A description of the fundamentals of the spectral element method

A DESCRIPTION OF THE FUNDAMENTALS OF THE SPECTRAL ELEMENT METHOD.

by

L.J.P. Timmermans[1]
J.K.M. Jansen[2]
F.N. van de Vosse[1]

[1] Department of Mechanical Engineering (Division WFW)
[2] Department of Mathematics & Computer Science
Eindhoven University of Technology
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SUMMARY.

Spectral element methods are high order weighted residual techniques for partial differential equations. They combine the geometric flexibility of the well known h– and p–type finite element method with the attractive convergence properties of spectral methods. Thus they are well suited for application of complex problems in complex geometries.

In the spectral element discretization the domain is divided into (spectral) elements, and the dependent and independent variables are approximated by high order polynomial expansions within the individual subdomains. In order to decrease the demands of continuity at element boundaries, the partial differential equation is written in its equivalent variational form. The variational formulation is taken as a basis for the discretization process, in which high order Gauss–type numerical quadrature plays an essential role. Convergence of the approximate solution to the exact solution of the partial differential equation is obtained in a similar way as in spectral methods, by increasing the order of the approximation. The number of elements is kept fixed.

In this report the discretization process of spectral element methods applied to a class of one–dimensional elliptic equations is described in detail. The final set of discrete equations will be presented and discussed. Also some approximation results will be given, that indicate the high accuracy of spectral element methods. Moreover, the application of spectral element methods to a non–linear testcase, the steady viscous Burgers equation is given. For both problems numerical results are presented.

LIST OF SYMBOLS.

Conventions.

A	matrix
$\underline{a}$	vector
Ω	one dimensional region
Γ	boundary of Ω
$\mathscr{C}^0(\overline{\Omega})$	space of continuous functions on $\overline{\Omega}$
$\mathscr{L}_w^2(\Omega)$	Lebesgue space, $\mathscr{L}_w^2(\Omega) = \{ v \mid [\int_\Omega v^2 w \, d\Omega]^{1/2} < \infty \}$
$\mathscr{H}_0^1(\Omega)$	Sobolev space, $\mathscr{H}_0^1(\Omega) = \{ v \in \mathscr{L}^2(\Omega) \mid v_x \in \mathscr{L}^2(\Omega), v\|_\Gamma = 0 \}$
$(.,.)_w$	continuous inner product in $\mathscr{L}_w^2(\Omega)$
$\| \cdot \|_w$	corresponding continuous norm
$\mathscr{P}_n$	space of algebraic polynomials of degree $\leq n$

Symbols.

$\tilde{u}_n(x)$	continuous spectral truncated series expansion of $u(x)$
$\tilde{u}_i(t)$	continuous spectral expansion coefficients
$\psi_i(x)$	spectral expansion or basisfunctions
λ_i	eigenvalues of a singular Sturm–Liouville problem
$w(x)$	weight function
$v(x)$	test function
$T_n(x)$	Chebyshev polynomial of order n
$L_n(x)$	Legendre polynomial of order n
$\overline{u}_n(x)$	discrete spectral interpolating polynomial of $u(x)$
$\overline{u}_i(t)$	discrete spectral expansion coefficients
x_j	Gauss–type collocation points
X_h	finite dimensional subspace of $\mathscr{H}_0^1(\Omega)$, $X_h = \mathscr{H}_0^1(\Omega) \cap Y_h$

Y_h	polynomial space, $Y_h = \{ v \in \mathscr{L}^2(\Omega) \mid v_{\mid \Omega_j} \in \mathscr{P}_n(\Omega_j) \}$
n	order of spectral (element) approximation
k	number of spectral elements
h	characteristic integer pair, $h = (k,n)$
Ω_j	spectral elements
e	standard spectral element $[-1,1]$
$\mathscr{C}^0(e,k)$	element space, $\mathscr{C}^0(e,k) = \{ v \in \mathscr{L}^2(e) \mid v_{\mid \overline{\Omega}_j} \in \mathscr{C}^0(\overline{\Omega}_j) \}$
$\mathscr{C}^1(e,k)$	element space, $\mathscr{C}^1(e,k) = \{ v \in \mathscr{L}^2(e) \mid v_{\mid \overline{\Omega}_j} \in \mathscr{C}^1(\overline{\Omega}_j) \}$
$(.,.)_{gl}$	discrete inner product in $\mathscr{C}^0(e,k)$
$\| \cdot \|_{gl}$	corresponding discrete norm
u_h	spectral element approximate solution
u_h^j	restriction of u_h to Ω_j
ξ_i	Legendre Gauss–Lobatto points
w_i	Legendre Gauss–Lobatto weights
ξ_{ij}	global Legendre Gauss–Lobatto points in Ω_j
w_{ij}	global Legendre Gauss–Lobatto weights in Ω_j
$\psi_p^j(\xi)$	local spectral element basisfunctions in Ω_j
$\boldsymbol{A}$	elliptic matrix
$\boldsymbol{B}$	elliptic matrix
$\boldsymbol{M}$	mass matrix
$\boldsymbol{S}$	stiffness matrix
$\boldsymbol{N}(\underline{u})$	convection matrix
$\boldsymbol{J}(\underline{u})$	Jacobian matrix of the residual
$\boldsymbol{I}(\underline{u})$	Jacobian matrix of the convection term
ν	kinematic viscosity
p_k	algebraic polynomials of degree k
$P_n u$	orthogonal projection of u(x) upon $\mathscr{P}_n$
$I_n u$	general interpolating polynomial of u(x)
$R_n u$	aliasing error

CONTENTS.

1. INTRODUCTION.

This report deals with the fundamentals of the spectral element method. Spectral element methods are high order weighted residual techniques for partial differential equations. They combine the geometric flexibility of the well known h–type finite element method with the attractive convergence properties of spectral methods. Thus they are well suited for application of complex problems in complex geometries.

The analysis will be started in chapter 2 with a short review on spectral approximation. Origin and basic properties will be discussed and several important approximation results of direct importance to the spectral element method will be presented. Moreover the similarities and differences with finite element methods will be discussed.

Chapter 3 will deal with spectral element methods. After a general introduction, a detailed theoretical formulation of the discretization process of one–dimensional elliptic equations will be analyzed. Some convergence results are presented for linear equations. Moreover we will give the application of the spectral element method to a non–linear testequation: the steady viscous Burgers equation. The set of discrete equations and the linearization process will be described. Numerical results are presented, both for linear and non–linear problems. Finally, conclusions will be drawn and some aspects of future research on the application of spectral element methods will be addressed.

The results presented in this report are part of a larger project, which deals with experimental and numerical research on the atherosclerotic disease in the carotid artery bifurcation. As regards the numerical aspects, it is the ultimate goal to describe the flow phenomena in the bifurcation, using a spectral element discretization of the three dimensional incompressible Navier–Stokes equations. This report must be viewed as a foundation for later research.

2. A SHORT REVIEW ON SPECTRAL APPROXIMATION.

2.1 Introduction.

Spectral methods are a class of discretization and solution procedures for partial differential equations, in which the approximate solution to the problem is expanded in a global high order Fourier or polynomial series. For problems with analytic solutions spectral methods obtain exponential convergence if the order of the expansion is increased. Thus they afford the possibility of obtaining highly accurate solutions with relatively few degrees of freedom. A major limitation however, is their intrinsic limitation to simple geometric configurations due to the global character of the expansion.

Spectral methods are a particular case of the more common, so called weighted residual approaches to partial differential equations. The key elements in this technique are the expansion functions and the testfunctions. The expansion functions are used as basisfunctions for a truncated series expansion of the solution. The testfunctions are used to ensure that the differential equation is satisfied as closely as possible by the truncated series expansion. This is done by minimizing the residual, due to truncation, with respect to a suitable norm. If we denote the partial differential equation by,

$$L\mathrm{u} = \mathrm{f}, \tag{2.1}$$

the weighted residual equation can formally be written as,

$$(L\mathrm{u}_n,\mathrm{v}) = (\mathrm{f},\mathrm{v}) \tag{2.2}$$

with $\mathrm{u}_n(\mathrm{x})$ the truncated series expansion, $\mathrm{v}(\mathrm{x})$ the testfunction and (.,.) denotes an inner product corresponding to a suitable norm. Equation (2.2) provides one or more relations for the unknown expansion coefficients. In other words, the coeffcients are determined by weighted residual projection of the continuous equation with respect to a suitable norm. The choice of the basisfunctions is the major feature that distinguishes spectral methods from low order methods like the h–type finite element technique. For a mathematical review the reader is referred to (Axelsson and Barker, 1984), for application of finite element methods to the Navier–Stokes equations to (Cuvelier et al, 1986) and (Girault and Raviart, 1986). In spectral methods the expansion functions are infinitely differentiable

high order global functions.

In this chapter a short review will be given of some important aspects of spectral approximation. In particular, the choice of basisfunctions in a spectral approach will be explained, and several theorems that indicate the attractive approximation results of spectral expansions will be presented. For an introduction to spectral methods the reader is referred to (Gottlieb and Orszag, 1977, Gottlieb et al, 1984), a more detailed recent description is found in (Canuto et al, 1988). Also some aspects of spectral interpolation or collocation will be addressed, see e.g. (Hussaini et al, 1989). Finally, in order to pave the way for the spectral element method, a comparison will be made with the well known h–type finite element method.

2.2 Spectral approximation.

In spectral methods the approximate solution $\tilde{u}_n(x)$ to the partial differential equation is expanded in a truncated series of expansion or basisfunctions,

$$\tilde{u}_n(x) = \sum_{i=0}^{n} \tilde{u}_i(t)\psi_i(x) \tag{2.3}$$

with $\tilde{u}_i(t)$ the expansion or spectral coefficients and $\psi_i(x)$ the basisfunctions. The essential point behind spectral methods is that they achieve convergence by letting $n \to \infty$, that is by increasing the order of the expansion (2.3). The convergence rate of spectral methods is therefore determined by the smoothness of the function being approximated.

For a special class of basisfunctions spectral methods have the property of exponential or spectral accuracy. In order to establish this property consider the expansion of a function u(x) in terms of an infinite sequence of orthogonal functions $\{\psi_i\}_{i=0}^{\infty}$,

$$u(x) = \sum_{i=0}^{\infty} \tilde{u}_i\psi_i(x) \tag{2.4}$$

If the system of orthogonal functions is complete in a suitable Hilbert space, relation (2.4) can be inverted. Thus the function u(x) can be described both through its values in fysical space and through its coefficients in spectral space, this relationship is called the continuous transform.

In the case of periodic functions the obvious expansions are Fourier series. A well

known result of the Fourier theory is the following theorem.

Theorem 2.1.
If the function u(x) is periodic and infinitely smooth and its derivatives are periodic as well, the ith coefficient of its Fourier series decays faster than any inverse power of i, i.e.

$$\tilde{u}_i < i^{-p} \qquad \forall\, p > 0 \qquad (i \to \infty) \tag{2.5}$$

This so called property of spectral accuracy can also be obtained for expansions in non–periodic functions. As will be deduced in the next paragraph, if the function u(x) is expanded in a series of eigenfunctions to a singular Sturm–Liouville problem spectral accuracy is also obtained. For reasons of efficiency polynomials solutions are of special importance. It will also be proven that the only polynomial solutions to singular Sturm–Liouville problems are encompassed in the class of Jacobi polynomials, the two most common applications of which are the Chebyshev and Legendre polynomials.

In this way spectral approximation is a special case of the expansion of a function u(x) in terms of an infinite sequence of general orthogonal polynomials, the theory of which is provided in appendix A.

2.3 Sturm–Liouville problems.

Consider the following eigenvalue problem on the domain (–1,1),

$$\begin{cases} -\left(p(x)\,(\psi_n)_x\right)_x + q(x)\,\psi_n = \lambda_n\, w(x)\,\psi_n & \text{in } (-1,1) \\ \text{plus homogenuous boundary conditions} \end{cases} \tag{2.6}$$

Problem (2.6) is a Sturm–Liouville eigenvalue problem on (–1,1). The solution ψ_n is called the eigenfunction to (2.6) with eigenvalue λ_n. The function p(x), q(x) and the weight function w(x) must be non–negative in (–1,1). The problem is called singular, or more specific, singular at the boundary points, if,

$$p(-1) = p(1) = 0 \tag{2.7}$$

As already indicated, those eigenfunctions that form a complete orthogonal polynomial set in a Hilbert space are of special importance. First, consider the property of completeness.

Therefore consider the Lebesgue space,

$$\mathscr{L}_w^2(-1,1) = \{ v \mid [\int_{-1}^{1} v^2 w(x)\, dx]^{1/2} < \infty \} \tag{2.8}$$

which is a Hilbert space with inner product $(.,.)_w$ defined by,

$$(u,v)_w = \int_{-1}^{1} w(x)\, u(x)\, v(x)\, dx \qquad \forall\, u,v \in \mathscr{L}_w^2(-1,1) \tag{2.9}$$

and associated norm $\| v \|_w^2 = (v,v)_w$, $v \in \mathscr{L}_w^2(-1,1)$.

Theorem 2.2.

The eigenfunction solutions to the Sturm–Liouville problem (2.6) $\{\psi_n\}_{n=0}^{\infty}$ form a complete set on the Lebesgue space $\mathscr{L}_w^2(-1,1)$ if and only if,

$$\lambda_n \to \infty \qquad (n \to \infty) \tag{2.10}$$

For a proof the reader is referred to (Courant and Hilbert, p. 424, 1953).

It will be seen later that the polynomial eigenfunctions that we are interested in have the property (2.10). Therefore it is allowed to assume in the further analysis that the eigenfunctions form a complete set in the Lebesgue space $\mathscr{L}_w^2(-1,1)$. The orthogonality property of the eigenfunctions requires that,

$$(\psi_l,\psi_k)_w = c_k\, \delta_{kl} \tag{2.11}$$

The following theorem motivates the choice of eigenfunctions of singular Sturm–Liouville problems as expansion functions in a spectral approximation.

Theorem 2.3.
Consider the spectral expansion (2.4) of an infinitely smooth function u(x). If the basisfunctions $\psi_i(x)$ are eigenfunctions of a singular Sturm–Liouville problem on (–1,1) then the expansion has the property of spectral accuracy, that is the coefficients $\tilde{u}_i$ decay faster than any inverse power of the corresponding eigenvalue of the Sturm–Liouville problem,

$$\tilde{u}_i < \lambda_i^{-p} \qquad \forall\, p > 0 \qquad (i \to \infty) \tag{2.12}$$

or equivalently,

$$\| u - \tilde{u}_n \|_w^2 < C\, \lambda_n^{-p} \qquad \forall\, p > 0 \qquad (n \to \infty) \tag{2.13}$$

with $\tilde{u}_n(x)$ the truncated series expansion (2.3).

For a proof the reader is referred to (Canuto et al, sec. 9.2 ,1988).

It should be noted here that this spectral accuracy is normally not obtained for eigenfunction expansions of a non–singular Sturm–Liouville problem (p(x) > 0 throughout (–1,1)), or even a half–singular Sturm–Liouville problem (p(–1) or p(1) equals 0). The expansion coefficients $\tilde{u}_i$ of an expansion in non–singular or half–singular eigenfunctions decay algebraically, unless the expanded function u(x) satisfies an infinite number of very special boundary conditions. Singular eigenfunction expansions converge at a rate governed by the smoothness of the function being expanded, not by any special boundary conditions satisfied by the function. A proof of these properties is given by Gottlieb and Orszag, (sec. 3, 1977), who consider Sturm–Liouville problems with zero boundary conditions and extend the analysis to the special cases of Chebyshev and Legendre expansions.

Consider now the second property that the spectral expansions must satisfy. Of specific interest are those orthogonal eigenfunctions of singular Sturm–Liouville problems, that are polynomials, because of the efficiency with which they can be evaluated and differentiated numerically.

Theorem 2.4.
The class of polynomial solutions to a singular Sturm–Liouville problem on (–1,1) is the class of Jacobi polynomials.

A proof of this theorem is given in appendix B.

In appendix B the class of Jacobi polynomials is defined. In practice only the two following applications of Jacobi polynomials are used in spectral methods, Chebyshev and Legendre polynomials, see e.g. (Abramowitz and Stegun, 1968). The Chebyshev polynomials $T_n(x)$ are plotted in Fig. 2.1 for n = 0,....4.

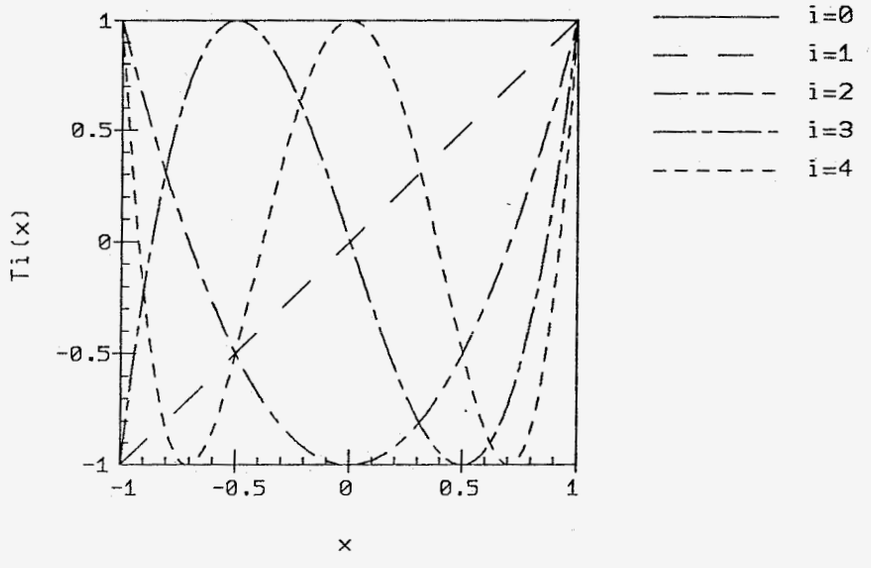

Fig. 2.1 Chebyshev polynomials.

They can be evaluated using the recurrence relation,

$$T_{n+1}(x) = 2x\ T_n(x) - T_{n-1}(x) \qquad n \geq 1 \qquad (2.14)$$

with $T_0(x) = 1$ and $T_1(x) = x$. The Legendre polynomials $L_n(x)$ n =0,....4, are plotted in Fig. 2.2.

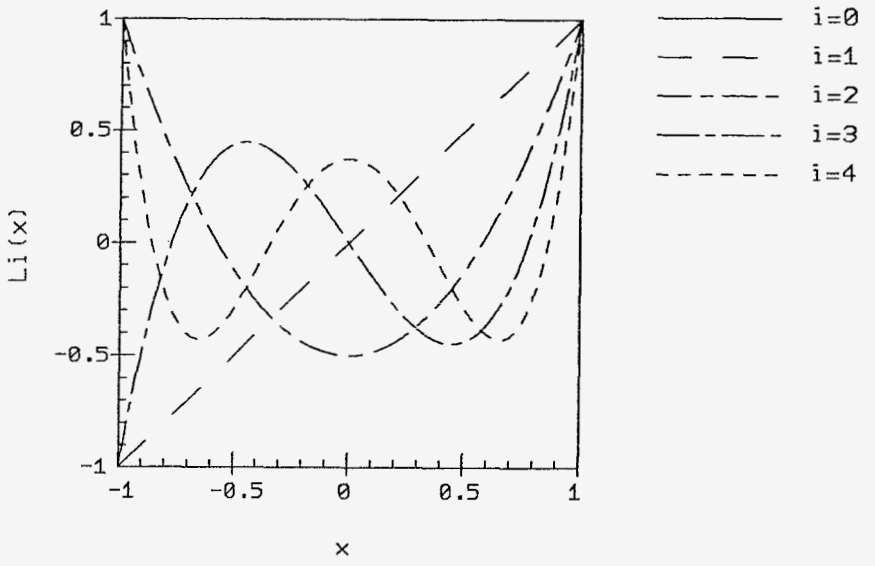

Fig. 2.2 Legendre polynomials.

The recurrence relation for Legendre polynomials is,

$$L_{n+1}(x) = \frac{2n+1}{n+1}\, x\, L_n(x) - \frac{n}{n+1}\, L_{n-1}(x) \qquad n \geq 2 \qquad (2.15)$$

with $L_0(x) = 1$ and $L_1(x) = x$.

2.4 Spectral collocation.

In the previous sections spectral approximation of a smooth function u(x) in terms of a series of high order polynomial expansion functions was discussed. This approximation defined a continuous transform, see equation (2.4). The function u(x) could be described both by its values in physical space and by the set of its continuous expansion coefficients $\{\tilde{u}_i\}_{i=0}^{\infty}$ in spectral space. However, the continuous expansion coefficients are seldom computed exactly, since they depend on all the values of the function u(x) in physical space.

In spectral collocation methods a set of approximate coefficients is calculated using the values of u(x) at a finite number of high order Gauss type quadrature or collocation points. This procedure defines a discrete transform between the values of u(x) at the collocation points and the set of its discrete coefficients $\{\bar{u}_i\}_{i=0}^{n}$. The finite series defined by the discrete coefficients is actually the interpolant polynomial of u(x) at the collocation points.

The fundamental representation of a smooth function u(x) defined on (–1,1) in spectral

collocation, is in terms of its values at the discrete set of Gauss type points. If the set of Gauss type collocation points is denoted by $\{x_j\}_{j=0}^n$, then the interpolant polynomial $\bar{u}_n(x)$ satisfies,

$$\bar{u}_n(x_j) = u(x_j) \qquad j = 0,\ldots,n \qquad (2.16)$$

Since it is a polynomial of degree n, it can be written as,

$$\bar{u}_n(x) = \sum_{i=0}^{n} \bar{u}_i \psi_i(x) \qquad (2.17)$$

where the ψ_i are the spectral expansion functions, eigenpolynomials of singular Sturm–Liouville problems on (–1,1). Obviously,

$$u(x_j) = \sum_{i=0}^{n} \bar{u}_i \psi_i(x_j) \qquad j = 0,\ldots,n \qquad (2.18)$$

Equation (2.18) gives the discrete transform between the values of u(x) at the set of collocation points and the set of its discrete coefficients. For more details on interpolating polynomials we refer to appendix A, in which the more general case of discrete transforms in terms of general orthogonal polynomials is described.

Naturally, it will be convenient if the property of spectral accuracy is retained in replacing the finite transform with the discrete transform in spectral Gauss type collocation. In appendix B it is deduced that the discrete coefficients can be expressed in the continous expansion coefficients, see equation (A.14). Since the continuous coefficients show spectral accuracy, the next theorem is obvious.

Theorem 2.5.
Consider the spectral collocation expansion (2.17) (or interpolant at the Gauss type points) of a smooth function u(x). If the basisfunctions $\psi_i(x)$ are eigenfunctions of a singular Sturm–Liouville problem on (–1,1) then the discrete expansion shows spectral accuray, or equivalently,

$$\| u - \bar{u}_n \|_w^2 < C\, \lambda_n^{-p} \qquad \forall\, p > 0 \qquad (n \to \infty) \qquad (2.19)$$

For a recent review on spectral collocation the reader is referred to (Hussaini et al, 1989).

2.5 A comparison to finite element methods.

Before the spectral element method is discussed in the next chapter, a comparison is made between spectral and h–type finite element methods. This is done, because as will be seen, both methods are integrated within the spectral element approach.

The most important similarity between the two methods is their weighted residual foundation, which provides the means to combine both methods. The differences occur when this weighted residual framework is expanded to the final set of discrete equations. As already seen in the previous sections, spectral methods use high order infinitely differentiable basisfunctions, mostly Chebyshev or Legendre polynomials. Finite element methods on the other hand use low order basisfunctions. Furthermore, the spectral basis is a global basis, defined over the whole domain of interest, whereas in finite elements (as in all domain decomposition methods), the basisfunctions are local in character.

The above difference has an important consequence, in spectral methods convergence of the approximate solution of the partial differential equation is obtained by increasing the order of the spectral expansion. In a finite element approach however, this is achieved by increasing the number of elements. Another major consequence is that whereas spectral expansions have the property of spectral or exponential accuracy, finite element methods achieve at most algebraic convergence. On the other hand the use of high order global approximations in spectral methods leads to full system matrices in the final set of discrete equations. Finite element systems are sparse due to the local character of the approximation.

The element division of the finite element method ensures that they are perfectly suited to handle complex geometries, contrary to spectral methods. Moreover the possibility of local mesh refining in finite element approximation provides the means for handling complex physical phenomena, e.g. strong discontinuities of the solution, an advantage over the global spectral methods shared by the spectral element method.

In summary, spectral methods are perfectly suited for problems in which the olution and data are highly regular and the domain not very complex. However, if either strong discontinuities in data or complexity in geometry occur, finite element methods are by far the more easier to implement.

3. FUNDAMENTALS OF THE SPECTRAL ELEMENT METHOD.

3.1 Introduction.

Spectral element methods, first proposed by Patera (1984), are high order weighted residual techniques for partial differential equations, that combine geometric flexible h–type, (Axelsson and Barker, 1984, Cuvelier et al, 1986, Girault and Raviart, 1986), and p–type, (Babuska and Dorr, 1981, Babuska et al, 1981) finite element methods, with the highly accurate spectral methods, (Canuto et al, 1988, Gottlieb and Orszag, 1977, Gottlieb et al, 1984) An extended theoretical foundation was presented recently by Maday and Patera (1989).

In the spectral element discretization the domain is divided into spectral elements, and the dependent and independent variables are approximated by high order polynomial expansions within the individual subdomains. In order to decrease the demands of continuity at element boundaries, the partial differential equation is written in its equivalent variational form. The variational formulation is taken as a basis for the discretization process, in which high order Gauss–type numerical quadrature plays an essential role, (Davis and Rabinowitz, 1984). Convergence of the approximate solution to the exact solution of the partial differential equation is obtained in a similar way as in spectral methods, by increasing the order of the approximation. The number of elements is kept fixed.

The element division of the spectral element methods ensures that even problems with complex geometries can be well approximated. Moreover in areas where rapid function variation or other physical complexities occur, the high order interpolation provides good results. Thus spectral element methods handle both geometric and physical complexity.

As spectral element approximation can be viewed as a spectral collocation approach in fixed elements, the convergence rate of spectral element methods is equivalent to that of spectral methods and depends on the smoothness of the function that is approximated. For analytic solutions spectral element methods obtain thus spectral or exponential convergence. Therefore, they are well suited for problems in which high order regularity is not excepted, e.g. incompressible fluid mechanics, (Korczak and Patera, 1986, Ronquist and Patera, 1988). If the regularity of the solution decreases spectral element methods may no longer be preferable to h–type finite element techniques. The same argument holds if the acceptable error level is relatively high.

In this chapter the discretization process of spectral element methods applied to a class of one–dimensional elliptic equations is described in detail. The final set of discrete equations will be presented and discussed. Also some approximation results will be given, that indicate the high accuracy of spectral element methods. Moreover, the application of spectral element methods to a non–linear testcase, the steady viscous Burgers equation is given. For both problems numerical results are presented. In the last section conclusions are drawn.

3.2 Discretization process of a class of one–dimensional elliptic problems.

In order to establish the spectral element concepts consider the following linear symmetric elliptic boundary value problem in one dimension,

$$\begin{cases} -(p(x)\,u_x)_x + q(x)\,u = f(x) & \text{in } \Omega = (a,b) \\ u(a) = u(b) = 0 \end{cases} \tag{3.1}$$

It is assumed that the functions p(x), q(x) and f(x) satisfy,

$$p(x),\, q(x) \in \mathscr{C}^0(\overline{\Omega}),\; f(x) \in \mathscr{L}^2(\Omega) \tag{3.2}$$

$$0 < p_0 < p(x) < p_1,\; 0 < q(x) < q_1 \tag{3.3}$$

where $\mathscr{C}^0(\overline{\Omega})$ is the space of continuous functions on $\overline{\Omega}$, and the Lebesgue space $\mathscr{L}^2(\Omega)$ is defined by,

$$\mathscr{L}^2(\Omega) = \{\, v \mid [\int_\Omega v^2\, dx\,]^{1/2} < \infty \,\} \tag{3.4}$$

Equation (3.1) is sufficiently simple to allow for a clear illustration of spectral element discretization of elliptic problems.

As already mentioned in the introduction to this chapter, the basis for the numerical scheme is the variational equivalent to (3.1). The space of acceptable solutions is defined to be the Sobolev space $\mathscr{H}_0^1(\Omega)$, given by,

$$\mathscr{H}_0^1(\Omega) = \{\, v \in \mathscr{L}^2(\Omega) \mid v_x \in \mathscr{L}^2(\Omega),\; v(a) = v(b) = 0 \,\} \tag{3.5}$$

This space is equipped with the following inner product,

$$(u,v) = \int_{\Omega} u(x)\, v(x)\, dx \qquad \forall\, u,v \in \mathscr{L}^2(\Omega) \qquad (3.6)$$

and associated norm $\| v \|^2 = (v,v)$. The variational formulation equivalent to the boundary value problem (3.1) can then be written as, find $u(x) \in \mathscr{H}_0^1(\Omega)$ such that,

$$a(u,v) = (f,v) \qquad \forall\, v \in \mathscr{H}_0^1(\Omega) \qquad (3.7)$$

where the bilinear continuous form a(.,.) is defined by,

$$a(u,v) = \int_{\Omega} p(x)\, u_x v_x\, dx + \int_{\Omega} q(x)\, uv\, dx \qquad u,v \in \mathscr{H}_0^1(\Omega) \qquad (3.8)$$

Theorem 3.1.
The variational problem (3.7) has a unique solution. This solution is the generalized solution of the original boundary value problem (3.1), i.e. if (3.1) has a classical solution then it must coincide with the solution to the variational formulation.

Proof.

The bilinear form a(.,.) is bounded and coercive (positive–definite) in $\mathscr{H}_0^1(\Omega)$. Application of the Lax–Milgram lemma, (Axelsson and Barker, sec. 3.2, 1984), immediately gives the required result.

(End proof).

The first step in the spectral element discretization is to choose an integer pair h = (k,n) and break up the domain Ω into k (spectral) elements, $\Omega_j = [\omega_j, \omega_{j+1}]$, of length l_j,

$$\Omega = \bigcup_{j=1}^{k} \Omega_j \qquad (3.9)$$

$$| \Omega_j | = l_j \qquad j = 1,\ldots,k \qquad (3.10)$$

The spectral elements are non–overlapping,

$$\Omega_i \cap \Omega_j = \phi \qquad i \neq j \tag{3.11}$$

The next step is to discretize the solution space $\mathscr{H}_0^1(\Omega)$. The space of approximation for the solution u(x) is taken to be the n–dimensional subspace X_h of the Hilbert space $\mathscr{H}_0^1(\Omega)$ consisting of all piecewise high order polynomials of degree $\leq n$,

$$X_h = Y_h \cap \mathscr{H}_0^1(\Omega) \tag{3.12}$$

with Y_h defined by,

$$Y_h = \{ \psi \in \mathscr{L}^2(\Omega) \mid \psi_{|\Omega_j} \in \mathscr{P}_n(\Omega_j) \} \tag{3.13}$$

where $\mathscr{P}_n(\Omega_j)$ denotes the space of polynomials on Ω_j of degree $\leq n$. The discrete variational problem now can be written as, find $u_h \in X_h$ such that,

$$a(u_h,v_h) = (f,v_h) \qquad \forall\, v_h \in X_h \tag{3.14}$$

Theorem 3.2.
There exists a unique solution $u_h \in X_h$ to the discrete problem (3.14).

Proof.
The space of approximation for the solution X_h is a finite dimensional subspace of a Hilbert space, and therefore itself a Hilbertspace. The Lax–Milgram lemma is now also applicable to the discrete problem.

(End proof).

The equation (3.14) cannot be implemented numerically without numerical quadrature. The choice of a numerical quadrature corresponds with the choice of a discrete inner product or norm. The high order discretization suggests a Gauss type quadrature formula. Moreover, since the integrals appearing in (3.14) have a natural weight function equal to 1, the quadrature should be a Legendre Gauss type quadrature. For reasons to be explained later, Legendre Gauss–Lobatto numerical integration is chosen. This quadrature is defined by,

$$\int_{-1}^{1} \psi(\xi)\, d\xi = \sum_{i=0}^{n} \psi(\xi_i)\, w_i \qquad \forall\, \psi \in \mathcal{P}_{2n-1}(-1,1) \tag{3.15}$$

where the ξ_i are the Legendre Gauss–Lobatto integration points given by,

$$\begin{aligned} &\xi_0 = -1 \\ &\xi_i \text{ zeroes of } L_n'(\xi),\ i = 1,....,n-1 \\ &\xi_n = 1 \end{aligned} \tag{3.16}$$

The w_i are the Legendre Gauss–Lobatto integration weights given by,

$$w_i = \frac{2}{n(n+1)\, L_n(\xi_j)^2} \qquad i = 0,....,n \tag{3.17}$$

In (3.16) and (3.17) $L_n(\xi)$ is the Legendre polynomial of order n, (Abramowitz and Stegun, 1968). For an extensive description of numerical quadrature the reader is referred to (Davis and Rabinowitz, 1984).

In order to be able to apply Legendre Gauss–Lobatto quadrature an affine transformation is used to map each spectral element $\Omega_j = [\omega_j, \omega_{j+1}]$ to the standard spectral element $e = [-1,1]$ ($x \in \Omega_j \rightarrow \xi \in e$). The corresponding global integration points and weights in the element Ω_j are then given by, ($i = 1,....,n$ $j = 1,....,k$),

$$\xi_{ij} = \omega_j + l_j\, (\xi_i + 1)/\, 2 \tag{3.18}$$

$$w_{ij} = l_j\, w_i/\, 2 \tag{3.19}$$

Applying the Legendre Gauss–Lobatto numerical integration to the discrete variational system (3.14) yields the following discrete problem, find $u_h \in X_h$ such that,

$$a_{gl}(u_h, v_h) = (f, v_h)_{gl} \qquad \forall\, v_h \in X_h \tag{3.20}$$

The corresponding discrete norm $\| \cdot \|_{gl}$ with inner product $(.,.)_{gl}$ is given by,

$$(\psi, \phi)_{gl} = \sum_{j=0}^{k} \sum_{i=0}^{n} \psi(\xi_{ij})\, \phi(\xi_{ij})\, w_{ij} \qquad \forall\, \psi, \phi \in \mathcal{C}^0(e,k) \tag{3.21}$$

where,

$$\mathscr{C}^0(e,k) = \{ \psi \in \mathscr{L}^2(e) \mid \psi|_{\overline{\Omega}_j} \in \mathscr{C}^0(\overline{\Omega}_j) \} \tag{3.22}$$

Compare the definition of the discrete inner product in appendix A, equation (A.13). Furthermore, the discrete bilinear form $a_{gl}(.,.)$ is given by,

$$a_{gl}(\psi,\phi) = \sum_{j=0}^{k} \sum_{i=0}^{n} p(\xi_{ij})\, \psi_x(\xi_{ij})\, \phi_x(\xi_{ij})\, w_{ij} + \sum_{j=0}^{k} \sum_{i=0}^{n} q(\xi_{ij})\, \psi(\xi_{ij})\, \phi(\xi_{ij})\, w_{ij} \qquad \forall\, \psi,\phi \in \mathscr{C}^1(e,k) \tag{3.23}$$

where,

$$\mathscr{C}^1(e,k) = \{ \psi \in \mathscr{L}^2(e) \mid \psi|_{\overline{\Omega}_j} \in \mathscr{C}^1(\overline{\Omega}_j) \} \tag{3.24}$$

Equation (3.20) can be regarded as the complete spectral element discretization of the original partial differential equation (3.1). The next theorem is a natural extension of the two previous ones.

Theorem 3.3.
There exists a unique solution $u_h \in X_h$ to the complete discrete spectral element problem (3.20).

The proof of theorem 3.3 involves a non–trivial result on numerical quadrature. Therefore it is not a standard application of the Lax–Milgram lemma. It is given in appendix C.

A detailed discussion on error estimates for the spectral discretization is given in (Maday and Patera, 1989). Here, the result for the case that p(x) and q(x) are constant is stated. The theorem involves the use of high order Sobolev spaces and norms, for a survey the reader is referred to (Axelsson and Barker, sec. 3.1, 1984).

Theorem 3.4.
Let k be a fixed integer and let γ be a non–negative real number such that,

$$l_j \geq \gamma \qquad j= 1,....,k \tag{3.25}$$

Let σ and τ be real numbers such that,

$$\sigma \geq 1, \tau > {}^1/_2 \tag{3.26}$$

Then, if $f(x) \in \mathscr{H}^{\tau}(\Omega)$ and $u(x) \in \mathscr{H}^{\sigma}(\Omega) \cap \mathscr{H}_0^1(\Omega)$, the discrete solution $u_h \in X_h$ satisfies,

$$\| u - u_h \|_1 \leq C \left(n^{1-\sigma} \| u \|_{\sigma} + n^{({}^1/_2)-\tau} \| f \|_{\tau}\right) \tag{3.27}$$

For a proof the reader is referred to (Maday and Patera, 1989).

The sources of the discretization error (3.27) are approximation errors, interpolation errors and quadrature errors. From (3.27) one can conclude that if the data are analytic, exponential convergence to the exact solution u(x) is obtained for u_h as $n \to \infty$.

In order to implement the discrete system (3.20) it is necessary to choose a basis for the polynomial space X_h. It should be noted that the choice of the basis does not effect the discrete error estimate. It does of course effect the form and conditioning of the discrete equations. Moreover, the choice of the basis is important as regards the inter elemental coupling. Therefore a Legendre Gauss–Lobatto Lagrangian interpolant basis is chosen to represent any function in X_h. The basisfunctions $\psi_p^j(\xi)$ (j = 1,....,k, p = 0,....,n) in each spectral element Ω_j with respect to the affine transformation (3.18) are polynomials of degree n. They satisfy,

$$\psi_p^j(\xi_{qj}) = \delta_{qp} \qquad q = 0,....,n \tag{3.28}$$

where the ξ_{ij} are the Legendre Gauss–Lobatto points ref. equation (3.18). Moreover,

$$\psi_p^j(\xi) = 0 \qquad \xi \in \omega_l \qquad l \neq j \tag{3.29}$$

An explicit expression for the basisfunctions is given by Gottlieb et al (1984),

$$\psi_p^j(\xi) = \frac{1}{n(n+1)\ L_n(\xi_{pj})\ (\xi - \xi_{pj})} (1 - \xi^2)\ L_n'(\xi) \tag{3.30}$$

with $L_n(\xi)$ the Legendre polynomial of order n. The approximate solution u_h in each spectral element Ω_j is now written,

$$u_h^j(\xi) = \sum_{p=0}^{n} u_p^j\ \psi_p^j(\xi) \tag{3.31}$$

where $u_p^j = u_h(\xi_{pj})$. To ensure continuity of the approximate solution over element boundaries ($u_h \in \mathscr{H}_0^1(\Omega)$), it is required that,

$$u_n^{j-1} = u_0^j \qquad j = 2,....,k \tag{3.32}$$

Moreover u_h must satisfy the Dirichlet boundary conditions,

$$u_0^1 = u_n^k = 0 \tag{3.33}$$

It will be convenient to represent the source term f(x) in terms of the basis in X_h. However f(x) need not satisfy the conditions (3.32) and (3.33). Therefore f(x) is approximated as,

$$f_h \in Y_h \tag{3.34}$$

Substitution of u_h, f_h and the testfunction v_h into equation (3.20) in the standard Galerkin way by choosing each testfunction v_h to be unity at only one global Legendre Gauss–Lobatto point, yields the final discrete matrix system,

$$(A + B)\ \underline{u} = M\ \underline{f} \tag{3.35}$$

or equivalently,

$$\sum_{j=0}^{k}{}' \sum_{p=0}^{n} (A_{qp}^j + B_{qp}^j)\ u_p^j = \sum_{j=0}^{k}{}' \sum_{p=0}^{n} M_{qp}^j\ f_p^j \tag{3.36}$$

with $f_p^j = f(\xi_{pj})$, (p = 0,....,n) and,

$$
\begin{cases}
A^j_{qp} = {}^4/_{l_j^2} \overset{n}{\underset{i=0}{\Sigma}} D^i_{ip}\, D^i_{iq}\, p(\xi_{ij})\, w_{ij} & q,p = 0,....,n \qquad (3.37) \\
B^j_{qp} = q(\xi_{pj})\, w_{pj}\, \delta_{pq} & q,p = 0,....,n \qquad (3.38) \\
M^j_{qp} = w_{pj}\, \delta_{pq} & q,p = 0,....,n \qquad (3.39) \\
D^i_{ip} = (\psi^j_p(\xi_{ij}))_x & i,p = 0,....,n \qquad (3.40)
\end{cases}
$$

Explicit values for the collocation derivative matrix D^i_{ip} are given by Gottlieb et al (1984). In equation (3.36) Σ' denotes direct stiffness summation, i.e. the contributions of the same global element boundary points are summed to ensure (3.32), and the Dirichlet boundary conditions (3.33) are taken into account by eliminating from the system those rows and columns corresponding to the boundary points (matrix condensation).

To conclude this section some remarks are made on the final matrix system (3.36). As already mentioned, because of the choice of Gauss–Lobatto interpolation elements couple only at boundary nodes. It is now clear that Gauss–Lobatto quadrature in combination with Gauss–Lobatto interpolation results in a diagonal elemental mass matrix. This property ensures a rapid evaluation of the right hand side of equation (3.36).

3.3 Application to the steady viscous Burgers equation.

In this section the spectral element discretization of a one–dimensional test problem, the steady viscous Burgers equation is discussed,

$$
-\nu\, u_{xx} + uu_x = f(x) \qquad (3.41)
$$

with ν the viscosity. There are several motivations to do this. First of all, this equation introduces a non–linear term uu_x. Moreover, Burgers equation offers the possibility to apply spectral methods to an equation 'close' to the Navier–Stokes equations, that will be investigated in a later report. It should also be stated that the viscous Burgers equation is often taken as a test problem for numerical methods, e.g. (Basdevant et al, 1986).

The spectral element discrete system corresponding to (3.41) can be written as,

$$
S\,\underline{u} + N(\underline{u})\,\underline{u} = M\,\underline{f} \qquad (3.42)
$$

The elemental matrices can be deduced analogous to the analysis in the previous section.

They are given by,

$$\begin{cases} S^j_{qp} = {}^4/_{l_j^2}\, \nu \sum_{i=0}^{n} D^i_{ip}\, D^i_{iq}\, w_{ij} & q,p = 0,\ldots,n \quad (3.43) \\ N(\underline{u})^j_{qp} = {}^2/_{l_j}\, w_{pj}\, \delta_{pq} \sum_{r=0}^{n} u^j_r\, D^j_{rp} & q,p = 0,\ldots,n \quad (3.44) \\ M^j_{qp} = w_{pj}\, \delta_{pq} & q,p = 0,\ldots,n \quad (3.45) \\ D^i_{ip} = (\psi^j_p(\xi_{ij}))_x & i,p = 0,\ldots,n \quad (3.46) \end{cases}$$

The non–linear convective term $N(\underline{u})\ \underline{u}$ is linearized by a standard Newton–Raphson iteration scheme, (Cuvelier et al, sec 5.6, 1986). Given a non–linear discrete system,

$$K(\underline{u})\ \underline{u} = \underline{g} \qquad (3.47)$$

an initial solution $\underline{u}^0$ is chosen and then for each i = 1,2,....,

$$J(\underline{u}^{i-1})\ (\underline{u}^i - \underline{u}^{i-1}) = R(\underline{u}^{i-1}) \qquad (3.48)$$

is solved, where $R(\underline{u}^{i-1})$ is the residual given by,

$$R(\underline{u}^{i-1}) = K(\underline{u}^{i-1})\ \underline{u}^{i-1} - M\,\underline{f} \qquad (3.49)$$

and $J(\underline{u}^{i-1})$ is the Jacobian matrix of the residual defined by,

$$J(\underline{u}^{i-1})_{jk} = \frac{\partial R_j(\underline{u}^{i-1})}{\partial(\underline{u}^{i-1})_k} \qquad (3.50)$$

Application to the discrete Burgers system (3.42) yields, in the ith iteration step solve,

$$(S + I(\underline{u}^{i-1}))\ (\underline{u}^i - \underline{u}^{i-1}) = R(\underline{u}^{i-1}) \qquad (3.51)$$

with $I(\underline{u}^{i-1})$ the Jacobian matrix of the non–linear term $N(\underline{u}^{i-1})$ defined by,

$$I(\underline{u}^{i-1})_{jk} = \frac{\partial N_j(\underline{u}^{i-1})}{\partial(\underline{u}^{i-1})_k} \tag{3.52}$$

and the residual $R(\underline{u}^{i-1})$ given by,

$$R(\underline{u}^{i-1}) = S\,\underline{u}^{i-1} + N(\underline{u}^{i-1})\,\underline{u}^{i-1} - M\,\underline{f} \tag{3.53}$$

3.4 Numerical results.

In this section some examples of spectral element simulation will be presented to demonstrate numerically the predicted spectral accuracy. First consider an application of the one–dimensional elliptic problem (3.1), with data,

$$p(x) = e^x \tag{3.54}$$
$$q(x) = 0 \tag{3.55}$$
$$f(x) = e^x\,(\cos x - \sin x) \tag{3.56}$$

on the domain $(0,\pi)$ for which the exact solution is,

$$u(x) = -\sin x \tag{3.57}$$

As all data are analytic on $(0,\pi)$, exponential convergence to the exact solution can be expected according to theorem 3.4. This is demonstrated in fig. 3.1, where the discrete maximum error in the approximate solution is plotted on a logaritmic scale as a function of the number of degrees of freedom in the mesh (number of nodes). For spectral element approximation the number of elements was kept fixed on 4.

The same problem was solved using finite element quadratic approximation, the results of which are also plotted in fig. 3.1. Here convergence is achieved by increasing the number of elements.

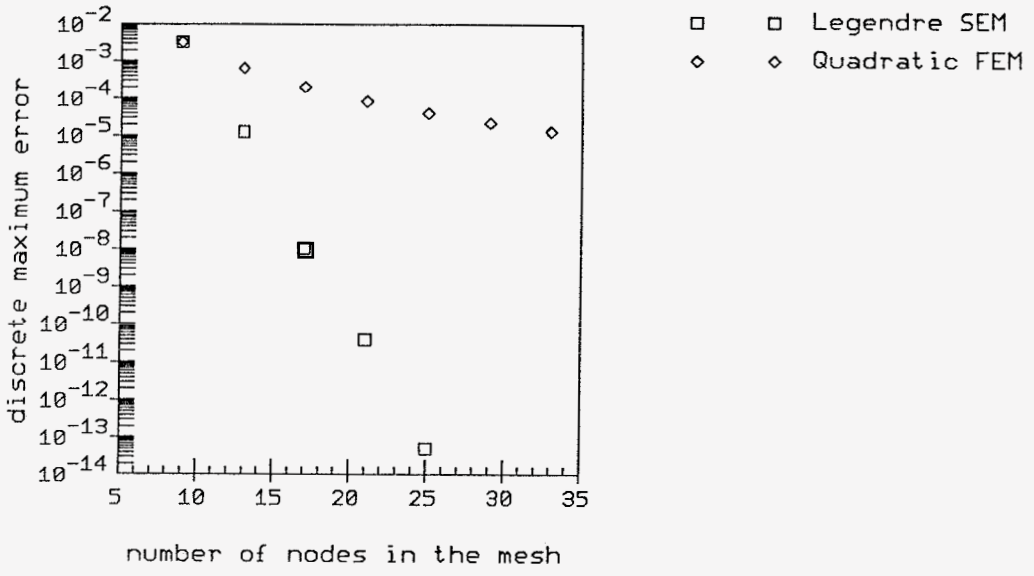

Fig. 3.1 Plot of the maximum pointwise error for the elliptic problem (3.1), with data (3.54)–(3.57).

It is also seen that finite element approximation achieves at most algebraic convergence. This demonstrates the advantage of p–type spectral approximation (increase of the approximation degree in a fixed number of elements) over h–type approximation (increase of the number of elements with a fixed approximation degree per element) for smooth linear problems.

Next consider an example of the steady viscous Burgers equation (3.41). Consider solutions $u^m(x)$ on the domain (0,1) that are written in a Fourier sine series,

$$u^m(x) = \sum_{k=0}^{m} \frac{\sin\,(2k+1)\pi x}{2k+1} \tag{3.58}$$

with data,

$$f^m(x) = \nu \sum_{k=0}^{m} \pi^2\,(2k+1)\sin\,(2k+1)\pi x + \sum_{k=0}^{m} \frac{\sin\,(2k+1)\pi x}{2k+1} \sum_{k=0}^{m} \pi \cos\,(2k+1)\pi x \tag{3.59}$$

In fig 3.2 the solutions $u^m(x)$ for m = 1,3,5 are plotted. Notice that the solution shows larger gradients if m increases.

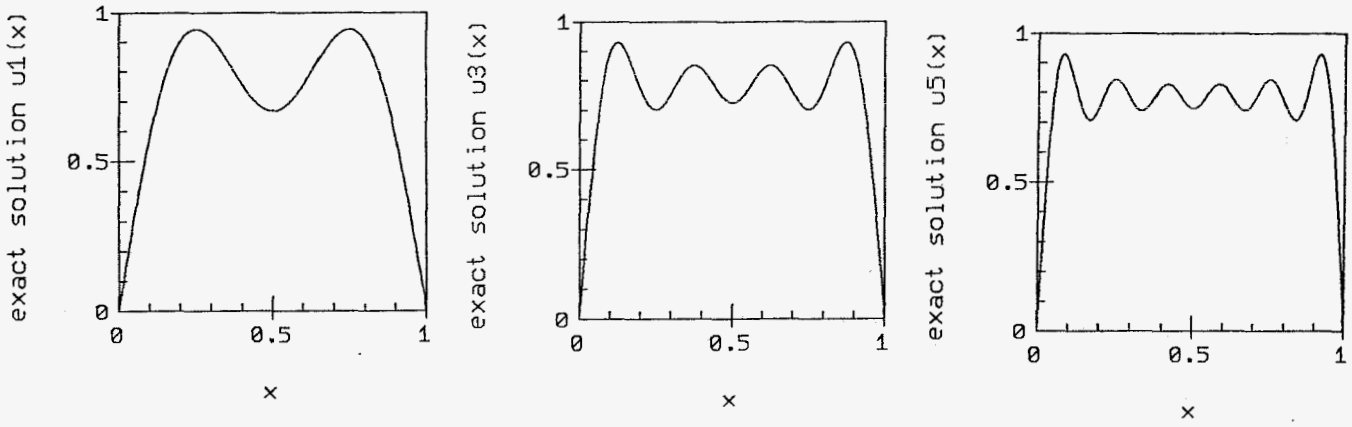

Fig. 3.2 Plot of the exact solution (3.58) to the steady viscous Burgers equation (3.41) for m = 1,3,5.

In order to analyze the influence of the non–linear convective term uu_x, the exact solution with m = 5 was numerically approximated, again using both spectral element and finite element approximation. Three testcases were examined, corresponding to a viscosity $\nu = 1$, $\nu = 0.01$ and $\nu = 0.001$. The number of elements in the spectral element approach was kept fixed on 4. Finite element approximation achieved convergence by increasing the number of elements. The initial solution to the Newton–Raphson linearization process was chosen to be $u^m(x) = 1$ on (0,1). The number of iterations needed for the process to converge with convergence criterion that the maximum pointwise error of the difference between the old and new solution should be smaller than 10^{-8}, slowly increased as the convective term gained influence. For decreasing viscosity respectively 3, 6 and 7 iterations were needed, both for spectral and finite element approximation.

In fig 3.3 the pointwise maximum error is plotted on a logaritmic scale as a function of the number of degrees of freedom in the mesh. In order to be able to compare the three testcases, all results are plotted in one figure.

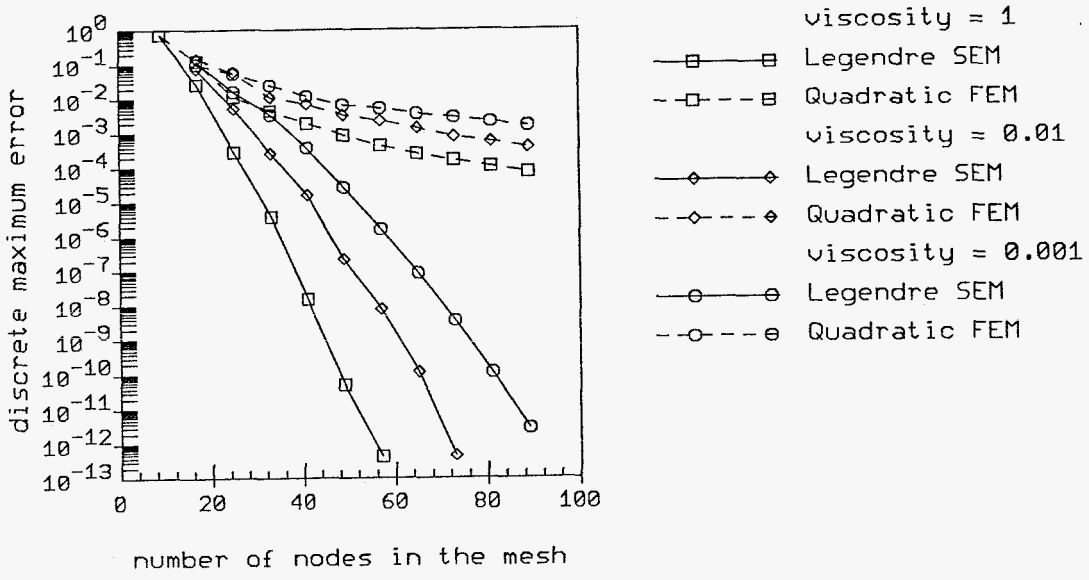

Fig. 3.3 Plot of the pointwise maximum error for the non–linear problem (3.41) with data (3.58)–(3.59) and with $m = 5$, for three testcases $\nu = 1, 0.01, 0.001$.

Notice that spectral element approximation converges exponentially for all three testcases, although more degrees of freedom, i.e. a higher polynomial degree of approximation per element, is needed if the convective term gains influence. On the other hand finite element approximation shows slower algebraic convergence with decreasing viscosity. From these testcases one may conclude that also for strongly non–linear problems spectral element approximation obtains exponential convergence to the exact solution, though at a slower rate if the convective term dominates the diffusive term.

3.5 Conclusions.

In this chapter the theoretical fundamentals of spectral element methods were described. It was seen that spectral element methods combine attractive properties of both h–type finite element methods and spectral methods. The spectral element method used a variational formulation equivalent to the partial differential equation. Legendre Gauss type quadrature applied to the variational formulation ensured that for smooth linear problems, spectral element approximation achieved exponential convergence to the exact solution of the problem with relatively few degrees of freedom. Furthermore, in order to couple elements only at boundary nodes a Legendre Gauss–Lobatto interpolant basis for the approximation was chosen, that in combination with Legendre Gauss–Lobatto quadrature

resulted in an attractive system of discrete equations, e.g. a diagonal mass matrix. The results for linear elliptic equations were demonstrated by a numerical example. Moreover spectral element methods were applicated to a non–linear testexample, the steady viscous Burgers equation. Numerical examples of this non–linear problem also showed the property of exponential convergence, even if the convective term strongly dominated the diffusive term, although a higher degree of approximation and more iterations were needed as the convective non–linear term gained influence. Both the linear and the non–linear problem also were approximated by finite element methods, which showed only slow algebraic convergence. It therefore can be concluded that for smooth problems spectral element approximation is preferable to finite element approximation as regards accuracy of the approximate solution.

Several aspects of spectral element methods are still open for research. At this moment the application of spectral element methods to time–dependent problems is investigated, in particular spectral element spatial discretization combined with Runga–Kutta Chebyshev time discretization methods, (Verwer, 1990). In a future stadium the application of spectral element approxomation to the incompressible two– and three–dimensional Stokes and Navier–Stokes equations will be analyzed. The results of the application of spectral element methods to the strongly convective non–linear Burgers equation indicate that good results can be expected for spectral element simulation of flows at moderate and high Reynolds number.

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Appendix A. Orthogonal systems of polynomials.

In chapter 2 the spectral approximation of an analytic function u(x) in terms of a series of orthogonal expansion functions was discussed. By using eigenfunctions of a singular Sturm–Liouville problem the expansion of the function u(x) showed so called spectral accuracy, that is the expansion coefficients decay faster than any inverse power of the eigenvalue of the corresponding Sturm–Liouville problem, see theorem 2.3. In this appendix the expansion of a function u(x) in terms of a system of orthogonal polynomials shall be discussed from a general point of view.

The expansion of a function u(x) in terms of an orthogonal system defines a transformation between u(x) and the set of expansion coefficients $\{\tilde{u}_k\}_{k=0}^{\infty}$. We call this the continuous transform between physical space and transform space (or spectral space in the case of a spectral expansion). If the system of expansion functions is complete in a Hilbert space, this transform can be inverted. Hence, the function u(x) can be described both in physical and transform space.

First define the space $\mathcal{P}_n$ to be the space of polynomials of degree $\leq n$. Let $\{p_k\}_{k=0}^{\infty}$ be a system of algebraic polynomials (with degree of $p_k = k$) in $\mathcal{P}_n$, that are mutually orthogonal over the interval $\Omega = (-1,1)$ with respect to a weight function w(x),

$$\int_{\Omega} p_k(x)\, p_l(x)\, w(x)\, dx = 0 \qquad k \neq l \qquad (A.1)$$

The classical Weirstrass theorem implies that the system $\{p_k\}$ is complete in the Lebesgue space $\mathcal{L}_w^2(-1,1)$,

$$\mathcal{L}_w^2(-1,1) = \{\, v \mid \left[\int_{\Omega} v^2(x)\, w(x)\, dx \right]^{1/2} < \infty \,\} \qquad (A.2)$$

with the inner product $(.,.)_w$ defined as follows,

$$(u,v)_w = \int_{\Omega} u(x)\, v(x)\, w(x)\, dx \qquad \forall\, u,v \in \mathcal{L}_w^2(-1,1) \qquad (A.3)$$

and associated norm $\| v \|_w^2 = (v,v)_w$.

A function $u(x) \in \mathcal{L}_w^2(-1,1)$ can be expanded in terms of the orthogonal system $\{p_k\}$. We denote this formal series expansion by,

$$\tilde{u}(x) = \sum_{k=0}^{\infty} \tilde{u}_k p_k(x) \tag{A.4}$$

where the expansion coefficients $\tilde{u}_k$ are defined as,

$$\tilde{u}_k = \frac{1}{(p_k,p_k)_w} \int_\Omega u(x)\, p_k(x)\, w(x)\, dx \tag{A.5}$$

Consider now the truncated series expansion $P_n u(x)$ defined for any integer $n > 0$ by,

$$P_n u(x) = \sum_{k=0}^{n} \tilde{u}_k p_k(x) \tag{A.6}$$

Equation (A.6) denotes the continuous transform between physical and transform space. Due to the orthogonality relation (A.1) $P_n u(x)$ is the orthogonal projection of $u(x)$ upon $\mathcal{P}_n$ with respect to the inner product (A.3), i.e.

$$(P_n u, v)_w = (u,v)_w \qquad \forall\, v \in \mathcal{P}_n \tag{A.7}$$

The completeness of the system $\{p_k\}_{k=0}^{\infty}$ in the space $\mathcal{L}_w^2(-1,1)$ is equivalent to the property that for any function $u(x) \in \mathcal{L}_w^2(-1,1)$,

$$\| u - P_n u \|_w \to 0 \qquad (n \to \infty) \tag{A.8}$$

Equation (A.8) is a well known result of classical functional analysis. It states that the function $u(x)$ is equivalent to its series expansion (A.4). Thus the function $u(x)$ can not only be described by its values in physical space, but also by the set of expansion coefficients $\{\tilde{u}_k\}_{k=0}^{\infty}$ in transform space.

The expansion coefficients depend on all the values of u(x) in physical space. Hence they can rarely be computed exactly. A finite number of approximate expansion coefficients can easily be computed using the values of u(x) at at finite number of selected points, the so called collocation points. They usually are the nodes of high precision Gauss type quadrature, (Davis and Rabinowitz, 1984). This procedure defines a discrete transform between the set of values of u(x) at the collocation points and the set of approximate coefficients. The finite series defined by the discrete transform is actually the interpolant of u(x) at the collocation nodes. If the properties of accuracy (in particular the spectral accuracy defined in chapter 2), are retained if the finite transform is replaced by the discrete transform, the interpolant series can be used instead of the truncated series in approximating functions. This is done in spectral element methods.

The nodes of Gauss type numerical quadrature play an important role in collocation or interpolation approximations. Let $\{p_k\}_{k=0}^{\infty}$ be the sequence of othogonal polynomials, and let $\{x_i\}_{i=0}^{n}$ and $\{w_i\}_{i=0}^{n}$ be the sets of Gauss type quadrature points and weights. In a collocation method an analytic function u(x) on (–1,1) is represented in terms of its values at the discrete Gauss type points. Derivatives of the function are approximated by analytic derivatives of the interpolating polynomial, which is an element of $\mathcal{P}_n$ and can be written as,

$$I_n u(x) = \sum_{k=0}^{n} \bar{u}_k p_k(x) \tag{A.9}$$

It is uniquely defined by,

$$I_n u(x_i) = u(x_i) \qquad i = 0,....,n \tag{A.10}$$

and consequently,

$$u(x_i) = \sum_{k=0}^{n} \bar{u}_k p_k(x_i) \qquad i = 0,....,n \tag{A.11}$$

Equation (A.11) denotes the discrete transform between physical space and transform space. The $\bar{u}_i$ are the set of discrete coefficients of u(x). They satisfy,

$$\bar{u}_k = \frac{1}{(p_k,p_k)_n} \sum_{i=0}^{n} u(x_i)\, p_k(x_i)\, w_i \tag{A.12}$$

with $(.,.)_n$ the discrete inner product defined by,

$$(u,v)_n = \sum_{i=0}^{n} u(x_i)\, v(x_i)\, w_i \qquad \forall\, u,v \in \mathcal{C}^0(-1,1) \tag{A.13}$$

compare equation (A.5). The discrete coefficients $\bar{u}_k$ can be expressed in terms of the continuous coefficients $\tilde{u}_k$ as follows,

$$\bar{u}_k = \tilde{u}_k + \frac{1}{(p_k,p_k)_n} \sum_{l>n} (p_l,p_k)_n\, \tilde{u}_l \tag{A.14}$$

see e.g. (Canuto et al, sec. 2.2, 1988). Equivalently, one can write,

$$I_n u = P_n u + R_n u \tag{A.15}$$

where,

$$R_n u = \sum_{k=0}^{n} \left(\frac{1}{(p_k,p_k)_n} \sum_{l>n} (p_l,p_k)_n\, \tilde{u}_l \right) p_k \tag{A.16}$$

is considered as the so called aliasing error due to interpolation. The aliasing error is orthogonal to the truncation or approximation error $u - P_n u$, so that,

$$\| u - I_n u \|_w^2 = \| u - P_n u \|_w^2 + \| R_n u \|_w^2 \tag{A.17}$$

In chapter 2 it was seen that in the case of a spectral expansion the aliasing error due to interpolation was of the same order as the truncation error. Therefore spectral collocation also had the property of spectral accuracy. For more information on orthogonal systems of polynomials the reader is referred to (Canuto et al, 1988).

Appendix B. Jacobi polynomials.

The class of Jacobi polynomials is defined as follows, Jacobi polynomials are solutions to the singular Sturm–Liouville problem on (–1,1),

$$-[\,(1-x)^{1+\sigma}(1+x)^{1+\tau}(\psi_n)_x\,]_x = \lambda_n\,(1-x)^{\sigma}(1+x)^{\tau}\psi_n \tag{B.1}$$

with $\sigma,\tau > -1$. The functions p(x), q(x) and w(x), see equation (2.5) are given by,

$$p(x) = (1-x)^{1+\sigma}(1+x)^{1+\tau} \tag{B.2}$$

$$w(x) = (1-x)^{\sigma}(1+x)^{\tau} \tag{B.3}$$

$$q(x) = 0 \tag{B.4}$$

The importance of Jacobi polynomials to spectral approximation was indicated in theorem 2.4. In this appendix this theorem is proven.

Theorem 2.4.
The class of polynomial solutions to a singular Sturm–Liouville problem on (–1,1) is the class of Jacobi polynomials.

Proof.
Let ψ_n be a polynomial solution of degree n to a singular Sturm–Liouville problem on (–1,1), then,

$$\psi_n = [-(p\,(\psi_n)_x)_x + q\,\psi_n\,] \,/\, \lambda_n\, w$$

$$= [-p\,(\psi_n)_x - p_x\,(\psi_n)_x + q\,\psi_n\,] \,/\, \lambda_n\, w$$

Consequently,

p/w is a polynomial of degree 2
q/w is a polynomial of degree 0

This implies that $q(x) = q_0\, w(x)$. Using now that $p(-1) = p(1) = 0$ gives,

$$p(x) = C_2\,(1-x)^{1+\sigma}(1+x)^{1+\tau} \tag{B.5}$$

$$w(x) = C_3\,(1-x)^{\sigma}(1+x)^{\tau} \tag{B.6}$$

with $\sigma,\tau > -1$. This completes the proof.

(End proof).

The two most common applications of Jacobi polynomials are Chebyshev and Legendre polynomials. The Chebyshev polynomials $T_n(x)$ have parameters $\sigma = \tau = -\,{}^1/_2$ and eigenvalues $\lambda_n = n^2$. They can be evaluated using the recurrence relation,

$$T_{n+1}(x) = 2x\; T_n(x) - T_{n-1}(x) \qquad n \geq 1 \tag{B.7}$$

with $T_0(x) = 1$ and $T_1(x) = x$. Another formula for Chebyshev polynomials is,

$$T_n(x) = \cos(n \arccos(x)) \tag{B.8}$$

The Legendre polynomials $L_n(x)$ have parameters $\sigma = \tau = 0$ and eigenvalues $\lambda_n = n(n+1)$. They satisfy the recurrence relation,

$$L_{n+1}(x) = \frac{2n+1}{n+1}\, x\; L_n(x) - \frac{n}{n+1}\, L_{n-1}(x) \qquad n \geq 2 \tag{B.9}$$

with $L_0(x) = 1$ and $L_1(x) = x$. For more properties of Chebyshev and Legendre polynomials the reader is referred to (Abramowitz and Stegun, 1968).

Appendix C. Uniqueness of the spectral element solution.

In section 3.2 the complete discrete spectral element problem to the elliptic boundary value problerm (3.1) was derived,

$$a_{gl}(u,v) = (f,v_h)_{gl} \qquad v_h \in X_h \qquad (C.1)$$

with $(.,.)_{gl}$ and $a_{gl}(.,.)$ the discrete inner product and bilinear form defined in (3.21) and (3.23).

Theorem 3.3.
There exists a unique solution $u_h \in X_h$ to the complete discrete spectral element problem.

Proof.
It must be proven that $a_{gl}(.,.)$ is bounded and coercive in X_h. Let $\psi,\phi \in X_h$,

$$a_{gl}(\psi,\phi) = \sum_{j=0}^{k} \sum_{i=0}^{n} p(\xi_{ij})\, \psi_x(\xi_{ij})\, \phi_x(\xi_{ij})\, w_{ij}$$

$$+ \sum_{j=0}^{k} \sum_{i=0}^{n} q(\xi_{ij})\, \psi(\xi_{ij})\, \phi(\xi_{ij})\, w_{ij}$$

From equation (3.3) it follows that, using the Cauchy–Schwartz inequality,

$$a_{gl}(\psi,\phi) \le (p_1 + q_1) \left[\sum_{j=0}^{k} \sum_{i=0}^{n} \psi_x^2(\xi_{ij})\, w_{ij} + \psi^2(\xi_{ij})\, w_{ij} \right]^{1/2}$$

$$\left[\sum_{j=0}^{k} \sum_{i=0}^{n} \phi_x^2(\xi_{ij})\, w_{ij} + \phi^2(\xi_{ij})\, w_{ij} \right]^{1/2}$$

Then, using the following result concerning Legendre Gauss–Lobatto numerical integration, there exist two constants c_1 and c_2 independent of n such that, $\forall\ \psi \in \mathcal{P}_n(-1,1)$

$$c_1 \int_{-1}^{1} \psi^2(\xi)\, d\xi \le \sum_{i=0}^{n} \psi^2(\xi_i)\, w_i \le c_2 \int_{-1}^{1} \psi^2(\xi)\, d\xi \qquad (C.2)$$

see (Canuto and Quarteroni, 1982), yields,

$$a_{gl}(\psi,\phi) \le c_2\,(p_1 + q_1) \left[\int_{-1}^{1} \psi_x^2(\xi) + \psi^2(\xi)\, d\xi \right]^{1/2} \left[\int_{-1}^{1} \phi_x^2(\xi) + \phi^2(\xi)\, d\xi \right]^{1/2}$$

from which it follows immediately that,

$$a_{gl}(\psi,\phi) \le C \,\|\, \psi \,\|_1 \,\|\, \phi \,\|_1 \qquad \psi,\phi \in \mathcal{H}^1(-1,1) \qquad (C.3)$$

where $\|\ .\ \|_1$ is the $\mathcal{H}^1$–norm defined by,

$$\|\, \psi \,\|_1 = \left[\int_{-1}^{1} \psi_x^2(\xi) + \psi^2(\xi)\, d\xi \right]^{1/2} \qquad \forall\, \psi \in \mathcal{H}^1(-1,1) \qquad (C.4)$$

This proves that the bilinear form $a_{gl}(\ ,\ )$ is bounded. Now it must be shown that $a_{gl}(\ ,\ )$ is coercive (or positive–definite) in X_h. Let $\psi \in X_h$,

$$a_{gl}(\psi,\psi) = \sum_{j=0}^{k} \sum_{i=0}^{n} p(\xi_{ij})\, \psi_x^2(\xi_{ij})\, w_{ij} + \sum_{j=0}^{k} \sum_{i=0}^{n} q(\xi_{ij})\, \psi^2(\xi_{ij})\, w_{ij}$$

Using again equation (3.3) yields,

$$a_{gl}(\psi,\psi) \ge p_0 \int_{-1}^{1} \psi_x^2(\xi)\, d\xi = p_0 \,\|\, \psi_x \,\|^2$$

with $\|\ .\ \|$ the $\mathcal{L}^2$–norm defined in (3.5). Now use Friedrichs' first inequality, (Axelsson and Barker, sec. 3, 1984),

$$\int_{-1}^{1} \psi_x^2(\xi)\, d\xi \geq \alpha \int_{-1}^{1} \psi^2(\xi)\, d\xi \qquad \psi \in \mathcal{H}^1(-1,1) \qquad (C.5)$$

where α is a positive constant, to show that,

$$a_{gl}(\psi,\psi) \geq {}^1/_2\, p_0 \int_{-1}^{1} \psi_x^2(\xi)\, d\xi + {}^1/_2\, p_0 \int_{-1}^{1} \psi_x^2(\xi)\, d\xi$$

$$\geq {}^1/_2\, p_0 \int_{-1}^{1} \psi_x^2(\xi)\, d\xi + {}^1/_2\, \alpha\, p_0 \int_{-1}^{1} \psi^2(\xi)\, d\xi$$

$$\geq C \int_{-1}^{1} \psi_x^2(\xi) + \psi^2(\xi)\, d\xi$$

with the positive constant $C = \min \{{}^1/_2\, p_0, {}^1/_2\, \alpha\, p_0\}$. It is now proven that,

$$a_{gl}(\psi,\psi) \geq C \parallel \psi \parallel_1^2 \qquad \psi \in \mathcal{H}^1(-1,1) \qquad (3.29)$$

i.e. the bilinear form $a_{gl}(.,.)$ is coercive. Application of the Lax–Milgram lemma completes the proof.

(End proof).