence of the fact that various problems can be modelled by one type of equation, regularized and mixed models can be considered too. However, only the introductory chapter sketches some physical background of this all; our main interest lies in the mathematics.
At a first glance, Hopper's equation is somewhat peculiar since a "free" function figures in it. The values of this function are not prescribed but it has to be analytic. A direct construction of solutions seems to be almost impossible because of this indeterminacy. That is why we try to find solutions by making an Ansatz. That is, we parametrize a rational conformal mapping with an arbitrary number of poles of arbitrary order, substitute this into a Hopper equation and obtain a finite set of differential equations for the time-dependent parameters. It is proved by means of complex analysis and the theory of ordinary differential equations that this set of equations has a local solution. We stress that exact solutions can thus be obtained; we do not make any mathematical approximations. Thus we have generalized and proved Hopper's conjecture on the existence of polynomial and partial fraction solutions. Moreover, we have established a number of conserved quantities. On the basis of these quanities, global existence of solutions for a class of problems is demonstrated.
These considerations show a deep relationship between Hopper equations and time-dependent rational solutions. We reveal this relationship by studying a particular type of partial differential equation that we call the extended Löwner-Kufareev equation. We deduce how singularities such as poles and branch points of solutions of such an equation propagate. This leads to some rules for the propagation of singularities of solutions of Hopper equations. Such propagation rules in turn imply the conservation of certain properties of solutions of the aforementioned problems. For example, a solution that is polynomial, rational or algebraic

at a certain instant, must be polynomial, rational or algebraic at all times.
We finally approach Hopper equations from the theory of semi-groups of operators. We sketch the lines of a proof of the assertion that Hopper equations can be solved uniquely under general conditions. An important ingredient to this proof is the result that a conformal mapping Ω on the unit disc is determined by the values of $\Omega'\overline{\Omega}$ on the boundary of the disc.
The other parts of this thesis concern closely related subjects. We mention a few of them.
It has already been remarked that a "free function" appears in Hopper's equation. The origin of this function lies in the possibility to represent solutions of Stokes' equations in two dimensions by a pair of analytic functions. We prove that solutions of Stokes' equations in three dimensions can be represented by a pair of left monogenic functions with values in a Clifford algebra. If the solution is axially symmetric, it can be represented by "deformed" analytic functions.
We also treat moving boundary problems in a general framework. It turns out that many of these problems can be modelled by a quasi-linear Löwner-Kufareev equation. We prove a theorem on the existence of solutions of the related initial value problems by an iteration method.

Samenvatting

Het uitgangspunt van deze studie is Hopper's vergelijking. Dit is een evolutievergelijking voor een tijdsafhankelijke conforme afbeelding van de eenheidsschijf naar een twee-dimensionaal domein dat ingenomen wordt door een visceuze vloeistof die voldoet aan Stokes' vergelijkingen en beweegt onder de invloed van de oppervlaktespanning. Generalisaties van deze vergelijking noemen we Hopper-vergelijkingen. We tonen aan dat met deze generalisaties ook andere bewegende-randproblemen uit de stromingsleer te behandelen zijn, zoals bijvoorbeeld de Hele-Shaw-stroming. Als gevolg van het feit dat uiteenlopende problemen met één type vergelijking gemodelleerd kunnen worden, zijn ook voor de hand liggende mengvormen behandelbaar. We gaan in het inleidende hoofdstuk echter maar kort in op de fysische achtergrond van dit alles; de aandacht richt zich vooral op de wiskunde.
De Hopper-vergelijking is op het eerste gezicht wat merkwaardig, aangezien er een "vrije" functie in voorkomt waarvan alleen het karakter bepaald is: deze functie is analytisch, maar heeft geen a priori voorgeschreven waarden. Deze onbepaaldheid maakt dat een rechtstreekse constructie van oplossingen onmogelijk lijkt. Daarom wordt geprobeerd oplossingen te vinden via een Ansatz. Dat wil zeggen, we parametriseren een rationale conforme afbeelding met een willekeurig aantal polen van een willekeurige orde, substitueren deze in de Hopper-vergelijking en vinden een stelsel differentiaalvergelijkingen voor de tijdsafhankelijke parameters. Met behulp van complexe analyse en de theorie van gewone differentiaalvergelijken tonen we aan dat dit stelsel uniek oplosbaar is. Het blijkt nu dat aldus exacte oplossingen verkregen worden. Daarmee hebben we Hopper's hypothese over het bestaan van polynomiale oplossingen veralgemeend en bewezen. Bovendien hebben we gaandeweg een aantal behouden grootheden gevonden. Hiervan uitgaande bewijzen we voor een klasse van problemen het bestaan van globale oplossingen.

Het blijkt aldus dat er een diep verband bestaat tussen Hopper-vergelijkingen en tijdsafhankelijke rationale conforme afbeeldingen. Dit verband wordt inzichtelijk gemaakt door een partiële-differentiaalvergelijking te beschouwen die wij de uitgebreide Löwner-Kufareev-vergelijking noemen. We leiden af hoe singulariteiten zoals polen en vertakkingspunten van oplossingen hiervan zich voortplanten. Dit leidt dan tot een aantal regels voor de voortplanting van singulariteiten van

oplossingen van Hopper-vergelijkingen. Uit deze regels blijkt vervolgens dat zekere eigenschappen van oplossingen van de bovengenoemde problemen behouden blijven: een oplossing die op een tijdstip polynomiaal, rationaal of algebraïsch is, is dat voor alle tijden.
Ten slotte worden Hopper-vergelijkingen benaderd vanuit de theorie van semigroepen van operatoren. We schetsen de contouren van een bewijs dat Hopper-vergelijkingen onder zeer algemene voorwaarden uniek oplosbaar zijn. Een belangrijk ingrediënt hierbij is het resultaat dat een conforme afbeelding Ω op de eenheidsschijf vrijwel geheel bepaald is door de waarden van $\Omega'\overline{\Omega}$ op de rand van de eenheidsschijf.

De overige delen van het proefschrift behandelen onderwerpen die nauw verwant zijn aan deze beschouwingen. We noemen enkele hiervan.
We hebben al opgemerkt dat in Hopper's vergelijking een functie verschijnt waarvan alleen bekend is dat zij analytisch is. De herkomst van deze functie ligt in de mogelijkheid oplossingen van Stokes' vergelijkingen in twee dimensies te representeren door een stel analytische functies. We bewijzen dat oplossingen van Stokes' vergelijkingen in drie dimensies te representeren zijn door een stel links-monogene functies met waarden in een Clifford-algebra. In het axiaalsymmetrische geval kan men volstaan met, wat wij noemen, gedeformeerd analytische functies.
Verder worden bewegende-randproblemen behandeld in een algemeen kader. Het blijkt dat veel van deze problemen gemodelleerd kunnen worden door een quasi-lineaire Löwner-Kufareev-vergelijking. We bewijzen een existentiestelling over het bijbehorende beginwaardeprobleem door een iteratie-techniek te gebruiken.

Curriculum Vitea

26 december 1966	geboren te Amsterdam
mei 1985	eindexamen Atheneum B, Bernadinus College te Heerlen
september 1985– februari 1991	studie natuurkunde, Universiteit van Amsterdam afstudeerrichting: theoretisch fysica afstudeerscriptie: "Kaluza-Klein cosmologies"
vanaf mei 1991	assistent in opleiding, Technische Universiteit Eindhoven onderzoeksterrein: complexe analyse en Clifford analyse toegepast op problemen uit de stromingsleer

Stellingen behorende bij het proefschrift

Moving Boundary Problems in relation with equations of Löwner-Kufareev type

van B. Klein Obbink

-1-

Laat Ω op $\{\zeta \in \mathbb{C} \mid |\zeta| < 1\} \times [0, \infty)$ een tijdsafhankelijke conforme afbeelding zijn die een oplossing beschrijft van het bewegende-randprobleem voor Stokes-stroming gedreven door oppervlaktespanning. Als Ω op $t = 0$ polynomiaal, rationaal, algebraïsch, stervormig of convex is, dan is Ω dat voor alle $t \geq 0$.
Zie ook hoofdstuk 5 van dit proefschrift.

-2-

In de uitleg van het ontstaan van cusps in Hele-Shaw-stroming zoals gegeven in [1], worden de begrippen domein en bereik verwisseld; deze uitleg is dan ook onjuist.

-3-

De wiskundige problemen die ontstaan door singulariteiten in Hele-Shaw-stroming kunnen worden omzeild door een gemodificeerd model te beschouwen dat gebaseerd is op een regularisatie van de bijbehorende Hopper-vergelijking.
Zie ook hoofdstuk 3 van dit proefschrift.

-4-

Laat D de eenheidsschijf zijn in $\mathbb{C}$ en laat $L_2^-(\partial D)$ de ruimte zijn van quadratisch integreerbare functies f waarvoor geldt

$$\int_0^{2\pi} e^{-in\theta} f(e^{i\theta})\, d\theta = 0 \qquad \text{voor alle } n \in \mathbb{N}_0,$$
$$\int_0^{2\pi} e^{i\theta} f(e^{i\theta})\, d\theta \in \mathbb{R}_+.$$

Voor alle functies $\Xi_- \in L_2^-(\partial D)$ bestaat er een $c_0 \in \mathbb{R}$ zodanig dat voor alle $c > c_0$ er één functie $\Xi_+ \in L_2^-(\partial D)^\perp$ en één functie Ω in de Sobolev-ruimte $H^1(\partial D)$ bestaat zodanig dat:

i). $\overline{\Omega'} \in L_2^-(\partial D)$,

ii). de voortzetting van Ω' op $\overline{D}$ heeft geen nulpunten in D,

iii). $\Omega'(\zeta)\overline{\Omega(\zeta)} = \Xi_+(\zeta) + \Xi_-(\zeta) + c\zeta$ voor bijna alle $\zeta \in \partial D$.

-5-

Het is puur conservatisme dat oliemaatschappijen sommige gevallen van „coning" niet tegengaan door olie de grond *in* te pompen.

-6-

Het verschil tussen de ijktheorieën van Chisholm en Farwell en andere theorieën over contravariante afgeleiden van spinoren, is enkel een verschil in interpretatie van het begrip ijktransformatie.
Vergelijk [2],[3].

-7-

Zij $Cl_{0,n}$ de Clifford-algebra behorend bij een n-dimensionale ruimte met een negatief definiet inproduct. Het inproduct van twee basiselementen e_A en e_B, waarbij A en B geordende, niet-lege deelverzamelingen zijn van $V = \{1, .., n\}$, wordt gegeven door

$$e_A \cdot e_B = *\Big(\varepsilon_1(a,b) * e_A \wedge e_B + \varepsilon_2(a,b) e_A \wedge *e_B\Big),$$

waarbij $a = \sharp A$, $b = \sharp B$, $*$ de Hodge-ster-afbeelding is en ε_1 en ε_2 gegeven worden door

$$\varepsilon_1(a,b) = \left(\frac{2-\delta_{a,b}}{2}\right)(-1)^{[(b+1)/2]+ab+a+b+1}$$
$$\varepsilon_2(a,b) = \left(\frac{2-\delta_{a,b}}{2}\right)(-1)^{[(a+1)/2]+ab+(n-1)(a+b)+1},$$

waar [ ] het entier aangeeft.
Zie ook [4],[5].

-8-

De toepasbaarheid van de theorie van monogene functies met waarden in een Clifford-algebra is voor het oplossen van problemen uit de stromingsleer beperkt, doordat de compositie-eigenschap ontbreekt wanneer meer dan twee variabelen een rol spelen.

-9-

Wie de richting van de tijd wil begrijpen, moet vroeger opstaan.
Vergelijk [6].

-10-

Nederland is buiten Amsterdam nog te leeg.

Referenties

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