On the minimal property of the Fourier projection

ON THE MINIMAL PROPERTY OF THE FOURIER PROJECTION

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Let C be the space of real 2π-periodic continuous functions normed with the supremum norm. Let P_n denote the subspace of trigonometric polynomials of degree $\leqq n$. It is known [1] that the Fourier projection F of C onto P_n is *minimal*; i.e., if A is a projection of C onto P_n then $\|F\| \leqq \|A\|$. We prove that F is the only minimal projection of C onto P_n. The proof is constructed by verifying the assertions listed below. Details will appear elsewhere.

ASSERTION. *If there exists a minimal projection different from F, then there exist minimal projections L and H, different from F such that* $\frac{1}{2}L+\frac{1}{2}H=F$.

The proof of this assertion utilizes Berman's equation,

$$F = \frac{1}{2\pi}\int_{-\pi}^{\pi} T_{-\lambda} A T_\lambda d\lambda,$$

which is valid for any projection A of C onto P_n. Here T_λ denotes the shift operator $(T_\lambda f)(x) = f(x+\lambda)$.

ASSERTION. *There is a function $K(x, t)$ of two variables such that*

(i) $K(x, \cdot) \in L^1$ *for each fixed* x,

(ii) $K(\cdot, t) \in P_n$ *for each fixed* t, *and*

(iii) $(Lf)(x) = \int f(t) K(x, t) dt$.

This is proved by extending A to its second adjoint, and applying the Radon-Nikodym theorem to the functionals $\phi(f) = (A^{**}f)(x)$.

Let D_n denote the Dirichlet kernel. The next assertion follows from an examination of the roots of K where K is considered as a function of x.

ASSERTION. *There is a function $g \in L^1$ such that $0 \leqq g \leqq 2$, and* $K(x, t) = g(t) D_n(x-t)$.

ASSERTION. (i) $(1-g) \perp P_{2n}$ *and* (ii) $(1-g) * |D_n| = 0$ *where* $*$ *denotes convolution.*

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Part (i) is immediate from the fact that L is a projection. The minimality of L is needed to prove part (ii).

Let $d(n, k) = \int |D_n(t)| e^{ikt} dt$.

ASSERTION. *$d(n, k) \neq 0$ for $|k| > 2n$.*

This result, when combined with the preceding assertion, will prove the theorem. The remainder of this paper pertains to proving that $d(n, k) \neq 0$.

ASSERTION.

$$d(n, k) = \frac{1}{\pi} \sum_{j=k-n}^{k+n} \frac{1}{j} \frac{\beta^j - 1}{\beta^j + 1}$$

where $\beta = e^{2\pi i/2n+1}$.

ASSERTION. *If $d(n, k) = 0$ then*

$$\sum_{j=k-n}^{k+n} \frac{1}{j} \sum_{t=1}^{2n} (-\beta^j)^t = 0.$$

Thus if $d(n, k) = 0$ we have a polynomial of degree $2n$ with rational coefficients which has β as a root. We next derive a relation which must be satisfied by the coefficients of such a polynomial. The final step is to show that in our case this relation is not even satisfied modulo a convenient prime. The existence of the convenient prime is a consequence of the following extension of the Sylvester-Schur theorem.

ASSERTION. *If n and k are integers satisfying $6 \leq k \leq n/2$, then at least two integers between $n-k+1$ and n possess prime factors exceeding k.*

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