

On the minimal property of the Fourier projection

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ON THE MINIMAL PROPERTY OF THE FOURIER PROJECTION

BY E. W. CHENEY,¹ C. R. HOBBY,¹ P. D. MORRIS,²
F. SCHURER³ AND D. E. WULBERT²

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Let C be the space of real 2π -periodic continuous functions normed with the supremum norm. Let P_n denote the subspace of trigonometric polynomials of degree $\leq n$. It is known [1] that the Fourier projection F of C onto P_n is *minimal*; i.e., if A is a projection of C onto P_n then $\|F\| \leq \|A\|$. We prove that F is the only minimal projection of C onto P_n . The proof is constructed by verifying the assertions listed below. Details will appear elsewhere.

ASSERTION. *If there exists a minimal projection different from F , then there exist minimal projections L and H , different from F such that $\frac{1}{2}L + \frac{1}{2}H = F$.*

The proof of this assertion utilizes Berman's equation,

$$F = \frac{1}{2\pi} \int_{-\pi}^{\pi} T_{-\lambda} A T_{\lambda} d\lambda,$$

which is valid for any projection A of C onto P_n . Here T_{λ} denotes the shift operator $(T_{\lambda}f)(x) = f(x + \lambda)$.

ASSERTION. *There is a function $K(x, t)$ of two variables such that*

- (i) $K(x, \cdot) \in L^1$ for each fixed x ,
- (ii) $K(\cdot, t) \in P_n$ for each fixed t , and
- (iii) $(Lf)(x) = \int f(t)K(x, t)dt$.

This is proved by extending A to its second adjoint, and applying the Radon-Nikodym theorem to the functionals $\phi(f) = (A^{**}f)(x)$.

Let D_n denote the Dirichlet kernel. The next assertion follows from an examination of the roots of K where K is considered as a function of x .

ASSERTION. *There is a function $g \in L^1$ such that $0 \leq g \leq 2$, and $K(x, t) = g(t)D_n(x - t)$.*

ASSERTION. (i) $(1 - g) \perp P_{2n}$ and (ii) $(1 - g) * |D_n| = 0$ where $*$ denotes convolution.

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Part (i) is immediate from the fact that L is a projection. The minimality of L is needed to prove part (ii).

Let $d(n, k) = \int |D_n(t)| e^{ikt} dt$.

ASSERTION. $d(n, k) \neq 0$ for $|k| > 2n$.

This result, when combined with the preceding assertion, will prove the theorem. The remainder of this paper pertains to proving that $d(n, k) \neq 0$.

ASSERTION.

$$d(n, k) = \frac{1}{\pi} \sum_{j=k-n}^{k+n} \frac{1}{j} \frac{\beta^j - 1}{\beta^j + 1}$$

where $\beta = e^{2\pi i/2n+1}$.

ASSERTION. If $d(n, k) = 0$ then

$$\sum_{j=k-n}^{k+n} \frac{1}{j} \sum_{t=1}^{2n} (-\beta^j)^t = 0.$$

Thus if $d(n, k) = 0$ we have a polynomial of degree $2n$ with rational coefficients which has β as a root. We next derive a relation which must be satisfied by the coefficients of such a polynomial. The final step is to show that in our case this relation is not even satisfied modulo a convenient prime. The existence of the convenient prime is a consequence of the following extension of the Sylvester-Schur theorem.

ASSERTION. If n and k are integers satisfying $6 \leq k \leq n/2$, then at least two integers between $n-k+1$ and n possess prime factors exceeding k .

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UNIVERSITY OF TEXAS, AUSTIN, TEXAS 78712

UNIVERSITY OF WASHINGTON, SEATTLE, WASHINGTON 98105

PENNSYLVANIA STATE UNIVERSITY, UNIVERSITY PARK, PENNSYLVANIA 16802

TECHNOLOGICAL UNIVERSITY, EINDHOVEN, NETHERLANDS

UNIVERSITY OF WASHINGTON, SEATTLE, WASHINGTON 98105