# Maximizing maximal angles for plane straight-line graphs

# Maximizing Maximal Angles for Plane Straight-Line Graphs*


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**Abstract.** Let $G = (S, E)$ be a plane straight-line graph on a finite point set $S \subset \mathbb{R}^2$ in general position. The *incident angles* of a point $p \in S$ in $G$ are the angles between any two edges of $G$ that appear consecutively in the circular order of the edges incident to $p$. A plane straight-line graph is called $\varphi$*-open* if each vertex has an incident angle of size at least $\varphi$. In this paper we study the following type of question: What is the maximum angle $\varphi$ such that for any finite set $S \subset \mathbb{R}^2$ of points in general position we can find a graph from a certain class of graphs on $S$ that is $\varphi$-open? In particular, we consider the classes of triangulations, spanning trees, and paths on $S$ and give tight bounds in most cases.


## 1 Introduction

Conditions on angles in plane straight-line graphs have been studied extensively in discrete and computational geometry. It is well known that Delaunay triangulations maximize the minimum angle over all triangulations, and that in a (Euclidean) minimum weight spanning tree each angle is at least $\frac{\pi}{3}$. In this paper we address the fundamental combinatorial question, what is the maximum value


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$\alpha$ such that for each finite point set in general position there exists a (certain type of) plane straight-line graph where each vertex has an incident angle of size at least $\alpha$. In other words, we consider $\min - \max - \min - \max$ problems, where we minimize over all finite point sets $S$ in general position in the plane, the maximum over all plane straight-line graphs $G$ (of the considered type), of the minimum over all $p \in S$, of the maximum angle incident to $p$ in $G$. We present bounds on $\alpha$ for three classes of graphs: spanning paths, (general and bounded degree) spanning trees, and triangulations. Most of the bounds we give are tight. In order to show that, we describe families of point sets for which no graph from the respective class can achieve a greater incident angle at each vertex.

***Background.*** Our motivation for this research stems from the investigation of "pseudo-triangulations", a straight-line framework which apart from deep combinatorial properties has applications in motion planning, collision detection, ray shooting and visibility; see [3,12,13,15,16] and references therein. Pseudo-triangulations with a minimum number of pseudo-triangles (among all pseudo-triangulations for a given point set) are called *minimum* (or *pointed*) pseudo-triangulations. They can be characterized as plane straight-line graphs where each vertex has an incident angle greater than $\pi$. Furthermore, the number of edges in a minimum pseudo-triangulation is maximal, in the sense that the addition of any edge produces an edge-crossing or negates the angle condition.

In comparison to these properties, we consider connected plane straight-line graphs where each vertex has an incident angle $\alpha$—to be maximized—and the number of edges is minimal (spanning trees) and the vertex degree is bounded (spanning trees of bounded degree and spanning paths). We further show that any planar point set has a triangulation in which each vertex has an incident angle which is at least $\frac{2\pi}{3}$. Observe that perfect matchings can be described as plane straight-line graphs where each vertex has an incident angle of $2\pi$ and the number of edges is maximal.

***Related Work.*** There is a vast literature on triangulations that are optimal according to certain criteria, cf. [2]. Similar to Delaunay triangulations which maximize the smallest angle over all triangulations for a point set, farthest point Delaunay triangulations minimize the smallest angle over all triangulations for a convex polygon [9]. If all angles in a triangulation are $\geq \frac{\pi}{6}$ then it contains the relative neighborhood graph as a subgraph [14]. The relative neighborhood graph for a point set connects any pair of points which are mutually closest to each other (among all points from the set). Edelsbrunner et al. [10] showed how to construct a triangulation that minimizes the maximum angle among all triangulations for a set of $n$ points in $O(n^2 \log n)$ time.

In applications where small angles have to be avoided by all means, a Delaunay triangulation may not be sufficient in spite of its optimality because even there arbitrarily small angles can occur. By adding so-called Steiner points one can construct a triangulation on a superset of the original points in which there is some absolute lower bound on the size of the smallest angle [7]. Dai et al. [8] describe several heuristics to construct minimum weight triangulations

(triangulations which minimize the total sum of edge lengths) subject to absolute lower or upper bounds on the occurring angles.

Spanning cycles with angle constraints can be regarded as a variation of the traveling salesman problem. Fekete and Woeginger [11] showed that if the cycle may cross itself then any set of at least five points admits a locally convex tour, that is, a tour in which the angle between any three consecutive points is positive. Arkin et al. [5] consider as a measure for (non-)convexity of a point set $S$ the minimum number of (interior) reflex angles (angles $> \pi$) among all plane spanning cycles for $S$. Aggarwal et al. [4] prove that finding a spanning cycle for a point set which has minimal total angle cost is NP-hard, where the angle cost is defined as the sum of direction changes at the points. Regarding spanning paths, it has been conjectured that each planar point set admits a spanning path with minimum angle at least $\frac{\pi}{6}$ [11]; recently, a lower bound of $\frac{\pi}{9}$ has been presented [6].

***Definitions and Notation.*** Let $S \subset \mathbb{R}^2$ be a finite set of points in general position, that is, no three points of $S$ are collinear. In this paper we consider plane straight-line graphs $G = (S, E)$ on $S$. The vertices of $G$ are the points in $S$, the edges of $G$ are straight-line segments that connect two points in $S$, and two edges of $G$ do not intersect except possibly at their endpoints. The *incident angles* of a point $p \in S$ in $G$ are the angles between any two edges of $G$ that appear consecutively in the circular order of the edges incident to $p$. We denote the *maximum incident angle* of $p$ in $G$ with $\mathrm{op}_G(p)$. For a point $p \in S$ of degree at most one we set $\mathrm{op}_G(p) = 2\pi$. We also refer to $\mathrm{op}_G(p)$ as the *openness* of $p$ in $G$ and call $p \in S$ *$\varphi$-open* in $G$ for some angle $\varphi$ if $\mathrm{op}_G(p) \geq \varphi$. Consider for example the graph depicted in Fig. 1. The point $p$ has four incident edges of $G$ and, therefore, four incident angles. Its openness is $\mathrm{op}_G(p) = \alpha$. The point $q$ has only one incident angle and correspondingly $\mathrm{op}_G(q) = 2\pi$.

Similarly we define the *openness* of a plane straight-line graph $G = (S, E)$ as $\mathrm{op}(G) = \min_{p \in S} \mathrm{op}_G(p)$ and call $G$ *$\varphi$-open* for some angle $\varphi$ if $\mathrm{op}(G) \geq \varphi$. In other words, a graph is $\varphi$-open if and only if every vertex has an incident angle of size at least $\varphi$. The *openness* of a class $\mathcal{G}$ of graphs is the supremum over all angles $\varphi$ such that for every finite point set $S \subset \mathbb{R}^2$ in general position there exists a $\varphi$-open connected plane straight-line graph $G$ on $S$ and $G$ is an embedding of some graph from $\mathcal{G}$. For example, the openness of minimum pseudo-triangulations is $\pi$.

Observe that without the general position assumption many of the questions become trivial because for a set of collinear points the non-crossing spanning tree is unique—the path that connects them along the line—and its interior points have no incident angle greater than $\pi$.

The convex hull of a point set $S$ is denoted with $CH(S)$. Points of $S$ on $CH(S)$ are called vertices of $CH(S)$. Let $a$, $b$, and $c$ be three points in the plane that are not collinear. With $\angle abc$ we denote the counterclockwise angle between the segment $(b, a)$ and the segment $(b, c)$ at $b$.

***Results.*** In this paper we study the openness of several well-known classes of plane straight-line graphs, such as triangulations (Section 2), (general and

**Table 1.** Openness of several classes of plane straight-line graphs. All given values except for paths on point sets in general position are tight.

| Triangulations | Trees | Trees with maxdeg. 3 | Paths (convex sets) | Paths (general) |
|---|---|---|---|---|
| $\frac{2\pi}{3}$ | $\frac{5\pi}{3}$ | $\frac{3\pi}{2}$ | $\frac{3\pi}{2}$ | $\frac{5\pi}{4}$ |

bounded degree) trees (Section 3), and paths (Section 4). The results are summarized in Table 1 above.

## 2 Triangulations

**Theorem 1.** *Every finite point set in general position in the plane has a triangulation that is $\frac{2\pi}{3}$-open and this is the best possible bound.*

*Proof.* Consider a point set $S \subset \mathbb{R}^2$ in general position. Clearly, $\mathrm{op}_G(p) > \pi$ for every point $p \in \mathrm{CH}(S)$ and every plane straight-line graph $G$ on $S$. We recursively construct a $\frac{2\pi}{3}$-open triangulation $T$ of $S$ by first triangulating $\mathrm{CH}(S)$; every recursive subproblem consists of a point set with a triangular convex hull.

Let $S$ be a point set with a triangular convex hull and denote the three points of $\mathrm{CH}(S)$ with $a$, $b$, and $c$. If $S$ has no interior points, then we are done. Otherwise, let $a'$, $b'$ and $c'$ be (not necessarily distinct) interior points of $S$ such that the triangles $\Delta a'bc$, $\Delta ab'c$ and $\Delta abc'$ are empty (see Fig. 2). Since the sum of the six exterior angles of the hexagon $ba'cb'ac'$ equals $8\pi$, the sum of the three angels $\angle ac'b$, $\angle ba'c$, and $\angle cb'a$ is at least $2\pi$. In particular, one of them, say $\angle cb'a$, is at least $\frac{2\pi}{3}$. We then recurse on the two subsets of $S$ that have $\Delta b'bc$ and $\Delta b'ab$ as their respective convex hulls.

The upper bound is attained by a set $S$ of $n$ points as depicted in Fig. 3. $S$ consists of a point $p$ and of three sets $S_a$, $S_b$, and $S_c$ that each contain $\frac{n-1}{3}$ points. $S_a$, $S_b$, and $S_c$ are placed at the vertices of an equilateral triangle $\Delta$ and $p$ is placed at the barycenter of $\Delta$. Any triangulation $T$ of $S$ must connect $p$

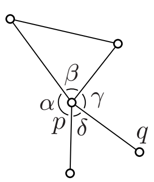


**Fig. 1.** The incident angles of $p$

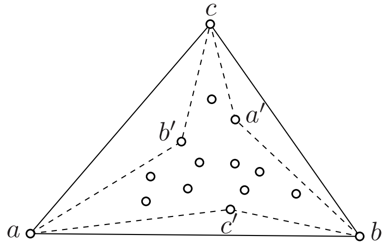


**Fig. 2.** Constructing a $\frac{2\pi}{3}$-open triangulation

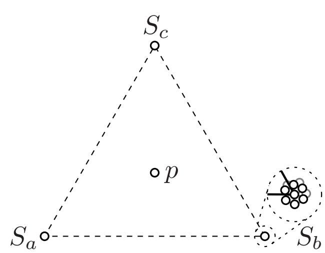


**Fig. 3.** The openness of triangulations of this point set approaches $\frac{2\pi}{3}$

with at least one point of each of $S_a$, $S_b$, and $S_c$ and hence $\mathrm{op}_T(p)$ approaches $\frac{2\pi}{3}$ arbitrarily close. □

# 3 Spanning Trees

In this section we give tight bounds on the $\varphi$-openness of two basic types of spanning trees, namely general spanning trees and spanning trees with bounded vertex degree. Consider a point set $S \subset \mathbb{R}^2$ in general position and let $p$ and $q$ be two arbitrary points of $S$. Assume w.l.o.g. that $p$ has smaller $x$-coordinate than $q$. Let $l_p$ and $l_q$ denote the lines through $p$ and $q$ that are perpendicular to the edge $(p, q)$. We define the *orthogonal slab* of $(p, q)$ to be the open region bounded by $l_p$ and $l_q$.

**Observation 1.** *Assume that $r \in S \setminus \{p, q\}$ lies in the orthogonal slab of $(p, q)$ and above $(p, q)$. Then $\angle qpr \leq \frac{\pi}{2}$ and $\angle rqp \leq \frac{\pi}{2}$. A symmetric observation holds if $r$ lies below $(p, q)$.*

Recall that the diameter of a point set is the distance between a pair of points that are furthest away from each other. Let $a$ and $b$ define the diameter of $S$ and assume w.l.o.g. that $a$ has a smaller $x$-coordinate than $b$. Clearly, all points in $S \setminus \{a, b\}$ lie in the orthogonal slab of $(a, b)$.

**Observation 2.** *Assume that $r \in S \setminus \{a, b\}$ lies above a diametrical segment $(a, b)$ for $S$. Then $\angle arb \geq \frac{\pi}{3}$ and hence at least one of the angles $\angle bar$ and $\angle rba$ is at most $\frac{\pi}{3}$. A symmetric observation holds if $r$ lies below $(a, b)$.*

## 3.1 General Spanning Trees

**Theorem 2.** *Every finite point set in general position in the plane has a spanning tree that is $\frac{5\pi}{3}$-open and this is the best possible bound.*

The upper bound is attained by the point set depicted in Fig. 6. Each of the sets $S_i, i \in 1, 2, 3$ consists of $\frac{n}{3}$ points. If a point $p \in S_1$ is connected to any other point from $S_1 \cup S_2$, then it can only be connected to a point of $S_3$ forming an angle of at least $\frac{\pi}{3} - \varepsilon$. As the same argument holds for $S_2$ and $S_3$, respectively, any connected graph, and thus any spanning tree on $S$ is at most $\frac{5\pi}{3}$-open.

The proof for the lower bound strongly relies on Observation 2 and can be found in the full paper.

## 3.2 Spanning Trees of Bounded Vertex Degree

**Theorem 3.** *Let $S \subset \mathbb{R}^2$ be a set of $n$ points in general position. There exists a $\frac{3\pi}{2}$-open spanning tree $T$ of $S$ such that every point from $S$ has vertex degree at most three in $T$. The angle bound is best possible, even for the much broader class of spanning trees of vertex degree at most $n - 2$.*

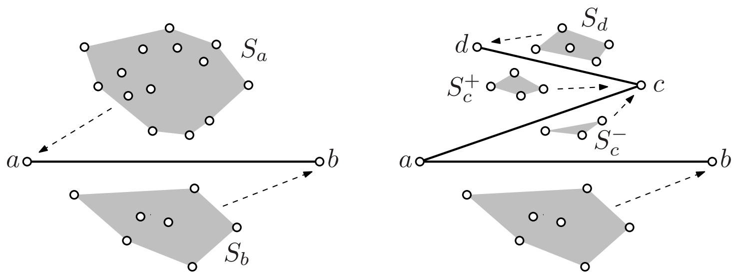


**Fig. 4.** Constructing a $\frac{3\pi}{2}$-open spanning tree with maximum vertex degree four

*Proof.* We show in fact that $S$ has a $\frac{3\pi}{2}$-open spanning tree with maximum vertex degree three. To do so, we first describe a recursive construction that results in a $\frac{3\pi}{2}$-open spanning tree with maximum vertex degree four. We then refine our construction to yield a spanning tree of maximum vertex degree three.

Let $a$ and $b$ define the diameter of $S$. W.l.o.g. $a$ has a smaller $x$-coordinate than $b$. The edge $(a, b)$ partitions $S \setminus \{a, b\}$ into two (possibly empty) subsets: the set $S_a$ of the points above $(a, b)$ and the set $S_b$ of the points below $(a, b)$. We assign $S_a$ to $a$ and $S_b$ to $b$ (see Fig 4). Since all points of $S \setminus \{a, b\}$ lie in the orthogonal slab of $(a, b)$ we can connect any point $p \in S_a$ to $a$ and any point of $q \in S_b$ to $b$ and by this obtain a $\frac{3\pi}{2}$-open path $P = \langle p, a, b, q\rangle$. Based on this observation we recursively construct a spanning tree of vertex degree at most four.

If $S_a$ is empty, then we proceed with $S_b$. If $S_a$ contains only one point $p$ then we connect $p$ to $a$. Otherwise consider a diametrical segment $(c, d)$ for $S_a$. W.l.o.g. $d$ has a smaller $x$-coordinate than $c$ and $d$ lies above $(a, c)$. Either $\angle adc$ or $\angle dca$ must be less than $\frac{\pi}{2}$. W.l.o.g. assume that $\angle dca < \frac{\pi}{2}$. Hence we can connect $d$ via $c$ to $a$ and obtain a $\frac{3\pi}{2}$-open path $P = \langle d, c, a, b\rangle$. The edge $(d, c)$ partitions $S_a$ into two (possibly empty) subsets: the set $S_d$ of the points above $(d, c)$ and the set $S_c$ of the points below $(d, c)$. The set $S_c$ is again partitioned by the edge $(a, c)$ into a set $S_c^+$ of points that lie above $(a, c)$ and a set $S_c^-$ of points that lie below $(a, c)$. We assign $S_d$ to $d$ and both $S_c^+$ and $S_c^-$ to $c$ and proceed recursively.

The algorithm maintains the following two invariants: $(i)$ at most two sets are assigned to any point of $S$, and $(ii)$ if a set $S_p$ is assigned to a point $p$ then $p$ can be connected to any point of $S_p$ and $\mathrm{op}_T(p) \geq \frac{3\pi}{2}$ for any resulting tree $T$.

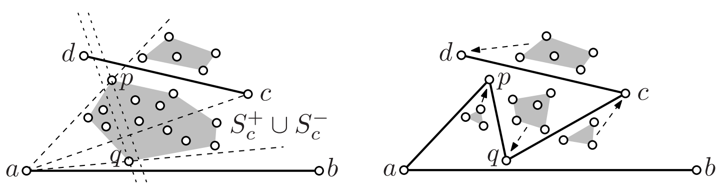


**Fig. 5.** Constructing a $\frac{3\pi}{2}$-open spanning tree with maximum vertex degree three

We now refine our construction to obtain a $\frac{3\pi}{2}$-open spanning tree of maximum vertex degree three. If $S_c^+$ is empty then we assign $S_c^-$ to $c$, and vice versa. Otherwise, consider the tangents from $a$ to $S_c$ and denote the points of tangency with $p$ and $q$ (see Fig. 5). Let $l_p$ and $l_q$ denote the lines through $p$ and $q$ that are perpendicular to $(a, c)$. W.l.o.g. $l_q$ is closer to $a$ than $l_p$. We replace the edge $(a, c)$ by the three edges $(a, p)$, $(p, q)$, and $(q, c)$. The resulting path is $\frac{3\pi}{2}$-open and partitions $S_c$ into three sets which can be assigned to $p$, $q$, and $c$ while maintaining invariant $(ii)$. The refined recursive construction assigns at most one set to every point of $S$ and hence constructs a $\frac{3\pi}{2}$-open spanning tree with maximum vertex degree three.

The upper bound is attained by the set $S$ of $n$ points depicted in Fig. 7. $S$ consists of $n-1$ near-collinear points close together and one point $p$ far away. In order to construct any connected graph with maximum degree at most $n-2$, one point of $S_1$ has to be connected to another point of $S_1$ and to $p$. Thus any spanning tree on $S$ with maximum degree at most $n-2$ is at most $\frac{3\pi}{2}$-open. $\square$

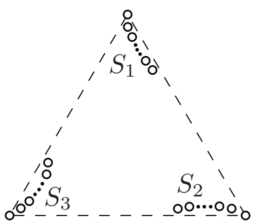

**Fig. 6.** Every spanning tree is at most $\frac{5\pi}{3}$-open

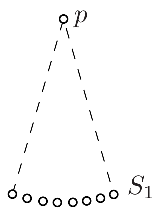

**Fig. 7.** Every spanning tree with vertex degree at most $n-2$ is at most $\frac{3\pi}{2}$-open

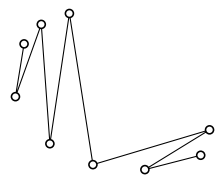
**Fig. 8.** A zigzag path

## 4 Spanning Paths

Spanning paths can be regarded as spanning trees with maximum vertex degree two. Therefore, the upper bound construction from Fig. 7 applies to paths as well. We will show below that the resulting bound of $\frac{3\pi}{2}$ is tight for points in convex position, even in a very strong sense: There exists a $\frac{3\pi}{2}$-open spanning path starting from any point.

### 4.1 Point Sets in Convex Position

Consider a set $S \subset \mathbb{R}^2$ of $n$ points in convex position. We can construct a spanning path for $S$ by starting at an arbitrary point $p \in S$ and recursively taking one of the tangents from $p$ to $\mathrm{CH}(S \setminus \{p\})$. As long as $|S| > 2$, there are two tangents from $p$ to $\mathrm{CH}(S \setminus \{p\})$: the left tangent is the oriented line $t_\ell$ through $p$ and a point $p_\ell \in S \setminus \{p\}$ (oriented in direction from $p$ to $p_\ell$) such that no point from $S$ is to the left of $t_\ell$. Similarly, the right tangent is the oriented line $t_r$ through $p$ and a point $p_r \in S \setminus \{p\}$ (oriented in direction from $p$ to $p_r$)

such that no point from $S$ is to the right of $t_r$. If we take the left and the right tangent alternately, see Fig. 8, we call the resulting path a *zigzag* path for $S$.

**Theorem 4.** *Every finite point set in convex position in the plane admits a spanning path that is $\frac{3\pi}{2}$-open and this is the best possible bound.*

*Proof.* As a zigzag path is completely determined by one of its endpoints and the direction of the incident edge, there are exactly $n$ zigzag paths for $S$. (Count directed zigzag paths: There are $n$ choices for the startpoint and two possible directions to continue in each case, that is, $2n$ directed zigzag paths and, therefore, $n$ (undirected) zigzag paths.)

Now consider a point $p \in S$ and sort all other points of $S$ radially around $p$, starting with one of the neighbors of $p$ along $\mathrm{CH}(S)$. Any angle that occurs at $p$ in some zigzag path for $S$ is spanned by two points that are consecutive in this radial order. Moreover, any such angle occurs in exactly one zigzag path because it determines the zigzag path completely. Since the sum of all these angles at $p$ is less than $\pi$, for each point $p$ at most one angle can be $\geq \frac{\pi}{2}$. Furthermore, if $p$ is an endpoint of a diametrical segment for $S$ then all angles at $p$ are $< \frac{\pi}{2}$. Since there is at least one diametrical segment for $S$, there are at most $n-2$ angles $> \frac{\pi}{2}$ in all zigzag paths together. Thus, there exist at least two spanning zigzag paths that have no angle $> \frac{\pi}{2}$, that is, they are $\frac{3\pi}{2}$-open.

To see that the bound of $\frac{3\pi}{2}$ is tight, consider again the point set shown in Fig. 7. □

A constructive proof for Theorem 4 is given in the full paper. There we also prove the following stronger statement.

**Corollary 1.** *For any finite set $S \subset \mathbb{R}^2$ of points in convex position and any $p \in S$ there exists a $\frac{3\pi}{2}$-open spanning path for $S$ which has $p$ as an endpoint.*

### 4.2 General Point Sets

The main result of this section is the following theorem about spanning paths of general point sets.

**Theorem 5.** *Every finite point set in general position in the plane has a $\frac{5\pi}{4}$-open spanning path.*

Let $S \subset \mathbb{R}^2$ be a set of $n$ points in general position. For a suitable labeling of the points of $S$ we denote a spanning path for (a subset of $k$ points of) $S$ with $\langle p_1, \ldots, p_k \rangle$, where we call $p_1$ the starting point of the path. Then Theorem 5 is a direct consequence of the following, stronger result.

**Theorem 6.** *Let $S$ be a finite point set in general position in the plane. Then*

*(1) For every vertex $q$ of the convex hull of $S$, there exists a $\frac{5\pi}{4}$-open spanning path $\langle q, p_1, \ldots, p_k \rangle$ on $S$ starting at $q$.*

*(2) For every edge $\overline{q_1 q_2}$ of the convex hull of $S$ there exists a $\frac{5\pi}{4}$-open spanning path starting at either $q_1$ or $q_2$ and using the edge $\overline{q_1 q_2}$, that is, a spanning path $\langle q_1, q_2, p_1, \ldots, p_k\rangle$ or $\langle q_2, q_1, p_1, \ldots, p_k\rangle$.*

*Proof.* For each vertex $p$ in a path $G$ the maximum incident angle $\mathrm{op}_G(p)$ is the larger of the two incident angles (except for start- and endpoint of the path). To simplify the case analysis we will consider the smaller angle at each point and prove that we can construct a spanning path such that it is at most $\frac{3\pi}{4}$. We denote with $(q, S)$ a spanning path for $S$ starting at $q$, and with $(\overline{q_1 q_2}, S)$ a spanning path for $S$ starting with the edge connecting $q_1$ and $q_2$. The *outer normal cone* of a vertex $y$ of a convex polygon is the region between two half-lines that start at $y$, are respectively perpendicular to the two edges incident at $y$, and are both in the exterior of the polygon.

We prove the statements (1) and (2) of Theorem 6 by induction on $|S|$. The base cases $|S| = 3$ are obviously true.

**Induction for (1):** Let $\mathcal{K} = CH(S \setminus \{q\})$.

**Case 1.1.** $q$ lies between the outer normal cones of two consecutive vertices $y$ and $z$ of $\mathcal{K}$, where $z$ lies to the right of the ray $\overrightarrow{qy}$.

Induction on $(\overline{yz}, S\setminus\{q\})$ results in a $\frac{5\pi}{4}$-open spanning path $\langle y, z, p_1, \ldots, p_k\rangle$ or $\langle z, y, p_1, \ldots, p_k\rangle$ of $S \setminus \{q\}$. Obviously $\angle qyz \leq \frac{\pi}{2} < \frac{3\pi}{4}$ and $\angle yzq \leq \frac{\pi}{2} < \frac{3\pi}{4}$, and thus we get a $\frac{5\pi}{4}$-open spanning path $\langle q, y, z, p_1, \ldots, p_k\rangle$ or $\langle q, z, y, p_1, \ldots, p_k\rangle$ for $S$ (see Fig. 9).

**Case 1.2.** $q$ lies in the outer normal cone of a vertex of $\mathcal{K}$.

Let $p$ be that vertex and let $y$ and $z$ be the two vertices of $\mathcal{K}$ adjacent to $p$, $z$ being to the right of the ray $\overrightarrow{py}$. The three angles $\angle qpz$, $\angle zpy$ and $\angle ypq$ around $p$ obviously add up to $2\pi$. We consider subcases according to which of the three angles is the smallest, the cases of $\angle qpz$ and $\angle ypq$ being symmetric (see Fig. 10).

**Case 1.2.1.** $\angle zpy$ is the smallest of the three angles.

Then, in particular, $\angle zpy < \frac{3\pi}{4}$. Assume without loss of generality that $\angle qpz$ is smaller than $\angle ypq$ and, in particular, that it is smaller than $\pi$. Since $q$ is in the normal cone of $p$, $\angle qpz$ is at least $\frac{\pi}{2}$, hence $\angle pzq$ is at most $\frac{\pi}{2} < \frac{3\pi}{4}$. Let $S' = S \setminus \{q, z\}$ and consider the path that starts with $q$ and $z$ followed by $(p, S')$, that is $\langle q, z, p, p_1, \ldots, p_k\rangle$. Note that $\angle zpp_1 \leq \angle zpy$.

**Case 1.2.2.** $\angle ypq$ is the smallest of the three angles.

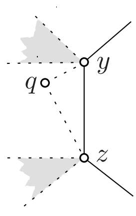


**Fig. 9.** Case 1.1

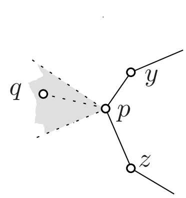


**Fig. 10.** Case 1.2

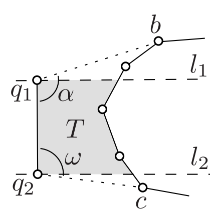


**Fig. 11.** Case 2

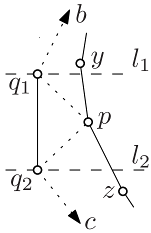


**Fig. 12.** Case 2.2.1

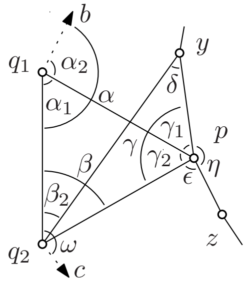


**Fig. 13.** Case 2.2.1.[1,2]

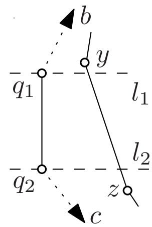


**Fig. 14.** Case 2.2.2

Then $\angle ypq < \frac{3\pi}{4}$. Moreover, in this case all three angles $\angle qpz$, $\angle ypq$ and $\angle zpy$ are at least $\frac{\pi}{2}$, the first two because $q$ lies in the normal cone of $p$, the latter because it is is not the smallest of the three angles. We have $\angle qyp < \frac{\pi}{2}$ because this angle lies in the triangle containing $\angle ypq \geq \frac{\pi}{2}$, and $\angle ypq < \frac{3\pi}{4}$ by assumption. We iterate on $(\overline{py}, S \setminus \{q\})$ and get a $\frac{5\pi}{4}$-open spanning path on $S \setminus \{q\}$ by induction, which can be extended to a $\frac{5\pi}{4}$-open spanning path on $S$, $\langle q, p, y, p_1, \ldots, p_k\rangle$ or $\langle q, y, p, p_1, \ldots, p_k\rangle$, respectively.

**Induction for (2):** Let $b$ and $c$ be the neighboring vertices of $q_1$ and $q_2$ on $CH(S)$, such that $CH(S)$ reads $\ldots, b, q_1, q_2, c, \ldots$ in ccw order (see Fig. 11).

**Case 2.1.** $\alpha < \frac{3\pi}{4}$ or $\omega < \frac{3\pi}{4}$ (see Fig. 11).
Without loss of generality assume that $\alpha < \frac{3\pi}{4}$. By induction on $(q_1, S \setminus \{q_2\})$ we get a $\frac{5\pi}{4}$-open spanning path $\langle q_1, p_1, \ldots, p_k\rangle$ on $S \setminus \{q_2\}$. As $\angle q_2 q_1 p_1 \leq \alpha < \frac{3\pi}{4}$ we get a $\frac{5\pi}{4}$-open spanning path $\langle q_2, q_1, p_1, \ldots, p_k\rangle$ on $S$.

**Case 2.2.** Both $\alpha$ and $\omega$ are at least $\frac{3\pi}{4}$.
Let $l_1$ and $l_2$ be the lines through $q_1$ and $q_2$, respectively, and orthogonal to $\overline{q_1 q_2}$. Further let $\mathcal{K} = CH(S \setminus \{q_1, q_2\})$ and with $T$ we denote the region bounded by $\overline{q_1 q_2}$, $l_1$, $l_2$ and the part of $\mathcal{K}$ closer to $\overline{q_1 q_2}$ (see Fig. 11).

**Case 2.2.1.** At least one vertex $p$ of $\mathcal{K}$ exists in $T$.
If there exist several vertices of $\mathcal{K}$ in $T$, then we choose $p$ as the one with smallest distance to $\overline{q_1 q_2}$ (see Fig. 12). Obviously the edges $\overline{q_1 p}$ and $\overline{q_2 p}$ intersect $\mathcal{K}$ only in $p$ and the angles $\alpha_1$ and $\beta$ are each at most $\frac{\pi}{2}$ (see Fig. 13).

**Case 2.2.1.1.** $\gamma_2 > \frac{\pi}{2}$ (see Fig.13).
By induction on $(p, S \setminus \{q_1, q_2\})$ we get a $\frac{5\pi}{4}$-open spanning path $\langle p, p_1, \ldots, p_k\rangle$ for $S \setminus \{q_1, q_2\}$. Moreover the smaller of $\angle q_2 p p_1$ and $\angle p_1 p q_1$ is at most $\frac{2\pi - \frac{\pi}{2}}{2} = \frac{3\pi}{4}$. Thus we get a $\frac{5\pi}{4}$-open spanning path $\langle q_1, q_2, p, p_1, \ldots, p_k\rangle$ or $\langle q_2, q_1, p, p_1, \ldots, p_k\rangle$ for $S$.

**Case 2.2.1.2.** $\gamma_2 \leq \frac{\pi}{2}$ (see Fig.13).
Let $y$ and $z$ be vertices of $\mathcal{K}$, with $y$ being the clock-wise neighbor of $p$ and $z$ being the counterclockwise one ($b$ might equal $y$ and $c$ might equal $z$). At least one of $\alpha_1$ or $\beta$ is $\geq \frac{\pi}{4}$. Without loss of generality assume that $\beta \geq \frac{\pi}{4}$, the other case is symmetric. Then $q_1, q_2, p, y$ form a convex four-gon because $\alpha \geq \frac{3\pi}{4}$ and

$\beta \geq \frac{\pi}{4}$ imply that $\angle bpq_2$ in the four-gon $b, q_1, q_2, p$ is less than $\pi$. Therefore also $\gamma \leq \angle bpq_2 < \pi$. We will show that all four angles $\alpha_1$, $\gamma_1$, $\beta_2$ and $\delta$ are at most $\frac{3\pi}{4}$. Then we apply induction on $(\overline{py}, S \setminus \{q_1, q_2\})$ and get a $\frac{5\pi}{4}$-open spanning path on $S \setminus \{q_1, q_2\}$, which can be completed to a $\frac{5\pi}{4}$-open spanning path for $S$, $\langle q_2, q_1, p, y, p_1, \ldots, p_k\rangle$ or $\langle q_1, q_2, y, p, p_1, \ldots, p_k\rangle$, respectively.

- Both $\alpha_1$ and $\beta_2 < \beta$ are clearly smaller than $\frac{\pi}{2}$, hence smaller than $\frac{3\pi}{4}$.
- For $\gamma_1$, observe that the supporting line of $\overline{yp}$ must cross the segment $\overline{q_1 b}$, so that we have $\alpha_2 + \gamma_1 < \pi$ (they are two angles of a triangle). Also, $\alpha_2 = \alpha - \alpha_1 \geq \frac{3\pi}{4} - \frac{\pi}{2} = \frac{\pi}{4}$, so $\gamma_1 < \frac{3\pi}{4}$.
- Analogously, for $\delta$, observe that the supporting line of $\overline{yp}$ must cross the segment $\overline{q_2 c}$, so that we have $\omega - \beta_2 + \delta < \pi$. Also $\omega - \beta_2 \geq \frac{\pi}{4}$, so $\delta < \frac{3\pi}{4}$.

**Case 2.2.2.** No vertex of $\mathcal{K}$ exists in $T$.

Both, $l_1$ and $l_2$, intersect the same edge $\overline{yz}$ of $\mathcal{K}$ (in $T$), with $y$ closer to $l_1$ than to $l_2$ (see Fig. 14). We will show that the four angles $\angle yzq_1$, $\angle q_2q_1z$, $\angle yq_2q_1$ and $\angle q_2yz$ are all smaller than $\frac{3\pi}{4}$. Then induction on $(\overline{yz}, S\setminus\{q_1, q_2\})$ yields a path that can be extended to a $\frac{5\pi}{4}$-open path $\langle q_2, q_1, z, y, p_1, \ldots, p_k\rangle$ or $\langle q_1, q_2, y, z, p_1, \ldots, p_k\rangle$. Clearly, the angles $\angle q_2q_1z$ and $\angle yq_2q_1$ are both smaller than $\frac{\pi}{2}$. The sum of $\angle q_2yz + \angle cq_2y$ is smaller than $\pi$ because the supporting line of $\overline{yz}$ intersects the segment $\overline{q_2c}$. Now, $\angle cq_2y$ is at least $\frac{\pi}{4}$ by the assumption that $\angle cq_2q_1 \geq \frac{3\pi}{4}$. So, $\angle q_2yz < \frac{3\pi}{4}$. The symmetric argument shows that $\angle yzq_1 < \frac{3\pi}{4}$. □

Note that for Theorem 6 it is essential that the predefined starting point of a $\frac{5\pi}{4}$-open path is an extreme point of $S$, as an equivalent result is in general not true for interior points. As a counter example consider a regular $n$-gon with an additional point in its center. It is easy to see that for sufficiently large $n$ starting at the central point causes a path to be at most $\pi + \varepsilon$-open for a small constant $\varepsilon$. Similar, non-symmetric examples already exist for $n \geq 6$ points, and analogously, if we require an interior edge to be part of the path, there exist examples bounding the openness by $\frac{4\pi}{3} + \varepsilon$ [17]. Despite these examples we conclude this section with the following conjecture.

**Conjecture 1.** *Every finite point set in general position in the plane has a $\frac{3\pi}{2}$-open spanning path.*

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