# Poisson processes and a Bessel function integral 

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# POISSON PROCESSES AND A BESSEL FUNCTION INTEGRAL* 

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#### Abstract

The probability of winning a simple game of competing Poisson processes turns out to be equal to the well-known Bessel function integral $J(x, y)$ (cf. Y. L. Luke, Integrals of Bessel Functions, McGraw-Hill, New York, 1962). Several properties of $J$, some of which seem to be new, follow quite easily from this probabilistic interpretation. The results are applied to the random telegraph process as considered by Kac [Rocky Mountain J. Math., 4 (1974), pp. 497-509].


Key words. Poisson process, Bessel function, random telegraph

1. Competing Poisson processes. Several problems can be described as follows: An object has to travel a distance $x$; it does so at unit speed, but it is obstructed at random moments and then held for a random period of time before it is allowed to continue. The object may be a particle moving between two electrodes, a person walking to a bus stop, or, as in [5, Problem 147], a book being read with random interruptions. The question is: What is the probability that the object reaches its destination at a moment not exceeding $x+y$ ? The situation may be modelled as a game of two competing (Poisson) renewal processes in the following way (see Fig. 1):

Let $X_{1}, Y_{1}, X_{2}, Y_{2}, \cdots$ be independent, exponentially distributed random variables with expectation one. Two persons, $X$ and $Y$, take turns drawing lengths $X_{j}$ and $Y_{j}$. Person $X$ starts, and wins if the sum of his $X_{j}$ exceeds $x$ before the sum of $Y$ 's $Y_{j}$ exceeds $y$.


Fig 1. $N_{x}=5, N_{y}=3 ; X$ loses.
More formally, if $N_{x}$ and $N_{y}$ are random variables defined by

$$
\begin{aligned}
& N_{x}=\min \left\{n ; X_{1}+\cdots+X_{n}>x\right\}, \\
& N_{y}=\min \left\{n ; Y_{1}+\cdots+Y_{n}>y\right\},
\end{aligned}
$$

then (remember that $X$ starts)

$$
\begin{equation*}
X \text { wins } \Leftrightarrow N_{x} \leqq N_{y} \Leftrightarrow X_{1}+\cdots+X_{N_{y}}>x . \tag{1}
\end{equation*}
$$

[^0]Remark. For our purposes the assumption that $E X_{j}=E Y_{j}=1$ for $j=1,2, \cdots$, is no restriction: replacing $X_{j}$ and $Y_{j}$ by $X_{j} / \lambda$ and $Y_{j} / \mu$, respectively, is equivalent to replacing $x$ and $y$ by $\lambda x$ and $\mu y$, respectively. The process $Z(t)$ depicted in Fig. 1, representing the distance travelled by the object at time $t$, would, of course, be changed by a transformation of the $X_{j}$ and $Y_{j}$.

We shall use the following two well-known facts: $N_{y}-1$ has a Poisson distribution with mean $y$, i.e.,

$$
\begin{equation*}
P\left(N_{y}=n\right)=e^{-y} \frac{y^{n-1}}{(n-1)!} \quad(n=1,2, \cdots), \tag{2}
\end{equation*}
$$

and $X_{1}+\cdots+X_{n}$ has a gamma distribution with density

$$
\begin{equation*}
\frac{d}{d x} P\left(X_{1}+\cdots+X_{n} \leqq x\right)=e^{-x} \frac{x^{n-1}}{(n-1)!} \quad(x>0) \tag{3}
\end{equation*}
$$

Now, let $J(x, y)$ be defined by (cf. Luke [4, p. 271])

$$
\begin{equation*}
J(x, y)=1-e^{-y} \int_{0}^{x} I_{0}(2 \sqrt{y t}) e^{-t} d t \tag{4}
\end{equation*}
$$

where $I_{0}$ is the modified Bessel function of order zero:

$$
\begin{equation*}
I_{0}(z)=\sum_{0}^{\infty} \frac{(z / 2)^{2 n}}{(n!)^{2}} \tag{5}
\end{equation*}
$$

Then we easily obtain
Proposition 1.

$$
\begin{equation*}
P\left(N_{x} \leqq N_{y}\right)=J(x, y) . \tag{6}
\end{equation*}
$$

Proof. By (1)-(5) we have

$$
\begin{aligned}
P\left(N_{x} \leqq N_{y}\right) & =1-P\left(N_{x}>N_{y}\right)=1-P\left(X_{1}+\cdots+X_{N_{y}} \leqq x\right) \\
& =1-\sum_{n=1}^{\infty} P\left(N_{y}=n, X_{1}+\cdots+X_{n} \leqq x\right) \\
& =1-\sum_{n=1}^{\infty} e^{-y} \frac{y^{n-1}}{(n-1)!} \int_{0}^{x} e^{-t} \frac{t^{n-1}}{(n-1)!} d t \\
& =1-e^{-y} \int_{0}^{x} I_{0}(2 \sqrt{y t}) e^{-t} d t=J(x, y)
\end{aligned}
$$

Remark. Srivastava and Kashyap [6, pp. 77, 78] consider an equivalent interpretation, in the context of a randomized random walk; there the interpretation remains implicit and is not pursued.
2. Properties of $J(x, y)$. Several properties of $J(x, y)$ follow immediately from (6). We list the following six together with their simple proofs.
(i) $J(0, y)=P\left(X_{1}>0\right)=1$,
(ii) $J(x, 0)=P\left(X_{1}>x\right)=e^{-x}$.

From (2) and its counterpart for $N_{x}$ (independent of $N_{y}$ ) it follows that

$$
\begin{aligned}
P\left(N_{x}=N_{y}\right) & =\sum_{1}^{\infty} P\left(N_{x}=n, N_{y}=n\right) \\
& =\sum_{1}^{\infty} e^{-x} \frac{x^{n-1}}{(n-1)!} e^{-y} \frac{y^{n-1}}{(n-1)!}=e^{-x-y} I_{0}(2 \sqrt{x y}) .
\end{aligned}
$$

From this we conclude using (6) that
(iii) $J(x, y)+J(y, x)=1+P\left(N_{x}=N_{y}\right)=1+e^{-x-y} I_{0}(2 \sqrt{x y})$,
and especially
(iv) $J(x, x)=\frac{1}{2}+\frac{1}{2} e^{-2 x} I_{0}(2 x)$.

Conditioning on $X_{1}=u$, with density $e^{-u}$, we have

$$
P\left(N_{x} \leqq N_{y}\right)=\int_{0}^{x}\left(1-P\left(N_{y} \leqq N_{x-u}\right)\right) e^{-u} d u+\int_{x}^{\infty} e^{-u} d u
$$

or in view of (5)

$$
\text { (v) } J(x, y)=1-\int_{0}^{x} J(y, x-u) e^{-u} d u
$$

which seems to be new. Rewriting (v) as

$$
e^{x} J(x, y)=e^{x}-\int_{0}^{x} J(y, v) e^{v} d v
$$

and differentiating with respect to $x$, using (4) we recover (iii):

$$
\text { (vi) } \frac{\partial}{\partial x} J(x, y)=1-J(x, y)-J(y, x)=-e^{-x-y} I_{0}(2 \sqrt{x y}) \text {. }
$$

Several other relations given in [4] are easily obtained from (i)-(vi). In §3 we collect some asymptotic results.
3. Asymptotics. From the probabilistic interpretation the following limit relations are quite obvious (it is easy to give estimates; also compare (v)):

$$
\begin{aligned}
& \lim _{x \rightarrow \infty} J(x, y)=\lim _{x \rightarrow \infty} P\left(N_{x} \leqq N_{y}\right)=0, \\
& \lim _{y \rightarrow \infty} J(x, y)=\lim _{y \rightarrow \infty} P\left(N_{x} \leqq N_{y}\right)=1 .
\end{aligned}
$$

For both $x$ and $y$ large we have the following very simple relation, which seems related to expansions in [2] involving the error function, but which seems to be new in this form. Its proof is a simple consequence of the asymptotic normality of Poisson random variables with large means.

Proposition 2. For $x \rightarrow \infty$ and $y \rightarrow \infty$

$$
\begin{equation*}
J(x, y)=\Phi\left(\frac{y-x+1 / 2}{\sqrt{x+y}}\right)+O\left(\frac{1}{\sqrt{x}}+\frac{1}{\sqrt{y}}\right) \tag{7}
\end{equation*}
$$

where $\Phi$ is the standard normal distribution function defined as

$$
\Phi(u)=(2 \pi)^{-1 / 2} \int_{-\infty}^{u} e^{-v^{2} / 2} d v
$$

Proof.

$$
J(x, y)=P\left(N_{x}-N_{y} \leqq 0\right)=P\left(N_{x}-N_{y}<\frac{1}{2}\right),
$$

where the $\frac{1}{2}$ is the usual "continuity correction". As $N_{x}-N_{y}$ is asymptotically normal with mean $x-y$ and variance $x+y$, it follows that

$$
\begin{equation*}
J(x, y)=P\left(\frac{N_{x}-N_{y}-x+y}{\sqrt{x+y}} \leqq \frac{y-x+1 / 2}{\sqrt{x+y}}\right) \approx \Phi\left(\frac{y-x+1 / 2}{\sqrt{x+y}}\right) . \tag{8}
\end{equation*}
$$

That $J(x, y)$ actually satisfies (7) follows easily from the Berry-Esseen version of the central limit theorem (Feller [1, p. 542]).

Remark. Relation (7), of course, also holds without the term $\frac{1}{2}$. In practice the approximation (8) is much better than is suggested by (7). For values of $x$ and $y$ of 10 and higher it yields a result correct to about three decimal places. Two examples: $x=10$ and $y=20$ yields $J(10,20)=0.974206$ and $\phi(10.5 \sqrt{30})=\Phi(1.917)=0.972$. For $x=y=50$ we find $J(50,50)=0.519972$ and $\Phi(0.5 / 10)=\Phi(0.05)=0.5199$. The abundance of tables of $\Phi$ makes the approximation (8) quite practical. To obtain good (proven) bounds is not so easy.
4. Relation with Kac's random telegraph model. In [3] Kac considers an (integrated) telegraph process $X(t)$ (in his formula (25) denoted by $x(t)$ ) that is closely related to the process $Z(t)$ of Fig. 1. The process $X(t)$ is constructed from the same $X_{j}$ and $Y_{j}$ as $Z(t)$; its graph is sketched in Fig. 2. Evidently, the processes $Z(t)$ and $X(t)$ are related by

$$
\begin{equation*}
Z(t)=\frac{1}{2}(X(t)+t) . \tag{9}
\end{equation*}
$$

From Fig. 1 we immediately see that

$$
Z(x+y)>x \Leftrightarrow N_{x} \leqq N_{y},
$$

and therefore by Proposition 1 we have, in view of (9),


Fig. 2
Proposition 3. Let $F(x, t)=P(X(t) \leqq x)$ be the distribution function of $X(t)$. Then for $0 \leqq x \leqq t$

$$
\begin{equation*}
F(x, t)=1-J\left(\frac{t+x}{2}, \frac{t-x}{2}\right) . \tag{10}
\end{equation*}
$$

From Proposition 2 we then obtain, not very surprisingly, Corollary.

$$
F(x, t) \sim \Phi\left(\frac{x-1 / 2}{\sqrt{t}}\right) \quad(t \rightarrow \infty)
$$

i.e., $X(t)$ is asymptotically normal with mean $\frac{1}{2}$ and variance $t$.

Remark 1. Of course, $X(t)$ is also asymptotically normal with mean zero and variance $t$; the $\frac{1}{2}$ will improve the approximation, though.

Remark 2. Since by (vi) (see also [4, p. 272]) $J$ satisfies $J_{x y}+J_{x}+J_{y}=0$, from (10) it follows that $F$ satisfies the "telegrapher's" equation: $F_{t t}=F_{x x}-2 F_{t}$ as is proved in [3] for a more general $F$.

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