Minimal representations of convex polyhedral sets

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Abstract Necessary and sufficient conditions are given for an inequality vz equality involved in a linear system to be redundant, or for an inequality to be an implicit equality. These conditions are used to prove well-known necessary and sufficient conditions for a representation of a convex polyhedral set to be minimal in the sense that it involves a minimum number of linear relations. Moreover, an explicit relation is given between two minimal representations of the same convex polyhedral set.

Introduction We consider in R^n a finite, consistent system of linear inequalities and equalities

$$(1) \qquad Ax \le a \;, \quad Bx = b$$

and its set of solutions, the non-empty convex polyhedral set

$$(2) \qquad V := \{x \mid Ax \le a, \;\; Bx = b\}.$$

The convex polyhedral set (2) is said to be represented by the linear system (1). Since such a representation is in general not unique, one may be interested in conditions that guarantee a minimal representation i.e. a representation involving a minimum number of linear restrictions. This problem has been solved essentially already by Luenberger [1], and recently also by Telgen [2] both using mainly geometrical arguments. The treatment in this paper is completely based on a few duality statements on redundant inequalities or equalities occuring in linear systems and on relations stated in the system as inequalities that are actually equalities. The latter correspond to Luenberger's null-variables. It proves also possible to specify an explicit relation between two mimimal representations of the same convex polyhedral set, implicitly used already by Telgen [2] in the proof of his main redundancy theorem.

Redundancy and implicit equalities in linear systems

Notation: if $a_{i.}' x \le a_i$ or $b_{i.}' x = b_i$ is an unequality vz an equality involved in system (1), then

$$(3.1) \qquad \bar{A}x \le \bar{a} \quad , \quad Bx = b$$

or

$$(3.2) \qquad Ax \le a \quad , \quad \bar{B}x = \bar{b}$$

is the linear system obtained by deleting this inequality vz equality from system (1).

Definition 1: The inequality $a_{i.}' x \le a$ vz the equality $b_{i.}' x = b_i$ is called redundant with respect to system (1) in which it is involved, if it is satisfied by all solutions of (3.1) or (3.2), respectively.

Definition 2: The linear relations involved in $Bx = b$ are called explicit equalities of system (1).

Definition 3: The linear inequality $a_{i.}' x \le a_i$, involved in $Ax \le a$, is called an implicit equality of system (1) if $a_{i.}' x = a_i$ holds for all solutions of (1).

Lemma 1: The linear inequality $a_{i.}' x \le a_i$ is redundant with respect to system (1) if and only if the linear system

$$(4) \qquad \begin{aligned} \bar{A}'\bar{u} + B'v &= a_{i.} \\ \bar{a}'\bar{u} + b'v &\le a_i \quad , \quad \bar{u} \ge 0 \end{aligned}$$

is consistent.

Proof: From definition 1, $a_{i.}'x \leq a_i$ is redundant with respect to system (1) if and only if

$$(5) \qquad \max \{a_{i.}'x \mid \bar{A}x \leq \bar{a} \;, \;\; Bx = b\} \leq a_i .$$

By the assumption that system (1) is consistent, the linear program in (5) is feasible and (5) states that it must have a finite solution. By the duality theorem for linear programs, this happens if and only if

$$\min \{\bar{a}'\bar{u} + b'v \mid \bar{A}'\bar{u} + B'v = a_{i.} , \;\; u \geq 0\} \leq a_i$$

hence if and only if (4) is satisfied.

Lemma 2: The linear inequality $a_{i.}'x \leq a_i$ is an implicit equality in system (1) if and only if the linear system

$$(6) \qquad \begin{aligned} &\bar{A}'\bar{u} + B'v + a_{i.} = 0 \\ &\bar{a}'\bar{u} + b'v + a_i = 0 \;, \quad \bar{u} \geq 0 \end{aligned}$$

is consistent.

Proof: The linear inequality $a_{i.}'x \leq a_i$ is an implicit equality in system (1) if and only if the reverse inequality $a_{i.}'x \geq a_i$, if adjoint to system (1), is redundant in the new system. By lemma 1, this is equivalent to the consistency of the linear system

$$(7) \qquad \begin{aligned} &A'u + B'v = -a_{i.} \\ &a'u + b'v \leq -a_i \;, \quad u \geq 0. \end{aligned}$$

Hence, for any solution x of (1) and any solution (u,v) of (7) we have

$$(8) \qquad u'(Ax - a) + v'(Bx - b) \geq -(a_{i.}'x - a_i) = 0.$$

Recalling $Ax \leq b$, $Bx = b$ and $u \geq 0$, the left-hand side of (8) is always non-positive. Hence in (8) and therefore also in (7) the equality sign holds for any solution of (7), which shows that (7) is equivalent to (6).

Corollary 1: [2]. If the set of inequalities $Ax \leq a$ in system (1) is not empty, this system contains implicit inequalities if and only if the linear system

$$(9) \qquad \begin{aligned} & A'u + B'v = 0 \\ & a'u + b'v = 0 \quad , \quad u \geq 0 \quad , \quad u \neq 0 \end{aligned}$$

is consistent.
The inequality $a_{i.}'x \leq a_i$ is an implicit equality of (1) if and only if system (9) has a solution (u,v) in which $u_i > 0$.

Proof: If x is any solution of (1) and (u,v) is any solution of (9), then

$$(10) \qquad u'(Ax - a) + v(Bx - b) = 0.$$

Since $Ax \leq a$, $Bx = b$ and $u \geq 0$, any term $u_i(a_{i.}'x - a_i)$ in the left-hand side of (10) must be zero.
If $a_{i.}'x \leq a_i$ is not an implicit equality in (1), hence if (1) has a solution x with $a_{i.}'x < a_i$, the corresponding component u_i of any solution (u,v) of (9) must be zero.
The remaining statements in the corollary are direct consequences of lemma 2.

Corollary 2: [3]. If the set of inequalities $Ax \leq a$ in system (1) is not empty, it can be partitioned uniquely into two subsets

$$(11) \qquad A_{in}x \leq a_{in} \quad , \quad A_{eq}x \leq a_{eq}$$

such that the linear system

$$A_{in}x \le a_{in} \quad , \quad A_{eq}x = a_{eq} \quad , \; Bx = b \tag{12}$$

is equivalent to system (1) and such that system (12) does not contain any implicit equality.
Moreover, the linear system

$$\begin{aligned} &A'_{eq}\, u_{eq} + B'v = 0 \\ &a'_{eq}\, u_{eq} + b'v = 0 \quad , \quad u_{eq} > 0 \end{aligned} \tag{13}$$

is consistent.

Proof: Obviously by lemma 2 and corollary 1, $A_{eq}x \le a_{eq}$ is the (possibly empty) set of implicit equalities of system (1).

Lemma 3: The linear equality $b'_{i.}\, x = b_i$ is redundant in system (1) if and only if the linear system

$$\begin{aligned} &A'_{eq}\, u_{eq} + \bar{B}'\bar{v} = b_{i.} \\ &a'_{eq}\, u_{eq} + \bar{b}'\bar{v} = b_i \end{aligned} \tag{14}$$

is consistent, hence if and only if it can be written as a linear combination of the explicit and implicit equalities of system (1).

Proof: For any solution x of (1) and any solution (u_{eq}, v) of (14) we have

$$u'_{eq}(A_{eq}x - a_{eq}) + v'(\bar{B}x - \bar{b}) = b_{i.}\, x - b_i \; . \tag{15}$$

Since $A_{eq}x = a_{eq}$ and $\bar{B}x = \bar{b}$ for any solution of (1), it follows that if $b'_{i.}\, x = b_i$ for any solution of (3.2), hence if $b_{i.}\, x = b_i$ is redundant in system (1), it is also redundant in system (12). But then $b_{i.}' x \geq b_i$ must be an implicit equality in the system

$$(16) \qquad A_{in} x \leq a_{in} \;,\; A_{eq} x = a_{eq} \;,\; b_{i.}' x \geq b_i \;,\; \bar{B}x = \bar{b}.$$

By lemma 2, this implies that the linear system

$$(17) \qquad \begin{aligned} &A_{in}' u_{in} + A_{eq}' u_{eq} + \bar{B}'\bar{v} = b_{i.} \\ &a_{in}' u_{in} + a_{eq}' u_{eq} + \bar{b}'\bar{v} = b_i \qquad u_{in} \geq 0 \end{aligned}$$

is consistent.

Since, however, system (12) does not contain implicit equalities it follows from corollary 1 that $u_{in} = 0$ for any solution $(u_{in}, u_{eq}, \bar{v})$ of (17). Hence, the consistency of (17) is equivalent to the consistency of (14).

Minimal representations The representation (1) of the convex polyhedral set (2) is called minimal if there is no other representation containing less linear restrictions then contained in (1).

Clearly a linear system containing redundant inequalities or redundant equalities is never a minimal representation of its solution set. However we also have

Lemma 4: The linear system (1) is not a minimal representation of the convex polyhedral set (2) if it contains an implicit equality.

Proof: If system (1) contains an implicit equality, then by corollary 2, the convex polyhedral set (2) may also be represented by the linear system (12), which contains the same number of linear restrictions as system (1). Again by corollary 2, the linear system (13) has a solution. Hence if $a_{i.}' x \leq a_i$ is an implicit equality of (1) it follows that the equality $a_{i.}' x = a_i$ in (12) can be expressed as a linear combination of the other implicit and explicit equalities. But then, by lemma 3, this equality is redundant in (12) proving that (12) and hence (1) is not a minimal representation of (2).

Theorem 1: [1], [2]. The linear system (1) is a minimal representation of the convex polyhedral set (2) if and only if it contains neither redundant equalities and inequalities nor implicit equalities.

Proof: The necessity part of theorem 1 follows immediately from the concept of redundancy and from lemma 4. The sufficiency part is also expressed in the first part of the following theorem 2.

Theorem 2: If the two systems

(18.1) $Ax \leq a \quad , \quad Bx = b$

an

(18.2) $Sx \leq s \quad , \quad Tx = t$

do not contain redundant inequalities and equalities nor implicit equalities, and they are both representations of the same convex polyhedral set, then it are both minimal representations of this set.
Moreover there exist

. a non-singular square matrix R such that

(19) $$B = RT \qquad b = Rt$$

. a diagonal matrix D with positive diagonal elements, a permutation matrix P and a matrix Q such that

(20) $$A = DPS + QT \qquad a = DPs + Qt \; .$$

If on the other hand such matrices R,D,P and Q exist and one of the systems (18) is a minimal representation of a convex polyhedral set then the other system is also a minimal representation of this set.

The second theorem expresses some intuitively clear geometrical properties of minimal representations, already indicated in the proof of theorem ' given by Telgen [2]:

a two minimal representations of the same convex polyhedral set contain both the same number of equalities and the same number of inequalities.

b any equality in the first representation is a linear combination of those in the second representation.

c there is a 1 - 1 relation between the inequalities in both systems such that any inequality in the first system is a positiv multiple of its associated inequality plus possibly a linear combination of the equalities of the second system.

Proof: To begin with the last part, it is easily checked that the existence of the matrices R,D and P as specified in the theorem imply that both systems (18.1) and (18.2) have the same solution set, hence are representations of the same convex polyhedral set. Moreover, they contain the same number of linear relations; hence if one of them is a minimal reoresentation then also the other one.

The remainder of the proof is based on the observation that any relation in one of the systems is redundant if it is adjoint to the other system. So $b_{i.}'x = b_i$ is redundant if adjoint to (18.2). Hence, by lemma 2, taking into account that system (18.2) does not contain implicit equalities, there exists a vector $r_{i.}$ such that

$$b_{i.}' = r_{i.}'T \qquad\qquad (21)$$
$$b_i = r_{i.}'t \; .$$

The vector $r_{i.}$ is unique since otherwise the set of equalities in (18.2) would show linear dependency, hence (18.2) would contain a redundant equality.

Doing this for all equalities of (18.1), denoting the matrix consisting of the row vectors $r_{i.}'$ by R, we get the relations (19). The rows of R are linearly independent since otherwise the equalities in system (18.1) would show dependency, hence redundancy. Finally, since (19) implies that the rows of B belong to the linear subspace spanned by the rows of T and by a similar reasoning we can show that the rows of T belong to the linear subspace spanned by the rows of B, the matrix R is square and non-singular.

Moreover, both systems must contain the same number of explicit equalities. Again, the inequality $a_{i.}'x \leq a_i$ is redundant if adjoint to system (18.2). By lemma 1, a non-negative vector $u_{i.}$ and a vector $q_{i.}$ exist such that

(22.1) $\quad a_{i.}' = u_{i.}'B + q_{i.}'T$

(22.2) $\quad a_i \geq u_{i.}'s + q_{i.}'t$.

Doing this for all i, denoting the matrix consisting of the row vectors $u_{i.}'$ and $q_{i.}'$ by U and Q, respectively, we get the relations

(23.1) $\quad A = US + QT$

(23.2) $\quad a \geq Us + Qt$

where U is a matrix with non-negative elements. Since also any inequality $s_{i.}'x \leq s_i$ is redundant if adjoint to system (18.1), a non-negative vector $y_{i.}$ and a vector $z_{i.}$ exist such that

(24.1) $\quad s_{i.}' = y_{i.}'A + z_{i.}'B$

(24.2) $\quad s_i \geq y_{i.}'a + z_{i.}'b$

Using (19) and the relations (22.1) we obtain.

(25.1) $\quad s'_{i.} = y'_{i.}US + (y'_{i.}Q + z'_{i.}R)T$

(25.2) $\quad s_i \geq y'_{i.}Us + (y'_{i.}Q + z'_{i.}R)t$

Denoting by $\bar{U},\bar{S}$ and $\bar{s}$ the matrices obtained from U,S and s by deleting the column $u_{.i}$, the row $\bar{s}_{i.}$ and the element s'_i, respectively, we may write

(26.1) $\quad (1 - y'_{i.}u_{.i})s'_{i.} = y_{i.}\bar{U}\bar{S} + (y'_{i.}Q + z'_{i.}R)T$

(26.2) $\quad (1 - y'_{i.}u_{.i})s_i \geqslant y_{i.}\bar{U}\bar{s} + (y'_{i.}Q + z'_{i.}R)t$.

Since system (14.2) does not contain implicit equalities it has a solution $\hat{x}$ such that

(27) $\quad S\hat{x} < s \quad , \quad T\hat{x} = t$

hence by (26)

(28) $\quad (1 - y'_{i.}u_{.i})(s'_{i.}\hat{x} - s_i) \quad y'_{i.}\bar{U}(\bar{S}\hat{x} - \bar{s}) \leq 0$

which implies by $y_{i.} \geq 0$ and $\bar{u} \geq 0$ that $1 - y'_{i.}u_{.i} \geq 0$.
If $1 - y'_{i.}u_{.i} > 0$ then it follows from (26) and lemma 1 that $s'_{i.}x \leq s_i$ would be redundant in system (14.2), which by assumption is excluded. Hence

(29.1) $\quad y'_{i.}w_{.i} = 1 \quad$ for all i.

But then, the lefthand side in (28) in zero and

(29.2) $\quad y'_{i.}u_{.j} = 0 \;$ if $i \neq j$.

The relations (29.1) and (29.2) imply that the first inequality sign in (28) must be an equality which is only possible if (24.2) is an equality for all i:

(30.1) $s'_{i.} = g'_{i.}A + z'_{i.}B$

(30.2) $s_i = y'_{i.}a + z'_{i.}b.$

By a similar reasoning, also the inequality (23.2) is actually an equality. From this we may conclude that for all i, at least one component of $y_{i.}$ must be positive since otherwise $s'_{i.}x \leq s_{i.}$ would be satisfied as an equality for all solutions of system (18.1), hence it would be an implicit equality of system (18.2) which, by assumption, is not possible.

We now prove that both systems (18.1) and (18.2) must contain the same number of inequalities. Assuming that system (18.2) contains more inequalities then system (18.1) the set vectors $y_{i.}$ must show linear dependency. Denoting the matrix consisting of the row vectors $y'_{i.}$ by Y, a vector $w \neq 0$ must then exist such that $w'Y = 0$.

However, writing the relations (29.1) and (29.2) in the matrix form

(31) $YU = I$

where I is a unitmatrix of proper dimensions, one would conclude that

$$w' = w'YU = 0$$

which is impossible for $w \neq 0$.

This proves that Y and U are square matrices. Since they must have non-negative elements, relation (31) can be satisfied only if both Y and U have the property that any row and any column contains exactly one positive element. This means that both matrices may be written as the product of a diagonal matrix with positive diagonal elements and a permutation matrix.

So, in particular, a diagonal matrix D and a permutation matrix P exist such that

(32) $U = DP.$

Since it has been remarked already that the equality sign must hold in (23.2), substitution of (32) in (24.1) and (24.2) delivers the relations (20).

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