Harmonic Rayleigh-Ritz for the multiparameter eigenvalue problem

HARMONIC RAYLEIGH–RITZ FOR THE MULTIPARAMETER EIGENVALUE PROBLEM

MICHIEL E. HOCHSTENBACH* AND BOR PLESTENJAK†

Abstract. We study harmonic and refined extraction methods for the multiparameter eigenvalue problem. These techniques are generalizations of their counterparts for the standard and generalized eigenvalue problem. The methods aim to approximate interior eigenpairs, generally more accurately than the standard extraction does. We study their properties and give Saad type theorems. The processes can be combined with any subspace expansion approach, for instance a Jacobi–Davidson type technique, to form a subspace method for multiparameter eigenproblems of high dimension.

Key words. Multiparameter eigenvalue problem, two-parameter eigenvalue problem, harmonic extraction, refined extraction, Rayleigh–Ritz, subspace method, Saad's theorem, Jacobi–Davidson.

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1. Introduction. We study harmonic and refined Rayleigh–Ritz techniques for the multiparameter eigenvalue problem (MEP). For ease of presentation we will focus on the two-parameter eigenvalue problem

$$\begin{aligned} A_1x_1 &= \lambda B_1x_1 + \mu C_1x_1, \\ A_2x_2 &= \lambda B_2x_2 + \mu C_2x_2, \end{aligned} \tag{1.1}$$

for given $n_1 \times n_1$ (real or complex) matrices A_1, B_1, C_1, and $n_2 \times n_2$ matrices A_2, B_2, C_2; we are interested in eigenpairs $((\lambda, \mu), x_1 \otimes x_2)$ where x_1 and x_2 have unit norm. The approaches for general multiparameter eigenproblems will be straightforward generalizations of the two-parameter case.

Multiparameter eigenvalue problems of this kind arise in a variety of applications [1], particularly in mathematical physics when the method of separation of variables is used to solve boundary value problems [25]; see [10] for several other applications.

Two-parameter problems can be expressed as two coupled generalized eigenvalue problems as follows. On the tensor product space $\mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2}$ of dimension n_1n_2, one defines the matrix determinants

$$\begin{aligned} \Delta_0 &= B_1 \otimes C_2 - C_1 \otimes B_2, \\ \Delta_1 &= A_1 \otimes C_2 - C_1 \otimes A_2, \\ \Delta_2 &= B_1 \otimes A_2 - A_1 \otimes B_2. \end{aligned} \tag{1.2}$$

The MEP is called *right definite* if all the A_i, B_i, C_i, $i = 1, 2$, are Hermitian and Δ_0 is (positive or negative) definite; in this case the eigenvalues are real and the eigenvectors can be chosen to be real. The MEP is called *nonsingular* if Δ_0 is nonsingular (without further conditions on the A_i, B_i, C_i). A nonsingular two-parameter eigenvalue problem is equivalent to the coupled generalized eigenvalue problems

$$\begin{aligned} \Delta_1 z &= \lambda \Delta_0 z, \\ \Delta_2 z &= \mu \Delta_0 z, \end{aligned} \tag{1.3}$$

*Version October 30, 2006. Department of Mathematics and Computing Science, Eindhoven University of Technology, PO Box 513, 5600 MB, The Netherlands. m.e.hochstenbach@tue.nl. The research of this author was supported in part by NSF grant DMS-0405387.

†Department of Mathematics, University of Ljubljana, Jadranska 19, SI-1000 Ljubljana, Slovenia, bor.plestenjak@fmf.uni-lj.si

where $z = x_1 \otimes x_2$, and $\Delta_0^{-1}\Delta_1$ and $\Delta_0^{-1}\Delta_2$ commute. Because of the product dimension $n_1 n_2$, the multiparameter eigenvalue problem is a computationally quite challenging problem.

There exist several numerical methods for the MEP. Blum and colleagues [2, 3, 4], Bohte [5], and Browne and Sleeman [6] proposed methods for computing one eigenvalue, hopefully the closest one to a given approximation. There are also methods which determine all eigenvalues. The first class is formed by *direct methods* for right definite MEPs [22, 13, 7] and for non right definite MEPs [10]; these methods are suitable for small (dense) matrices only since their complexity is $\mathcal{O}((mn)^3)$. The second class consists of continuation methods [21, 18, 19] that are asymptotically somewhat cheaper than direct methods, but are so far often not very competitive for small problems in practice; for larger problems their computational cost is still enormous.

Fortunately, in applications often only a few relevant eigenpairs are of interest, for instance those corresponding to the largest eigenvalues, or the eigenvalues closest to a given target. Recently some subspace methods for the MEP have been proposed [11, 10] that are suitable for finding some selected eigenpairs. These methods combine a subspace approach with one of the mentioned dense methods as solver for the projected MEP. The approaches are also suitable for multiparameter problems where the matrices are large and sparse, although convergence to the wanted eigenpairs may sometimes remain an issue of concern. In particular, in [11] it was observed that finding interior eigenvalues was one of the challenges for the Jacobi–Davidson type method. It was left as an open question how to generalize the harmonic Rayleigh–Ritz approach for the MEP. This paper addresses this issue, and also introduces a refined Ritz method.

The rest of the paper has been organized as follows. In Section 2 we review the harmonic Rayleigh–Ritz method for the generalized eigenproblem, after which this method is generalized for the MEP in Section 3. In Section 4 we present two Saad type theorems for the standard and harmonic extraction. Section 5 proposes a refined Rayleigh–Ritz method for the MEP. We conclude with experiments and a conclusion in Sections 6 and 7.

2. Harmonic Rayleigh–Ritz for the generalized eigenvalue problem. We first briefly review the harmonic Rayleigh–Ritz for the generalized eigenvalue problem

$$Ax = \lambda Bx.$$

Suppose we would like to compute an approximation (θ, u) to the eigenpair (λ, x), where the approximate eigenvector u should be in a given search space $\mathcal{U}_k$ of low dimension k, and θ should be in the neighborhood of the target $\tau \in \mathbb{C}$.

Since $u \in \mathcal{U}_k$, we can write $u = U_k c$, where the columns of U_k form an orthonormal basis for $\mathcal{U}_k$, and c is a vector in $\mathbb{C}^k$ of unit norm. The standard Ritz–Galerkin condition on the residual r is (cf. [17])

$$r := Au - \theta Bu \perp \mathcal{U}_k,$$

which implies that (θ, c) should be a primitive Ritz pair (terminology from Stewart [24]), an eigenpair of the projected generalized eigenproblem

$$U_k^* A U_k c = \theta\, U_k^* B U_k c.$$

It follows that if $u^*Bu \neq 0$, then $\theta = \frac{u^*Au}{u^*Bu}$; the case $u^*Bu = 0$ is an exceptional case where the Ritz value is infinite (if $u^*Au \neq 0$) or undefined (if $u^*Au = 0$). If B is Hermitian positive definite, then we can define the B^{-1}-norm of the residual by $\|z\|_{B^{-1}}^2 = z^*B^{-1}z$, and one can show that this θ minimizes $\|r\|_{B^{-1}}$.

However, the problem with this standard Rayleigh–Ritz approach is that even if there is a Ritz value $\theta \approx \tau$, we do not have the guarantee that the two-norm $\|r\|$ is small, which reflects the fact that the approximate eigenvector may be poor. As a remedy, the harmonic Rayleigh–Ritz was proposed by Morgan [15], Paige, Parlett, and Van der Vorst [16] for the standard eigenproblem, and by Stewart [24] for the generalized eigenproblem; see also Fokkema, Sleijpen, and Van der Vorst [9]. Assuming $A - \tau B$ is nonsingular, the idea is to consider a spectral transformation

$$(A - \tau B)^{-1} Bx = (\lambda - \tau)^{-1} x. \tag{2.1}$$

Thus, the interior eigenvalues $\lambda \approx \tau$ are exterior eigenvalues of $(A-\tau B)^{-1}B$ for which a Galerkin condition usually works well in practice. To avoid working with $(A-\tau B)^{-1}$, the inverse of a large sparse matrix, we impose a Petrov–Galerkin condition

$$(A - \tau B)^{-1} Bu - (\theta - \tau)^{-1} u \perp (A - \tau B)^*(A - \tau B)\mathcal{U}_k,$$

or, equivalently,

$$Au - \theta Bu = (A - \tau B)u - (\theta - \tau)Bu \perp (A - \tau B)\mathcal{U}_k, \tag{2.2}$$

leading to the projected eigenproblem

$$U_k^*(A - \tau B)^*(A - \tau B)U_k c = (\theta - \tau)\, U_k^*(A - \tau B)^* BU_k c. \tag{2.3}$$

Here we are interested in the primitive harmonic Ritz pair(s) (θ, c) with θ closest to τ. This approach has two motivations:

- if an exact eigenvector is in the search space, $x = U_k c$, then the eigenpair (λ, x) satisfies (2.3) (this implies that exact eigenvectors in the search space will be detected unless we are in the special circumstance of the presence of multiple harmonic Ritz values);
- a harmonic Ritz pair (θ, u) satisfies [24]

  $$\|Au - \theta Bu\| \leq |\theta - \tau| \cdot \|Bu\| \leq |\theta - \tau| \cdot \|BU_k\|$$

  which motivates the choice of the harmonic Ritz value closest to τ.

The harmonic Rayleigh–Ritz approach was generalized for the polynomial eigenproblem in [12].

3. Harmonic Rayleigh–Ritz for the multiparameter eigenvalue problem. For the MEP (1.1) it is natural to make use of two search spaces, $\mathcal{U}_k$ and $\mathcal{V}_k$, for the vectors x and y, respectively. Let the columns of U_k and V_k form orthonormal bases for $\mathcal{U}_k$ and $\mathcal{V}_k$. We look for an approximate eigenpair $((\theta, \eta),\ u \otimes v) \approx ((\lambda, \mu), x \otimes y)$, where $u \otimes v$ is of the form $Uc \otimes Vd$, where both $c, d \in \mathbb{C}^k$ are of unit norm. The standard extraction,

$$\begin{aligned} (A_1 - \theta B_1 - \eta C_1)\, U_k c &\perp \mathcal{U}_k, \\ (A_2 - \theta B_2 - \eta C_2)\, V_k d &\perp \mathcal{V}_k \end{aligned} \tag{3.1}$$

was introduced in [11]. As is also experienced for the standard eigenvalue problem, the standard extraction for the MEP works well for exterior eigenvalues, but is generally less favorable for interior ones [11].

Now suppose we are interested in a harmonic approach to better approximate eigenpairs near the target (σ, τ). One obstacle is that MEPs do not seem to allow for a straightforward generalization of the spectral transformation (2.1). Therefore we generalize (2.2) and impose the two Galerkin conditions

$$\begin{aligned}(A_1 - \theta B_1 - \eta C_1)\, u &\perp (A_1 - \sigma B_1 - \tau C_1)\, \mathcal{U}_k,\\ (A_2 - \theta B_2 - \eta C_2)\, v &\perp (A_2 - \sigma B_2 - \tau C_2)\, \mathcal{V}_k,\end{aligned} \tag{3.2}$$

or, equivalently,

$$\begin{aligned}(A_1 - \sigma B_1 - \tau C_1)\, U_k c - (\theta - \sigma)\, B_1 U_k c - (\eta - \tau)\, C_1 U_k c &\perp (A_1 - \sigma B_1 - \tau C_1)\, \mathcal{U}_k,\\ (A_2 - \sigma B_2 - \tau C_2)\, V_k d - (\theta - \sigma)\, B_2 V_k d - (\eta - \tau)\, C_2 V_k d &\perp (A_2 - \sigma B_2 - \tau C_2)\, \mathcal{V}_k.\end{aligned}$$

We call this the harmonic Rayleigh–Ritz extraction for the MEP. Introduce the following reduced QR-decompositions

$$(A_1 - \sigma B_1 - \tau C_1)\, U_k = Q_1 R_1, \qquad (A_2 - \sigma B_2 - \tau C_2)\, V_k = Q_2 R_2, \tag{3.3}$$

which we can compute incrementally, i.e., one column per step, during the subspace method. This is done for computational efficiency, as well as stability: cross products of the form $(A_i - \sigma B_i - \tau C_i)^*(A_i - \sigma B_i - \tau C_i)$ with a potential high condition number are avoided. Then, computationally, the extraction amounts to the projected two-parameter eigenproblem

$$\begin{aligned}R_1\, c &= (\theta - \sigma)\, Q_1^* B_1 U_k c + (\eta - \tau)\, Q_1^* C_1 U_k c,\\ R_2\, d &= (\theta - \sigma)\, Q_2^* B_2 V_k d + (\eta - \tau)\, Q_2^* C_2 V_k d.\end{aligned}$$

We now compute the smallest eigenpair(s) $((\xi_1, \xi_2),\ c \otimes d)$ (in absolute value sense, that is, with minimal $|\xi_1|^2 + |\xi_2|^2$) of the low-dimensional MEP

$$\begin{aligned}R_1\, c &= \xi_1\, Q_1^* B_1 U_k c + \xi_2\, Q_1^* C_1 U_k c,\\ R_2\, d &= \xi_1\, Q_2^* B_2 V_k d + \xi_2\, Q_2^* C_2 V_k d,\end{aligned} \tag{3.4}$$

which can be solved by existing low-dimensional techniques as mentioned in the introduction.

As in the case for the generalized eigenproblem, there are two justifications for the harmonic approach for the MEP:

- we have the following upper bounds for the residual norms:

  $$\begin{aligned}\|(A_1 - \sigma B_1 - \tau C_1)\, u\| &\le |\xi_1|\, \|B_1 u\| + |\xi_2|\, \|C_1 u\| \le |\xi_1|\, \|B_1 U_k\| + |\xi_2|\, \|C_1 U_k\|,\\ \|(A_2 - \sigma B_2 - \tau C_2)\, v\| &\le |\xi_1|\, \|B_2 v\| + |\xi_2|\, \|C_2 v\| \le |\xi_1|\, \|B_2 V_k\| + |\xi_2|\, \|C_2 V_k\|,\end{aligned}$$

  so to minimize the residual norms it is clear that it is sensible to select the smallest (ξ_1, ξ_2);
- if the search spaces contain an eigenvector, $x = U_k c$, $y = V_k d$, then the pair $((\lambda, \mu), x \otimes y)$ satisfies (3.2); this means that this pair is a harmonic Ritz pair unless the harmonic Ritz value (θ, η) is not simple. We will prove a more precise statement in the next section.

In conclusion, the harmonic approach for the MEP tries to combine two desirable properties: it will generally find an exact eigenpair present in the search spaces, while it will also try to detect approximate eigenpairs with small residual norm.

4. Saad type theorems. In this section, we derive Saad type theorems for both the standard and harmonic extraction for the MEP. This type of theorem expresses the quality of the approximate vectors in terms of the quality of the search spaces. The original theorem by Saad [20, Thm. 3.6] was for the standard extraction for the standard Hermitian eigenvalue problem. A generalization for non-Hermitian matrices and eigenspaces was given by Stewart [23], while an extension for the harmonic extraction for the standard eigenvalue problem was presented by Chen and Jia [8].

4.1. Saad type theorem for the standard extraction. Let $w := u \otimes v$ be a Ritz vector corresponding to Ritz value (θ, η), and $[w \ \ W \ \ W_\perp]$ be an orthonormal basis for $\mathbb{C}^{n_1 n_2}$ such that

$$\mathrm{span}([w \ \ W]) = \mathcal{U}_k \otimes \mathcal{V}_k.$$

Define, for $i = 0, 1, 2$,

$$E_i = [w \ \ W]^* \Delta_i [w \ \ W]. \tag{4.1}$$

We assume that E_0 is invertible, which is guaranteed if Δ_0 is definite as in the case of a right definite MEP. From (3.1) we have that the Ritz pairs are of the form $((\theta,\eta), U_k c \otimes V_k d)$ where $((\theta,\eta), c \otimes d)$ are the eigenvalues of

$$\begin{aligned} U_k^* A_1 U_k c &= \theta\, U_k^* B_1 U_k c + \eta\, U_k^* C_1 U_k c, \\ V_k^* A_2 V_k d &= \theta\, V_k^* B_2 V_k d + \eta\, V_k^* C_2 V_k d. \end{aligned}$$

The three matrix determinants of this projected MEP (cf. (1.2)) are of the form $Q^* E_i Q$, where Q is the orthonormal basis transformation matrix that maps $U_k \otimes V_k$ coordinates to $[w \ \ W]$ coordinates: $Q = [w \ \ W]^* (U_k \otimes V_k)$. For instance, we have

$$\begin{aligned} U_k^* B_1 U_k \otimes V_k^* C_2 V_k - U_k^* C_1 U_k \otimes V_k^* B_2 V_k &= (U_k \otimes V_k)^* \Delta_0 (U_k \otimes V_k) \\ &= Q^* [w \ \ W]^* \Delta_0 [w \ \ W] Q = Q^* E_0 Q. \end{aligned}$$

Therefore, the components θ_j and η_j of the Ritz values (θ_j, η_j) are eigenvalues of

$$E_0^{-1} E_1 \quad \text{and} \quad E_0^{-1} E_2,$$

respectively (cf. (1.3)). In particular we know that

$$(E_1 - \theta E_0)\, e_1 = 0$$

so $E_0^{-1} E_1$ is of the form

$$E_0^{-1} E_1 = \begin{bmatrix} \theta & f_1^* \\ 0 & G_1 \end{bmatrix}, \tag{4.2}$$

where the precise expression for G_1 is not so much of importance as well as the fact that its eigenvalues are the $k^2{-}1$ θ_j-values other than θ. (Here, "other than" is different from "not equal to": even if (θ, η) is not a multiple Ritz value, its first coordinate may still be equal to the first coordinate of other Ritz values.) Similarly, $E_0^{-1} E_2$ is of the form

$$E_0^{-1} E_2 = \begin{bmatrix} \eta & f_2^* \\ 0 & G_2 \end{bmatrix}, \tag{4.3}$$

where the eigenvalues of G_2 are the k^2-1 η_j-values other than η. Using these quantities, we can now prove the following theorem, which extends [20, Thm. 3.6], [23, Thm. 2], and [8, Thm. 3].

THEOREM 4.1. *Let $((\theta,\eta),\ u\otimes v)$ be a Ritz pair and $((\lambda,\mu),\ x\otimes y)$ an eigenpair. Let $E_i=[w\ \ W]^*\Delta_i[w\ \ W]$ for $i=0,1,2$ and assume E_0^{-1} is invertible. Then*

$$\varphi(\sin(u,x),\,\sin(v,y))\le\min\left\{\left(1+\frac{\gamma_1^2}{\delta_1^2}\right),\left(1+\frac{\gamma_2^2}{\delta_2^2}\right)\right\}\cdot\varphi(\sin(\mathcal{U}_k,x),\,\sin(\mathcal{V}_k,y)) \tag{4.4}$$

where

$$\begin{aligned}
\varphi(a,b)&=a^2+b^2-a^2b^2,\\
\gamma_1&=\|E_0^{-1}\|\,\|P_{\mathcal{U}_k\otimes\mathcal{V}_k}(\Delta_1-\lambda\Delta_0)(I-P_{\mathcal{U}_k\otimes\mathcal{V}_k})\|,\\
\gamma_2&=\|E_0^{-1}\|\,\|P_{\mathcal{U}_k\otimes\mathcal{V}_k}(\Delta_2-\mu\Delta_0)(I-P_{\mathcal{U}_k\otimes\mathcal{V}_k})\|,\\
\delta_1&=\sigma_{\min}(G_1-\lambda I)\le\min_{\theta_j\ne\theta}|\theta_j-\lambda|,\\
\delta_2&=\sigma_{\min}(G_2-\mu I)\le\min_{\eta_j\ne\eta}|\eta_j-\mu|,
\end{aligned}$$

(θ_j,η_j) range over all k^2-1 Ritz values other than (θ,η), and $P_{\mathcal{U}_k\otimes\mathcal{V}_k}$ is the orthogonal projection onto $\mathcal{U}_k\otimes\mathcal{V}_k$.

Proof. From $\Delta_1 z=\lambda\Delta_0 z$ we get with a change of variables

$$[w\ \ W\ \ W_\perp]^*(\Delta_1-\lambda\Delta_0)[w\ \ W\ \ W_\perp][a_1\ \ a_2\ \ a_3]^T=0,$$

where

$$[a_1\ \ a_2\ \ a_3]^T=[w\ \ W\ \ W_\perp]^*z.$$

Writing out the first and second (block) equation gives

$$(E_1-\lambda E_0)\begin{bmatrix}a_1\\a_2\end{bmatrix}=-[w\ \ W]^*(\Delta_1-\lambda\Delta_0)W_\perp a_3.$$

Left-multiplying by E_0^{-1} and using (4.2) yield

$$\begin{bmatrix}\theta-\lambda & f_1^*\\ 0 & G_1-\lambda I\end{bmatrix}\begin{bmatrix}a_1\\a_2\end{bmatrix}=-E_0^{-1}[w\ \ W]^*(\Delta_1-\lambda\Delta_0)W_\perp\,a_3.$$

Hence,

$$\begin{aligned}
\sigma_{\min}(G_1-\lambda I)\,\|a_2\|&\le\|(G_1-\lambda I)\,a_2\|\\
&\le\|E_0^{-1}[w\ \ W]^*(\Delta_1-\lambda\Delta_0)W_\perp\,a_3\|\\
&\le\|E_0^{-1}\|\,\|[w\ \ W]^*(\Delta_1-\lambda\Delta_0)W_\perp\|\,\|a_3\|.
\end{aligned}$$

This gives us the bound $\|a_2\|\le(\gamma_1/\delta_1)\,\|a_3\|$. Similarly, if we start from $\Delta_2 z=\mu\Delta_0 z$ and use (4.3), we eventually get the bound $\|a_2\|\le(\gamma_2/\delta_2)\,\|a_3\|$.

Since $\|a\| = 1$, with some calculations we see that

$$
\begin{aligned}
\|a_1\|^2 &= \cos^2(u,x)\,\cos^2(v,y),\\
\|a_1\|^2 + \|a_2\|^2 &= \cos^2(\mathcal{U}_k,x)\,\cos^2(\mathcal{V}_k,y),\\
\|a_2\|^2 + \|a_3\|^2 &= 1 - \cos^2(u,x)\,\cos^2(v,y)\\
&= \sin^2(u,x) + \sin^2(v,y) - \sin^2(u,x)\,\sin^2(v,y),\\
\|a_3\|^2 &= 1 - \cos^2(\mathcal{U}_k,x)\,\cos^2(\mathcal{V}_k,y)\\
&= \sin^2(\mathcal{U}_k,x) + \sin^2(\mathcal{V}_k,y) - \sin^2(\mathcal{U}_k,x)\,\sin^2(\mathcal{V}_k,y).
\end{aligned}
$$

The result now follows from substituting the bound for $\|a_2\|$ in terms of $\|a_3\|$ in the expression $\|a_2\|^2 + \|a_3\|^2$. $\square$

The main significance of the theorem is the following. If both $\sin(\mathcal{U}_k, x) \to 0$ and $\sin(\mathcal{V}_k, y) \to 0$, then the right-hand side of (4.4) goes to 0. Since φ is nonnegative on the domain $[-1,1] \times [-1,1]$ and has has only one zero in the origin, we know that both $\sin(u,x) \to 0$ and $\sin(v,y) \to 0$, so there is a Ritz vector $u \otimes v$ converging to the eigenvector $x \otimes y$—unless *both* δ_1 and δ_2 are zero, which means that (θ,η) must be a multiple Ritz value.

Moreover, as expected, to have an exact eigenpair in the search spaces, we see that it is generally not sufficient if just one of $\sin(\mathcal{U}_k, x)$ and $\sin(\mathcal{V}_k, y)$ tends to zero.

4.2. Saad type theorem for the harmonic extraction. We have a similar theorem for the harmonic extraction, *mutatis mutandis*, which means that the harmonic extraction is also asymptotically accurate. Define the quantities

$$
\begin{aligned}
\widetilde{A}_i &= (A_i - \sigma B_i - \tau C_i)^* A_i, \qquad & \widetilde{\Delta}_0 &= \widetilde{B}_1 \otimes \widetilde{C}_2 - \widetilde{C}_1 \otimes \widetilde{B}_2,\\
\widetilde{B}_i &= (A_i - \sigma B_i - \tau C_i)^* B_i, \qquad & \widetilde{\Delta}_1 &= \widetilde{A}_1 \otimes \widetilde{C}_2 - \widetilde{C}_1 \otimes \widetilde{A}_2,\\
\widetilde{C}_i &= (A_i - \sigma B_i - \tau C_i)^* C_i, \qquad & \widetilde{\Delta}_2 &= \widetilde{B}_1 \otimes \widetilde{A}_2 - \widetilde{A}_1 \otimes \widetilde{B}_2.
\end{aligned}
$$

Let $\widetilde{w} := \widetilde{u} \otimes \widetilde{v}$ be a harmonic Ritz vector corresponding to the harmonic Ritz value $(\widetilde{\theta}, \widetilde{\eta})$, and let $[\widetilde{w}\ \widetilde{W}\ W_\perp]$ be an orthonormal basis for $\mathbb{C}^{n_1 n_2}$ such that $\operatorname{span}([\widetilde{w}\ \widetilde{W}]) = \mathcal{U}_k \otimes \mathcal{V}_k$.

Similar to (4.1), define, $E_i = [\widetilde{w}\ \widetilde{W}]^* \widetilde{\Delta}_i [\widetilde{w}\ \widetilde{W}]$ for $i = 0,1,2$. Then the components $\widetilde{\theta}_j$ and $\widetilde{\eta}_j$ of the harmonic Ritz values $(\widetilde{\theta}_j, \widetilde{\eta}_j)$ are eigenvalues of $\widetilde{E}_0^{-1}\widetilde{E}_1$ and $\widetilde{E}_0^{-1}\widetilde{E}_2$, respectively. Since $(\widetilde{E}_1 - \widetilde{\theta}\widetilde{E}_0)\,e_1 = 0$ we know that $\widetilde{E}_0^{-1}\widetilde{E}_1$ is of the form

$$
\widetilde{E}_0^{-1}\widetilde{E}_1 = \begin{bmatrix} \widetilde{\theta} & \widetilde{f}_1^* \\ 0 & \widetilde{G}_1 \end{bmatrix},
$$

where the eigenvalues of $\widetilde{G}_1$ are the k^2-1 $\widetilde{\theta}_j$-values other than $\widetilde{\theta}$. Similarly, $\widetilde{E}_0^{-1}\widetilde{E}_2$ is of the form

$$
\widetilde{E}_0^{-1}\widetilde{E}_2 = \begin{bmatrix} \widetilde{\eta} & \widetilde{f}_2^* \\ 0 & \widetilde{G}_2 \end{bmatrix},
$$

where the eigenvalues of $\widetilde{G}_2$ are the k^2-1 $\widetilde{\eta}_j$-values other than $\widetilde{\eta}$. Analogous to Theorem 4.1 we can prove the following result.

THEOREM 4.2. *Let $((\widetilde{\theta},\widetilde{\eta}),\ \widetilde{u}\otimes\widetilde{v})$ be a harmonic Ritz pair and $((\lambda,\mu),\ x\otimes y)$ be an eigenpair. Let $E_i=[\widetilde{w}\ \ \widetilde{W}]^*\Delta_i[\widetilde{w}\ \ \widetilde{W}]$ for $i=0,1,2$ and assume $\widetilde{E}_0^{-1}$ is invertible. Then*

$$\varphi(\sin(\widetilde{u},x),\,\sin(\widetilde{v},y))\le\min\left\{\left(1+\frac{\widetilde{\gamma}_1^2}{\widetilde{\delta}_1^2}\right),\left(1+\frac{\widetilde{\gamma}_2^2}{\widetilde{\delta}_2^2}\right)\right\}\cdot\varphi(\sin(\mathcal{U}_k,x),\,\sin(\mathcal{V}_k,y))$$

where $\varphi(a,b)$ is as before and

$$\begin{aligned}
\gamma_1&=\|\widetilde{E}_0^{-1}\|\,\|P_{\mathcal{U}_k\otimes\mathcal{V}_k}(\widetilde{\Delta}_1-\lambda\widetilde{\Delta}_0)(I-P_{\mathcal{U}_k\otimes\mathcal{V}_k})\|,\\
\gamma_2&=\|\widetilde{E}_0^{-1}\|\,\|P_{\mathcal{U}_k\otimes\mathcal{V}_k}(\widetilde{\Delta}_2-\mu\widetilde{\Delta}_0)(I-P_{\mathcal{U}_k\otimes\mathcal{V}_k})\|,\\
\delta_1&=\sigma_{\min}(\widetilde{G}_1-\lambda I)\le\min_{\widetilde{\theta}_j\ne\widetilde{\theta}}|\widetilde{\theta}_j-\lambda|,\\
\delta_2&=\sigma_{\min}(\widetilde{G}_2-\mu I)\le\min_{\widetilde{\eta}_j\ne\widetilde{\eta}}|\widetilde{\eta}_j-\mu|,
\end{aligned}$$

$(\widetilde{\theta}_j,\widetilde{\eta}_j)$ range over all k^2-1 harmonic Ritz values other than $(\widetilde{\theta},\widetilde{\eta})$.

This means that if $\sin(\mathcal{U}_k,x)\to 0$ and $\sin(\mathcal{V}_k,y)\to 0$, then there is a Ritz vector $\widetilde{u}\otimes\widetilde{v}$ converging to the eigenvector $x\otimes y$—unless $(\widetilde{\theta},\widetilde{\eta})$ is a multiple harmonic Ritz value.

Comparing Theorems (4.1) and (4.2), we see that both the standard and harmonic extraction are asymptotically accurate: up to multiple (harmonic) Ritz values, they will recognize an eigenvector which is in the search space.

5. Refined extraction for the multiparameter problem. The refined extraction is an alternative approach that minimizes the residual norm over a given search space. It was popularized for the standard eigenvalue problem by Jia [14]. We now generalize this approach for the MEP.

Given σ and τ, for instance a tensor Rayleigh quotient [18] of the form

$$\begin{aligned}
\sigma&=\frac{(u\otimes v)^*\Delta_1(u\otimes v)}{(u\otimes v)^*\Delta_0(u\otimes v)}=\frac{(u^*A_1u)(v^*C_2v)-(u^*C_1u)(v^*A_2v)}{(u^*B_1u)(v^*C_2v)-(u^*C_1u)(v^*B_2v)},\\
\tau&=\frac{(u\otimes v)^*\Delta_2(u\otimes v)}{(u\otimes v)^*\Delta_0(u\otimes v)}=\frac{(u^*B_1u)(v^*A_2v)-(u^*A_1u)(v^*B_2v)}{(u^*B_1u)(v^*C_2v)-(u^*C_1u)(v^*B_2v)},
\end{aligned}\tag{5.1}$$

or a target, the refined extraction determines an approximate eigenvector $\widehat{u}\otimes\widehat{v}$ by minimizing the residual norms over $\mathcal{U}_k$ and $\mathcal{V}_k$, respectively:

$$\begin{aligned}
\widehat{u}&=\underset{u\in\mathcal{U}_k,\ \|u\|=1}{\operatorname{argmin}}\ \|(A_1-\sigma B_1-\tau C_1)\,u\|,\\
\widehat{v}&=\underset{v\in\mathcal{V}_k,\ \|v\|=1}{\operatorname{argmin}}\ \|(A_2-\sigma B_2-\tau C_2)\,v\|.
\end{aligned}\tag{5.2}$$

In practice, it is often sensible to take the target in the beginning of the process, and switch to the Rayleigh quotient when the residual norm is under a certain threshold, because this indicated that the Rayleigh quotient is of sufficient quality. Computationally, (5.2) is best determined by two incremental QR-decompositions (3.3), followed by singular value decompositions of R_1 and R_2.

The following theorem is a generalization of [24, Thm. 4.10].

THEOREM 5.1. *For the residuals of the refined Ritz vector* (5.2) *we have*

$$\|(A_1 - \sigma B_1 - \tau C_1)\,\widehat{u}\| \le \frac{|\lambda-\sigma|\,\|B_1\| + |\mu-\tau|\,\|C_1\| + \|(A_1-\sigma B_1-\tau C_1)\|\,\sin(\mathcal{U}_k, x)}{\sqrt{1-\sin^2(\mathcal{U}_k, x)}},$$
$$\|(A_2 - \sigma B_2 - \tau C_2)\,\widehat{v}\| \le \frac{|\lambda-\sigma|\,\|B_2\| + |\mu-\tau|\,\|C_2\| + \|(A_2-\sigma B_2-\tau C_2)\|\,\sin(\mathcal{V}_k, y)}{\sqrt{1-\sin^2(\mathcal{V}_k, y)}}.$$

Proof. Decompose $x = \gamma_U x_U + \sigma_U e_U$, where $x_U := UU^*x \,/\, \|UU^*x\|$ is the orthogonal projection of x onto $\mathcal{U}$, $\|x_U\| = \|e_U\| = 1$, $\gamma_U = \cos(\mathcal{U}_k, x)$, and $\sigma_U = \sin(\mathcal{U}_k, x)$. Since $x_U = (x - \sigma_U e_U)/\gamma_U$, we have by the definition of a refined Ritz vector (5.2)

$$\begin{aligned}\|(A_1 - \sigma B_1 - \tau C_1)\,\widehat{u}\| &\le \|(A_1 - \sigma B_1 - \tau C_1)\,x_U\| \\ &\le \|(\lambda-\sigma)B_1 x + (\mu-\tau)C_1 x + \sigma_U\,(A_1 - \sigma B_1 - \tau C_1)\,e_U\|/\gamma_U,\end{aligned}$$

from which the first result follows. The derivation of the second result is similar. □

Similar to the refined extraction for the standard eigenvalue problem we see that, in contrast to the situation for the standard and harmonic extraction methods (Theorems 4.1 and 4.2), the conditions $\sin(\mathcal{U}_k, x) \to 0$ and $\sin(\mathcal{V}_k, y) \to 0$ are no longer sufficient for convergence of the refined vectors to eigenvectors; we also need $\sigma \to \lambda$ and $\tau \to \mu$.

In principle we may vary σ and τ during the process, but this is computationally more expensive, since each change makes the computation of new QR-decompositions (3.3) necessary. This may make the refined approach less attractive in practice.

Additionally, the refined extraction seems to be not tailored for computing more than one eigenpair. The reason for this is that this extraction, in contrast to the standard and harmonic variants, renders only one approximate eigenvector. (In principle we can do (5.2) for multiple shifts, but this is computationally unattractive.) In the MEP, a standard deflation technique is not feasible [11, 10]. Avoiding previously computed eigenvectors is ensured by examining their angles with new potential approximate vectors. However, this selection principle seems unsuitable for combination with the principle of the refined extraction.

We note that the refined extraction for the MEP shares this disadvantage with the refined extraction for the polynomial eigenproblem [12] because of related deflation issues. The refined extraction for the standard and generalized eigenproblem do not suffer from this problem.

For the reasons mentioned above, we will exclude the refined approach from the numerical experiments in the next section.

6. Numerical experiments. The numerical results in this section were obtained with Matlab 7.0.

EXPERIMENT 6.1. In the first example we consider a random right definite two-parameter eigenvalue problems with known eigenpairs, which enables us to check the obtained results. We take the matrices

$$A_i = S_i F_i S_i^*, \quad B_i = S_i G_i S_i^*, \quad C_i = S_i H_i S_i^*, \tag{6.1}$$

where F_i, G_i, and H_i are diagonal matrices and S_i are banded matrices generated in Matlab by `2*speye(1000)+triu(tril(sprandn(1000,1000,d),b),-b)`, where d

is the density and b is the bandwidth for $i = 1, 2$. We select the diagonal elements of F_1, F_2, G_2, and H_1 as normally distributed random numbers with mean zero, variance one and standard deviation one, and the diagonal elements of G_1 and H_2 as normally distributed random numbers with mean 5, variance one and standard deviation one. In this way, the problem is right definite and the eigenvalues can be computed exactly from diagonal elements of F_i, G_i, and H_i, see [11] for details.

We are interested in approximating the innermost eigenpair $((\lambda, \mu), x \otimes y)$, i.e., the pair of which the eigenvalue (λ, μ) is closest to the arithmetic mean of the eigenvalues. For the search subspace $\mathcal{U}_1$ we take the span of $\widetilde{x}$, which is a perturbation of the eigenvector component x, and nine additional random vectors. The search space $\mathcal{U}_2$ is formed similarly.

We test with different perturbations which affect the quality of the 10-dimensional search spaces $\mathcal{U}_1$ and $\mathcal{U}_2$. We compare the approximations for the innermost eigenpair obtained from the Ritz extraction and the harmonic Ritz extraction. The results are in Table 6.1. Let (θ, η) be a standard or harmonic Ritz value that approximates (λ, μ) and let $U_1 c_1 \otimes U_2 c_2$ be the corresponding standard or harmonic Ritz vector, where the orthogonal columns of U_i span $\mathcal{U}_i$ for $i = 1, 2$. The rows in Table 6.1 are:

- *subspace*: $\angle(x \otimes y, \mathcal{U}_1 \otimes \mathcal{U}_2)$, the angle between the exact eigenvector $x \otimes y$ and the search subspace $\mathcal{U}_1 \otimes \mathcal{U}_2$; this quantity indicates the best result any extraction method can obtain.
- *vector*: $\angle(U_1 c_1 \otimes U_2 c_2,\ x \otimes y)$, the angle between the exact eigenvector and the (harmonic) Ritz vector.
- *residual*: the norm of the associated residual

$$\left(\|(A_1 - \theta B_1 - \eta C_1) U_1 c_1\|_2^2 + \|(A_2 - \theta B_2 - \eta C_2) U_2 c_2\|_2^2\right)^{1/2}.$$

- *value*: $(|\lambda - \theta|^2 + |\mu - \eta|^2)^{1/2}$, the difference between the (harmonic) Ritz value (θ, η) and the exact eigenvalue (λ, μ).
- *RQ value*: the difference between the tensor Rayleigh quotient (5.1) of the standard or harmonic Ritz vector and the exact eigenvalue (λ, μ).
- *RQ residual*: the norm of the residual

$$\left(\|(A_1 - \widetilde{\theta} B_1 - \widetilde{\eta} C_1) U_1 c_1\|_2^2 + \|(A_2 - \widetilde{\theta} B_2 - \widetilde{\eta} C_2) U_2 c_2\|_2^2\right)^{1/2},$$

  where we take the tensor Rayleigh quotient instead of the (harmonic) Ritz value.

TABLE 6.1

A comparison of the Ritz and harmonic Ritz extraction from three different subspaces for a right definite two-parameter eigenvalue problem.

	subspace=3.6e-3		subspace=4.0e-5		subspace=4.2e-7	
	Ritz	harmonic	Ritz	harmonic	Ritz	harmonic
vector	1.9e-2	4.4e-3	3.7e-4	4.2e-5	1.6e-6	4.3e-7
value	1.2e-5	2.0e-2	1.2e-8	4.0e-5	2.5e-14	1.4e-8
residual	1.4e-1	1.8e-1	2.6e-3	4.2e-4	1.2e-5	3.0e-6
RQ value	1.2e-5	4.6e-6	1.2e-8	8.8e-9	2.5e-14	1.5e-14
RQ residual	1.4e-1	3.4e-2	2.6e-3	3.9e-4	1.2e-5	3.0e-6

From Table 6.1 we see that the harmonic extraction returns eigenvector approximations that are almost optimal, i.e., they are very close to the orthogonal projections of the exact eigenvectors onto the search subspace. On the other hand, as is also usual for the standard eigenvalue problem, the harmonic Ritz values are less favorable approximations to the exact eigenvalues than the Ritz values. However, if we use the harmonic Ritz vector with its tensor Rayleigh quotient as approximation to the eigenvalue, we get better approximations and smaller residuals than in the standard extraction.

EXPERIMENT 6.2. In the second example we take a random non right definite two-parameter eigenvalue problem of the same dimensions as in Example 6.1. Here we select the diagonal elements of F_1, F_2, G_1, G_2, H_1, and H_2 as complex numbers where both the real and the imaginary part are uniformly distributed random numbers from the interval $(-0.5, 0.5)$.

Table 6.2 contains the results of similar numerical experiments as in Example 6.1.

TABLE 6.2
A comparison of the Ritz and harmonic Ritz extraction from three different subspaces for a non right definite two-parameter eigenvalue problem.

	subspace=2.6e-3		subspace=4.8e-5		subspace=5.0e-7	
	Ritz	harmonic	Ritz	harmonic	Ritz	harmonic
vector	7.4e-3	2.6e-3	1.8e-4	4.9e-5	1.0e-5	5.1e-7
value	2.7e-3	1.5e-4	4.9e-5	4.7e-6	1.4e-6	4.1e-8
residual	1.8e-2	5.8e-3	4.5e-4	1.3e-4	2.9e-5	1.5e-6
RQ value	2.7e-3	3.0e-3	4.9e-5	3.8e-5	1.4e-6	1.6e-6
RQ residual	1.8e-2	7.5e-3	4.5e-4	1.3e-4	2.9e-5	3.1e-6

As in the previous example, the harmonic extraction returns almost optimal eigenvector approximations which are clearly better than the results of the standard extraction. Since this problem is non right definite, the error in the tensor Rayleigh quotient is linear in the eigenvector approximation error, compared to quadratic for the right definite problem in the previous example; see [11].

EXPERIMENT 6.3. We take the right definite examples from Example 6.1 with density $d = 0.01$ and bandwidth $b = 80$. We compare the eigenvalues obtained by the Jacobi-Davidson method [11] where we apply the standard and harmonic Ritz extraction, respectively. We start with the same initial vectors and we test various numbers of inner GMRES steps for approximately solving the correction equation. We use the second order correction equation with the oblique projections, for the details see [11]. The maximum dimension of the search spaces is 14, after which we restart with three-dimensional spaces.

For the target (σ, τ) we take the arithmetic mean of the eigenvalues. In the extraction phase the standard or harmonic Ritz value is selected that is closest to a given target. Subsequently, we take the corresponding eigenvector approximation and take its tensor Rayleigh quotient as an approximate eigenvalue. As one can see in Examples 6.1 and 6.2, this gives a better approximation for the eigenvalue when we use the harmonic extraction, whereas it–naturally–does not change the eigenvalue approximation in the Ritz extraction.

We compute 100 eigenvalues, where we note that with a total number of 10^6 eigenvalues this problem cannot be considered a toy problem. The criterion for the convergence is that the norm of the residual is below $5 \cdot 10^{-7}$. In the correction equation we use preconditioning with an incomplete LU factorization of the matrix $A_i - \sigma B_i - \tau C_i$ with a drop tolerance 10^{-3} for $i = 1, 2$.

The values in Table 6.3 are:

- *iter*: the number of outer iterations,
- *time*: time in seconds,
- *in 50, in 100*: the number of the computed eigenvalues that are among the 50 and 100 closest eigenvalues to the target, respectively.

TABLE 6.3
Comparison of the standard and harmonic Ritz extraction for the JD type method for a right definite two-parameter eigenvalue problem.

	Ritz extraction				Harmonic Ritz extraction			
GMRES	iter	time	in 50	in 100	iter	time	in 50	in 100
4	2270	373	50	96	917	234	50	96
8	913	191	50	98	379	118	50	97
16	370	99	50	93	179	72	46	80
32	278	107	50	98	140	76	50	88
64	305	129	50	92	131	86	49	85

The results in Table 6.3 show that the harmonic Ritz extraction is faster and only slightly less accurate than the Ritz extraction. Solving the projected problems is more expensive for the harmonic Ritz extraction because they are not right definite, which is in contrast to the projected problems in the standard Ritz extraction. However, as the harmonic Ritz extraction requires far fewer outer iterations, it computes the eigenvalues faster than the standard extraction. On the other hand, the standard Ritz extraction returns a bit more accurate results, for instance, in all cases we get all 50 closest eigenvalues to the target. Both methods are suitable for this right definite two-parameter eigenvalue problem and based on the results we give the harmonic Ritz extraction a slight preference over the Ritz extraction.

As one can see in the next example, the difference between the Ritz and the harmonic extraction may be much more in favor of the harmonic Ritz extraction when we consider a non definite two-parameter eigenvalue problem.

EXPERIMENT 6.4. In this example we take a random non right definite two-parameter eigenvalue problem with matrices of size 1000×1000 from Example 6.2. We perform similar experiments as in Example 6.3, the only difference is that now we use the two-sided Ritz extraction as well. As discussed in [10], this is a natural approach when we have a non definite two-parameter eigenvalue problem.

We limit the computation to 2500 outer iterations or to 50 extracted eigenvalues. Neither the one-sided standard nor the two-sided standard Ritz extraction is able to compute the required number of eigenvalues in the prescribed number of outer iterations. This is not an issue for the harmonic Ritz extraction which is a clear winner in this example. The results are presented in Table 6.4.

Figure 6.1 shows the convergence graphs for the two-sided Ritz extraction (a) and the harmonic extraction (b) for the first 50 outer iterations, in both cases we take 2 GMRES steps in the inner iteration. One can see that the convergence is more erratic when we use the standard extraction and smoother (almost monotonous) if we use the harmonic extraction.

EXPERIMENT 6.5. Using the same non right definite problem as in Example 6.3 we test how many eigenvalues can we extract with a limited number of matrix-vector multiplications, i.e., we fix the product of inner and outer iterations. The limit is 3200.

The results in Table 6.5 show that for a low number of inner iterations we get

Table 6.4

Comparison of one-sided Ritz, two-sided Ritz, and harmonic extraction methods for the JD type method for a non right definite two-parameter eigenvalue problem.

	One-sided Ritz			Two-sided Ritz			Harmonic Ritz				
GMRES	eigs	in 10	in 30	eigs	in 10	in 30	iter	time	in 10	in 30	in 50
4	9	9	9	8	8	8	594	286	10	30	47
8	17	10	17	12	9	12	226	127	10	30	46
16	19	10	19	19	10	19	106	79	10	30	44
32	20	10	20	22	10	22	89	97	10	29	40
64	22	10	22	30	10	29	93	136	10	28	40

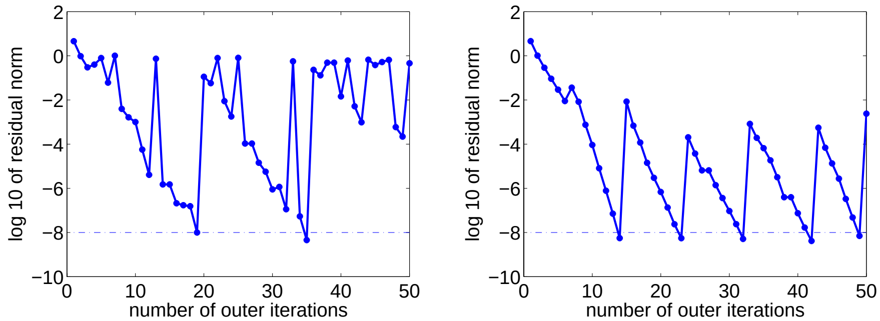

Fig. 6.1. *Comparison of convergence using the two-sided Ritz extraction (a) and the harmonic extraction (b)*

fewer eigenvalues and spend more time, but the possibility to compute an unwanted eigenvalue is smaller. If we use more inner iterations we get many unwanted eigenvalues, but spend less time. The optimal combination is to take a moderate number of inner iterations, in this example this would be between 16 and 32 inner steps.

Besides the matrix-vector multiplications, the most time consuming operation is to solve the projected low-dimensional two-parameter eigenvalue problem in each outer step. If the search spaces are of size k, then we need $\mathcal{O}(k^6)$ flops to solve these projected problems. Since this is relatively expensive compared to the work for the matrix-vector multiplications, it is a good idea to use several GMRES steps (but not too many to avoid convergence to an unwanted eigenvalue) to try to reduce the number of outer iterations.

Experiment 6.6. In this example we take a non right definite two-parameter eigenvalue problem where A_i, B_i, and C_i are random complex banded matrices of size 500×500 generated by the Matlab commands

```
M=sparse(triu(tril(randn(500)+i*randn(500),5),-5));
```

where M is respectively equal to A_1, B_1, C_1, A_2, B_2, and C_2.

We look for the eigenvalues closest to the origin and for a preconditioner we take $M_i = A_i$ for $i = 1, 2$. We limit the computation to 1000 outer iteration or 15 extracted eigenvalues.

From the results in Table 6.6 one can see that the harmonic extraction clearly extracts more eigenvalues than the Ritz extraction. Both the one-sided and the two-sided standard Ritz extraction fail to compute 15 eigenvalues in 1000 outer iterations; therefore, the number of iterations is displayed only for the harmonic extraction, the

Table 6.5

Eigenvalues obtained using the harmonic extraction and 3200 inner iterations.

GMRES	all	in 25	in 50	time (sec)
4	59	25	50	381
8	68	25	50	231
16	75	25	49	154
32	52	24	40	97
64	24	18	21	63

Table 6.6

Comparison of Ritz and harmonic extraction for a non right definite two-parameter eigenvalue problem.

	One-sided Ritz extraction			Two-sided Ritz extraction			Harmonic Ritz extraction			
GMRES	eigs	time	in 5	eigs	time	in 5	iter	time	in 5	in 10
4	0	417	0	3	552	3	657	282	5	10
8	0	475	0	3	697	3	645	331	3	8
16	1	648	1	4	1016	4	292	195	5	8
32	0	745	0	3	1552	3	236	246	4	7
64	0	1065	0	3	1799	4	165	269	2	6

number of iterations for both the one-sided and two-sided Ritz extraction is 1000. The one-sided Ritz extraction is particularly poor in view of the fact that it manages to compute only one eigenvalue in five examples. The two-sided Ritz extraction computes more eigenvalues, but falls considerably short of the required 15. For this example the harmonic extraction is clearly the suggested method.

Experiment 6.7. We take the two-parameter eigenvalue problem from Example 8.4 in [10]. The problem is non right definite and the matrices are of size 1000×1000. We used this problem in [10] to demonstrate that the two-sided Ritz extraction may give better results than the one-sided Ritz extraction. We limit the computation to 500 outer iteration or 30 extracted eigenvalues. For the target we take the arithmetic mean of the eigenvalues.

The results in Table 6.7 show that the harmonic extraction is a substantial improvement to the standard Ritz extraction. As in the previous example, both the one-sided and two-sided Ritz extraction fail to compute the required number of eigenvalues in the available number of outer iterations. The number of iterations is displayed in Table 6.7 only for the harmonic extraction, the number of iterations for one-sided and two-sided Ritz extraction is 500.

Table 6.7

Comparison of Ritz and harmonic extraction for a non right definite two-parameter eigenvalue problem.

	One-sided Ritz extraction			Two-sided Ritz extraction			Harmonic Ritz extraction			
GMRES	eigs	time	in 10	eigs	time	in 10	iter	time	in 10	in 20
10	11	659	4	15	1022	6	116	280	10	17
20	11	942	4	21	1533	9	95	305	10	18
30	10	1143	3	20	2062	8	110	399	10	19

7. Conclusions. It was observed in [11] that the multiparameter eigenvalue problem is a challenge, especially with respect to the task of finding interior eigenvalues. The concept of a harmonic extraction technique for the MEP was left as an open question, and dealt with in this paper. We have seen that, although there seems to be no straightforward generalization of a spectral transformation (2.1) for the MEP, the harmonic approach can be generalized to the MEP, with a corresponding elegant and intuitive generalization of Saad's theorem. We also gave a generalization of the refined extraction, which seems to be less suited for this problem.

Based on the theory and the numerical results, our recommendations for the numerical computation of interior eigenvalues of a MEP are the following. For right definite MEPs we use the one-sided Jacobi–Davidson method [11] for the subspace expansion. The harmonic extraction presented in this paper is at least very competitive with the standard extraction described in [11].

For non right definite problems, the one-sided approach [11] combined with the harmonic extraction, described in this paper, is both faster and more accurate than the two-sided approach proposed in [10], which on its turn is more accurate than the one-sided approach with standard extraction [11].

It is important to realize that for the MEP solving the projected problems is itself already a computationally non-negligible task in view of the $\mathcal{O}(k^6)$ costs. Therefore it is advisable to invest in solving the correction equations relatively accurately to minimize these costs. Hence, although just a few steps of GMRES may give more accurate results because we compute fewer unwanted eigenpairs, this may be much more time demanding.

For exterior eigenvalues we opt for the standard Ritz extraction, combined with a one-sided approach for right definite MEPs [11], or a two-sided approach for non right definite MEPs [10].

Finally, we remark that a two-sided harmonic approach is possible, but much less effective since the correspondence between right and left approximate eigenvectors is lost; we will not go into further details. The methods in this paper can be generalized to MEPs with more than two parameters in a straightforward way.

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