Markov processes without a finite invariant measure

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by

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0. Introduction

Spring 1975 at the Technological University of Eindhoven a group of people studied the chapter on finite invariant measures in Foguel's book on the ergodic theory of Markov processes [3]. This memorandum is a summary of the discussed topics, and it contains known results, or slight extensions of known results of which we were not able to discover them in literature.
The material is divided in two parts. The first part deals with properties of Markov operators which do not admit a finite invariant measure, and the second part gives some applications to the theory of measurable transformations.
The author is very much indebted to the members of the group: L. Groenewegen, K. van Hee, K van Harn, A. Nijst, D. Overdijk, B. Schaaf and F. Steutel. Without the stimulating discussions these notes would not have been written.

1. Markov operators

We follow the terminology in Foguel [3]. A Markov process P is a quadruple (X,Σ,m,P), where (X,Σ,m) is a probability space and P a positive linear operator in $\mathcal{L}_\infty$ which satisfies $P1 \le 1$ and which is σ-additive, i.e.

$$P\left(\sum_{n=1}^{\infty} f_n\right) = \sum_{n=1}^{\infty} Pf_n, \text{ if } f_n \in \mathcal{L}_\infty \ (n = 1,2,\dots) \text{ and } \sum_{n=1}^{\infty} f_n \in \mathcal{L}_\infty .$$

Such a Markov operator in $\mathcal{L}_\infty$ is the adjoint operator of a positive linear contraction in $\mathcal{L}_1$, and conversely. We shall denote the $\mathcal{L}_1$-operator also by P, but, in order to distinguish, in this situation we shall write the operator symbol P to the right of the function symbol. The relationship is then given by

$$\langle uP,f\rangle = \langle u,Pf\rangle, \quad u \in \mathcal{L}_1,\ f \in \mathcal{L}_\infty ,$$

where

$$\langle u,f\rangle = \int uP\, dm, \quad u \in \mathcal{L}_1,\ f \in \mathcal{L}_\infty .$$

The domain of the operator P in $\mathcal{L}_1$ and $\mathcal{L}_\infty$ can be extended to M^+, the class of the nonnegative extended real valued measurable functions. For these extensions also the relation $\langle uP,f\rangle = \langle u,Pf\rangle$, $u \in M^+$, $f \in M^+$ holds.

We say that the Markov process P admits a finite invariant measure if there exists a function $u \in \mathcal{L}_1^+$ with $u \not\equiv 0$ such that $uP = u$.

The results of this section are collected in the next theorem:

Theorem 1.1. Let P be a Markov process on (X,Σ,m). The following statements are equivalent:

a) there does not exist a finite invariant measure.

b) for every $\varepsilon > 0$ there exists a function h with $0 < h \leq 1$ and $\langle 1,h\rangle > 1-\varepsilon$ such that

$$\liminf_{n\to\infty} \langle 1,P^n h\rangle = 0 .$$

c) for every $\varepsilon > 0$ there exists a function h with $0 \leq h \leq 1$ and $\langle 1,h\rangle > 1-\varepsilon$ and a sequence of integers $n_0 = 0 < n_1 < n_2 < \dots$ such that

$$\sum_{i=0}^{\infty} P^{n_i} h \leq 1 .$$

d) for every $\varepsilon > 0$ there exists a function h with $0 \leq h \leq 1$ and $\langle 1,h\rangle > 1-\varepsilon$ such that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} P^k h = 0 \quad \text{uniformly on } X .$$

If the process satisfies the nonsingularity condition $1P > 0$, then each of these statements is also equivalent to any of the following ones:

c') for every $\varepsilon > 0$ there exists a function h with $0 < h \leq 1$ and $\langle 1,h\rangle > 1-\varepsilon$ and a sequence of integers $n_0 = 0 < n_1 < \dots$ such that

$$\sum_{i=0}^{\infty} P^{n_i} h \leq 1 .$$

d') for every $\varepsilon > 0$ there exists a function h with $0 < h \leq 1$ and $\langle 1,h\rangle > 1-\varepsilon$ such that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} P^k h = 0 \quad \text{uniformly on } X .$$

Proof. We shall proceed along the following lines:

$$a) \Rightarrow b) \begin{cases} \Rightarrow c) \Rightarrow d) \Rightarrow a) & \\ \Rightarrow c') \Rightarrow d') \Rightarrow a) & \text{if } 1P > 0 \,. \end{cases}$$

a) ⇒ b). The equivalence of a) and b) is due to Neveu [9] and can also be found in Foguel [3]. We shall give a sketch of the proof of the implication a) ⇒ b).

Let L be a Banach limit (see e.g. [12], or [3], p. 33), and define $\lambda(h) = L(\langle 1,P^n h\rangle)$ for all $h \in \mathcal{L}_\infty$. Then $\lambda(\cdot)$ is a bounded positive linear functional on $\mathcal{L}_\infty$ satisfying $\lambda(Ph) = \lambda(h)$, and therefore there exists a non-negative charge λ with $\lambda P = \lambda$ and $\lambda << m$ such that

$$\lambda(h) = \int h \, d\lambda \quad \text{for all } h \in \mathcal{L}_\infty \,.$$

Now we use a result due to Calderon [1] which says that $\lambda = \mu + \lambda_0$, where μ is the largest σ-additive measure such that $\mu(A) \le \lambda(A)$ for all $A \in \Sigma$, and λ_0 is a pure charge.

From $\lambda P = \lambda$ we conclude $\lambda = \mu P + \lambda_0 P$, and since μP is σ-additive, we have $\mu P \le \mu$.

On the other hand, using $\lambda P = \lambda$ and $P1 \le 1$, we have

$$\mu P(X) = \mu(P1) = \lambda(P1) - \lambda_0(P1) \ge \lambda(1) - \lambda_0(1) = \mu(1) = \mu(X) \,,$$

and we obtain $\mu P = \mu$. Since P does not admit a finite invariant measure, we have $\mu \equiv 0$ and therefore $\lambda = \lambda_0$. Now because of $\lambda << m$ and the fact that λ is a pure charge, there exists a partition (mod m) $X_1, X_2, \ldots$, of X such that $\lambda(X_n) = 0$ for all n. Let (α_n) be a sequence with $0 < \alpha_n \le 1$ for all n and $\alpha_n \downarrow 0$ if $n \to \infty$, and define

$$h = \sum_{n=1}^{\infty} \alpha_n 1_{X_n} \,.$$

Note that $0 < h \le 1$, and that we can obtain $\langle 1,h\rangle > 1 - \varepsilon$ by choosing sufficiently many $\alpha_n = 1$. Since every step function f with $0 \le f \le h$ is positive on subsets of finitely many X_n, we have $\lambda(h) = 0$, hence $L(\langle 1,P^n h\rangle) = 0$. Statement b) now follows from the observation that

$$L(\langle 1,P^n h\rangle) \ge \liminf_{n\to\infty} \langle 1,P^n h\rangle \,.$$

b) ⇒ c). Condition c) is also due to Neveu [9]. The proof of the implication b) ⇒ c) can for instance be found in Foguel [3], p. 40, 41, where the function f has to be chosen such that $<1,f> > 1 - \frac{\varepsilon}{2}$, and the constant c such that $0 < c < \frac{\varepsilon}{2}$. The function g obtained in this proof then satisfies the conditions.

b) ⇒ c'). The proof of Neveu for the implication b) ⇒ c) to which we have referred in the book of Foguel [3], p. 40, 41 now needs a slight adaption. For convenience of the reader we shall write out the proof in detail. We start with some preliminaries.

Lemma 1.1. Let $A \in \Sigma$ and $h \in \mathcal{L}_\infty^+$ be given. For every sequence $n_0 = 0 < n_1 < n_2 < \ldots$ the following statements are equivalent:

i) there exists a function $u \in \mathcal{L}_1^+$ such that $\{u > 0\} = A$ and $\lim_{i\to\infty} <u,P^{n_i}h> = 0$,

ii) for all functions $u \in \mathcal{L}_1^+$ with $\{u > 0\} = A$ we have

$$\lim_{i\to\infty} <u,P^{n_i}h> = 0 .$$

Proof. ii) ⇒ i) is obvious. Suppose i) holds and take $v \in \mathcal{L}_1^+$ such that $\{v > 0\} = A$. Define $B_N = \{Nu < v\}$, then $v \le 1_{B_N} v + Nu$,

$$<v,P^{n_i}h> \le <v1_{B_N},P^{n_i}h> + N<u,P^{n_i}h>$$

$$\le \|h\|_\infty <v1_{B_N},1> + N<u,P^{n_i}h>$$

$$0 \le \limsup_{i\to\infty} <v,P^{n_i}h> \le \|h\|_\infty \cdot \int_{B_N} v\, dm .$$

Since $B_N \downarrow \emptyset$ if $N \to \infty$, it follows that

$$\lim_{i\to\infty} <v,P^{n_i}h> = 0 .$$

Lemma 1.2. Suppose $1P > 0$. If for some $h \in \mathcal{L}_\infty^+$ and some sequence $n_0 = 0 < n_1 < n_2 < \ldots$ we have

$$\lim_{i\to\infty} \langle 1, P^{n_i} h\rangle = 0 ,$$

then for all $u \in \mathcal{L}_1^+$ and all $n \in \mathbb{Z}$ we have

$$\lim_{i\to\infty} \langle u, P^{n_i+n} h\rangle = 0 \quad (\text{put } \langle u, P^{n_i+n} h\rangle = 0 \quad \text{if } n_i + n < 0) .$$

Proof. First assume $n \geq 0$. From lemma 1.1 we conclude

$$\lim_{i\to\infty} \langle uP^n, P^{n_i} h\rangle = \lim_{i\to\infty} \langle u, P^{n_i+n} h\rangle = 0 .$$

Now assume $n < 0$. From

$$\lim_{i\to\infty} \langle 1P^{-n}, P^{n_i+n} h\rangle = \lim_{i\to\infty} \langle 1, P^{n_i} h\rangle = 0$$

and the fact that $1P^{-n} > 0$ we conclude with lemma 1.1

$$\lim_{i\to\infty} \langle u, P^{n_i+n} h\rangle = 0 \quad \text{for every } u \in \mathcal{L}_1^+ .$$

Lemma 1.3. Suppose $1P > 0$. If for some $f \in \mathcal{L}_\infty^+$, $0 < \| f \|_\infty \leq 1$, and some sequence (n_i) we have $\lim_{i\to\infty} \langle 1, P^{n_i} f\rangle = 0$, then for every $\varepsilon > 0$ there exists a function $h \in \mathcal{L}_\infty^+$, $\| h \|_\infty > 0$, $0 \leq h \leq f$, $\langle 1, f - h\rangle > \varepsilon$ and a subsequence (n_j^*) of (n_i) such that $\sum_{j=0}^{\infty} P^{n_j^*} h \leq 1$.

Proof. The proof is practically idential to the proof of lemma C, chapter IV as given in Foguel [3]. The function h we are looking for will be of the type

$$h = \Big(f - \sum_{j=0}^{\infty} \sum_{i=0}^{j} P^{n_{j+1}^* - n_i^*} f\Big)^+$$

where the sequence (n_j^*) still has to be chosen. Note that for any choice of this sequence we have

$$0 \le f - h \le \sum_{j=0}^{\infty} \sum_{i=0}^{j} P^{n_{j+1}^* - n_i^*} f$$

$$= \sum_{j=0}^{\infty} \left(\sum_{i=0}^{j} P^{n_j^* - n_i^*} \right) P^{n_{j+1}^* - n_j^*} f ,$$

and therefore

$$\langle 1, f - h \rangle \le \sum_{j=0}^{\infty} \langle u_j, P^{n_{j+1}^* - n_j^*} f \rangle, \text{ where } u_j = \sum_{i=0}^{j} 1 P^{n_j^* - n_i^*} .$$

Without loss of generality we may assume $0 < \varepsilon < \langle 1, f \rangle$. We shall construct the subsequence (n_j^*) such that $\langle 1, f - h \rangle > \varepsilon$. Then necessarily we have $h \not\equiv 0$. Put $n_0^* = 0$. Suppose $n_0^* < \ldots < n_j^*$ have been constructed such that

$$\langle u_{i-1}, P^{n_i^* - n_{i-1}^*} f \rangle < \frac{\varepsilon}{2^i} \quad (1 \le i < j) .$$

Then because of lemma 1.2 we have

$$\lim_{i \to \infty} \langle u_j, P^{n_i - n_j^*} f \rangle = 0 ,$$

hence there exists a $n_{j+1}^* > n_j^*$ in the sequence (n_i) such that

$$\langle u_j, P^{n_{j+1}^* - n_j^*} f \rangle < \frac{\varepsilon}{2^{j+1}} .$$

For the subsequence (n_j^*) constructed in this way we indeed have

$$\langle 1, f - h \rangle < \sum_{j=0}^{\infty} \frac{\varepsilon}{2^{j+1}} = \varepsilon .$$

It remains to show $\sum_{j=0}^{\infty} P^{n_j^*} h \le 1$. To this end it suffices to prove by induction on k that for every $i \ge 0$ we have

$$\sum_{j=i}^{i+k} P^{n_j^* - n_i^*} h \le 1 .$$

For $k = 0$ the statement is obvious. Suppose the formula holds for some $k \ge 0$.

$$\sum_{j=i}^{i+k+1} P^{n_j^*-n_i^*} h = h + P^{n_{i+1}^*-n_i^*} \sum_{j=i+1}^{i+k+1} P^{n_j^*-n_{i+1}^*} h .$$

If $h(x) = 0$, then the statement follows from the induction hypothesis and the fact that $\| P^{n_{i+1}^*-n_i^*} \| \leq 1$. If $h(x) > 0$, then

$$h(x) = f(x) - \sum_{j=0}^{\infty} \sum_{i=0}^{j} P^{n_{j+1}^*-n_i^*} f(x) ,$$

hence

$$\sum_{j=i}^{i+k+1} P^{n_j^*-n_i^*} h(x) = h(x) + \sum_{j=i}^{i+k} P^{n_{j+1}^*-n_i^*} h(x) \leq h(x) + \sum_{j=i}^{i+k} P^{n_{j+1}^*-n_i^*} f(x)$$

$$\leq h(x) + \sum_{j=0}^{\infty} \sum_{i=0}^{j} P^{n_{j+1}^*-n_i^*} f(x) = f(x) \leq 1 .$$

We now continu the proof of the implication b) $\Rightarrow$ c'). Choose $\varepsilon > 0$, and take some function h_0 with $0 < h_0 \leq 1$, $<1,h_0> > 1 - \frac{\varepsilon}{2}$, and a sequence $n_0^0 = 0 < n_1^0 < n_2^0 < \ldots$ such that

$$\lim_{i\to\infty} <1,P^{n_i^0} h_0> = 0 .$$

By lemma 1.3 there exists a function h_1' with $0 \leq h_1' \leq h_0$ such that $<1,h_0 - h_1'> < \frac{\varepsilon}{4}$ and a subsequence (n_i^1) of (n_i^0) such that

$$\sum_{i=0}^{\infty} P^{n_i^1} h_1' \leq 1 .$$

Determine α such that $0 < \alpha < 1$, and $<1,\alpha h_1'> > 1 - \varepsilon$. Put $h_1 = \alpha h_1'$, then

$$\sum_{i=0}^{\infty} P^{n_i^1} h_1 \leq \alpha < 1 .$$

Define $A_1 = \{h_1 > 0\}$. Using lemma 1.3 we construct by an exhaustion procedure a sequence of disjoint sets $A_2, A_3, \ldots$, all also disjoint with A_1 and such that $X = \bigcup_{n=1}^{\infty} A_n$, a sequence of nonnegative functions $h_2, h_3, \ldots$ such that $\{h_k > 0\} = A_k$ and $0 \leq h_k \leq h_0$, and a sequence of sequences (n_i^k) such that for all $k \geq 2$ (n_i^k) is a subsequence of (n_i^{k-1}) and

$$\sum_{i=0}^{\infty} P^{n_i^k} h_k \leq \frac{1 - \alpha}{2^k} .$$

Without loss of generality we may assume that for every $k \geq 0$

$$(n_0^k, \ldots, n_k^k) = (n_0^{k+1}, \ldots, n_k^{k+1}) .$$

(Otherwise we add the missing terms and multiply h_k by a suitable constant.) Finally, define $h = \sum_{k=1}^{\infty} h_k$ and $n_k = n_k^k$, then we have $h > 0$, $\langle 1,h\rangle \geq \langle 1,h_1\rangle = 1 - \varepsilon$ and

$$\sum_{n=0}^{\infty} P^{n_k} h \leq 1 .$$

c) $\Rightarrow$ d) and c') $\Rightarrow$ d'). Both proofs are immediate consequences of the following lemma of which the proof is taken from Foguel [3], p. 42, 43.

Lemma 1.4. If for a function h with $0 \leq h \leq 1$ there exists a sequence (n_i) such that $\sum_{i=0}^{\infty} P^{n_i} h \leq 1$, then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} P^k h = 0 \quad \text{uniformly on } X .$$

Proof.

1) In the usual product topology, the space $[0,1]^\Sigma$ (i.e the set of all mappings from the σ-algebra Σ into $[0,1]$) is a compact Hausdorff space and therefore sequentially compact. In this space the class of all nonnegative charges λ with $\lambda(X) \leq 1$ is closed, and therefore sequentially compact. In fact, if $\mu_0 \in [0,1]^\Sigma$ is not finitely additive, there exists sets $A_0, A_1, \ldots, A_n$ in Σ such that A_0 is the disjoint union of $A_1, \ldots, A_n$, and

$$\mu_0(A_0) \neq \sum_{i=1}^{n} \mu_0(A_i) .$$

Put

$$\varepsilon = \left|\mu_0(A) - \sum_{i=1}^{n} \mu_0(A_i)\right|$$

and consider

$$U = \{\mu \in [0,1]^\Sigma \mid |\mu(A_i) - \mu_0(A_i)| < \frac{\varepsilon}{n+1} , 0 \leq i \leq n\} .$$

The set U is an open neighborhood of μ_0 which does not contain a nonnegative charge.

2) Let λ be a charge with $\lambda(X) \leq 1$, and suppose λ is invariant, i.e. $\lambda P = \lambda$. Then for every r we have

$$1 \geq \lambda\left(\sum_{i=0}^{r} P^{n_i} h\right) = (r + 1)\lambda(h) ,$$

hence $\lambda(h) = 0$.

3) Fix $\delta > 0$ and define

$$A_k = \{\frac{1}{k} \sum_{i=0}^{k-1} P^i h \geq \delta\} .$$

Suppose there exists a sequence (k_i) such that $m(A_{k_i}) > 0$ for every i. Define

$$\mu_i(B) = \frac{m(A_{k_i} \cap B)}{m(A_{k_i})} ,$$

then μ_i is a probability with support A_{k_i}. Put

$$\lambda_i = \frac{1}{k_i} \sum_{j=0}^{k_i - 1} \mu_i P^j ,$$

then λ_i is a measure with $\lambda_i(X) \leq 1$, and

$$\lambda_i(h) = \langle \mu_i, \frac{1}{k_i} \sum_{j=0}^{k_i - 1} P^j h \rangle \geq \delta \mu_i(A_{k_i}) = \delta .$$

Let λ be a limit point in $[0,1]^{\Sigma}$ of the sequence (λ_i), then λ is a nonnegative finitely additive set function. Without loss of generality we may assume $\lambda_i \to \lambda$ if $i \to \infty$, which implies $\lambda_i(B) \to \lambda(B)$ for all $B \in \Sigma$, and therefore $\lambda_i(f) \to \lambda(f)$ for all $f \in \mathcal{L}_\infty^+$. It follows that $\lambda(h) \geq \delta$.

4) If we can show that $\lambda P = \lambda$, then the results in 2) and 3) give a contradiction, and therefore the assumption that $m(A_k) > 0$ for infinitely many k will be wrong. This implies that

$$\frac{1}{n} \sum_{k=0}^{n-1} P^k h \to 0 \quad \text{uniformly on } X .$$

In order to prove $\lambda P = \lambda$, note that for all $B \in \Sigma$

$$|\lambda_i(B) - \lambda_i P(B)| = \frac{1}{k_i} |\mu_i(B) - \mu_i P^{k_i}(B)| \leq \frac{2}{k_i} .$$

Now let $i \to \infty$, then we obtain $\lambda P(B) = \lambda(B)$ for all $B \in \Sigma$.

d) ⇒ a) and d') ⇒ a). Since obviously d') ⇒ d) it remains to show that $uP = u$ for some $u \in \mathcal{L}_1^+$ implies $u \equiv 0$ if condition d) holds. Put $A = \{u > 0\}$ and suppose $m(A) = \alpha > 0$. Take a function h which satisfies condition c) with $\varepsilon = \alpha$ then $\langle u,h \rangle > 0$. On the other hand we have

$$\langle u,h \rangle = \langle \frac{1}{n} \sum_{k=0}^{n-1} uP^k, h \rangle = \langle u, \frac{1}{n} \sum_{k=0}^{n-1} P^k h \rangle \to 0 \quad \text{if } n \to \infty .$$

Contradiction, hence $\alpha = 0$, $u \equiv 0$.

2. Measurable transformations

In this section we want to apply the previous results on the Markov process induced by a measurable transformation. We shall also discuss a recent extension of Jones and Krengel [8] of the weakly wandering set theorem of Hajian-Kakutani [4].

Let T be a nonsingular transformation on a probability space (X,Σ,m), i.e. T is a mapping of (almost all of) X into itself such that for all $A \in \Sigma$ we have $T^{-1}A \in \Sigma$, and $m(A) = 0$ iff $m(T^{-1}A) = 0$.

For every $f \in \mathcal{L}_\infty$ we can define Pf by

$$Pf = f \circ T .$$

It is easily verified that P is a Markov process on (X,Σ,m) satisfying $P1 = 1$ and $1P > 0$.

A finite measure $\mu \ll m$ is said to be invariant under T if $\mu(T^{-1}A) = \mu(A)$ for all $A \in \Sigma$.

Let $\mu \ll m$ be a finite measure, and put $u = \frac{d\mu}{dm}$. Then μ is invariant under T if and only if

$$\mu(T^{-1}A) = \int u(1_A \circ T)\,dm = \int u\, 1_A dm = \mu(A) \quad \text{for all } A \in \Sigma ,$$

hence if and only if

$$\int_A (uP)\,dm = \int_A u\, dm \quad \text{for all } A \in \Sigma ,$$

$$uP = u .$$

Now let the transformation T have the property that there exists no positive invariant finite measure $\mu \ll m$. Then there is no $u \in \mathcal{L}_1^+$ with $uP = u$, and $u \not\equiv 0$, and by theorem 1.1 there exists a function h with $0 < h \leq 1$ such that

$$\frac{1}{n}\sum_{k=0}^{n-1} P^k h \to 0 \quad \text{uniformly} .$$

Define $A = \{x \mid h(x) > \delta\}$, then it follows

$$\frac{1}{n}\sum_{k=0}^{n-1} 1_A \circ T^k \to 0 \quad \text{uniformly} .$$

By choosing δ sufficiently small, we can get A arbitrary close to X. Since obviously for any set $B \subset A$ we also have

$$\frac{1}{n}\sum_{k=0}^{n-1} 1_B \circ T^k \to 0 \quad \text{uniformly}$$

we completed the proof in one direction of the following theorem which is a slight extension of a theorem of Dowker [2]. (See also Foguel [3], chapter IV, theorem E).

Theorem 2.1. Let T be a nonsingular measurable transformation on a probability space (X,Σ,m). Then there exists no positive finite invariant measure $\mu \ll m$ if and only if the class of sets $A \in \Sigma$ for which

$$\frac{1}{n}\sum_{k=0}^{n-1} 1_A \circ T^k \to 0 \quad \text{uniformly on } X$$

is dense in Σ.

The proof of this theorem in the other direction is an immediate consequence of the ergodic theorem (cf. [5], p. 18) or of theorem 1.1.

Definition 2.1. A set $E \in \Sigma$ is said to be a sweep-out set if $\sum_{n=0}^{\infty} T^{-n}E = X$.

Note that if E is a sweep-out set, then since $X = \bigcup_{k=n}^{\infty} T^{-k}E$, also $T^{-n}E$ is a sweep-out set for every n. For later reference we state the following lemma here.

Lemma 2.1. If T admits no positive finite invariant measure $\mu \ll m$, then for every $\varepsilon > 0$ and every p there exists a sweep-out set E such that E, $T^{-1}E,\dots,T^{-p}E$ are disjoint and

$$m\left(\bigcup_{k=0}^{p} T^{-k}E\right) < \varepsilon .$$

Proof. Since on the periodic part of X there exists a positive finite invariant measure (cf. [7]), it follows that T is aperiodic. Hence, by [7], theorem 2.1 there exists a sweep-out set A such that $A,T^{-1}A,\dots,T^{-\ell}A$ are disjoint, where ℓ is chosen such that $\ell > \frac{p+1}{\varepsilon}$.
Then it follows that for at least one $n < \ell - p$ we must have

$$m\left(\bigcup_{k=n}^{n+p} T^{-k}A\right) < \varepsilon .$$

The set $E = T^{-n}A$ now satisfies the conditions.

Remark. There also exist some results related to theorem 2.1. In [11] it is shown that there exists a sweep-out set B with

$$\frac{1}{n}\sum_{k=0}^{n-1} 1_B \circ T^k \to 0 \qquad \text{(not necessarily uniformly)}$$

if and only if T does not admit a positive finite invariant measure $\mu \ll m$. It is also shown in [11] that in this case the sweep-out set B may be chosen arbitrary small, and it follows that for every $A \in \Sigma$ for every $\alpha \in [0,1]$ and for every $\varepsilon > 0$, there exists a set $A' \in \Sigma$ with $m(A \Delta A') < \varepsilon$ such that

$$\frac{1}{n}\sum_{k=0}^{n-1} 1_{A'} \circ T^k \to \alpha .$$

We now turn to the Hajian-Kakutani theorem.

Definition 2.2. A set $W \in \Sigma$ is said to be weakly wandering if there exists a sequence $n_0 = 0 < n_1 < n_2 < \dots$ such that $W,T^{-n_1}W,\dots$ are disjoint.

Theorem 2.2 (Hajian-Kakutani [4]). Let T be a nonsingular measurable transformation on a probability space (X,Σ,m). Then there exists no positive finite invariant measure $\mu \ll m$ if and only if the class of weakly wandering sets is dense in Σ.

Proof. Since every subset of a weakly wandering set again is weakly wandering, it suffices to show that there exist weakly wandering sets arbitrary close to X.

By theorem 1.1 there exists a function h with $0 < h \le 1$ and a sequence (n_i) such that $\sum_{i=0}^{\infty} P^{n_i} h \le 1$. Put $A = \{x \mid h(x) > \frac{2}{3}\}$, then $1_A < \frac{3}{2} h$, and

$$\sum_{i=0}^{\infty} P^{n_i} 1_A = \sum_{i=0}^{\infty} 1_A \circ T^{n_i} < \frac{3}{2} .$$

Since $\sum_{i=0}^{\infty} 1_A \circ T^{n_i}$ is integer valued, it follows that $\sum_{i=0}^{\infty} 1_A \circ T^{n_i} = \{^0_1$ on X, and therefore the sets $A, T^{-n_1}A, \ldots$ are disjoint, A is weakly wandering. Since the function h can be chosen arbitrary close to 1, it follows that the set A can be constructed arbitrary close to X.

Conversely, let B be an invariant set on which a positive finite invariant measure $\mu \approx m$ (on B) exists. Then because of the finiteness of μ every weakly wandering subset A of B has μ-measure, and therefore m-measure 0. Since the wandering sets are dense in Σ it follows $m(B) = 0$, hence $\mu(B) = 0$.

Definition 2.3. A set $W \in \Sigma$ is said to be exhaustive weakly wandering if there exists a sequence $n_0 = 0 < n_1 < n_2 < \ldots$ such that $W, T^{-n_1}W, T^{-n_2}W, \ldots$ are disjoint and

$$\bigcup_{i=0}^{\infty} T^{-n_i} W = X .$$

Lemma 2.2. For every exhaustive weakly wandering set W we have $W \in \bigcap_{n=0}^{\infty} T^{-n}\Sigma$.

Proof. Let W be an exhaustive weakly wandering set. Then it follows from the definition that $W \in T^{-n_1}\Sigma$, i.e. there exists a set W_1 such that $W = T^{-n_1}W_1$. Hence

$$X = \bigcup_{i=0}^{\infty} T^{-n_i} W = \bigcup_{i=0}^{\infty} T^{-n_i} T^{-n_1} W_1 = T^{-n_1} \bigcup_{i=0}^{\infty} T^{-n_i} W_1 .$$

Since T is nonsingular, it follows that $X = \bigcup_{i=0}^{\infty} T^{-n_i} W_1$, and the sets $T^{-n_i}W_1$ are disjoint $(i = 0,1,\ldots)$.

The set W_1 is therefore exhaustive weakly wandering under the same sequence $n_0 = 0, n_1, n_2, \ldots$. Repeating this argumentation, we construct a sequence $W_1, W_2, \ldots$ of exhaustive weakly wandering sets such that $W_{k-1} = T^{-n_1} W_k$ for $k = 1,2,\ldots$. Hence $W = T^{-kn_1} W_k \in T^{-kn_1}\Sigma$ for all k. Since the sequence $(T^{-n}\Sigma)$ is a decreasing sequence of σ-algebra's it follows that

$$W \in \bigcap_{k=0}^{\infty} T^{-kn_1}\Sigma = \bigcap_{n=0}^{\infty} T^{-n}\Sigma .$$

Recently, Jones and Krengel [8] have shown that, under the condition that T is invertible, there exists no positive finite invariant measure $\mu \ll m$ if and only if the class of exhaustive weakly wandering sets is dense in Σ.
We shall show that we can replace the condition that T is invertible by the condition $T^{-1}\Sigma = \Sigma$ (which is hardly a weakening). Since by the previous lemma exhaustive weakly wandering sets are elements of the tail σ algebra $\Sigma_\infty = \bigcap_{n=0}^{\infty} T^{-n}\Sigma$, there exist arbitrary large exhaustive weakly wandering sets if and only if the transformation T does not admit a finite invariant measure on the measure space (X,Σ_∞,m). Obviously on (X,Σ_∞,m) the condition $T^{-1}\Sigma_\infty = \Sigma_\infty$ is satisfied.
In [10] an example is given of a dissipative transformation with a trivial tail σ-algebra, hence of a transformation without a finite invariant measure for which no exhaustive weakly wandering sets exist.

We shall now give a modified proof of the theorem of Jones and Krengel.

Theorem 2.3 (Jones-Krengel [8]). Let T be a nonsingular measurable transformation on a probability space (X,Σ,m) such that $T^{-1}\Sigma = \Sigma$. Then there exists no positive finite invariant measure $\mu \ll m$ if and only if the class of exhaustive weakly wandering sets is dense in Σ.

Proof. Since exhaustive weakly wandering sets are weakly wandering, one direction of the proof is immediate.
For the proof in the other direction we need some preliminaries.

From $T^{-1}\Sigma = \Sigma$ and the nonsingularity of T we conclude that for every $A \in \Sigma$ there exists a (mod m) unique set $B \in \Sigma$ such that $A = T^{-1}B$. We shall denote this set by TA. Note that, while $T^{-1}A$ is the set of all points which are mapped by T into A, the set TA is in general not the set of all images of points of A.

The following properties are easily verified:

i) $TT^{-1}A = T^{-1}TA = A$ for all $A \in \Sigma$

ii) if $A_1, A_2, \ldots$ are disjoint, then $TA_1, TA_2, \ldots$ are disjoint and

$$T\left(\bigcup_{n=1}^{\infty} A_n\right) = \bigcup_{n=1}^{\infty} TA_n .$$

It follows that if we define the operator P^{-1} on $\mathcal{L}_\infty$ by

$$P^{-1} 1_A = 1_{TA} \quad \text{for all } A \in \Sigma ,$$

then P^{-1} induces a Markov process on (X,Σ,m) satisfying

$$PP^{-1} = P^{-1}P = I .$$

Actually, P is the forward process and P^{-1} is the backward process associated with the transformation T, cf [6].

The proof of theorem 2.3 is based on the existence of arbitrary small sweep-out sets, with arbitrary many disjoint preimages (lemma 2.1), and the following result.

Lemma 2.3. If $T^{-1}\Sigma = \Sigma$ and there does not exist a positive finite invariant measure $\mu \ll m$, then for every $\varepsilon > 0$ there exist a set $A \in \Sigma$ and an integer p such that A, T^pA and $T^{-p}A$ are disjoint and $m(A) > 1 - \varepsilon$.

Proof. From the fact that $uP = u$ implies $u = 0$, we conclude that also P^{-1} and P^2 do not admit positive finite invariant measures. In fact, if $uP^{-1} = u$, then $u = uP^{-1}P = uP$, hence $u = 0$, and if $uP^2 = u$, then $(u + uP)P = uP + u$, $u + uP = 0$, $u = 0$.

Hence by theorem 1.1 there exist functions $h_1 > 0$, $h_2 > 0$, $h_3 > 0$ such that

$$\frac{1}{n}\sum_{k=0}^{n-1} P^k h_1 \to 0 \quad \text{uniformly}$$

$$\frac{1}{n}\sum_{k=0}^{n-1} P^{-k} h_2 \to 0 \quad \text{uniformly}$$

$$\frac{1}{n}\sum_{k=0}^{n-1} P^{2k} h_3 \to 0 \quad \text{uniformly} .$$

Put $h = \min(h_1, h_2, h_3)$, then $h > 0$ and

$$\frac{1}{n}\sum_{k=0}^{n-1} (P^k h + P^{-k} h + P^{2k} h) \to 0 \quad \text{uniformly} .$$

Define $A' = \{x \mid h(x) > \delta\}$, where δ is chosen such that $\delta > 0$ and $m(A) > 1 - \frac{\varepsilon}{2}$. It follows that

$$\frac{1}{n}\sum_{k=0}^{n-1} (1_{T^{-k}A'} + 1_{T^{k}A'} + 1_{T^{-2k}A'}) \to 0 \quad \text{uniformly}$$

$$\frac{1}{n}\sum_{k=0}^{n-1} (m(T^{-k}A') + m(T^{k}A') + m(T^{-2k}A')) \to 0 .$$

Hence there exists an integer p such that

$$m(T^{-p}A') + m(T^{p}A') + m(T^{-2p}A') < \frac{\varepsilon}{2} .$$

Put $B = T^{-p}A' \cup T^{p}A' \cup T^{-2p}A'$, and $A = A' \setminus B$, then $m(A) > 1 - \varepsilon$. Since $T^pA \subset B$ and $T^{-p}A \subset B$, we have $A \cap T^pA = \emptyset$ and $A \cap T^{-p}A = \emptyset$. Finally from $T^{-p}(T^pA \cap T^{-p}A) = A \cap T^{-2p}A \subset A \cap B = \emptyset$ we conclude $T^pA \cap T^{-p}A = \emptyset$, since T and therefore T^p is nonsingular.

The rest of the proof of theorem 2.3 is rather technical. We first give a rough scetch before writing the proof out in detail. In this scetch the notation $A \sim A'$ will stand for A and A' differ as little as we want.
Start with some set $A \in \Sigma$, fix an integer $L > 0$ and consider $\bigcup_{n=0}^{L} T^{-n}A$. In step 2, using a technical result given in step 1, and lemma 2.3, a set $A' \sim A$ is obtained and an integer $p > L$ such that

$$\bigcup_{n=0}^{L} T^{-n}A',\ T^{p}\Big(\bigcup_{n=0}^{L} T^{-n}A'\Big),\ T^{-p}\Big(\bigcup_{n=0}^{L} T^{-n}A'\Big) \text{ are disjoint} ,$$

$$\bigcup_{n=0}^{L} T^{-n}A' \sim \bigcup_{n=0}^{L} T^{-n}A,\ T^{p}(\bigcup_{n=0}^{L} T^{-n}A') \sim \emptyset,\ T^{-p}(\bigcup_{n=0}^{L} T^{-n}A') \sim \emptyset .$$

This enables us by means of an exchange procedure (step 3) to construct a set $A'' \sim A'$ and a sequence $n_0 = 0 < n_1 < \ldots < n_k$ such that $A'', T^{-n_1}A'', \ldots, T^{-n_k}A''$ are disjoint and

$$\bigcup_{n=0}^{L} T^{-n}A \sim \bigcup_{n=0}^{L} T^{-n}A' \sim \bigcup_{i=0}^{k} T^{-n_i}A'' .$$

Because of the sweep-out set lemma 2.1 we may suppose that A'' thus obtained is a sweep-out set (step 4), and therefore

$$\bigcup_{n=0}^{L'} T^{-n}A'' \sim X$$

if L' is chosen sufficiently large. Repeating the construction from A to A'', but now starting with A'', yields a set $A_1 \sim A$ such that $A_1, T^{-n_1}A_1, \ldots, T^{-n_{k_1}}A_1$ are disjoint and $\sum_{i=0}^{k_1} m(T^{-n_i}A_1) > 1 - \varepsilon$. In step 5 finally this construction is extended to the construction of an exhaustive weakly wandering set $B \sim A$.

We shall now perform each of these steps in detail. Throughout, we shall assume $T^{-1}\Sigma = \Sigma$ and T does not admit a finite positive measure $\mu << m$.

<u>Step 1</u>. Let the set $A' \in \Sigma$ and the integer L be given. Then for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for every $B \in \Sigma$ with $m(B) < \delta$ there exists a set $A' \subset A$ with $m(A \setminus A') < \varepsilon$ and $\bigcup_{n=0}^{L} T^{-n}A' \cap B = \emptyset$.

<u>Proof</u>. Put $A' = A \setminus \bigcup_{n=0}^{L} T^{n}(T^{-n}A \cap B)$, then we have $\bigcup_{n=0}^{L} T^{-n}A' \cap B = \emptyset$.

Since T is nonsingular, the measures m and mT^n are equivalent. This implies that for every $\varepsilon > 0$ there exists a $\delta_n > 0$ such that if $m(B) < \delta_n$, we have $mT^n(B) < \frac{\varepsilon}{L+1}$.
Take $\delta = \min(\delta_0, \ldots, \delta_L)$, then $m(A \setminus A') < \varepsilon$ if $m(B) < \delta$.

Step 2. Let $A \in \Sigma$ and the integer L be given. Then for every $\varepsilon > 0$ there exists a set $A' \subset A$ and an integer p such that $m(A \setminus A') < \varepsilon$,

$$\bigcup_{n=0}^{L} T^{-n}A', \; T^{p}(\bigcup_{n=0}^{L} T^{-n}A') \text{ and } T^{-p}(\bigcup_{n=0}^{L} T^{-n}A') \text{ are disjoint}$$

and

$$mT^{p}(\bigcup_{n=0}^{L} T^{-n}A') < \varepsilon, \; mT^{-p}(\bigcup_{n=0}^{L} T^{-n}A') < \varepsilon .$$

Proof. Take $\varepsilon > 0$, and determine $\delta > 0$ as in step 1. We may assume $\delta < \varepsilon$. By lemma 2.3 there exists a set E and an integer p such that $m(E) > 1 - \delta$, and $E, T^{p}E, T^{-p}E$ are disjoint.
Put $B = \bigcup_{n=0}^{L} T^{-n}A \setminus E$, then $m(B) < \delta$. Finally let A' be as in step 1, then A' satisfies the conditions.

Step 3 (Exchange procedure). Let there be given a set $A \in \Sigma$, integers $n_0 = 0 < n_1 < \ldots < n_k$, $0 < m_1 < \ldots < m_\ell$, and sets $A_1, \ldots, A_\ell$ all contained in A, and real numbers $\alpha_0, \ldots, \alpha_k$, $\beta_1, \ldots, \beta_\ell$ such that

i) $A, T^{-n_1}A, \ldots, T^{-n_k}A, T^{-m_1}A_1, \ldots, T^{-m_\ell}A_\ell$ are disjoint

ii) $$m(T^{-n_i}A) > \alpha_i, \; 0 \le i \le k$$

$$m(T^{-m_i}A_i) > \beta_i, \; 1 \le i \le \ell .$$

Then for every $\varepsilon > 0$ there exists a set $A' \in \Sigma$ with $m(A \,\Delta\, A') < \varepsilon$ and an integer $n_{k+1} > n_k$ such that

i') $A', T^{-n_1}A', \ldots, T^{-n_k}A', T^{-m_1}(A_1 \cap A'), \ldots, T^{-m_{\ell-1}}(A_{\ell-1} \cap A'), T^{-n_{k+1}}A'$ are disjoint.

ii') $$m(T^{-n_i}A') > \alpha_i, \; 0 \le i \le k$$

$$m(T^{-m_i}(A_i \cap A')) > \beta_i, \; 0 \le i \le \ell-1$$

$$m(T^{-n_{k+1}}A') > \beta_\ell .$$

Proof. Fix $L \geq n_k + m_\ell$. Because of step 2 and the equivalency of the measures m, mT^{-n_i} and mT^{-m_i} there exists a set $A_0 \subset A$ with $m(A \backslash A_0) < \frac{\varepsilon}{2}$ such that the conditions i) and ii) hold with A replaced by A_0, and moreover

$$\bigcup_{n=0}^{L} T^{-n}A_0,\ T^p(\bigcup_{n=0}^{L} T^{-n}A_0),\ T^{-p}(\bigcup_{n=0}^{L} T^{-n}A_0) \text{ are disjoint },$$

$$m(T^p(\bigcup_{n=0}^{L} T^{-n}A_0)) < \frac{\varepsilon}{2},\ m(T^{-p}(\bigcup_{n=0}^{L} T^{-n}A_0)) < \frac{\varepsilon}{2} \quad \text{for some } p > L .$$

Then define $A' = A_0 \cup T^{p-m_\ell}(A_\ell \cap A_0)$. From $T^{p-m_\ell}(A_\ell \cap A_0) \subset T^p(\bigcup_{n=0}^{L} T^{-n}A_0)$ we conclude $mT^{p-m_\ell}(A_\ell \cap A_0) < \frac{\varepsilon}{2}$, hence $m(A' \Delta A) < \varepsilon$.
Since $A_\ell \subset A$, we obtain the following survey:

α) $$T^{p-m_\ell}(A_\ell \cap A_0), T^{p-m_\ell-n_1}(A_\ell \cap A_0), \ldots, T^{p-m_\ell-n_k}(A_\ell \cap A_0)$$

are disjoint subsets of $T^p(\bigcup_{n=0}^{L} T^{-n}A_0)$.

β) $$A_0, T^{-n_1}A_0, \ldots, T^{-n_k}A_0 ,$$

$$T^{-m_1}(A_1 \cap A_0), \ldots, T^{-m_{\ell-1}}(A_{\ell-1} \cap A_0), T^{-p}(T^{p-m_\ell}(A_\ell \cap A_0))$$

are disjoint subsets of $\bigcup_{n=0}^{L} T^{-n}A_0$.

γ) $$T^{-p}A_0 \text{ is a subset of } T^{-p}(\bigcup_{n=0}^{L} T^{-n}A_0) .$$

Now using $A_i \cap A' = A_i \cap A_0$ for $1 \leq i \leq \ell$ and the fact that condition ii) holds for A replaced by A_0, the verification of the condition i') and ii') for A' with $p = n_k + 1$ is straightforward.

Step 4. Let $A \in \Sigma$ and suppose $A, T^{-n_1}A, \ldots, T^{-n_k}A$ are disjoint. Fix $L > n_k$. If $m(T^{-n_i}A) > \alpha_i$ $(0 \leq i \leq k)$ and $m(\bigcup_0^L T^{-n}A) > \beta$, then for every $\varepsilon > 0$ there exists a sweep-out set A' with $m(A \Delta A') < \varepsilon$ and integers $n_{k+1}, \ldots, n_{k+p}$ such that

$$A', T^{-n_1}A', \ldots, T^{-n_k}A', T^{-n_{k+1}}A', \ldots, T^{-n_{k+p}}A' \text{ are disjoint}$$

$$m(T^{-n_i}A') > \alpha_i \quad (0 \le i \le k) \text{ and } \sum_{i=0}^{k+p} m(T^{-n_i}A') > \beta .$$

Proof. It is easy to verify that for a suitable choice of the subsets $A_1,\dots,A_p$ of A and the integers $m_1,\dots,m_p$ we have

$$\bigcup_{n=0}^{L} T^{-n}A = A \cup T^{-n_1}A \cup \dots \cup T^{-n_k}A \cup T^{-m_1}A_1 \cup \dots \cup T^{-m_p}A_p ,$$

where the sets on the right hand side are disjoint.
If we apply the exchange procedure p times, we obtain a set A" with $m(A\Delta A'') < \frac{\varepsilon}{3}$ and integers $n_{k+1},\dots,n_{k+p}$ such that

$$A'',T^{-n_1}A'',\dots,T^{-n_{k+p}}A'' \text{ are disjoint ,}$$

$$m(T^{-n_i}A'') > \alpha_i \text{ for } 0 \le i \le k, \text{ and } \sum_{i=0}^{k+p} m(T^{-n_i}A'') > \beta .$$

The only thing we still have to show is the sweep-out property.
By step 1 and the equivalence of the measures $m, mT^{-n_1},\dots,mT^{-n_{k+p}}$ there exists a $\delta > 0$ such that if $m(B) < \delta$, there exists a set $A''' \subset A''$ with

$$m(A''\backslash A''') < \frac{\varepsilon}{3} \quad A''',T^{-n_1}A''',\dots,T^{-n_{k+p}}A''',B \text{ disjoint ,}$$

$$m(T^{-n_i}A''') > \alpha_i \quad (0 \le i \le k), \quad \sum_{i=0}^{k+p} m(T^{-n_{k+p}}A''') > \beta .$$

By lemma 2.1 there exists a sweep-out set E such that $E,T^{-1}E,\dots,T^{-n_{k+p}}E$ are disjoint and $m(\bigcup_{n=0}^{n_{k+p}} T^{-n}E) < \min(\delta, \frac{\varepsilon}{3})$. If we define $B = \bigcup_{n=0}^{n_{k+p}} T^{-n}E$ and $A' = A''' \cup E$, then A' satisfies the conditions.

Step 5. Choose $A \in \Sigma$ and $\varepsilon > 0$. Let E_0 be a sweep-out set with $m(E_0) < \frac{\varepsilon}{2}$, and put $A_0 = A \cup E_0$, then A_0 is a sweep-out set. Put $n_0 = 0$.
We now proceed by induction. Suppose after step p ($p \ge 0$) we have found a sweep-out set A_p with $m(A_p \Delta A_{p-1}) < \frac{\varepsilon}{2^{p+1}}$ ($A_{-1} = A$), and a sequence $n_0 = 0 < \dots < n_{k_1} < \dots < n_{k_p}$ such that the sets $T^{-n_i}A_p$ are disjoint for $0 \le i \le k_p$, and

$$\sum_{i=0}^{k_j} m(T^{-n_i}A_p) > 1 - \frac{1}{j} \quad \text{for } 1 \le j \le p .$$

Then by 4) there exists a sweep-out set A_{p+1} and integers $n_{k_p+1},\dots,n_{k_{p+1}}$ such that $m(A_{p+1}\Delta A_p) < \frac{\varepsilon}{2^{p+2}}$, the sets $T^{-n_i}A_{p+1}$ are disjoint for $0 \le i \le k_{p+1}$, and

$$\sum_{i=0}^{k_j} m(T^{-n_i}A_{p+1}) > 1 - \frac{1}{j} \quad \text{for } 1 \le j \le p+1 .$$

Finally we shall show that the set

$$B = \bigcap_{q=1}^{\infty} \bigcup_{p=q}^{\infty} A_p$$

is an exhaustive weakly wandering set under the sequence $n_0 = 0, n_1, \dots, n_{k_p}, \dots$ $\dots, n_{k_{p+1}}, \dots$ with $m(A\Delta B) < \varepsilon$.

From $\quad A\Delta B \subset \bigcup_{p=0}^{\infty} A_{p-1}\Delta A_p$

we conclude $m(A\Delta B) < \varepsilon$.

Fix p. Then for all $q \ge p$ we have

$$\sum_{i=0}^{k_p} m(T^{-n_i}A_q) > 1 - \frac{1}{p} ,$$

and therefore for all $q \ge p$

$$\sum_{i=0}^{k_p} mT(^{-n_i}(\bigcup_{p=q}^{\infty} A_p) > 1 - \frac{1}{p} .$$

Hence, if $q \to \infty$

$$\sum_{i=0}^{k_p} m(T^{-n_i}B) \ge 1 - \frac{1}{p} .$$

It follows that

$$\sum_{i=0}^{\infty} m(T^{-n_i}B) = 1 .$$

It remains to show that $T^{-n_i}B \cap T^{-n_j}B = \emptyset$ if $i \neq j$. Choose $\varepsilon > 0$, and take $\delta > 0$ such that if $m(E) < \delta$, we have

$$m(T^{-n_i}E \cup T^{-n_j}E) < \varepsilon \ .$$

Since

$$(A_p \Delta B) \subset \bigcup_{q=p}^{\infty} (A_{q+1} \Delta A_q) \ ,$$

there exists an integer p such that $m(A_p \Delta B) < \delta$. Hence

$$m((T^{-n_i}B \Delta T^{-n_i}A_p) \cup (T^{-n_j}B \Delta T^{-n_j}A_p)) < \varepsilon \ ,$$

and because of $T^{-n_i}A_p \cap T^{-n_j}A_p = \emptyset$, we conclude

$$m(T^{-n_i}B \cap T^{-n_j}B) < \varepsilon \ .$$

Since ε is arbitrary, we have

$$m(T^{-n_i}B \cap T^{-n_j}B) = 0 \ .$$

This completes the proof of the theorem.

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