

## A note on continuous bisimulation and stability

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# A note on Continuous Bisimulation and Stability

P.J.L. Cuijpers

## 1 Introduction

Recently, the notion of *bisimilarity* as an equivalence on systems behavior, has been transferred from computer science to control science [5]. It has been used, for example, to reduce the complexity of linear differential equations, while preserving reachability notions. In [3] it was argued that, if we want to use bisimulation for the preservation of control science notions, we need continuity conditions in addition to the usual preservation of transitions. This idea was independently explored in [6, 4] where a modal logic was extended with topological operators, to be able to reason about robustness of a control strategy for embedded systems. In [4], it was shown that so-called upper semi-continuous bisimulation relations preserve this logic, and hence the stated robustness conditions.

The work on preservation of control properties, however, has not ended with that. In this report, we will show that upper semi-continuous bisimulation is not sufficient to preserve the well-known idea of *stability of a state* [1]. As a direct, but rather unexpected consequence, we conclude that stability is therefore not expressible in the logic of [4]. Even more surprising, it turns out that even bi-continuous bisimulation (i.e. a symmetric simulation relation that is both upper and lower semicontinuous) only preserves stability in one direction (the image of a stable set is again stable). To preserve stability in the other direction (if the image of a set is stable, then the set itself is stable) we need functionality conditions in addition.

We start this report by a short discussion of the mathematical preliminaries used to model a systems behavior in Section 2, and we define the notions of equivalence that we use to reason about systems. Then, in Section 3, we discuss our definition of the notion of stability, and we give theorems, proofs and counterexamples to show how this property is preserved by several notions of equivalence. Finally, in section 4, we discuss the consequences of our findings for hybrid systems theory.

## 2 Mathematical Preliminaries

We start out by giving a number of definitions that fix our way of thinking about systems behaviour.

The first definition, is that of a *topology* [7]. A topology is a mathematical structure that describes when two elements of a set are in each others vicinity.

**Definition 1 (Topological space)** A tuple,  $\langle X, \mathcal{T} \rangle$  is called a topological space, and  $\mathcal{T} \subseteq 2^X$  of subsets of  $X$  is a topology on  $X$ , iff

- $X, \emptyset \in \mathcal{T}$ ;
- $U, V \in \mathcal{T}$  implies  $U \cap V \in \mathcal{T}$ ;
- $U_i \in \mathcal{T}$  for all  $i \in I$  implies  $\bigcup_{i \in I} U_i \in \mathcal{T}$ .

The elements  $U \in \mathcal{T}$  are called open sets, their complements  $\overline{U}$  are closed sets.

One popular way to define a topology, is for example by using a *distance function* or *metric*  $d : X \times X \rightarrow \mathbb{R}$ , satisfying the well known triangular equations:  $d(x, x) = 0$  and  $d(x, y) + d(y, z) \geq d(x, z)$ . The topology is then given by all arbitrary unions of sets of the form  $\{x \in X \mid d(x, y) < \epsilon\}$ , with  $y \in X$  and  $\epsilon > 0$ .

The second definition, is that of a *labelled transition system*. A labelled transition system is a mathematical structure that describes how a system behaves when going from one state to the other.

**Definition 2 (Labelled transition system)** A labelled transition system is a tuple  $\langle X, A, \rightarrow \rangle$  where

- $X$  is called the state space
- $A$  is the set of actions, and
- $\rightarrow \subseteq X \times A \times X$  is a transition relation.

We use  $\langle x \rangle \xrightarrow{a} \langle x' \rangle$  to denote  $(x, a, x') \in \rightarrow$ .

In this report, we will always assume that we have a labelled transition system that is also equipped with a topology on the state space. This allows us to reason about small variations in behavior that are not captured in the transition relation.

**Definition 3 (Topological transition system)** *A topological transition system is a tuple  $\langle X, A, \rightarrow, \mathcal{T} \rangle$  where*

- $X$  is called the state space
- $A$  is the set of actions,
- $\rightarrow \subseteq X \times A \times X$  is a transition relation, and.
- $\mathcal{T} \subseteq 2^X$  is a topology on  $X$ .

When analysing a systems behaviour, is often useful to transform a it into a simpler system, while preserving the properties that are the focus of our analysis. In this report, we will assume without loss off generality, that the two systems we want to compare are both part of a common topological transition system, and we will simply make a comparison between the states of this topological transition system. The comparison is carried out by a relation  $\mathcal{R} \subseteq X \times X$ , with the following notational conventions.

**Definition 4 (Relations)** *Given a relation  $\mathcal{R} \subseteq X \times X$ , we use*

- $x\mathcal{R}y$  to denote  $(x, y) \in \mathcal{R}$ ;
- $x\overline{\mathcal{R}}y$  to denote  $(x, y) \notin \mathcal{R}$ ;
- $y\mathcal{R}^{-1}x$  to denote  $(x, y) \in \mathcal{R}$ ;
- $\mathcal{R}^l(S)$  to denote the lower image  $\{y \in X \mid \exists_{x\mathcal{R}y} x \in S\}$ ;
- $\mathcal{R}^u(S)$  to denote the upper image  $\{y \in X \mid \forall_{x\mathcal{R}y} x \in S\}$ ;
- $\mathcal{R}^{-l}(S)$  to denote the lower inverse  $\{x \in X \mid \exists_{x\mathcal{R}y} y \in S\}$ ;
- $\mathcal{R}^{-u}(S)$  to denote the upper inverse  $\{x \in X \mid \forall_{x\mathcal{R}y} y \in S\}$ ;

The first transformation that is used to compare to systems, is that of a *simulation relation*. It is defined for labelled transition systems, but is of course easily extended to topological transition systems. A state  $x$  is said to simulate a state  $y$ , if  $x$  can mimick all the transitions of  $y$ , and if the state  $x'$  that is reached by mimicking a transition to  $y'$  also simulates  $y'$ .

**Definition 5 (Simulation)** Let  $\langle X, A, \rightarrow \rangle$  and be a labelled transition system. A relation  $\mathcal{R} \subseteq X \times X$  on the state space of this system is a simulation iff for all  $x, y \in X$  we find that  $x\mathcal{R}y$  and  $\langle x \rangle \xrightarrow{a} \langle x' \rangle$  implies there exists  $y'$  such that  $\langle y \rangle \xrightarrow{a} \langle y' \rangle$  and  $x'\mathcal{R}y'$ .

If both  $\mathcal{R}$  and  $\mathcal{R}^{-1}$  are simulation relations, then  $\mathcal{R}$  is called a *bisimulation relation*.

The notion of simulation preserves transition relations. The notion of *continuity* is usually used to preserve a topology. However, preserving open sets by relations can be done in two ways, as was first observed by Kwiatkowsky [2]. Either, we preserve openness of lower inverse images, this is called a *lower-semicontinuous* relation, or we preserve openness of upper inverse images, this is called an *upper-semicontinuous* relation. If we preserve both, we speak of a *continuous* relation.

**Definition 6 (Continuity)** Let  $\langle X, \mathcal{T}_X \rangle$  be a topological space. A relation  $\mathcal{R} \subseteq X \times X$  is:

- lower semi-continuous iff for every  $U \in \mathcal{T}$  we have  $\mathcal{R}^{-l}(U) \in \mathcal{T}$ .
- upper semi-continuous iff for every  $U \in \mathcal{T}$  we have  $\mathcal{R}^{-u}(U) \in \mathcal{T}$ .
- continuous iff we have both.

As a peculiar note, it was shown in [4] that upper semi-continuity has striking similarities with the notion of simulation when phrased in a slightly different way.

Finally, as will turn out at the end of the next section, it is sometimes important to preserve the size of sets. This is done by demanding functionality.

**Definition 7 (Functionality)** A relation  $\mathcal{R} \subseteq X \times X$  is functional iff for each  $x, y, z \in X$  we have that  $x\mathcal{R}y$  and  $x\mathcal{R}z$  implies  $y = z$ .

### 3 Stability

Stability is a fundamental notion from control science, stating that small deviations will not trigger large variations in behaviour. Stability of a set of states signifies that any behaviour that begins in an arbitrarily close neighbourhood of this set, will remain close to this set, regardless of the transitions that are being taken. Formally, the definition of stability requires a way to reason about consecutive transitions in a topological transition system. To this end, we introduce the notion of concatenation closure.

**Definition 8 (Concatenation closure)** *Given a topological transition system  $\langle X, A, \rightarrow, \mathcal{T} \rangle$ , the concatenation closure  $\rightarrow_{\subseteq} X \times X$  of the transition relation  $\rightarrow$  is the smallest relation such that:*

- $\langle x \rangle \xrightarrow{a} \langle y \rangle$  implies  $x \rightarrow y$ ;
- $x \rightarrow y$  and  $y \rightarrow z$  implies  $x \rightarrow z$ .

Note, that  $\rightarrow(A)$  denotes the set of all states that are reachable from  $A$ . Furthermore, it is a standard theorem from computer science that if  $\mathcal{R}$  is a simulation relation, then  $\rightarrow^l(\mathcal{R}^{-l}(A)) \subseteq \mathcal{R}^{-l}(\rightarrow^l(A))$  for any  $A \subseteq X$ .

Using the definition of concatenation closure, stability is defined as follows.

**Definition 9 (Stable set)** *Given a topological transition system  $\langle X, A, \rightarrow, \mathcal{T} \rangle$ , a set  $S \subseteq X$  of states is stable if for every open set  $U \in \mathcal{T}$  with  $S \subseteq U$ , there exists an open set  $V \in \mathcal{T}$  such that  $S \subseteq V$  and  $\rightarrow^l(V) \subseteq U$ .*

According to [1], stability is preserved under so-called *conjugacies*, i.e. under symmetric continuous simulation functions.

**Theorem 1** *Let  $\langle X, A, \rightarrow, \mathcal{T} \rangle$  be a topological transition system, and let  $\mathcal{R} \subseteq X \times X$  be a symmetric continuous functional simulation relation. Then, a set  $S \subseteq X$  is stable if and only if its lower image  $\mathcal{R}^l(S)$  is stable.*

**Proof** This is a standard result from control science. See, for example, [1]. □

When we drop functionality, symmetry and part of the continuity, we still preserve stability of sets in one way.

**Theorem 2** *Let  $\langle X, A, \rightarrow, \mathcal{T} \rangle$  be a topological transition system, and let  $\mathcal{R} \subseteq X \times X$  be a relation such that  $\mathcal{R}$  is upper semi-continuous,  $\mathcal{R}^{-1}$  is lower semi-continuous and  $\mathcal{R}^{-1}$  is a simulation. Then, if a set  $S \subseteq X$  is stable, its lower image  $\mathcal{R}^l(S)$  is also stable.*

**Proof** Adapted, slightly, from [3].

Assume that the set  $S$  is stable. Now, take an open set  $U \in \mathcal{T}$  such that  $\mathcal{R}^l(S) \subseteq U$ . It is a standard property of upper inverses that  $S \subseteq \mathcal{R}^{-u}(\mathcal{R}^l(S)) \subseteq \mathcal{R}^{-u}(U)$  (see [2]). By upper semi-continuity, the set  $\mathcal{R}^{-u}(U)$  is therefore open around  $S$ . Because  $S$  is stable, we can construct an open set  $V$  around  $S$  such that

$$S \subseteq V \subseteq \rightarrow^l(V) \subseteq \mathcal{R}^{-u}(U).$$

The fact that  $\mathcal{R}^{-1}$  is a simulation relation then gives us

$$\mathcal{R}^l(S) \subseteq \mathcal{R}^l(V) \subseteq \rightarrow(\mathcal{R}^l(V)) \subseteq \mathcal{R}^l(\rightarrow(V)) \subseteq \mathcal{R}^l(\mathcal{R}^{-u}(U)).$$

Finally, the set  $\mathcal{R}^l(V)$  is open because  $\mathcal{R}^{-1}$  is lower semi-continuous, and another standard property of upper inverses gives us  $\mathcal{R}^l(\mathcal{R}^{-u}(U)) \subseteq U$  (see, again, [2]). This concludes the proof.  $\square$

However, and this is most surprising, if we want to preserve stability in the other direction as well, i.e. if we really want stability of  $S$  *if and only if* we have stability of  $\mathcal{R}(S)$ , it does not suffice to have a bi-continuous bisimulation relation. This is illustrated by the following example.

**Example** Consider the topological transition system  $\langle X, A, \rightarrow, \mathcal{T} \rangle$ , with  $X = [0, 1] \cup \{\perp\}$ ,  $A = \{a\}$ , and  $\mathcal{T}$  the sum of the usual topology on  $[0, 1]$  and the singleton topology on  $\{\perp\}$ . The transition relation  $\rightarrow$  is defined by  $\langle x \rangle \xrightarrow{a} \langle \sqrt{x} \rangle$ , for all  $x \in \mathbb{R}$  and  $\langle \perp \rangle \xrightarrow{a} \langle \perp \rangle$ .

Clearly, the set  $\{0\} \subseteq \mathbb{R}$  is not stable in this transition system, because any deviation from 0 will trigger a sequence of transitions that converges to 1. The set  $\{\perp\}$  is stable, because there are no small deviations from this singleton.

Finally, observe that the sets  $[0, 1]$  and  $\{\perp\}$  are open, closed and compact in the chosen topology, which means that the relation  $\mathcal{R} = [0, 1] \times \{\perp\}$ , is an upper- and lower-semicontinuous simulation relation, and so is  $\mathcal{R}^{-1}$  (i.e. it is a bicontinuous bisimulation relation). The counterexample now lies in the observation that  $\mathcal{R}^l(\{0\}) = \{\perp\}$  is stable, while  $\{0\}$  is not.

From this, we may already conclude that stability cannot be expressed in the logic of [4], because that logic is preserved by relations on which the only requirement is that they are upper semi-continuous simulation both ways. To make the impossibility even clearer, a slight adaptation of the previous example shows that those relations are not even sufficient to preserve stability in one direction.

**Example** Consider, the topological transition system of the previous example again, but this time consider the relation  $\mathcal{S} = \{(0, \perp), (\perp, 0)\}$ . Observe, that it is an upper-semicontinuous simulation relation, and so is  $\mathcal{S}^{-1}$ . But, despite upper-semicontinuity,  $\mathcal{R}^l(\{0\}) = \mathcal{R}^{-l}(\{0\}) = \{\perp\}$  is stable and  $\{0\} = \mathcal{R}^l(\{\perp\}) = \mathcal{R}^{-l}(\{\perp\})$  is not.

## 4 Conclusions

We have shown, by two simple counterexamples, that bi-upper semicontinuous bisimulation is not sufficient to preserve stability of sets. Additional lower semicontinuity constraints help in preserving it from a set to its lower image, but functionality constraints are needed to preserve stability both ways. We suspect, that this phenomenon is caused by the fact that stability is not preserved when taking subsets. I.e. if  $S$  is a stable set, then a subset  $S' \subseteq S$  is not necessarily stable. The precise reason, however, needs more research.

An important consequence of our research is that stability cannot be expressed in the logic of [4], because this logic is preserved under bi-upper semicontinuous bisimulation. This observation justifies further research from a control / hybrid systems point of view, in the direction of expressibility of topological and intuitionistic modal logics, and their relationship to topological labelled transition systems.

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