

The acoustics of a kettledrum

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THE ACOUSTICS OF A KETTLEDRUM

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december 1989



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THE ACOUSTICS OF A KETTLEDRUM

by S.W. Rienstra

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Abstract

A mathematical model is analysed of a kettledrum, consisting of a hard-walled open ended finite cylinder, closed by an ideal membrane, connected to a semi-infinite space by a hard-walled flange, and provided with a small vent hole in the bottom. The primary aim of the analysis is to determine the spectrum of the kettledrum, i.e. the eigenfrequencies. These frequencies are complex because due to the radiation to infinity any unforced solution will decay in time. By decomposing the acoustic field into circumferential Fourier components (m -modes, where m is an integer) the problem is reduced to a quasi two-dimensional problem per m . By means of suitable Greens functions the field inside and outside the cylinder are written as a function of the motion of the membrane, so that the membrane equation, including the field pressure difference as a forcing term, can be written as an integro-differential equation. Its solution can formally be written as a sum of vacuum modes, by which the equation can be rewritten as a matrix equation, which is then solved numerically by iteration, applying standard eigenvalue techniques.

From the studied examples it is found that two types of modes can be distinguished: vacuum mode-like, and cavity mode-like. This is not a strict classification. When we vary a problem parameter a mode may go over from one type into another. A property of these vacuum-type modes is that the membrane vibrations are relatively strong so that the energy is radiated away quickly, whereas with the cavity-type mode the motion is trapped in the cavity, with a relatively long decay time.

1. Introduction

A theoretical analysis of the acoustics of a musical instrument is an intrinsically difficult problem. This is not because the basic mechanisms are unknown or poorly understood, but because the sound produced is meant to be perceived by human ears, rather than measuring instruments. Not only are human ears extremely sensitive, with their range (in acoustic energy) spanning a ratio of the order of 10^{14} , but also is the appreciation of beauty in a very subtle and subjective manner dependent on the spectral components of the sound in a way not easily represented by a formula.

This observation couldn't be more true for the kettledrum, the instrument that will be considered in the present report. The basic mechanisms (vibrating flanged membrane, resonating cavity) are known for more than a century ([1]), but still sound quality aspects as pitch and decay time are affected by secondary resonances and dissipation, which are all known in principle ([2]), but unknown in any practical situation.

Therefore, it is both impossible and useless to model a kettledrum theoretically: we do not really know what to model, and if we had a model the results would be as difficult to interpret as a real experiment. So the crucial first step to understand the acoustics of a kettledrum and to bring order in experimental results is to model the basic elements in such a way, that we can *quantify* (numerically) our qualitative knowledge, predict trends, and assist our intuition in those cases where a subtle interaction between equally important effects occurs.

The present report deals with such a model of a kettledrum, describing the vibrations of a circular homogeneous flanged membrane backed by a cylindrical cavity in a medium of air. The cavity is hard-walled and has a small opening in the bottom. In vacuum the modes of vibration of the membrane would be independent of the presence of a cavity. With air the frequencies of these vacuum-modes are decreased by the air loading (although there is an additional effect of increasing by the presence of the cavity), while at the same time these modes decay in time by the effect of radiation of acoustic energy out to infinity. A further effect of the cavity is a coupling of the membrane vibration with the cavity modes.

The rotational symmetry, the cylindrical kettle geometry, and the artefact of the flange are simplifications made to further assist use of the model: they allow a much more detailed mathematical analysis, making the numerical solution efficient, compact, and fast, so that tracing the spectrum as a function of various parameters is possible even on a small computer. Alternatives, without these simplifications, are in principle possible, but only at the expense of expensive very massive, cumbersome, program packages, only executable at high speed computers. Another approximation, exploiting a small air/membrane density ratio, is reported in [3], but this is not applicable here, since this ratio is not small. The approximation of a small membrane-wave velocity/sound speed ratio, including a compact source (small ratio of membrane diameter/acoustic wave length), which is essentially the approximation in [4,5], is only useful in a

proper matched asymptotic expansion setting. We have not exploited that further, but found the present approach the best combination of flexibility and accuracy, as experiments have confirmed [6].

The essence of the model adopted, and the corresponding mathematical analysis, has already been published in [6]. In this reference one may find also the (very favourable) comparison with experiments. Our analysis adds to these results: the effect of the bottom opening, a very detailed analysis of a central integral greatly improving the efficiency of the calculations, and (in the presentation of the results) a more careful treatment of the cavity-modes. In [6] all the modes are considered to be variations of vacuum-membrane modes. We will show that there are also modes close to cavity modes, which were, indeed, actually not taken into consideration in the comparison with experiments.

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2. The Model

An inviscid, compressible, stationary, and uniform medium, provided with a cylindrical coordinate system (r, θ, z) , allows acoustic perturbations described by (cf. [7,8])

$$\nabla^2 \phi - \frac{1}{c_a^2} \frac{\partial^2}{\partial t^2} \phi = 0 \quad (2.1)$$

in the half space $z > L$ and in the cylinder $r < a$, $0 < z < L$. See figure 1. ϕ is the velocity potential, $c_a = (\gamma p_a / \rho_a)^{1/2}$, p_a , and ρ_a are the mean sound speed, pressure and density respectively, with γ the specific-heat ratio (for air $\gamma = 1.4$). The pressure perturbation is given by

$$p = -\rho_a \frac{\partial}{\partial t} \phi. \quad (2.2)$$

Between cylinder and halfspace, at $z = 0$, $r \leq a$, a membrane of surface density σ is stretched with uniform tension T , allowing linear deflections $z = L + \eta(r, \theta, t)$ described by

$$T \nabla_0^2 \eta - \sigma \frac{\partial^2}{\partial t^2} \eta = p(r, \theta, L+, t) - p(r, \theta, L-, t) \quad (2.3)$$

where the right hand side denotes the pressure difference across the membrane. Naturally, ∇_0 denotes the two-dimensional gradient in the plane $z = L$. From continuity of particle displacement, the vertical velocities of air and membrane must match, so that

$$\frac{\partial}{\partial t} \eta(r, \theta, t) = \frac{\partial}{\partial z} \phi(r, \theta, L, t) \quad (2.4)$$

(for $0 \leq r \leq a$) both for $z = L+$ and $z = L-$. The propagation speed c_M of the (transversal) waves in the membrane is given by

$$c_M = (T / \sigma)^{1/2}.$$

Except for a small opening $z = 0$, $r < d$ in the bottom of the cylinder, the boundaries are hard walls, giving the boundary conditions

$$\frac{\partial}{\partial z} \phi = 0 \quad \text{at } z = L, r > a$$

$$\text{and } z = 0, d < r < a$$

$$\frac{\partial}{\partial r} \phi = 0 \quad \text{at } r = a, 0 < z < L.$$

The bottom opening is modelled as a small ($d \ll$ any acoustic wave length) orifice in an infinitely thin wall, so that the diffraction effects are acoustically equivalent to a dipole source, of which the strength is determined by the incompressible flow through the orifice (the inner region in a matched asymptotic expansion formulation). This flow is on its turn driven by the pressure of the incident acoustic wave. The resulting relation, which goes back to Rayleigh and can be found in for example [7], is one between the rate of change of volume velocity through and the pressure

at the hole:

$$p(0,t) = \frac{\rho_a}{2d} \frac{d}{dt} \int_0^{2\pi} \int_0^d \frac{\partial}{\partial z} \phi(r, \theta, 0, t) r dr d\theta \quad (d \rightarrow 0). \quad \text{I}$$

As we do not accept sources at infinity radiating inwards we have for $r \rightarrow \infty, z \rightarrow \infty$ the radiation condition of only outward radiating waves (Sommerfeld's condition for harmonic waves, causality condition for initial value problems). As is customary with diffraction problems with sharp edges ([9]), the solution of the problem may be not unique without additional constraints describing the behaviour near the edge: the so-called edge condition. Without this condition non-acceptable solutions may be added to the sought solution by allowing point or line sources at these edges. This edge condition is therefore usually formulated as the finiteness of energy stored in any finite neighbourhood of the edge. Especially near the edge of the membrane $r = a, z = L$ we may have to take some care; that is: the anticipated series expansion of the solution will have to converge fast enough. Unfortunately, the numerical procedure we adopt does not give us very much handles to control this property of the solution. On the other hand, however, it is common practice that this type of numerical approach selects "automatically" the solution with the correct, i.e. mildest, singularity. So we will not consider this point here any further. The solutions we are interested in are free vibrations, i.e. without a source non-zero for all t . Since the problem is linear we may consider these solutions per frequency, so we set

$$\phi(r, \theta, z, t) = \phi(r, \theta, z) e^{-i\omega t}$$

and similarly for p and η . For convenience, the exponent $e^{-i\omega t}$ will be suppressed throughout from here on in the formulas. It appears that only discrete values of the frequency ω are possible. The corresponding solution will be called a mode or eigensolution. The modal frequency may also be called eigenfrequency or resonance frequency. It is this frequency which is usually of primary interest since it is in a musical context the most important property of the vibration. Note that, due to the loss of energy by radiation into the far field, a mode decays in time. This implies that the frequency ω is a complex number with $\text{Im}(\omega) < 0$. Furthermore, we will always assume $\text{Re}(\omega) > 0$.

Another observation that can be made concerns the cylindrical symmetry (i.e., periodicity in θ). In view of this, a solution can always be expanded in a Fourier series in θ , say $\phi = \sum \phi_m e^{im\theta}$. So we can always look for modes of the type $\phi(r, z) e^{im\theta - i\omega t}$, and count them per m : $\phi = \phi_{mn}, \omega = \omega_{mn}$.

3. Ideal modes

3.1. Vacuum modes

In the absence of air ($\rho_a = 0$) the acoustic field vanishes ($\phi \equiv 0$), and the solution reduces to the vibrations of the membrane alone:

$$\eta_{mn}^{(0)}(r, \theta) = \frac{J_m(x_{mn} r/a) e^{im\theta}}{a \sqrt{\pi} J_m'(x_{mn})} \quad (3.1)$$

$$\omega_{mn}^{(0)} = x_{mn} c_M/a \quad (3.2)$$

where J_m is the m -th order Besselfunction of the first kind ([10]), with $J_m(x_{mn}) = 0$. For later use, $\eta_{mn}^{(0)}$ is normalized, such that

$$\int_0^a \int_0^{2\pi} \eta_{mn}^{(0)}(r, \theta) \eta_{m'n'}^{(0)*}(r, \theta) r d\theta dr = 1 \quad (3.3)$$

if $m = m'$ and $n = n'$, and zero otherwise.

3.2. Cavity modes

When the density of the membrane tends to infinity, the deflection η vanishes, and the cylinder field is decoupled from the outer field. In view of the infinite domain and the radiation condition this outer field also vanishes, so the solution reduces to the resonances of the cylindrical cavity. When the bottom opening is closed ($d = 0$), these modes are simply

$$\phi_{mnl}^{(c)}(r, \theta) = J_m(y_{mn} r/a) \cos(l \pi z/L) e^{im\theta} \quad (3.4)$$

$$\omega_{mnl}^{(c)} = \frac{c_a}{a} (y_{mn}^2 + (l \pi a/L)^2)^{1/2} \quad (3.5)$$

with $n = 1, 2, 3, \dots$, $l = 0, 1, 2, \dots$, and $J_m'(y_{mn}) = 0$. (Note that $y_{m1} > m$, except for $y_{01} = 0$.)

4. Analysis

4.1. General solution

The approach of solution we will follow is, as in [6], to describe the acoustic field, both inside the cylinder and in the outside region, as if it were driven by the membrane displacements. On the other hand, the field just below and above the membrane can be considered as a driving force to the membrane (eq. (2.3)). So we can formulate an integral equation for the membrane displacements. This equation is then rewritten in matrix form by a suitable expansion in vacuum modes $\eta_{mn}^{(0)}$. This matrix equation may then be solved numerically.

A most appropriate way to describe the acoustic field is by means of a Greens function. Define

$$(\nabla^2 + \omega^2/c_a^2) G(\mathbf{r}; \mathbf{r}') = -4\pi \delta(\mathbf{r} - \mathbf{r}') \quad (4.1)$$

with boundary conditions in the cylinder

$$\begin{aligned} \frac{\partial}{\partial r} G_{in} &= 0 \quad \text{at } r = a \\ \frac{\partial}{\partial z} G_{in} &= 0 \quad \text{at } z = 0, z = L \end{aligned}$$

and in the outer region

$$\frac{\partial}{\partial z} G_{out} = 0 \quad \text{at } z = L.$$

The result for G_{in} is then ([6])

$$\begin{aligned} G_{in}(\mathbf{r}; \mathbf{r}') &= -\frac{4}{a} \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} e^{im(\theta-\theta')} \frac{J_m(\gamma_{mn} r/a) J_m(\gamma_{mn} r'/a)}{(1-m^2/\gamma_{mn}^2) J_m^2(\gamma_{mn})} \\ &\quad \cdot \frac{\cos(\gamma_{mn} z_{<}/a) \cos(\gamma_{mn}(L-z_{>})/a)}{\gamma_{mn} \sin(\gamma_{mn} L/a)} \end{aligned} \quad (4.2)$$

where $z_{<} = \min(z, z')$, $z_{>} = \max(z, z')$ and $\gamma_{mn} = \gamma(\gamma_{mn})$ where the complex function

$$\gamma(\lambda) = \sqrt{k^2 - \lambda^2},$$

with $k = \omega a/c_a$, is defined as the following product of principal branch square roots (see figure 2)

$$\gamma(\lambda) = i \sqrt{i(\lambda - k)} \cdot \sqrt{-i(\lambda + k)}. \quad (4.3)$$

The solution G_{out} for the outer region is

$$G_{out}(\mathbf{r}; \mathbf{r}') = \frac{1}{R} e^{ikR/a} + \frac{1}{R} e^{ik\bar{R}/a} \quad (4.4)$$

where

$$R^2 = r^2 + r'^2 - 2 r r' \cos(\theta - \theta') + (z - z')^2$$

$$\bar{R}^2 = r^2 + r'^2 - 2 r r' \cos(\theta - \theta') + (z + z' - 2L)^2.$$

By applying Green's theorem ([7]) we obtain for the outside field a form of the Rayleigh integral ([7])

$$\phi(r, \theta, z) = \frac{i\omega}{4\pi} \int_0^a \int_0^{2\pi} G_{out}(r, \theta, z; r', \theta', L) \eta(r', \theta') r' dr' d\theta'. \quad (4.5)$$

Similarly, we find for the field in the cavity (note the bottom opening)

$$\begin{aligned} \phi(r, \theta, z) &= \frac{1}{4\pi} \int_0^a \int_0^{2\pi} G_{in}(r, \theta, z; r', \theta', L) \frac{\partial}{\partial z'} \phi(r', \theta', L) r' dr' d\theta' \\ &\quad - \frac{1}{4\pi} \int_0^a \int_0^{2\pi} G_{in}(r, \theta, z; r', \theta', 0) \frac{\partial}{\partial z'} \phi(r', \theta', 0) r' dr' d\theta' \\ &= -\frac{i\omega}{4\pi} \int_0^a \int_0^{2\pi} G_{in}(r, \theta, z; r', \theta', L) \eta(r', \theta') r' dr' d\theta' \\ &\quad + \frac{1}{2\pi} \int_0^a \int_0^{2\pi} G_{in}(r, \theta, z; 0, 0, 0) \phi(0, 0, 0). \end{aligned} \quad (4.6)$$

It may be noted that $G_{in}(r; \mathbf{0})$ is symmetric (only $m = 0$ modes), and singular in $r = \mathbf{0}$, which is indeed a result of the fact that this expression is, with respect to the effect of the orifice, approximate and only valid for $d \ll a$, $d \ll c_a/\omega$, and $|r| \gg d$. Furthermore, since the second term of $O(d)$ is a correction, the value $\phi(\mathbf{0})$ to be substituted is effectively the one obtained for $d = 0$. So we end up with

$$\begin{aligned} \phi(r, \theta, z) &= -\frac{i\omega}{4\pi} \int_0^a \int_0^{2\pi} \eta(r', \theta') \left[G_{in}(r, \theta, z; r', \theta', L) + \right. \\ &\quad \left. \frac{d}{2\pi} G_{in}(r, \theta, z; \mathbf{0}) G_{in}(\mathbf{0}; r', \theta', L) \right] r' dr' d\theta'. \end{aligned} \quad (4.7)$$

After substitution in eq. (2.3) we finally obtain the equation for η

$$\begin{aligned} T \nabla_0^2 \eta + \omega^2 \sigma \eta &= -\frac{\omega^2 \rho_a}{4\pi} \int_0^a \int_0^{2\pi} \left[G_{out}(r, \theta, L; r', \theta', L) \right. \\ &\quad \left. + G_{in}(r, \theta, L; r', \theta', L) + \frac{d}{2\pi} G_{in}(r, \theta, L; \mathbf{0}) G_{in}(\mathbf{0}; r', \theta', L) \right] \eta(r', \theta') r' dr' d\theta'. \end{aligned} \quad (4.8)$$

Substitute the expansion

$$\eta(r, \theta) = \sum_{m'=-\infty}^{\infty} \sum_{n'=1}^{\infty} a_{m'n'} \eta_{m'n'}^{(0)}(r, \theta) \quad (4.9)$$

into eq. (4.8), multiply left- and righthand side by $\eta_{mn}^{(0)}(r, \theta)^* r$, and integrate over the membrane surface to obtain

$$\begin{aligned} (\omega^2 - \omega_{mn}^{(0)2}) \sigma a_{mn} = & -\frac{\omega^2 \rho_a}{4\pi} \sum_{m'=-\infty}^{\infty} \sum_{n'=1}^{\infty} a_{m'n'} \int_0^{2\pi} \int_0^a \int_0^a \int_0^a \\ & \left[G_{out}(r, \theta, L; r', \theta', L) + G_{in}(r, \theta, L; r', \theta', L) + \frac{d}{2\pi} G_{in}(r, \theta, L; \mathbf{0}) G_{in}(\mathbf{0}; r', \theta', L) \right] \\ & \cdot \eta_{m'n'}^{(0)}(r', \theta') \eta_{mn}^{(0)}(r, \theta)^* r' r dr' d\theta' dr d\theta. \end{aligned}$$

We further evaluate

$$\int_0^{2\pi} \int_0^a G_{in}(r, \theta, L; \mathbf{0}) \eta_{mn}^{(0)}(r, \theta)^* r dr d\theta = 8\sqrt{\pi} \delta_{0,m} x_{0n} S_n$$

where $\delta_{p,q} = 1$ if $p = q$ and 0 otherwise, and

$$S_n = \sum_{n''=1}^{\infty} \left[(x_{0n}^2 - y_{0n''}^2) \gamma_{0n''} \sin(\gamma_{0n''} L/a) J_0(y_{0n''}) \right]^{-1}$$

so that

$$\begin{aligned} -\frac{\omega^2 \rho_a}{4\pi} \sum_{m'=-\infty}^{\infty} \sum_{n'=1}^{\infty} a_{m'n'} \int_0^{2\pi} \int_0^a \int_0^a \int_0^a \frac{d}{2\pi} G_{in}(r, \theta, L; \mathbf{0}) G_{in}(\mathbf{0}; r', \theta', L) \cdot \\ \cdot \eta_{m'n'}^{(0)}(r', \theta') \eta_{mn}^{(0)}(r, \theta)^* r' r dr' d\theta' dr d\theta = \\ -\frac{8}{\pi} \omega^2 \rho_a d \delta_{0,m} x_{0n} S_n \sum_{n'=1}^{\infty} a_{0n'} x_{0n'} S_{n'}. \end{aligned}$$

Next we have

$$\begin{aligned} -\frac{\omega^2 \rho_a}{4\pi} \sum_{m'=-\infty}^{\infty} \sum_{n'=1}^{\infty} a_{m'n'} \int_0^{2\pi} \int_0^a \int_0^a \int_0^a G_{in}(r, \theta, L; r', \theta', L) \cdot \\ \cdot \eta_{m'n'}^{(0)}(r', \theta') \eta_{mn}^{(0)}(r, \theta)^* r' r dr' d\theta' dr d\theta = \end{aligned}$$

$$4\omega^2 \rho_a a \sum_{n'=1}^{\infty} a_{mn'} x_{mn} x_{mn'} C_{mnn'},$$

where

$$C_{mnn'} = \sum_{n''=1}^{\infty} \frac{\cotg(\gamma_{mn''} L/a)}{\gamma_{mn''}(x_{mn}^2 - y_{mn''}^2)(x_{mn'}^2 - y_{mn''}^2)(1 - m^2/y_{mn''}^2)}$$

For the final term, with G_{out} , we use the representation

$$\begin{aligned} G_{out}(r, \theta, L; r', \theta', L) &= \frac{2}{R_0} e^{ikR_0/a} \\ &= \frac{2i}{a} \int_0^{\infty} \frac{\lambda J_0(\lambda R_0/a)}{\gamma(\lambda)} d\lambda \\ &= \frac{2i}{a} \sum_{m''=-\infty}^{\infty} e^{im''(\theta-\theta')} \int_0^{\infty} \frac{\lambda J_{m''}(\lambda r/a) J_{m''}(\lambda r'/a)}{\gamma(\lambda)} d\lambda \end{aligned}$$

where $R_0^2 = r^2 + r'^2 - 2rr' \cos(\theta - \theta')$, and the complex λ integration contour is indented *under* around the branch cut from $\lambda = k$ (see figure 2). Then we have

$$\begin{aligned} -\frac{\omega^2 \rho_a}{4\pi} \sum_{m'=-\infty}^{\infty} \sum_{n'=1}^{\infty} a_{m'n'} \int_0^{2\pi} \int_0^a \int_0^a G_{out}(r, \theta, L; r', \theta', L) \cdot \\ \cdot \eta_{m'n'}^{(0)}(r', \theta') \eta_{mn}^{(0)}(r, \theta)^* r' r dr' d\theta' dr d\theta = \\ -2i \omega^2 a \rho_a \sum_{n'=1}^{\infty} a_{mn'} x_{mn} x_{mn'} I_{mnn'} \end{aligned}$$

with

$$I_{mnn'} = \int_0^{\infty} \frac{\lambda J_m^2(\lambda)}{\gamma(\lambda) (\lambda^2 - x_{mn}^2) (\lambda^2 - x_{mn'}^2)} d\lambda$$

Altogether we have the (infinite) set of equations

$$(\omega^2 - \omega_{mn}^{(0)2}) \sigma a_{mn} = 4 \omega^2 \rho_a a \sum_{n'=1}^{\infty} a_{mn'} x_{mn} x_{mn'} [C_{mnn'} - \frac{1}{2} i I_{mnn'} - \frac{2d}{\pi a} \delta_{0,m} S_n S_{n'}]$$

where ω , and a corresponding vector (a_{m1}, a_{m2}, \dots) , is to be found. This set of equations can be cast in matrix form as follows

$$\begin{bmatrix} c_M \\ k c_a \end{bmatrix}^2 \mathbf{a}_m = A_m(k) \mathbf{a}_m$$

where $\mathbf{a}_m = (a_{m1}, a_{m2}, a_{m3}, \dots)^T$ and $A_m(k) = (A_{mnn'}(k))$ with

$$\text{if } n = n' : A_{mnn} = \frac{1}{x_{nn}^2} - \frac{4 \rho_a a}{\sigma} \left[C_{mnn} - \frac{1}{2} i I_{mnn} - \frac{2d}{\pi a} \delta_{0,m} S_n^2 \right]$$

$$\text{if } n \neq n' : A_{mnn'} = -\frac{4 \rho_a a}{\sigma} \frac{x_{mn'}}{x_{mn}} \left[C_{mnn'} - \frac{1}{2} i I_{mnn'} - \frac{2d}{\pi a} \delta_{0,m} S_n S_{n'} \right].$$

Note that $(A_{mnn'} x_{mn} / x_{mn'})$ is symmetric.

4.2. Numerical method

The numerical approach adopted is one in which use is made of the shape the final equation has in matrix form, namely that of an eigenvalue problem. If $A_m(k)$ is independent of k , the eigenvalues μ_1, μ_2, \dots of this A_m would yield the solutions $k_j = c_M / c_a \mu_j^{1/2}$. This observation suggests the iterative scheme

1. calculate eigenvalues (μ_j) of $A_m(k_n)$
2. calculate the corresponding $\bar{k}_j = c_M / c_a \mu_j^{1/2}$
3. select the $j = j_0$ where $|k_n - \bar{k}_{j_0}|$ is minimal, and set $k_{n+1} = \bar{k}_{j_0}$
4. return to 1 until $|k_n - k_{n+1}|$ is small enough.

To control the iteration, a relaxation parameter $\eta \in [0, 1)$ is introduced: $k_{n+1} = \eta k_n + (1 - \eta) \bar{k}_{j_0}$.

To accelerate the convergence, the present iteration is written as an Atkinson iteration ([11]), giving the terms every other iteration an extra correction by extrapolation:

$$k_{n+2} := k_n - (k_{n+1} - k_n)^2 / (k_{n+2} - 2k_{n+1} + k_n).$$

Since the calculation of the integral $I_{mnn'}$ is particularly expensive, these integrals are not recalculated every iteration, but only every fourth step, and at each other step the matrix elements are only updated approximately. For the same reason, the integral $I_{mnn'}$ is analysed carefully (the results of which are presented in the next paragraph) so that, together with the saving of the expensive Besselfunction evaluations, the integrals $I_{mnn'}$ are calculated efficiently.

For the calculation of the eigenvalues use is made of the public domain package "EISPACK". Theory and description of in- and output may be found in [12,13].

4.3. Analysis of $I_{mnn'}$

A direct numerical integration of

$$I_{mnn'} = \int_0^{\infty} \frac{\lambda J_m^2(\lambda)}{\gamma(\lambda) (\lambda^2 - x_{mn}^2) (\lambda^2 - x_{mn'}^2)} d\lambda$$

requires a relatively large interval of integration, especially when the index n or n' is large. To facilitate the numerical integration we therefore propose the following reformulation.

Observe that

$$J_n(\lambda) = \frac{1}{2} (H_n^{(1)}(\lambda) + H_n^{(2)}(\lambda))$$

$J_n(\lambda) H_n^{(\cdot)}(\lambda)$ is integrable in $\lambda = 0$

$$H_n^{(1)}(\lambda) \sim e^{i\lambda} \quad (\lambda \rightarrow \infty)$$

$$H_n^{(2)}(\lambda) \sim e^{-i\lambda} \quad (\lambda \rightarrow \infty)$$

$$H_n^{(1)}(it) = \frac{2}{\pi} (-i)^{n+1} K_n(t) \quad (t > 0)$$

$$H_n^{(2)}(-it) = \frac{2}{\pi} i^{n+1} K_n(t) \quad (t > 0)$$

$$J_n(it) = i^n I_n(t)$$

where $H_n^{(1)}$ and $H_n^{(2)}$ are Hankel functions, and I_n and K_n are modified Besselfunctions of the first and second kind ([10]).

We split up the integrand in a part convergent in the upper and one convergent in the lower complex half plane, and deform the contours of integration accordingly, taking into account the branch cut of γ , and a pole in $\lambda = x_{mn}$ if $n = n'$.

$$\begin{aligned} I_{mnn'} &= \int_0^{\infty} \frac{1}{2} \frac{\lambda J_m(\lambda) (H_n^{(1)}(\lambda) + H_n^{(2)}(\lambda))}{\gamma(\lambda) (\lambda^2 - x_{mn}^2) (\lambda^2 - x_{mn'}^2)} d\lambda \\ &= -\frac{2i}{\pi} \int_1^{\infty} \frac{t I_n(t) K_n(t)}{\sqrt{t^2 + k^2} (t^2 + x_{mn}^2) (t^2 + x_{mn'}^2)} dt \\ &\quad + \int_i^k \frac{\lambda J_n(\lambda) H_n^{(1)}(\lambda)}{\gamma(\lambda) (\lambda^2 - x_{mn}^2) (\lambda^2 - x_{mn'}^2)} d\lambda \\ &\quad + \frac{\delta_{n,n'}}{2 x_{mn}^2 \gamma(x_{mn})}. \end{aligned}$$

The integration contour, deformed into the upper half plane, runs from $\lambda = 0$ to $\lambda = i$, then back to $\lambda = k$, to $\lambda = i$, and further to $\lambda = i\infty$ (see figure 3). The point $\lambda = i$ is rather arbitrarily selected such, that the contour will not be close to a singularity if $x_{mn} < \text{Re}(k)$.

After having noted that

$$t I_n(t) K_n(t) \rightarrow \frac{1}{2} \quad (t \rightarrow \infty)$$

we may transform the t^{-5} -behaviour at infinity of the t -integral to linear behaviour near the origin by: $t = z^{-1/2}$.

Finally, we can remove the square root singularity in $\lambda = k$.

$$\int_i^k \frac{\lambda J_n(\lambda) H_n^{(1)}(\lambda)}{\gamma(\lambda) (\lambda^2 - x_{mn}^2) (\lambda^2 - x_{mn'}^2)} d\lambda = \int_i^{\operatorname{Re}(k)} + \int_{\operatorname{Re}(k)}^k (id.) d\lambda$$

and each integral can be written as

$$\frac{2}{\pi} (\Theta_1 - \Theta_0) \int_0^1 \frac{G I_n(G) K_n(G)}{(G^2 + x_{mn}^2) (G^2 + x_{mn'}^2)} dh$$

with

$$\lambda = i G = k \sin \Theta, \text{ so that } \gamma(\lambda) = k \cos \Theta,$$

$$\Theta = \Theta_0 + h(\Theta_1 - \Theta_0).$$

5. Numerical examples

To illustrate the present theory, we calculated for a typical kettledrum the spectrum as a function of the problem parameters tension, volume, and hole diameter. These examples are primarily meant for illustration, and to visualize trends and coupling mechanisms.

The parameter values for cavity, membrane, and air are, to ease a comparison, the same as in [6], and given by

$$\begin{aligned} \text{membrane diameter} &= 2a = 0.656 \text{ m} \\ \text{membrane density} &= \sigma = 0.2653 \text{ kg/m}^2 \\ \text{membrane tension} &= T = 3990 \text{ N/m} \\ \text{kettle volume} &= \pi a^2 L = 0.14 \text{ m}^3 \\ \text{air density} &= \rho_a = 1.21 \text{ kg/m}^3 \\ \text{air sound speed} &= c_a = 344.0 \text{ m/s} \\ \text{hole diameter} &= 2d = 0.028 \text{ m} \end{aligned}$$

(We note that we found a discrepancy with [6] with respect to the vacuum modes, so, assuming a printing error, we changed the reported $\sigma = 0.262$ into 0.2653.)

These figures yield

$$L = 0.4142 \text{ m}$$

$$c_M = 122.6 \text{ m/s}$$

vacuum modes frequencies

m	n	[Hz]
0	1	143.10
0	2	328.48
0	3	514.95
0	4	701.67
1	1	228.01
1	2	417.47
1	3	605.39
2	1	305.60
2	2	500.88
2	3	691.46
3	1	379.66
3	2	580.84
4	1	451.56
4	2	658.42

5	1	521.96
6	1	591.26
7	1	659.71

cavity modes frequencies

$m = 0$:	415.2
		639.6
$m = 1$:	307.3
		516.6
$m = 2$:	509.8
		657.5
$m = 3$:	701.3

Guided by these "ideal" values we found (with a matrix dimension of 6×6) the following eigen-frequencies

$m = 0$:	126.883 Hz ,	$t_{60} =$	0.787 sec.
		252.315		1.307
		415.298		2.971
		476.693		0.341
		614.132		2.524
$m = 1$:	150.918		9.101
		311.116		326.888
		351.415		1.392
		507.574		64.225
		566.704		0.470
$m = 2$:	227.704		27.980
		412.031		6.191
		524.376		5.422
		600.767		4.822
		690.786		0.880
$m = 3$:	300.174		101.227
		492.940		10.522
		678.371		11.370
$m = 4$:	370.642		385.520
		570.712		23.807
$m = 5$:	439.826		1506.148
		646.494		61.380
$m = 6$:	508.096		1425.619

$$m = 7 \quad : \quad 575.721 \qquad 2859.600$$

We recall that the frequency in Hertz is $\text{Re}(\omega/2\pi) = \text{Re}(k c_a/2\pi a)$, while t_{60} , the time necessary for the sound pressure level to be attenuated by 60 dB $= -20 \log_{10}(e^{\text{Im}(\omega)t_{60}})$, is given by

$$t_{60} = -3 / \text{Im}(\omega) \log_{10}(e).$$

The very long decay times of some modes are of course due to the idealization of the model: inviscid air without any dissipation from humidity, no absorbing walls, ideal membrane, etcetera. It may be observed that these long decay times mostly occur for higher m -modes, where it is the inefficient radiation of the membrane which holds the vibrational energy in the system (the membrane is for not too high frequencies acoustically equivalent to some high order multipole). The long decay times occurring for lower m -modes indicate a mode close to a cavity resonance mode. In this case the energy is trapped in the cavity with only little motion of the membrane.

In figures 4a–i we have plotted the present spectrum for respectively $m = 0$ to 7 with tension varying between 3000 N/m and 5000 N/m, which are realistic values. We see that in general the frequencies of modes close to a vacuum mode increase steadily with tension. A mode close to a cavity mode, however, is somewhat reluctant to increase. It seems to be locked until the tension is high enough to drag it away from this cavity mode level, and another mode from below takes its place.

This phenomenon is further worked out in figure 5, where for $m = 1$ the modes are traced along a very much larger tension range, so that the various levels corresponding to cavity modes are clearly distinguishable.

In the case of a varying volume (figure 6) we see the opposite effect: now the cavity modes are directly affected by the change of volume, and so are the corresponding frequencies of our problem, and the other frequencies remain longer near the vacuum mode frequencies.

In figure 7 we see the effect of the hole in the bottom. It is clear that, at least for the present configuration, the difference from the closed bottom case is very small.

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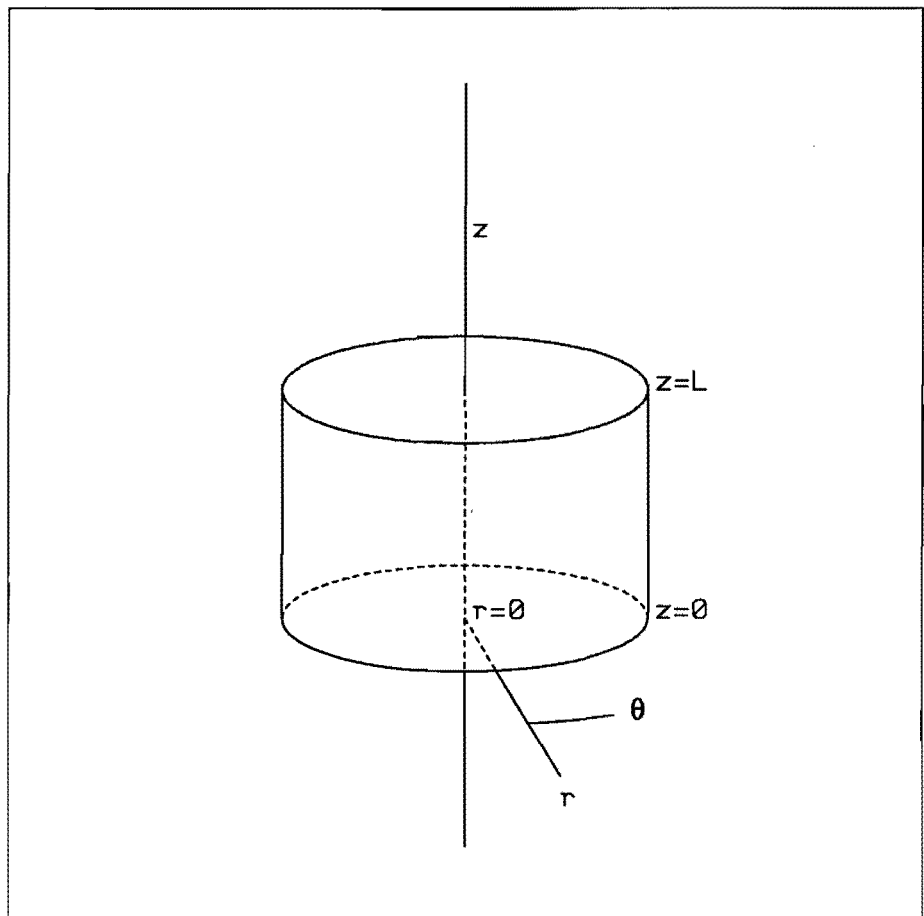


Figure 1. Sketch of geometry and coordinate system

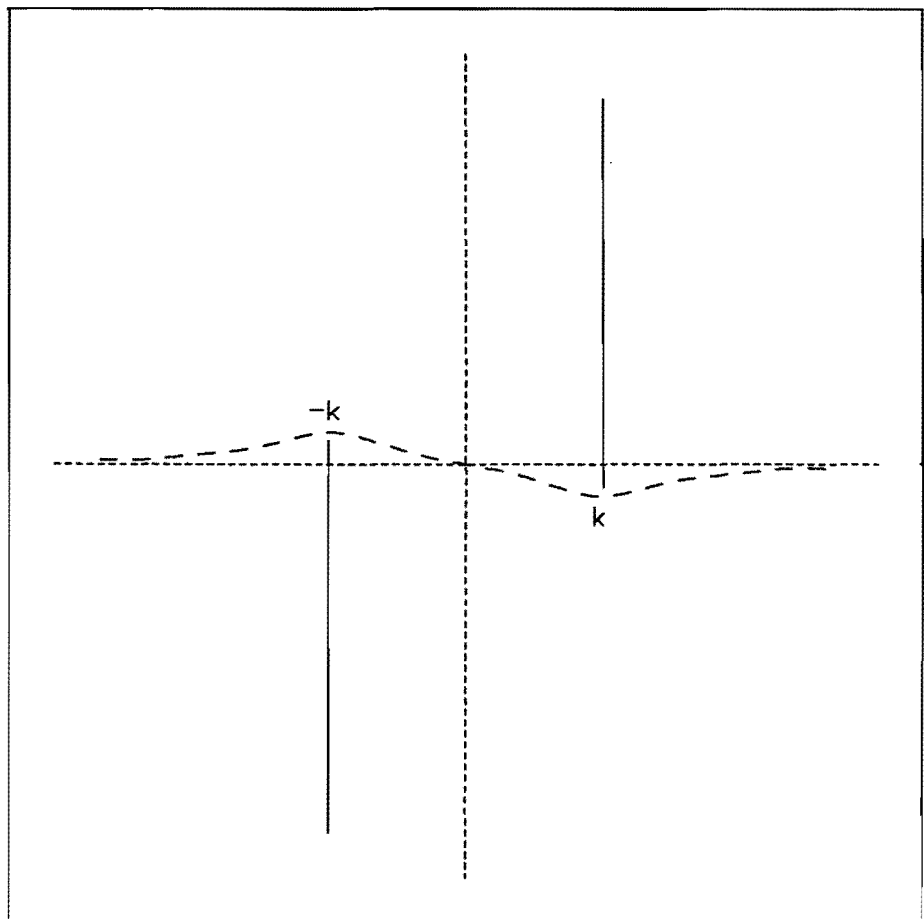


Figure 2. Complex plane with branch cuts and integration contour

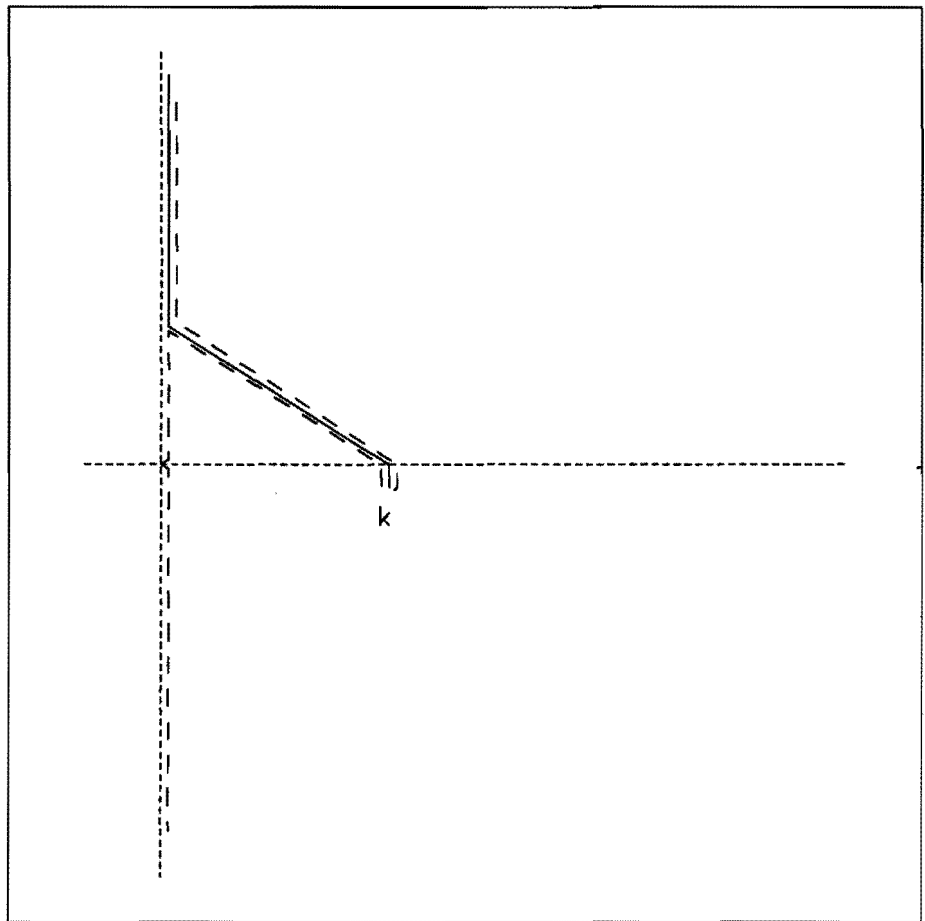


Figure 3. Complex plane with deformed branch cut and modified integration contours

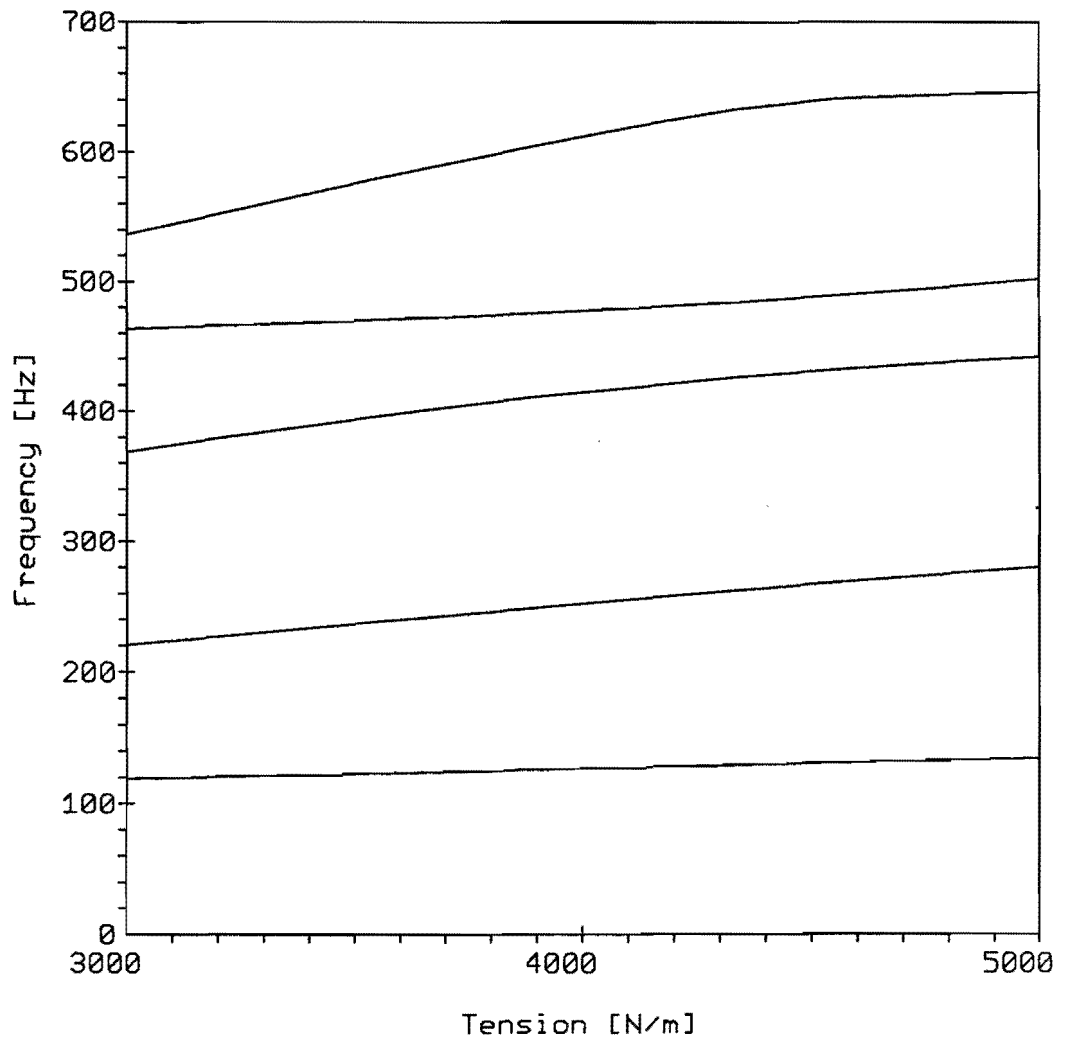


Figure 4a. $m = 0$ modes

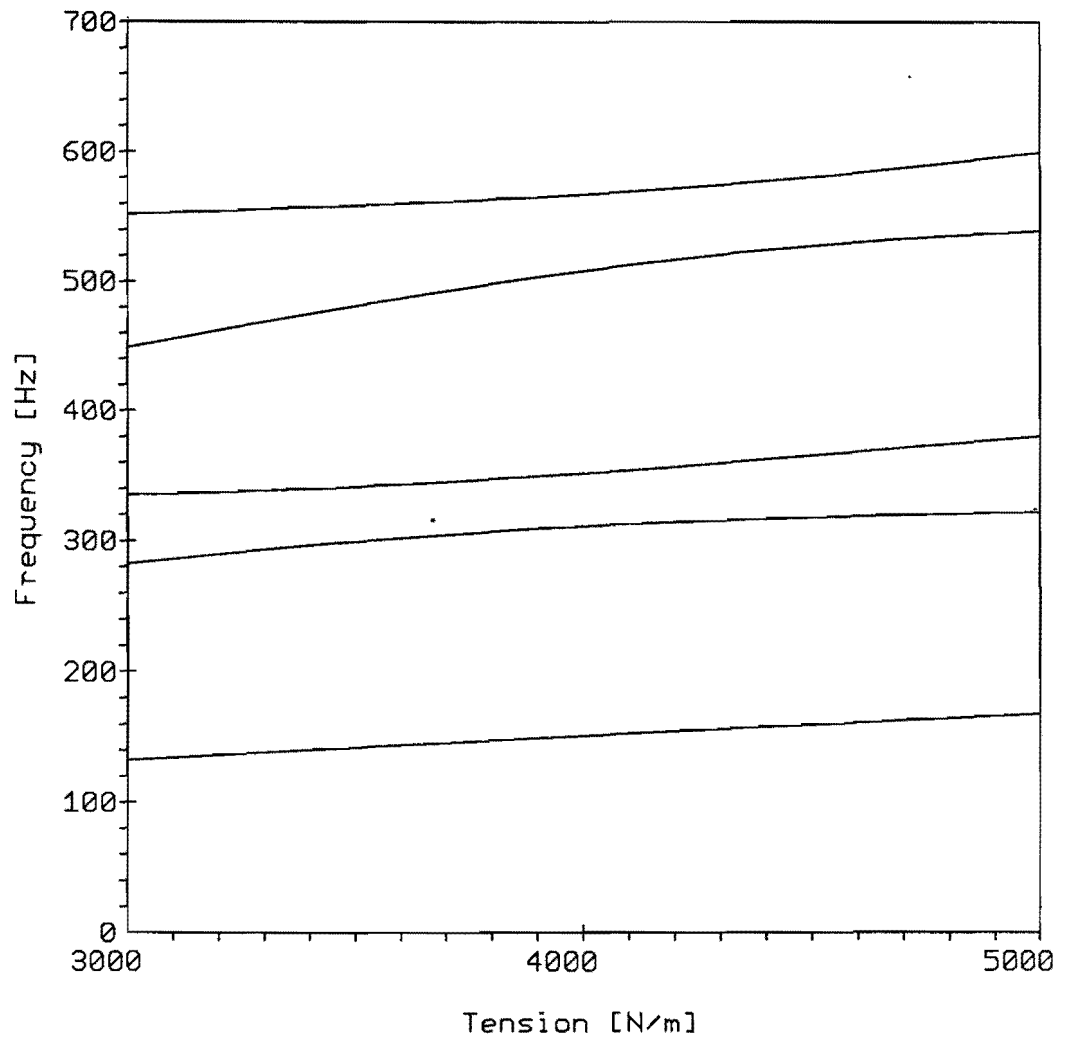


Figure 4b. $m = 1$ modes

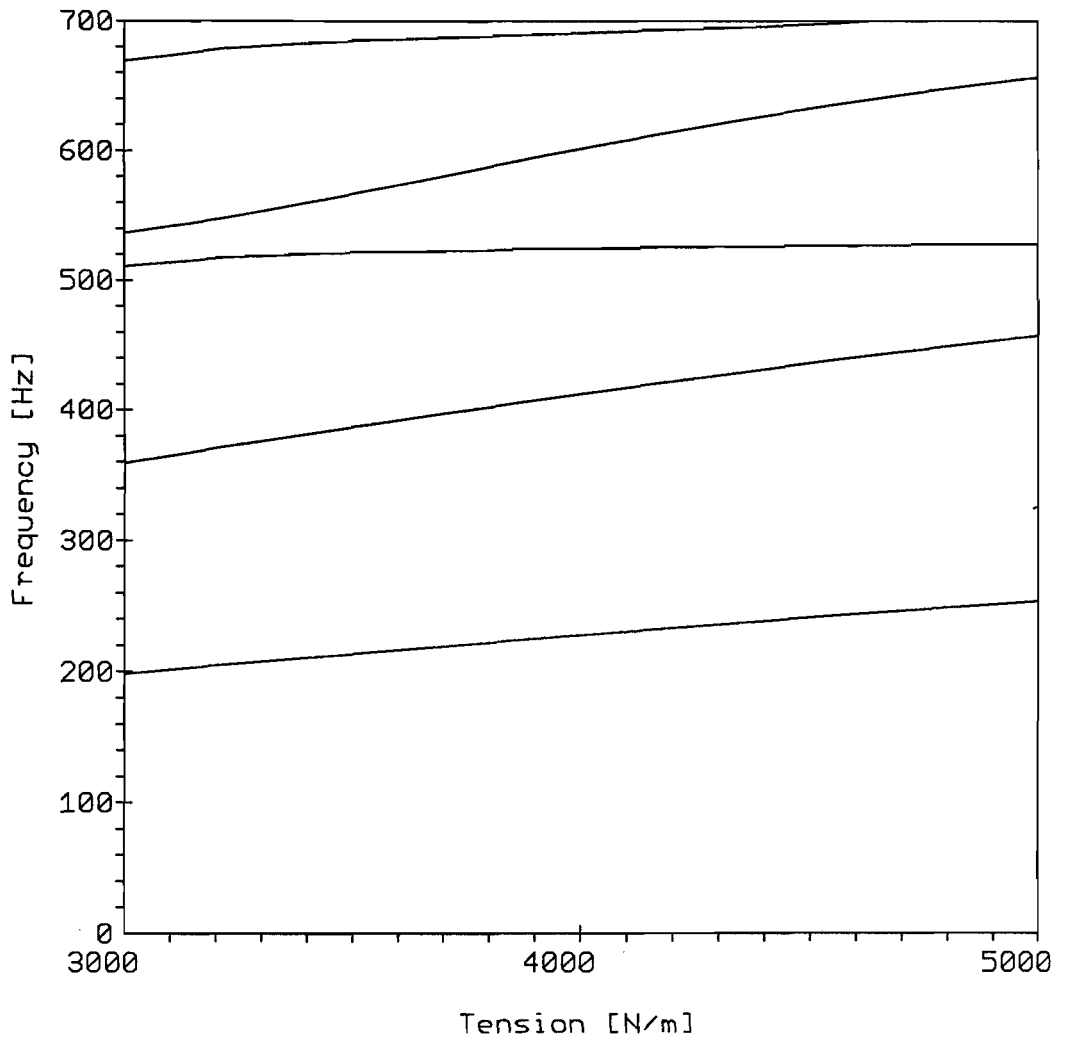


Figure 4c. $m = 2$ modes

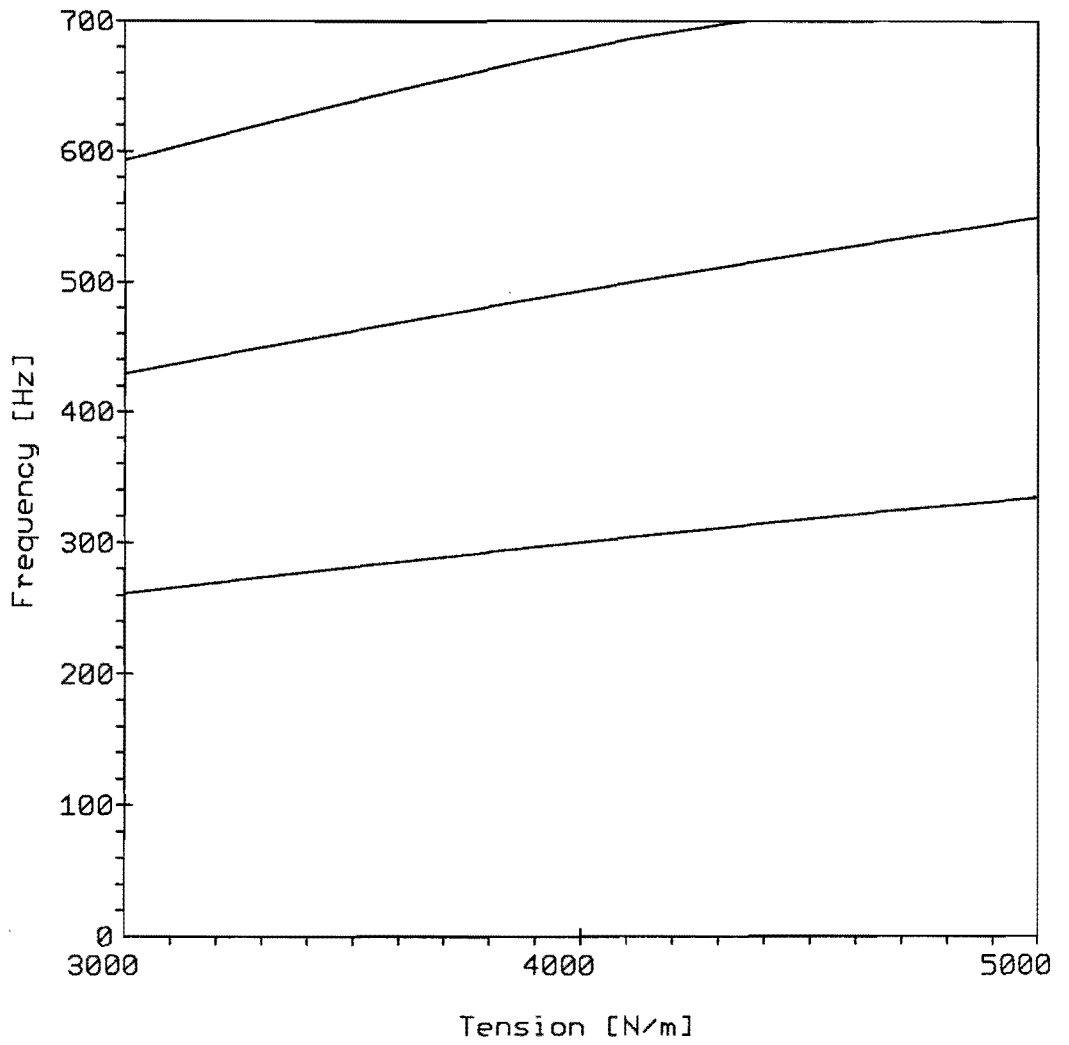


Figure 4d. m = 3 modes

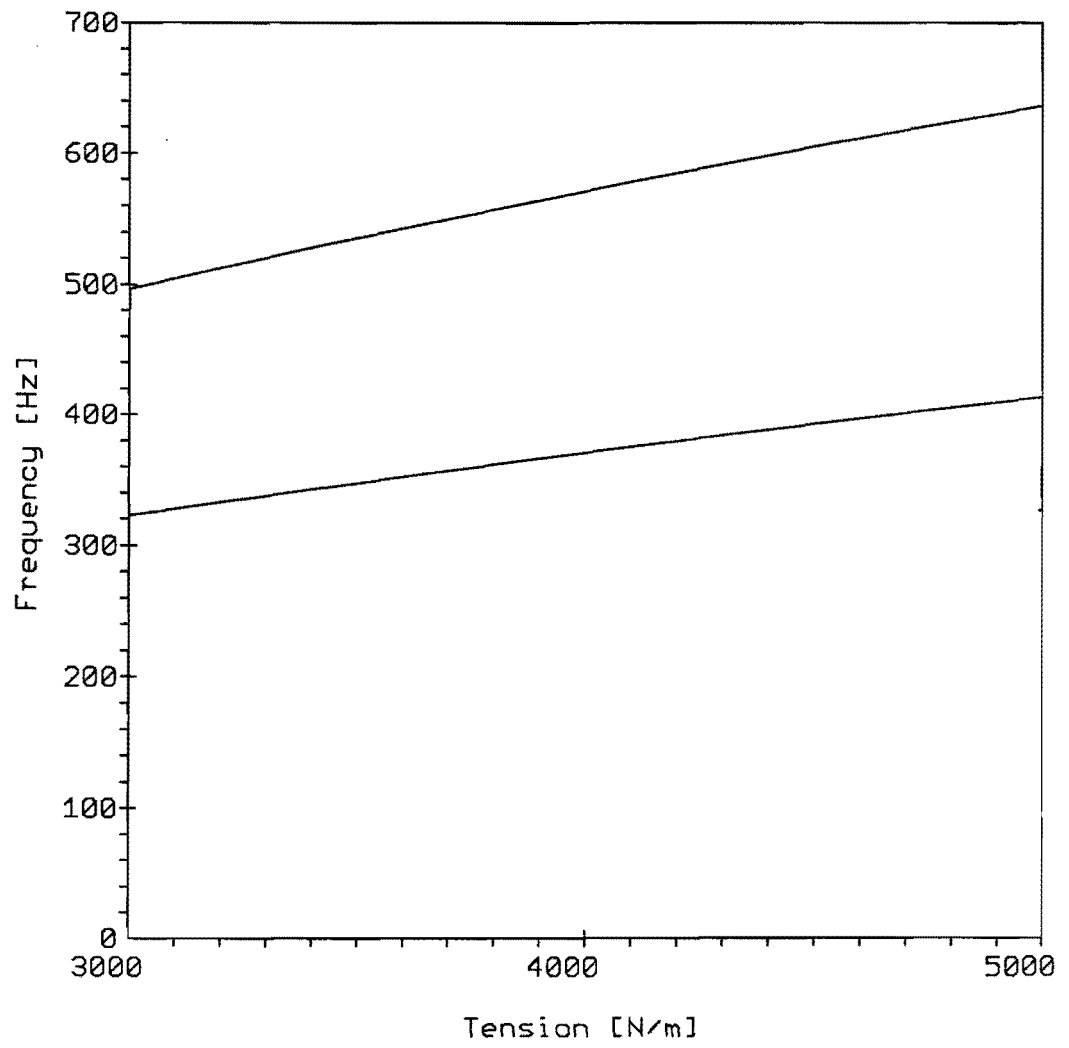


Figure 4e. m = 4 modes

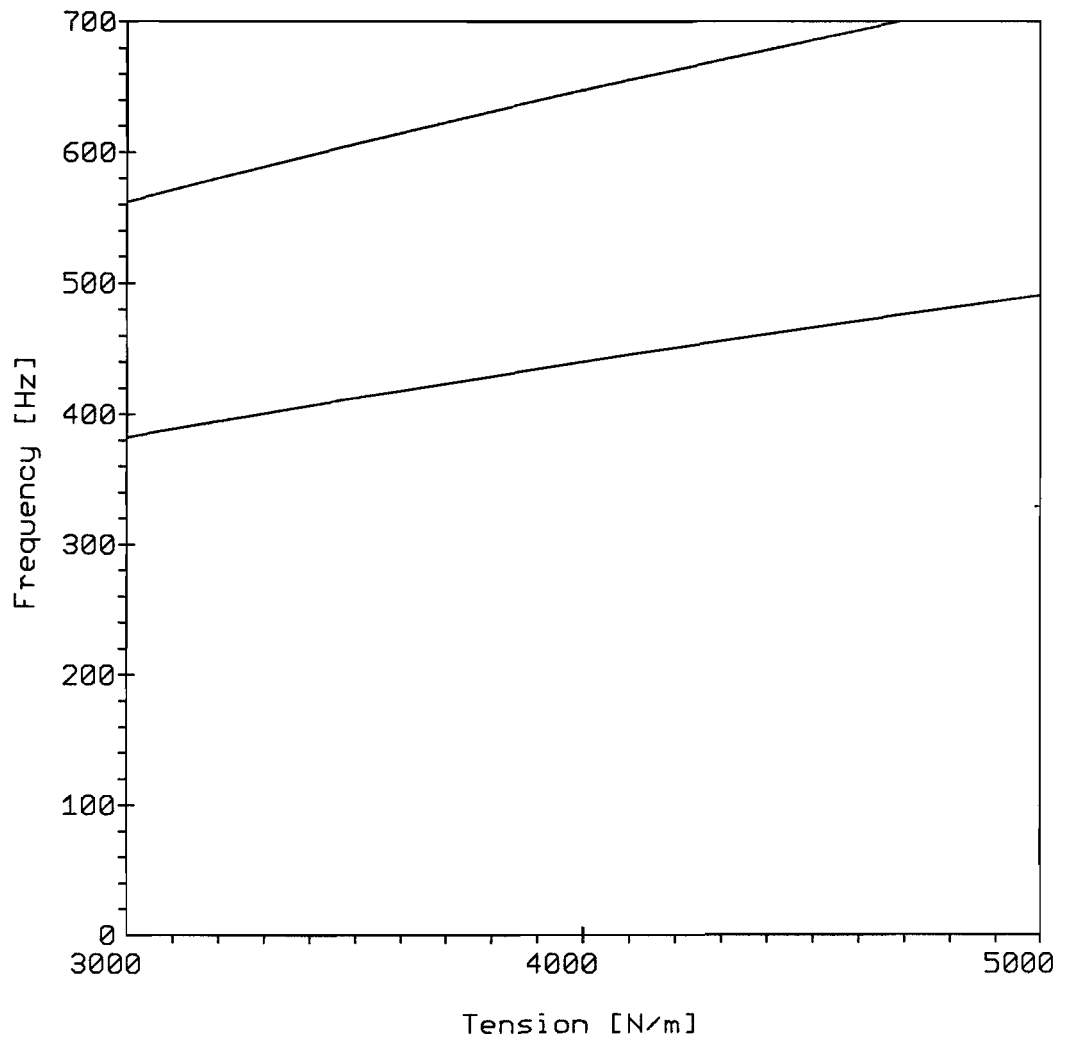


Figure 4f. m = 5 modes

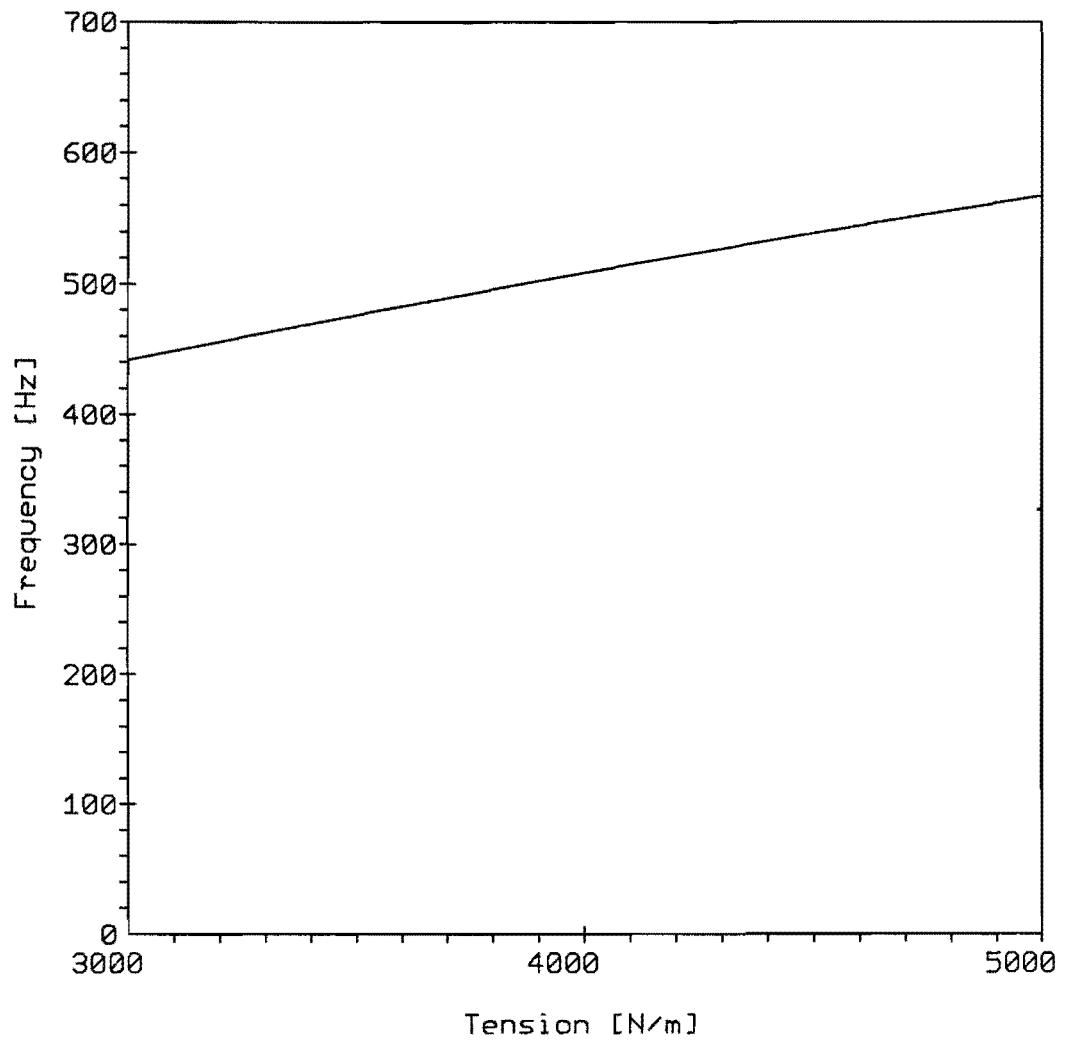


Figure 4g. m = 6 modes

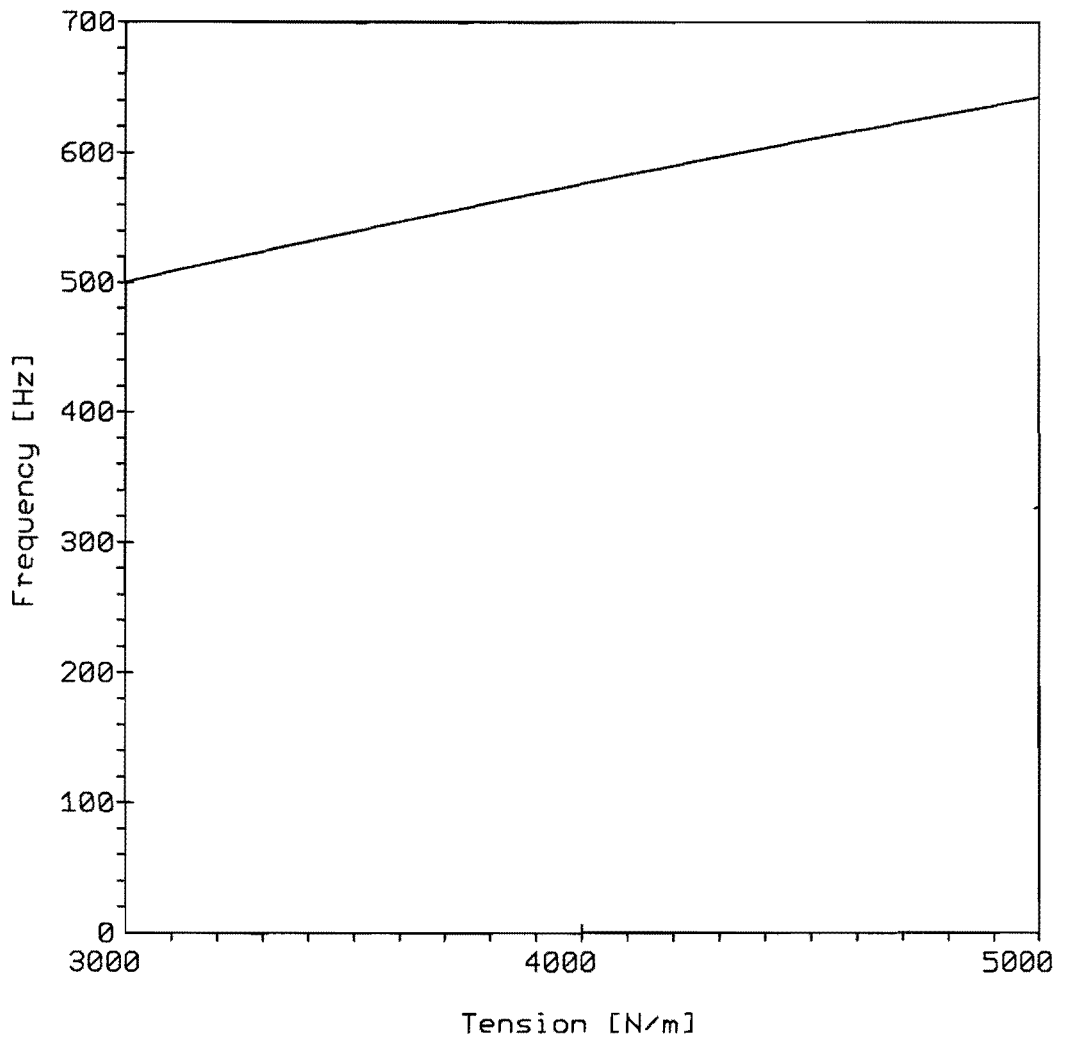


Figure 4h. $m = 7$ modes

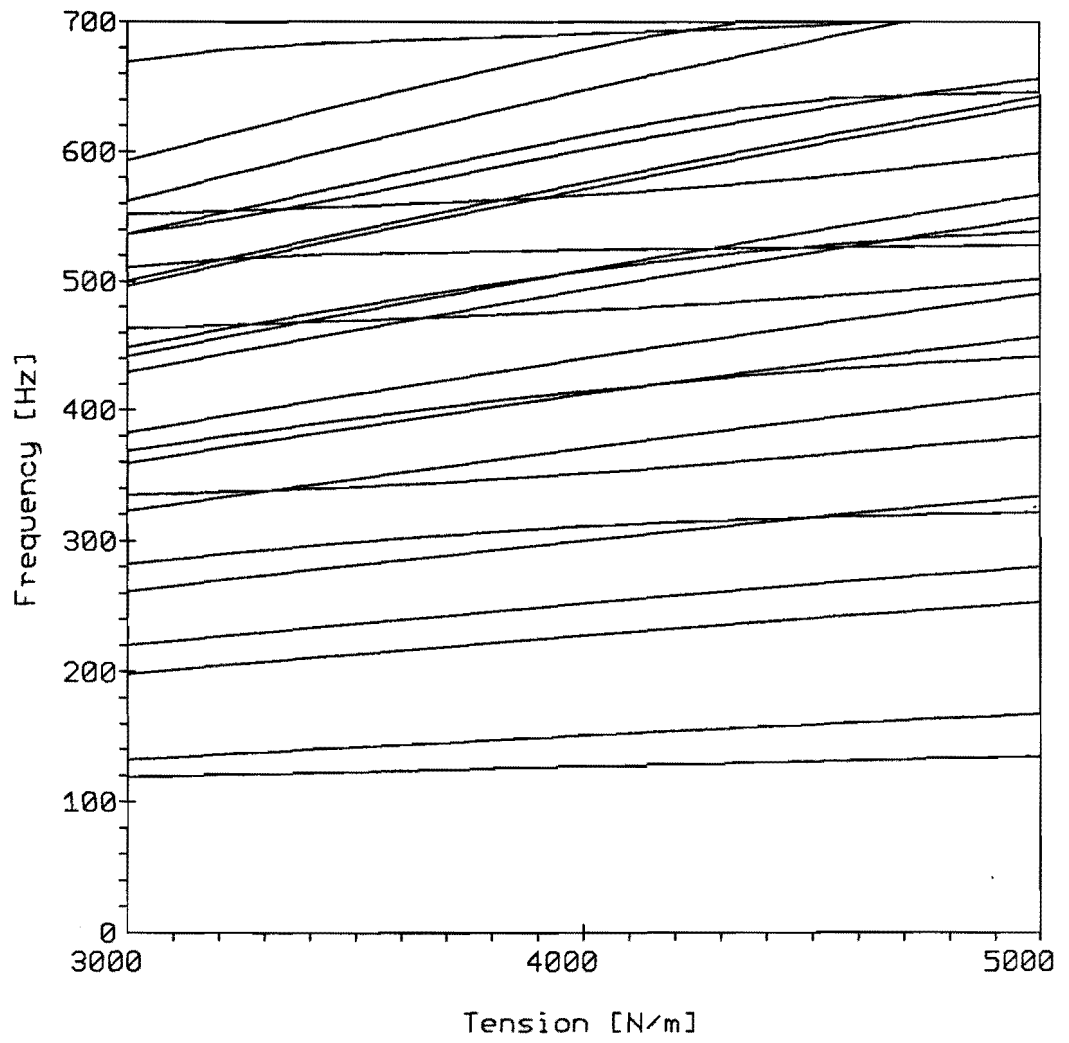


Figure 4i. m = 0-7 modes

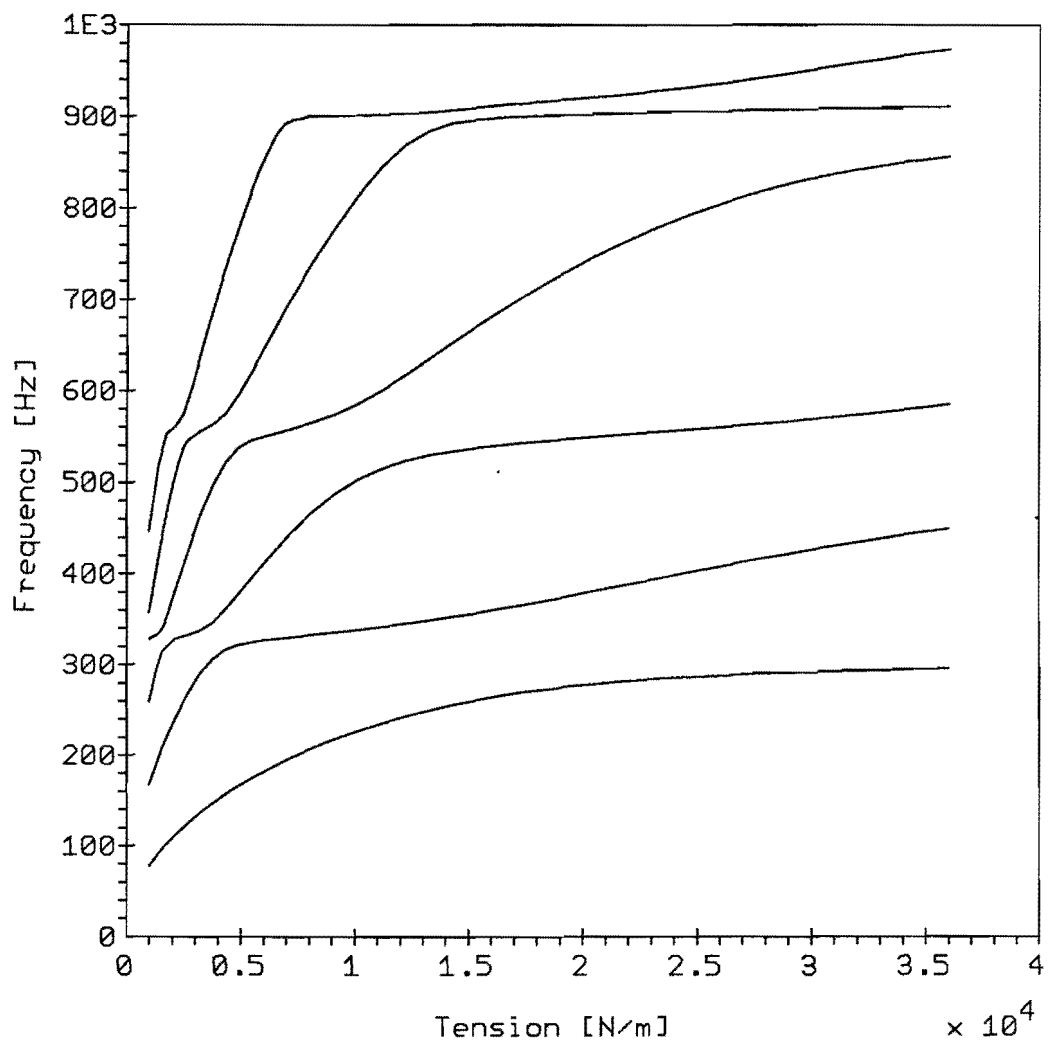


Figure 5. $m = 1$ modes

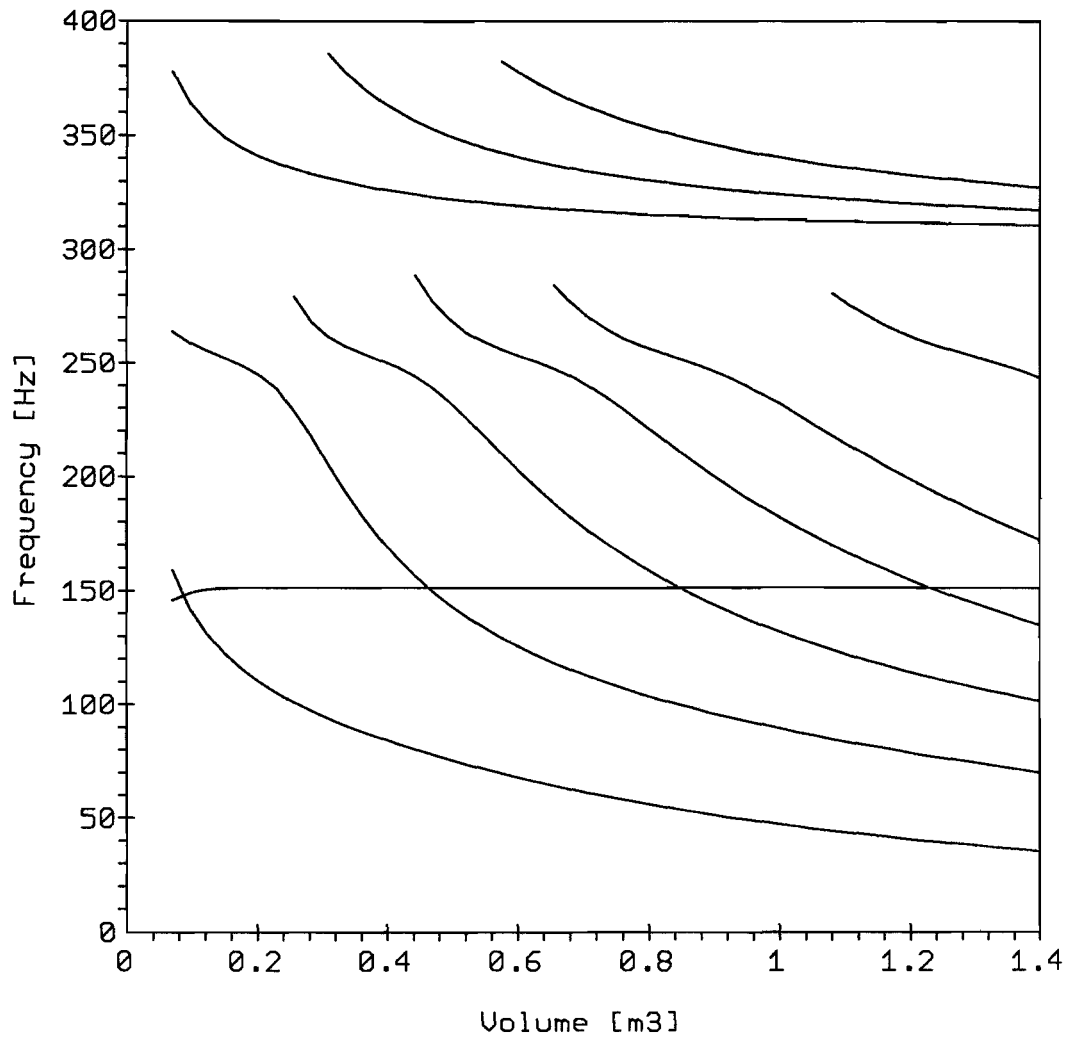


Figure 6. m = 0-1 modes

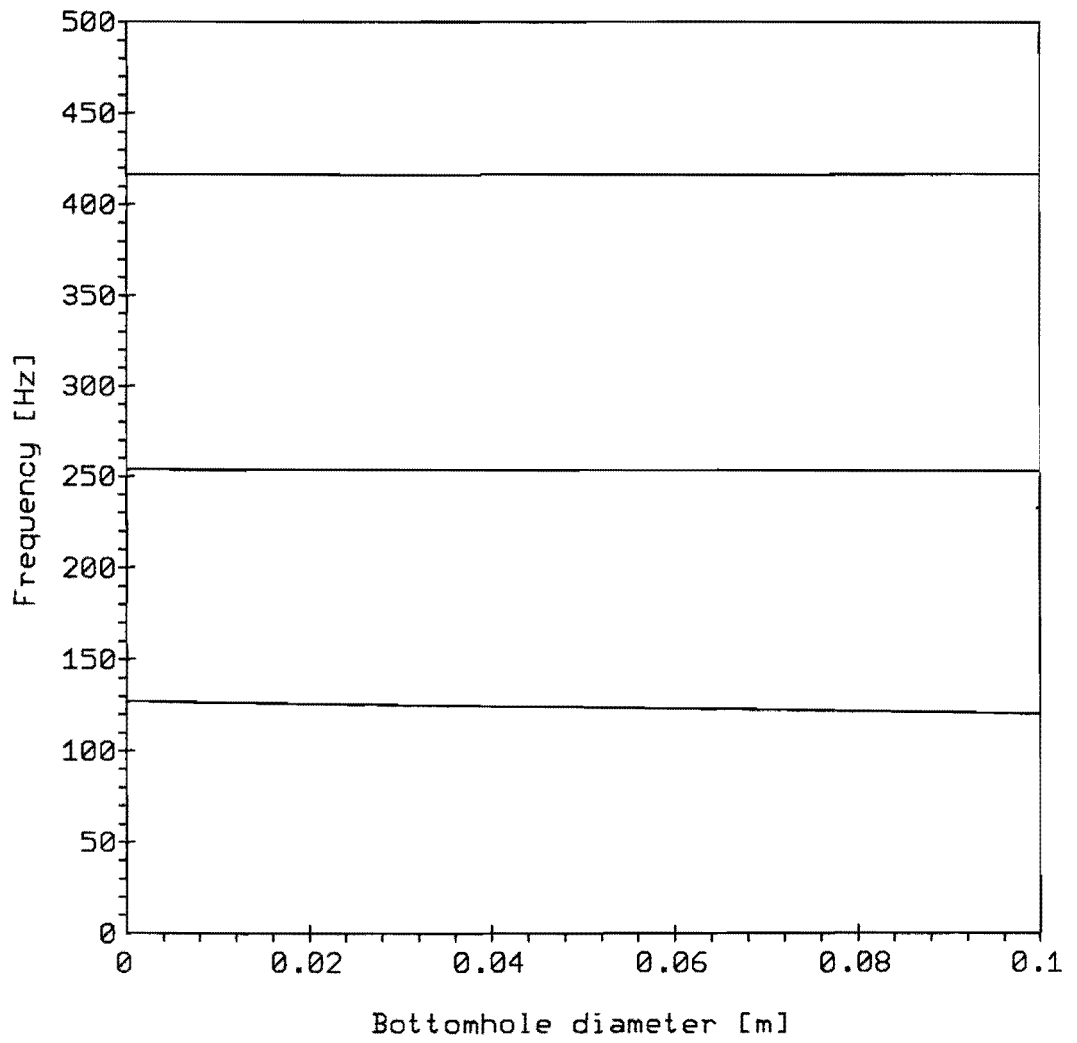


Figure 7. $m = 0$ modes