The singular minimum entropy H_∞ control problem

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Abstract

In this paper we search for controllers which minimize an entropy function of the closed loop transfer matrix under the constraint of internal stability and under the constraint that the closed loop transfer matrix has H_∞ norm less than some a priori given bound γ. We find an explicit expression for the infimum. Moreover, we give a charaterization when the infimum is attained (contrary to the regular case, for the singular minimum entropy H_∞ control problem the infimum is not always attained).

Keywords H_∞ control, algebraic Riccati equation, quadratic matrix inequality, minimum entropy.

1 Introduction

The H_∞ control problem has been investigated extensively in the last decade. Via several techniques a complete solution is now available (see e.g. [3, 4, 8, 9]). However, most of these results were derived under two kinds of essential assumptions:

- Two subsystems do not have invariant zeros on the imaginary axis.
- Two direct feedthrough matrices are injective and surjective respectively.

These assumptions were removed in [14, 15, 17, 18].
The minimum entropy H_∞ control problem is defined as the problem of minimizing an entropy function under the constraint of internal stability of the closed loop system and under the constraint of an upper bound γ on the H_∞ norm of the closed loop transfer matrix. Under the standard assumptions mentioned above, this problem was solved in [10]. It was shown that there always exists a minimizing controller which is often called the "central controller". The latter is due to the fact that the parametrization (given in [3]) of all internally stabilizing controllers which yield a closed loop system with H_∞ norm less than γ, is centered around this specific controller.
It should be noted that the interest in minimizing this entropy function is related to the fact that the entropy function is an upper bound to the cost function used in the Linear Quadratic Gaussian control problem. Therefore, it is hoped that by minimizing this entropy function, one is minimizing the LQG cost criterion at the same time. This has, as far as we

know, never been proven but this is the reason why the problem we discuss in this paper, is sometimes referred to as the mixed LQG/H_∞ control problem. The reason for investigating this mixed problem is the well-known fact that controllers which are optimal for the LQG control problem are in general not robust, i.e. they are sensitive for perturbations on the system parameters. By using H_∞ constraints it is hoped that this sensitivity is reduced.
In this paper we extend the results of [10] to so-called singular systems, i.e. systems which do not necessarily satisfy the above mentioned assumptions on the direct feedthrough matrices. We still exclude invariant zeros on the imaginary axis. This paper is in essence a combination of the results in [10, 19]. Note that the singular LQG control problem has been investigated in [20].

2 Problem formulation and results

Consider the linear time-invariant system:

$$\Sigma : \begin{cases} \dot{x} = Ax + Ew + Bu, \\ z = C_1 x + D_1 u, \\ y = C_2 x + D_2 w, \end{cases} \tag{2.1}$$

Here A, B, E, C_1, C_2, D_1 and D_2 are real matrices of suitable dimension. Let G be a strictly proper real rational matrix which has no poles on the imaginary axis and which is such that

$$\|G\|_\infty := \sup_{\omega \in \mathcal{R}} \|G(i\omega)\| < \gamma.$$

where $\|\cdot\|$ denotes the largest singular value. For such a transfer matrix G, we define the following entropy function:

$$\mathcal{J}(G,\gamma) := -\frac{\gamma^2}{2\pi} \int_{-\infty}^{\infty} \ln\det\left(I - \frac{1}{\gamma^2} G^\sim(i\omega) G(i\omega)\right) d\omega \tag{2.2}$$

where $G^\sim(s) := G^{\mathrm{T}}(-s)$. The following equality is easily derived using the Lebesgue dominated convergence theorem:

$$\mathcal{J}(G,\gamma) := \lim_{s\to\infty} -\frac{\gamma^2}{2\pi} \int_{-\infty}^{\infty} \ln\det\left(I - \frac{1}{\gamma^2} G^\sim(i\omega) G(i\omega)\right) \left(\frac{s^2}{s^2+\omega^2}\right) d\omega$$

The latter expression was used in [10]. The minimum entropy H_∞ control problem for given γ is then defined as:

> infimize $\mathcal{J}(G_{cl},\gamma)$ over all controllers which yield a strictly proper, internally stable closed loop transfer matrix G_{cl} with H_∞ norm strictly less than γ.

We will investigate controllers of the form:

$$\Sigma_F : \begin{cases} \dot{p} = Kp + Ly, \\ u = Mp + Ny. \end{cases} \tag{2.3}$$

It should be noted that the above class of controllers is restrictive since it will be shown that in general for the singular problem the infimum is attained only by a non-proper controller.
A central role in our study of the above problem will be played by the *quadratic matrix inequality.* For any $\gamma > 0$ and matrix $P \in \mathcal{R}^{n\times n}$ we define the following matrix:

$$F_\gamma(P) := \begin{pmatrix} A^{\mathrm T}P + PA + C_1^{\mathrm T}C_1 + \gamma^{-2}PEE^{\mathrm T}P & PB + C_1^{\mathrm T}D_1 \\ B^{\mathrm T}P + D_1^{\mathrm T}C_1 & D_1^{\mathrm T}D_1 \end{pmatrix}. \tag{2.4}$$

We also define a dual version of this quadratic matrix inequality. For any $\gamma > 0$ and matrix $Q \in \mathcal{R}^{n\times n}$ we define the matrix:

$$G_\gamma(Q) := \begin{pmatrix} AQ + QA^{\mathrm T} + EE^{\mathrm T} + \gamma^{-2}QC_1^{\mathrm T}C_1Q & QC_2^{\mathrm T} + ED_2^{\mathrm T} \\ C_2Q + D_2E^{\mathrm T} & D_2D_2^{\mathrm T} \end{pmatrix}. \tag{2.5}$$

In addition to these two matrices we define two polynomial matrices, whose role is again completely dual:

$$L_\gamma(P,s) := \begin{pmatrix} sI - A - \gamma^{-2}EE^{\mathrm T}P & -B \end{pmatrix} \tag{2.6}$$

$$M_\gamma(Q,s) := \begin{pmatrix} sI - A - \gamma^{-2}QC_1^{\mathrm T}C_1 \\ -C_2 \end{pmatrix} \tag{2.7}$$

We note that $L_\gamma(P,s)$ and $M_\gamma(Q,s)$ are the controllability and observability pencil respectively of related systems. Finally, we define the following two transfer matrices:

$$G_{ci}(s) := C_1\,(sI - A)^{-1}\,B + D_1, \tag{2.8}$$

$$G_{di}(s) := C_2\,(sI - A)^{-1}\,E + D_2. \tag{2.9}$$

In the formulation of our main result we also require the concept of *invariant zero* of the system $\Sigma = (A,B,C,D)$. These are all $s \in \mathcal{C}$ such that

$$\operatorname{rank}\begin{pmatrix} sI - A & -B \\ C & D \end{pmatrix} < \operatorname{normrank}\begin{pmatrix} sI - A & -B \\ C & D \end{pmatrix}. \tag{2.10}$$

Here "*normrank*" denotes the rank of a matrix as a matrix with entries in the field of rational functions. Moreover let $\mathcal{C}^+$ ($\mathcal{C}^0$, $\mathcal{C}^-$) denote all $s \in \mathcal{C}$ such that Re $s > 0$ (Re $s = 0$, Re $s < 0$). Finally, let $\rho(M)$ denotes the spectral radius of the matrix M. We first formulate the main result from [18]:

Theorem 2.1 : *Consider the system (2.1). Assume that both the system (A,B,C_1,D_1) as well as the system (A,E,C_2,D_2) have no invariant zeros on the imaginary axis. Then the following two statements are equivalent:*

(*i*) *For the system (2.1) there exists a time-invariant, finite-dimensional dynamic compensator Σ_F of the form (2.3) such that the resulting closed loop system, with transfer matrix G_{cl}, is internally stable and has H_∞ norm less than γ, i.e. $\|G_{cl}\|_\infty < \gamma$.*

(*ii*) *There exist positive semi-definite solutions P,Q of the quadratic matrix inequalities $F_\gamma(P) \geq 0$ and $G_\gamma(Q) \geq 0$ satisfying $\rho(PQ) < \gamma^2$, such that the following rank conditions are satisfied*

(*a*) *rank $F_\gamma(P) =$ normrank G_{ci},*

(b) *rank* $G_\gamma(Q) =$ *normrank* G_{di},

(c) *rank* $\begin{pmatrix} L_\gamma(P,s) \\ F_\gamma(P) \end{pmatrix} = n +$ *normrank* G_{ci} $\qquad \forall\, s \in \mathcal{C}^0 \cup \mathcal{C}^+$,

(d) *rank* $\begin{pmatrix} M_\gamma(Q,s) & G_\gamma(Q) \end{pmatrix} = n +$ *normrank* G_{di} $\qquad \forall\, s \in \mathcal{C}^0 \cup \mathcal{C}^+$. $\square$

It has also been shown in [18] that P and Q satisfying the conditions in part (ii) are unique and can be calculated by solving reduced order Riccati equations. Note that the existence of such P and Q guarantees that the system Σ is detectable from y and stabilizable by u. To present our main result we need another definition:

Definition 2.2 : *We define the detectable strongly controllable subspace* $\mathcal{T}_g(A,B,C,D)$ *as the smallest subspace* $\mathcal{T}$ *of* $\mathcal{R}^n$ *for which there exists a linear mapping* G *such that* $A+GC \mid \mathcal{R}^n/\mathcal{T}$ *is asymptotically stable and such that the following subspace inclusions are satisfied:*

$$(A+GC)\mathcal{T} \subseteq \mathcal{T}, \tag{2.11}$$

$$Im\,(B+GD) \subseteq \mathcal{T}. \tag{2.12}$$

We also define the stabilizable weakly unobservable subspace $\mathcal{V}_g(A,B,C,D)$ *as the largest subspace* $\mathcal{V}$ *for which there exists a mapping* F *such that* $A+BF \mid \mathcal{V}$ *is asymptotically stable and such that the following subspace inclusions are satisfied:*

$$(A+BF)\mathcal{V} \subseteq \mathcal{V}, \tag{2.13}$$

$$(C+DF)\mathcal{V} = \{0\}. \tag{2.14}$$

We can now formulate the main result from this paper:

Theorem 2.3 : *Consider the system (2.1). Let* $\gamma > 0$ *be given. Assume that the systems* (A,B,C_1,D_1) *and* (A,E,C_2,D_2) *have no invariant zeros on the imaginary axis and assume that there exists a controller which is such that the closed loop system is internally stable and has* H_∞ *norm strictly less than* γ. *The infimum of (2.2), over all internally stabilizing controllers of the form (2.3) which are such that the closed system has* H_∞ *norm strictly less than* γ, *is equal to:*

$$\text{Trace}\; E^{\mathrm{T}}PE + \text{Trace}\,(A^{\mathrm{T}}P + PA + C_1^{\mathrm{T}}C_1 + \gamma^{-2}PEE^{\mathrm{T}}P)(\gamma^2 I - QP)^{-1}Q$$

where P *and* Q *are such that part (ii) of theorem 2.1 is satisfied. The infimum is attained if and only if*

(i) $Im\,E_Q \subseteq (I-\gamma^{-2}QP)\mathcal{V}_g(\Sigma_{ci}) + B\,Ker\,D_1$

(ii) $Ker\,C_{1,P} \supseteq (I-\gamma^{-2}QP)^{-1}\mathcal{T}_g(\Sigma_{di}) \cap C_2^{-1}\,Im\,D_2$

(iii) $(I-\gamma^{-2}QP)\mathcal{V}_g(\Sigma_{ci}) \supseteq \mathcal{T}_g(\Sigma_{di})$

where $\Sigma_{ci} = (A + \gamma^{-2}EE^{\mathrm{T}}P, B, C_{1,P}, D_P)$ *and* $\Sigma_{di} = (A + \gamma^{-2}QC_2^{\mathrm{T}}C_2, E_Q, C_2, D_Q)$. *The matrices* $C_{1,P}, D_P, E_Q, D_Q$ *are arbitrary matrices satisfying*

$$F_\gamma(P) = \begin{pmatrix} C_{1,P}^{\mathrm{T}} \\ D_P^{\mathrm{T}} \end{pmatrix} \begin{pmatrix} C_{1,P} & D_P \end{pmatrix}, \quad G_\gamma(Q) = \begin{pmatrix} E_Q \\ D_Q \end{pmatrix} \begin{pmatrix} E_Q^{\mathrm{T}} & D_Q^{\mathrm{T}} \end{pmatrix}. \tag{2.15}$$

□

Remarks:

(*i*) In the system (2.1) we have two direct feedthrough matrices which are identical to zero. We can extend the above result to the more general case with all direct feedthrough matrices possibly unequal to zero by loop-shifting arguments (see [13, 19]). The entropy function we investigate in this paper behaves neatly under loop-shifting as can be seen from lemma 3.5.

(*ii*) It is straightforward to prove that the conditions (i)-(iii) are independent of the particular factorizations chosen in (2.15). It can also be shown that conditions (i)-(iii) are automatically satisfied if D_1 and D_2 are injective and surjective respectively.

(*iii*) We will only prove this result for $\gamma = 1$. The general result can then easily be derived by scaling.

3 Properties of the entropy function

In this section we recall some basic properties of the entropy function as defined in (2.2). These properties were derived in [10] but we give seperate proofs because we only investigate what is called in [10] "entropy at infinity". This enables us to derive more straightforward proofs. We first define the property of being inner.

Definition 3.1 : *A proper rational transfer matrix* G *is called inner if* G *is a stable square rational matrix such that* $G^{\sim}G = I$. *A system* Σ *is called inner if the system is internally stable and its transfer matrix is inner.* □

We can derive the following properties of our entropy function (2.2):

Lemma 3.2 : *Let* G *be a strictly proper, stable rational matrix and let* γ *be such that* $\|G\|_\infty \leq \gamma$. *Then we have*

- $\mathcal{J}(G,\gamma) \geq 0$ *and* $\mathcal{J}(G,\gamma) = 0$ *implies* $G = 0$.
- $\mathcal{J}(G,\gamma) = \mathcal{J}(G^{\sim},\gamma) = \mathcal{J}(G^{\mathrm{T}},\gamma)$. □

Proof : Straightforward. ∎

Next, we relate our entropy function to the LQG cost-criterion. First we define the LQG cost-criterion:

Definition 3.3 : *Let Σ be given by*

$$\Sigma : \begin{cases} dx = Ax\,dt + C dw, \\ z \ = Cx. \end{cases} \tag{3.1}$$

Assume that A is stable. Let w be a standard Wiener process and define the solution to the first equation (which is a stochastic differential equation) via Wiener integrals. Then the associated LQG cost is defined as:

$$\mathcal{C}(G) := \lim_{s\to\infty} \mathcal{E} \left\{ \frac{1}{s} \int_0^s z^{\mathrm{T}}(t) z(t) dt \right\},$$

where $\mathcal{E}$ denotes the expectation with respect to the noise. □

Using the above definition of the LQG cost we find:

Lemma 3.4 : *Let Σ be defined by (3.1). Assume that A is stable and let G be the transfer matrix from dw to z. For $\gamma \geq \|G\|_\infty$, the function $\mathcal{J}(G,\gamma)$ is a monotonically decreasing function of γ such that*

$$\mathcal{J}(G,\gamma) \downarrow \mathcal{C}(G) \qquad \text{as } \gamma \to \infty.$$ □

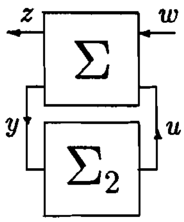
Proof : It is well known that the LQG cost is equal to Trace $B^{\mathrm{T}}\tilde{X}B$ where $\tilde{X}$ is the unique solution of the following Lyapunov equation:

$$\tilde{X}A + A^{\mathrm{T}}\tilde{X} + C^{\mathrm{T}}C = 0$$

A proof of lemma 3.4 can then be based upon corollary 3.7, by showing that $X \geq \tilde{X}$. ∎

Next, we give two key lemmas. Of the first lemma, the first part stems from [3, 19] while the second part originates from [10]. We give a seperate proof of the second part.

Lemma 3.5 : *Suppose that two systems Σ and Σ_2, both described by some state space representation, are interconnected in the following way:*



(3.2)

Assume that the system Σ is inner. Moreover, assume that its transfer matrix G has the following decomposition:

$$G\begin{pmatrix} w \\ u \end{pmatrix} =: \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix}\begin{pmatrix} w \\ u \end{pmatrix} = \begin{pmatrix} z \\ y \end{pmatrix} \tag{3.3}$$

such that $G_{21}^{-1} \in H_\infty$ and such that G_{11}, G_{22} are strictly proper. Under the above assumptions the following two statements are equivalent:

(*i*) *The closed loop system (3.2) is internally stable and its closed loop transfer matrix G_{cl} has H_∞ norm less than* 1.

(*ii*) *The system Σ_2 is internally stable and its transfer matrix G_2 has H_∞ norm less than* 1.

Moreover, G_{cl} is strictly proper if and only if G_2 is strictly proper. Finally, if (i) holds and G_2 is strictly proper then the following relation between the entropy functions for the different transfer matrices is satisfied:

$$\mathcal{J}(G_{cl}, 1) = \mathcal{J}(G_{11}, 1) + \mathcal{J}(G_2, 1). \tag{3.4}$$

□

Proof : The first claim that the statements (i) and (ii) are equivalent, has been shown in [3, 19]. We know that G_{11} and G_{22} are strictly proper. Combined with the fact that G is inner, this implies that G_{12} and G_{21} are bicausal. Using this, it is trivially checked that G_2 is strictly proper if and only if G_{cl} is strictly proper. Remains to show (3.4). The following equality is easily derived using the property that Σ is inner:

$$I - G_{cl}^{\sim} G_{cl} = G_{21}^{\sim} \left(I - G_2^{\sim} G_{22}^{\sim}\right)^{-1} \left(I - G_2^{\sim} G_2\right) \left(I - G_{22} G_2\right)^{-1} G_{21}$$

Therefore, we find that

$$\ln\det\left(I - G_{cl}^{\sim} G_{cl}\right) = \ln\det\left(I - G_{11}^{\sim} G_{11}\right) + \ln\det\left(I - G_2^{\sim} G_2\right) - 2\ln\det\left(I - G_{22} G_2\right) \tag{3.5}$$

Moreover, if statement (i) is satisfied and if G_2 is strictly proper then we have

$$\mathcal{J}(G_2, 1) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} \ln\det\left(I - G_2^{\sim}(i\omega) G_2(i\omega)\right) d\omega, \tag{3.6}$$

$$\mathcal{J}(G_{11}, 1) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} \ln\det\left(I - G_{11}^{\sim}(i\omega) G_{11}(i\omega)\right) d\omega. \tag{3.7}$$

Using the fact that G_2 is strictly proper, stable and has H_∞ norm strictly less than 1 and the fact that also G_{22} is stable, strictly proper and has H_∞ norm less than or equal to 1, we know there exists a constant M such that

$$|\ln\det\left(I - G_{22}(s) G_2(s)\right)| < \frac{M}{|s|^2} \qquad \forall s \in \mathcal{C}^0 \cup \mathcal{C}^+$$

This implies, using Cauchy's theorem, that

$$\int_{-\infty}^{\infty} \ln\det\left(I - G_{22}(i\omega) G_2(i\omega)\right) d\omega = 0 \tag{3.8}$$

Combining (3.5),(3.6),(3.7) and (3.8) we find (3.4). ■

The following lemma is an essential tool of actually calculating the entropy function for some specific system:

Lemma 3.6 : *Assume that a rational matrix G is given which has a detectable and stabilizable realization (A, B, C, D) with $\det D = 1$. Finally, assume that $G, G^{-1} \in H_\infty$ and G has H_∞ norm equal to 1. Then we have:*

$$\int_{-\infty}^{\infty} \ln|\det G(i\omega)|\, d\omega = -\pi \text{ Trace } BD^{-1}C \tag{3.9}$$

□

Proof : Denote the integral in (3.9) by $\mathcal{K}$ and define $a = -$ Trace $BD^{-1}C$. We have (remember that $\ln|z| = \text{Re}\ln z$):

$$\mathcal{K} = \text{Re}\left(\int_{-\infty}^{\infty} \ln\det G(i\omega) - \frac{a}{1+i\omega}\, d\omega\right) + a\int_{-\infty}^{\infty} \frac{d\omega}{1+\omega^2} \tag{3.10}$$

Next, it is easily checked that $p(s) := \ln\det G(s) - \frac{a}{1+s}$ is an bounded analytic function in C^+ such that $p(s) = O(1/s^2)$ ($|s| \to \infty$, Re $s \geq 0$). Hence using Cauchy's theorem we find

$$\int_{-\infty}^{\infty} p(i\omega)\, d\omega = 0. \tag{3.11}$$

Combining (3.10) and (3.11) yields (3.9). ■

Corollary 3.7 : *Let G be a strictly proper, stable transfer matrix with H_∞ norm strictly less than γ and with stabilizable and detectable realization $(A, B, C, 0)$. Then we have:*

$$\mathcal{J}(G, \gamma) = \text{ Trace } B^{\mathrm{T}}XB \tag{3.12}$$

where X is the unique solution of the algebraic Riccati equation:

$$XA + A^{\mathrm{T}}X + \gamma^{-2}XBB^{\mathrm{T}}X + C^{\mathrm{T}}C = 0$$

such that $A + \gamma^{-2}BB^{\mathrm{T}}X$ is asymptotically stable. □

Proof : The existence and uniqueness of X is a well-known result (see e.g. [22]). It is easily checked that the transfer matrix M with realization $(A, B, -\gamma^{-2}B^{\mathrm{T}}X, I)$ satisfies:

$$I - \gamma^{-2}G^{\sim}G = M^{\sim}M$$

Moreover, $M, M^{-1} \in H_\infty$, i.e. M is a spectral factor of $I - \gamma^{-2}G^{\sim}G$. We have

$$\mathcal{J}(G, \gamma) = \frac{-\gamma^2}{\pi}\int_{-\infty}^{\infty} \ln|\det M(i\omega)|\, d\omega$$

and therefore (3.12) is a direct consequence of applying lemma 3.6 to the above equation. ■

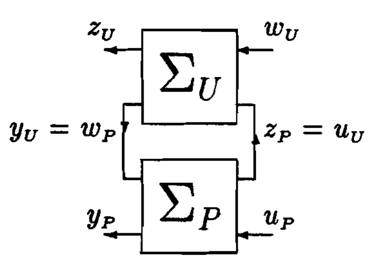
4 A system transformation

Throughout this section we assume that $\gamma = 1$ and that there exist matrices P and Q satisfying the conditions in theorem 2.1 for $\gamma = 1$. Note that this is no restriction when proving theorem 2.3. The assumption $\gamma = 1$ can be easily removed by scaling while the existence of such P and Q is implied by our assumption that there exists an internally stabilizing controller which makes the H_∞ norm strictly less than 1. We use a technique from [18] of transforming the system twice such that the problem of minimizing the entropy function for the original system is equivalent to minimizing the entropy function for the new system we thus obtain. In the next section we will show that this new system satisfies some desirable properties which enables us to solve the minimum entropy H_∞ control problem for this new system and hence also for the original system.
We factorize $F(P)$ as in (2.15). This can be done since $F(P) \geq 0$. We define the following system:

$$\Sigma_P : \begin{cases} \dot{x}_P = A_P x_P + E w_P + B u_P, \\ z_P = C_{1,P} x_P + D_P u_P, \\ y_P = C_{2,P} x_P + D_2 w_P \end{cases}, \tag{4.1}$$

where $A_P := (A + EE^{\mathrm{T}}P)$ and $C_{2,P} := (C_2 + D_2 E^{\mathrm{T}} P)$.

Lemma 4.1 : *Let Σ and Σ_P be defined by (2.1) and (4.1), respectively. For any system Σ_U of suitable dimensions consider the following interconnection:*



(4.2)

and decompose the transfer matrix U of Σ_U as follows:

$$U \begin{pmatrix} w_U \\ u_U \end{pmatrix} =: \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \begin{pmatrix} w_U \\ u_U \end{pmatrix} = \begin{pmatrix} z_U \\ y_U \end{pmatrix},$$

compatible with the sizes of u_U, w_U, y_U, and z_U. Then the following holds: there exists a system Σ_U of suitable dimensions such that:

(*i*) *The system Σ_U is inner*

(*ii*) *The transfer matrix U_{21}^{-1} is well-defined and stable*

(*iii*) *The transfer matrices U_{11} and U_{22} are strictly proper*

(*iv*) $\mathcal{J}(U_{11}, 1) =$ *Trace* $E^{\mathrm{T}}PE$

(v) The system Σ and the interconnection in (4.2) have the same transfer matrix.

(vi) The interconnection in (4.2) is detectable from y_P and stabilizable by u_P. □

Proof : In [18] a system Σ_U is constructed which satisfies all of the above conditions. Note that U_{21} is a spectral factor for $I - U_{11}^{\sim} U_{11}$ which yields (iv) by using the state space realization for U_{21} given in [18] and by applying lemma 3.6. ■

Remark: A state space realization for a system Σ_U satisfying the conditions of lemma 4.1 in case D_1 and D_2 are injective and surjective respectively is given by:

$$\Sigma_U : \begin{cases} \dot{x}_U = \tilde{A} x_U + E w_U + B u_U, \\ z_U = \tilde{C}_1 x_U + u_U, \\ y_U = -E^{\mathrm{T}} P x_U + w_U. \end{cases}$$

where $\tilde{A} = A - B(D_1^{\mathrm{T}} D_1)^{-1}(B^{\mathrm{T}} P + D_1^{\mathrm{T}} C_1)$ and $\tilde{C}_1 = C_1 - D_1(D_1^{\mathrm{T}} D_1)^{-1}(B^{\mathrm{T}} P + D_1^{\mathrm{T}} C_1)$.

Combining lemmas 3.5 and 4.1, we find the following theorem:

Theorem 4.2 : *Let the systems (2.1) and (4.1) be given. Moreover, let a compensator Σ_F of the form (2.3) be given. The following two conditions are equivalent:*

- *Σ_F is internally stabilizing for Σ such that the closed loop transfer matrix G_{cl} is strictly proper and has H_∞ norm strictly less than 1.*
- *Σ_F is internally stabilizing for Σ_P such that the closed loop transfer matrix $G_{cl,P}$ is strictly proper and has H_∞ norm strictly less than 1.*

Moreover, if Σ_F satisfies the above conditions then we have

$$\mathcal{J}(G_{cl}, 1) = \mathcal{J}(G_{cl,P}, 1) + \text{ Trace } E^{\mathrm{T}} P E.$$ □

Next, we make another transformation from Σ_P to $\Sigma_{P,Q}$. This transformation is exactly dual to the transformation from Σ to Σ_P. We know there exists a controller which is internally stabilizing for Σ_P which makes the H_∞ norm of the closed loop system strictly less than 1. Therefore if we apply theorem 2.1 to Σ_P we find that that there exists a unique matrix Y such that $\bar{G}(Y) \geq 0$ and

(*i*) rank $\bar{G}(Y) = \text{rank}_{\mathcal{R}(s)} \bar{G}_{di}$,

(*ii*) rank $\begin{pmatrix} \bar{M}(Y,s) & \bar{G}(Y) \end{pmatrix} = n + \text{rank}_{\mathcal{R}(s)} \bar{G}_{di}, \qquad \forall\, s \in C^0 \cup C^+,$

where

$$\begin{aligned}
\bar{G}(Y) &:= \begin{pmatrix} A_P Y + Y A_P^{\mathrm{T}} + EE^{\mathrm{T}} + YC_{1,P}^{\mathrm{T}} C_{1,P} Y & YC_{2,P}^{\mathrm{T}} + ED_2^{\mathrm{T}} \\ C_{2,P} Y + D_2 E^{\mathrm{T}} & D_2 D_2^{\mathrm{T}} \end{pmatrix}, \\
\bar{M}(Y,s) &:= \begin{pmatrix} sI - A_P - YC_{1,P}^{\mathrm{T}} C_{1,P} \\ -C_{2,P} \end{pmatrix}, \\
\bar{G}_{di}(s) &:= C_{2,P} (sI - A_P)^{-1} E + D_2.
\end{aligned}$$

In [18] it has been shown that $Y := (I - QP)^{-1} Q$ satisfies the above conditions. We factorize $\bar{G}(Y)$:

$$\bar{G}(Y) = \begin{pmatrix} E_{P,Q} \\ D_{P,Q} \end{pmatrix} \begin{pmatrix} E_{P,Q}^{\mathrm{T}} & D_{P,Q}^{\mathrm{T}} \end{pmatrix}. \tag{4.3}$$

where $E_{P,Q}$ and $D_{P,Q}$ are matrices of suitable dimensions. We define the following system:

$$\Sigma_{P,Q} : \begin{cases} \dot{x}_{P,Q} = A_{P,Q} x_{P,Q} + E_{P,Q} w + B_{P,Q} u_{P,Q}, \\ z_{P,Q} = C_{1,P} x_{P,Q} + D_P u_{P,Q}, \\ y_{P,Q} = C_{2,P} x_{P,Q} + D_{P,Q} w, \end{cases} \tag{4.4}$$

where $A_{P,Q} := A_P + YC_{1,P}^{\mathrm{T}} C_{1,P}$ and $B_{P,Q} := B + YC_{1,P}^{\mathrm{T}} D_P$. Using theorem 4.2 and a dualized version for the transformation from Σ_P to $\Sigma_{P,Q}$ we can derive the following corollary:

Corollary 4.3 : *Let the systems (2.1) and (4.4) be given. Moreover, let a compensator Σ_F of the form (2.3) be given. The following two conditions are equivalent:*

- *Σ_F is internally stabilizing for Σ such that the closed loop transfer matrix G_{cl} is strictly proper and has H_∞ norm strictly less than* 1.
- *Σ_F is internally stabilizing for $\Sigma_{P,Q}$ such that the closed loop transfer matrix $G_{cl,P,Q}$ is strictly proper and has H_∞ norm strictly less than* 1.

Moreover, if Σ_F satisfies the above conditions then we have

$$\mathcal{J}(G_{cl}, 1) = \mathcal{J}(G_{cl,P,Q}, 1) + \text{ Trace } E^{\mathrm{T}} P E + \text{ Trace } C_{2,P} Y C_{2,P}. \qquad \square$$

From this corollary it is immediate that it is sufficient to investigate $\Sigma_{P,Q}$ to prove the results in our main theorem 2.3. This is done in the next section.

5 (Almost) Disturbance Decoupling and minimum entropy

The first thing we would like to know is what we gain by our transformation from Σ to $\Sigma_{P,Q}$. It turns out that we obtain in this way a system of the form (2.1) such that part (ii) of theorem 2.1 is satisfied for $P = 0$ and $Q = 0$. This implies that:

$$\operatorname{rank}\begin{pmatrix} sI - A_{P,Q} & -B_{P,Q} \\ C_{1,P} & D_P \end{pmatrix} = n + \operatorname{rank}\begin{pmatrix} C_{1,P} & D_P \end{pmatrix}, \qquad \forall\, s \in C^0 \cup C^+ \tag{5.1}$$

$$\operatorname{rank}\begin{pmatrix} sI - A_{P,Q} & -E_{P,Q} \\ C_{2,P} & D_{P,Q} \end{pmatrix} = n + \operatorname{rank}\begin{pmatrix} E_{P,Q} \\ D_{P,Q} \end{pmatrix}, \qquad \forall\, s \in C^0 \cup C^+. \tag{5.2}$$

By using these conditions we can derive the following theorem:

Theorem 5.1 : *Let $\Sigma_{P,Q}$ be defined by (4.4) such that (5.1) and (5.2) are satisfied. Then there exists a sequence of controllers $\Sigma_{F,n}$ of the form (2.3) such that the closed loop systems are internally stable for all n and the closed loop transfer matrices $G_{cl,n}$ are strictly proper and satisfy:*

$$\|G_{cl,n}\|_\infty \;\to\; 0 \qquad as\ n \to \infty \tag{5.3}$$

$$\|G_{cl,n}\|_2 \;\to\; 0 \qquad as\ n \to \infty \tag{5.4}$$

where $\|\cdot\|_2$ denotes the $L_2(-i\infty, i\infty)$-norm. □

Proof : In [18] a sequence of compensators is constructed which are internally stabilizing and which are such that L_1-norm of the closed loop impulse response tends to zero as $n \to \infty$. Moreover the closed loop transfer matrices are strictly proper. In [18] is referred to theorem 3.36 of [21]. By using theorem 3.25 of [21] instead it is straightforward to show that for this sequence of controllers also the L_2 norm of the closed loop impulse response matrix tends to zero as $n \to \infty$.
The L_1 norm of the closed loop impulse response matrix is an upper bound for the H_∞ norm of the closed loop transfer matrix. This yields (5.3). Since the L_2 norm of the closed loop impulse response matrix is equal to the $L_2(-i\infty, i\infty)$-norm of the closed loop transfer matrix by Parseval's theorem, we find (5.4). ■

This theorem yields the following corollary:

Corollary 5.2 : *Consider the system (2.1). Assume that the systems (A, B, C_1, D_1) and (A, E, C_2, D_2) have no invariant zeros on the imaginary axis and assume that there exists a controller which is such that the closed loop system is internally stable and has H_∞ norm strictly less than 1. The infimum of (2.2), over all internally stabilizing controllers of the form (2.3) which are such that the closed system has H_∞ norm strictly less than 1, is equal to:*

$$\mathit{Trace}\ E^{\mathrm{T}}PE + \mathit{Trace}\ (A^{\mathrm{T}}P + PA + C_1^{\mathrm{T}}C_1 + PEE^{\mathrm{T}}P)(I - QP)^{-1}Q \tag{5.5}$$

where P and Q are such that part (ii) of theorem 2.1 is satisfied. □

Proof : By corollary 4.3 and the non-negativity of the entropy function, the infimum is always larger than or equal to (5.5). Next, choose a sequence of controllers $\Sigma_{F,n}$ satisfying the conditions of theorem 5.1. These controllers applied to $\Sigma_{P,Q}$ yield internally stable closed loop systems and it is straightforward to check that the closed loop transfer matrices $\tilde{G}_{cl,n,P,Q}$ satisfy:

$$\mathcal{J}(\tilde{G}_{cl,n,P,Q},1) \to 0$$

as $n \to \infty$. By applying corollary 4.3 we find that if we apply the controllers $\Sigma_{F,n}$ to Σ then we find closed loop systems which are internally stable and the closed loop transfer matrices $G_{cl,n}$ are strictly proper, have H_∞ norm less than 1 and satisfy

$$\mathcal{J}(G_{cl,n},1) \to \text{Trace } E^{\mathrm{T}}PE + \text{Trace } (A^{\mathrm{T}}P + PA + C_1^{\mathrm{T}}C_1 + PEE^{\mathrm{T}}P)(I - QP)^{-1}Q$$

as $n \to \infty$, which completes the proof of this corollary. ■

Corollary 5.2 is in fact the main part of theorem 2.3. Remains to investigate when the infimum of the entropy function is attained. From the theory of almost disturbance decoupling (see [21]), it is well known that lemma 5.1 implies that there always exists an in general non-proper controller which attains the infimum. However, we would like to know when it is possible to attain the infimum by a *proper* controller. Note that the infimum is attained if and only if we can find a controller for $\Sigma_{P,Q}$ which makes the closed loop system internally stable and the closed loop transfer matrix $\tilde{G}_{cl}$ has H_∞ norm less than 1 and its entropy is equal to 0. However, the entropy is 0 if and only if $\tilde{G}_{cl} = 0$ by lemma 3.2. This reduces our original problem to what is often called the disturbance decoupling problem with measurement feedback and internal stability (DDPMS), i.e. the problem of finding a stabilizing controller which makes the closed loop transfer matrix equal to 0. The following theorem is a generalization of the results in [7, 16, 23].

Theorem 5.3 : *Let Σ be given of the form (2.1). There exists a controller of the form (2.3) such that the closed loop system is internally stable and the closed loop transfer matrix is equal to* 0 *if and only if (A,B) is stabilizable, (C_2,A) is detectable and*

$$\begin{aligned} Im\ E &\subseteq \mathcal{V}_g(A,B,C_1,D_1) + B\ Ker\ D_1 \\ Ker\ C_1 &\supseteq \mathcal{T}_g(A,E,C_2,D_2) \cap C_2^{-1}\ Im\ D_2 \\ \mathcal{T}_g(A,E,C_2,D_2) &\subseteq \mathcal{V}_g(A,B,C_1,D_1) \end{aligned}$$ □

After some extensive calculations the following two equalities can be derived:

$$\begin{aligned} \mathcal{V}_g(A + EE^{\mathrm{T}}P, B, C_{1,P}, D_P) &= \mathcal{V}_g(A_{P,Q}, B_{P,Q}, C_{1,P}, D_P) \\ \mathcal{T}_g(A + QC_2^{\mathrm{T}}C_2, E_Q, C_2, D_Q) &= (I - QP)\mathcal{T}_g(A_{P,Q}, E_{P,Q}, C_{2,P}, D_{P,Q}) \end{aligned}$$

Applying theorem 5.3 completes the proof of the results in theorem 2.3.

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6 Conclusion

In this paper we have given a complete treatment of the singular minimum entropy H_∞ control problem. We have an explicit formula for the infimum. Moreover, we can characterize when the infimum is attained. The construction of a controller can be done using some tools from the geometric approach to control theory. However, this method is too extensive (though straightforward) to discuss in this paper. Finally we would like to note that this paper gives a nice structured approach to "entropy at infinity" with less technicalities than [10] where entropy at infinity is simply a special yet important case.
A main open problem remains the problem of invariant zeros on the imaginary axis. Another interesting extension is the case where we have two kinds of disturbances and two kinds of to be controlled outputs. From one disturbance input to one of the outputs to be controlled we want to satisfy an H_∞ norm bound. For the closed loop transfer matrix from the other disturbance to the other output to be controlled we want to minimize an entropy function. In this way we can relate performance criteria to robustness criteria with much more arbitrary structure for the parameter uncertainty.

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