# Existence of general equilibria in economies with natural exhaustible resources and an infinite horizon

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EXISTENCE OF GENERAL EQUILIBRIA IN
ECONOMIES WITH NATURAL EXHAUSTIBLE
RESOURCES AND AN INFINITE HORIZON.

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*ABSTRACT*

This paper establishes the existence of a general equilibrium for economies with natural exhaustible resources and an infinite horizon. It is argued that the traditional methods for proving existence in economies with an infinite dimensional commodity space cannot be invoked here and an alternative proof is provided.

## 1. Introduction

The present paper deals with the existence of a general competitive equilibrium in an infinite horizon discrete time model. The economic context is given by exploitation and use of exhaustible natural resources. It turns out that this framework gives rise to several problems if one tries to apply standard results from the vast existing literature on economies with an infinite dimensional commodity space. In order to clarify this it is convenient to sketch briefly the type of models we have in mind.

The economics of exhaustible natural resources (see eg. Dasgupta and Heal (1979) and Withagen (1985) for surveys) addresses a large variety of questions such as the optimal rate of exploitation under different technological constraints and the pricing of exhaustible resource commodities under different market structures. The latter type of questions is mostly considered in a partial equilibrium framework. Indeed the number of contributions dealing with general equilibrium is relatively small (see van Geldrop van Withagen (1988) for an exhaustive account). Most studies in this area assume the existence of a general equilibrium and aim at a characterization, or make rather specific assumptions implying that existence is more or less trivial. The model we present here is far more general than the existing ones. We consider an economy in which there are several resource stocks of different quality, meaning that different costs are to be incurred when exploitation takes places. The (homogeneous) raw material serves as an input in production processes undertaken by an arbitrary number of firms which also employ capital as an input to provide the consumers with a consumption commodity and the mining firms with capital needed to extract the resources. Initial endowments consist of the resource stocks and capital stock and are owned by the consumers, who also hold shares in the firms' profits.

Since we are looking at an economy with an infinite horizon the commodity space is of an infinite dimension. Our model bears some resemblance to Mitra's (1980), who gives an existence result working along "traditional" methods as developed by eg. Koopmans (1965), Gale (1967), Brock (1970) and McKenzie (1968). However he employs a homogeneous production function and a bounded utility function which appears to be crucial in his proof. Another approach to existence problems has been set out in a seminal paper by Bewley (1972), on which has been elaborated by many authors. We refer to Zame (1987) or Jones (1986) for an excellent survey. Unfortunately none of the results from that literature applies to our model. Let us consider the Bewley paper as a first reference. Two of his assumptions are not necessarily satisfied in our model. Since our production functions are allowed to be strictly concave, his adequacy assumption would mean here that the initial endowments are strictly positive in all periods of time. However the resource stocks are given at the beginning of time and no new stocks become available in future periods. So adequacy does not hold. The same is the case with the so-called boundedness assumption. Even with exhaustible resources one can imagine that production is unbounded, as the following example shows. $y = k^{2/3} z^{1/3}$ where $y$ is output, $k$ is capital input and $z$ is raw material input. It is

easily seen that with $\int_0^\infty z(t)\, dt \le s_0$ and $y = \dot{k}$, production can be expanded without bound.

Admittedly Bewley's conditions have been relaxed considerably in recent years but, as the discussion of many contributions in Zame (1987) shows, compactness (in some topology) remains required. It is also clear from recent work that when the production set is not a cone one needs conditions such as boundedness of the marginal efficiency of production (Zame (1987)), uniform properness (Mas-Colell (1985), Richard (1986)) or universal technical substitution (Zame (1987)). With respect to preferences and consumption one introduces conditions such as extreme desirability of a consumption vector smaller than the initial endowment vector which conditions are not fulfilled with for example Bernouilli-type instantaneous utility functions and endowments of the type we postulate.

As a consequence, and this is the root of all evil, there is a problem with the choice of the price space. It is usual, and reasonable, to choose price systems in the topological dual of the commodity space, because then, in a continuous way, all bundles have a finite value. However in an economy with production the definition of a general competitive equilibrium, denoted by $(\pi, \bar{x}, \bar{y})$, requires, besides feasibility, only that in all production sets $Y_j$ profits are maximized by the finite value $\pi \cdot \bar{y}$; and that in all consumption sets $X_i$ utility is maximized over the set $\pi \cdot x \le w_i$ where $w_i$ is the finite income of consumer $i$. In this interpretation one could, if necessary, allow for price systems assigning an infinite value to a large set of commodities and of course without any continuity property. Everybody is free to choose his or her own degree of appreciation of this way of reasoning. In order to make our point more explicit, consider the following example.

The commodity bundles are in $L_\infty(M, 2^M, \mu)$ where $M = \{t \in Z \mid t \ge -1\}$, $2^M$ is the set of all subsets of $M$ and $\mu$ is the counting measure.

There is one producer and the production set is

$$Y = \{y \in L_\infty \mid y(-1) \le 0, y(t) \ge 0, t \ge 0, \sum_{t=-1}^{\infty} y(t) \le 0\} \ .$$

There is one consumer and the consumption set is

$$X = \{x \in L_\infty \mid x(t) \ge 0, \text{all } t\} \ .$$

Preferences are induced by

$$U(x) := \sum_{t=-1}^{\infty} 2\beta^t x(t)^{\frac{1}{2}} \ , \ 0 < \beta < 1 \ .$$

The initial endowment is given by $\omega$ where $\omega(-1) = 1/\beta^2(1 - \beta^2)$, $\omega(t) = 0$, $t \ge 0$. Of course the only share equals unity.

This economy satisfies conditions (i)-(vi) of theorem 3 in Bewley (1972). Hence, if there is an equilibrium with a price system in the topological dual of $L_\infty$ (with normtopology), then there is an equilibrium, denoted by $(\pi, \bar{x}, \bar{y})$ where $\pi > 0$ is an element of $L_1$. So $\pi$ may be considered as a sequence $p(t) \big|_{t=-1}^{\infty}$ where $p(t) \ge 0$ for all $t$.

It is clear that $p(-1) > 0$, $\pi \cdot \bar{y} = 0$ and that the consumer maximizes $U(x)$ over the set

$$\sum_{t=-1}^{\infty} p(t) x(t) \le w \ ,$$

where $w = p(-1) \omega(-1) > 0$. Hence $p(t) > 0$ for all $t$, $x(t) > 0$ for all $t$, $y(t) > 0$ for all $t$ and, as a consequence, $p(t) = p > 0$ for all $t \ge 0$. So $\pi \notin L_1$ and the economy has no equilibrium. But

$$p(t) = 1, \ t \ge -1; \ x(t) = \beta^{2t}, \ t \ge -1; \ y(-1) = \frac{1}{\beta^2 - 1}; \ y(t) = x(t), \ t \ge 0$$

is an equilibrium in the wider sense. It is an easy exercise to verify that this equilibrium may be considered as the limit for $T \to \infty$ of the truncated (up to horizon $T$) economy.

Of course the problem lies in the fact that the initial endowment is not an interior point of $X$. As a consequence the adequacy assumption fails to hold here. Unfortunately modeling economies with exhaustible natural resources gives rise to such situations, as already mentioned. So, if one wishes to establish the existence of an equilibrium the wider class of price systems should be taken for granted.

The outline of the present paper is as follows. In section two we sketch an example of an economy with exhaustible resources. It catches the basic features of a far more general model to be discussed in section 5. We show the existence of a general equilibrium for each finite horizon economy of this type. In section 3 it is proved that equilibrium allocations and equilibrium prices are uniformly bounded and some other properties are derived. The results are then used in section 4 in establishing the existence of a general equilibrium for the infinite horizon economy. Section 5 elaborates on some straightforward generalizations, points at possible directions of future research and concludes. As a final remark it should be stressed here that we focus our attention on an existence theorem and not on the description of possibly interesting features or characteristics of an equilibrium.

## 2. The model

In this section first an outline is given of the finite horizon discrete time version of the model. Existence of a general equilibrium is shown. Next the infinite horizon model is introduced.

In the economy there are two consumers, four production sectors and four physically distinguishable commodities. The economy lasts for $T + 1$ periods, indexed by $t = 0, 1, \cdots, T$.

- Commodities.

There is a non-resource commodity, which serves as the only consumer commodity and as input and output of the production processes to be described. Furthermore there are two types of resource stocks from which a homogeneous raw material is extracted.

- Production.

The non-resource commodity is produced in the first two production sectors, using the extracted raw material and the non-resource commodity as inputs, according to technology $F^f$, $f = 1, 2$. Formally the production sets are described by

$$Y_f = \{(-s_f^1, -s_f^2, -k_f(0), -z_f(0), v_f(1) + k_f(0) - k_f(1), -z_f(1), \cdots$$

$$\cdots, v_f(T) + k_f(T-1) - k_f(T), -z_f(T)) \mid s_f^1 \geq 0, s_f^2 \geq 0 ,$$

$$k_f(t) \geq 0, z_f(t) \geq 0, v_f(t) \leq F^f(k_f(t-1), z_f(t-1)))\}, f = 1, 2 .$$

This is to be interpreted as follows. $s_f^1$ and $s_f^2$ are the inputs of the non-extracted resource commodities. The only motivation for introducing them is to have equal dimensions of all production sets. $k_f(t)$ and $z_f(t)$ are the non-resource input and resource input respectively in period $t$. It is assumed that production takes time so that output becomes available one period hence.

Output then consists of net output, denoted by $v_f(t + 1)$, and the non-resource input of the previous period $k_f(t)$. So it is assumed that capital does not depreciate. However, depreciation can be introduced at no cost. About $F^1$ and $F^2$ the following assumptions are made (we omit the index $f$ when there is no danger of confusion).

A1    $F$ is defined on $\mathbb{R}_+^2$, is continuous, concave and weakly monotonous increasing.

A2    $F(k, 0) = F(0, z) = 0$.

A3.1    $F(k, z) \leq q\, z$ for all $(k, z)$ for some given $q > 0$.

or    ·

A3.2    $\lim\limits_{k \to \infty} \dfrac{F(k,z)}{k} = 0$ for all $z$.

$\lim\limits_{k \to 0} \dfrac{F(k,z)}{k} = \infty$ for all $z > 0$.

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A1 is quite standard and needs no further comment. A2 incorporates the necessity of both inputs. A3.1 implies that the average product of the raw material is bounded. In case of A3.2 the reader may recognize some elements from neoclassical growth models.

Extraction requires the input of the non-resource commodity (capital), which is not lost during the production process. The technology is linear. It is assumed that at the outset of period zero each extraction sector has to acquire an amount of the non-extracted resource stock, sufficiently large to extract the entire planned production from it. This will be commented on below. The production sets are given by

$$Y_f = \{(-s_f^1, -s_f^2, -k_f(0), e_f(0), k_f(0) - k_f(1), e_f(1), \cdots ,$$

$$\cdots , k_f(T-1) - k_f(T), e_f(T)) \mid s_f^1 \geq 0, s_f^2 \geq 0, k_f(t) \geq 0, e_t(t) \geq 0 ,$$

$$\sum_{t=0}^{T} e_f(t) \leq s_f^{-2}, e_f(t) \leq k_f(t)/a_{f-2}\} , f = 3, 4 .$$

Here $a_1$ and $a_2$ are positive constants.

- Consumption.

There are two consumers, indexed by $h = 1, 2$. The consumption sets are

$$X_h = \{(0, 0, x_h(0), 0, x_h(1), 0, \cdots , x_h(T), 0) \mid x_h(t) \geq 0\} , h = 1, 2 .$$

So the consumers do not consume resource stocks nor raw materials. Endowments are

$$\omega_h = (\bar{s}_h^1, \bar{s}_h^2, \bar{k}_h, 0, 0, \cdots ), h = 1, 2 .$$

It will be assumed that each consumer holds positive initial capital

A4    $\bar{k}_1 > 0, \bar{k}_2 > 0$ .

The consumers hold shares in the production sectors, given by $\delta_{hf} \geq 0$ $(h = 1, 2; f = 1, 2, 3, 4)$ with $\sum_h \delta_{hf} = 1$.

The preference relations are described by

$$U_h( ) = \sum_{t=0}^{T} (\frac{1}{1+\rho_h})^t u_h(x_h(t)) , h = 1, 2 ,$$

where $\rho_h$ denotes the constant rate of time preference and $u_h$ is the instantaneous utility function. $u_h$ satisfies

A5    $u_h$ is continuous, concave, strictly increasing; $u'_h(0) = \infty$, $h = 1, 2$.

- Prices.

A price vector consists of prices for each period for each commodity

$$\pi = (\eta^1, \eta^2, p(0), \xi(0), p(1), \xi(1), \cdots , p(T), \xi(T)) .$$

**Remark**

One can think of this economy as a two-country world where each country possesses a stock of an exhaustible resource. The stocks of the countries have different extraction costs. Each country also has the technology to convert the raw material together with capital into a commodity that can be used for consumption and investment purposes. The factors of production are perfectly mobile. Each economy aims at the maximization of a utilitarian welfare functional. The current accounts are not required to equilibrate but of course total discounted expenditures should not exceed total discounted income. In the present interpretation of the model $\delta_{1f} = 1$ and $\delta_{2f} = 1$ for sector $f$ belonging to the first and the second economy respectively.

Let $T$ be fixed and let $y_f$ and $x_h$ denote a production vector for sector $f$ and a consumption vector for household $h$ respectively.

**Definition 2.1**

$(\hat{\pi}, \hat{x}_1, \hat{x}_2, \hat{y}_1, \hat{y}_2, \hat{y}_3, \hat{y}_4)$ with $\hat{\pi} > 0$ constitutes a general competitive equilibrium if

i)  $\hat{y}_f$ maximizes $\hat{\pi} y_f$ over $Y_f$ ($f = 1, 2, 3, 4$)

ii) $\hat{x}_h$ maximizes $U_h(x_h)$ subject to $\hat{\pi} \cdot x_h \leq \hat{\pi} \cdot \omega_h + \sum_{f=1}^{4} \delta_{hf} \hat{\pi} \cdot \hat{y}_f$ ($h = 1, 2$)

iii) $\sum_h \hat{x}_h \leq \sum_f \hat{y}_f + \sum_h \omega_h$

iv) $\hat{\pi} \cdot (\sum_h \hat{x}_h - \sum_f \hat{y}_f - \sum_h \omega_h) = 0$ .

It should be clear that we employ here the Arrow/Debreu dated commodity framework. In such a world the assumption that trade in resource stocks only takes place at the outset is obviously innocuous. It is even convenient since no explicit account has to be taken of savings and investments.

**Theorem 2.1**

For the economy described above there exists a general competitive equilibrium.

**Proof**

This theorem can be proved using fairly standard techniques (see e.g. Arrow and Hahn (1972)). The economy satisfies all the standard conditions sufficient for the existence of a general equilibrium except for $\omega_h \in \text{int} X_h$, $h = 1, 2$. But there obviously exists a compensated or quasi-equilibrium without this assumption. In order to prove the theorem then it is sufficient to show that imputed incomes of the consumers are positive.

This is a simple exercise. It makes use of the fact that in a compensated equilibrium not all prices are zero and that such an equilibrium is Pareto efficient.                                                                    ▯

## 3. Properties of the finite horizon general equilibria

In this section some properties of the general equilibria described in the previous section will be established. The focus is on the uniform boundedness of several equilibrium prices and quantities.

In order to economize on space it is convenient to work here with an aggregate production set, $Y^T$, which also incorporates some of the feasibility constraints. This set is defined as follows

$$y = (y(-2), y(-1), y(0), y(1), \cdots, y(T)) \in Y^T \Longleftrightarrow$$

$$y(-2) \le 0, y(-1) \le 0, y(0) \le 0, y(t) \ge 0 \quad (t = 1, \cdots, T) \ , \tag{3.1}$$

and there are $k_1(t-1) \ge 0, k_2(t-1) \ge 0, k(t-1) \ge 0, z_1(t-1) \ge 0, z_2(t-1) \ge 0, e_1(t-1) \ge 0$ and $e_2(t-1) \ge 0$ $(t = 1, 2, \cdots, T)$ such that

$$k(t) + y(t) \le k(t-1) + F^1(k_1(t-1), z_1(t-1)) + F^2(k_2(t-1), z_2(t-1)), \ t = 1, 2, \cdots, T \tag{3.2}$$

$$k(0) + y(0) \le 0 \tag{3.3}$$

$$k_1(t-1) + k_2(t-1) + a_1 e_1(t-1) + a_2 e_2(t-1) \le k(t-1), \ t = 1, 2, \cdots, T \tag{3.4}$$

$$z_1(t-1) + z_2(t-1) \le e_1(t-1) + e_2(t-1), \ t = 1, 2, \cdots, T \tag{3.5}$$

$$\sum_{t=1}^{T} e_i(t-1) + y(-i) \le 0 \ , \ i = 1, 2 \ . \tag{3.6}$$

A few comments are in order here.

- $k_3$ and $k_4$ from $Y_3$ and $Y_4$ of section 2 are replaced by $a_1 e_1$ and $a_2 e_2$ respectively since in general equilibrium they will obviously be equal.

- $z_i(T)$, $e_i(T)$ and $k_i(T)$ $(i = 1, 2)$ do not appear in $Y^T$ because production in period $T$ does not yield any profits. So they can be set equal to zero from the outset.

- $Y^T$ "contains" $Y_3$ and $Y_4$.

- $y(t)$ $(t = 1, 2, \cdots, T)$ are to be interpreted as net outputs coming available in period $t$ for consumption purposes. $y(-2)$ and $y(-1)$ are resource stock inputs and $y(0)$ is initial capital stock input.

- through the introduction of $Y^T$ the dimension of the commodity space is reduced. Therefore also the consumption sets, initial endowment vectors and the price vector should be redefined. This is straightforward to do.

- there may seem to occur a problem with respect to share holdings. It is clear however that the economy has an equilibrium for any distribution of shares. The share of consumer $h$ in total profits will be denoted by $\delta_h$.

- formally taking $T = \infty$, one obtains the infinite-horizon set $Y$, which is $\sigma(L_\infty, L_1)$ closed. Each $Y^T$ is a truncation of $Y$.

In the sequel the price vector will be denoted by $\pi^T = (p^T(-2), p^T(-1), p^T(0), \cdots, p^T(T))$. For a variable $x$, $x^T(t)$ is the value of that variable in period $t$ along a general equilibrium of an economy with horizon $T$.

In a general equilibrium consumer $h$ maximizes

$$\sum_{t=0}^{T} (\frac{1}{1+\rho_h})^t \, u_h(x_h(t)) \, , \tag{3.7}$$

subject to

$$\pi^T \cdot x_h \le \pi^T \cdot (\omega_h + \delta_h \, y) \, . \tag{3.8}$$

Since the $Y_f$'s are convex, individual profit maximization implies joint profit maximization:

$$y \text{ maximizes } \pi^T \cdot y \text{ over } Y^T \, . \tag{3.9}$$

**Lemma 3.1**

For all $T$ and for both $h$:

i) $\quad p^T(t) > 0, 0 \le t \le T$

ii) $\quad x_h^T(t) > 0, 0 \le t \le T$

iii) $\quad$ there exists a sequence $r^T(t) \ge 0$ such that

$$p^T(t+1) = p^T(t)/(1 + r^T(t+1)), \ 0 \le t \le T - 1$$

$$u'_h(x_h^T(t+1)) = \frac{1+\rho_h}{1+r^T(t+1)} \, u'_h(x_h^T(t)), \ 0 \le t \le T-1 \, .$$

**Proof**

i) $\quad u_h$ is strictly increasing. A zero price would cause infinite demand, which cannot be met.

ii) $\quad$ this follows from $A\,5: u'_h(0) = \infty$.

iii) $\quad$ in an equilibrium (3.8) holds with equality. Furthermore $x_h^T(t) > 0$. Hence there exist $\phi_1^T$ and $\phi_2^T$, which are constant for all $t \ge 0$, such that

$$(\frac{1}{1+\rho_h})^t \, u'_h(x_h^T(t)) = \phi_h^T p^T(t), \ 0 < t \le T \, .$$

Suppose next that there exist $T_1$ and $t_1 < T_1$ such that $p^T(t_1 + 1) > p^T(t_1)$. Then profits are not maximized. To see this just increase $k(t_1)$ at the cost of less net output $y(t_1)$. Additional profits can be made by selling the increment of $k$ in period $t_1 + 1$.

These two arguments together yield iii. $\qquad\qquad\qquad\qquad\qquad\qquad$ []

**Lemma 3.2**

For all $T$

$$y^T(0) < 0, y^T(t) > 0, 1 \le t \le T \, .$$

**Proof**

In equilibrium $\sum_h x_h^T(0) < \bar{k}_1 + \bar{k}_2$ because otherwise there would be no future consumption in view of the necessity of capital inputs in production. But

$$\Sigma\, x_h^T(0) = y(0) + \bar{k}_1 + \bar{k}_2 \ .$$

$y^T(t) > 0$ is obvious. □

## Lemma 3.3

There exists $B > 0$ such that for all $T$ and both $h$

$$\sup_t x_h^T(t) \le B \quad \text{and} \quad \sup_t y^T(t) \le B \ .$$

## Proof

Define $\bar{s} := \sum_j \sum_h \bar{s}_h^j$ .

i)  If $F_i(k, z) \le q\, z$ (A3.1) for $i = 1, 2$, where $q > 0$ is a given constant, then

$$\sum_{t=1}^{T} y^T(t) \le \bar{k}_1 + \bar{k}_2 + q\, \bar{s}$$

from (3.2) and the result follows.

ii)  If A3.1 does not hold, A3.2 holds.

There is a uniform upper bound for $y^T(1)$.

It follows from profit maximization that in equilibrium all $r^T(t) > 0$, so (3.2) and (3.4) hold with equality. Profit maximization also implies that there exists $\eta^T(t) \ge 0$, a Lagrangean parameter, such that

$$F_i(k_i^T(t-1), z_i^T(t-1)) - r^T(t)\, k_i(t-1) - \eta^T(t)\, z_i(t-1) \ge F_i(k, z) - r^T(t)\, k - \eta^T(t)\, z$$

for all $k \ge 0, z \ge 0$; $i = 1, 2$; $t = 1, 2, \cdots, T$. Now choose $\bar{r} := \min\,(\rho_1, \rho_2)$ and define $k_i^*$ by

$$F_i(k_i^*, \bar{s}) = \bar{r}\, k_i^* \ .$$

By A3.2 $k_i^*$ exists and we furthermore have

$$r^T(t) \ge \bar{r} \text{ implies } k_i(t-1) \le \max_i k_i^* =: k^* \ .$$

Since (3.4) holds with equality it follows that

$$r^T(t) \ge \bar{r} \text{ implies } y^T(t) \le 2k^* + (a_1 + a_2)\, \bar{s} + F_1(k^*, \bar{s}) + F_2(k^*, \bar{s}) \ .$$

When $r^T(t) < \bar{r}$ then it follows from lemma 3.1 part iii) and the concavity of $u_h$ that $x_h^T(t) < x_h^T(t-1), h = 1, 2$. Also

$$x_h^T(0) \le \bar{k}_1 + \bar{k}_2 \ .$$

So, for all $t$ with $r^T(t) \ge \bar{r}$, $y^T(t)$ and hence $x_h^T(t)$ are bounded (uniformly). And, for all $t$ with $r^T(t) < \bar{r}$, $x_h^T(t)$ is smaller than $x_h^T(t-1)$. This completes the proof. □

**Lemma 3.4**

There exist $\beta > 0$ and $V > 0$ such that for all $T$

i) $u'_h(x_h^T(0)) \le V$, $h = 1, 2$.

ii) $p^T(t) \le \beta p^T(0)$, $t < 0$.

iii) $\pi^T \cdot y^T \le V p^T(0)$.

**Proof**

Define

$$H := \sum_{t=0}^{\infty} (\frac{1}{1+\rho_1})^t u_1(B) + \sum_{t=0}^{\infty} (\frac{1}{1+\rho_2})^t u_2(B) \ ,$$

where $B$ is the uniform upper bound of $y^T(t)$.

$$H \ge \sum_{t=0}^{T} (\frac{1}{1+\rho_1})^t u_1(x_1^T(t)) + \sum_{t=0}^{T} (\frac{1}{1+\rho_2})^t u_2(x_2^T(t))$$

$$\ge \sum_{t=0}^{T} (\frac{1}{1+\rho_1})^t u'_1(x_1^T(t)) \, x_1^T(t) + \sum_{t=0}^{T} (\frac{1}{1+\rho_2})^t u'_2(x_2^T(t)) \, x_2^T(t)$$

$$= \phi_1^T \sum_{t=0}^{T} \pi^T(t) \, x_1^T(t) + \phi_2^T \sum_{t=0}^{T} \pi^T(t) \, x_2^T(t)$$

$$= \phi_1^T \, (\pi^T \cdot \omega_1 + \delta_1 \pi^T \cdot y^T) + \phi_2^T (\pi^T \cdot \omega_2 + \delta_2 \pi^T \cdot y^T) \ ,$$

where use has been made of the concavity of $u_h$, the necessary conditions for utility maximization and (3.8).

Since $\phi_h^T = \dfrac{u'_h(x_h^T(0))}{p^T(0)}$ we find

$$H \ge u'_1(x_1^T(0)) \, (\frac{\pi^T \cdot \omega_1}{p^T(0)} + \delta_1 \frac{\pi^T \cdot y^T}{p^T(0)}) + u'_2(x_2^T(0)) \, (\frac{\pi^T \cdot \omega_2}{p^T(0)} + \delta_2 \frac{\pi^T \cdot y^T}{p^T(0)})$$

$$\ge \omega_1(0) \, u'_1(x_1^T(0)) + \omega_2(0) \, u'_2(x_2^T(0)) \ .$$

Since $\omega_h(0) = \bar{k}_h > 0$ part $i$ of the lemma follows. Parts ii) and iii) are now immediate. □

## 4. Existence of an infinite horizon general equilibrium

Let $(\pi^T, x_1^T, x_2^T, y^T)$ be an equilibrium of the economy with horizon $T$. These vectors can be written as

$$x_h^T = (0, 0, x_h^T(0), \cdots, x_h^T(T), 0, 0, \cdots) \, , h = 1, 2$$

$$y^T = (y^T(-2), y^T(-1), y^T(0), y^T(1), \cdots, y^T(T), 0, 0, \cdots)$$

$$\pi^T = (p^T(-2), p^T(-1), p^T(0), p^T(1), \cdots, p^T(T), 0, 0, \cdots) \, .$$

Production and consumption sets for finite $T$ can easily be extended so as to have zeros after $T + 3$ coordinates.

We normalize prices by setting $p^T(0) := 1$. This is motivated by the considerations given when we were discussing the example of the introduction.

In view of Alaoglu's theorem, lemma 3.3 and lemma 3.4 ii there are

$$x_h = (0, 0, x_h(0), x_h(1), \cdots) \in X_h, h = 1, 2 \, ,$$

$$y = (y(-2), y(-1), y(0), y(1), \cdots) \in Y \, ,$$

$$\pi = (p(-2), p(-1), p(0), p(1), \cdots) \in L_\infty, p(0) = 1 \, ,$$

and a subsequence $T_n \to \infty$ $(n \to \infty)$ such that

$$x_h^{T_n} \overset{*}{\to} x_h \, , h = 1, 2 \, ,$$

$$y^{T_n} \overset{*}{\to} y \, ,$$

$$\pi^{T_n} \overset{*}{\to} \pi \, ,$$

for $n \to \infty$, where the convergence is weak star in the $\sigma(L_\infty, L_1)$ topology.

It will be shown in this section that $(\pi, x_1, x_2, y)$ is a general competitive equilibrium for the infinite horizon economy.

Remark first that the convergence is a least pointwise so that there exist $B > 0$ and $\beta > 0$ such that

$$\|y\|_\infty \leq B \, , \|x_h\|_\infty \leq B \, , p(t) \leq \beta p(0) \text{ for } t \geq -2 \, . \tag{4.1}$$

Moreover

$$\sum_{h=1}^{2} x_h(t) \leq y(t) \, ; t \geq 1 \tag{4.2}$$

$$\sum_{h=1}^{2} x_h(0) = \sum_{h=1}^{2} \omega_h(0) + y(0) \, . \tag{4.3}$$

For $\bar{y} \in Y$ we define $\pi \cdot \bar{y} := \sum_{t=-2}^{\infty} p(t) \bar{y}(t) \leq \infty.$

## Lemma 4.1

$\pi \cdot \bar{y} \le V$ for all $\bar{y} \in Y$, where $V$ is defined in Lemma 3.4.

## Proof

Suppose there exists $\bar{y} \in Y$ such that $\pi \cdot \bar{y} > V$. There exists $T^*$ such that

$$\sum_{t=-2}^{T^*} p(t) \bar{y}(t) > V ,$$

implying that

$$\lim_{n \to \infty} \sum_{t=-2}^{T^*} p^{T_n}(t) \bar{y}(t) > V .$$

Hence for $T_n \ge T^*$ and $n$ large enough

$$\sum_{t=-2}^{T^*} p^{T_n}(t) \bar{y}(t) > V$$

and also

$$\sum_{t=-2}^{T_n} p^{T_n}(t) \bar{y}(t) > V ,$$

which contradicts lemma 3.4iii. $\qquad \Box$

## Lemma 4.2

i) $\quad \lim\limits_{n \to \infty} \pi^{T_n} \cdot x_h^{T_n} = \pi \cdot x_h , h = 1, 2.$

ii) $\quad \lim\limits_{n \to \infty} \pi^{T_n} \cdot y^{T_n} = \pi \cdot y.$

iii) $\quad \pi \cdot x_h = \pi \cdot (\omega_h + \delta_h y) , h = 1, 2.$

iv) $\quad \sum\limits_{h=1}^{2} x_h \le y + \sum\limits_{h=1}^{2} \omega_h.$

## Proof

i)  $0 \le x_h(t) \le y(t), t \ge 1$ (4.2). So $\pi \cdot x_h < \infty$ by the previous lemma. It follows from lemma 3.1 that

$$\frac{u'_h(x_h^T(t))}{(1+\rho_h)^t} = u'_h(x_h^T(0)) p^T(t) , 0 \le t \le T .$$

In view of lemma 3.3 we have

$$u_h(B) \ge u_h(x_h^T(t)) \ge u'(x_h^T(t)) x_h^T(t)$$

so that

$$\sum_{t=0}^{T} p^T(t) \, x_h^T(t) \le \frac{1}{u'_h(x_h^T(0))} \sum_{t=0}^{T} \frac{u_h(B)}{(1+\rho_h)^t} \ .$$

Also from lemma 3.4

$$0 < \frac{1}{V} \le \frac{1}{u'_h(x_h^T(0))} \le \frac{1}{u'_h(B)} \ .$$

Hence

$$\sum_{t=0}^{T} p^T(t) \, x_h^T(t) \le \bar{\psi}_h \sum_{t=0}^{T} \frac{u_h(B)}{(1+\rho_h)^t} \ , \ \bar{\psi}_h = 1 \, / \, u'_h(B) \ .$$

Let $\varepsilon > 0$ be given. Take $T^*$ such that

$$\sum_{t=T^*+1}^{\infty} p(t) \, x_h(t) < \frac{\varepsilon}{3}, \ \psi_h \sum_{t=T^*+1}^{\infty} \frac{u_h(B)}{(1+\rho_h)^t} < \frac{\varepsilon}{3} \ .$$

Then, for $T_n > T^*$ and large enough, we have

$$\left| \sum_{t=0}^{T^*} (p(t) \, x_h(t) - p^{T_n}(t) \, x_h^{T_n}(t)) \right| < \frac{\varepsilon}{3} \ .$$

The proof is completed by the identity

$$\pi \cdot x_h - \pi^{T_n} \cdot x_h^{T_n} = \sum_{t=0}^{T^*} (p(t) \, x_h(t) - p^{T_n}(t) \, x^{T_n}(t)) + \sum_{t=T^*+1}^{\infty} p(t) \, x(t) - \sum_{t=T^*+1}^{T_n} p^{T_n}(t) \, x_h^{T_n}(t) \ .$$

ii)   Since $y(t) = \sum_{h=1}^{2} x_h(t)$ for $t \ge 1$ and $\pi \cdot y \le V$ we can copy the proof of i) in order to prove ii)

by taking into account that

$$\sum_{t=0}^{T} p^T(t) \, y^T(t) \le \sum_{t=0}^{T} \sum_{h=1}^{2} \frac{u_h(B)}{(1+\rho_h)^t} \ .$$

iii)   This follows directly from i) and ii).

iv)   This follows from (4.2) and (4.3).                                                                                          []

In the sequel $P$ and $R$ will denote strict preference and weak preference respectively.

**Lemma 4.3**

If $x R_h x_h$ and $\bar{y} \in Y$ then

$$\pi \cdot x \ge \pi \cdot (\omega_h + \delta_h \bar{y}) \ .$$

**Proof**

The proof is given in three steps.

<u>Step 1</u>.   For all $\varepsilon > 0$ and all $\bar{x} \in L_\infty^+$ there are $x' \in L_\infty^+$ and $T^* \in I\!N$ such that

$$p(0)\,x'(0) + \cdots + p(T^*)\,x'(T^*) < \varepsilon + p(0)\,\bar{x}(0) + \cdots + p(T^*)\,\bar{x}(T^*)$$

$$x'(t) = 0 \text{ for } t > T^*$$

$$\sum_{t=0}^{T^*} \frac{u_h(x'(t))}{(1+\rho_h)^t} > \sum_{t=0}^{\infty} \frac{u_h(\bar{x}(t))}{(1+\rho_h)^t} \ .$$

**Proof of step 1.**

Choose $0 < \eta < \varepsilon / \sum_{t=0}^{\infty} p(t)\,2^{-t}$. Define

$$x''(t) := \bar{x}(t) + \eta\,2^{-t} \,, \ t \geq 0 \ .$$

Then $x'' \, P_h \, \bar{x}$. So there exists $T^*$ such that

$$\sum_{t=0}^{T^*} \frac{u_h(x''(t))}{(1+\rho_h)^t} > \sum_{t=0}^{\infty} \frac{u_h(\bar{x}(t))}{(1+\rho_h)^t} \ .$$

Define $x'(t) := x''(t)$ for $0 \leq t \leq T^*$ and $x'(t) := 0$ for $t > T^*$.

Then $x'(t)$ has the desired properties.

<u>Step 2.</u> For all $\varepsilon > 0$ and all $\bar{y} \in Y$, $\pi \cdot \bar{y} \leq \varepsilon + \pi^{T_n} \cdot \bar{y}$ for $n$ large enough.

Proof of step 2.

Take $T^*$ such that

$$\sum_{t=T^*+1}^{\infty} p(t)\,\bar{y}(t) < \tfrac{1}{2}\,\varepsilon \ .$$

This is possible from lemma 4.1. Take $T_n > T^*$ large enough to ensure that

$$\sum_{t=-2}^{T^*} (p(t) - p^{T_n}(t))\,\bar{y}(t) < \tfrac{1}{2}\,\varepsilon \ .$$

Then

$$\pi \cdot \bar{y} - \pi^{T_n} \cdot \bar{y} = \sum_{t=-2}^{T^*} (p(t) - p^{T_n}(t))\,\bar{y}(t) - \sum_{t=T^*+1}^{T_n} p^{T_n}(t)\,\bar{y}(t) + \sum_{t=T^*+1}^{\infty} p(t)\,\bar{y}(t)$$

$$< \tfrac{1}{2}\,\varepsilon + \tfrac{1}{2}\,\varepsilon + 0 \ .$$

<u>Step 3.</u>

Let $\varepsilon > 0$ be given. Take $T^*$ and $x'$ with $x'_h \, P_h \, x$ and with

$$p(0)\,x'(0) + \cdots + p(T^*)\,x'(T^*) < \varepsilon + p(0)\,x(0) + \cdots + p(T^*)\,x(T^*)$$

$$x'(t) = 0 \text{ for } t > T^* \ .$$

So, for $T_n > T^*$ and $n$ large enough we have $x'\,P_h\,x_h^{T_n}$, because $x\,P_h\,x_h$. Hence

$$\pi^{T_n} \cdot x' > \pi^{T_n} \cdot x_h^{T_n} = \pi^{T_n} \cdot (\omega_h + \delta_h y^{T_n}) \ge \pi^{T_n} \cdot (\omega_h + \delta_h \bar{y})$$

where the (strict) first inequality occurs from the fact that $x'$ could have been chosen by consumer $h$ in the economy with horizon $T_n$.

On the other hand

$$\pi^{T_n} \cdot x' = \sum_{t=0}^{T^*} p^{T_n}(t) x'(t) < \sum_{t=0}^{T^*} p(t) x'(t) + \varepsilon < 2\varepsilon + \sum_{t=0}^{T^*} p(t) x(t) \le 2\varepsilon + \sum_{t=0}^{\infty} p(t) x(t) \ .$$

Moreover $\pi^{T_n} \cdot \bar{y} \ge \pi \cdot \bar{y} - \varepsilon.$

Hence

$$\pi \cdot \omega_h + \delta_h \pi \cdot \bar{y} < 3\varepsilon + \pi \cdot x \text{ for all } \varepsilon > 0$$

or

$$\pi \cdot x \ge \pi \cdot \omega_h + \delta_h \pi \cdot \bar{y} \ . \qquad \qquad \square$$

## Lemma 4.4

For all $\bar{y} \in Y$, $\pi \cdot \bar{y} \le \pi \cdot y$.

## Proof

It follows from the previous lemma that $\pi \cdot x_h \ge \pi \cdot \omega_h + \delta_h \pi \cdot \bar{y}$, $h = 1, 2$. Then also

$$\sum_h \pi \cdot x_h \ge \sum_h \pi \cdot \omega_h + \pi \cdot \bar{y} \ .$$

Application of lemma 4.2iii gives the desired result. $\qquad \qquad \square$

## Lemma 4.5

For $t \ge 0$ and $h = 1, 2$

$$\frac{u'_h(x_h(t))}{(1+\rho_h)^t} = u'_h(x_h(0)) p(t) \ .$$

## Proof

It follows from lemma 3.4i that there exists $m > 0$ such that $0 < m \le x_h^T(0)$ for all $T$. For $n$ large enough we have

$$u'_h(x_h^{T_n}(t + 1)) \le (1 + \rho_h) u'_h(x_h^{T_n}(t)) \ .$$

Hence

$$u'_h(x_h(t + 1)) \le (1 + \rho_h) u'_h(x_h(t)) \ .$$

By induction on $t$ we derive $x_h(t) > 0$ for all $t$. Moreover, since $\|x_h^T\|_\infty \le B$ we conclude that

$$r(t+1) := \lim_{n \to \infty} r^{T_n}(t+1)$$

exists, where $r^{T_n}$ has been defined in lemma 3.1. So

$$u'_h(x_h(t+1)) = \frac{1+\rho_h}{1+r(t+1)} \, u'_h(x_h(t))$$

$$p(t+1) = \frac{p(t)}{1+r(t+1)}, \, t \geq 0$$

$$p(0) = 1 \, . \qquad\qquad\qquad []$$

## Lemma 4.5

$$\pi \cdot \overline{x} \leq \pi \cdot (\omega_h + \delta_h y) \Rightarrow x_h \, R_h \, \overline{x} \, .$$

## Proof

$$u_h(\overline{x}(t)) - u_h(x_h(t)) \leq u'_h(x_h(t+1)) \, (\overline{x}(t) - x_h(t))$$

$$\frac{u_h(\overline{x}(t))}{(1+\rho_h)^t} \leq \frac{u_h(x_h(t))}{(1+\rho_h)^t} + u'_h(x_h(0)) \, (p(t) \, \overline{x}(t) - p(t) \, x_h(t))$$

$$\sum_{t=0}^{\infty} \frac{u_h(\overline{x}(t))}{(1+\rho_h)^t} \leq \sum_{t=0}^{\infty} \frac{u_h(x(t))}{(1+\rho_h)^t} + u'_h(x_h(0)) \sum_{t=0}^{\infty} (p(t) \, \overline{x}(t) - p(t) \, x_h(t))$$

$$\leq \sum_{t=0}^{\infty} \frac{u_h(x_h(t))}{(1+\rho_h)^t}$$

since $\pi \cdot \overline{x} \leq \pi \cdot (\omega_h + \delta_h y) \leq \pi \cdot x_h$. $\qquad\qquad []$

## Theorem 4.1

$(\pi, x_1, x_2, y)$ is a general equilibrium for the infinite horizon economy.

## Proof

Combine lemmata 4.4, 4.5 and 4.2iii. $\qquad\qquad []$

.

## 5. Conclusions

The central issue of the present paper has been the existence of a general competitive equilibrium in a model with exhaustible natural resources. It may seem that the model employed is rather special and therefore one may tempted to conclude that the approach proposed is of rather minor general importance. However this conjecture is false. In section 4 use has been made of the lemmata of section 3 only. And there are obviously may models which exhibit the desired properties. We list a few examples. The number of consumers (countries) can be extended from two to an arbitrary number. Without altering the basic technological features the same holds for the number of types of resource stocks, as long as a homogeneous raw material is extracted. Also the number of non-resource producing sectors can arbitrarily be increased if one sticks to the convex technologies described in A1-A3. We have been postulating a linear extraction technology. Clearly the only thing that matters is the convexity of the aggregate production set. This can be achieved by a variety of non-linear extraction technologies. Moreover, it is fairly straightforward to generalize the model with respect to the number of intermediate commodities, which can be defined as produced inputs that do not enter into the utility functions. It should be admitted however that increasing the number of "goods" is less easy to implement. This is subject to further research. Nonetheless it is our conviction that our existence proof applies to a rather broad class of economically meaningful models whereas other methods of existence proofs would employ assumptions which are implausible and/or undesired from an economic point of view.

Two other subjects for further research are worth mentioning. First one can ask for a characterization of a general equilibrium. It would be too far-going to elaborate on that issue here. Obviously the more specific one is on production functions and preferences, the more can be said about the properties of a general equilibrium. For a rather complete characterization for the case of homogeneous production functions we refer to van Geldrop and Withagen (1988).

Finally, some remarks should be made about the continuous-time case. In the present paper we have largely benefitted from the existence of a general equilibrium for any finite horizon. With discrete time existence with a finite horizon is due to the finiteness of the number of commodities. But with continuous-time even for a finite horizon the commodity space is of infinite dimension. So there is a problem, which is subject to further research.

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| M 88-03 | January | B.G. Hansen, E. Willekens | The generalized logarithmic series distribution. |
| M 88-04 | January | J. van Geldrop, C. Withagen | A general equilibrium model of international trade with exhaustible natural resource commodities. |
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