Analyticity spaces, trajectory spaces, and linear mappings between them

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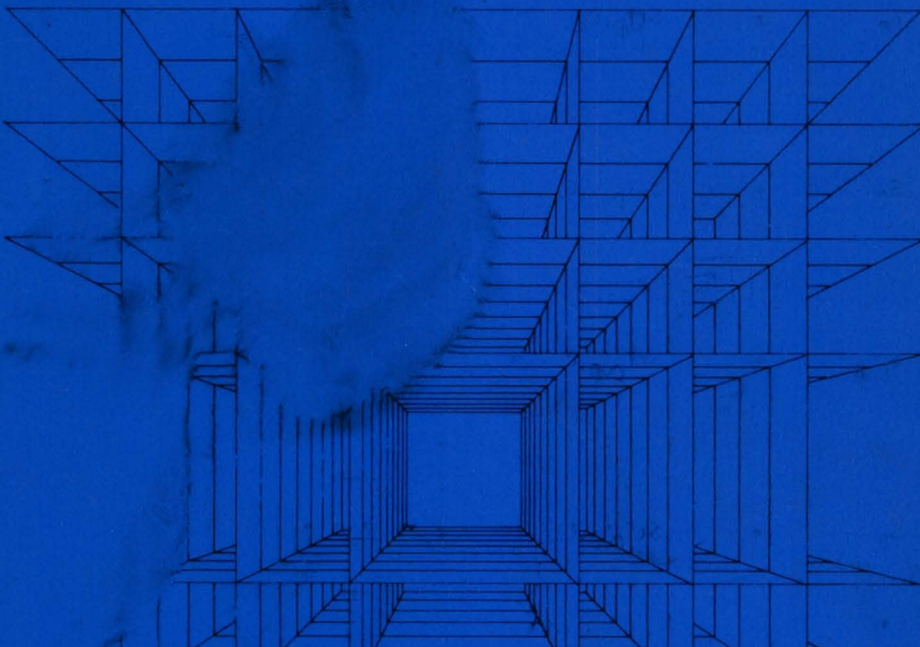

s.j.l. van eijndhoven

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PROEFSCHRIFT

TER VERKRIJGING VAN DE GRAAD VAN DOCTOR IN DE TECHNISCHE WETENSCHAPPEN AAN DE TECHNISCHE HOGESCHOOL EINDHOVEN, OP GEZAG VAN DE RECTOR MAGNIFICUS, PROF.DR. S.T.M. ACKERMANS, VOOR EEN COMMISSIE AANGEWEZEN DOOR HET COLLEGE VAN DEKANEN IN HET OPENBAAR TE VERDEDIGEN OP DINSDAG 24 MEI 1983 TE 16.00 UUR

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stephanus jacobus louis van eijndhoven

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door de promotoren

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en

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CONTENTS

PROLOGUE

The introduction of generalized functions has considerably advanced mathematical analysis, in particular harmonic analysis and the theory of partial differential equations. In a non-rigorous way, electrical engineers and physicists have been using generalized functions for almost a century. But it took some time before mathematical justification of the use of improper functions such as the Heaviside step function and the Dirac delta function has been taken up.

The first mathematical concepts which started up a theory of generalized functions were the finite parts of divergent integrals used by Hadamard and the Riemann-Liouville integrals due to Riesz. Later Sobolev defined generalized derivatives by means of integration by parts, and Bochner developed the theory of the Fourier transform for functions increasing as some power of their argument. Many of these results were unified by Schwartz in his monograph Théorie des Distributions. Here the unifying concept is the notion of locally convex topological vector space. Generalized functions (distributions) are continuous linear functionals on such spaces of well behaved functions.

Later on, also Gelfand and Shilov defined many classes of generalized functions. But more importantly, they showed how to use generalized functions in mathematical analysis. It turned out that generalized functions connect many aspects of analysis, of functional analysis, of the theory of partial differential equations and of the representation theory of locally compact Lie groups.

Thus, generalized functions have gained wide popularity among mathematicians.

The theories of Schwartz and of Gelfand-Shilov can be described as follows. One starts with a vector space S of 'good' functions for instance the set $\mathcal{D}$ of infinitely differentiable functions with compact support or the set S of infinitely differentiable functions of rapid decrease. This vector space is called the test space. The test space S carries a suitable Hausdorff topology which makes S into a locally convex, topological vector space. The choice of the topology is not arbitrary; an extra condition will be imposed. A generalized function is a continuous linear functional on S. Equivalently, the space of generalized functions is the topological dual S' of S. Thus the space of generalized functions gains a natural weak topology. To justify the name generalized function we construct a space S^* that can be identified with S' and contains S. Therefore, let X be a Hilbert space (e.g. $L_2(\mathbb{R})$ or a Sobolev space) such that S is a dense subspace of X and such that the embedding of S in X is continuous. Then by means of the inner product of X, the subspace S of X induces the weak Hausdorff topology $\sigma(X,S)$ on X. Next, one considers the sequential completion S^* of X with this topology. The mentioned extra condition one has to impose on the topology of S is the following: each member of S' can be represented by an element of S^* by means of the canonical pairing of S and S^*. So S' and S^* can be identified. Since $S \subset X \subset S^*$ and since the members of S are functions, S^*, and hence S' can be regarded as a space of improper functions. Thus, $\mathcal{D}'$ can be interpreted as a space of improper functions which are derivatives of some order of continuous functions on the real line.

Even Lighthill's more classical approach can be described in this functional analytic set up. One considers so-called regular sequences in S which converge in a weak sense. It turns out that a sequence is regular if it converges in $\sigma(X,S)$. Two regular sequences are equivalent if the difference of these sequences is a null-sequence in $\sigma(X,S)$. A generalized function in the sense of Lighthill is just an equivalence class of regular sequences. So the theory based on the triplet $S \subset X \subset S^*$ and the theory based on regular sequences are equivalent.

In an inspiring paper [B], De Bruijn proposed a new theory of generalized functions, which was developed further in Janssen's thesis [J]. In [B] three kinds of functions occur: smooth functions, smoothed functions and generalized functions. A function is said to be smooth if it belongs to Gelfand-Shilov's space $S_{\frac{1}{2}}^{\frac{1}{2}}$, a special class of entire functions. A smoothed function f is derived from a smooth function g by application to g of an operator from a set of smoothing operators. The set of smoothing operators is a one-parameter semigroup denoted by $(N_\alpha)_{\alpha>0}$. De Bruijn proved that each smooth function is smoothed and that each smoothed function is smooth.

Now, a generalized function is a mapping F from $(0,\infty)$ into the set of smooth functions that satisfies $N_\alpha F(\beta) = F(\alpha+\beta)$ for all positive α and β. Although De Bruijn establishes a pairing between the spaces of smoothed functions and of generalized functions, no topologies are introduced for these spaces and questions about duality and continuity of linear mappings can be linked to sequential convergence only.

In [G], De Graaf generalizes De Bruijn's theory considerably by treating it on a functional analytic level. The paper [G] contains a theory of the two types of topological vector spaces $S_{X,A}$ and $T_{X,A}$ which are generated by a holomorphic semigroup with infinitesimal generator A in the Hilbert space X. In this thesis $S_{X,A}$ will be called an analyticity space and $T_{X,A}$ a trajectory space. If we take a suitable operator A in a Hilbert space $X = L_2(M,\mu)$, the trajectory space $T_{X,A}$ contains generalized functions on the measure space M.

The space $S_{X,A}$ is an inductive limit. This inductive limit is non-strict. So the general theory on inductive limits, which assumes strictness, can not be applied. In my opinion, the main feature in [G] is the introduction of the function algebra $\mathcal{B}(\mathbb{R})$. Each element of $\mathcal{B}(\mathbb{R})$ agrees with a seminorm on $S_{X,A}$. Together these seminorms generate the inductive limit topology. This important observation has led to complete characterizations of null sequences, of bounded subsets and of compact subsets of $S_{X,A}$ just as for strict inductive limits. Furthermore, large pieces of Hilbert space theory can be inserted into the theory. For instance, in [G] this has led to a detailed exposition of continuous linear mappings, of topological tensor products and of so-called Kernel theorems, all with respect to analyticity spaces and trajectory spaces. Considerations of this type are not current in distribution theory.

The main source of inspiration for the present work has been the systematic functional analytic approach in [G] to continuous linear mappings, which is absent in other distribution theories. During the research, we got the firm expectation that more, interesting results would be obtained by applying

Hilbert space techniques as already mentioned. This became a second motive for this thesis. Furthermore, any theory of generalized functions should contain some spectral theory. It should tell whether continuous self-adjoint operators on an analyticity space $S_{X,A}$ admit generalized eigenfunctions in $T_{X,A}$. Finally, we have had the ambition to interprete parts of the formalism of quantum theory in terms of analyticity spaces and trajectory spaces because in such an interpretation these spaces seem more appropriate than Hilbert spaces.

Summarized, motivation for this thesis has been the wish to develop the purely functional analytic theory [G], to translate various concepts of classical distribution theory into the language of [G] and to give a mathematical interpretation of some quantum physics.

The second part of this prologue is devoted to a short survey of the contents of this thesis.

For a nonnegative, self-adjoint operator A in a Hilbert space X the analyticity space $S_{X,A}$ is the dense subspace of X defined by

$$S_{X,A} = \bigcup_{t>0} e^{-tA}(X) \ .$$

On $S_{X,A}$ a non-strict inductive limit topology is imposed. The trajectory space $T_{X,A}$ consists of all mappings $F : (0,\infty) \to X$ which satisfy

$$\forall_{t>0} \forall_{\tau>0} : F(t+\tau) = e^{-\tau A} F(t) \ .$$

Examples of such trajectories are $t \mapsto A^m e^{-tA} x$ with $x \in X$ and $m \geq 0$. A suitable choice of seminorms turns $T_{X,A}$ into a Frêchet space. The Hilbert space X is embedded in $T_{X,A}$ by means of the mapping emb : $X \to T_{X,A}$ given by

$$\text{emb}(\omega) : t \mapsto e^{-tA}\omega \quad , \quad \omega \in X, \; t > 0 .$$

Thus we obtaine the triplet $S_{X,A} \subset X \subset T_{X,A}$.

It is clear that for each $f \in S_{X,A}$ there exists $\tau > 0$ such that $e^{\tau A} f \in X$. So it makes sense to define a pairing between $S_{X,A}$ and $T_{X,A}$ as follows,

$$\langle f,G\rangle = (e^{\tau A} f, G(\tau)) \quad , \quad f \in S_{X,A}, \; G \in T_{X,A}$$

with $(\cdot,\cdot)$ the usual inner product in X. Due to the trajectory property of the elements of $T_{X,A}$, the definition of $\langle\cdot,\cdot\rangle$ does not depend on the choice of $\tau > 0$. With this pairing the spaces $S_{X,A}$ and $T_{X,A}$ can be seen as each other's strong dual spaces.

The theory on the spaces $S_{X,A}$ and $T_{X,A}$ forms a functional analytic description of a new kind of distribution theory. If $X = L_2(M,\mu)$ for some measure space M, then $T_{X,A}$ consists of improper functions on M.

The paper [G] contains a detailed discussion of several topological features of analyticity and trajectory spaces, and of the duality between them. Moreover, it contains a detailed discussion of continuous linear mappings, which is new in distribution theory. In [G] five types of morphisms are discussed and also four Kernel theorems. A Kernel theorem gives conditions such that all continuous linear mappings arise from the elements (kernels) out of a suitable topological tensor product.

In Chapter one of this thesis we shall summarize the results in De Graaf's paper. In addition, this chapter contains some examples of analyticity spaces, which can be characterized in classical analytic terms. Further, we discuss a relation between representation theory of Lie groups and the theory presented here.

In order to obtain the appropriate topological tensor product of the spaces $S_{X,A}$ and $T_{Y,B}$ and of the spaces $T_{X,A}$ and $S_{Y,B}$, the spaces Σ'_A and Σ'_B are brought up in [G]. In Chapter two we shall shed more light on these rather obscure spaces. With the introduction of two new types of analyticity/trajectory spaces, we obtain a unifying approach to all spaces which occur in [G]. It is possible to describe the intersection of Σ'_A and Σ'_B in terms of these new spaces. This description leads to a Kernel theorem for the extendable linear mappings, i.e. the continuous linear mappings on an analyticity space with a continuous linear extension on the corresponding trajectory space.

If the space $S_{X,A}$ or the space $S_{Y,B}$ is nuclear, then one of the Kernel theorems says that Σ'_A comprises all continuous linear mappings from $S_{X,A}$ into $S_{Y,B}$. Chapter three contains the explicit formulation of the four Kernel theorems of [G] and of the Kernel theorem for the extendable linear mappings. Subsequently, we study the following operator algebras: the algebra T^A of continuous linear mappings from $S_{X,A}$ into itself, the algebra T_A of continuous linear mappings from $T_{X,A}$ into itself and the algebra E_A of extendable linear mappings. In our research we involve the relation between algebraic structures and topological structures. We use the algebra E_A as a mathematical model for the description of parts of quantum statistics.

The remaining part of Chapter three is devoted to matrices. If $S_{X,A}$ is a nuclear space, then to every continuous linear mapping on $S_{X,A}$ there can be associated an infinite matrix. We shall derive a simple characterization of the infinite matrices corresponding to the elements of T^A, T_A and E_A. In a separate section we treat the continuous linear mappings whose matrices consist of only one non-zero (co)diagonal. These mappings are usually called weighted shift. In fact, weighted shifts and their finite combinations appear frequently in applied mathematics and in the theory of special functions. At the end of this chapter, the matrix calculus is applied in the construction of nuclear analyticity spaces $S_{X,A}$ on which a finite number of bounded linear operators on X and, also, a finite number of commuting self-adjoint operators in X act continuously.

Chapter four is the self-contained part of this thesis, in which we shall develop a theory of generalized functions in terms of our distribution theory. For a self-adjoint operator P which is continuous on a nuclear analyticity space $S_{X,A}$ there exist generalized eigenvectors in $T_{X,A}$ for almost every point of the spectrum $\sigma(P)$. In the proof of this result nuclearity seems to play an essential role.

The remaining part of Chapter four is devoted to a mathematical interpretation of Dirac's formalism. A reinterpretation of Dirac's bracket notion leads to a mathematical theory which involves Fourier expansion of kets, orthogonality of complete sets of eigenkets and matrices of unbounded linear mappings, all in the spirit of Dirac.

We conclude this thesis with an epilogue. The study of analyticity spaces and trajectory spaces has raised questions and consequently has brought up results. This thesis cannot contain all of them. So we have made a selection. In the epilogue we shall point at related results.

I. ANALYTICITY SPACES, TRAJECTORY SPACES AND LINEAR MAPPINGS BETWEEN THEM

1. The space $S_{X,A}$

Let A be a nonnegative, self-adjoint operator in a Hilbert space X. Then the semigroup $(e^{-tA})_{t\geq 0}$ consists of bounded linear operators on X. In order that this semigroup is smoothing, A is supposed to be unbounded. The test space $S_{X,A}$ is the dense linear subspace of X consisting of smooth elements $e^{-tA}h$, where $h \in X$ and $t > 0$. We have

$$S_{X,A} = \bigcup_{t>0} e^{-tA}(X) = \bigcup_{n\in\mathbb{N}} e^{-\frac{1}{n}A}(X) .$$

Since each subspace $e^{-tA}(X)$ of X can be given its obvious Hilbert space structure, $S_{X,A}$ can be looked upon as a union of Hilbert spaces. We note that for each $f \in S_{X,A}$ there exist $\tau > 0$ such that $e^{\tau A} f$ makes sense as an element of X.

The strong topology in $S_{X,A}$ is the finest locally convex topology on $S_{X,A}$ for which the injections i_t: $e^{-tA}(X) \to S_{X,A}$, $t > 0$, are all continuous. In other words, we impose on $S_{X,A}$ the inductive limit topology with respect to the spaces $e^{-tA}(X)$, $t > 0$. We note that this inductive limit is not strict.

The function algebras $B(\mathbb{R})$ and $B_+(\mathbb{R})$ are defined as follows:

- $B(\mathbb{R})$ consists of all everywhere finite, real valued Borel functions ψ on $\mathbb{R}$ such that for all $t > 0$ the function $x \mapsto \psi(x)e^{-tx}$ is bounded on $[0,\infty)$.
- $B_+(\mathbb{R})$ consists of all $\psi \in B(\mathbb{R})$ with $\psi(x) \geq \varepsilon > 0$, $\varepsilon \in \mathbb{R}$.

By the spectral theorem for self-adjoint operators, the operators $\psi(A)$, $\psi \in B(\mathbb{R})$ are well defined, and the operators $\psi(A)e^{-tA}$, $t > 0$, are all bounded. Further for $f \in S_{X,A}$ and $\psi \in B(\mathbb{R})$

$$\psi(A)f = e^{-\tau A}(\psi(A)\, e^{-(t-\tau)A})e^{+tA} f \in S_{X,A}$$

if $t > 0$ sufficiently small and $0 < \tau < t$.

On $S_{X,A}$ the seminorms p_ψ are well-defined by

(1.1) $$p_\psi(f) = \|\psi(A)\, f\|$$

where $\|\cdot\|$ denotes the usual norm in X. Then the following very fundamental theorem can be proved.

(1.2) <u>Theorem.</u>

The seminorms p_ψ of (1.1) are continuous on $S_{X,A}$ and they generate the strong topology on $S_{X,A}$.

Although the inductive limit is not strict, because of Theorem (1.2) most results for strict inductive limits are also valid in our $S_{X,A}$ space. In [G] the following results have been proved with ad hoc arguments.

(1.3) Theorem.

A subset $B \subset S_{X,A}$ is bounded iff there is $t > 0$ such, that B is a bounded subset of $e^{-tA}(X)$.

(1.4) Theorem.

A subset $K \subset S_{X,A}$ is compact iff there is $t > 0$ such, that K is a compact subset of $e^{-tA}(X)$.

(1.5) Theorem.

A sequence (f_n) in $S_{X,A}$ is Cauchy iff (f_n) is a Cauchy sequence in some $e^{-tA}(X)$.

Hence $S_{X,A}$ is sequentially complete, because each $e^{-tA}(X)$ is complete. The elements of $S_{X,A}$ can be characterized as follows.

(1.6) Lemma.

Let $f \in X$, and suppose $f \in D(\psi(A))$ for all $\psi \in B_+(\mathbb{R})$. Then $f \in S_{X,A}$. Employing the standard terminology of topological vector spaces, the properties of $S_{X,A}$ are the following.

(1.7) Theorem.

I $S_{X,A}$ is complete.

II $S_{X,A}$ is bornological.

III $S_{X,A}$ is barreled.

IV $S_{X,A}$ is Montel, iff for every $t > 0$ the operator e^{-tA} is compact on X.

V $S_{X,A}$ is nuclear iff for every $t > 0$ the operator e^{-tA} is Hilbert-Schmidt on X.

2. The space $\mathcal{T}_{X,A}$

In X consider the evolution equation

$$(2.1) \qquad \frac{dF}{dt} = -A F .$$

A solution F of (2.1) is called a trajectory if F satisfies

$$(2.2.i) \qquad \forall_{t>0} \forall_{\tau>0} : e^{-\tau A} F(t) = F(t+\tau)$$

$$(2.2.ii) \quad \forall_{t>0} : F(t) \in X .$$

We emphasize that $\lim_{t \downarrow 0} F(t)$ does not necessarily exist in X-sense. The complex vector space of all trajectories is denoted by $\mathcal{T}_{X,A}$. For $F \in \mathcal{T}_{X,A}$ we have $F(t) \in \mathcal{S}_{X,A}$, $t > 0$. The Hilbert space X can be embedded in $\mathcal{T}_{X,A}$. To this end, define emb: $X \to \mathcal{T}_{X,A}$ by

$$(2.3) \qquad \text{emb}(x)\ (t) = e^{-tA} x \quad , \quad x \in X.$$

Thus X can be considered as a subspace of $\mathcal{T}_{X,A}$, and we have

$$\mathcal{S}_{X,A} \subset X \subset \mathcal{T}_{X,A} .$$

The characterization of the elements of $\mathcal{T}_{X,A}$ is as follows.

(2.4) Theorem.

Let $F \in \mathcal{T}_{X,A}$. Then there exists $\omega \in X$ and $\psi \in \mathcal{B}_+(\mathbb{R})$ such that $F(t) = \psi(A) e^{-tA} \omega$, $t > 0$.

The strong topology in $\mathcal{T}_{X,A}$ is the locally convex topology induced by the seminorms

(2.5) $\rho_n(F) = \| F(\frac{1}{n}) \|$, $n \in \mathbb{N}$.

With this topology $T_{X,A}$ becomes a Frêchet space, i.e. a metrizable and complete space.

It is not hard to see that $S_{X,A}$ is dense in $T_{X,A}$. For $F \in T_{X,A}$ just take the sequence $(F(\frac{1}{n})) \subset S_{X,A}$. This sequence converges to F in the strong topology of $T_{X,A}$. Further in [G], ch. II, the following results have been proved:

(2.6) Theorem.

A set $B \subset T_{X,A}$ is bounded iff each of the sets $\{F(t) \mid F \in B\}$, $t > 0$, is bounded in X.

(2.7) Theorem.

A set $K \subset T_{X,A}$ is compact iff each of the sets $\{F(t) \mid F \in K\}$, $t > 0$, is compact in X.

With the aid of the standard terminology of topological vector spaces $T_{X,A}$ can be described as follows.

(2.8) Theorem.

I $T_{X,A}$ is bornological.

II $T_{X,A}$ is barreled.

III $T_{X,A}$ is Montel iff the operators e^{-tA} are compact on X for all $t > 0$.

IV $T_{X,A}$ is nuclear iff the operators e^{-tA} are Hilbert-Schmidt on X for all $t > 0$.

3. The pairing of $S_{X,A}$ and $T_{X,A}$

On $S_{X,A} \times T_{X,A}$ the sesquilinear form $\langle \cdot , \cdot \rangle$ is defined by

$$(3.1) \qquad \langle g , F \rangle := (e^{tA} g , F(t)) ,$$

where as usual $(\cdot , \cdot)$ denotes the inner product of X. We note that this definition makes sense for $t > 0$ sufficiently small, and does not depend on the choice of $t > 0$ because of the trajectory property (2.2.ii) satisfied by F.

The spaces $S_{X,A}$ and $T_{X,A}$ can be considered as the strong topological dual spaces of each other by this pairing. So we have

(3.2) Theorem.

I Let ℓ be a linear functional on $S_{X,A}$. Then ℓ is continuous iff there exists $F \in T_{X,A}$ such, that $\ell(h) = \langle h , F \rangle$, $h \in S_{X,A}$.

II Let m be a linear functional on $T_{X,A}$. Then m is continuous iff there exists $f \in S_{X,A}$ such, that $m(G) = \overline{\langle f , G \rangle}$, $G \in T_{X,A}$.

As usual, the linear functionals of $S_{X,A}$ resp. $T_{X,A}$ induce the weak topology on $T_{X,A}$ resp. $S_{X,A}$ in the following way:

(3.3.i) The weak topology on $S_{X,A}$ is the topology induced by the seminorms, $p_F(h) = |\langle h , F \rangle|$, $F \in T_{X,A}$.

(3.3.ii) The weak topology on $T_{X,A}$ is the topology induced by the seminorms $\rho_f(G) = |\langle f , G \rangle|$, $f \in S_{X,A}$.

A simple argument [CH], II. §22, shows, that $S_{X,A}$ and $T_{X,A}$ are reflexive both in the strong and the weak topology.

(3.4) Theorem. (Banach-Steinhaus)

Weakly bounded sets in $S_{X,A}$ resp. $T_{X,A}$ are strongly bounded.

In the next two theorems weak convergence of sequences in $S_{X,A}$ as well as in $T_{X,A}$ are characterized.

(3.5) Theorem.

$f_n \to 0$ in the weak topology of $S_{X,A}$ iff

$$\exists_{t>0}: (f_n) \subset e^{-tA}(X) \text{ and } f_n \to 0, \text{ weakly, in } e^{-tA}(X) .$$

As a corollary it immediately follows that strong convergence of a seqence in $S_{X,A}$, implies its weak convergence. Further, any bounded sequence in $S_{X,A}$ has a weakly convergent subsequence.

(3.6) Theorem.

$F_n \to 0$ weakly in $T_{X,A}$ iff $\forall_{t>0}$: $F_n(t) \to 0$ weakly in X.

So again it follows that strongly converging sequences in $T_{X,A}$ are weakly convergent. By a diagonal argument it can be proved that any bounded sequence in $T_{X,A}$ has a weakly converging subsequence.

When are weakly convergent sequences always strongly convergent? The next theorem deals with this question.

(3.7) Theorem.

The following three statements are equivalent:

I For each $t > 0$, the operator e^{-tA} is compact on X.

II Each weakly convergent sequence in $S_{X,A}$ converges strongly in $S_{X,A}$.

III Each weakly convergent sequence in $T_{X,A}$ converges strongly in $T_{X,A}$.

4. Characterization of continuous linear mappings between the spaces $S_{X,A}$, $T_{X,A}$, $S_{Y,B}$ and $T_{Y,B}$

Let B be a non-negative self-adjoint operator in the separable Hilbert space Y. In this section we give conditions implying continuity of linear mappings $S_{X,A} \to S_{Y,B}$, $S_{X,A} \to T_{Y,B}$, $T_{X,A} \to T_{Y,B}$ and $T_{X,A} \to S_{Y,B}$. Further, there are given conditions on a linear operator in X such that it can be extended to a continuous linear mapping on $T_{X,A}$. The next theorem is an immediate consequence of the fact that $S_{X,A}$ is bornological.

(4.1) Theorem.

Let R be an arbitrary locally convex topological vector space. A linear mapping $\mathcal{L}: S_{X,A} \to R$ is continuous iff

I for each $t > 0$ the mapping $\mathcal{L}\, e^{-tA}: X \to R$ is continuous.

II for each null sequence $(u_n) \subset S_{X,A}$, the sequence $(\mathcal{L}\, u_n)$ is a null sequence in R.

In [G], De Graaf gives several equivalent conditions on linear mappings of one of the mentioned types to be continuous. Each of these conditions is useful in its own context. The next theorem deals with continuous linear mappings from $S_{X,A}$ into $S_{Y,B}$.

(4.2) Theorem.

Suppose $P: S_{X,A} \to S_{Y,B}$ is a linear mapping. Then P is continuous iff one of the following conditions is satisfied

I $f_n \to 0$ strongly in $S_{X,A}$ implies $Pu_n \to 0$ strongly in $S_{Y,B}$.

II For each $t > 0$ the operator Pe^{-tA} is continuous from X into Y.

III For each $t > 0$ there exists $s > 0$ such that $Pe^{-tA}(X) \subset e^{-sB}(Y)$ and $e^{sB}Pe^{-tA}$ is a bounded linear operator from X into Y.

IV There exists a dense linear subspace $\Xi \subset Y$ such that for each fixed $y \in \Xi$ the linear functional $\ell_{P,y}(f) = (Pf, y)_Y$ is continuous on $S_{X,A}$.

V For each $t > 0$ the adjoint $(Pe^{-tA})^*$ of Pe^{-tA} is continuous from Y into X.

The next corollary is important for applications.

(4.3) Corollary.

Let Q be a densely defined closable operator: $X \to Y$. If $D(Q) \supset S_{X,A}$ and $Q(S_{X,A}) \subset S_{Y,B}$, then Q maps $S_{X,A}$ continuously into $S_{Y,B}$.

(4.4) Theorem.

Let $K: S_{X,A} \to T_{Y,B}$ be a linear mapping. Then K is continuous iff

I For each $t > 0$, $s > 0$ the operator $e^{-sB}Ke^{-tA}$ is continuous from X into Y.

II For each $s > 0$ the mapping $e^{-sB}K$ is continuous from $S_{X,A}$ into $S_{Y,B}$.

(4.5) Theorem.

Let V: $T_{X,A} \to S_{Y,B}$ be a linear mapping, and let V_r: X → Y denote its restriction to X. Then V is continuous iff one of the following conditions is satisfied

I $V_r^*(Y) \subset S_{X,A}$.

II There exists t > 0 such that $V_r^*(Y) \subset e^{-tA}(X)$ and $e^{tA} V_r^*$ is bounded as an operator from Y into X.

III There exists t > 0 such that $V_r e^{tA}$ with domain $e^{-tA}(X) \subset X$ is bounded as an operator from X into Y.

IV There exists t > 0 and a continuous linear mapping Q: $S_{X,A} \to S_{Y,B}$ such, that $V = Q e^{-tA}$.

(4.6) Theorem.

Let Φ: $T_{X,A} \to T_{Y,B}$ be a linear mapping. Let Φ_r: X → $T_{Y,B}$ denote the restriction of Φ to X. Then Φ is continuous iff one of the following conditions is satisfied.

I For each g ∈ $S_{Y,B}$ the linear functional $F \mapsto \overline{\langle y , \Phi F \rangle}$ is continuous on $T_{X,A}$.

II For each s > 0 the linear mapping $e^{-sB}\Phi$ is continuous from $T_{X,A}$ into $S_{Y,B}$.

III For each s > 0 $(e^{-sB}\Phi_r)^*(Y) \in S_{X,A}$.

IV For each s > 0 there exists t > 0 such that $e^{-sB}\Phi_r e^{tA} = e^{-sB}\Phi e^{tA}$ on the domain $e^{-tA}(X)$ is bounded as an operator form X into Y.

An interesting class of densely defined linear operators is established by those operators in X which can be extended to continuous linear mappings from $T_{X,A}$ into $T_{Y,B}$. This class is characterized as follows.

(4.7) Theorem.

Let E be a densely defined linear operator from X into Y. E can be extended to a continuous linear mapping $\bar{E}$: $T_{X,A} \to T_{Y,B}$ iff E has a densely defined adjoint E^*: $D(Q^*) \supset S_{Y,B} \to X$ with $E^*(S_{Y,B}) \subset S_{X,A}$.

As a corollary of this theorem it follows that a continuous linear mapping Q: $S_{X,A} \to S_{Y,B}$ can be extended to a continuous mapping $\bar{Q}$: $T_{X,A} \to T_{Y,B}$ iff its adjoint Q^* satisfies $D(Q^*) \supset S_{Y,B}$ and $Q^*(S_{Y,B}) \subset S_{X,A}$.

5. Topological tensor products and Kernel theorems

Let $X \otimes Y$ denote the set of Hilbert-Schmidt operators from X into Y. $X \otimes Y$ is a Hilbert space, which can be regarded as a complete topological tensor product of the Hilbert spaces X and Y. Further, in $X \otimes Y$ the operator $A \boxplus B$ is defined to be the unique self-adjoint extension of the operator $A \otimes I + I \otimes B$ which is well defined on the algebraic tensor product $D(A) \otimes_a D(B)$. We have $e^{-t(A \boxplus B)} = e^{-tA} \otimes e^{-tB}$, $t > 0$. So $(e^{-t(A \boxplus B)})_{t>0}$ is a semigroup of smoothing operators on $X \otimes Y$.

Now, according to section 1 and 2, we introduce the spaces $S_{X \otimes Y, A \boxplus B}$ and $T_{X \otimes Y, A \boxplus B}$. They can be regarded as topological completions of the algebraic tensor products $S_{X,A} \otimes_a S_{Y,B}$ c.q. $T_{X,A} \otimes_a T_{Y,B}$.

An element $J \in S_{X \otimes Y, A \boxplus B}$ can be considered as a linear operator J: $S_{X,A} \to S_{Y,B}$ in the following way: Let $F \in T_{X,A}$. Define JF by

$$JF = e^{-\varepsilon B} (e^{\varepsilon B} J e^{\varepsilon A}) \; F(\varepsilon) \; .$$

For $\varepsilon > 0$ and sufficiently small this definition makes sense and does not depend on the choice of ε.

(5.1) Kernel theorem.

If for each $t > 0$ at least one of the operators e^{-tA}, e^{-tB} is Hilbert-Schmidt, then $S_{X\otimes Y,A\boxplus B}$ comprises all continuous linear mappings from $T_{X,A}$ into $S_{Y,B}$.

An element $K \in T_{X\otimes Y,A\boxplus B}$ can be considered as a linear operator K: $S_{X,A} \to T_{Y,B}$ in the following way: Let $f \in S_{X,A}$. Define $Kf \in T_{Y,B}$ by

$$(Kf)(t) := e^{-(t-\varepsilon)B} K(\varepsilon)\, e^{\varepsilon A} f, \quad t > 0.$$

For any $f \in S_{X,A}$ and $t > 0$ this definition makes sense for $\varepsilon > 0$ sufficiently small. Moreover $(Kf)(t)$ does not depend on the choice of ε.

(5.2) Kernel theorem.

If for each $t > 0$ at least one of the operators e^{-tA}, e^{-tB} is Hilbert-Schmidt, then $T_{X\otimes Y,A\boxplus B}$ comprises all continuous linear mappings from $S_{X,A}$ into $T_{Y,B}$.

Next, in order to describe continuous linear mappings P: $S_{X,A} \to S_{Y,B}$ and Φ: $T_{X,A} \to T_{Y,B}$ De Graaf introduces two more topological tensor products:

The subspace Σ_A' of $T_{X\otimes Y,A\otimes I}$ defined by

$$\Sigma_A' := \{P \mid P \in T_{X\otimes Y,A\otimes I}\ ,\ \forall_{t>0}: P(t) \in S_{X\otimes Y,A\boxplus B}\}\ .$$

This is a topological completion of $T_{X,A} \otimes_a S_{Y,B}$.

The subspace Σ_{B}' of $T_{X\otimes Y, I\otimes B}$ defined by

$$\Sigma_{B}' := \{\Phi \mid \Phi \in T_{X\otimes Y, I\otimes B} , \forall_{t>0}: \Phi(t) \in S_{X\otimes Y, A\boxplus B}\}.$$

Σ_{B}' is a topological completion of $S_{X,A} \otimes_a T_{Y,B}$.

On the spaces Σ_{A}' and Σ_{B}' complete sets of seminorms are introduced. An element $P \in \Sigma_{A}'$ can be considered as a linear operator $P: S_{X,A} \to S_{Y,B}$ as follows: For $f \in S_{X,A}$ define $Pf \in S_{Y,B}$ by

$$Pf = P(\varepsilon)\, e^{\varepsilon A} f \; .$$

Then $Pf \in S_{Y,B}$, because $P(\varepsilon) \in S_{X\otimes Y, A\boxplus B}$. The definition makes sense for $\varepsilon > 0$ sufficiently small and does not depend on the choice of ε.

(5.3) Kernel theorem.

If for each $t > 0$ at least one of the operators e^{-tA}, e^{-tB} is Hilbert-Schmidt, then Σ_{A}' comprises all continuous linear mappings from $S_{X,A}$ into $S_{Y,B}$.

Finally, an element $\Phi \in \Sigma_{B}'$ can be considered as a linear operator $\Phi: T_{X,A} \to T_{Y,B}$ in the following way: For $F \in T_{X,A}$ define $\Phi F \in T_{Y,B}$ by

$$(\Phi F)(t) := \Phi(t)\, e^{\varepsilon(t) A} F(\varepsilon(t)) \; .$$

This definition makes sense for each $t > 0$ and $\varepsilon(t) > 0$ sufficiently small. The result does not depend on the specific choice of $\varepsilon(t)$.

(5.4) Kernel theorem.

If for each $t > 0$ at least one of the operators e^{-tA}, e^{-tB} is Hilbert-Schmidt, then Σ_B' comprises all continuous linear mappings from $T_{X,A}$ into $T_{Y,B}$.

For more details and proofs the reader is referred to [G], Ch. VI. In Ch. II the spaces Σ_A' and Σ_B' will be defined in a more elegant way and discussed in a wider context. Further investigations in this theory of generalized functions led to a fifth Kernel theorem for those continuous linear mappings from $S_{X,A}$ into $S_{Y,B}$, which can be extended to a continuous linear mapping from $T_{X,A}$ into $T_{Y,B}$, the so called extendable linear mappings.

6. Examples of $S_{X,A}$-spaces

(1) The S_α^β-spaces of Gelfand-Shilov

De Bruijn's theory of generalized function is based on the test function space $S_{L_2(\mathbb{R}),H}$, where H is the Hamiltonian operator of the harmonic oscillator,

$$H = \frac{1}{2}\left(-\frac{d^2}{dx^2} + x^2 + 1\right) .$$

The space $S_{L_2(\mathbb{R}),H}$ consists of entire analytic functions f satisfying

$$|f(x+iy)| \le C \exp(-\tfrac{1}{2}Ax^2 + \tfrac{1}{2}By^2) \quad , \quad x,y \in \mathbb{R},$$

where A, B en C are some positive constants only dependent on f. The space $S_{L_2(\mathbb{R}),H}$ equals the space $S_{\frac{1}{2}}^{\frac{1}{2}}$ introduced in the books of Gelfand-Shilov [GS_2].

Recently, it has been proved that the Gelfand-Shilov spaces $S^{k/k+1}_{1/k+1}$, $k \in \mathbb{N}$, are $S_{X,A}$-type spaces. (see [EGP]). To this end, put

$$\mathcal{B}_k = \left(-\frac{d^2}{dx} + x^{2k}\right)^{k+1/2k} .$$

Then $S^{k/k+1}_{1/k+1} = S_{L_2(\mathbb{R}),\mathcal{B}_k}$. By applying the Fourier transform it easily follows that

$$S^{1/k+1}_{k/k+1} = S_{L_2(\mathbb{R}),\tilde{\mathcal{B}}_k}$$

where $\tilde{\mathcal{B}}_k = \left(\left(-\frac{d^2}{dx^2}\right)^k + x^2\right)^{k+1/2k}$.

We conjecture that a great number of Gelfand-Shilov spaces S^{β}_{α} are of type $S_{X,A}$.

(2) Hankel invariant distribution spaces

For $\alpha > -1$, the Hankel transform $\mathbb{H}_\alpha$ is formally defined by

$$(\mathbb{H}_\alpha f)(x) = \int_0^\infty \mathcal{J}_\alpha(xy)\sqrt{xy}\, f(y)dy \quad , \quad x > 0,$$

where $\mathcal{J}_\alpha$ is the Bessel function of order α. The Hankel transform extends to a unitary operator on $Z = L_2(0,\infty)$. The generalized Laguerre functions $\mathcal{L}^{(\alpha)}_n$, $n \in \mathbb{N} \cup \{0\}$,

$$\mathcal{L}^{(\alpha)}_n(x) = \left(\frac{2\,\Gamma(n+1)}{\Gamma(n+\alpha+1)}\right)^{\frac{1}{2}} x^{\alpha+\frac{1}{2}}\, e^{-\frac{1}{2}x^2}\, L^{(\alpha)}_n(x^2) \quad , \quad x > 0,$$

where $L^{(\alpha)}_n$ is the n-th generalized Laguerre polynomial of type α, satisfy

$$\mathbb{H}_\alpha \mathcal{L}^{(\alpha)}_n = (-1)^n \mathcal{L}^{(\alpha)}_n .$$

They establish a complete orthonormal basis of eigenfunctions in Z for the positive self-adjoint operator A_α

$$A_\alpha : -\frac{d^2}{dx^2} + x^2 + \frac{\alpha^2 - \frac{1}{4}}{x^2} - 2\alpha \ .$$

Their respective eigenvalues are $4n+2$, $n \in \mathbb{N} \cup \{0\}$.

By routine methods it can be shown that the space S_{Z,A_α} is invariant under the unitary operator $\mathbb{H}_\alpha$. So $\mathbb{H}_\alpha$ extends to a continuous bijection on the distribution space T_{Z,A_α}. In [E_2], [EG] the elements of S_{Z,A_α} are characterized as follows

$f \in S_{Z,A_\alpha}$ iff

(i) $z \mapsto z^{-(\alpha+\frac{1}{2})} f(z)$ extends to an entire analytic and even function

and (ii) there are positive constants A, B and C such that

$$|z^{-(\alpha+\frac{1}{2})} f(z)| \le C \exp(-\tfrac{1}{2}Ax^2 + \tfrac{1}{2}By^2)$$

where $z = x + iy$.

(3) Nuclear $S_{X,A}$-spaces for given sets of operators in X

In Ch. III, there will be given a matrix calculus for the continuous linear mappings from a nuclear $S_{X,A}$ space into itself. With the aid of this calculus we have been able to construct a nuclear $S_{X,A}$ space for a finite number of bounded linear operators on a Hilbert space X, and also for a finite number of commuting, self-adjoint operators in X. The existence of such nuclear $S_{X,A}$ space is very important for our theory of generalized eigenfunctions and our interpretation of Dirac's formalism (see Ch. IV).

7. Analytic vectors

In $[Ne_1]$, Nelson introduced the notion analytic vector. Let A be a self-adjoint operator in X. Then $f \in X$ is an analytic vector for A iff

$$\| A^n f \| \leq a b^n n! \quad , \quad n = 0,1,2,\ldots$$

for some fixed constants a, b only dependent on f. The space of analytic vectors for A is denoted by $C^{\omega}(A)$, and called the analyticity domain of A. Nelson showed that for a nonnegative, self-adjoint operator A the vector $f \in C^{\omega}(A)$ can be written as $f = e^{-tA} w$ where $t > 0$ and $w \in X$. Hence $C^{\omega}(A) = S_{X,A}$.

The notion analytic vector was also introduced for unitary representations of Lie groups (see $[Ne_1]$, [Wa], [Go] and [Na]):

Let G be a finite dimensional Lie group. A unitary representation U of G is a mapping

$$g \mapsto U(g) \quad , \quad g \in G$$

from G into the unitary operators on some Hilbert space X.

A vector $f \in X$ is called an analytic vector for the representation U, if the mapping

$$g \mapsto U(g) f$$

is analytic on G. We shall denote the space of analytic vectors for U by $C^{\omega}(U)$.

Let $A(G)$ denote the Lie algebra of the Lie group G, and let $\{p_1,\ldots,p_d\}$ be a basis for $A(G)$. Then for every $p \in A(G)$

$$s \mapsto U(\exp(sp))$$

is a one parameter group of unitary operators on X. By Stone's theorem its infinitesimal generator, denoted by $\partial U(p)$, is skew-adjoint. Thus the Lie algebra $A(G)$ is represented by skew-adjoint operators in X. Put

$$\Delta = I - \sum_{k=1}^{d} (\partial U(p_k))^2.$$

Nelson, [Ne_1], has proved that the operator Δ can be uniquely extended to a positive, self-adjoint operator in X. Denote its extension by Δ, also. Then we have (see [Ne_1], [Go])

(7.1) Theorem.

The space of analytic vectors for the representation U, $C^{\omega}(U)$ equals the space $S_{X,\Delta^{\frac{1}{2}}}$.

The following result tells something about the action of $\partial U(p)$, $p \in A(G)$ on the space $S_{X,\Delta^{\frac{1}{2}}}$.

(7.2) Theorem.

The linear operators $\partial U(p)$, $p \in A(G)$, are continuous as linear mappings from $S_{X,\Delta^{\frac{1}{2}}}$ into itself.

Proof. Let $p \in A(G)$.
Following [Go], proposition 2.1, the operator $\partial U(p)$ maps $S_{X,\Delta^{\frac{1}{2}}}$ into itself. Since $\partial U(p)$ is skew-adjoint, continuity follows from section 4, Theorem 4.2. □

In several cases the space $S_{X,\Delta^{\frac{1}{2}}}$ is nuclear. Here we mention the following cases. Possibly, other cases can be found in the book of Warner,

[Wa]. For a proof we refer to [Na].

$S_{X,\Delta^{\frac{1}{2}}}$ is nuclear if U is an irreducible unitary representation of G on X and one of the following statements is satisfied:

(i) G is semi-simple with finite center.

(ii) G is the semi-direct product of $A \otimes K$ where A is an abelian invariant subgroup and K is a compact subgroup, e.g. the Euclidian groups.

(iii) G is nilpotent.

Again we note that nuclearity of $S_{X,\Delta^{\frac{1}{2}}}$ is very important for our theory of generalized functions and our interpretation of Dirac's formalism.

II. ANALYTICITY SPACES AND TRAJECTORY SPACES BASED ON A PAIR OF COMMUTING, HOLOMORPHIC SEMIGROUPS

Introduction

A main result in the theory on analyticity and trajectory spaces is the validity of four Kernel theorems for four types of continuous linear mappings which appear in this theory. A Kernel theorem provides conditions such that all linear mappings of a specific kind arise from the elements (kernels) out of a suitable topological tensor product.

In order to prove a Kernel theorem for the continuous linear mappings from $S_{X,A}$ into $S_{Y,B}$, resp. from $T_{X,A}$ into $T_{Y,B}$ the rather curious spaces Σ_A' and Σ_B' are brought up in [G]. The space Σ_A' is a topological tensor product of $T_{X,A}$ and $S_{Y,B}$ and the space Σ_B' of $S_{X,A}$ and $T_{Y,B}$.

In the third chapter of this thesis we shall explicitly formulate the mentioned Kernel theorems within the framework of a thorough discussion of continuous linear mappings on analyticity and trajectory spaces.

During the investigations which led to the third chapter of this thesis, we needed a clearer view on those remarkable spaces Σ_A' and Σ_B'.

To this end we studied two new types of spaces, namely $S(T_{Z,C},\mathcal{D})$ and $T(S_{Z,C},\mathcal{D})$ with C and $\mathcal{D}$ commuting, nonnegative, self-adjoint operators in a Hilbert space Z. We shall present them here. Up to now these spaces have no other than an abstract use. However, the space $S(T_{Z,C},\mathcal{D})$ can be regarded as the 'analyticity domain' of the operator $\mathcal{D}$ in $T_{Z,C}$, cf. Ch. I, Section 7. The space $T(S_{Z,C},\mathcal{D})$ contains all trajectories of $T_{Z,\mathcal{D}}$ through $S_{Z,C}$. We mention the following relations

$$\Sigma'_A = T(S_{X\otimes Y,I\otimes B},A\otimes I) \quad , \quad \Sigma_A = S(T_{X\otimes Y,I\otimes B},A\otimes I),$$

$$\Sigma'_B = T(S_{X\otimes Y,A\otimes I},I\otimes B) \quad , \quad \Sigma_B = S(T_{X\otimes Y,A\otimes I},I\otimes B).$$

The first section is concerned with the analyticity space $S(T_{Z,C},\mathcal{D})$. This space is a countable union of Frêchet spaces

$$S(T_{Z,C},\mathcal{D}) = \bigcup_{s>0} e^{-s\mathcal{D}}(T_{Z,C}) = \bigcup_{s>0} T_{e^{-s\mathcal{D}}(Z),C} .$$

For the strong topology we take the inductive limit topology. We shall produce an explicit system of seminorms which generates this topology, and characterize the elements of $S(T_{Z,C},\mathcal{D})$. We looked for a characterization of null-sequences, bounded subsets and compact subsets of $S(T_{Z,C},\mathcal{D})$ and for the proof of its completeness; however, without success. The second section is devoted to the trajectory space $T(S_{Z,C},\mathcal{D})$. With the introduction of a 'natural' topology, the space $T(S_{Z,C},\mathcal{D})$ becomes a complete topological vector space. Here we have been more successful. The elements, the bounded and the compact subsets, and the null-sequences of $T(S_{Z,C},\mathcal{D})$ will be described completely. Since $T_{X,A}$ is a special $T(S_{Z,C},\mathcal{D})$-space the latter results extend the theory on the topological structure of $T_{X,A}$. Cf.[G], ch.II. In Section 3 we shall introduce a pairing between $S(T_{Z,C},\mathcal{D})$ and $T(S_{Z,C},\mathcal{D})$. With this pairing they can be regarded as each other's strong dual spaces. Further we note that for both spaces a Banach-Steinhaus theorem will be proved.

The extendable linear mappings establish a fifth type of mappings in the theory. They are continuous from $S_{X,A}$ into $S_{Y,B}$, and can be 'extended' to continuous linear mappings from $T_{X,A}$ into $T_{Y,B}$. In order to describe the class of extendable linear mappings it is natural to look for a description of the intersection of Σ'_A and Σ'_B, or, more generally, of $T(S_{Z,C},\mathcal{D})$ and $T(S_{Z,\mathcal{D}},C)$. Therefore in Section 4 we introduce the nonnegative, self-adjoint operators $C \wedge \mathcal{D} = \max(C,\mathcal{D})$ and $C \vee \mathcal{D} = \min(C,\mathcal{D})$. To these both the theory in [G] and the theory of Sections 1-3 apply. The operators $C \wedge \mathcal{D}$ and $C \vee \mathcal{D}$ enable us to represent intersections and algebraic sums of the spaces $S_{Z,C}$, $S_{Z,\mathcal{D}}$, $T_{Z,C}$, $T_{Z,\mathcal{D}}$, $S(T_{Z,C},\mathcal{D})$, etc., as spaces of one of our types. It will lead to a fifth Kernel theorem in the following chapter.

The spaces which appear in our theory are ordered by inclusion. In the final section we discuss the inclusion scheme. Since each space can be considered as a space of continuous linear mappings of a specific kind the scheme illustrates the interdependence of these types.

1. The space $S(T_{Z,C},\mathcal{D})$

Let C and $\mathcal{D}$ denote two commuting, nonnegative, self-adjoint operators in a Hilbert space Z. We take them fixed throughout this part of the paper. Suppose $C,\mathcal{D}$ admit spectral resolutions $(G_\lambda)_{\lambda\in\mathbb{R}}$ and $(H_\mu)_{\mu\in\mathbb{R}}$, such that

$$C = \int_{\mathbb{R}} \lambda \, d G_\lambda \quad , \quad \mathcal{D} = \int_{\mathbb{R}} \mu \, d H_\mu .$$

Then for every pair of Borel sets Δ_1, Δ_2 in $\mathbb{R}$

$$G(\Delta_1)\, H(\Delta_2) = H(\Delta_2)\, G(\Delta_1) \; .$$

Since the operators $e^{-s\mathcal{D}}$, $s > 0$, and e^{-tC}, $t > 0$, consequently commute, for each fixed $s > 0$ the linear mapping $e^{-s\mathcal{D}}$ is continuous on the trajectory space $T_{Z,C}$ (Cf. Ch. I, Section 4). We now introduce the space $S(T_{Z,C},\mathcal{D})$ as follows.

(1.1) <u>Definition</u>

$$S(T_{Z,C},\mathcal{D}) = \bigcup_{s>0} e^{-s\mathcal{D}}(T_{Z,C}) = \bigcup_{n\in\mathbb{N}} e^{-\frac{1}{n}\mathcal{D}}(T_{Z,C}) \; .$$

We note that $e^{-s\mathcal{D}}(T_{Z,C}) \subset e^{-\sigma\mathcal{D}}(T_{Z,C})$ for $0 < \sigma < s$. Since the operator $e^{-s\mathcal{D}}$ is injective on $S_{Z,C}$, the space $e^{-s\mathcal{D}}(T_{Z,C})$ is dense in $T_{Z,C}$ by duality. Hence $S(T_{Z,C},\mathcal{D})$ is a dense subspace of $T_{Z,C}$. In the space $e^{-s\mathcal{D}}(T_{Z,C}) = T_{e^{-s\mathcal{D}}(Z),C}$, the strong topology is the topology generated by the seminorms $q_{s,n}$, $n \in \mathbb{N}$,

$$q_{s,n}(h) = \| e^{s\mathcal{D}}\, h(\tfrac{1}{n}) \|_Z \quad , \quad h \in e^{-s\mathcal{D}}(T_{Z,C})$$

We remark that $e^{-s\mathcal{D}}(T_{Z,C})$ is a Frêchet space.

(1.2) <u>Definition</u>

The strong topology on $S(T_{Z,C},\mathcal{D})$ is the inductive limit topology, i.e.

the finest locally convex topology for which all injections

$$i_s : e^{-s\mathcal{D}}(T_{Z,C}) \to S(T_{Z,C},\mathcal{D})$$

are continuous.

Note that the inductive limit is not strict!

A subset $\Omega \subset S(T_{Z,C},\mathcal{D})$ is open if and only if the intersection $\Omega \cap e^{-s\mathcal{D}}(T_{Z,C})$ is open in $e^{-s\mathcal{D}}(T_{Z,C})$ for each $s > 0$.
In this section we shall produce a system of seminorms in $S(T_{Z,C},\mathcal{D})$ which induces a locally convex topology equivalent to the strong topology of (1.2). Therefore we introduce the set of functions $F(\mathbb{R}^2)$.

(1.3) Definition

Let θ be an everywhere finite real-valued Borel function on $\mathbb{R}^2$. Then $\theta \in F(\mathbb{R}^2)$ if and only if

$$\forall_{s>0}\ \exists_{t>0} : \sup_{\substack{\lambda\geq 0\\ \mu\geq 0}} \left(|\theta(\lambda,\mu)|\, e^{-\mu s}\, e^{\lambda t}\right) < \infty.$$

Further, $F_+(\mathbb{R}^2)$ denotes the subset of all functions $F(\mathbb{R}^2)$ which are positive on $\{(\lambda,\mu) \mid \lambda \geq 0,\ \mu \geq 0\}$.

For $\theta \in F(\mathbb{R}^2)$ the operator $\theta(C,\mathcal{D})$ in X is defined by

$$\theta(C,\mathcal{D}) = \iint_{\mathbb{R}^2} \theta(\lambda,\mu)\ dG_\lambda H_\mu.$$

Here $dG_\lambda H_\mu$ denotes the operator-valued measure on the Borel subsets of $\mathbb{R}^2$ related to the spectral projections of C and $\mathcal{D}$. On the domain

$$D(\theta(C,\mathcal{D})) = \{w \in Z \mid \iint_{\mathbb{R}^2} |\theta(\lambda,\mu)|^2 \, d(G_\lambda H_\mu w, w) < \infty\}$$

$\theta(C,\mathcal{D})$ is self-adjoint.

The operators $\theta(C,\mathcal{D})$, $\theta \in F(\mathbb{R}^2)$, are continuous linear mappings from the space $S(T_{Z,C},\mathcal{D})$ into Z. This can be seen as follows. Let $h \in S(T_{Z,C},\mathcal{D})$. Then define

$$\theta(C,\mathcal{D})h = (e^{tC}\theta(C,\mathcal{D})e^{-s\mathcal{D}})e^{s\mathcal{D}}(h(t)).$$

Since there exists $s > 0$ such, that $e^{s\mathcal{D}}\,h(t) \in Z$ for all $t > 0$, and since for each $s > 0$ there exists $t > 0$ such, that the operator $e^{tC}\theta(C,\mathcal{D})e^{-s\mathcal{D}}$ is bounded on Z (cf. Definition (1.3)), the vector $\theta(C,\mathcal{D})h$ is in Z. Hence the following definition makes sense.

(1.4) Definition

For each $\theta \in F_+(\mathbb{R}^2)$ the seminorm p_θ is defined by

$$p_\theta(h) = \|\theta(C,\mathcal{D})h\|_Z , \quad h \in S(T_{Z,C},\mathcal{D}) ,$$

and the set $U_{\theta,\varepsilon}$, $\varepsilon > 0$, by

$$U_{\theta,\varepsilon} = \{h \in S(T_{Z,C},\mathcal{D}) \mid \|\theta(C,\mathcal{D})h\|_Z < \varepsilon\}.$$

The next theorem is the generalization of Theorem (1.4) in [G] to the type of spaces $S(T_{Z,C},\mathcal{D})$.

(1.5) Theorem

I. For each $\theta \in F_+(\mathbb{R}^2)$ the seminorm p_θ is continuous in the strong topology of $S(T_{Z,C},\mathcal{D})$.

II. Let a convex set $\Omega \subset S(T_{Z,C},\mathcal{D})$ have the property that for each $s > 0$ the set $\Omega \cap e^{-s\mathcal{D}}(T_{Z,C})$ contains a neighbourhood of 0 in $e^{-s\mathcal{D}}(T_{Z,C})$. Then Ω contains a set $U_{\theta,\varepsilon}$ for well-chosen $\theta \in F_+(\mathbb{R}^2)$ and $\varepsilon > 0$. Hence the strong topology in $S(T_{Z,C},\mathcal{D})$ is induced by the seminorms p_θ.

Proof.

I. In order to prove that p_θ is a continuous seminorm on $S(T_{Z,C},\mathcal{D})$ we have to show that $\theta(C,\mathcal{D})$ is a continuous linear mapping from $S(T_{Z,C},\mathcal{D})$ into Z. Therefore, let $s > 0$. Then there is $t > 0$ such that $\|e^{tC}\theta(C,\mathcal{D})e^{-s\mathcal{D}}\| < \infty$. So $\theta(C,\mathcal{D})$ is continuous on $e^{-s\mathcal{D}}(T_{Z,C})$ (cf. Ch. I, Section 4). Since $s > 0$ is arbitrarily taken, it implies that $\Theta(C,\mathcal{D})$ is continuous on $S(T_{Z,C},\mathcal{D})$.

II. We introduce the projections P_{nm}, $n,m \in \mathbb{N}$,

$$P_{nm} = \int_{n-1}^{n}\int_{m-1}^{m} d\,G_\lambda H_\mu \,.$$

Then $P_{nm}(\Omega)$ contains an open neighbourhood of 0 in $P_{nm}(Z)$. (We note that $P_{nm}(S(T_{Z,C},\mathcal{D})) \subset P_{nm}(Z)$.) So the following definition makes sense,

$$r_{nm} = \sup\{\rho \mid (h \in P_{nm}(Z) \wedge \|P_{nm}h\| < \rho) \Rightarrow h \in P_{nm}(\Omega)\} \,.$$

Next we define the function θ as follows

$$\theta(\lambda,\mu) = \frac{n^2m^2}{r_{nm}} \quad , \quad \lambda \in (n-1,n] \ , \ \mu \in (m-1,m],$$

$$\theta(\lambda,0) = \theta(\lambda,\tfrac{1}{2}), \quad \lambda > 0 \ ,$$

$$\theta(0,\mu) = \theta(\tfrac{1}{2},\mu), \quad \mu > 0 \ ,$$

$$\theta(\lambda,\mu) = 0 \quad , \quad \lambda < 0 \vee \mu < 0 \ .$$

We shall prove that $\theta \in F(\mathbb{R}^2)$. To this end, let $s > 0$. Then there are $t > 0$ and $\varepsilon > 0$ such that

$$\{h \mid \int_0^\infty \int_0^\infty e^{\mu s} d(G_\lambda H_\mu h(t),h(t)) < \varepsilon^2\} \subset \Omega \cap e^{-\frac{1}{2}s\mathcal{D}}(T_{Z,C}).$$

because $\Omega \cap e^{-\frac{1}{2}s\mathcal{D}}(T_{Z,C})$ contains an open neighbourhood of 0 by assumption. So we derive

$$r_{nm} > \varepsilon\, e^{(n-1)t}\, e^{-\frac{1}{2}ms} \ , \ n,m \in \mathbb{N} \ .$$

With $\lambda \in (n-1,n]$, $\mu \in (m-1,m]$ it follows that

$$\theta(\lambda,\mu)\, e^{\frac{1}{2}\lambda t}\, e^{-\mu s} < \frac{n^2m^2}{r_{nm}}\, e^{\frac{1}{2}nt}\, e^{-(m-1)s}$$

$$\leq \frac{n^2m^2}{\varepsilon}\, e^{-\frac{1}{2}nt}\, e^{-\frac{1}{2}(m-1)s}\, e^{\frac{1}{2}(s+t)} \ .$$

So $\quad \sup_{\substack{\lambda\geq 0\\ \mu\geq 0}} (e^{\frac{1}{2}\lambda t}\, e^{-\mu s}\, \theta(\lambda,\mu) < \infty.$

We claim that

(*) $\qquad \|\theta(C,\mathcal{D})h\| < 1 \ \Rightarrow \ h \in \Omega.$

Suppose $h \in e^{-s\mathcal{D}}(\mathcal{T}_{Z,\mathcal{C}})$ for some $s > 0$. Then for all $t > 0$

$$\sum_{n,m} \| e^{s\mathcal{D}} e^{-t\mathcal{C}} P_{nm} h \|^2 < \infty$$

and for σ, $0 < \sigma < s$, fixed and every $\tau > t$

$$(**) \qquad \| e^{\sigma\mathcal{D}} e^{-\tau\mathcal{C}} P_{nm} h \| \le e^{-(m-1)s-\sigma)} e^{-(n-1)(\tau-t)} \| e^{s\mathcal{D}} e^{-t\mathcal{C}} P_{nm} h \| .$$

Because of assumption $(*)$

$$\| P_{nm} h \| < (n^2 m^2)^{-1} \, r_{nm} .$$

Hence $n^2 m^2 P_{nm} h \in \Omega \cap e^{-\sigma\mathcal{D}}(\mathcal{T}_{Z,\mathcal{C}})$ for every $n,m \in \mathbb{N}$. In $e^{-\sigma\mathcal{D}}(\mathcal{T}_{Z,\mathcal{C}})$ we represent h by

$$h = \sum_{n,m}^{N,M} \frac{1}{n^2 m^2}(n^2 m^2 P_{nm} h) + \left(\sum_{(n>N)\vee(m>M)} \frac{1}{n^2 m^2} \right) h_{NM}$$

where

$$h_{NM} = \left(\sum_{(j>N)\vee(i>M)} \frac{1}{i^2 j^2} \right)^{-1} \left(\sum_{(n>N)\vee(m>M)} P_{nm} h \right) .$$

With $(**)$ we calculate

$$\| e^{\sigma\mathcal{D}} e^{-\tau\mathcal{C}} h_{NM} \|^2 \le$$

$$\le \left(N^4 \sum_{n=N+1}^{\infty} \sum_{m=1}^{\infty} + M^4 \sum_{n=1}^{\infty} \sum_{m=M+1}^{\infty} \right) (\| e^{\sigma\mathcal{D}} e^{-\tau\mathcal{C}} P_{nm} h \|^2)$$

$$\le \left(N^4 e^{-2N(\tau-t)} + M^4 e^{-2M(s-\sigma)} \right) \| e^{s\mathcal{D}} e^{-t\mathcal{C}} h \|^2 .$$

Hence $h_{NM} \to 0$ in $e^{-\sigma\mathcal{D}}(\mathcal{T}_{Z,C})$ because both $t > 0$ and $\tau > t$ are taken arbitrarily. So for sufficiently large N,M we have $h_{NM} \in [\Omega \cap e^{-\sigma\mathcal{D}}(\mathcal{T}_{Z,C})]$. Since h is a sub-convex combination of elements in the convex set $\Omega \cap e^{-\sigma\mathcal{D}}(\mathcal{T}_{Z,C})$ the result $h \in \Omega$ follows. □

Similar to Ch. I, Section 1, we should like to characterize bounded subsets, compact subsets, and sequential convergence in $S(\mathcal{T}_{Z,C},\mathcal{D})$. However, we think that this requires a method of constructing functions in $F_+(\mathbb{R}^2)$ similar to the construction of functions in $B_+(\mathbb{R})$ in the proofs of the characterizations given in [G], Ch.I. Up to now, our attempts to solve this problem were not successful.

Remark. As in Ch. I the set $B_+(\mathbb{R})$ consists of all everywhere finite Borel function φ on $\mathbb{R}$ which are strictly positive and satisfy

$$\forall_{\varepsilon>0} : \sup_{x>0} (\varphi(x)e^{-\varepsilon x}) < \infty.$$

Finally, we characterize the elements of $S(\mathcal{T}_{Z,C},\mathcal{D})$.

(1.6) Lemma

$h \in S(\mathcal{T}_{Z,C},\mathcal{D})$ iff there are $\psi \in B_+(\mathbb{R})$, $w \in Z$ and $s > 0$ such that $h = e^{-s\mathcal{D}}\psi(C)w$.

Proof. The proof is an immediate consequence of the following equivalence:

$$F \in \mathcal{T}_{Z,C} \Leftrightarrow \exists_{\psi\in B_+(\mathbb{R})} \exists_{w\in Z} : F = \psi(C)w$$ □

As in [G], Ch.I,it can be proved that $S(\mathcal{T}_{Z,C},\mathcal{D})$ is bornological and barreled.

2. The space $\mathcal{T}(S_{Z,C},\mathcal{D})$

The elements of $\mathcal{T}_{Z,\mathcal{D}}$ are called trajectories, i.e. functions F from $(0,\infty)$ into Z with the following property:

$$\forall_{s>0}\ \forall_{\sigma>0} : F(s+\sigma) = e^{-\sigma\mathcal{D}} F(s) .$$

Now the subspace $\mathcal{T}(S_{Z,C},\mathcal{D})$ of $\mathcal{T}_{Z,\mathcal{D}}$ is defined as follows:

(2.1) Definition

The space $\mathcal{T}(S_{Z,C},\mathcal{D})$ contains all elements $G \in \mathcal{T}_{Z,\mathcal{D}}$ which satisfy

$$\forall_{s>0} : G(s) \in S_{Z,C} .$$

Remark. $\mathcal{T}(S_{Z,C},\mathcal{D})$ consists of trajectories of $\mathcal{T}_{Z,\mathcal{D}}$ through $S_{Z,C}$. The space $\mathcal{T}(S_{Z,C},\mathcal{D})$ is not trivial. The embedding of Z into $\mathcal{T}_{Z,\mathcal{D}}$ maps $S_{Z,C}$ into $\mathcal{T}(S_{Z,C},\mathcal{D})$, because the bounded operators $e^{-s\mathcal{D}}$, $s > 0$ and $e^{-t\mathcal{D}}$, $t > 0$, commute.

In $\mathcal{T}(S_{Z,C},\mathcal{D})$ we introduce the seminorms $p_{\psi,s}$, $\psi \in \mathcal{B}_+(\mathbb{R})$, $s > 0$, by

(2.2) $$p_{\psi,s} = \|\psi(C)F(s)\|_Z \quad , \ F \in \mathcal{T}(S_{Z,C},\mathcal{D}) .$$

The strong topology in $\mathcal{T}(S_{Z,C},\mathcal{D})$ is the locally convex topology induced by the seminorms $p_{\psi,s}$.

The bounded subsets of $\mathcal{T}(S_{Z,C},\mathcal{D})$ can be fully characterized with the aid of the function algebra $\mathcal{F}_+(\mathbb{R}^2)$. To this end we first prove the following lemma.

(2.3) Lemma

The subset B in $\mathcal{T}(S_{Z,C},\mathcal{D})$ is bounded iff for each $s > 0$ there exists $t > 0$ such that the set $\{F(s) | F \in B\}$ is bounded in the Hilbert space $e^{-tC}(Z)$.

Proof. B is bounded in $\mathcal{T}(S_{Z,C},\mathcal{D})$ iff each seminorm $p_{\psi,s}$ is bounded on B iff the set $\{F(s) | F \in B\}$ is bounded in $S_{Z,C}$ for each $s > 0$. From Ch. I, Section 1, the assertion follows.

Because of Definition (1.3) for every $\theta \in F_+(\mathbb{R}^2)$ and each $w \in Z$ the vector $\theta(C,\mathcal{D})e^{-s\mathcal{D}}w$ is in $S_{Z,C}$. So the trajectory $s \mapsto \theta(C,\mathcal{D})e^{-s\mathcal{D}}w$ is an element of $\mathcal{T}(S_{Z,C},\mathcal{D})$ and it will be denoted by $\theta(C,\mathcal{D})w$.

(2.4) Theorem

The set $B \subset \mathcal{T}(S_{Z,C},\mathcal{D})$ is bounded iff there exists $\theta \in F_+(\mathbb{R}^2)$ and a bounded subset V of Z such that $B = \theta(C,\mathcal{D})(V)$

Proof.

$\Leftarrow$) Let $s > 0$. Then there exists $t > 0$ such that

$$\| e^{tC}\theta(C,\mathcal{D})e^{-s\mathcal{D}}w \| \leq \| e^{tC}\theta(C,\mathcal{D})e^{-s\mathcal{D}}\| \| w\| .$$

Hence B is a bounded subset by Lemma (2.3).

$\Rightarrow$) Let $n,m \in \mathbb{N}$. Define

$$P_{nm} = \int_{n-1}^{n} \int_{m-1}^{m} dG_\lambda H_\mu ,$$

and put $r_{nm} = \sup_{G\in B} (\| P_{nm}G\|)$. Let $s > 0$. Then there are $t > 0$ and $K_{s,t} > 0$ such that

$$r_{nm}^2 = \sup_{G\in B} \left(\int_{n-1}^{n} \int_{m-1}^{m} d(G_\lambda H_\mu\, G, G)\right) \le$$

$$\le e^{2ms}\, e^{-2(n-1)t} \sup_{G\in B} \left(\int_{n-1}^{n} \int_{m-1}^{m} e^{-2\mu s} e^{2\lambda t} d(G_\lambda H_\mu\, G, G)\right) \le$$

$$\le e^{2ms}\, e^{-2(n-1)t} \sup_{G\in B} \| e^{tC}\, G(s) \|^2 \le e^{2ms}\, e^{-2nt}\, K_{s,t}^2 .$$

Thus we obtain the following

$$\forall_{s>0}\, \exists_{t>0}\, \exists_{K>0}\, \forall_{n,m\in \mathbb{N}} \;:\; nm\, r_{nm}\, e^{-ms}\, e^{nt} \le K .$$

Define θ on $\mathbb{R}^2$ by

$$\theta(\lambda,\mu) = nm\, r_{nm} \text{ if } r_{nm} \ne 0,\; n-1 \le \lambda < n,\; m-1 \le \mu < m,$$

$$\theta(\lambda,\mu) = e^{-n} \quad \text{ if } r_{nm} = 0 ,$$

$$\theta(\lambda,\mu) = 0 \quad \text{ if } \lambda < 0 \text{ or } \mu < 0 .$$

Then $\theta \in F_+(\mathbb{R}^2)$. To show this, let $s > 0$. Then there are $0 < t < 1$ and $K > 0$ such that for all $\lambda \in [n-1,n)$ and $\mu \in [m-1,m)$

$$\theta(\lambda,\mu)\, e^{\lambda t}\, e^{-\mu s} \le nm\, r_{nm}\, e^{nt}\, e^{-(m-1)s} \le e^{s}\, K_{s,t}$$

if $r_{nm} \ne 0$, and if $r_{nm} = 0$,

$$\theta(\lambda,\mu)\, e^{\lambda t}\, e^{-\mu s} \le e^{-n}\, e^{nt} < 1 .$$

For each $G \in B$ define W by

$$W = \theta(C,\mathcal{D})^{-1} G = \sum_{r_{nm}\ne 0} \left(\frac{r_{nm}^{-1}}{nm}\; P_{nm}\, G\right) .$$

Then we calculate as follows

$$\|w\|_Z^2 = \sum_{r_{nm}\neq 0} n^{-2}m^{-2}(r_{nm}^{-2}\|P_{nm}G\|^2) < \sum_{n,m} n^{-2}m^{-2} = (\frac{\pi^2}{6})^2$$

Hence $w \in Z$ with $\|w\| < \frac{\pi^2}{6}$, and the set $V = \theta(C,\mathcal{D})^{-1}(B)$ is bounded in Z. □

Since $T_{X,A}$ is a special $T(S_{Z,C},\mathcal{D})$ space, Theorem (2.4) yields a characterization of the bounded subsets of $T_{X,A}$.

(2.5) Corollary

Let $B \subset T_{X,A}$. Then B is bounded iff there exists $\psi \in B_+(\mathbb{R})$ and a bounded subset V in X such that $B = \psi(A)(V)$.

Special bounded subsets of $T(S_{Z,C},\mathcal{D})$ are the sets consisting of one single point. This observation leads to the following.

(2.6) Corollary

Let $H \in T(S_{Z,C},\mathcal{D})$. Then there are $w \in Z$ and $\theta \in F_+(\mathbb{R}^2)$ such that $H = \theta(C,\mathcal{D})w$. (Cf. Ch. I, Section 2).

Similar to Lemma (2.3) strong convergence in $T(S_{Z,C},\mathcal{D})$ can be characterized.

(2.7) Lemma

Let (H_ℓ) be a sequence in $T(S_{Z,C},\mathcal{D})$. Then $H_\ell \to 0$ in $T(S_{Z,C},\mathcal{D})$ iff $\forall_{s>0}\ \exists_{t>0} : \|e^{tC} H_\ell(s)\| \to 0$.

Proof. (H_ℓ) is a null sequence in $T(S_{Z,C},D)$ iff $(H_\ell(s))$ is a null sequence in $S_{Z,C}$ for each $s > 0$. From Ch. I, Section 1 the assertion follows. □

(2.8) Theorem

(H_ℓ) is a null sequence in $T(S_{Z,C},D)$ iff there exists a null sequence (ω_ℓ) in Z and $\theta \in F_+(\mathbb{R}^2)$ such that $H_\ell = \theta(C,D)\omega_\ell$.

Proof. The sequence (H_ℓ) is bounded in $T(S_{Z,C},D)$. Then construct $\theta \in F_+(\mathbb{R}^2)$ as in Theorem (2.4):

$$\theta(\lambda,\mu) = nm\, r_{nm} \quad \text{if } r_{nm} \neq 0,\ n-1 \leq \lambda < n,\ m-1 \leq \mu < m\,,$$

$$\theta(\lambda,\mu) = e^{-n} \quad \text{if } r_{nm} = 0\,,$$

$$\theta(\lambda,\mu) = 0 \quad \text{if } \lambda < 0 \text{ or } \mu < 0$$

where $r_{nm} = \max_{\ell\in\mathbb{N}} (\| P_{nm} H_\ell \|)$.

Let $\varepsilon > 0$. Then there are $N,M \in \mathbb{N}$ such that

$$\sum_{(n>N)\vee(m>M)} \frac{1}{n^2 m^2} < (\varepsilon/2)^2.$$

Define $\omega_\ell = \theta(C,D)^{-1} H_\ell = \sum_{r_{nm}\neq 0} \frac{r_{nm}^{-1}}{nm} P_{nm} H_\ell$, $\ell \in \mathbb{N}$. Then for all $\ell \in \mathbb{N}$

$$(*) \qquad \sum_{(n>N)\vee(m>M)} n^{-2}m^{-2}\left(r_{nm}^{-2} \| P_{nm}H_\ell \|^2\right) < (\varepsilon/2)^2.$$

Further, there exist $t > 0$ and $\ell_0 \in \mathbb{N}$ such that for all $\ell > \ell_0$

$$(**) \qquad \sum_{(n\leq N)\wedge(m\leq M)\wedge r_{nm}\neq 0} (n^{-2}m^{-2}\, r_{nm}^{-2} \| P_{nm} H_\ell \|^2) \leq$$

$$\leq e^{2M} \max_{(n\leq N)\wedge(m\leq M)\wedge r_{nm}\neq 0} \left[(r_{nm}^{-2}) \| e^{tC} H_\ell(1) \|^2 \right] < (\varepsilon/2)^2 .$$

A combination of (*) and (**) yields the result

$$\|w_\ell\| < \varepsilon \quad \text{for all } \ell > \ell_0 \qquad \square$$

Since the choice of $\theta \in F_+(\mathbb{R}^2)$ in the proof of the previous theorem has to do only with the boundedness of the sequence (H_ℓ) in $T(S_{Z,C},\mathcal{D})$, Theorem (2.8) implies the following.

(2.9) Corollary

(F_ℓ) is a Cauchy sequence in $T(S_{Z,C},\mathcal{D})$ iff there exists $\theta \in F_+(\mathbb{R}^2)$ and a Cauchy sequence (w_ℓ) in Z such that $F_\ell = \theta(C,\mathcal{D})w_\ell$, $\ell \in \mathbb{N}$. Hence every Cauchy sequence in $T(S_{Z,C},\mathcal{D})$ converges to a limit point.

Further, we have the following extension of the theory in [C].

(2.10) Corollary

(F_ℓ) is a null (Cauchy) sequence in $T_{X,A}$ if there exists a null (Cauchy) sequence (w_ℓ) in X and $\psi \in B_+(\mathbb{R})$ with $F_\ell = \psi(A)w_\ell$, $\ell \in \mathbb{N}$.

Finally we characterize the compact subsets of $T(S_{Z,C},\mathcal{D})$.

(2.11) Theorem

Let $K \subset T(S_{Z,C},\mathcal{D})$. Then K is compact iff there exists $\theta \in F_+(\mathbb{R}^2)$ and a compact subset $W \subset Z$ such that $K = \theta(C,\mathcal{D})(W)$.

Proof.

$\Rightarrow$) Since K is compact, K is bounded in $T(S_{Z,C},\mathcal{D})$. So construct $\theta \in F_+(\mathbb{R}^2)$ and the bounded subset W of Z as in the proof of Theorem (2.4). We shall prove that W is compact. Let (w_ℓ) be a sequence in W. Then $(\theta(C,\mathcal{D})w_\ell)$ is a sequence in K. Since K is compact there exists a subsequence (w_{ℓ_k}) and $w \in Z$ such that

$$\theta(C,\mathcal{D})(w_{\ell_k}-w) \to 0 \quad \text{in } T(S_{Z,C},\mathcal{D}) .$$

The same arguments which led to Theorem (2.8) yield $w_{\ell_k} \to w$ in Z. Hence W is compact in Z.

$\Leftarrow$ Since $\theta(C,\mathcal{D}) : Z \to T(S_{Z,C},\mathcal{D})$ is continuous for each $\theta \in F_+(\mathbb{R}^2)$, the compact set $W \subset Z$ has a compact image $\theta(C,\mathcal{D})(W)$ in $T(S_{Z,C},\mathcal{D})$ for each $\theta \in F_+(\mathbb{R}^2)$ □

(2.12) Corollary

$K \subset T(S_{Z,C},\mathcal{D})$ is compact iff K is sequentially compact.

(2.13) Corollary

$K \subset T_{X,A}$ is compact iff there exists a compact $W \subset X$ and $\psi \in B_+(\mathbb{R})$ such, that $K = \psi(A)(W)$.

(2.14) Theorem

$T(S_{Z,C},\mathcal{D})$ is complete.

Proof. Let (F_α) be a Cauchy net in $T(S_{Z,C},\mathcal{D})$. Then for each $s > 0$ the net $(F_\alpha(s))$ is Cauchy in $S_{Z,C}$. Completeness of $S_{Z,C}$ yields $F(s) \in S_{Z,C}$ with $F_\alpha(s) \to F(s)$. Since $(e^{-s\mathcal{D}})_{s\geq 0}$ is a semigroup of continuous linear mappings on $S_{Z,C}$, the function $s \mapsto F(s)$ is a trajectory of $T(S_{Z,C},\mathcal{D})$. □

Finally,we prove the following result.

(2.15) Lemma

$S_{Z,C}$ is sequentially dense in $T(S_{Z,C},\mathcal{D})$.

Proof. Let $H \in T(S_{Z,C},\mathcal{D})$. Then $H(\frac{1}{n}) \in S_{Z,C}$, $n \in \mathbb{N}$ and $H(\frac{1}{n}) \to H$ in $T(S_{Z,C},\mathcal{D})$.

3. The pairing of $S(T_{Z,C},\mathcal{D})$ and $T(S_{Z,C},\mathcal{D})$

In this section we introduce a pairing of $S(T_{Z,C},\mathcal{D})$ and $T(S_{Z,C},\mathcal{D})$. It is shown that $S(T_{Z,C},\mathcal{D})$ and $T(S_{Z,C},\mathcal{D})$ can be regarded as each other's strong dual spaces.

(3.1) Definition

Let $h \in S(T_{Z,C},\mathcal{D})$ and let $F \in T(S_{Z,C},\mathcal{D})$. Then the number $\langle\langle h,F\rangle\rangle$ is defined by

$$\langle\langle h,F\rangle\rangle = \overline{\langle F(s),\ e^{s\mathcal{D}} h\rangle}\ .$$

Here $\langle\cdot,\cdot\rangle$ denotes the usual pairing of $S_{Z,C}$ and $T_{Z,C}$.

We note that the above definition makes sense for $s > 0$ sufficiently small and that it does not depend on the choice of $s > 0$ because of the trajectory property of F.

(3.2) Theorem

I. Let $F \in T(S_{Z,C},\mathcal{D})$. Then the functional

$$h \mapsto \langle\langle h,F\rangle\rangle$$

is continuous on $S(T_{Z,C},\mathcal{D})$.

II. Let ℓ be a continuous linear functional on $S(T_{Z,C},\mathcal{D})$. Then there exists $G \in T(S_{Z,C},\mathcal{D})$ such that

$$\ell(h) = \langle\!\langle h,G \rangle\!\rangle \quad , \quad h \in S(T_{Z,C},\mathcal{D}) \ .$$

III. Let $h \in S(T_{Z,C},\mathcal{D})$. Then the functional

$$F \mapsto \overline{\langle\!\langle h,F \rangle\!\rangle}$$

is continuous on $T(S_{Z,C},\mathcal{D})$.

IV. Let m be a continuous linear functional on $T(S_{Z,C},\mathcal{D})$. Then there exists $g \in S(T_{Z,C},\mathcal{D})$ such that

$$m(F) = \overline{\langle\!\langle g,F \rangle\!\rangle} \quad , \quad F \in T(S_{Z,C},\mathcal{D}) .$$

<u>Proof</u>.

I. For every $W \in T_{Z,C}$ and every $s > 0$

$$\langle\!\langle e^{-s\mathcal{B}}W,F \rangle\!\rangle = \langle F(s),W \rangle \ ,$$

and $W_n \to 0$ in $T_{Z,C}$ implies $\langle F(s),W_n \rangle \to 0$. Hence the functional $h \mapsto \langle\!\langle h,F \rangle\!\rangle$ is strongly continuous on $S(T_{Z,C},\mathcal{D})$.

II. Because of the definition of inductive limit topology, each linear functional $\ell \circ e^{-s\mathcal{D}}$ is continuous on $T_{Z,C}$. So there exists $G(s) \in S_{Z,C}$ with $(\ell \circ e^{-s\mathcal{D}})(W) = \overline{\langle G(s),W \rangle}$, $W \in T_{Z,C}$, $s > 0$. Since $(e^{-s\mathcal{D}})_{s \geq 0}$ is a semigroup of continuous linear mappings on $S_{Z,C}$ it follows that

$$G(s+\sigma) = e^{-\sigma\mathcal{D}}G(s) \quad , \quad s,\sigma \geq 0 \ .$$

So $s \mapsto G(s)$ is in $T(S_{Z,C},\mathcal{D})$ and

$$\ell(h) = \overline{\langle G(s),e^{s\mathcal{D}}h \rangle} = \langle\!\langle h,G \rangle\!\rangle \ , \ h \in S(T_{Z,C},\mathcal{D}).$$

III. Following Lemma (1.6), there are $w \in Z$, $s > 0$ and $\psi \in B_+(\mathbb{R})$ with $h = e^{-s\mathcal{D}}\psi(C)w$. Hence the inequality

$$|\langle\!\langle h,F\rangle\!\rangle| = |\langle w,\psi(C)F(t)\rangle| \le \|w\|\|\psi(C)F(t)\|$$

the continuity follows.

IV. The strong topology in $T(S_{Z,C},\mathcal{D})$ is generated by the seminorms $p_{\psi,s}$ where $s > 0$ and $\psi \in B_+(\mathbb{R})$. Since m is strongly continuous on $T(S_{Z,C},\mathcal{D})$ there are $\sigma > 0$ and $\varphi \in B_+(\mathbb{R})$ such that

$$|m(F)| \le p_{\varphi,\sigma}(F) = \|\varphi(C)F(\sigma)\|, \ F \in T(S_{Z,C},\mathcal{D}).$$

So the linear functional $m \circ \varphi(C)^{-1}e^{\sigma\mathcal{D}}$ is norm continuous on the dense linear subspace $\varphi(C)e^{-\sigma\mathcal{D}}(T(S_{Z,C},\mathcal{D})) \subset Z$. It therefore can be extended to a continuous linear functional on Z. So there exists $w \in Z$ with

$$(m \circ \varphi(C)^{-1}e^{\sigma\mathcal{D}})(\varphi(C)F(\sigma)) = (\varphi(C)F(\sigma),w).$$

Put $g = \varphi(C)e^{-\sigma\mathcal{D}}w \in S(T_{Z,C},\mathcal{D})$.

<u>Definition</u>

The weak topology on $S(T_{Z,C},\mathcal{D})$ is the topology generated by the seminorms $u_F(h) = |\langle\!\langle h,F\rangle\!\rangle|$, $h \in S(T_{Z,C},\mathcal{D})$.
The weak topology on $S(T_{Z,C},\mathcal{D})$ is the topology generated by the seminorms $u_h(F) = |\langle\!\langle h,F\rangle\!\rangle|$, $F \in T(S_{Z,C},\mathcal{D})$.

A standard argument [Ch], II,§22 shows that the weakly continuous linear functionals on $S(T_{Z,C},\mathcal{D})$ are all obtained by pairing with elements of $T(S_{Z,C},\mathcal{D})$ and vice versa. So it follows that $S(T_{Z,C},\mathcal{D})$ and $T(S_{Z,C},\mathcal{D})$ are reflexive both in the strong and the weak topology.

(3.4) <u>Theorem</u> (Banach-Steinhaus)

I. Let $W \subset T(S_{Z,C},\mathcal{D})$ be weakly bounded. Then W is strongly bounded.

II. Let $V \subset S(T_{Z,C},\mathcal{D})$ be weakly bounded. Then V is strongly bounded.

<u>Proof</u>.

I. Let $s > 0$, and let $\psi \in B_+(\mathbb{R})$. Then following Lemma (1.6) $e^{-s\mathcal{D}}\psi(C)w \in$ $\in S(T_{Z,C},\mathcal{D})$ for each $w \in Z$ and by assumption there exists $N_w > 0$ such that $|\langle\langle e^{-s\mathcal{D}}\psi(C)w,F\rangle\rangle| = |(w,\psi(C)F(s))| \le N_w$, $F \in W$.

By the Banach-Steinhaus theorem for Hilbert spaces there exists $\alpha_{s,\psi} > 0$ such that

$$\|\psi(C)F(s)\| < \alpha_{s,\psi} .$$

With Lemma (2.3) the proof is finished.

II. Let $\theta \in F_+(\mathbb{R}^2)$. Then for each $w \in Z$, $\theta(C,\mathcal{D})w \in T(S_{Z,C},\mathcal{D})$.

By assumption there exists $M_w > 0$ such that

$$|(\theta(C,\mathcal{D})h,w)| \le M_w$$

for each $w \in Z$. Hence for all $h \in V$

$$\|\theta(C,\mathcal{D})h\| \le \alpha_\theta$$

for some $\alpha_\theta > 0$. □

The next theorem characterizes weakly converging sequences in $T(S_{Z,C},\mathcal{D})$.

(3.5) <u>Theorem</u>

$F_\ell \to 0$ in the weak topology of $T(S_{Z,C},\mathcal{D})$ iff there exists a sequence (w_ℓ) in Z with $w_\ell \to 0$ weakly in Z, and a function $\theta \in F_+(\mathbb{R}^2)$ such that $F_\ell = \theta(C,\mathcal{D})w_\ell$, $\ell \in \mathbb{N}$.

Proof

$\Leftarrow$) Trivial

$\Rightarrow$) The null sequence (F_ℓ) is weakly bounded. So by Theorem (3.4) it is a strongly bounded sequence in Z. As in Theorem (2.8) define r_{nm} for $n,m \in \mathbb{N}$ by

$$r_{nm} = \sup_{\ell \in \mathbb{N}} (\| P_{nm} F_\ell \|) .$$

Then $\forall_{s>0} \exists_{t>0} : \sup_{n,m} (nm\, r_{nm}\, e^{-ms} e^{nt}) < \infty$, and the function θ defined by

$$\theta(\lambda,\mu) = n\, m\, r_{nm} \quad \text{if } r_{nm} \neq 0,\ n-1 \le \lambda < n,\ m-1 \le \mu < m ,$$

$$\theta(\lambda,\mu) = e^{-n} \quad \text{if } r_{nm} = 0 ,$$

$$\theta(\lambda,\mu) = 0 \quad \text{elsewhere} ,$$

is in $F_+(\mathbb{R}^2)$. Put $w_\ell = \theta(C,D)^{-1} F_\ell = \sum_{r_{nm} \neq 0} n^{-1} m^{-1} r_{nm}^{-1} P_{nm} F_\ell$, $\ell \in \mathbb{N}$.

Let $u \in Z$, and let $\varepsilon > 0$ and $N,M \in \mathbb{N}$ so large that

$$\sum_{(n>N) \vee (n>M)} (n^{-2} m^{-2}) < (\varepsilon/2)^2 .$$

Then

$$\left| \sum_{\substack{(n>N) \vee (m>M) \\ r_{nm} \neq 0}} (u, P_{nm} w_\ell) \right| \le \| u \| \left(\sum_{\substack{(n>N \vee (m>M) \\ r_{nm} \neq 0}} \| P_{nm} w_\ell \|^2 \right)^{\frac{1}{2}}$$

$$\le \| u \| \left(\sum_{\substack{(n>N) \vee (m>M) \\ r_{nm} \neq 0}} n^{-2} m^{-2} (r_{nm}^{-2} \| P_{nm} F_\ell \|^2) \right)^{\frac{1}{2}} .$$

$$< \varepsilon/2 \| u \| .$$

Further, since $P_{nm}u \in S(T_{Z,C},\mathcal{D})$ for all n,m $\in \mathbb{N}$, there exists $\ell_0 \in \mathbb{N}$ such that for all $\ell > \ell_0$

$$\left| \sum_{\substack{(n\leq N)\wedge(m\leq M),\\ r_{nm}\neq 0}} (u, P_{nm}w_\ell) \right| \leq \left| \left\langle\!\left\langle \left\{ \sum_{\substack{(n\leq N)\wedge(m\leq M),\\ r_{nm}\neq 0}} n^{-1}m^{-1}r_{nm}^{-1}P_{nm}u \right\}, F_\ell \right\rangle\!\right\rangle \right| < \varepsilon/2 .$$

Hence, for each $\varepsilon > 0$ and $u \in Z$ there exists $\ell_0 \in \mathbb{N}$ such that for all $\ell > \ell_0$

$$\left|(u,w_\ell)\right| \leq \left| \sum_{\substack{(n>N)\vee(m>M),\\ r_{nm}\neq 0}} (u,P_{nm}w_\ell) \right| + \left| \sum_{\substack{(n\leq N)\wedge(m\leq M),\\ r_{nm}\neq 0}} (u,P_{nm}w_\ell) \right| < \varepsilon.$$

Thus we have proved that $w_\ell \to 0$ weakly in Z, and

$$F_\ell = \theta(C,\mathcal{D})w_\ell . \qquad \square$$

(3.6) <u>Corollary</u>

I. Strong convergence of a sequence in $T(S_{Z,C},\mathcal{D})$ implies its weak convergence.

II. Any bounded sequence in $T(S_{Z,C},\mathcal{D})$ has a weakly converging subsequence.

(3.7) <u>Corollary</u>

(F_ℓ) is a weakly converging null sequence in $T_{X,A}$ iff there exists a weakly converging null sequence (w_ℓ) in X and a function $\psi \in B_+(\mathbb{R})$ such that $F_\ell = \psi(A)w_\ell$, $\ell \in \mathbb{N}$.

<u>Remark</u>: From Theorem (2.4) and Definition (3.2) it follows that the strong topology in $S(T_{Z,C},\mathcal{D})$ equals the so-called Mackey topology (Cf.[Tr],p.369).

4. Spaces related to the operators $C \vee \mathcal{D}$ and $C \wedge \mathcal{D}$

As in the previous sections, $(G_\lambda)_{\lambda\in\mathbb{R}}$ and $(H_\mu)_{\mu\in\mathbb{R}}$ denote the spectral resolutions of C and $\mathcal{D}$. The orthogonal projection P, defined by

$$P = \iint_{\lambda\geq\mu} dG_\lambda H_\mu$$

commutes with C as well as $\mathcal{D}$.

(4.1) Definition

The nonnegative, self-adjoint operator $C \wedge \mathcal{D}$ is defined by

$$C \wedge \mathcal{D} = PCP + (I - P)\mathcal{D}(I - P) .$$

The nonnegative, self-adjoint operator $C \vee \mathcal{D}$ is defined by

$$C \vee \mathcal{D} = (I - P)C(I - P) + P\mathcal{D}P.$$

Remark: The operators $C \wedge \mathcal{D}$ and $C \vee \mathcal{D}$ are also given by

$$C \wedge \mathcal{D} = \iint_{\mathbb{R}^2} \max(\lambda,\mu)dG_\lambda H_\mu \ , \ C \vee \mathcal{D} = \iint_{\mathbb{R}^2} \min(\lambda,\mu)dG_\lambda H_\mu \ .$$

The spaces $S_{Z,C\vee\mathcal{D}}$, $S_{Z,C\wedge\mathcal{D}}$, $T_{Z,C\vee\mathcal{D}}$ and $T_{Z,C\wedge\mathcal{D}}$ are well-defined by Ch. I, Section 1 and 2. With the aid of these spaces sums and intersections of $S_{Z,C}$, $S_{Z,\mathcal{D}}$, $T_{Z,C}$, and $T_{Z,\mathcal{D}}$ can be described.

(4.2) Theorem

I. $S_{Z,C\wedge\mathcal{D}} = S_{Z,C+\mathcal{D}} = S_{Z,C} \cap S_{Z,\mathcal{D}}$

II. $S_{Z,C\vee\mathcal{D}} = S_{Z,C} + S_{Z,\mathcal{D}}$

III. $T_{Z,C\wedge\mathcal{D}} = T_{Z,C+\mathcal{D}} = T_{Z,C} + T_{Z,\mathcal{D}}$

IV. $T_{Z,C\vee D} = T_{Z,C} \cap T_{Z,D}$.

(In II, + denotes the usual sum in Z, and in III the usual sum in $T_{Z,C+D}$.)

Proof. From the definition of the projection P we derive easily that for all $t > 0$ the operators $Pe^{-tC}e^{tD}P$ and $(I-P)e^{-tD}e^{tC}(I-P)$ are bounded in Z.

I. Let $f \in S_{Z,C\wedge D}$. Then there are $t > 0$ and $w \in Z$ such that

$$f = e^{-t(C\wedge D)}w = Pe^{-tC}Pw + (I-P)e^{-tD}(I-P)w .$$

So $f = e^{-tC}\tilde{w}$ with $\tilde{w} = Pw + (I-P)e^{tC}e^{-tD}(I-P)w \in Z$, and hence $f \in S_{Z,C}$. Similarly it follows that $f \in S_{Z,D}$.

On the other hand, let $g \in S_{Z,C} \cap S_{Z,D}$. Then for some $w, v \in Z$ and $t > 0$,

$$g = e^{-tC}w \quad \text{and} \quad g = e^{-tD}v .$$

So g can be written as

$$g = Pg + (I-P)g = Pe^{-tC}Pw + (I-P)e^{-tD}(I-P)v =$$

$$= e^{-t(C\wedge D)}(Pw + (I-P)v) \in S_{Z,C\wedge D}.$$

Finally, we prove that $S_{Z,C\wedge D} = S_{Z,C+D}$.

Since $C + D \geq C \wedge D$ it is obvious that $S_{Z,C+D} \subset S_{Z,C\wedge D}$.

Now let $f \in S_{Z,C\wedge D}$. Then $f = (Pe^{-tC}P + (I-P)e^{-tD}(I-P))w$ for certain $t > 0$ and $w \in Z$. Thus we find

$$f = e^{-\frac{1}{2}t(C+D)}[Pe^{-\frac{1}{2}tC}e^{\frac{1}{2}tD}P + (I-P)e^{\frac{1}{2}tD}e^{\frac{1}{2}tC}(I-P)]w, \text{ and}$$

and hence $f \in S_{Z,C+D}$.

II. Let $f \in S_{Z,C\vee D}$. Then there are $w \in Z$ and $t > 0$ such that

$$f = e^{-t(C\vee D)}w = Pe^{-tD}Pw + (I-P)e^{-tC}(I-P)w .$$

So $f \in S_{Z,C} + S_{Z,\mathcal{D}}$. On the other hand let $u,v \in Z$ and let $t > 0$. Put $g = e^{-tC}u + e^{-t\mathcal{D}}v$. Then

$$g = e^{-t(C\vee\mathcal{D})}[e^{t(C\vee\mathcal{D})}e^{-tC}u + e^{t(C\vee\mathcal{D})}e^{-t\mathcal{D}}v].$$

Since $C \vee \mathcal{D} \le C$ and $C \vee \mathcal{D} \le \mathcal{D}$, this yields $g \in S_{Z,C\vee\mathcal{D}}$.

III. Let $G \in T_{Z,C\wedge\mathcal{D}}$. Then $w \in Z$ and $\varphi \in B_+(\mathbb{R})$ are such that $G = \varphi(C \wedge \mathcal{D})w$. Since $\varphi(C \wedge \mathcal{D}) = \varphi(C)P + \varphi(\mathcal{D})(I - P)$,

$$G = \varphi(C)Pw + \varphi(\mathcal{D})(I - P)w \in T_{Z,C} + T_{Z,\mathcal{D}}.$$

On the other hand let $\varphi,\psi \in B_+(\mathbb{R})$ and let $u,v \in Z$. Put

$$G = \varphi(C)u + \psi(\mathcal{D})v .$$

Since the operators $\varphi(C)e^{-t(C\wedge\mathcal{D})}$ and $\psi(\mathcal{D})e^{-t(C\wedge\mathcal{D})}$, $t > 0$, are bounded on Z, for all $t > 0$

$$e^{-t(C\wedge\mathcal{D})}G = (e^{-t(C\wedge\mathcal{D})}\varphi(C)u + e^{-t(C\wedge\mathcal{D})}\psi(\mathcal{D})v) \in Z .$$

Hence $G \in T_{Z,C\wedge\mathcal{D}}$. Because $S_{Z,C\wedge\mathcal{D}} = S_{Z,C+\mathcal{D}}$ also topologically, it is clear that $T_{Z,C\wedge\mathcal{D}} = T_{Z,C+\mathcal{D}}$.

IV. Let $H \in T_{Z,C} \cap T_{Z,\mathcal{D}}$. Then there are $\psi,\chi \in B_+(\mathbb{R})$ and $v,w \in Z$ such that $H = \psi(C)w$ and $H = \chi(\mathcal{D})v$. So H can be written as

$$H = \psi(C)(I - P)w + \chi(\mathcal{D})Pv ,$$

and $e^{-t(C\vee\mathcal{D})}H = e^{-tC}\psi(C)(I - P)w + e^{-t\mathcal{D}}\chi(\mathcal{D})Pv \in Z$. This implies $H \in T_{Z,C\vee\mathcal{D}}$. Since $C \vee \mathcal{D} \le C$ and $C \vee \mathcal{D} \le \mathcal{D}$ we have

$$T_{Z,C\vee\mathcal{D}} \subset T_{Z,C} \text{ and } T_{Z,C\vee\mathcal{D}} \subset T_{Z,\mathcal{D}}.$$

□

It is obvious that the operators $C \wedge \mathcal{D}$ and $C \vee \mathcal{D}$ commute. So the spaces $S(T_{C\wedge\mathcal{D}}, C \vee \mathcal{D})$, $S(T_{C\vee\mathcal{D}}, C \wedge \mathcal{D})$, $T(S_{C\wedge\mathcal{D}}, C \vee \mathcal{D})$, $T(S_{C\vee\mathcal{D}}, C \wedge \mathcal{D})$ are well defined. Here, for convenience, we have omitted the subscript Z. Similar to Theorem (4.2) we shall prove the following.

(4.3) Theorem

I. $S(T_C,\mathcal{D}) \cap S(T_{\mathcal{D}},C) = S(T_{C\vee\mathcal{D}}, C \wedge \mathcal{D})$,

II. $S(T_C,\mathcal{D}) + S(T_{\mathcal{D}},C) = S(T_{C\wedge\mathcal{D}}, C \vee \mathcal{D})$,

III. $T(S_C,\mathcal{D}) \cap T(S_{\mathcal{D}},C) = T(S_{C\wedge\mathcal{D}}, C \vee \mathcal{D})$,

IV. $T(S_C,\mathcal{D}) + T(S_{\mathcal{D}},C) = T(S_{C\vee\mathcal{D}}, C \wedge \mathcal{D})$.

Proof

I. Let $k \in S(T_C,\mathcal{D}) \cap S(T_{\mathcal{D}},C)$. Then there are $\varphi,\psi \in B_+(\mathbb{R})$, $t > 0$ and $u,v \in Z$ such that $k = e^{-tC}\varphi(\mathcal{D})u$ and $k = e^{-t\mathcal{D}}\psi(C)v$.

Put $\chi = \max(\varphi,\psi)$. Then $\chi \in B_+(\mathbb{R})$ and k is given by

$$k = e^{-tC}\chi(\mathcal{D})\tilde{u} \text{ and } k = e^{-t\mathcal{D}}\chi(C)\tilde{v}$$

with $\tilde{u} = \chi^{-1}(\mathcal{D})\varphi(\mathcal{D})u \in Z$ and $\tilde{v} = \chi^{-1}(C)\psi(C)v \in Z$. So

$$k = Pk + (I-P)k = Pe^{-tC}\chi(\mathcal{D})\tilde{u} + (I-P)e^{-t\mathcal{D}}\chi(C)\tilde{v}$$

$$= e^{-t(C\wedge\mathcal{D})}\chi(C \vee \mathcal{D})[P\tilde{u} + (I-P)\tilde{v}] .$$

This yields $k \in S(T_{C\vee\mathcal{D}}, C \wedge \mathcal{D})$.

On the other hand, let $\varphi \in B_+(\mathbb{R})$ and let $w \in Z$, $t > 0$. Then for $h = \varphi(C \vee \mathcal{D})e^{-t(C\wedge\mathcal{D})}w$,

$$h = \varphi(C)e^{-t\mathcal{D}}(\varphi(C)^{-1}\varphi(C \vee \mathcal{D})e^{t\mathcal{D}}e^{-t(C\wedge\mathcal{D})}w).$$

Hence $h \in S(T_C,\mathcal{D})$. Similarly it can be shown that $h \in S(T_{\mathcal{D}},C)$.

II. Let $h \in S(T_{\mathcal{C}},\mathcal{D}) + S(T_{\mathcal{D}},\mathcal{C})$. Then there are $w,\nu \in Z$, $t > 0$ and $\chi \in B_+(\mathbb{R})$, such that

$$h = e^{-t\mathcal{C}}\chi(\mathcal{D})w + e^{-t\mathcal{D}}\chi(\mathcal{C})\nu \ .$$

Hence h can be written as

$$h = e^{-t(\mathcal{C}\vee\mathcal{D})}\chi(\mathcal{C} \wedge \mathcal{D})[e^{t(\mathcal{C}\vee\mathcal{D})}e^{-t\mathcal{C}}\chi^{-1}(\mathcal{C} \wedge \mathcal{D})\chi(\mathcal{D})w +$$

$$+ e^{t(\mathcal{C}\vee\mathcal{D})}e^{-t\mathcal{D}}\chi^{-1}(\mathcal{C} \wedge \mathcal{D})\chi(\mathcal{C})\nu] \ .$$

Since $\mathcal{C} \vee \mathcal{D} \leq \mathcal{C},\mathcal{D}$ and $\mathcal{C} \wedge \mathcal{D} \geq \mathcal{C},\mathcal{D}$, this yields $h \in S(T_{\mathcal{C}\wedge\mathcal{D}},\mathcal{C} \vee \mathcal{D})$.

In order to prove the other inclusion, assume that $g \in S(T_{\mathcal{C}\wedge\mathcal{D}},\mathcal{C} \vee \mathcal{D})$. Then there are $w \in Z, t > 0$ and $\varphi \in B_+(\mathbb{R})$ such, that

$$g = e^{-t(\mathcal{C}\vee\mathcal{D})}\varphi(\mathcal{C} \wedge \mathcal{D})w =$$

$$= e^{-t\mathcal{D}}\varphi(\mathcal{C})Pw + e^{-t\mathcal{C}}\varphi(\mathcal{D})(I - P)w \in S(T_{\mathcal{C}},\mathcal{D}) + S(T_{\mathcal{D}},\mathcal{C}).$$

III. Let $Q \in T(S_{\mathcal{C}},\mathcal{D}) \cap T(S_{\mathcal{D}},\mathcal{C})$ and let $t > 0$. Then there exists $s > 0$ such, that $e^{s\mathcal{C}}e^{-t\mathcal{D}}Q \in Z$ and $e^{s\mathcal{D}}e^{-t\mathcal{C}}Q \in Z$.
Hence $Pe^{s\mathcal{C}}e^{-t\mathcal{D}}PQ \in Z$ and $(I - P)e^{s\mathcal{D}}e^{-t\mathcal{C}}(I - P)Q \in Z$ which implies $e^{s(\mathcal{C}\wedge\mathcal{D})}e^{-t(\mathcal{C}\vee\mathcal{D})}Q \in Z$.

On the other hand, let $R \in T(S_{\mathcal{C}\wedge\mathcal{D}},\mathcal{C} \vee \mathcal{D})$, and let $t > 0$. Then take $s > 0$ such, that $e^{s(\mathcal{C}\wedge\mathcal{D})}e^{-t(\mathcal{C}\vee\mathcal{D})}R \in Z$. This yields

$$e^{s\mathcal{D}}e^{-t\mathcal{C}}R = [Pe^{s\mathcal{D}}e^{-t\mathcal{C}}P + (I - P)e^{s\mathcal{D}}e^{-t\mathcal{C}}(I - P)]R$$

$$= [Pe^{(s+t)\mathcal{D}}e^{-(s+t)\mathcal{C}}P + (I - P)][e^{s(\mathcal{C}\wedge\mathcal{D})}e^{-t(\mathcal{C}\vee\mathcal{D})}]R \ .$$

So R can be seen as an element of $T(S_{\mathcal{D}},\mathcal{C})$, and similarly as an element of $T(S_{\mathcal{C}},\mathcal{D})$.

IV. Let $Q \in \mathcal{T}(S_{\mathcal{C}},\mathcal{D}) + \mathcal{T}(S_{\mathcal{D}},\mathcal{C})$. Then there are $Q_1 \in \mathcal{T}(S_{\mathcal{C}},\mathcal{D})$ and $Q_2 \in \mathcal{T}(S_{\mathcal{D}},\mathcal{C})$ such that $Q = Q_1 + Q_2$ with the sum understood in $\mathcal{T}_{\mathcal{C}+\mathcal{D}}$. Let $t > 0$. Then there is $s > 0$ such that

$$e^{s\mathcal{C}}e^{-t\mathcal{D}}Q_1 \in Z \text{ and } e^{s\mathcal{D}}e^{-t\mathcal{C}}Q_2 \in Z .$$

Hence $e^{s(\mathcal{C}\vee\mathcal{D})}e^{-t(\mathcal{C}\wedge\mathcal{D})}Q =$

$$= (\mathcal{P}e^{(t+s)\mathcal{D}}e^{-(t+s)\mathcal{C}}\mathcal{P} + (\mathcal{I} - \mathcal{P}))e^{s\mathcal{C}}e^{-t\mathcal{D}}Q_1 +$$

$$+ (\mathcal{P} + (\mathcal{I} - \mathcal{P})e^{(t+s)\mathcal{C}}e^{-(t+s)\mathcal{D}}(\mathcal{I} - \mathcal{P}))e^{s\mathcal{D}}e^{-t\mathcal{C}}Q_2 ,$$

so that $Q \in \mathcal{T}(S_{\mathcal{C}\vee\mathcal{D}},\mathcal{C} \wedge \mathcal{D})$.

Finally, let $R \in \mathcal{T}(S_{\mathcal{C}\vee\mathcal{D}},\mathcal{C} \wedge \mathcal{D})$ and let $t > 0$. Then there is $s > 0$ with

$$e^{s(\mathcal{C}\vee\mathcal{D})}e^{-t(\mathcal{C}\wedge\mathcal{D})}R \in Z.$$

Hence $R = \mathcal{P}R + (\mathcal{I} - \mathcal{P})R$ and $e^{s\mathcal{D}}e^{-t\mathcal{C}}\mathcal{P}R =$

$$= \mathcal{P}e^{s(\mathcal{C}\vee\mathcal{D})}e^{-t(\mathcal{C}\wedge\mathcal{D})}R \in Z \text{ and similarly } e^{s\mathcal{C}}e^{-t\mathcal{D}}R \in Z.$$

Thus we have shown $R \in \mathcal{T}(S_{\mathcal{C}},\mathcal{D}) + \mathcal{T}(S_{\mathcal{D}},\mathcal{C})$. □

The preceding theorems play a major role in the inclusion scheme which we give in Section 5. The results of Theorem (4.3) will lead to a fifth Kernel theorem in the following chapter.

5. The inclusion scheme

The spaces which are introduced in [G] and in the previous sections fit into an inclusion scheme. Here we shall give some properties of the spaces in this scheme. The reader may as well skip the proofs. They are added for completeness. Let $\tilde{C}$ and $\tilde{\mathcal{D}}$ denote two commuting, nonnegative, self-adjoint operators in Z.

(5.1) Lemma

Let $\tilde{C} \geq \tilde{\mathcal{D}}$. Then

$$S(T_{\tilde{\mathcal{D}}},\tilde{C}) = S_{\tilde{C}} \quad \text{and} \quad T(S_{\tilde{\mathcal{D}}},\tilde{C}) = T_{\tilde{C}}.$$

Proof. It is clear that $S_{\tilde{C}} \subset S(T_{\tilde{\mathcal{D}}},\tilde{C})$ and $T(S_{\tilde{\mathcal{D}}},\tilde{C}) \subset T_{\tilde{C}}$.

So let $f \in S(T_{\tilde{\mathcal{D}}},\tilde{C})$. Then there are $t > 0$ and $\varphi \in B_+(\mathbb{R})$ and $w \in Z$ such that $f = e^{-t\tilde{C}}\varphi(\tilde{\mathcal{D}})w$. Hence

$$f = e^{-t/2\tilde{C}}(\varphi(\tilde{\mathcal{D}})e^{-t/2\tilde{C}}w) \in S_{\tilde{C}} ,$$

because $\varphi(\tilde{\mathcal{D}})e^{-t/2\tilde{C}}$ is a bounded operator on Z.

Similarly, $T_{\tilde{C}} \subset T(S_{\tilde{\mathcal{D}}},\tilde{C})$ can be proved. □

(5.2) Lemma

$S(T_{\tilde{\mathcal{D}}},\tilde{C}) \subset T(S_{\tilde{C}},\tilde{\mathcal{D}})$.

Proof. Let $h \in S(T_{\tilde{\mathcal{D}}},\tilde{C})$. Then h can be written as

$$h = e^{-t\tilde{C}}\varphi(\tilde{\mathcal{D}})w \quad ,$$

where $t > 0$, $\varphi \in B_+(\mathbb{R})$ and $w \in Z$. Hence, for all $s > 0$,

$$e^{-s\tilde{\mathcal{D}}}e^{t\tilde{C}}h = \varphi(\tilde{\mathcal{D}})e^{-s\tilde{\mathcal{D}}}w \in Z.$$

With $\text{emb}(h) : s \mapsto e^{-s\tilde{\mathcal{D}}}h$, the proof is complete. □

$$\begin{array}{ccccccc}
S_{\mathcal{C}\vee\mathcal{D}} & \subset & S(T_{\mathcal{C}\wedge\mathcal{D}},\mathcal{C}\vee\mathcal{D}) & \subset & T(S_{\mathcal{C}\vee\mathcal{D}},\mathcal{C}\wedge\mathcal{D}) & = & T_{\mathcal{C}\wedge\mathcal{D}} \\
\Vert & & \cup & & \cup & & \cup \\
S_{\mathcal{C}\vee\mathcal{D}} & \subset & S(T_{\mathcal{D}},\mathcal{C}\vee\mathcal{D}) & \subset & T(S_{\mathcal{C}\vee\mathcal{D}},\mathcal{D}) & = & T_{\mathcal{D}} \\
\cup & & \cup & & \cup & & \Vert \\
S_{\mathcal{C}} & \subset & S(T_{\mathcal{D}},\mathcal{C}) & \subset & T(S_{\mathcal{C}},\mathcal{D}) & \subset & T_{\mathcal{D}} \\
\Vert & & \cup & & \cup & & \cup \\
S_{\mathcal{C}} & = & S(T_{\mathcal{C}\vee\mathcal{D}},\mathcal{C}) & \subset & T(S_{\mathcal{C}},\mathcal{C}\vee\mathcal{D}) & \subset & T_{\mathcal{C}\vee\mathcal{D}} \\
\cup & & \cup & & \cup & & \Vert \\
S_{\mathcal{C}\wedge\mathcal{D}} & = & S(T_{\mathcal{C}\vee\mathcal{D}},\mathcal{C}\wedge\mathcal{D}) & \subset & T(S_{\mathcal{C}\wedge\mathcal{D}},\mathcal{C}\vee\mathcal{D}) & \subset & T_{\mathcal{C}\vee\mathcal{D}} \\
\cap & & \cap & & \cap & & \Vert \\
S_{\mathcal{D}} & = & S(T_{\mathcal{C}\vee\mathcal{D}},\mathcal{D}) & \subset & T(S_{\mathcal{D}},\mathcal{C}\vee\mathcal{D}) & \subset & T_{\mathcal{C}\vee\mathcal{D}} \\
\Vert & & \cap & & \cap & & \cap \\
S_{\mathcal{D}} & \subset & S(T_{\mathcal{C}},\mathcal{D}) & \subset & T(S_{\mathcal{D}},\mathcal{C}) & \subset & T_{\mathcal{C}} \\
\cap & & \cap & & \cap & & \Vert \\
S_{\mathcal{C}\vee\mathcal{D}} & \subset & S(T_{\mathcal{C}},\mathcal{C}\vee\mathcal{D}) & \subset & T(S_{\mathcal{C}\vee\mathcal{D}},\mathcal{C}) & = & T_{\mathcal{C}} \\
\Vert & & \cap & & \cap & & \cap \\
S_{\mathcal{C}\vee\mathcal{D}} & \subset & S(T_{\mathcal{C}\wedge\mathcal{D}},\mathcal{C}\vee\mathcal{D}) & \subset & T(S_{\mathcal{C}\vee\mathcal{D}},\mathcal{C}\wedge\mathcal{D}) & = & T_{\mathcal{C}\wedge\mathcal{D}}
\end{array}$$

Fig. (5.3) The inclusion scheme

A row in the inclusion scheme (5.3) is of the form

(5.4) $$S_{\tilde{\mathcal{C}}} \subset S(T_{\tilde{\mathcal{D}}},\tilde{\mathcal{C}}) \subset T(S_{\tilde{\mathcal{C}}},\tilde{\mathcal{D}}) \subset T_{\tilde{\mathcal{D}}}.$$

(5.5) Theorem

In (5.4) all embeddings are continuous and have dense ranges.

Proof. We proceed in three steps.

(i) $S_{\tilde{\mathcal{C}}} \subset S(T_{\tilde{\mathcal{D}}},\tilde{\mathcal{C}})$

Let (w_n) be a null sequence in $S_{\tilde{\mathcal{C}}}$. Then there is $t > 0$ such that

$e^{t\widetilde{C}} w_n \to 0$ in Z. So for all $s > 0$

$$e^{t\widetilde{C}} \mathrm{emb}(w_n)(s) = e^{t\widetilde{C}} e^{-s\widetilde{\mathcal{D}}} w_n \to 0$$

in X. This proves that the embedding emb : $S_{\widetilde{C}} \hookrightarrow S(T_{\widetilde{\mathcal{D}}},\widetilde{C})$ is continuous. To show that $S_{\widetilde{C}}$ is dense in $S(T_{\widetilde{\mathcal{D}}},\widetilde{C})$, let $H \in T(S_{\widetilde{\mathcal{D}}},\widetilde{C})$ with $\langle\langle f,H\rangle\rangle = 0$ for all $f \in S_{\widetilde{C}}$. Then $\langle f,H\rangle = 0$ for all $f \in S_{\widetilde{C}}$. So $H = 0$, and $S_{\widetilde{C}}$ is dense in $S(T_{\widetilde{\mathcal{D}}},\widetilde{C})$.

(ii) $S(T_{\widetilde{\mathcal{D}}},\widetilde{C}) \subset T(S_{\widetilde{C}},\widetilde{\mathcal{D}})$.

First we remind that in Lemma (5.2) we showed how $S(T_{\widetilde{\mathcal{D}}},\widetilde{C})$ can be embedded in $T(S_{\widetilde{C}},\widetilde{\mathcal{D}})$. The embedding is continuous. To show this, let $s > 0$ and $\psi \in B_+(\mathbb{R})$. Then the seminorm

$$h \to \|\psi(\widetilde{C})e^{-s\widetilde{\mathcal{D}}}h\|$$

is continuous on $S(T_{\widetilde{\mathcal{D}}},\widetilde{C})$.

Now let $g \in S(T_{\widetilde{C}},\widetilde{\mathcal{D}})$, the dual of $T(S_{\widetilde{C}},\widetilde{\mathcal{D}})$. Then g can be written as $g = \varphi(\widetilde{C})u$ where $u \in S_{\widetilde{\mathcal{D}}}$ and $\varphi \in B_+(\mathbb{R})$. Suppose

$$\langle\langle g,h\rangle\rangle = 0 \quad , \quad h \in S(T_{\widetilde{\mathcal{D}}},\widetilde{C}).$$

Then for all $f \in S_{\widetilde{C}}$ and all $\chi \in B_+(\mathbb{R})$

$$(\varphi(\widetilde{C})f \ , \ \chi(\widetilde{\mathcal{D}})u) = 0 \ .$$

Hence $u = 0$, and $S(T_{\widetilde{\mathcal{D}}},\widetilde{C})$ is dense in $T(S_{\widetilde{C}},\widetilde{\mathcal{D}})$.

(iii) $T(S_{\widetilde{C}},\widetilde{\mathcal{D}}) \subset T_{\widetilde{\mathcal{D}}}$.

The continuity of the embedding follows from the continuity of the seminorms

$$t \to \|H(t)\| \quad , \ t > 0 \ ,$$

on $T(S_{\widetilde{C}},\widetilde{\mathcal{D}})$.

Further, let $f \in S_{\tilde{\mathcal{D}}}$ and suppose $\langle f,H\rangle = 0$ for all $H \in T(S_{\tilde{\mathcal{C}}},\tilde{\mathcal{D}})$. Then $(f,h) = 0$ for all $h \in S_{\tilde{\mathcal{C}}}$. So $f = 0$. □

Consider the inclusion subscheme of (5.3).

(5.6) $$S_{C\wedge\mathcal{D}} \subset S_{C} \subset S_{C\vee\mathcal{D}} .$$

Then similar to Theorem (5.5) we show

(5.7) Theorem

In (5.6) all embeddings are continuous and have dense ranges.

Proof. We proceed in two steps.

(i) Let (f_n) be a null sequence in $S_{C\wedge\mathcal{D}}$. Then there is $t > 0$ such that $\|e^{t(C\wedge\mathcal{D})}f_n\| \to 0$. Hence

$$\|e^{tC}f_n\| \le \|e^{tC}e^{-t(C\wedge\mathcal{D})}\|\,\|e^{t(C\wedge\mathcal{D})}f_n\| \to 0.$$

Further, let $G \in T_C$ and suppose for all $f \in S_{C\wedge\mathcal{D}}$,

$$\langle f,G\rangle = 0 .$$

So for all $\chi \in Z$ and $t > 0$, $(\chi, e^{-t(C\wedge\mathcal{D})}G) = 0$. This implies $G = 0$, and hence $S_{C\wedge\mathcal{D}}$ is dense in S_C.

(ii) $S_C \subset S_{C\vee\mathcal{D}}$:

Follows from (i) because $C = (C \vee \mathcal{D}) \wedge C$. □

(5.8) Corollary

In the inclusion scheme

$$T_{C\vee\mathcal{D}} \subset T_C \subset T_{C\wedge\mathcal{D}}$$

all embeddings are continuous and have dense ranges.

Proof. Follows from Theorem (5.7) by duality.

Finally we consider the inclusion subscheme.

(5.9) $$T(S_{C\wedge D}, C \vee D) \subset T(S_C, C \vee D) \subset T(S_C, D) \ .$$

We prove

(5.10) Theorem

In (5.9) all embeddings are continuous and have dense ranges.

Proof. We proceed in two steps.

(i) Since the seminorms

$$F \to \|\varphi(C)e^{-t(C\vee D)}F\| \quad , \quad t > 0, \ \varphi \in B_+(\mathbb{R})$$

are continuous in $T(S_{C\wedge D}, C \vee D)$, the embedding of $T(S_{C\wedge D}, C \vee D)$ in $T(S_C, C \vee D)$ is continuous. Further, $S_{C\wedge D} \subset T(S_{C\wedge D}, C \vee D)$ is dense in S_C, and S_C is dense in $T(S_C, C \vee D)$. So $T(S_{C\wedge D}, C \vee D)$ is dense in $T(S_C, C \vee D)$. (See Lemma (1.16)).

(ii) The seminorms

$$G \to \|\varphi(C)e^{-tD}G\| \quad , \quad t > 0, \ \varphi \in B_+(\mathbb{R}) \quad ,$$

are continuous in $T(S_C, C \vee D)$. So the embedding from $T(S_C, C \vee D)$ into $T(S_C, D)$ is continuous. Further we note that S_C is dense both in $T(S_C, C \vee D)$ and in $T(S_C, D)$ by Theorem (2.15). Hence $T(S_C, C \vee D)$ is dense in $T(S_C, D)$. □

(5.11) Corollary

In the inclusion scheme

$$S(T_{C\wedge D}, C \vee D) \supset S(T_C, C \vee D) \supset S(T_C, D)$$

all embeddings are continuous and have dense ranges.

Finally, the main result of this section will be given.

(5.12) Theorem

In (5.3) all embeddings are continuous and have dense ranges.

Proof. Follows from Theorem (5.5), (5.7) and (5.10), and from Corollary (5.8) and (5.11). □

III. ON CONTINUOUS LINEAR MAPPINGS BETWEEN ANALYTICITY AND TRAJECTORY SPACES

Introduction

Here X and Y will denote Hilbert spaces, and A will be a nonnegative self-adjoint operator in X and B a nonnegative self-adjoint operator in Y. In [G], the fourth chapter contains a detailed discussion of the four types of continuous linear mappings:

$S_{X,A} \to S_{Y,B}$, $S_{X,A} \to T_{Y,B}$, $T_{X,A} \to S_{Y,B}$, $T_{X,A} \to T_{Y,B}$. Cf. Ch. I, Section 4.

In order to prove a Kernel theorem for each of these types, in addition to the topological tensor products $S_{X\otimes Y, A\boxplus B}$ and $T_{X\otimes Y, A\boxplus B}$, the spaces Σ'_A and Σ'_B have been introduced. Σ'_A and Σ'_B are topological tensor products of $T_{X,A}$ and $S_{Y,B}$ and of $S_{X,A}$ and $T_{Y,B}$.

In order to gain a deeper understanding of the topological structure of these spaces Σ'_A and Σ'_B, we have introduced the more general type of spaces $T(S_{Z,C}, D)$ and $S(T_{Z,C}, D)$, where C and D are commuting nonnegative self-adjoint operators in the Hilbert space Z. The following relations have been mentioned:

$$\Sigma'_A = T(S_{X\otimes Y, I\otimes B}, A\otimes I) \quad , \quad \Sigma_A = S(T_{X\otimes Y, I\otimes B}, A\otimes I) \quad ,$$

$$\Sigma'_B = T(S_{X\otimes Y, A\otimes I}, I\otimes B) \quad , \quad \Sigma_B = S(T_{X\otimes Y, A\otimes I}, I\otimes B) \quad .$$

So obviously results in Ch. II apply to the spaces Σ'_A, Σ'_B, Σ_A and Σ_B.

Thus, the intersection of Σ_A' and Σ_B' is a space of type $T(S_{Z,C},\mathcal{D})$. This observation leads to a Kernel theorem for so-called extendable mappings. Cf. Ch. I, Section 4.

Precise formulations of the above-mentioned five Kernel theorems can be found in Section 1. In the remaining sections we consider the case X = Y and $A = B$. Hence, we investigate the spaces

$$T^A = T(S_{X\otimes X,I\otimes A},A\otimes I) \text{ and } T_A = T(S_{X\otimes X,A\otimes I},I\otimes A).$$

In Section 2 we shall prove that T^A and T_A admit an algebraic structure and that they are homeomorphic. The homeomorphism is denoted by ^{c}. The mapping ^{c} is also a homeomorphism from the space $S_A = S(T_{X\otimes X,A\otimes I},I\otimes A)$ onto $S^A = S(T_{X\otimes X,I\otimes A},A\otimes I)$. Put $E_A = T^A \cap T_A$. Then E_A is an algebra and it inherits several properties of the algebras T^A and T_A. The mapping ^{c} is an involution on E_A. The strong dual E_A' equals the algebraic sum $S_A + S^A$. We shall extend ^{c} to E_A' in a natural way.

In the sequel we shall confine our attention to nuclear analyticity spaces $S_{X,A}$. Then, because of the Kernel theorems the space $T^A(T_A)$ comprises all continuous linear mappings from $S_{X,A}(T_{X,A})$ into itself. Inspired by operator theory for Hilbert spaces, we introduce the topology of pointwise and weak pointwise convergence in $T^A(T_A)$. These topologies correspond to the strong and weak operator topology for Von Neumann algebras, while the weak and strong topology of $T^A(T_A)$ correspond to the ultra-weak and uniform operator topology.

In Sections 3 and 4 we study the relations between the algebraic and the topological structure of T^A and T_A. It appears that separate multiplication is continuous in all mentioned topologies. The effects of the results of the previous sections on the algebra E_A and its strong dual E'_A are investigated in Section 5.

In Section 6 we indicate possibilities to interprete parts of quantum statistics by means of the mathematical apparatus developed for the spaces E_A and E'_A. They seem to be more appropriate than any operator algebra on a Hilbert space, because in general E_A contains unbounded, self-adjoint operators. However, we emphasize that we consider it as an Ansatz only. We are not fully aware of all consequences of such redescription.

If the Kernel theorem holds true, each continuous linear mapping from $S_{X,A}$ into itself has a well-defined infinite matrix. Section 7 of this paper is devoted to a thorough description of this kind of matrices. There are manageable, necessary and sufficient conditions on the entries of an infinite matrix, such, that its corresponding linear mapping is continuous on $S_{X,A}$. The thus obtained identification between T^A and a class $M(T^A)$ of well-specified infinite matrices enables us to construct a large variety of elements in T^A. Particularly, we note here that the matrix calculus will be of great importance in a forthcoming paper on one-parameter (semi-)groups of elements of T^A. In Section 8 we treat a subclass of $M(T^A)$, the class of unbounded weighted shifts. Weighted shifts are the simplest, non-trivial operators in T^A.

In the final section our matrix calculus yields the construction of nuclear analyticity spaces on which a prescribed set of linear operators act continuously.

1. Kernel theorems

In this section we shall recall the four Kernel theorems introduced in [G], ch.VI, and we shall add one to them.

The Hilbert space $X \otimes Y$ of all Hilbert-Schmidt operators from X into Y can be regarded as a topological tensor product of X and Y. Let A and B denote nonnegative self-adjoint operators in X and Y. Let $w \in D(A)$. Then for all $v \in Y$, we define

$$A \otimes I\,(w \otimes v) = (Aw) \otimes v \ .$$

With the aid of linear extension, the operator $A \otimes I$ is well-defined on the algebraic tensor product $D(A) \otimes_a Y$. It can be proved that $A \otimes I$ with domain $D(A) \otimes_a Y$ is nonnegative and essentially self-adjoint. Cf.[W],[G]. Similarly $I \otimes B$ with domain $X \otimes_a D(B)$ is nonnegative and essentially self-adjoint in $X \otimes Y$. Further, the operators $A \otimes I$ and $I \otimes B$ commute, i.e., their spectral projections commute. So the operator $A \boxplus B = A \otimes I + I \otimes B$ with domain

$$\{W \in X \otimes Y \,|\, \int_{\mathbb{R}^2} (\lambda + \mu)^2 d((E_\lambda \otimes F_\mu) W, W) < \infty\}$$

is self-adjoint and nonnegative. Consequently the spaces $S_{X \otimes Y, A \boxplus B}$ and $T_{X \otimes Y, A \boxplus B}$ are well-defined. In [G] it is proved that $S_{X \otimes Y, A \boxplus B}$ is a topological tensor product of $S_{X,A}$ and $S_{Y,B}$, and $T_{X \otimes Y, A \boxplus B}$ a topological tensor product of $T_{X,A}$ and $T_{Y,B}$. We note that $e^{-t(A \boxplus B)} = e^{-tA} \otimes e^{-tB}$, $t \geq 0$.

Case (a). Continuous linear mappings from $T_{X,A}$ into $S_{Y,B}$.
An element $\theta \in S_{X \otimes Y, A \boxplus B}$ induces a linear mapping $T_{X,A} \to S_{Y,B}$ in the following way. Let $F \in T_{X,A}$. Then θF is defined by

(a) $$\theta F = e^{-\varepsilon B}(e^{\varepsilon B}\theta e^{\varepsilon A})F(\varepsilon)$$

where $\varepsilon > 0$ has to be taken sufficiently small.

(1.1) Theorem

I. For each $\theta \in S_{X\otimes Y, A\boxplus B}$, the linear operator $\theta: T_{X,A} \to S_{Y,B}$ as defined by (a) is continuous.

II. For $\theta \in S_{X\otimes Y, A\boxplus B}$, $F \in T_{X,A}$ and $G \in T_{Y,B}$,

$$\langle \theta F, G\rangle_Y = \langle \theta, F\otimes G\rangle_{X\otimes Y} .$$

III. If for each $t > 0$ at least one of the operators e^{-tA}, e^{-tB} is Hilbert-Schmidt, then $S_{X\otimes Y, A\boxplus B}$ comprises all continuous linear mappings from $T_{X,A}$ into $S_{Y,B}$.

IV. $S_{X\otimes X, A\boxplus A}$ comprises all continuous linear mappings from $T_{X,A}$ into $S_{X,A}$, iff for each $t > 0$ the operator e^{-tA} is Hilbert-Schmidt.

Proof. Cf.[G], Theorem 6.1. □

Case (b). Continuous linear mappings from $S_{X,A}$ into $T_{Y,B}$.

Let $\Phi \in T_{X\otimes Y, A\boxplus B}$. For $f \in S_{X,A}$ we define $\Phi f \in T_{Y,B}$ by

(b) $$(\Phi f)(t) = e^{-(t-\varepsilon)B}\Phi(\varepsilon)e^{\varepsilon A}f \quad , \quad t > 0 ,$$

where $\varepsilon > 0$ has to be taken sufficiently small.

(1.2) Theorem

I. For each $\Phi \in T_{X\otimes Y, A\boxplus B}$ the linear mapping $\Phi: S_{X,A} \to T_{Y,B}$ defined by (b) is continuous.

II. For each $\Phi \in T_{X\otimes Y,A\boxplus B}$, $f \in S_{X,A}$ and $g \in S_{Y,B}$

$$\langle g,\Phi f\rangle_Y = \langle f\otimes g,\Phi\rangle_{X\otimes Y}.$$

III. If for each $t > 0$ at least one of the operators e^{-tA}, e^{-tB} is H.S. then $T_{X\otimes Y,A\boxplus B}$ comprises all continuous minear mappings from $S_{X,A}$ into $T_{Y,B}$.

IV. $T_{X\otimes X,A\boxplus A}$ comprises all continuous linear mappings from $S_{X,A}$ into $T_{X,A}$ iff for each $t > 0$ the operator e^{-tA} is H.S.

Proof. Cf.[G], Theorem 6.2. □

In [G], Ch.V, the spaces Σ'_A and Σ'_B are introduced as follows.

$$\Sigma'_A = \{P \in T_{X\otimes Y,A\otimes I} \mid \forall_{t>0}: P(t) \in S_{X\otimes Y,A\boxplus B}\} ,$$

$$\Sigma'_B = \{K \in T_{X\otimes Y,I\otimes B} \mid \forall_{t>0}: K(t) \in S_{X\otimes Y,A\boxplus B}\} .$$

It is not hard to prove that Σ'_A equals the space $T(S_{X\otimes Y,I\otimes B},A\otimes I)$ and Σ'_B the space $T(S_{X\otimes Y,A\otimes I},I\otimes B)$ both set theoretically and topologically. Cf. Ch.II, Section 2; [G], Ch.V.

Let $F \in T_{X,A}$ and $g \in S_{Y,B}$. Then $F\otimes g$ is defined as the trajectory

$$F\otimes g:\ t \to F(t)\otimes g.$$

Since $F(t)\otimes(e^{\varepsilon B}g) \in X\otimes Y$ for $\varepsilon > 0$ sufficiently small and all $t > 0$, the trajectory $F\otimes g$ is an element of $T(S_{X\otimes Y,I\otimes B},A\otimes I)$. So the algebraic tensor product of $T_{X,A}$ and $S_{Y,B}$ is contained in $T(S_{X\otimes Y,I\otimes B},A\otimes I)$. De Graaf proves that $T(S_{X\otimes Y,I\otimes B},A\otimes I)$ is a complete topological tensor product of $T_{X,A}$ and $S_{Y,B}$. Moreover, for $F \in T_{X,A}$ and $g \in S_{Y,B}$ the tensor product $F\otimes g$ is an element of $S(T_{X\otimes Y,A\otimes I},I\otimes B)$, because there exists $\varepsilon > 0$ fixed such that

$$(I \otimes e^{\varepsilon B})(F \otimes g) = F \otimes (e^{\varepsilon B} g) \in T_{X\otimes Y, A\otimes I} .$$

So the algebraic tensor product $T_{X,A} \otimes_a S_{Y,B}$ is also contained in $S(T_{X\otimes Y, A\otimes I}, I\otimes B)$. By similar arguments it follows that the space $T(S_{X\otimes Y, A\otimes I}, I\otimes B)$ is a complete topological tensor product of the spaces $S_{X,A}$ and $T_{Y,B}$. The algebraic tensor product $S_{X,A} \otimes_a T_{Y,B}$ is contained in $S(T_{X\otimes Y, I\otimes B}, A\otimes I)$. We note that $S(T_{X\otimes Y, A\otimes I}, I\otimes B)$ is included in $T(S_{X\otimes Y, I\otimes B}, A\otimes I)$, and that $S(T_{X\otimes Y, I\otimes B}, A\otimes I)$ is included in $T(S_{X\otimes Y, A\otimes I}, I\otimes B)$, Cf. Ch. II, Section 5.

<u>Case</u> c. Continuous linear mappings from $S_{X,A}$ into $S_{Y,B}$.
Let $P \in T(S_{X\otimes Y, I\otimes B}, A\otimes I)$. Then for $f \in S_{X,A}$ we define Pf by

(c) $$P(f) = P(\varepsilon) e^{\varepsilon A} f,$$

where $\varepsilon > 0$ has to be taken sufficiently small. We note that (c) does not depend on the choice of $\varepsilon > 0$. Since $P(\varepsilon) \in S_{X\otimes Y, I\otimes B}$ we have $Pf \in S_{Y,B}$.

(1.3) <u>Theorem</u>

I. For each $P \in T(S_{X\otimes Y, I\otimes B}, A\otimes I)$ the linear operator $P: S_{X,A} \to S_{Y,B}$ defined by (c) is continuous.

II. For each $P \in T(S_{X\otimes Y, I\otimes B}, A\otimes I)$, $f \in S_{X,A}$ and $G \in T_{Y,B}$

$$\overline{\langle Pf, G\rangle_Y} = \langle\langle f \otimes G, P\rangle\rangle_{X\otimes Y} .$$

III. If for each $t > 0$ at least one of the operators e^{-tA}, e^{-tB} is H.S. then $T(S_{X\otimes Y, I\otimes B}, A\otimes I)$ comprises all continuous linear mappings from $S_{X,A}$ into $S_{Y,B}$.

IV. $\mathcal{T}(S_{X\otimes Y,I\otimes A},A\otimes I)$ comprises all continuous linear mappings from $S_{X,A}$ into itself iff for each $t > 0$ the operator e^{-tA} is H.S.

Proof. Cf.[G], Theorem 6.3. □

Case (d). Continuous linear mappings from $\mathcal{T}_{X,A}$ into $\mathcal{T}_{Y,B}$.

Let $K \in \mathcal{T}(S_{X\otimes Y,A\otimes I},I\otimes B)$. For $F \in \mathcal{T}_{X,A}$, define $KF \in \mathcal{T}_{Y,B}$ by

(d) $$(KF)(t) = K(t)e^{\varepsilon A})F(\varepsilon) .$$

This definition makes sense for all $t > 0$ and for each $\varepsilon > 0$ sufficiently small. We have $(KF)(t) \in S_{Y,B}$, because $K \in \mathcal{T}_{X\otimes Y,I\otimes B}$.

(1.4) Theorem.

I. For each $K \in \mathcal{T}(S_{X\otimes Y,A\otimes I},I\otimes B)$, the linear mapping $K: \mathcal{T}_{X,A} \to \mathcal{T}_{Y,B}$ defined in (d), is continuous.

II. For each $K \in \mathcal{T}(S_{X\otimes Y,A\otimes I},I\otimes B)$, $F \in \mathcal{T}_{X,A}$, $g \in S_{Y,B}$

$$\langle g,KF\rangle_Y = \langle\langle F\otimes g,K\rangle\rangle_{X\otimes Y}.$$

III. If for each $t > 0$ at least one of the operators e^{-tA}, e^{-tB} is H.S., then $\mathcal{T}(S_{X\otimes Y,A\otimes I},I\otimes B)$ comprises all continuous linear mappings from $\mathcal{T}_{X,A}$ into $\mathcal{T}_{Y,B}$.

IV. $\mathcal{T}(S_{X\otimes X,A\otimes I},I\otimes A)$ comprises all continuous linear mappings from $\mathcal{T}_{X,A}$ into itself iff the operator e^{-tA} is Hilbert-Schmidt for all $t > 0$.

Proof. Cf.[G], Theorem 6.4. □

(1.5) Definition

A continuous linear mapping E from $S_{X,A}$ into $S_{Y,B}$ is called extendable, if E can be extended to a continuous linear mapping from $\mathcal{T}_{X,A}$ into $\mathcal{T}_{Y,B}$.

In [G], necessary and sufficient conditions are given in order that a linear mapping on $S_{X,A}$ is extendable, cf. Ch. I, Section 4.
In Ch. II for a pair of commuting, nonnegative, self-adjoint operators we have defined the operator $C \wedge D$ by

$$C \wedge D = \iint_{\mathbb{R}^2} \max(\lambda,\mu)\,dG_\lambda H_\mu,$$

and the operator $C \vee D$ by

$$C \vee D = \iint_{\mathbb{R}^2} \min(\lambda,\mu)\,dG_\lambda H_\mu.$$

where $(G_\lambda)_{\lambda\in\mathbb{R}}$ and $(H_\mu)_{\mu\in\mathbb{R}}$ are the spectral resolutions of C and D.
Moreover, we have shown that

$$T(S_{Z,C},D) \cap T(S_{Z,D},C) = T(S_{Z,C\wedge D},C\vee D).$$

Applying this result to the spaces $T(S_{X\otimes Y,I\otimes B},A\otimes I)$ and $T(S_{X\otimes Y,A\otimes I},I\otimes B)$, we find that their intersection equals the space $T(S_{X\otimes Y,A\circledS B},A\oslash B)$ with

$$A\circledS B = (A\otimes I) \wedge (I\otimes B) \text{ and } A\oslash B = (A\otimes I) \vee (I\otimes B).$$

(1.6) <u>Definition</u>

The canonical mapping emb: $S_{X,A}\otimes_a S_{Y,B} \to T(S_{X\otimes Y,A\circledS B},A\oslash B)$ is defined by

$$\text{emb}(f\otimes g) : t \mapsto e^{-t(A\oslash B)}(f\otimes g)\ .$$

It is obvious that $\text{emb}(f\otimes g) \in T(S_{X\otimes Y,A\circledS B},A\oslash B)$.
The space $T(S_{X\otimes Y,A\circledS B},A\oslash B)$ is a complete topological tensor product of the spaces $S_{X,A}$ and $S_{Y,B}$. By this we mean

(1.7) Theorem

I. $T(S_{X\otimes Y,A\otimes B},A\otimes B)$ is complete.

II. The mapping $\otimes : S_{X,A} \times S_{Y,B} \to T(S_{X\otimes Y,A\otimes B},A\otimes B)$ is continuous.

III. $S_{X,A}\otimes_a S_{Y,B}$ is dense in $T(S_{X\otimes Y,A\otimes B},A\otimes B)$.

Proof.

I. All spaces of this kind are complete. Cf. Ch. II, Section 2.

II. It is sufficient to check continuity at [0;0]. Let $\psi \in B_+(\mathbb{R})$, and let t > 0. Then

$$\|\psi(A\otimes B)e^{-t(A\otimes B)}(f\otimes g)\|_{X\otimes Y} \le$$

$$\le \|\psi(A)f\|_X\|g\|_Y + \|f\|_X\|\psi(B)g\|_Y < \varepsilon,$$

as soon as $\|\psi(A)f\|$ and $\|\psi(B)g\|$ are small enough. Cf.[G], Ch.I.

III. Following [G], Ch.V, the space $S_{X,A}\otimes_a S_{Y,B}$ is dense in $S_{X\otimes Y,A\boxplus B}$. From Ch. II, Section 5, it follows that $S_{X\otimes Y,A\boxplus B}$ is dense in $T(S_{X\otimes Y,A\otimes B},A\otimes B)$. □

The strong dual space of $T(S_{X\otimes Y,A\otimes B},A\otimes B)$ is equal to the space $S(T_{X\otimes Y,A\otimes B},A\otimes B)$, where

$$S(T_{X\otimes Y,A\otimes B},A\otimes B) = S(T_{X\otimes Y,A\otimes I},I\otimes B) + S(T_{X\otimes Y,I\otimes B},A\otimes I) .$$

Hence, for all $f \in S_{X,A}$, $g \in S_{Y,B}$ and all $F \in T_{X,A}$, $G \in T_{Y,B}$

$$f\otimes G + F\otimes g \in S(T_{X\otimes Y,A\otimes B},A\otimes B) .$$

Case (e). Extendable linear mappings from $S_{X,A}$ into $S_{Y,B}$.
Let $E \in T(S_{X\otimes Y,A\otimes B},A\otimes B)$. Then for $f \in S_{X,A}$ we define Ef by

(e_1) $\quad Ef = e^{\varepsilon(A\otimes B)}[(e^{-\varepsilon A}\otimes I)(E(\varepsilon))]e^{\varepsilon A}f$,

where $\varepsilon > 0$ has to be taken sufficiently small. Definition (e_1) does not depend on the choice of ε. Further $Ef \in S_{Y,B}$ because $e^{\tau(A\otimes B)}(e^{-\tau A}\otimes I)$ is a bounded operator on $X\otimes Y$, and because $E(\tau) \in S_{X\otimes Y,A\otimes B} \subset S_{X\otimes Y,I\otimes B}$.
Let $F \in T_{X,A}$. We define the extension $\overline{E}$ on $T_{X,A}$ by

(e_2) $\quad (\overline{E}F)(t) = e^{t(A\otimes B)}(I\otimes e^{-tB})(E(t)e^{\varepsilon A})(F(\varepsilon))$, $\quad t > 0$.

where each $\varepsilon > 0$ has to be chosen sufficiently small. We have $\overline{E}F \in T_{Y,B}$, because the operator $e^{tA\otimes B}(I\otimes e^{-tB})$ is bounded on $X\otimes Y$ for all $t > 0$, and because $E(t) \in S_{X\otimes Y,A\otimes B} \subset S_{X\otimes Y,A\otimes I}$.

<u>Remark</u>: If $E \in T(S_{X\otimes Y,A\otimes B},A\otimes B)$ then E can be embedded in $T(S_{X\otimes Y,I\otimes B},A\otimes I)$ as follows

$$\text{emb}_1(E) : t \mapsto e^{t(A\vee B)}(e^{-tA}\otimes I)(E(t)) ,$$

and in $T(S_{X\otimes Y,A\otimes I},I\otimes B)$ as

$$\text{emb}_2(E) : t \mapsto e^{t(A\otimes B)}(I\otimes e^{-tB})(E(t)) .$$

Cf. Ch. II, Section 4.

The proof of the next theorem will be omitted; it is an immediate corollary of Theorem (1.3) and (1.4).

(1.8) <u>Theorem</u>

I. By (e_1) and (e_2), each element of $T(S_{X\otimes Y,A\otimes B},A\otimes B)$ provides a continuous and extendable linear mapping from $S_{X,A}$ into $S_{Y,B}$.

II. For each $E \in \mathcal{T}(S_{X\otimes Y,A\otimes B},A\otimes B)$, $f \in S_{X,A}$, $g \in S_{Y,B}$, $F \in \mathcal{T}_{X,A}$ and $G \in \mathcal{T}_{Y,B}$,

$$\langle\!\langle f \otimes G + F \otimes g, E \rangle\!\rangle = \langle Ef, G\rangle + \langle g, \overline{E}F\rangle \ .$$

III. If for each $t > 0$ at least one of the operators e^{-tA} or e^{-tB} is Hilbert-Schmidt, then $\mathcal{T}(S_{X\otimes Y,A\otimes B},A\otimes B)$ comprises all extendable linear mappings from $S_{X,A}$ into $S_{Y,B}$.

IV. $\mathcal{T}(S_{X\otimes X,A\otimes A},A\otimes A)$ comprises all extendable linear mappings iff the operator e^{-tA} is Hilbert Schmidt for all $t > 0$.

By Theorem (1.8) we have given the space of extendable linear mappings the structure of a space of type $\mathcal{T}(S_{Z,C},\mathcal{D})$, if at least one of the spaces $S_{X,A}$ and $S_{Y,B}$ is nuclear.

2. The algebras $\mathcal{T}^A$, $\mathcal{T}_A$ and E_A

The space $\mathcal{T}^A = \mathcal{T}(S_{X\otimes X,I\otimes A},A\otimes I)$ comprises all continuous linear mappings from $S_{X,A}$ into itself if and only if the operator e^{-tA} is Hilbert-Schmidt for all $t > 0$. So in this case $\mathcal{T}^A$ admits an algebraic structure. If the space $S_{X,A}$ is not nuclear, then it is less natural that $\mathcal{T}^A$ is an algebra. Yet it is true. To show this, let $P_1,P_2 \in \mathcal{T}^A$. Then by the previous section for each $f \in S_{X,A}$ by definition,

$$P_1(P_2f) = P_1(\tau_1)e^{\tau_1 A}(P_2(\tau_2)e^{\tau_2 A}f)$$

where $\tau_1,\tau_2 > 0$ have to be taken sufficiently small. Thus to the product P_1P_2 there corresponds the trajectory (P_1P_2) in $\mathcal{T}^A$

$$(P_1P_2): t \to P_1(\tau)e^{\tau A}P_2(t)$$

where for each $t > 0$ we have to take $\tau > 0$ so small that $e^{\tau A}P_2(t) \in X \otimes X$. With the above-derived multiplication $(P_1,P_2) \to (P_1P_2)$, T_A is an algebra. Similarly, there exists a multiplication operation on $T_A = T(S_{X\otimes X,A\otimes I}, I\otimes A)$, $(K_1,K_2) \to (K_1K_2)$, where

$$(K_1K_2): t \to K_1(t)e^{\tau A}K_2(\tau).$$

(2.1) Definition

The linear mapping ^{c} on $T_{X\otimes X,A\boxplus A}$ is defined by

$$\Phi^c : t \to \Phi(t)^* \quad , \Phi \in T_{X\otimes X,A\boxplus A}.$$

Remark: Φ^c is called the adjoint of Φ.

(2.2) Lemma

The mapping ^{c} is a strongly continuous bijection on $T_{X\otimes X,A\boxplus A}$ with $\Phi^{cc} = \Phi$.

Proof. The lemma is a natural consequence of the definition of ^{c}, and of the strong topology in $T_{X\otimes X,A\boxplus A}$. □

Since T^A, T_A can be seen as subspaces of $T_{X\otimes X,A\boxplus A}$, the mapping ^{c} is well-defined on T^A and T_A. It is not difficult to see that for $P \in T^A$ its adjoint P^c is given by $P^c: t \to P(t)^*$. Here we note that $t \mapsto P(t)$ is a trajectory in T^A.

(2.3) Lemma

The mapping ^{c} is a bijection from T^A onto T_A.

Proof. Let $t > 0$, and let $P \in T^A$. Then there is $\tau > 0$ such that

$$e^{\tau A}P(t) \in X\otimes X$$

or, equivalently,

$$P(t) \in \mathcal{D}(I \otimes e^{\tau A}) .$$

So its adjoint $P(t)^*$ is in $\mathcal{D}(e^{\tau A} \otimes I)$, which yields $P^c \in \mathcal{T}_A$.
Similarly for $K \in \mathcal{T}_A$ we derive $K^c \in \mathcal{T}^A$. Hence ^{c} is a bijection. □

(2.4) Theorem

The mapping $^c: \mathcal{T}^A \to \mathcal{T}_A$ is a homeomorphism.

Proof. It is clear that ^{c} is a bijection satisfying $(P_1P_2)^c = P_2^c P_1^c$. Further, each seminorm on $\mathcal{T}^A$ transforms into a seminorm on $\mathcal{T}_A$ by the mapping ^{c}. In particular, for all $P \in \mathcal{T}^A$,

$$\| \psi(A) P(t) \|_{X \otimes X} = \| (I \otimes \psi(A)) P(t) \|_{X \otimes X} = \| (\psi(A) \otimes I) P(t)^* \|_{X \otimes X},$$

where $\psi \in \mathcal{B}_+(\mathbb{R})$ and $t > 0$. Thus the result is established. Cf. Ch. II, Section 2. □

(2.5) Corollary

The mapping $^c: \mathcal{T}_A \to \mathcal{T}^A$ is a homeomorphism.

The definitions (a) - (d) of the preceding section, which indicate how the elements of each of the four tensor products induce continuous linear mappings, lead to the following

(2.6) Lemma

Let $f,g \in S_{X,A}$, and let $F,G \in \mathcal{T}_{X,A}$. Then

$$\overline{\langle f,\Phi g\rangle} = \langle g,\Phi^c f\rangle \quad , \quad \Phi \in T_{X\otimes X,A\boxplus A} \; ,$$

$$\langle Pf,G\rangle = \langle f,P^c G\rangle \quad , \quad P \in T^A \; ,$$

$$\langle g,KF\rangle = \langle K^c g,F\rangle \quad , \quad K \in T_A \; ,$$

$$\overline{\langle \theta F,G\rangle} = \langle \theta^c G,F\rangle \quad , \quad \theta \in S_{X\otimes X,A\boxplus A} \; ,$$

We note that P^c is the representant in T_A of P' and K^c the representant in T^A of K', where P' and K' denote the dual mappings of P and K.

Following Ch. II, Section 2, each element $H \in T(S_{Z,C},D)$ can be written as $H = \Theta(C,D)w$, where $w \in Z$ and $\theta \in F_+(\mathbb{R}^2)$, i.e. a function from $\mathbb{R}^2$ into $\mathbb{R}^+$ satisfying

$$\forall_{s>0}\ \exists_{t>0} : \sup_{\lambda\geq 0,\mu\geq 0} (\theta(\lambda,\mu)e^{-t\lambda}e^{s\mu}) < \infty.$$

Applying this result to T^A we can write for $P \in T^A$

$$P = \theta(I\otimes A, A\otimes I)(W) \; ,$$

for a well-chosen $W \in X\otimes X$ and $\theta \in F_+(\mathbb{R}^2)$. Then it is obvious that

$$P(t)^* = (I\otimes e^{-tA})\theta(A\otimes I, I\otimes A)(W^*) \; .$$

Hence $P^c = \theta(A\otimes I, I\otimes A)(W^*)$. Similarly for $K \in T_A$, $K = \chi(A\otimes I, I\otimes A)(V)$, where $V \in X\otimes X$ and $\chi \in F_+(\mathbb{R}^2)$,

$$K^c = \chi(I\otimes A, A\otimes I)(V^*).$$

The strong dual spaces S_A of T_A and S^A of T^A are given by

$$S_A = S(T_{X\otimes X, A\otimes I}, I\otimes A)$$

and

$$S^A = S(T_{X\otimes X, A\otimes I}, I\otimes A) \ .$$

As already observed by De Graaf, we have $S_A \subset T^A$ and $S^A \subset T_A$. The mapping ^{c} is a continuous bijection from S_A onto S^A, and even a homeomorphism $S_A \to S^A$ because of the equalities

$$\| \Theta(A\otimes I, I\otimes A)(\theta) \|_{X\otimes X} = \| \Theta(I\otimes A, A\otimes I)(\theta^c) \|_{X\otimes X},$$

for all $\Theta \in F_+(\mathbb{R}^2)$ and for all $\theta \in S_A$. Cf. Ch. II, Section 1.

The elements S_A and S^A are characterized as follows.

$$\Psi \in S^A \Leftrightarrow \exists_{\psi \in B_+(\mathbb{R})} \exists_{t>0} \exists_{W \in X\otimes X} : \Psi = \psi(A) W e^{-tA}$$

$$\Phi \in S_A \Leftrightarrow \exists_{\varphi \in B_+(\mathbb{R})} \exists_{t>0} \exists_{V \in X\otimes X} : \Phi = e^{-tA} V \varphi(A) \ .$$

Thus, it easily follows that

$$\Psi^c = e^{-tA} W^* \psi(A) \in S_A$$

$$\Phi^c = \varphi(A) V^* e^{-tA} \in S^A$$

The weak topology for T^A is the coarsest topology in which all linear functionals on T^A obtained by pairing with elements of S^A are continuous Hence, the weak topology is generated by the seminorms

$$s_\Phi(P) = |\langle\langle \Phi, P \rangle\rangle| \quad , \quad P \in T^A$$

where $\Phi \in S^{A}$. Similarly the weak topology for T_A is generated by

$$r_{\Psi}(K) = |\ll \Psi, K \gg| \quad , \quad K \in T_A,$$

where $\Psi \in S_A$. The following lemma shows that ^{c} is weakly continuous.

(2.7) <u>Lemma</u>

Let $P \in T^{A}$ and let $\Phi \in S^{A}$. Then

$$\overline{\ll \Phi, P \gg} = \ll \Phi^c, P^c \gg .$$

<u>Proof</u>. There are $W, V \in X \otimes X$, and $\Theta \in F_+(\mathbb{R}^2)$, $\psi \in B_+(\mathbb{R})$ and $t > 0$ such that $P = \Theta(I \otimes A, A \otimes I)(W)$ and $\Phi = \psi(A) V e^{-tA}$. So employing spectral integrals with respect to the spectral resolution $(E_\lambda \otimes E_\mu)_{(\lambda,\mu) \in \mathbb{R}^2}$ of $I \otimes I$, we may write

$$\ll \Phi, P \gg = \iint_{\mathbb{R}^2} \Theta(\lambda,\mu) e^{-t\lambda} \psi(\mu) d(E_\mu V E_\lambda, W)_{X \otimes X} .$$

Since $\overline{(E_\mu V E_\lambda, W)}_{X \otimes X} = (E_\lambda V^* E_\mu, W^*)_{X \otimes X}$, we derive

$$\overline{\ll \Phi, P \gg} = \iint_{\mathbb{R}^2} \Theta(\mu,\lambda) e^{-t\lambda} \psi(\mu) d(E_\lambda V^* E_\mu, W^*) =$$

$$= \iint_{\mathbb{R}^2} \Theta(\lambda,\mu) e^{-t\mu} \psi(\lambda) d(E_\mu V^* E_\lambda, W^*) =$$

$$= \ll e^{-tA} V^* \psi(A),\ \Theta(A \otimes I, I \otimes A)(W^*) \gg =$$

$$= \ll \Phi^c, P^c \gg \qquad \square$$

(2.8) Theorem

I. The mapping ${}^c: \mathcal{T}^A \to \mathcal{T}_A$ resp. $\mathcal{T}_A \to \mathcal{T}^A$ is weakly continuous.
II. The mapping ${}^c: \mathcal{S}^A \to \mathcal{S}_A$ resp. $\mathcal{S}_A \to \mathcal{S}^A$ is weakly continuous.

The algebra $\mathcal{E}_A$ is defined as $\mathcal{E}_A = \mathcal{T}^A \cap \mathcal{T}_A$; it consists of extendable linear mappings from $\mathcal{S}_{X,A}$ into itself. In Section 1 we have shown that

$$\mathcal{E}_A = \mathcal{T}(\mathcal{S}_{X\otimes X, A\otimes A}, A\otimes A) \ .$$

Naturally, the strong topology of $\mathcal{E}_A$ is generated by the seminorms

$$s_{\psi,t}(E) = \|\psi(A\otimes A)e^{-t(A\otimes A)}(E)\|_{X\otimes X} \quad , \ E \in \mathcal{E}_A \ .$$

where $t > 0$ and $\psi \in \mathcal{B}_+(\mathbb{R})$. The seminorms $s_{\psi,t}$ are equivalent to the seminorms $u_{\psi,t}$ and $v_{\psi,t}$,

$$u_{\psi,t}(E) = \psi(A)Ee^{-tA} \quad , \ E \in \mathcal{E}_A,$$

$$v_{\psi,t}(E) = e^{-tA}E\psi(A) \quad , \ E \in \mathcal{E}_A.$$

So the embeddings $\mathcal{E}_A \hookrightarrow \mathcal{T}^A$ and $\mathcal{E}_A \hookrightarrow \mathcal{T}_A$ are continuous if the spaces carry their strong topology.
The dual space $\mathcal{E}_A'$ of $\mathcal{E}_A$ is expressed by the algebraic sum

$$\mathcal{E}_A' = \mathcal{S}^A + \mathcal{S}_A \qquad (+ \text{ in } \mathcal{T}_{X\otimes X, A\boxplus A}) \ .$$

Hence, the weak topology of $\mathcal{E}_A$ is equivalent to the topology induced by the weak topologies of $\mathcal{T}^A$ and $\mathcal{T}_A$. Put differently, the embeddings $\mathcal{E}_A \hookrightarrow \mathcal{T}^A$ and $\mathcal{E}_A \hookrightarrow \mathcal{T}_A$ are continuous if the spaces carry their weak topology.
The mapping ^{c} is a continuous bijection from $\mathcal{E}_A$ onto itself. Since

$E'_A \subset T_{X\otimes X, A\boxplus A}$, the mapping ^{c} is well defined on E'_A. We should like to write

$$(\Phi + \Psi)^c = \Phi^c + \Psi^c \ , \ \Phi \in S^A, \ \Psi \in S_A.$$

However, the choice of Φ and Ψ is not unique, because $S_A \cap S^A = S_{X\otimes X, A\boxplus A}$. In order to show the independence of the specific choice of Φ and Ψ in the wanted equality, suppose

$$\Phi_1 + \Psi_1 = \Phi_2 + \Psi_2$$

where $\Phi_1, \Phi_2 \in S^A$ and $\Psi_1, \Psi_2 \in S_A$. Then $\Phi_1 - \Phi_2 = \Psi_2 - \Psi_1$. Hence $\Phi_1 - \Phi_2 \in S^A \cap S_A = S_{X\otimes X, A\boxplus A}$. This implies

$$\Phi_1^c - \Phi_2^c = \Psi_2^c - \Psi_1^c \in S_{X\otimes X, A\boxplus A},$$

which yields

$$\Phi_1^c + \Psi_2^c = \Phi_2^c + \Psi_2^c \ .$$

The above-mentioned result leads to the following theorem

(2.9) Theorem

I. The mapping ^{c} is a strongly and weakly continuous linear bijection from E_A onto itself. It satisfies

$$E^{cc} = E \ , \ (E_1E_2)^c = E_2^cE_1^c \ , \ E_1, E_2, E \in E_A.$$

Hence, ^{c} is an involution on E_A.

II. The mapping ^{c} is a strongly and weakly continuous bijection from E'_A onto itself with $\theta^{cc} = \theta$, $\theta \in E'_A$.

III. Let $E \in E_A$. Then $E = \Theta(A\otimes A, A\otimes A)(W)$ for $\Theta \in F_+(\mathbb{R}^2)$ and $W \in X\otimes X$. We have $E^c = \Theta(A\otimes A, A\otimes A)(W^*)$.

IV. For $E \in E_A$ and $\theta \in E_A'$

$$\overline{\langle\langle \theta, E \rangle\rangle} = \langle\langle \theta^c, E^c \rangle\rangle .$$

If the Kernel theorem holds true, the algebra T^A comprises all continuous linear mappings from $S_{X,A}$ into itself. So T^A can be identified with the algebra of all continuous linear mappings from $S_{X,A}$ into itself.
As a space of linear mappings, T^A obtains some natural topologies from its domain space $S_{X,A}$, such as the topology of pointwise convergence and the topology of weak pointwise convergence. Similar constructions exist in the algebras T_A and E_A.
In the following chapters we shall deepen the topological structure of the algebras T^A, T_A and E_A. We shall investigate their affiliation with the respective algebraic structures.

3. The topological structure of the algebra T^A.

In the remaining part of this paper we assume that the space $S_{X,A}$ is nuclear. Equivalently, we assume that T^A comprises all continuous linear mappings from $S_{X,A}$ into itself. Then, besides its weak and its strong topology denoted by τ_s and τ_w in the sequel, we introduce the topologies τ_p and τ_{wp} for T^A.

(3.1) Definition. (The topology of pointwise convergence)
The topology τ_p is the locally convex topology for T^A induced by the seminorms $u_{f,\psi}$,

$$u_{f,\psi} = \| \psi(A) P f \| \quad , \quad P \in \mathcal{T}^A ,$$

where $f \in S_{X,A}$ and $\psi \in B_+(\mathbb{R})$.

The net (P_α) in $\mathcal{T}_A$ is τ_p-convergent if and only if the net $(P_\alpha f)$ in $S_{X,A}$ is strongly convergent for all $f \in S_{X,A}$. The topology τ_p is the coarsest topology for which the linear mappings $\mathcal{T}^A \to S_{X,A}$,

$$P \mapsto Pf \ , \ P \in \mathcal{T}^A ,$$

are strongly continuous for all $f \in S_{X,A}$.

The following result is remarkable. In fact, the strong topology of $\mathcal{T}^A$ is not introduced as a specific operator topology. Yet, it is one.

(3.2) Lemma

The topology τ_s is equivalent to the topology of uniform convergence on bounded subsets of $S_{X,A}$.

Proof. Let (P_α) be a strongly convergent net in $\mathcal{T}^A$ with limit P and let B be a bounded subset of $S_{X,A}$. Then there is $t > 0$ so that the set $e^{tA}(B)$ is bounded in X. For all $f \in B$, all $\psi \in B_+(\mathbb{R})$ and all α

$$\| \psi(A)(P_\alpha - P) f \| \le \| \psi(A)(P_\alpha(t) - P(t) \| \, \| e^{tA} f \| \ .$$

On the other hand, let $\varepsilon > 0$ and let $t > 0$. Suppose

$$P_\alpha f \to Pf$$

strongly in $S_{X,A}$ and uniformly on the bounded subset $\{e^{-tA} w \mid \|w\| = 1\}$. Then for each $\psi \in B_+(\mathbb{R})$ there is α_1, such that

$$\| \psi(A)(P_\alpha(t) - P(t)) w \| < \varepsilon/2 \ ,$$

for all $\alpha > \alpha_1$ and all $w \in X$ with $\|w\| = 1$. Hence,

$$\|\psi(A)(P_\alpha(t) - P(t))\| \leq \varepsilon/2 < \varepsilon . \qquad \square$$

Remark: In the proof of Lemma (3.2) we employed the norm $\|\cdot\|$ of the Banach algebra $\mathcal{B}(X)$ instead of the Hilbert-Schmidt norm $\|\cdot\|_{X\otimes X}$. However, this is allowed because of the following relation

$$\|P(t)\| \leq \|P(t)\|_{X\otimes X} \leq \|P(t/2)\| \; \|e^{-t/2A}\|_{X\otimes X} \; , P \in \mathcal{T}^A.$$

(3.3) <u>Definition</u>. (The topology of weak pointwise convergence)

The topology τ_{wp} is the locally convex topology generated by the semi-norm $u_{f,G}$,

$$u_{f,G}(P) = |\langle Pf,G\rangle| \; , \; P \in \mathcal{T}^A ,$$

where $f \in \mathcal{S}_{X,A}$ and $G \in \mathcal{T}_{X,A}$.

The net (P_α) in $\mathcal{T}^A$ converges to $P \in \mathcal{T}^A$ in τ_{wp}-sense if and only if $\langle (P_\alpha - P)f,G\rangle \to 0$ for all $f \in \mathcal{S}_{X,A}$ and $G \in \mathcal{T}_{X,A}$. The topology τ_{wp} is the coarsest topology for which the linear mappings

$$P \mapsto \langle Pf,G\rangle \; , \; P \in \mathcal{T}^A$$

are all continuous. τ_p is the topology of uniform weak pointwise convergence on bounded subsets of $\mathcal{T}_{X,A}$. The latter proposition is an immediate consequence of the characerization of bounded subsets of $\mathcal{T}_{X,A}$. The above introduced topologies for $\mathcal{T}^A$ are ordered as follows

(3.4)
$$\begin{array}{ccccc} & & \tau_s & & \\ & \subset & & \supset & \\ \tau_w & & & & \tau_p \\ & \supset & & \subset & \\ & & \tau_{wp} & & \end{array}$$

Here $\subset$ means 'coarser than'.

(3.5) Theorem. (Principle of uniform boundedness)

Let B be a subset of T^A. Then the following statements are equivalent

I. B is τ_s-bounded.

II. B is τ_w-bounded.

III. B is τ_p-bounded

IV. B is τ_{wp}-bounded.

Proof. The equivalence I $\Leftrightarrow$ II follows from Ch. II, Section 3. Further, it is clear that I $\Rightarrow$ III $\Rightarrow$ IV.

IV $\Rightarrow$ III: Each weakly bounded set in $S_{X,A}$ is strongly bounded, cf. Ch. I, Section 3. From this observation the assertion follows.

III $\Rightarrow$ I: For all $\psi \in B_+(\mathbb{R})$, $t > 0$ and $w \in X$, there exists $\alpha(t,\psi,w)$ such that the set $\{\psi(A)Pe^{-tA} | P \in B\}$ is strongly bounded in $B(X)$. Hence, the uniform boundedness for $B(X)$ yields $\alpha(t,\psi) > 0$ with $\|\psi(A)Pe^{-tA}\| \le \alpha(t,\psi)$. Thus we derive

$$\|\psi(A)Pe^{-tA}\|_{X\otimes X} \le \alpha(t/_2,\psi)\|e^{-t/2A}\|_{X\otimes X},\ P \in B.$$ □

(3.5) Lemma

Let (P_n) be a sequence in T^A such that $\lim_{n\to\infty} P_n f$ exists in $S_{X,A}$ for each $f \in S_{X,A}$. Then $P : f \mapsto \lim_{n\to\infty} P_n f$ is continuous, i.e., $P \in T^A$.

Proof. By Theorem (3.5) the sequence (P_n) is τ_s-bounded. So for each $t > 0$ there is $\alpha_t > 0$ such that $\|P_n(t)\| \le \alpha_t$, $n \in \mathbb{N}$. It is obvious that P is a linear mapping from $S_{X,A}$ into itself. Further, for all $w \in X$, $\|w\| = 1$ and for all $t > 0$

$$\| Pe^{-tA}\omega\| \leq \| (P - P_n)e^{-tA}\omega\| + \alpha_t \leq \alpha_t + 1$$

for $n \in \mathbb{N}$ sufficiently large. Hence $P \in T^A$ by Ch. I, Section 4. □

(3.7) Theorem

T^A is sequentially τ_p-complete and, similarly, sequentially τ_{wp}-complete

Proof. The proof is an immediate consequence of Lemma (3.5) and the (weak) sequential completeness of $S_{X,A}$. □

In the remaining part of this section we investigate the relation between the topological structure of T^A and its algebraic structure.

First we have the following result.

(3.8) Theorem

Joint multiplication is strongly sequentially continuous in T^A.

Proof. Let (P_n) and (T_n) be two converging sequences in T^A with $P_n \to P$ and $T_n \to T$. Let $t > 0$, and let $\phi \in B_+(\mathbb{R})$. Then there exists $\varepsilon > 0$ and $C > 0$ such that

$$\| e^{\varepsilon A} T_n(t) \| < C \ , \ n \in \mathbb{N} \ ,$$

and

$$\| e^{\varepsilon A}(T_n(t) - T(t))\| \to 0$$

because the sequence $(T_n(t))$ converges to $T(t)$ strongly in $S_{X\otimes X, I\otimes A}$.

Hence the inequality

$$\| \phi(A)(P_n T_n - PT)(t) \| \leq$$

$$\leq \|\psi(A)(P_n - P)(\varepsilon)\| \, \|e^{\varepsilon A}T_n(t)\| + \|\psi(A)P(\varepsilon)\| \, \|e^{\varepsilon A}(T_n - T)(t)\|$$

for all $n \in \mathbb{N}$, yields the desired result. □

As observed by De Graaf $S_A \subset T^A$, we have the following stronger result.

(3.9) <u>Lemma</u>

S_A is a proper two-sided ideal in T^A.

<u>Proof</u>. From the characterization of the elements of S_A we obtain the equivalence $\Phi \in S_A \Leftrightarrow \Phi$ represents a continuous linear mapping from $S_{X,A}$ into $e^{-tA}(X)$ for some $t > 0$.
Let $P_1, P_2 \in T^A$ and let $\Phi \in S_A$. Then Φ maps $S_{X,A}$ into some $e^{-\alpha A}(X)$ and further P_1 maps $e^{-\alpha A}(X)$ into $e^{-\beta A}(X)$ for some $\beta > 0$ (cf. Ch. I, Section 4).
So $P_1 \Phi P_2$ maps $S_{X,A}$ into $e^{-\beta A}(X)$ continuously, and hence $P_1 \Phi P_2 \in S_A$.
Since $I \notin S_A$, the ideal S_A is proper. □

(3.10) <u>Corollary</u>

S^A is a proper, two-sided ideal in T_A.

<u>Proof</u>. Follows directly from the properties of the adjoint mapping ^{c} and Lemma (3.9). □

(3.11) <u>Corollary</u>

Let $\Phi \in S^A$ and $P \in T^A$. Then

$$\langle\langle \Phi, P \rangle\rangle = \langle\langle P^c \Phi, I \rangle\rangle = \overline{\langle\langle \Phi^c P, I \rangle\rangle}$$

and

$$\langle\!\langle \Phi, P \rangle\!\rangle = \overline{\langle\!\langle \Phi^c, P^c \rangle\!\rangle} = \overline{\langle\!\langle P\Phi^c, I \rangle\!\rangle} = \langle\!\langle \Phi P^c, I \rangle\!\rangle .$$

(Note that $\langle\!\langle \Phi P^c, I \rangle\!\rangle = \text{trace } (\Phi P^c)$).

Proof. The proof is an application of Lemma (2.2) and Corollary (3.9). □

(3.12) Definition

The algebra Σ with topology τ is called locally convex, if

- (Σ,τ) is a locally convex, topolocical vector space.
- Separate multiplication is continuous in (Σ,τ).

(3.13) Theorem

The algebra $\mathcal{T}^A$ is locally convex if it carries any of the topologies τ_s, τ_w, τ_p and τ_{wp}.

Proof. We shall only prove the continuity of separate multiplication.

I. $(\mathcal{T}^A, \tau_s)$.

Let $P \in \mathcal{T}^A$ be fixed. Then for all $T \in \mathcal{T}^A$

$$\|\phi(A)(TP)(t)\|_{X\otimes X} \le \|\phi(A)T(\varepsilon)\|_{X\otimes X} \|e^{\varepsilon A}P(t)\|$$

for $\varepsilon > 0$ sufficiently small. Hence $T \mapsto TP$ is continuous. To show the continuity of $P \to TP$, let $T \in \mathcal{T}^A$ be fixed, and let $\varepsilon > 0$. Further, let $t > 0$ and let $\phi \in B_+(\mathbb{R})$. Then there is an open null-neighbourhood Ω in $S_{X,A}$ such, that

$$\|\phi(A)Tf\| < \varepsilon/2$$

as soon as $f \in \Omega$. The existence of Ω follows from the continuity of T. Let (P_α) be a net in $\mathcal{T}^A$ that converges strongly to P. Then there exists α_1 such that for all $f \in \{e^{-tA}w \,|\, \|w\| \le 1\}$ uniformly

$$(P_\alpha - P)f \in \Omega$$

if $\alpha > \alpha_1$. So α_1 does not depend on the choice of f. (Lemma (3.2)). Hence, if $\alpha > \alpha_1$, then

$$\|\psi(A)T(P_\alpha - P)f\| < \varepsilon/2$$

for all $f \in S_{X,A}$ with $\|e^{tA}f\| \le 1$. The latter observation leads to the result

$$\|\psi(A)T(P_\alpha - P)(t)\| \le \varepsilon/2 < \varepsilon$$

if $\alpha > \alpha_1$. This finishes the proof.

II. (T^A, τ_w) .

Let $P_1, P_2 \in T^A$. Then for each $\Phi \in S^A$

$$\langle\langle \Phi, P_1TP_2 \rangle\rangle = \langle\langle P_1^c\Phi P_2^c, T \rangle\rangle$$

and hence

$$T \mapsto |\langle\langle \Phi, P_1TP_2 \rangle\rangle|$$

is a weakly continuous seminorm on T^A.

III. (T^A, τ_p).

Let $T_\alpha f \to Tf$ for all $f \in S_{X,A}$.

Then $T_\alpha P_2 f \to TP_2 f$ and hence by continuity of P_1, $P_1T_\alpha P_2 f \to P_1TP_2 f$.

This completes the proof.

IV. (T^A, τ_{wp}).

The seminorm

$$T \mapsto |\langle T(P_2 f),\ P_1^c G \rangle|$$

is τ_{wp}-continuous for each $f \in S_{X,A}$ and each $G \in T_{X,A}$. □

4. The topological structure of the algebra T_A

As we have already assumed in Section 3, T_A comprises all continuous linear mappings from $T_{X,A}$ into itself. The strong topology and the weak topology of T_A will be denoted respectively by σ_w and σ_s. In correspondence with the topologies τ_p and τ_{wp} of T_A we first introduce the topologies σ_p and σ_{wp}.

(4.1) Definition (The topology of pointwise convergence)

The topology σ_p is the locally convex topology of T_A induced by the seminorms $v_{F,t}$

$$v_{F,t}(R) = \|(RF)(t)\| \ , \quad R \in T_A$$

where $F \in T_{X,A}$ and $t > 0$.

The net (R_α) in T_A converges to $R \in T_A$ in σ_p-sense if and only if $R_\alpha F \to RF$ strongly for all $F \in T_{X,A}$. The topology σ_p is the coarsest topology for which the linear mappings $T_A \to T_{X,A}$

$$R \leftrightarrow RF \quad , \quad R \in T_A \ ,$$

are all continuous.

(4.2) Lemma

The topology σ_s is equivalent to the topology of uniform convergence on bounded subsets of $T_{X,A}$.

Proof. Let (R_α) be a strongly convergent net in T_A with limit R. Let B be a strongly bounded subset of $T_{X,A}$. Then there exists $\psi \in B_+(\mathbb{R})$ and a bounded subset W of X such that $B = \psi(A)(W)$ (Cf. Ch. II, Section 2). Hence for all $w \in W$

$$\| e^{-tA}(R_\alpha - R)\psi(A)w\| \le \| (R_\alpha(t) - R(t))\psi(A)\| \|w\| \quad .$$

On the other hand, let $\varepsilon > 0$ and let $\psi \in B_+(\mathbb{R})$. Suppose $R_\alpha F \to RF$ strongly in $T_{X,A}$ and uniformly for $F \in \{\psi(A)w \mid \|w\| \le 1\}$. Then for each $t > 0$ there is α_1 such that

$$\| (R_\alpha(t) - R(t))\psi(A)w\| < \varepsilon/2$$

for all $\alpha \ge \alpha_1$ and all $w \in X$ with $w \le 1$. Hence

$$\| (R_\alpha(t) - R(t))\psi(A)\| \le \varepsilon/2 < \varepsilon \quad . \qquad \square$$

(Remember the remark after Lemma (3.2).)

(4.3) Definition (The topology of weak pointwise convergence).

The topology τ_{wp} is the locally convex topology induced by the seminorms

$$v_{G,f}(R) = |\langle f,RG\rangle| \quad , \quad R \in T_A \ ,$$

where $f \in S_{X,A}$ and $G \in T_{X,A}$.

The net (R_α) converges to R in (T_A,τ_{wp}) if and only if $\langle f,(R_\alpha - R)G\rangle \to 0$ for all $f \in S_{X,A}$ and $G \in T_{X,A}$. The topology τ_{wp} is the coarsest topology for which the linear mappings $T_A \to \mathbb{C}$

$$R \mapsto \langle f,RG\rangle \quad , \quad R \in T_A \ ,$$

are all continuous. The topology σ_p is the topology of uniform, weak pointwise convergence on bounded subsets of $S_{X,A}$.

The above introduced topologies are ordered as follows

(4.4)
$$\begin{array}{ccccc} & & \sigma_s & & \\ & \subset & & \supset & \\ \sigma_w & & & & \sigma_p \\ & \supset & & \subset & \\ & & \sigma_{wp} & & \end{array}$$

(4.5) Theorem (Principle of uniform boundedness).

Let B be a subset of T^A. Then the following statements are equivalent

I. B is σ_s-bounded ;

II. B is σ_p-bounded ;

III. B is σ_w-bounded ;

IV. B is σ_{wp}-bounded.

Proof. We shall only prove the implication II $\Rightarrow$ I. The other implications are trivial or easy corollaries of other structure theorems.

II $\Rightarrow$ I: For all $t > 0$, $w \in X$ and $\phi \in B_+(\mathbb{R})$, we thus assume that the set

$$\{e^{-tA}R\phi(A)w \mid R \in B\}$$

is strongly bounded in $B(X)$. Hence, the uniform boundedness principle for $B(X)$ yields $\alpha(t,\phi) > 0$ with $\| e^{-tA}R\phi(A)\| \leq \alpha(t,\phi)$, $R \in B$. Hence

$$\| e^{-tA}R\phi(A)\|_{X\otimes X} \leq \alpha(\tfrac{1}{2}t,\phi)\| e^{-\frac{1}{2}tA}\|_{X\otimes X}, \quad R \in B. \qquad \square$$

(4.6) Lemma

Let (R_n) be a sequence in T_A such that $\lim_{n\to\infty} R_n F$ exists in $T_{X,A}$ for each

$F \in T_{X,A}$. Then R: $F \mapsto \lim_{n\to\infty} R_n F$ is continuous, i.e. $R \in T_A$.

Proof. By the preceding theorem the sequence (R_n) is τ_s-bounded. So for each $t > 0$ there exists $\beta_t > 0$ such that $\|R_n(t)\| \le \beta_t$, $n \in \mathbb{N}$. It is clear that R maps $T_{X,A}$ into itself. Further, for all $w \in X$ with $\|w\| = 1$, and for all $t > 0$

$$\|e^{-tA}Rw\| \le \|e^{-tA}(R - R_n)w\| + \beta_t \le \beta_t + 1$$

for $n \in \mathbb{N}$ sufficiently large. Hence $R \in T_A$ by Ch. I, Section 4. □

(4.7) Theorem

T_A is sequentially σ_p- and σ_{wp}-complete.

In Section 2 we have proved that the mapping ^{c} from T^A onto T_A is $\tau_s \leftrightarrow \sigma_s$ and $\tau_w \leftrightarrow \sigma_w$ continuous, and its inverse ^{c} is $\sigma_s \leftrightarrow \tau_s$ and $\sigma_w \leftrightarrow \tau_w$ continuous. We do not know whether the mapping ^{c} is $\tau_p \leftrightarrow \sigma_p$ continuous and whether its inverse is $\sigma_p \leftrightarrow \tau_p$ continuous. However, for $f \in S_{X,A}$ and $G \in T_{X,A}$,

$$|\langle Pf,G\rangle| = |\langle f,P^cG\rangle| \quad , \quad P \in T^A.$$

So it follows that $P \mapsto P^c$, $P \in T^A$, is $\tau_{wp} \leftrightarrow \sigma_{wp}$ continuous and $R \mapsto R^c$, $R \in T_A$, is $\sigma_{wp} \leftrightarrow \tau_{wp}$ continuous.

With the above observed kinds of continuity of the mapping ^{c} and the mentioned properties of ^{c} the following results are straightforward corollaries of Theorem (3.8) and Theorem (3.13).

(4.8) Theorem

- Joint multiplication is sequentially continuous in T_A.
- The algebra T_A is locally convex if it carries one of the

topologies σ_s, σ_w and σ_{wp}.

Completing this section we prove the following.

(4.9) Theorem

The algebra T^A with topology τ_p is locally convex.

Proof. Let $R_\alpha F \to RF$ for all $F \in T_{X,A}$. Then for $S_1, S_2 \in T_A$, $R_\alpha S_2 F \to RS_2 F$ and hence by continuity of S_1, $S_1 R_\alpha S_2 F \to S_1 RS_2 F$. This completes the proof. □

5. The topological structure of the algebra E_A

Because of the assumption in Section 3 that $S_{X,A}$ is nuclear, E_A comprises all continuous linear mappings from $S_{X,A}$ into itself which are extendable to $T_{X,A}$. In Section 3 we observed that the strong and the weak topology of E_A, denoted by ρ_s and ρ_w in the sequel, admit the following characterizations

- ρ_s is the coarsest locally convex topology on E_A for which the embeddings $E_A \hookrightarrow T^A$ and $E_A \hookrightarrow T_A$ are continuous with respect to the strong topology of T^A resp. T_A.
- ρ_w is the coarsest locally convex topology on E_A for which the embeddings $E_A \hookrightarrow T^A$ and $E_A \hookrightarrow T_A$ are continuous with respect to the weak topology of T^A resp. T_A.

Similarly we introduce the topologies ρ_p and ρ_{wp}.

(5.1) Definition

The topology ρ_p is the coarsest locally convex topology on E_A for which the embeddings $E_A \hookrightarrow T^A$ and $E_A \hookrightarrow T_A$ are continuous with respect to τ_p

resp. σ_p. The net (E_α) in $\mathcal{E}_A$ converges to E if and only if $E_\alpha f \to Ef$ strongly in $\mathcal{S}_{X,A}$ for all $f \in \mathcal{S}_{X,A}$ as well as $E_\alpha G \to EG$ strongly in $\mathcal{T}_{X,A}$ for all $G \in \mathcal{T}_{X,A}$.

(5.2) <u>Lemma</u>

The topology ρ_s is equivalent to the topology of uniform τ_p- and σ_p-convergence on bounded sets in $\mathcal{S}_{X,A}$ resp. $\mathcal{T}_{X,A}$.

<u>Proof</u>. Cf. Lemma (3.2) and (4.2). □

(5.3) <u>Definition</u>

The topology ρ_{wp} is the coarsest locally convex topology on $\mathcal{E}_A$ for which the embeddings $\mathcal{E}_A \hookrightarrow \mathcal{T}^A$ and $\mathcal{E}_A \hookrightarrow \mathcal{T}_A$ are continuous with respect to τ_{wp} resp. σ_{wp}. The net (E_α) in $\mathcal{E}_A$ converges to E if and only if $E_\alpha f \to Ef$ weakly in $\mathcal{S}_{X,A}$ for all $f \in \mathcal{S}_{X,A}$ as well as $E_\alpha G \to EG$ weakly in $\mathcal{T}_{X,A}$ for all $G \in \mathcal{T}_{X,A}$.

The above introduced topologies of $\mathcal{T}^A$ are ordered as follows.

(5.4)
$$\begin{array}{ccccc} & & \rho_s & & \\ & \subset & & \supset & \\ \rho_w & & & & \rho_p \\ & \supset & & \subset & \\ & & \rho_{wp} & & \end{array}$$

(5.5) <u>Theorem</u> (Principle of uniform boundedness)

Let $\mathcal{B}$ be a subset of $\mathcal{E}_A$. Then the following statements are equivalent.

I. $\mathcal{B}$ is ρ_s-bounded;

II. $\mathcal{B}$ is ρ_w-bounded;

III. $\mathcal{B}$ is ρ_p-bounded;

IV. B is ρ_{wp}-bounded.

Proof. Cf. Theorem (3.5) and (4.5). □

(5.6) Theorem

E_A is sequentially complete in ρ_p- and ρ_{wp}-sense.

Proof. Cf. Theorem (3.7) and (4.7). □

The adjoint mapping ^{c} becomes an involution on the algebra E_A. From the previous sections it follows that ^{c} is ρ_s-, ρ_w- and ρ_{wp}-continuous. From Theorem (3.13), (4.8) and (4.9) we obtain immediately

(5.7) Theorem

- Joint multiplication is strongly sequentially continuous in E_A.
- Separate multiplication is ρ_s-, ρ_w-, ρ_p- and ρ_{wp}-continuous.

The dual space E_A' of E_A can be represented by the algebraic sum of the spaces S_A and S^A. So every continuous linear functional ℓ on E_A can be written as

$$\ell: E \mapsto \langle\langle K_1, E\rangle\rangle_{S_A} + \langle\langle K_2, E\rangle\rangle_{S^A},$$

where $K_1 \in S_A$ and $K_2 \in S^A$. The choice of K_1 and K_2 is not unique because $S_A \cap S^A = S_{X\otimes X, A\boxplus A}$, cf. Ch. II, Section 4.

(5.8) Proposition

The space $S_{X\otimes X, A\boxplus A}$ is a proper, two-sided ideal in E_A.

Proof. S_A and S^A are proper, two-sided ideals in T^A resp. T_A. Hence $S_{X\otimes X, A\boxplus A} = S_A \cap S^A$ is a proper two-sided ideal in $T_A \cap T^A = E_A$. □

Let $E_1, E_2 \in \mathcal{E}_A$. Then for all $(K_1 + K_2) \in \mathcal{E}_A'$, define

$$E_1(K_1 + K_2)E_2 := E_1K_1E_2 + E_1K_2E_2 \ .$$

Then $E_1(K_1 + K_2)E_2$ is a well-defined element of $\mathcal{E}_A'$ by Lemma (3.9) and Corollary (3.10). In order to prove this, we have to show that the definition of $E_1(K_1 + K_2)E_2$ does not depend on the choice of K_1 and K_2. So let $K_1 + K_2 = 0$. Then $K_1 = -K_2 \in \mathcal{S}_A \cap \mathcal{S}^A = \mathcal{S}_{X \otimes X, A \boxplus A}$. By Proposition (5.8), $E_1K_1E_2 = -E_1K_2E_2 \in \mathcal{S}_{X \otimes X, A \boxplus A}$. Hence, $E_1K_1E_2 + E_1K_2E_2 = 0$, which completes the proof.

These observations imply the following.

(5.9) Lemma

Let $K \in \mathcal{E}_A'$ and $E \in \mathcal{E}_A$. Then

$$\langle\langle K,E \rangle\rangle = \overline{\langle\langle K^c,E^c \rangle\rangle}$$

$$\langle\langle K,E \rangle\rangle = \langle\langle E^cK,I \rangle\rangle$$

$\langle\langle EK,I \rangle\rangle = \langle\langle KE,I \rangle\rangle$ or equivalently trace (EK) = trace(KE).

Proof. Cf. Corollary (3.11). □

In a forthcoming paper we shall give a complete description of two subalgebras of $\mathcal{E}_A$, where we no longer assume that $\mathcal{S}_{X,A}$ is nuclear. There we shall treat two topological algebras, the commutant of $\{A\}'$ and the double commutant $\{A\}''$. Inspired by the thesis of Pijls [Pij], we have been able to prove that $\{A\}'' \subset \mathcal{E}_A$ is a commutative GW^*-algebra, i.e. a commutative generalized Von Neumann algebra. The notion of GW^*-algebra has been introduced by Allan,[Al].

6. Applications to quantum statistics

In this section we consider a quantum mechanical system in which the dynamics is determined by a Hamiltonian operator H, i.e. a self-adjoint operator in some appropriate Hilbert space X. We assume the almost inevitable condition that there can be found a nuclear analyticity space $S_{X,A}$ such that H and each of the unitary operators $e^{i\alpha H}, \alpha \in \mathbb{R}$, are continuous linear mappings on $S_{X,A}$. Further, for the states of the quantum system we take the one-dimensional subspaces of the trajectory space $T_{X,A}$. In Ch. IV we have proved that $T_{X,A}$ contains almost all (generalized) eigenvectors of H.

In this section we adopt the terminology and notation of Dirac. The elements of $T_{X,A}$ are called kets and they are denoted by $|F\rangle$. Conjugate to the kets are the bras, denoted by $\langle F|$. The bra space is also a trajectory space, it has an antilinear structure. In Ch. IV we have interpreted Dirac's bracket notion so that the expression

$$\langle F|G\rangle$$

makes sense for arbitrary kets and bras. In fact, $\langle F|G\rangle$ denotes the function

$$\langle F|G\rangle : s \mapsto \overline{\langle |F\rangle(s), |G\rangle\rangle}$$

The elements of $S_{X,A}$ are called test kets. The bras conjugated to them are called test bras. In this section we shall only consider the bracket of a test bra $\langle g|$ and a ket $|F\rangle$ resp. of a bra $\langle G|$ and a test ket $|f\rangle$. Then for their brackets we may take the ordinary numbers $\langle g|F\rangle(0)$ and $\langle G|f\rangle(0)$.

At a certain instant the dynamical system is supposed to be in one or other of a number of possible states according to some given probability law. Following Dirac, [Di], these states may establish a discrete set, a continuous range or both together. Here we look at the discrete case. Suppose that the possible states are given by normalized test kets $|m\rangle$, $m \in \mathbb{N}$. Let p_m denote the probability that the system is in the m-th state. Then we define the quantum density operator ρ by

(6.1) $$\rho = \sum_{m=1}^{\infty} p_m |m\rangle\langle m| \quad , \quad \sum_{m=1}^{\infty} p_m = 1 \quad , \quad p_m \geq 0 \ ,$$

where, according to Dirac $|m\rangle\langle m| = |m\rangle \otimes |m\rangle$.

In Schrödinger's picture the kets will evoluate in time in accordance with Schrödinger's equation

$$i\hbar \frac{d}{dt}|F\rangle = H|F\rangle$$

and the bras with the hermitean conjugate of this equation. Since without disturbance the system remains in the same state, corresponding to a ket which satisfies Schrödinger's equation, the p_m's are constant in time. We therefore have the following equation

(6.2) $$\begin{aligned} i\hbar\dot{\rho} &= \sum_m p_m \left(H|m\rangle\langle m| - |m\rangle\langle m|H\right) \\ &= H\rho - \rho H = [H,\rho]. \end{aligned}$$

For convenience we shall take $\hbar = 1$ in the sequel.

In our interpretation, the observables of the quantum system are represented by self-adjoint operators in X, which maps $S_{X,A}$ continuously into itself. Or, equivalently, by the symmetric elements of E_A with a self-adjoint extension in X.

If the system is in the m-th state, the expectation value $\langle\beta\rangle$ of any observable β equals

$$\langle\beta\rangle = \langle m|\beta|m\rangle.$$

Hence, if we insert the distribution law of the system corresponding to the above-introduced density operator ρ, then the average expectation value $\langle\beta\rangle$ is given by

$$\langle\beta\rangle = \sum_m p_m \langle m|\beta|m\rangle = \langle\!\langle \rho,\beta \rangle\!\rangle = \mathrm{tr}(\rho\beta), \tag{6.3}$$

whenever $\rho \in E_A'$. Put $\beta = I$. Then it follows that

$$\langle I\rangle = \sum_m p_m = 1 .$$

The solution of equation (5.2) is given by

$$\rho(t) = e^{-itH}\rho_0 e^{itH}, \quad t \geq 0 ,$$

where $\rho(0)$ is ρ_0. Since the unitary operators $e^{i\alpha H}$, $\alpha \in \mathbb{R}$, are extendable, and since E_A' remains invariant under right and left multiplication by elements of E_A. (See Lemma (5.2)), we have $\rho(t) \in E_A'$, $t \geq 0$ iff $\rho_0 \in E_A'$.

Let β_0 be any observable. Then the average expectation value at time t equals

$$\langle\beta_0\rangle(t) = \langle\!\langle \rho(t), \beta_0(t) \rangle\!\rangle = \langle\!\langle \rho_0, e^{itH}\beta_0(t)e^{-itH} \rangle\!\rangle$$

where we have written $\beta_0(t)$ to indicate that the observable β_0 can intrinsically depend on t. Put $\beta(t) = e^{itH}\beta_0(t)e^{-itH}$. Then

$$\dot{\beta} = i[H,\beta] + \frac{\partial\beta}{\partial t} \tag{6.4.a}$$

(6.4.b) $$\frac{d}{dt}(\langle\beta\rangle) = i\langle[H,\beta]\rangle + \langle\frac{\partial\beta}{\partial t}\rangle$$

where $\frac{\partial\beta}{\partial t}(\tau) = e^{i\tau H}\frac{d\beta_0}{dt}(\tau)e^{-i\tau H}$. The differential equations (6.4.a) and (6.4.b) determine the evolution of the observables in the Heisenberg picture.

Now we are in a position to describe a quantum mechanical system in terms of observables out of some suitably chosen space E_A, and 'states' in its corresponding strong dual E'_A. We emphasize that the notion of state will get a meaning different from the one in the beginning of this section.

(6.5) Definition

A symmetric element $P \in E_A$ is called positive if $\langle f|P|f\rangle \geq 0$ for all test kets $|f\rangle$.

A positive element P of E_A leads to a positive, density defined, symmetric operator $\tilde{P}$ in X. This operator $\tilde{P}$ admits a so-called Friedrichs extension P_F in X, cf.[Fa]. The operator P_F is positive and self-adjoint in X. Hence, at least every positive element of E_A is an observable.

(6.6) Definition

Let $\sigma \in E'_A$. Then σ is called real if $\sigma(P) \in \mathbb{R}$ for all $P \in E_A$ with $P = P^c$.

From Section 5 we obtain the following characterization.

(6.7) Theorem

$\sigma \in \mathcal{E}_A'$ is real iff $\sigma^c = \sigma$.

Proof. Let $P \in \mathcal{E}_A$ be symmetric. Then by Section 5

$$\langle\!\langle \sigma, P \rangle\!\rangle = \overline{\langle\!\langle \sigma^c, P^c \rangle\!\rangle}\,.$$

This leads to the following equivalences

$$\langle\!\langle \sigma, P \rangle\!\rangle \in \mathbb{R} \text{ for all } P \in \mathcal{E}_A \text{ with } P = P^c \Leftrightarrow$$

$$\Leftrightarrow \langle\!\langle \sigma, P \rangle\!\rangle = \langle\!\langle \sigma^c, P \rangle\!\rangle \text{ for all } P \in \mathcal{E}_A \text{ with } P = P^c \Leftrightarrow$$

$$\Leftrightarrow \quad \sigma = \sigma^c.$$

The latter equivalence is due to the fact that every $E \in \mathcal{E}_A$ is a combination of two symmetric elements, $E = \frac{E + E^c}{2} + i\left(\frac{E - E^c}{2i}\right)$ □

Remark: Let $\sigma \in \mathcal{E}_A'$ with $\sigma = \sigma^c$. Then $\sigma = s_1 + s_2$ with $s_1 \in S^A$ and $s_2 \in S_A$. (Cf. Section 5). Put $s = \frac{s_1 + s_2^c}{2}$. Then $s \in S^A$ and $\sigma = s + s^c$.

(6.8) Definition

Let $\sigma \in \mathcal{E}_A'$ be a real functional. Then σ is called a state if

- $\sigma(P) \geq 0$ for all positive $P \in \mathcal{E}_A$;
- $\sigma(I) = 1$, i.e. a state is always normalized.

In order to characterize the states in $\mathcal{E}_A'$ we prove the following.

(6.9) Lemma

Let $E \in \mathcal{E}_A$, and let Π_n denote the orthogonal projection onto the linear span of the first n eigenvectors of A. Then the sequence $\{\Pi_n E \Pi_n\}$ converges to E in $\mathcal{E}_A$.

Proof. Let t > 0. Then we can take τ > 0 such, that both

$$\| e^{2\tau A} E e^{-\frac{1}{2}tA} \|_{X\otimes X} < \infty$$

and

$$\| e^{-\frac{1}{2}tA} E e^{2\tau A} \|_{X\otimes X} < \infty .$$

Now we compute as follows

$$\| e^{\tau A}(E - \Pi_n E \Pi_n) e^{-tA} \|_{X\otimes X} \leq$$

$$\leq \| e^{\tau A}(I - \Pi_n) E \Pi_n e^{-tA} \|_{X\otimes X} + \| e^{\tau A} E (I - \Pi_n) e^{-tA} \|_{X\otimes X} \leq$$

$$\leq (\| (I - \Pi_n) e^{-\tau A} \| + \| (I - \Pi_n) e^{-\frac{1}{2}tA} \|) \| e^{2\tau A} E e^{-\frac{1}{2}tA} \|_{X\otimes X} .$$

Hence, $\| e^{\tau A}(E - \Pi_n E \Pi_n) e^{-tA} \|_{X\otimes X} \to 0$ for $n \to \infty$.

Similarly we can prove

$$\| e^{-tA}(E - \Pi_n E \Pi_n) e^{\tau A} \|_{X\otimes X} \to 0 \text{ for } n \to \infty .$$

So the assertion has been shown. □

Remark: Let $P \in \mathcal{E}_A$ be positive. Then for each $n \in \mathbb{N}$, the operator $\Pi_n P \Pi_n$ is an element of $\mathcal{E}_A$. In fact $\Pi_n P \Pi_n$ is a positive self-adjoint Hilbert-Schmidt operator. So there exists $f_j^{(n)} \in \Pi_n(X)$, $j = 1,\dots,n$, such that

$$\Pi_n P \Pi_n = \sum_{j=1}^{n} \mu_j^{(n)} | f_j^{(n)} \rangle\langle f_j^{(n)} |$$

with $\mu_j \geq 0$. It leads to the following characterization.

(6.10) Theorem

Let $\sigma \in \mathcal{E}_A'$ be real. Then σ is a state iff

$$\langle\langle \sigma, |f\rangle\langle f| \rangle\rangle \geq 0$$

for all test kets $|f\rangle$.

Proof

$\Rightarrow$) Trivial. The projections $P_{|f\rangle} = |f\rangle\langle f|$ are elements of $\mathcal{E}_A$ and they are positive for all test kets $|f\rangle$.

$\Leftarrow$) Let $P \in \mathcal{E}_A$ be positive. Let the projection Π_n, $n \in \mathbb{N}$, be as in Lemma (6.9). The functional $E \mapsto \langle\langle \sigma, E\rangle\rangle$ is strongly continuous on $\mathcal{E}_A$. Hence

$$\langle\langle \sigma, P\rangle\rangle = \lim_{n\to\infty} \langle\langle \sigma, \Pi_n P \Pi_n \rangle\rangle .$$

With the above remark it can be easily seen that for all $n \in \mathbb{N}$ $\langle\langle \sigma, \Pi_n P \Pi_n \rangle\rangle \geq 0$. Hence $\langle\langle \sigma, P\rangle\rangle \geq 0$.

Thus we have shown that σ is a state. □

Remark: Since $\sigma \in \mathcal{E}_A' \subset \mathcal{T}_{X\otimes X, A\boxplus A}$, and $|f\rangle\langle f| \in \mathcal{S}_{X\otimes X, A\boxplus A}$ we derive $\langle\langle \sigma, |f\rangle\langle f| \rangle\rangle = \langle f|\sigma|f\rangle$. (See [Di]).

Special elements of $\mathcal{E}_A'$ are the pure states. Here is the definition.

(6.11) Definition

A state ρ is called pure if there exists a normalized test ket $|f\rangle$ with $\rho = |f\rangle\langle f|$.

Of course, one might wonder why we don't take normalizable kets in Definition (6.8), i.e. kets in the Hilbert space X. The following lemma shows the answer.

(6.12) Lemma

Let $|\omega\rangle$ be a ket. Then

$$|\omega\rangle\langle\omega| \in E'_A \Leftrightarrow |\omega\rangle \text{ is a test ket.}$$

Proof

$\Rightarrow$) Suppose $|\omega\rangle \notin S_{X,A}$. Then there exists $\psi \in B_+(\mathbb{R})$ such that $|\omega\rangle \notin D(\psi(A))$. The operator $\psi(A)^2$ is in E_A, but

$$\langle\langle |\omega\rangle\langle\omega|, \psi(A)^2 \rangle\rangle = \infty .$$

Hence $|\omega\rangle\langle\omega| \notin E'_A$.

$\Leftarrow$) Trivial. □

The pure states admit the folllowing characterization.

(6.13) Theorem

A state ρ is pure if and only if $\rho \in S^A$ (or S_A) with $\rho^2 = \rho$.

Proof. If ρ is pure, $\rho = |f\rangle\langle f|$ for some test ket $|f\rangle$. Hence $\rho \in S_{X\otimes X, A\boxplus A} = S^A \cap S_A$, and ρ is a projection. On the other hand, $\rho \in S^A$ and ρ is a state yield $\rho = \rho^c \in S_A$. Hence $\rho \in S_{X\otimes X, A\boxplus A}$; ρ is a Hilbert-Schmidt projection with $\mathrm{tr}(\rho) = 1$. So there exists a normalized $|f\rangle \in X$ with $\rho = |f\rangle\langle f|$. By Lemma (6.12) $|f\rangle$ is a test ket. □

(6.14) Theorem

Every pure state in E'_A is an extreme point in the set of states.

Proof. Let $|f\rangle$ be a normalized test ket, and Π_n, $n \in \mathbb{N}$, denote the projection as introduced in Lemma (6.9). Suppose there exist states $\sigma, \sigma_2 \in E'_A$ and $0 < \alpha < 1$ such that

$$|f\rangle\langle f| = \alpha\sigma_1 + (1-\alpha)\sigma_2 .$$

Then for all $n \in \mathbb{N}$ with $\Pi_n|f\rangle \neq 0$

$$\frac{\Pi_n|f\rangle\langle f|\Pi_n}{\|\Pi_n|f\rangle\|^2} = \frac{\alpha\sigma_1(\Pi_n)}{\|\Pi_n|f\rangle\|^2}\left[\frac{\Pi_n\sigma_1\Pi_n}{\sigma_1(\Pi_n)}\right] + \frac{(1-\alpha)\sigma_2(\Pi_n)}{\|\Pi_n|f\rangle\|^2}\left[\frac{\Pi_n\sigma_2\Pi_n}{\sigma_2(\Pi_n)}\right]$$

Take $k \in \mathbb{N}$ fixed, with $\Pi_k|f\rangle\langle f|\Pi_k \neq 0$. Then $\dfrac{\Pi_k|f\rangle\langle f|\Pi_k}{\|\Pi_k|f\rangle\|^2}$ is an extreme point in the set of states in $\Pi_k(X)\otimes\Pi_k(X)$. Hence, we may assume

$$\Pi_k|f\rangle\langle f|\Pi_k = \Pi_k\sigma_1\Pi_k .$$

Since $\Pi_k\Pi_\ell = \Pi_k$ for all $\ell \geq k$ we derive

$$\forall_{n\in\mathbb{N}} : \Pi_n|f\rangle\langle f|\Pi_n = \Pi_n\sigma_1\Pi_n .$$

By Lemma (6.9) the sequences $\{\Pi_n|f\rangle\langle f|\Pi_n\}$ and $\{\Pi_n\sigma_1\Pi_n\}$ converge to $|f\rangle\langle f|$ resp. σ_1 weakly. Hence $\sigma_1 = |f\rangle\langle f|$. □

In the following theorem we prove that the pure states are the only extreme points in the set of states.

(6.15) Theorem

Let ρ be an extreme point in the set of states. Then ρ is a pure state

Proof. Since $\rho \neq 0$, there exists a normalized test ket $|f\rangle$ such that

$$\rho(|f\rangle\langle f|) \neq 0 .$$

Remark: The following implication can be shown rather easily:

$$\left(\forall_{|f>\in S_{X,A}} : \rho(|f><f|) = 0\right) \Rightarrow (\rho = 0) \ .$$

Put $P_{|f>} = |f><f|$. Then ρ can be written as

$$\rho = \rho \circ P_{|f>} + \rho \circ (I - P_{|f>})$$

where $(\rho \circ P_{|f>})(E) = \rho(P_{|f>}E)$, $E \in E_A$. So $(\rho \circ P_{|f>})(I) = \rho(P_{|f>}) \neq 0$.

1) Suppose $\rho \circ (I - P_{|f>}) \neq 0$, and consequently $\rho(I - P_{|f>}) \neq 0$. Then we can write $\rho = \alpha\rho_1 + (1-\alpha)\rho_2$, where

$$\rho_1 = \frac{\rho \circ P_{|f>}}{\rho(P_{|f>})} \ , \quad \rho_2 = \frac{\rho \circ (I - P_{|f>})}{1 - \rho(P_{|f>})} \ ,$$

$$\alpha = \rho(P_{|f>}) .$$

The functionals ρ_1 and ρ_2 are states. This can be seen as follows

$$\rho_1(I) = \frac{\rho(P_{|f>})}{\rho(P_{|f>})} = 1 \ ,$$

and

$$\rho_1(E) = (\rho(P_{|f>}))^{-1}\rho(P_{|f>}E) = (\rho(P_{|f>}))^{-1}\rho(P_{|f>}EP_{|f>}) .$$

For the latter equality see Lemma (6.9) and observe that $P^2_{|f>} = P_{|f>}$. Thus we derive $\rho_1(E) \in \mathbb{R}$ for all $E \in E_A$ with $E = E^c$ and $\rho_1(E) \geq 0$ for all positive $E \in E_A$. Similarly, ρ_2 is a state. But now we have got a contradiction, because ρ is extreme. Hence $\rho \circ (I - P_{|f>}) = 0$, and consequently $\rho = \rho \circ P_{|f>}$ and $\rho(P_{|f>}) = 1$. Further, it easily follows that for all test kets $|g>$

$$\rho(|g\rangle\langle g|) = |\langle f|g\rangle|^2 .$$

Employing the projections Π_n, $n \in \mathbb{N}$, as introduced in Lemma (6.9), we find that for each symmetric $E \in \mathcal{E}_A$ and for each $n \in \mathbb{N}$ there exists $\mu_j^{(n)} \in \mathbb{R}$ and $|f_j^{(n)}\rangle \in \Pi_n(X)$ such that

$$\Pi_n E \Pi_n = \sum_{j=1}^{n} \mu_j^{(n)} |f_j^{(n)}\rangle\langle f_j^{(n)}|$$

and

$$\rho(\Pi_n E \Pi_n) = \rho\left(\sum_{j=1}^{n} \mu_j^{(n)} |f_j^{(n)}\rangle\langle f_j^{(n)}|\right) =$$

$$= \sum_{j=1}^{n} \mu_j^{(n)} |\langle f|f_j^{(n)}\rangle|^2 =$$

$$= \langle f|\Pi_n E \Pi_n|f\rangle .$$

Letting $n \to \infty$, by Lemma (6.9) we obtain

$$\rho(\Pi_n E \Pi_n) \to \rho(E)$$

and

$$\langle f|\Pi_n E \Pi_n|f\rangle \to \langle f|E|f\rangle .$$

Hence for all symmetric $E \in \mathcal{E}_A$, $\rho(E) = \langle f|E|f\rangle$.
This yields $\rho = |f\rangle\langle f|$. □

1) Remark: Let $\rho \in \mathcal{E}_A'$ be a real positive functional, i.e. $\rho(P) \geq 0$ for all positive $P \in \mathcal{E}_A$. Let $n \in \mathbb{N}$, and let $E \in \mathcal{E}_A$. Then the following inequality is immediate from the finite-dimensional case

$$|\rho(\Pi_n E \Pi_n)|^2 \leq \rho(\Pi_n)\rho(\Pi_n E^c E \Pi_n) \ .$$

So in the limit $n \to \infty$ we obtain

$$|\rho(E)|^2 \leq \rho(I)\rho(E^c E).$$

Consequently $\rho(I) = 0 \Leftrightarrow \rho = 0$.

(6.16) Theorem

The linear span of the pure states is dense in E_A'.

Proof. We assume that $P \in E_A$ and $<f|P|f> = 0$ for all test kets $|f>$. Then $<f+g|P|f+g>$ and $<f+ig|P|f+ig> = 0$, and hence, $\mathrm{Re}(<f|P|g>) = 0$ and $\mathrm{Im}(<f|P|g>) = 0$ for all test kets $|f>$ and test bras $<g|$. So $P = 0$. □

Finally we shall characterize the states in S^A (or S_A) or equivalently the states in $S_{X\otimes X, A\boxplus A}$.

(6.17) Theorem

Let $\rho \in S_{X\otimes X, A\boxplus A}$. Then the following statements are equivalent.

(1) ρ is a state.

(2) ρ is positive and self-adjoint with $\mathrm{tr}(\rho) = 1$.

(3) There exist normalized $|j> \in S_{X,A}$ and positive numbers p_j satisfying

$$\exists_{s>0} \ : \ \sum_{j=1}^{\infty} p_j^2 \| e^{sA}|j> \|^2 < \infty \ ,$$

and $\sum_j p_j = 1$ such that

$$\rho = \sum_j p_j |j><j| .$$

Proof. The proof proceeds as follows: (1) ⇒ (2) ⇒ (3) ⇒ (1).

(1) ⇒ (2):

From Theorem (6.10) it follows that ρ is a positive operator on $S_{X,A}$. Since ρ is Hilbert Schmidt and $\rho^c = \rho$, ρ is a positive, self-adjoint operator on X with $tr(\rho) = 1$.

(2) ⇒ (3):

By definition, there exists $s > 0$ such that $\rho = e^{-sA}We^{-sA}$ for some $W \in X \otimes X$ with $W \geq 0$. Since $\rho \in X \otimes X$ and $\rho \geq 0$, there exists an orthonormal basis $(|j\rangle)$ in X, and positive numbers p_j such that

$$\rho = \sum_j p_j |j\rangle\langle j| \text{ with } \sum_j p_j = 1 \ .$$

Further, since We^{-sA} is Hilbert Schmidt and $We^{-sA}|j\rangle = p_j e^{sA}|j\rangle$,

$$\sum_{j=1}^{\infty} \| We^{-sA}|j\rangle \|^2 = \sum_{j=1}^{\infty} p_j^2 \| e^{sA}|j\rangle \|^2 < \infty \ .$$

(3) ⇒ (1)

Note first that $\langle\!\langle \rho, I \rangle\!\rangle = \sum_{j=1}^{\infty} p_j \langle j|j\rangle = \sum_{j=1}^{\infty} p_j = 1$.

Let $s > 0$ as indicated. Then

$$\rho|j\rangle = p_j|j\rangle \ .$$

Put $W = e^{sA}\rho e^{sA}$. Then $We^{-sA}|j\rangle = p_j e^{sA}|j\rangle$.

Hence We^{-sA} is Hilbert-Schmidt and thus we find that

$$\rho = e^{-sA}We^{-sA} \in S_{X \otimes X, A \boxplus A} \ .$$

If $E \in \mathcal{E}_A$ is symmetric then $\langle j|E|j\rangle \in \mathbb{R}$ and hence $\rho(E) \in \mathbb{R}$. If $E \in \mathcal{E}_A$ is positive, then $\langle j|E|j\rangle \geq 0$ and hence $\rho(E) \geq 0$. Thus it is clear that ρ is a state. □

As a rule the dynamical state of a quantum system at a certain instant cannot be represented by one single ket, but we have a statistical mixture of kets. Therefore, in the beginning of this section we introduced the quantum density ρ (cf.(6.1)). According to the probability law determined by ρ, the quantum system is in one or other of a number of possible states. So it makes sense to define ρ to be the state of the quantum system at a given time.

If at $t = 0$ the quantum system is in the state ρ_0, at $t = \tau$ the system is in the state $\rho(\tau)$ with

$$\rho(\tau) = e^{-i\tau H}\rho_0 e^{i\tau H}.$$

So ρ satisfies the evolution equation (cf. (6.2))

$$\dot{\rho} = -i[H,\rho] \ .$$

In order to arrive at a mathematical rigorous theory, we only consider $\rho_0 \in \mathcal{E}'_A$. Then for every $t > 0$, $\rho(t) \in \mathcal{E}'_A$, because $e^{itH} \in \mathcal{E}_A$ for all $t \in \mathbb{R}$. (See Section 4). At every time t we can compute the expectation value $\langle\beta\rangle$ with respect to ρ of the observable $\beta \in \mathcal{E}_A$,

$$\langle\beta\rangle(t) = \langle\!\langle \rho(t),\beta \rangle\!\rangle,$$

where for convenience we have assumed that β is constant in time.

Now in general we shall assume that any state in $\mathcal{E}'_A$ as defined in Definition (6.8) represents an initial state of the quantum system in the above indicated way. A state σ_0 evoluates in time according to

$$e^{-itH}\sigma_0 e^{itH}, \quad t > 0 .$$

So the statistical mixture determined by the quantum density operator ρ is a particular kind of state; states such as ρ have an immediate physical interpretation. From (6.17) we obtain that every state $\rho_0 \in S_{X\otimes X, A\boxplus A}$ induces a statistical mixture. The pure states are special types of statistical mixtures; one knows with certainty that the system is in a state determined by one test ket.

We conclude this section with a short discussion of the three possible types of dynamical quantum systems.

(1) The Hamiltonian operator H admits a purely discrete spectrum

This case is the easiest one to treat and it probably contains the most promising results.

Let H be a Hamiltonian operator in X with eigenvalues $E_1 \le E_2 \le \ldots$, and corresponding normalized eigenkets $|E_1\rangle, |E_2\rangle, \ldots$. Then the eigenkets $|E_i\rangle$ of H establish a complete orthonormal basis for X. Define the positive numbers λ_n, $n \in \mathbb{N}$, as follows

$$\lambda_1 = E_1 \quad \lambda_n = \max(\lambda_{n-1} + 1, |E_n|), \quad n > 1 ,$$

and the self-adjoint operator A by

$$A|E_n\rangle = \lambda_n |E_n\rangle$$

followed by linear extension and unique self-adjoint extension to X. Then the analyticity space $S_{X,A}$ is nuclear because $\sum_{n=1}^{\infty} e^{-\lambda_n t} < \infty$ for all $t > 0$.

Further, H is continuous on $S_{X,A}$ because $\sup_{n\in\mathbb{N}} (|E_n| e^{-\lambda_n t}) < \infty$. Hence, $H \in E_A$. Similarly if follows that the unitary operators $e^{i\alpha H}$, $\alpha \in \mathbb{R}$, are elements of E_A. So the space $S_{X,A}$ satisfies the required conditions.

An important example of a statistical mixture is given by the state

$$\rho_0 = \sum_{n=1}^{\infty} p_n \; |E_n\rangle\langle E_n| \;, \; p_n \geq 0, \; \sum_{n=1}^{\infty} p_n = 1 \;.$$

Then ρ is represented by a diagonal matrix, and seen as a bounded operator on X, ρ clearly commutes with A and H. Since $\rho \in E'_A$, it satisfies

$$\exists_{\alpha>0} \forall_{a>0} \exists_{M>0} \forall_{n\in\mathbb{N}} : \; (p_n e^{-a\lambda_n} \, e^{\alpha\lambda_n}) < M \;.$$

Hence $p_n = \mathcal{O}(e^{-\alpha\lambda_n})$, and $\rho \in S_{X\otimes X, A\boxplus A}$. It is obvious that without disturbance the state ρ does not depend on the time t. We note that it is obvious that every term $|E_n\rangle\langle E_n|$ of the series does not depend on t, i.e. the system remains in a stationary state as long as disturbances do not occur.

In general a state ρ is given by

$$\rho = \sum_{n,m} \rho_{nm} |E_n\rangle\langle E_m| \;.$$

However, in many physically realistic cases the non-diagonal elements can be neglected.

An example for class (1) is given by the one dimensional harmonic oscillator where $H = \frac{1}{2}(\frac{-d^2}{dx^2} + x^2 + 1)$. Then H is self-adjoint in $L_2(\mathbb{R})$

with $E_n = n$, $n \in \mathbb{N}$ as its eigenvalues and the Hermite functions as its eigenfunctions. Hence, we can take $A = H$. We note that the space $S_{L_2(\mathbb{R}),H}$ is equal to the space $S_{\frac{1}{2}}^{\frac{1}{2}}$ of Gelfand-Shilov. Well-defined observables are the momentum operator $i\frac{d}{dx}$ and the position operator x.

(2) The Hamilton operator H admits a purely continuous spectrum

This is a harder case. We are able to construct a nuclear analyticity space $S_{X,A}$ such that H is continuous on $S_{X,A}$ (cf. Section 9). Then to almost every point in the spectrum of H there corresponds on eigenket in the trajectory space $T_{X,A}$. However, it is not clear whether the unitary operators $e^{i\alpha H}$, $\alpha \in \mathbb{R}$, are continuous on $S_{X,A}$, and this problem has not been solved yet. Of course, we could weaken the conditions on $S_{X,A}$ and skip nuclearity. Then the analyticity space $S_{X,|H|}$ with$(H) = (H^2)^{\frac{1}{2}}$ would be ideal. But nuclearity seems to play an essential role both in the discussions of this section and in our interpretation of Dirac's formalism.

There is another approach. Sometimes iH is one of the skew-adjoint generators of a unitary Lie group representation on X with nuclear analyticity space. We shall explain this to some extent. Let G be a finite dimensional Lie group with Lie algebra $A(G)$. Let U be a representation of G into the space of unitary operators on X, and ∂U the corresponding infinitesimal representation of $A(G)$ in X. Then for every $a \in A(G)$ the operator $\partial U(a)$ is skew-adjoint in X, by Stone's theorem.

Our first assertion is the following one.

- There exists $a_1 \in A(G)$ such that $iH = \partial U(a_1)$.

Since G has dimension $d < \infty$ there are $a_2, \ldots, a_d \in A(G)$ such that $\{a_1, \ldots, a_d\}$ generates the Lie group G in the usual way. Following Nelson, [Ne], the analyticity space corresponding to the unitary representation U is equal to

$$S_{X,\Delta^{\frac{1}{2}}}$$

where $\Delta = 1 - ((\partial U(a_1))^2 + (\partial U(a_2))^2 + \ldots + (\partial U(a_d))^2)$.

Then our second assumption is

- $S_{X,\Delta^{\frac{1}{2}}}$ is nuclear.

In Ch. I, Section 7, we have given several cases of unitary representations of Lie groups G with a nuclear analyticity space $S_{X,\Delta^{\frac{1}{2}}}$. Moreover, we have proved that both the unitary operators $U(g)$, $g \in G$ and the skew-adjoint operators $\partial U(a_j)$, $j = 1, \ldots, d$, are all continuous on $S_{X,\Delta^{\frac{1}{2}}}$. So under the above-mentioned assumptions the nuclear analyticity space $S_{X,\Delta^{\frac{1}{2}}}$ has the desired properties.

An example for this type of operators is the Hamiltonian operator of the free particle in one dimension,

$$H = -\frac{d^2}{dx^2}\ .$$

An appropriate algebra is the six-dimensional algebra generated by

$$i\frac{d^2}{dx^2}\ ,\ i(\frac{d}{dx}x + x\frac{d}{dx}),\ ix^2,\ ix,\ \frac{d}{dx}\ ,\ i\ .$$

It corresponds to the infinitesimal representation belonging to the unitary representation of the Schrödinger groups on $L_2(\mathbb{R})$. The Schrödinger group is obtained as a semidirect product of $SL(2,\mathbb{R})$ and of W_1, the Weyl group. We note that the Schrödinger group is the symmetry group of the Schrödinger equation of the free particle (see [Mi]).

(3) <u>The Hamiltonian operator H admits a discrete/continuous spectrum</u>

In many applications the intersting part of the spectrum of H is the discrete one. So we split X into the direct sum $X = X_d \oplus X_c$ such that H_d, the restriction of H to X_d, acts invariantly in X_d and H_d is a self-adjoint operator in X_d with discrete spectrum,and such that H_c, the restriction of H to X_c, acts invariantly in X_c and H_c is a self-adjoint operator in X_c with a purely continuous spectrum.
An example for this case is the Hamiltonian operator of the hydrogen atom.

7. The matrices of the elements of T_A and T^A

As in Section 3 we still assume that $S_{X,A}$ is a nuclear space. So in $S_{X,A}$ there exists an orthonormal basis (v_j) for X consisting of eigenvectors of A with eigenvalues λ_j, $\lambda_1 < \lambda_2 < \ldots$ satisfying

$$\sum_{j=1}^{\infty} e^{-\lambda_j t} < \infty$$

for all $t > 0$. Then the space T^A contains all linear mappings from $S_{X,A}$ into itself, and T_A all linear mappings from $T_{X,A}$ into itself. Let $L \in T^A$. Then to L there can be associated the well-defined matrix (L_{ij}) as follows

$$L_{ij} = (Lv_j, v_i), \quad i,j = 1,2\ldots \quad .$$

This section is devoted to the kind of infinite matrices which arises in this way. We shall produce necessary and sufficient conditions on a matrix (Q_{ij}) in order that its associated linear operator Q is a continuous linear mapping on $S_{X,A}$. We emphasize that there are neither elegant nor applicable conditions on infinite matrices which imply boundedness of its associated operator in X (see [Ha], Ch.IV).

Since the linear mapping L is continuous on $S_{X,A}$, it satisfies

$$\forall_{t>0} \exists_{s>0} \exists_{C>0} : \| e^{sA} L e^{-tA} \|_{X \otimes X} \leq C$$

where $\|\cdot\|_{X\otimes X}$ denotes the norm in $X \otimes X$. This implies that the columns Lv_j, $j \in \mathbb{N}$, of the matrix (L_{ij}) satisfy

(7.1) $$\forall_{t>0}\exists_{s>0}\exists_{C>0}\forall_{i\in\mathbb{N}} : \| e^{sA} L v_i \|_X \le C e^{\lambda_i t} .$$

Put $b_i = Lv_i$, $i \in \mathbb{N}$. Then the vectors b_i span the range $L(S_{X,A})$ and from (7.1) it follows that there exists $s > 0$ such that $b_i \in e^{-sA}(X)$, $i \in \mathbb{N}$. Define the trajectory $\hat{L} : (0,\infty) \to X \otimes X$ by

$$\hat{L}(t) = \sum_{i=1}^{\infty} e^{-\lambda_i t}(v_i \otimes b_i), \quad t > 0.$$

Then $\hat{L}(t) \in S_{X\otimes X, I\otimes A}$. To show this let $0 < t_1 < t$, and choose $s > 0$ and $C > 0$ such that

$$\| e^{sA} b_i \| \le C e^{\lambda_i t_1} , \; i \in \mathbb{N} .$$

Then

$$\| e^{sA}\hat{L}(t)\|_{X\otimes X} = \| \sum_{i=1}^{\infty} e^{-\lambda_i t} v_i \otimes (e^{sA} b_i)\|_{X\otimes X} \le$$

$$\le \sum_{i=1}^{\infty} e^{-\lambda_i t} \| e^{sA} b_i \|_X \le C \sum_{i=1}^{\infty} e^{-\lambda_i (t-t_1)} < \infty .$$

Hence $\hat{L}(t) \in S_{X\otimes X, I\otimes A}$. It is obvious that

$$\hat{L}(t_1 + t_2) = (e^{-t_1 A} \otimes I)\hat{L}(t_2) \; , \; t_1, t_2 > 0 .$$

So $\hat{L} \in T^{A}$. Since for all $f \in S_{X,A}$

$$\hat{L}f = \sum_{i=1}^{\infty} (f, v_i) b_i = \sum_{i=1}^{\infty} (f, v_i) L v_i = Lf,$$

the linear mapping L is represented by the series

$$\sum_{i=1}^{\infty} v_i \otimes b_i$$

with convergence in T^A.

On the other hand, let there be given $b_1, b_2, \ldots$ in $S_{X,A}$ satisfying

(7.2) $$\forall_{t>0} \exists_{\tau>0} \exists_{C>0} \forall_{i\in\mathbb{N}} : \| e^{\tau A} b_i \| \le C e^{\lambda_i t} .$$

Then it is obvious that the series $\sum_{i=1}^{\infty} v_i \otimes b_i$ converges in T^A, and represents the linear mapping

$$f \mapsto \sum_{i=1}^{\infty} (f, v_i) b_i \ , \ f \in S_{X,A}.$$

So the following characterization holds true.

(7.3) Characterization (the columns)

Let W be a linear operator in X with domain containing the linear span $\langle v_1, v_2, \ldots \rangle$. Then W maps $S_{X,A}$ continuously into itself iff the $b_i = Wv_i$, $i \in \mathbb{N}$, satisfy condition (7.2). W is represented in T^A by the series $\sum_{i=1}^{\infty} v_i \otimes (Wv_i)$.

The conjugate L^c of L is an element of T_A. Hence, as a continuous linear mapping from $T_{X,A}$ into itself L^c satisfies the following condition

$$\forall_{t>0} \exists_{s>0} \exists_{C>0} : \| e^{-tA} L^c e^{sA} \|_{X \otimes X} \le C.$$

Put $B_j = L^c v_j \in T_{X,A}$. Then they satisfy

(7.4) $$\forall_{t>0} \exists_{s>0} \exists_{C>0} \forall_{j\in\mathbb{N}} : \| B_j(t) \|_X \le C e^{-s\lambda_j} .$$

The trajectories B_j span $L^c(T_{X,A})$, and

$$B_j = \sum_{i=1}^{\infty} \bar{L}_{ji} v_i, \quad j \in \mathbb{N}$$

where the series converges in $T_{X,A}$. Hence B_j represents the j-th row of the matrix (L_{ij}). Define the trajectory $\tilde{L}$ by

$$\tilde{L}(t) = \sum_{j=1}^{\infty} B_j(t) \otimes v_j \quad , \; t > 0.$$

Then for each $t > 0$, $s_0 > 0$ can be chosen such that

$$\| B_j(t) \|_X \le C e^{-\lambda_j s_0} \; , \; j \in \mathbb{N} \, ,$$

and for $0 < s < s_0$,

$$\| e^{sA} \tilde{L}(t) \|_{X \otimes X} \le \| \sum_{j=1}^{\infty} B_j(t) \otimes (e^{sA} v_i) \|_{X \otimes X} \le$$

$$\le C \sum_{j=1}^{\infty} e^{-\lambda_j (s_0 - s)} < \infty \, .$$

Hence, $\tilde{L}(t) \in S_{X \otimes X, I \otimes A}$, $t > 0$, and $\tilde{L} \in T^A$. Since

$$\tilde{L} f = \sum_{j=1}^{\infty} \langle f, B_j \rangle \, v_j \;\; = \sum_{j=1}^{\infty} (Lf, v_j) v_j = Lf, \; f \in S_{X,A},$$

the mapping L is represented by the series $\sum_{j=1}^{\infty} B_j \otimes v_j$ with convergence in T^A.

On the other hand, let there be given $B_1, B_2, \ldots$ satisfying condition (7.4), then similarly it can be shown that the series $\sum_{j=1}^{\infty} B_j \otimes v_j$ represents the linear mapping

$$f \mapsto \sum_{j=1}^{\infty} \langle f, B_j \rangle v_j \ , \quad f \in S_{X,A}$$

in T_A. Thus we obtain a second characterization of the elements in T^A.

(7.5) Characterization (the rows)

Let W be a linear operator in X with domain containing the linear span $\langle v_1, v_2, \ldots \rangle$, and put $B_j = \sum_{i=1}^{\infty} (\overline{Wv_i, v_j}) v_i$. Then W is continuous on $S_{X,A}$ iff $B_j \in T_{X,A}$, $j \in \mathbb{N}$, with

$$\forall_{t>0} \exists_{s>0} \exists_{C>0} \forall_{j \in \mathbb{N}} : \|B_j(t)\|_X \leq Ce^{-\lambda_j s} \ .$$

We have $W = \sum_{j=1}^{\infty} B_j \otimes v_j$.

A complete characterization of the rows and columns of the matrices of elements in T^A is already quite interesting. A characterization of the entries is much more useful. The following theorem characterizes the entries.

(7.6) Theorem

Let the infinite matrix (L_{ij}) satisfy

(7.7) $$\forall_{t>0} \exists_{s>0} : \sup_{i,j \in \mathbb{N}} (e^{-\lambda_j t} e^{\lambda_i s} |L_{ij}|) < \infty \ .$$

Then L defined by

$$L = \sum_{i,j} L_{ij} v_j \otimes v_i$$

is in T^A, and conversely.

Proof.

$\Rightarrow$) Let t > 0. Then there are s > 0 and C > 0 such that

$$(e^{-\frac{1}{2}\lambda_j t} e^{\frac{3}{2}\lambda_i s} |L_{ij}|) < C, \quad i,j \in \mathbb{N}.$$

This yields the following estimate

$$\| e^{sA} L e^{-tA} \|^2_{X \otimes X} = \sum_{i,j} e^{-2\lambda_j t} e^{2\lambda_i s} |L_{ij}|^2 \leq$$

$$\leq C^2 \sum_{i,j} e^{-\lambda_j t} e^{-\lambda_i s} < \infty.$$

Since t > 0 has been taken arbitrarily, the result $L \in T^A$ follows.

$\Leftarrow$) Let $L \in T^A$. Then $\forall_{t>0} \exists_{s>0}$:

$$\sup_{i,j} (e^{-\lambda_j t} e^{\lambda_i s} |L_{ij}|) \leq \| e^{sA} L e^{-tA} \|_{X \otimes X} < \infty$$

where $L_{ij} = (Lv_j, v_i)$. □

We shall often employ condition (7.7). It is of great help in the construction of examples and counterexamples. In the sequel, we shall identify the space T^A with the space $M(T^A)$ of infinite matrices which satisfy condition (7.7).

The following lemma shows that the product in T^A corresponds to the matrix product in $M(T^A)$.

(7.8) Lemma

Let $R, S \in T^A$. Then the matrix of $R \circ S$ is given by

$$(R \circ S)_{ij} = \sum_{\ell=1}^{\infty} R_{i\ell} S_{\ell j} \quad , \quad i,j \in \mathbb{N}$$

where each of the series converges absolutely.

Proof. Let $t > 0$, $i,j \in \mathbb{N}$. Following Theorem (6.6) there are $s, s_0 > 0$ such that

$$S_{\ell j} \leq C_S \, e^{\lambda_j t} e^{-\lambda_\ell s_0}$$

and

$$R_{i\ell} \leq C_R \, e^{\frac{1}{2}\lambda_\ell s_0} \, e^{-\lambda_i s}$$

for some $C_S, C_R > 0$. This leads to the following estimate

$$\left| e^{\lambda_i s} \left(\sum_{\ell=1}^{\infty} R_{i\ell} S_{\ell j} \right) e^{-\lambda_j t} \right| \leq$$

$$\leq \sum_{\ell=1}^{\infty} \left(|e^{\lambda_i s} R_{i\ell} \, e^{-\frac{1}{2}\lambda_\ell s_0}| \, |e^{\lambda_\ell s_0} S_{\ell j} e^{-\lambda_j t}| e^{-\frac{1}{2}\lambda_\ell s_0} \right)$$

$$\leq C_S \, C_R \left(\sum_{\ell=1}^{\infty} e^{-\frac{1}{2}\lambda_\ell s_0} \right) .$$

Thus $\left(\sum_{\ell=1}^{\infty} R_{i\ell} S_{\ell j} \right)$ is an element of $M(T^A)$. Finally we have

$$\sum_{i,j} \left(\sum_{\ell} R_{i\ell} S_{\ell j} \right) v_j \otimes v_i =$$

$$= \sum_{i,j} \left(\sum_{\ell,k} R_{i\ell} S_{kj} (v_k, v_\ell) \right) v_j \otimes v_i$$

$$= \Big(\sum_{i,\ell} R_{i\ell}\ v_\ell \otimes v_i\Big) \cdot \Big(\sum_{j,k} S_{kj}\ v_j \otimes v_k\Big)$$

$$= R \circ S.$$

The conjugation ${}^c : T^A \to T_A$ induces a conjugation on $M(T^A)$. The precise result is given in the following lemma.

(7.9) Lemma

Let $L \in T^A$. Then $L^c \in T_A$, and

$$L^c = \sum_{i,j} \bar{L}_{ji}(v_j \otimes v_i)$$

where convergence of the series is in T_A.

Proof. From Theorem (7.8) we obtain

$$L(t) = \sum_{i,j} e^{-\lambda_j t} L_{ij} v_j \otimes v_i \ , \ t > 0 \ ,$$

with convergence in $S_{X\otimes X, I\otimes A}$ for each $t > 0$. Hence we find

$$L(t)^* = \sum_{i,j} e^{-\lambda_j t} \bar{L}_{ji}\ v_i \otimes v_j =$$

$$= \sum_{i,j} e^{-\lambda_i t} \bar{L}_{ji}\ v_j \otimes v_i \ , \ t > 0$$

with convergence in $S_{X\otimes X, A\otimes I}$ for each $t > 0$. □

If $L \in T^A$, then the matrix elements $\bar{L}_{ji}$ satisfy $\forall_{t>0} \exists_{s>0}$:

$$\sup_{i,j} (e^{-\lambda_i t} e^{\lambda_j s} |\bar{L}_{ji}|) < \infty \ .$$

Conversely, if the matrix (Q_{ij}) satisfies, $\forall_{t>0}\exists_{s>0}$:

$$\sup_{i,j} e^{-\lambda_i t} e^{\lambda_j s} |Q_{ij}| < \infty ,$$

then $(\bar{Q}_{ji})$ is the matrix of an elements in T^A.

Thus we arrive at the following theorem.

(7.10) <u>Theorem</u>

Let (Q_{ij}) be an infinite matrix. Then

$$Q = \sum_{i,j} Q_{ij}\ v_j \otimes v_i$$

is an element of T_A iff the matrix elements Q_{ij}, $i,j \in \mathbb{N}$, satisfy

(7.11) $$\forall_{t>0}\exists_{s>0} : \sup_{i,j} (e^{-\lambda_i t} e^{\lambda_j s} |Q_{ij}|) < \infty .$$

We note that $Q_{ij} = \overline{\langle v_i, Q v_j \rangle}$.

As a corollary of Theorem (7.6) and (7.10) we derive the following

(7.12) <u>Corollary</u>

The matrix (E_{ij}) represents an element of E_A if and only if it satisfies the condition (7.7) and (7.11).

In the following section we introduce the class of weighted shift operators. This kind of operators plays an important role in a lot of computations in mathematical physics (cf. the annihilation- and creation operator in a suitable representation). Further, because of their simple structure, the above-mentioned class provides the necessary illustrations of the theory.

8. The class of weighted shifts

For convenience we first introduce a set $\mathcal{D}_A$ of diagonal operators. A diagonal operator D is a linear operator in X which is well-defined on the linear span $\langle v_1, v_2, \ldots \rangle$, and which operates on this span as follows:

$$Dv_j = \delta_j v_j, \quad j \in \mathbb{N},$$

with $\delta_j \in \mathbb{C}$. Hence, the matrix of D is diagonal. Following Theorem (7.6), $D \in \mathcal{T}^A$ if and only if

$$\forall_{t>0}: \sup_j \left(|\delta_j| e^{-\lambda_j t}\right) < \infty .$$

Hence, D^c is also in $\mathcal{T}^A$, and D is extendable.

(8.1) Definition

$\mathcal{D}_A \subset \mathcal{E}_A$ denotes the set of diagonal operators D in X which satisfy

$$\forall_{t>0}: \sup_{j \in \mathbb{N}} |\delta_j| e^{-\lambda_j t} < \infty$$

where δ_j, $j \in \mathbb{N}$, are the diagonal entries of the matrix of D.

This section contains a first investigation of the special class of elements of $\mathcal{T}^A$ established by the weighted shift operators or, shortly, weighted shifts. A weighted shift W is a linear operator in X which is well defined on the linear span $\langle v_1, v_2, \ldots \rangle$, and which operates as follows

$$Wv_j = \omega_j v_{j+1}, \quad j \in \mathbb{N},$$

with $\omega_j \in \mathbb{C}$, $j \in \mathbb{N}$. Hence, W is uniquely determined by its matrix with respect to the basis (v_j) given by

$$W_{ij} = \omega_j \delta_{i,j+1} \ , \ i,j \in \mathbb{N} \ ,$$

where $\delta_{\ell,k}$ denotes Kronecker's delta. Then following Theorem (7.6) the linear mapping $W \in T^A$ if and only if

$$\forall_{t>0} \exists_{s>0} : \sup_j \left(|\omega_j| e^{-\lambda_j t} e^{\lambda_{j+1} s} \right) < \infty \tag{8.2}$$

and $W^c \in T^A$ if and only if

$$\forall_{t>0} \exists_{s>0} : \sup_{j>1} \left(|\omega_{j-1}| e^{-\lambda_j t} e^{\lambda_{j-1} s} \right) < \infty .$$

Since $\lambda_{j-1} \leq \lambda_j$ it is clear that continuity of W implies continuity of W^c. Hence, a continuous weighted shift is extendable.

Condition (8.2) can be rewritten into

$$\forall_{t>0} \exists_{s>0} : \sup_{j \in \mathbb{N}} |\omega_j| \exp\left\{ -\lambda_j \left(t - \frac{\lambda_{j+1}}{\lambda_j} s \right) \right\} < \infty \ .$$

In the remaining part of this section we impose the following condition on the eigenvalues of A.

$$\exists_M \forall_{j \in \mathbb{N}} : \frac{\lambda_{j+1}}{\lambda_j} \leq M \ . \tag{8.3}$$

This condition is not very severe; it implies the following order estimate, $\lambda_j = O(M^j)$. If Condition (8.3) is dropped, then there exists a subsequence (λ_{j_k}) such that $\frac{\lambda_{j_k+1}}{\lambda_{j_k}} \to \infty$ as $k \to \infty$. Let U be the unilateral shift given by $Uv_j = v_{j+1}$, $j \in \mathbb{N}$. So U is a bounded operator on X. Suppose $U \in T^A$. Then there should be $s > 0$ such that

$$\infty > \sup_{j\in\mathbb{N}} (e^{(\lambda_{j+1}s-\lambda_j)}) > \sup_{k\in\mathbb{N}} e^{\lambda_{j_k+1}\left(s-\frac{\lambda_{j_k+1}}{\lambda_{j_k}}\right)} .$$

Since $\lambda_j \to \infty$ and $\frac{\lambda_{j_k}}{\lambda_{j_k+1}} \to 0$ the assumption $U \in T^A$ yields a contradiction. Hence $U \notin T^A$. If the eigenvalues λ_j do not satisfy Condition (8.3), it is possible that there only occur Hilbert-Schmidt operators in E_A. Because of Condition (8.3) it follows that (8.2) reduces to

(8.4) $$\forall_{t>0} \sup_{j\in\mathbb{N}} (|\omega_j| e^{-\lambda_j t}) < \infty .$$

So the following characterization is an immediate consequence of Definition (8.1) and (8.4).

(8.5) Characterization

Let W be a weighted shift. Then $W \in T^A$ iff there exists a $D \in \mathcal{D}_A$ such that $W = UD$.

The following definition generalizes the notion of weighted shifts.

(8.6) Definition

A linear operator $W^{(n)}$ in X is called a weighted n-shift, $n \in \mathbb{N} \cup \{0\}$ if $W^{(n)}$ satisfies

$$W^{(n)} v_j = \omega_j^{(n)} v_{j+n} , \quad j \in \mathbb{N}$$

with $\omega_j^{(n)} \in \mathbb{C}$.

Hence, a weighted 0-shift is a diagonal operator, a weighted 1-shift is an ordinary weighted shift. Let $W^{(n)}$ be a weighted n-shift with weight sequence $(\gamma_j^{(n)})$. Then $W^{(n)} \in T^A$ if and only if

(8.7) $$\forall_{t>0}\exists_{s>0}: \sup_{j\in\mathbb{N}} (|\gamma_j^{(n)}|e^{-\lambda_j t}e^{\lambda_{j+n}s}) < \infty .$$

Because of (8.3) there exists M > 0 such that

$$\frac{\lambda_{j+n}}{\lambda_j} \leq M^n , \quad j \in \mathbb{N} .$$

So (8.7) is equivalent to

(8.8) $$\forall_{t>0}: \sup_{j\in\mathbb{N}} (|\gamma_j^{(n)}|e^{-\lambda_j t}) < \infty .$$

This yields the following characterization.

(8.9) <u>Characterization</u>

Let $W^{(n)}$ be a weighted n-shift, $n \in \mathbb{N} \cup \{0\}$. Then $W^{(n)} \in T^A$ iff there exists $D \in \mathcal{D}_A$ such that $W^{(n)} = U^n D$.

Since $U \in E_A$ and $D \in E_A$ for all $D \in \mathcal{D}_A$, from (8.9) we derive that every weigthed n-shift, $n \in \mathbb{N} \cup \{0\}$, is extendable.

(8.10) <u>Definition</u>

The operator $W^{(-n)}$, $n \in \mathbb{N}$, is called a weighted (-n)-shift if

$$W^{(-n)} v_j = \omega_{j-n}^{(-n)} v_{j-n} , \; j > n, \; j \in \mathbb{N}$$

with $\omega_j^{(-n)} \in \mathbb{C}$, and $W^{(-n)} v_j = 0$, $1 \leq j \leq n$.

If the linear mapping $W^{(-n)} \in T^A$ then it satisfies

$$\forall_{t>0}\exists_{s>0}: \sup_{\substack{j\in\mathbb{N}\\ j>n}} (|\omega_{j-n}^{(-n)}| e^{-\lambda_j t} e^{\lambda_{j-n} s}) < \infty ,$$

or equivalently

(8.11) $$\forall_{t>0}: \sup_{j\in\mathbb{N}} (|\omega_j^{(-n)}| e^{-\lambda_{j+n} t}) < \infty ,$$

since $\lambda_{j-n} < \lambda_j$ for $j > n$, $j \in \mathbb{N}$. The latter condition is equivalent to

(8.12) $$\forall_{t>0}: \sup_{j\in\mathbb{N}} (|\omega_j^{(-n)}| e^{-\lambda_j t}) < \infty .$$

The implication (8.12) ⇒ (8.11) is trivial. In order to prove that (8.11) implies (8.12), let $t > 0$. Then

$$\sup_{j\in\mathbb{N}} (|\omega_j^{(-n)}| e^{-\lambda_j t}) = \sup_{j\in\mathbb{N}} \left(|\omega_j^{(-n)}| e^{-(\lambda_j/\lambda_{j+1}\cdots\lambda_{j+n-1}/\lambda_{j+n})\lambda_{j+n} t}\right)$$

$$\leq \sup_{j\in\mathbb{N}} (|\omega_j^{(-n)}| e^{-\lambda_{j+n} t M^{-n}}) < \infty .$$

with $M > 0$ such that $\frac{\lambda_{j+1}}{\lambda_j} < M$, $j \in \mathbb{N}$.

So similar to (8.9) the weighted (-n)-shifts in T^A are characterized by

(8.13) <u>Characterization</u>

Let $W^{(-n)}$ be a weighted (-n)-shift. Then $W^{(-n)} \in T^A$ iff there exists $D \in \mathcal{D}_A$ such that $W^{(-n)} = D(U^*)^n$.

Since U^* and $D \in \mathcal{D}_A$ both are extendable, each $W^{(-n)}$ is extendable. Further, the product $W^{(k_1)}W^{(k_2)}$ with $k_1,k_2 \in \mathbb{Z}$ is a weighted (k_1+k_2)-shift and the conjugate $(W^{(k_1)c})$ is a $(-k_1)$-shift. So the weighted k-shifts $k \in \mathbb{Z}$, establish an involutive semi-group in E_A.

The weighted k-shifts, $k \in \mathbb{Z}$, span the algebra T^A in a very special way.

(8.14) Theorem

Let $L \in T^A$ with matrix (L_{ij}). Define the weighted k-shifts $W^{(k)}$ by

$$W^{(k)} v_j = L_{j+k,j} \quad v_j \ , \ j > \max\{0,-k\}, \quad j \in \mathbb{N},$$

where $k \in \mathbb{Z}$. Then $W^{(k)} \in E_A$ and $\sum_{k \in \mathbb{Z}} W^{(k)}$ represents L. This series converges absolutely.

Proof. The eigenvalues λ_j of A satisfy the following estimates

For $n \in \mathbb{N} \cup \{0\}$,

$$(*) \qquad e^{\lambda_{j+n} s} \leq e^{-\lambda_n (s_0 - s)} e^{\lambda_{j+n} s_0}$$

with $j \in \mathbb{N}$, $s_0 > 0$, and $0 < s < s_0$. For $n \in \mathbb{N}$,

$$(**) \qquad e^{-\lambda_j t} \leq e^{-\lambda_n (t - t_0)} e^{-\lambda_j t_0}$$

with $j \in \mathbb{N}$, $j > n$, $t_0 > 0$ and $t > t_0$.

First note that it is obvious that each $W^{(k)}$, $k \in \mathbb{Z}$, is continuous and hence extendable (cf.(8.9) and (8.13)). So we only prove the second assertion. Let $t > 0$. Then there exists $s > 0$ such that

$$\| e^{2sA} L e^{-\frac{1}{2} tA} \|_{X \otimes X} < \infty.$$

For $n \in \mathbb{N} \cup \{0\}$ by $(*)$ we have

$$\| e^{sA_W(n)} e^{-tA} \|_{X\otimes X} \le e^{-\lambda_n s}\Big(\sum_{j=1}^{\infty} |e^{2s\lambda_{n+j}} L_{n+j,j} e^{-t\lambda_j}|^2\Big)^{1/2}$$

$$\le e^{-\lambda_n s} \| e^{2sA} L e^{-\frac{1}{2}tA} \|_{X\otimes X} .$$

For $n \in \mathbb{N}$ by (**) we have

$$e^{sA_W(-n)} e^{-tA} {}_{X\otimes X} \le e^{-\frac{1}{2}\lambda_n t}\Big(\sum_{j=n+1}^{\infty} |e^{s\lambda_{j-n}} L_{j-n,j} e^{-\frac{1}{2}t\lambda_j}|^2\Big)^{1/2}$$

$$\le e^{-\frac{1}{2}\lambda_n t} \| e^{2sA} \; e^{-\frac{1}{2}tA} \|_{X\otimes X} .$$

A combination of the above results yields for all $N_1, N_2 \in \mathbb{N}$

$$\sum_{k=-N_1}^{N_2} \| e^{sA_W(k)} e^{-tA} \|_{X\otimes X} \le$$

$$\le \| e^{2sA} L e^{-\frac{1}{2}tA} \|_{X\otimes X} \Big(\sum_{n=1}^{N_1} e^{-\frac{1}{2}\lambda_n t} + \sum_{n=0}^{N_2} e^{-\lambda_n s} \Big)$$

Hence, the series $\sum_{k\in\mathbb{Z}} e^{sA_W(k)} e^{-tA}$ converges absolutely in $X\otimes X$.
Since $X\otimes X$ is a Hilbert space absolute convergence implies convergence and therefore

$$e^{sA} L e^{-tA} = \sum_{k\in\mathbb{Z}} e^{sA_W(k)} e^{-tA} .$$

Thus we have proved the second assertion. □

Since all weighted k-shifts, $k \in \mathbb{Z}$, are extendable, the following corollary is immediate.

(8.15) Corollary

The space T^A in Theorem (8.14) can be replaced by T_A.

For the weighted k-shifts $W^{(k)}$ spectral properties can be discussed in detail and eigenvectors in $T_{X,A}$ and $S_{X,A}$ can be constructed. This may be a subject for further investigation.

9. Construction of an analyticity space $S_{X,A}$ for some given operators in X

Given a finite number of linear operators in a Hilbert space X, the question arises whether there can be constructed nuclear analyticity spaces on which these operators are continuous linear mappings. In this section we shall show that for a finite number of bounded operators on X, resp. for a finite number of commuting self-adjoint operators in X, such a construction is indeed possible. The proof of the results of this section is closely related to the theory on matrices of elements in T^A (cf. Section 7).

Let P be a bounded, self-adjoint operator on X. Following [Ha], p.201, P can be represented by a Jacobi matrix, i.e. there exists an orthonormal basis (e_r) in X such that the matrix of P satisfies

$$(Pe_r, e_j) = 0 \text{ if } |r-j| < 1 \ , \ r,j \in \mathbb{N} \ .$$

If we define the positive self-adjoint operator A in X by

$$Ae_j = je_j \ , \ j \in \mathbb{N} \ ,$$

followed by linear and unique self-adjoint extension, then we have the following result.

(9.1) Lemma

The self-adjoint operator P is an element of T_A.

Proof. Following Theorem (7.6) we have to show

$$\forall_{t>0}\exists_{s>0}: \sup_{,j} (e^{-jt}e^{rs}|(Pe_j,e_r)|) < \infty.$$

Let $t > 0$, and let $0 < s < t$. Then

$$\sup_{r,j} e^{-jt}e^{rs} \; |(Pe_j,e_r)| \le \|P\| e^{-jt+(j+1)s} < e^s\|P\| ,$$

where $\|P\|$ denotes the norm of P in $B(X)$ □

With the aid of Lemma (9.1) the more general case of an unbounded self-adjoint operator T can be solved. To this end let $(F_\lambda)_{\lambda\in\mathbb{R}}$ denote the spectral resolution of the identity for T and Π_ℓ, $\ell \in \mathbb{N}$, the spectral projection

$$\Pi_\ell = \left(\int_{\ell-1}^{\ell} + \int_{-\ell}^{-\ell+1}\right)dF_\lambda .$$

Then X is decomposed into

$$X = \bigoplus_{i=1}^{\infty} \Pi_\ell(X)$$

where in each invariant subspace $\Pi_\ell(X)$ the estimate

$$\|Tf_\ell\| \le \ell\|f_\ell\| , \; f_\ell \in \Pi_\ell(X) ,$$

holds true. So if we put $T_\ell = \Pi_\ell T \Pi_\ell$, then T_ℓ is bounded on X , and there exists an orthonormal basis $(e_j^{(\ell)})$ such that $\left(\left(T_\ell e_j^{(\ell)}, e_r^{(\ell)}\right)\right)$ is a Jacobi matrix.

Define the positive self-adjoint operator A by

$$A e_j^{(\ell)} = (j+\ell)\, e_j^{(\ell)}, \quad j \in \mathbb{N}, \ \ell \in \mathbb{N}$$

followed by linear and unique self-adjoint extension. Then the eigenvalues of A are the numbers $\lambda_n = n+1$ with multiplicity n, $n \in \mathbb{N}$. So all the operators e^{-tA}, $t > 0$, are Hilbert-Schmidt and the analiticity space $S_{X,A}$ is nuclear.
Put $f_j^{(n)} = e_j^{(n+1-j)}$, $j = 1,\dots,n$. Then the vectors $f_j^{(n)}$ are the eigenvectors of A with eigenvalue λ_n. Enumerating the $f_j^{(n)}$'s in the usual way, we have constructed a complete orthonormal basis (g_k) for X, which yields the following theorem.

(9.2) <u>Theorem</u>

The operator T maps $S_{X,A}$ continuously into itself.

<u>Proof</u>. Let $t > 0$, and let $0 < s < t$. Then

$$\sup_{\ell,k} |(e^{sA} T e^{-tA} g_\ell, g_k)| =$$

$$= \sup_{r,n} \sup_{j,m} \left\{ e^{(r+n)s} e^{-(j+m)t} |(T e_j^{(m)}, e_r^{(n)})| \right\} =$$

$$= \sup_{m} (e^{-m(t-s)} \sup_{r,j} (e^{rs} e^{-jt} |(T_m e_j^{(m)}, e_r^{(m)})|)) \le$$

$$\le \sup_{m} (m e^{-(t-s)}) \sup_{|r-j|\le 1} (e^{rs} e^{-jt}) < \infty . \qquad \square$$

In order to establish a similar result for N bounded operators $B_1, B_2, \dots, B_N$ on X, we shall construct an orthonormal basis in X such that the matrix of each B_ν, $\nu = 1,\dots,N$, is column finite, i.e. for every $j \in \mathbb{N}$ there exists $r_0 \in \mathbb{N}$ such that

$$(B_\nu)_{rj} = 0 \text{ for } r > r_0.$$

To this end, let (f_r) be an orthonormal basis in X. Put $e_1 = f_1$. There exists an orthonormal set $\{e_2,e_3,\dots,e_{k_1}\} \perp \{e_1\}$ with $k_1 \leq (n+1) + 1$, such that

$$B_\nu e_1 \in \langle e_1,\dots,e_{k_1}\rangle, \quad \nu = 1,\dots,N$$

and

$$f_2 \in \langle e_1,\dots,e_{k_1}\rangle .$$

Similarly, there exists an orthonormal set $\{e_{k_1+1},\dots,e_{k_2}\}\perp\{e_1,\dots,e_{k_1}\}$, $k_2 \leq 2(n+1) + 1$, such that

$$B_\nu e_2 \in \langle e_1,\dots,e_{k_2}\rangle , \quad \nu = 1,\dots,N$$

and

$$f_3 \in \langle e_1,\dots,e_{k_2}\rangle.$$

Continuing in this way we derive sets $\{e_{k_{\ell-1}+1},\dots,e_{k_\ell}\}$ with $k_\ell \leq \ell(n+1) + 1$ and with $\{e_{k_{\ell-1}+1},\dots,e_{k_\ell}\} \perp \{e_1,\dots,e_{k_{\ell-1}}\}$ such that

$$B_\nu e_\ell \in \langle e_1,\dots,e_{k_\ell}\rangle , \quad \nu = 1,\dots,N$$

and

$$f_{\ell+1} \in \langle e_1,\dots,e_{k_\ell}\rangle .$$

Thus we obtain an orthonormal basis (e_r) in X. This basis is complete because $b_\ell \in \langle e_1, e_2, \ldots, e_{k_{\ell+1}} \rangle$, $\ell \in \mathbb{N}$. The matrix of each B_ν, $1 \le \nu \le N$, is column finite, because

$$(B_\nu e_j, e_r) = 0 \text{ if } r > j(N+1) + 1.$$

Now define the positive self-adjoint operator A by

$$Ae_j = je_j \ , \ j \in \mathbb{N} ,$$

followed by linear and unique self-adjoint extension. Then

(9.3) <u>Theorem</u>

The linear operators $B_1, \ldots, B_N$ map the nuclear analyticity space $S_{X,A}$ continuously into itself.

<u>Proof</u>. Let $\nu \in \{1,\ldots,N\}$, and let $t > 0$, $s > 0$ with $0 < s < \frac{t}{N+1}$. Then

$$\sup_{r,j} |(B_\nu e_j, e_r)| e^{-jt} e^{rs} =$$

$$= \sup_{1 \le r \le j(n+1)+1} (|(B_\nu e_j, e_r)| e^{-jt} e^{rs}) \le$$

$$\le \|B_\nu\| e^s \sup_{j \in \mathbb{N}} e^{-j(t-(N+1)s)} \le e^s \|B_\nu\| . \qquad \square$$

With the aid of Theorem (9.3) we can extend the result of Theorem (9.2) to hold true for a finite number of commuting self-adjoint operators in X. Let $T_1, T_2, \ldots, T_N$ be N commuting self-adjoint operators in X with resolutions of identity $(F_\lambda^{(\nu)})$, $\nu = 1,\ldots,N$. So their spectral projections commute, i.e. $F^{(\nu)}(\Delta_\nu)F^{(\mu)}(\Delta_\mu) = F^{(\mu)}(\Delta_\mu)F^{(\nu)}(\Delta_\nu)$ where Δ_ν, Δ_μ denote Borel sets in $\mathbb{R}$. Let Π_ℓ, $\ell \in \mathbb{N}^N$, denote the projection

$$\Pi_\ell = F^{(1)}(\ell_1 - 1 \le |\lambda| < \ell_1) \circ\circ\circ \ F^{(N)}(\ell_N - 1 \le |\lambda| < \ell_N) \ .$$

Then for all $f_\ell \in \Pi_\ell(X)$, $T_\nu f_\ell \in \Pi_\ell(X)$ and $\|T_\nu f_\ell\| \le \ell_\nu \| f_\ell\|$.
Further, $X = \bigoplus_{\ell \in \mathbb{N}^N} \Pi_\ell(X)$.

Since each operator $T_\nu | \Pi_\ell(X)$ is bounded, there exists an orthormal basis $(e_j^{(\ell)})$ in $\Pi_\ell(X)$ such that for all $\nu = 1,\dots,N$,

$$(T_\nu e_j^{(\ell)}, e_r^{(\ell)}) = 0 \text{ if } r > j(N+1) + 1.$$

Define the positive,self-adjoint operator A in X by

$$A e_j^{(\ell)} = (j + |\ell|) e_j^{(\ell)} \ , \ j \in \mathbb{N} \ , \ \ell \in \mathbb{N}^N \ ,$$

followed by the usual extensions (Note that $|\ell| = \ell_1 + \dots + \ell_N$). Then the eigenvalues of A are the numbers $\lambda_p = N+p$, $p \in \mathbb{N}$, with multiplicity $\binom{N+p-1}{N}$. Hence, the analyticity space $S_{X,A}$ is nuclear.
Renumerating the orthonormal basis $(e_j^{(\ell)})$ yields an orthonormal basis $(g_n)_{n\in\mathbb{N}}$ for X. We have

(9.4) <u>Theorem</u>

Each of the operators T_ν, $\nu = 1,\dots,N$ is a continuous linear mapping from $S_{X,A}$ into itself.

<u>Proof</u>. Let $\nu = 1,\dots,N$, and let $0 < s < \frac{t}{N+1}$. Then

$$\sup_{n,m} |(e^{sA} T_\nu e^{-tA} g_m, g_n)| =$$

$$= \sup_{r,j\in\mathbb{N}} \ \sup_{k,\ell\in\mathbb{N}^N} \left(e^{-(|\ell|+j)t} e^{(|k|+r)s} \left| \left(T_\nu e_j^{(\ell)}, e_r^{(k)} \right) \right| \right) \le$$

$$\le e^r \sup_{\ell\in\mathbb{N}^n} \left(\ell_\nu e^{-|\ell|(t-s)} \right) \sup_{j\in\mathbb{N}} \left(e^{-j(t-(N+1)s)} \right) < \infty \ . \qquad \square$$

IV. GENERALIZED EIGENFUNCTIONS WITH APPLICATIONS TO DIRAC'S FORMALISM

Abstract.

In the first part of this chapter a theory of generalized eigenfunctions is developed which is based on the theory of generalized functions introduced by De Graaf. For a finite number of commuting self-adjoint operators the existence of a complete set of simultaneous generalized eigenfunctions is proved. A major role in the construction of the proof is played by the commutative multiplicity theory.

The second part is devoted to an Ansatz for a mathematical interpretation of Dirac's formalism. Instead of employing rigged Hilbert space theory Dirac's bracket notion is reinterpreted and extended to the generalized function space $T_{X,A}$. In this way, the concepts of the Fourier expansion of kets, of the orthogonality of complete sets of eigenkets and of matrices of unbounded linear mappings, all in the spirit of Dirac, fit into a mathematical rigorous theory.

Preliminaries.

The introduction of a theory of generalized eigenfunctions is closely related to a theory of generalized functions, of course. In [GeVi], ch. I, to this end the theory of rigged Hilbert spaces is introduced. Here we employ De Graaf's theory of generalized functions, see [G]. In these preliminaries the main features of this theory will be given.

In a Hilbert space X consider the evolution equation

(p.1) $$\frac{du}{dt} = -Au$$

where A is a positive, unbounded self-adjoint operator. A solution u of (p.1) is called a trajectory if u satisfies

(p.2.i) $\forall_{t>0} : u(t) \in X$

(p.2.ii) $\forall_{t>0}\forall_{\tau>0} : e^{-\tau A}u(t) = u(t+\tau)$.

We emphasize that $\lim_{t\downarrow 0} u(t)$ does not necessarily exist in X-sense. The complex vector space of all trajectories is denoted by $\mathcal{T}_{X,A}$. The space $\mathcal{T}_{X,A}$ is considered as a space of generalized functions in [G].
The analyticity space $\mathcal{S}_{X,A}$ is defined to be the dense linear subspace of X consisting of smooth elements of the form $e^{-tA}h$ where $h \in X$ and $t > 0$. Hence $\mathcal{S}_{X,A} = \bigcup_{t>0} e^{-tA}(X)$. For each $f \in \mathcal{S}_{X,A}$, there exists $\tau > 0$ such that $e^{\tau A} f \in \mathcal{S}_{X,A}$. Further, for each $F \in \mathcal{T}_{X,A}$ we have $F(t) \in \mathcal{S}_{X,A}$ for all $t > 0$. $\mathcal{S}_{X,A}$ is the test function space in De Graaf's theory. In $\mathcal{T}_{X,A}$ we take the topology induced by the seminorms

(p.3) $F \mapsto \|F(t)\|$, $F \in \mathcal{T}_{X,A}$.

Because of the trajectory property (p.2.ii) of elements in $T_{X,A}$, it is a Frêchet space with this topology. In $S_{X,A}$ we take the inductive limit topology. In [G], a set of seminorms on $S_{X,A}$ is produced which generates the inductive limit topology.

The pairing between $S_{X,A}$ and $T_{X,A}$ is defined by

$$\text{(p.4)} \qquad \langle g,F\rangle = (e^{\tau A} g, F(\tau)) \ , \ g \in S_{X,A},\ F \in T_{X,A}.$$

Here $(\cdot,\cdot)$ denotes the inner product in X. Definition (p.4) makes sense for $\tau > 0$ sufficiently small. Due to the trajectory property (p.2.ii) it does not depend on the choice of τ.

The space $S_{X,A}$ is nuclear if and only if A generates a semigroup of Hilbert-Schmidt operators on X. In this case A has an orthonormal basis (v_k) of eigenvectors with respective eigenvalues λ_k, say. Further, for all $t > 0$ the series $\sum_{k=1}^{\infty} e^{-\lambda_k t}$ converges. It can be shown that $f \in S_{X,A}$ if and only if there exists $\tau > 0$ such that

$$\text{(p.5)} \qquad (f,v_k) = \mathcal{O}(e^{-\lambda_k \tau})$$

and $F \in T_{X,A}$ if and only if for all $t > 0$

$$\text{(p.6)} \qquad \langle v_k,F\rangle = \mathcal{O}(e^{\lambda_k t}) \ .$$

A topological tensor product $S_{X,A} \otimes S_{X,A}$ is given by $S_{X\otimes X, A\boxplus A}$ and, similarly, $T_{X,A} \otimes T_{X,A}$ by $T_{X\otimes X, A\boxplus A}$. Here $A\boxplus A$ denotes the positive, self-adjoint operator $A\otimes I + I\otimes A$. Since $S_{X,A}$ is nuclear, the Kernel theorems of [G], Ch. VI, apply. So $S_{X\otimes X,A\boxplus A}$ comprises the kernels of all continuous linear mappings from $T_{X,A}$ into $S_{X,A}$ and $T_{X\otimes X,A\boxplus A}$ the kernels of all continuous linear mappings from $S_{X,A}$ into $T_{X,A}$.

0. Introduction

First I want to give an illustrative example for the general theory of this chapter. Therefore, let $S_{X,A}$ be the test function space with $X = L_2(\mathbb{R})$ and $A = \frac{1}{2}\left(- \frac{d^2}{dx^2} + x^2 + 1 \right)$, the Hamiltonian operator of the harmonic oscillator. This $S_{X,A}$-space is one of the examples discussed in [G], and in Ch. I. It is well-known that the Hermite functions ψ_k, $k = 0,1,\ldots$ are the eigenfunctions of A with eigenvalues $k + 1$. So for each $t > 0$, the operator e^{-tA} is Hilbert-Schmidt, and the spaces $S_{X,A}$ and $T_{X,A}$ are nuclear. The self-adjoint operator Q

$$(Qf)(x) = x\, f(x) \quad , \quad x \in \mathbb{R} ,$$

maps $S_{X,A}$ continuously into itself, and can be extended to a continuous linear mapping on $T_{X,A}$, denoted by Q, also.
The linear functional δ_{x_0}, given by

$$\delta_{x_0} : f \mapsto f(x_0)$$

is an eigenfunctional of Q with eigenvalue x_0. The question arises whether $\delta_{x_0} \in T_{X,A}$. The space $S_{X,A}$ consists of entire analytic functions. So for each $f \in S_{X,A}$, $f(x_0)$ exists, and can be written as

$$f(x_0) = \sum_{k=0}^{\infty} (f,\psi_k)\psi_k(x_0) .$$

Hence $\delta_{x_0} \in T_{X,A}$ if and only if the series

$$\delta_{x_0}(t) = \sum_{k=0}^{\infty} e^{-(k+1)t}\psi_k(x_0)\psi_k$$

converges in X for all $t > 0$. Because of the growth properties of $|\psi_k(x_0)|$ for large k, this is true in this special case.

In this chapter only nuclear $S_{X,A}$ spaces are considered. This implies that all the operators e^{-tA}, $t > 0$, have to be Hilbert-Schmidt. So A has an orthonormal basis of eigenvectors $v_1, v_2, \dots$ with respective eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \dots$ satisfying $\sum_{i=1}^{\infty} e^{-\lambda_i t} < \infty$ for all $t > 0$.

Let T be a self-adjoint operator in X which is continuous on $S_{X,A}$. Since T is self-adjoint, T can always be represented as a multiplication operator in a countably direct sum of L_2-spaces. For convenience in this introduction, we shall consider the special case that T is unitarily equivalent to multiplication by the identity function in $L_2(\mathbb{R},\mu)$ for some finite Borel measure μ. In other words, a unitary operator $U : X \to L_2(\mathbb{R},\mu)$ exists, such that $Q = UTU^*$ is given by

$$(Q\mathfrak{f})(x) = x\,\mathfrak{f}(x)$$

on its domain $D(Q) = U(D(T))$. U maps $S_{X,A}$ continuously onto $S_{Y,B}$, where

$$Y = L_2(\mathbb{R},\mu) \quad \text{and} \quad B = UAU^* .$$

Put $\varphi_k = Uv_k$, $k = 1,2,\dots$. Then the φ_k's establish an orthonormal basis in Y and they are the eigenvectors of B with eigenvalues $\lambda_1, \lambda_2, \dots$.

Let $x_0 \in \sigma(T)$, the spectrum of T. It is obvious that x_0 is a (generalized) eigenvalue of T if and only if the linear functional $\Delta_{x_0} : f \mapsto f(x_0)$ is continuous on $S_{Y,B}$. This continuity condition is equivalent to the condition

$$(0.1) \qquad t \mapsto \sum_{k=1}^{\infty} e^{-\lambda_k t} \varphi_k(x_0) \varphi_k \in T_{Y,B} .$$

Of course, there is a problem here. In general $\mathfrak{f}(x_0)$ has no meaning for L_2-functions. Formula (0.1) makes sense only, if we can choose a representant from each equivalence class $\langle\varphi_k\rangle$ in a unique way. In case

$S_{Y,\mathcal{B}} \subset L_\infty(\mathbb{R},\mu)$ we could employ the lifting theory of Ionescu Tulcea (see [IT]). But in general $S_{Y,\mathcal{B}}$ is not contained in $L_\infty(\mathbb{R},\mu)$.
We shall prove that a unique choice of representants $\hat{\varphi}_k$ in the classes $\langle\varphi_k\rangle$, k = 1,2,... , implies a unique choice of representants in all classes $\langle f\rangle$ of $S_{Y,\mathcal{B}}$, just by defining

$$(0.2) \qquad \hat{f} := \sum_{k=1}^{\infty} (f,\varphi_k)\hat{\varphi}_k \ .$$

Here we take

$$(0.3) \qquad \hat{\varphi}_k \ : \ x \mapsto \lim_{h\downarrow 0}\left\{ \mu(Q_h(x))^{-1} \int_{Q_h(x)} \varphi_k \, d\mu \right\}$$

where $Q_h(x) = [x-h\,,\,x+h]$. It is clear that Definition (0.3) does not depend on the choice of $\hat{\varphi}_k \in \langle\varphi_k\rangle$.
The general case that T is equivalent to multiplication by the identity function in a countably direct sum of L_2-spaces can be dealt with similarly.

In section 1 we shall show the existence of generalized eigenfunctions for a continuous self-adjoint operator T on $S_{X,A}$. In section 2 excerpts of the commutative multiplicity theory are given. For this theory we refer to Nelson ([Ne_2]) and Brown ([Br]). The main theorem in section 3 states that we can a priori remove a set of measure zero N out of the spectrum $\sigma(T)$ of T such, that for all points in $\sigma(T)\setminus N$ with multiplicity m, $0 \le m \le \infty$, there exist precisely m independent generalized eigenfunctions. Section 4 is devoted to a sketchy proof of the result that in an adapted form the conclusions of section 3 remain valid for an n-tuple of commuting self-adjoint operators. Finally, in section 5 an Ansatz is given for a mathematical interpretation of Dirac's formalism.

1. The existence of generalized eigenfunctions

In the sequel A will denote a positive self-adjoint operator in X which generates a semigroup of Hilbert-Schmidt operators. So A has an orthonormal basis of eigenvectors $v_1, v_2, \ldots$ with respective eigenvalues $\lambda_1, \lambda_2, \ldots$ satisfying $\sum_{i=1}^{\infty} e^{-\lambda_i t} < \infty$ for all $t > 0$. Further, T will denote a self-adjoint operator in X, which maps $S_{X,A}$ continuously into itself. The spectral resolution of T is denoted by $(H_\lambda)_{\lambda\in\mathbb{R}}$.
For $\mathfrak{f} \in X$, the subspace $X_{\mathfrak{f}}$ of X is defined to be the closure of the linear span of the set $\{H(\Delta)\mathfrak{f} \mid \Delta \subset \mathbb{R}$ a Borel set$\}$. Here $H(\Delta)$ denotes the spectral projection $\int_\Delta dH_\lambda$.

(1.1) Lemma

The subspace $X_{\mathfrak{f}}$ of X is unitarily equivalent to $L_2(\mathbb{R},\rho_{\mathfrak{f}})$, where $\rho_{\mathfrak{f}}$ denotes the positive, finite Borel measure $(H_\lambda\mathfrak{f},\mathfrak{f})_{\lambda\in\mathbb{R}}$.

Proof

The proof will be sketchy. It is taken from [Br].
Let $g \in X_{\mathfrak{f}}$. Then there exist sequences $(\alpha_j^{(n)})_{j\in\mathbb{N}}$ and $(\Delta_j^{(n)})_{j\in\mathbb{N}}$ such that

$$(*) \qquad \lim_{n\to\infty} \| g - \sum_{j=1}^{j_n} \alpha_j^{(n)} H(\Delta_j^{(n)})\mathfrak{f} \| = 0 \ .$$

So we may conclude that the finite series

$$\sum_{j=1}^{j_n} \alpha_j^{(n)} H(\Delta_j^{(n)})\mathfrak{f} \quad , \quad n \in \mathbb{N} ,$$

are uniformly bounded. Then $\phi = \lim_{n\to\infty} \sum_{j=1}^{j_n} \alpha_j^{(n)} \Delta_j^{(n)}$ exists and because of the completeness of $L_2(\mathbb{R},\rho_{\mathfrak{f}})$,

$$\int_{\mathbb{R}} |\psi|^2 \, d\rho_{\mathfrak{f}} < \infty .$$

By (*) g can be expressed as $g = \psi(T)\mathfrak{f}$ with $\|g\| = \|\psi\|_{L_2}$. On the other hand, if $\psi \in L_2(\mathbb{R},\rho_{\mathfrak{f}})$, then

$$\psi = \lim_{n\to\infty} \sum_{j=1}^{j_n} \alpha_j^{(n)} \Delta_j^{(n)}$$

with the limit taken in L_2-sense. So obviously $g = \psi(T)\mathfrak{f}$.

The following equivalence holds

$$g \in X_{\mathfrak{f}} \Leftrightarrow \exists_{\psi\in L_2(\mathbb{R},\rho_{\mathfrak{f}})} : g = \psi(T)\mathfrak{f} .$$

The operator $U : X_{\mathfrak{f}} \to L_2(\mathbb{R},\rho_{\mathfrak{f}})$,

$$Ug = U(\psi(T)\mathfrak{f}) = \psi$$

is unitary. This completes the proof. □

(1.2) Notation

P denotes the set of $x \in \mathbb{R}$ which satisfy

$$\rho_{\mathfrak{f}}([x-\varepsilon , x+\varepsilon]) > 0$$

for every $\varepsilon > 0$.

For each $x \in P$, define

$$(1.3) \qquad G_{t,h}(x) := \text{emb}\left\{ [\rho_{\mathfrak{f}}(Q_h(x))]^{-1} \int_{Q_h(x)} dH_\lambda \, \mathfrak{f} \right\}(t) \quad , \quad t > 0 .$$

Here emb is the continuous linear mapping from X into $T_{X,A}$,

$$\mathrm{emb}(\omega) \;:\; t \mapsto e^{-tA}(\omega) \quad , \quad \omega \in X ,$$

and $Q_h(x)$ the closed interval $[x-h, x+h]$.

Since $(v_k)_{k\in\mathbb{N}}$ is an orthonormal basis of eigenvectors of A the Fourier expansion of $G_{t,h}(x)$ is given by

$$G_{t,h}(x) = \sum_{k=1}^{\infty} e^{-\lambda_k t} \left\{ \frac{\int_{Q_h(x)} d(H_\lambda \delta, v_k)}{\int_{Q_h(x)} d(H_\lambda \delta, \delta)} \right\} v_k \quad , \quad t > 0,\; h > 0 \; .$$

By Lemma (1.1) for each $k \in \mathbb{N}$ there exists $\varphi_k \in L_2(\mathbb{R}, \rho_\delta)$ such that

$$\int_{Q_h(x)} d(H_\lambda \delta, v_k) = \int_{Q_h(x)} \overline{\varphi_k}\, d\rho_\delta \quad , \quad h > 0 \; .$$

With the aid of Theorem 10.49 in [WZ] we can prove that there exists a null set $N_{1,k}$ for each $k \in \mathbb{N}$ such that the limit

$$\overline{\hat{\varphi}_k(x)} = \lim_{h\downarrow 0} \rho_\delta(Q_h(x))^{-1} \int_{Q_h(x)} \overline{\varphi_k}\, d\rho_\delta$$

exists for every $x \in P\backslash N_{1,k}$ and $\overline{\hat{\varphi}_k}$ can be interpreted as a representant of the L_2-class $\langle\overline{\varphi_k}\rangle$ in the usual way.

Furthermore, let $t > 0$. Then the function $\sum_{k\in\mathbb{N}} e^{-\lambda_k t} |\varphi_k|^2$ belongs to $L_1(\mathbb{R}, \rho_\delta)$. Hence applying Theorem 10.49 of [WZ] for the second time, we obtain a null set $N_{2,t}$ such that for all $x \in P\backslash N_{2,t}$.

$$\left(\sum_{k\in\mathbb{N}} e^{-\lambda_k t} |\varphi_k|^2\right)(x) = \lim_{h\downarrow 0} \rho_\delta(Q_h(x))^{-1} \left(\int_{Q_h(x)} \left(\sum_{k\in\mathbb{N}} e^{-\lambda_k t} |\varphi_k|^2\right) d\rho_\delta \right) .$$

Put $N = \left(\bigcup_{k\in\mathbb{N}} N_{1,k}\right) \cup \left(\bigcup_{n\in\mathbb{N}} N_{2,\frac{1}{n}}\right)$. Then N is a null set with respect to $\rho_{\mathcal{J}}$.

For each $x \in P\backslash N$ we have derived the following

$$(1.4.i) \qquad \forall_{k\in\mathbb{N}} : \overline{\hat{\varphi}_k(x)} = \lim_{h\downarrow 0} \rho_{\mathcal{J}}(Q_h(x))^{-1} \int_{Q_h(x)} \overline{\varphi_k}\, d\rho_{\mathcal{J}} .$$

$$(1.4.ii) \qquad \forall_{n\in\mathbb{N}} : \sum_k e^{-\frac{1}{n}\lambda_k} |\hat{\varphi}_k(x)|^2 = \lim_{h\downarrow 0} \rho_{\mathcal{J}}(Q_h(x))^{-1} \int_{Q_h(x)} \left(\sum_{k\in\mathbb{N}} e^{-\lambda_k t} |\varphi_k|^2\right) d\rho_{\mathcal{J}} .$$

Since for each $t > 0$ there exists $n \in \mathbb{N}$ with $0 < \frac{1}{n} < t$, we find

$$\sum_{k\in\mathbb{N}} e^{-\lambda_k t} |\hat{\varphi}_k(x)|^2 \le \sum_{k\in\mathbb{N}} e^{-\frac{1}{n}\lambda_k} |\hat{\varphi}_k(x)|^2 < \infty .$$

The latter observation leads to the following definition. The element $G_{t,x}$ of X is defined by

$$(1.5) \qquad G_{t,x} := \sum_{k\in\mathbb{N}} e^{-\lambda_k t}\, \overline{\hat{\varphi}_k(x)}\, v_k \quad , \quad t > 0 .$$

Then $t \mapsto G_{t,x}$ is an element of $T_{X,A}$.

Let $h \in S_{X,A}$, and put

$$\hat{h} : x \mapsto \sum_{k\in\mathbb{N}} (h,v_k)\, \hat{\varphi}_k(x) \in L_2(\mathbb{R},\rho_{\mathcal{J}}) .$$

Then $|\hat{h}(x)| < \infty$ for all $x \in P\backslash N$. This can be seen as follows:

$$|\hat{h}(x)| \le \left(\sum_{k\in\mathbb{N}} e^{2\lambda_k t} |(h,v_k)|^2\right)^{\frac{1}{2}} \left(\sum_{k\in\mathbb{N}} e^{-2\lambda_k t} |\hat{\varphi}_k(x)|^2\right)^{\frac{1}{2}} < \infty$$

for $t > 0$ small enough.

We now come to the main theorem of this section. It says that out of a null set N_δ with respect to the measure ρ_δ, each $x \in P$ admits a generalized eigenvector in $T_{X,A}$.

(1.6) Theorem

For each $x \in P$, $h > 0$ and $t > 0$, define

$$G_{t,h}(x) := \text{emb}\left\{\rho_\delta(Q_h(x))^{-1} \int_{Q_h(x)} dH_\lambda \delta\right\}(t) .$$

Then there exists a null set N_δ with respect to ρ_δ such that

(i) $G_{t,x} = \lim_{h \downarrow 0} G_{t,h}(x)$ exists in X for all $x \in P \backslash N_\delta$ and all $t > 0$.

(ii) $G_x : t \mapsto G_{t,x} \in T_{X,A}$, and $G_x \neq 0$ for all $x \in P \backslash N_\delta$.

(iii) $T G_x = x G_x$.

Proof

(1.5.i) Let $t > 0$, and let $\varepsilon > 0$. Let $x \in P \backslash N$ where N is the null set as defined in (1.4). Then there exists $n \in \mathbb{N}$ such that $0 < \frac{1}{n} < 2t$. Put $M_{x,n} = \left(\sum_{k \in \mathbb{N}} e^{-\frac{1}{n}\lambda_k} |\hat{\varphi}_k(x)|^2\right)^{\frac{1}{2}}$. Fix $k_0 \in \mathbb{N}$ so large that

$$e^{-(t-\frac{1}{n})\lambda_k} < \varepsilon(M_{x,n} + 1)^{-1} \quad , \quad k \geq k_0 .$$

Then

$$(*) \qquad \| \sum_{k=k_0+1} e^{-\lambda_k t} \overline{\hat{\varphi}_k(x)} v_k \|^2 = \sum_{k=k_0+1} e^{-2\lambda_k t} |\hat{\varphi}_k(x)|^2 \leq$$

$$\leq e^{-(t-\frac{1}{n})\lambda_{k_0}} M^2_{x,n} < \varepsilon^2 .$$

Furthermore, by (4.i) and (4.ii) we can choose $h > 0$ so small that

$$\left|\rho_{\delta}(Q_h(x))^{-1} \int_{Q_h(x)} \overline{\varphi_k}\, d\rho_{\delta} - \hat{\varphi}_k(x)\right| < \varepsilon \quad , \quad k = 1,\dots,k_0$$

and also

$$\rho_{\delta}(Q_h(x))^{-1} \int_{Q_h(x)} \left(\sum_{k\in\mathbb{N}} e^{-\frac{1}{n}\lambda_k} |\varphi_k|^2\right) d\rho_{\delta} < (M_{x,n} + 1)^2 .$$

Then

$$(**) \qquad \left\| \sum_{k=1}^{k_0} e^{-\lambda_k t}\left[\rho_{\delta}(Q_h(x))^{-1} \int_{Q_h(x)} \overline{\varphi_k}\, d\rho_{\delta} - \overline{\hat{\varphi}_k(x)}\right] v_k \right\| < \varepsilon \| e^{-tA}\|_{X\otimes X}$$

and

$$(***) \qquad \left\| \sum_{k=k_0+1} e^{-\lambda_k t} \rho_{\delta}(Q_h(x))^{-1} \left(\int_{Q_h(x)} \overline{\varphi_k}\, d\rho_{\delta}\right) v_k \right\|^2 =$$

$$= \sum_{k=k_0+1} e^{-2\lambda_k t} \left|\rho_{\delta}(Q_h(x))^{-1} \int_{Q_h(x)} \overline{\varphi_k}\, d\rho_{\delta}\right|^2 \le$$

$$\le e^{-2(t-\frac{1}{n})\lambda_{k_0}} \left\{\sum_{k\in\mathbb{N}} e^{-\frac{2}{n}\lambda_k} \rho_{\delta}(Q_h(x))^{-1} \int_{Q_h(x)} |\varphi_k|^2\, d\rho_{\delta}\right\} < \varepsilon^2 .$$

A combination of the estimates (*), (**) and (***) yields the result

$$\left\| \operatorname{emb} \rho_{\delta}(Q_h(x))^{-1} \left(\int_{Q_h(x)} d H_{\lambda}\delta\right)(t) - G_{t,x} \right\| < \varepsilon(2 + \| e^{-tA}\|_{X\otimes X}$$

for h small enough where $G_{t,x}$ is defined by (1.5)

(1.5.ii) If G_x is defined by $G_x : t \to G_{t,x}$ it is obvious that G_x in $T_{X,A}$. Let Γ_0 be the set of all $x \in P\backslash N$ for which $G_x = 0$. We shall show that Γ_0 is a null set with respect to ρ_{δ} . Note first that $G_x = 0$ implies $\hat{\varphi}_k(x) = 0$

for all $k \in \mathbb{N}$. Hence Γ_0 is a Borel set. Put $\gamma = \int_{\Gamma_0} d\,H_\lambda \mathfrak{f}$ and let $k \in \mathbb{N}$. Then

$$(\gamma, v_k) = \int_{\Gamma_0} d(H_\lambda \mathfrak{f}, v_k) = \int_{\Gamma_0} \hat{\varphi}_k \, d\rho_{\mathfrak{f}} = 0 \,.$$

Hence $\gamma = 0$ and Γ_0 is a null set with respect to $\rho_{\mathfrak{f}}$.

If we put $N_{\mathfrak{f}} = N \cup \Gamma_0$, then $N_{\mathfrak{f}}$ is a null set with respect to $\rho_{\mathfrak{f}}$ and for all $x \in P \backslash N_{\mathfrak{f}}$ we have $G_x \in T_{X,A}$ and $G_x \neq 0$.

(1.5.iii) Let $x \in P \backslash N_{\mathfrak{f}}$. We have to show $T\,G_x = x\,G_x$. Since $T - xI$ is continuous on $T_{X,A}$,

$$(*) \qquad (T - xI) \lim_{h \downarrow 0} \rho_{\mathfrak{f}}(Q_h(x))^{-1} \int_{Q_h(x)} d\,H_\lambda \mathfrak{f} =$$

$$= \lim_{h \downarrow 0} (T - xI) \left[\rho_{\mathfrak{f}}(Q_h(x))^{-1} \int_{Q_h(x)} d\,H_\lambda \mathfrak{f} \right].$$

We shall show that the latter limit tends to null in $T_{X,A}$ for $h \downarrow 0$. To this end, let $t > 0$. Then we compute as follows

$$\lim_{h \downarrow 0} \operatorname{emb}\left\{ (T - xI) \left[\rho_{\mathfrak{f}}(Q_h(x))^{-1} \int_{Q_h(x)} d\,H_\lambda \mathfrak{f} \right] \right\}(t) =$$

$$= \lim_{h \downarrow 0} \left\{ \sum_{k \in \mathbb{N}} e^{-\lambda_k t} \rho_{\mathfrak{f}}(Q_h(x))^{-1} \left(\int_{Q_h(x)} d(H_\lambda \mathfrak{f}, (T - xI) v_k) \right) v_k \right\} =$$

$$= \lim_{h \downarrow 0} \left\{ \sum_{k \in \mathbb{N}} e^{-\lambda_k t} \rho_{\mathfrak{f}}(Q_h(x))^{-1} \left(\int_{Q_h(x)} (\lambda - x) \overline{\varphi_k(\lambda)} \, d\rho_{\mathfrak{f}} \right) v_k \right\} .$$

This expression can be treated as follows

$$\sum_{k\in\mathbb{N}} e^{-2\lambda_k t} \left|\rho_{\delta}(Q_h(x))^{-1} \int_{Q_h(x)} (\lambda - x)\, \overline{\varphi_k(\lambda)}\, d\rho_{\delta}\right|^2 \leq$$

$$\leq \sum_{k \in \mathbb{N}} e^{-2\lambda_k t} \rho_{\delta}(Q_h(x))^{-1} \left(\int_{Q_h(x)} |\varphi_k(\lambda)|^2\, d\rho_{\delta}\right) \cdot$$

$$\cdot\ \rho_{\delta}(Q_h(x))^{-1} \left(\int_{Q_h(x)} |\lambda - x|^2\, d\rho_{\delta}\right) \leq$$

$$\leq h^2 (M_{x,n} + 1)^2$$

for h small enough and $n \in \mathbb{N}$ so large that $0 < \frac{1}{n} < 2t$.
So the limit (*) is null and also (1.5.iii) is proved. □

2. Commutative multiplicity theory

The commutative multiplicity theorem enables us to set up a theory, which ensures that the notion 'multiplicity of an eigenvalue' also makes sense for generalized eigenvalues. The so-called multiplicity theory which leads to this theorem is mainly measure theoretical. It is very well described by Nelson [Ne_2], ch. VI, and by Brown in [Br].

(2.1) Definition

Let ρ be a positive, finite Borel measure on $\mathbb{R}$. Then the support of ρ, $\mathrm{supp}(\rho)$, is defined by

$$\mathrm{supp}(\rho) := \{r \in \mathbb{R} \mid \forall_{\varepsilon>0} : \rho([r-\varepsilon, r+\varepsilon]) > 0\} .$$

(2.2) Lemma

Let ρ be a positive, finite Borel measure on $\mathbb{R}$. Then the complement of supp(ρ), supp$(\rho)^*$, is a set of measure zero with respect to ρ.

Proof

For each $x \in \text{supp}(\rho)^*$, define the set $Q_{x,\varepsilon} := [x - \varepsilon , x + \varepsilon]$ with $\varepsilon > 0$ taken so that $\rho(Q_{x,\varepsilon}) = 0$. Then

$$\text{supp}(\rho)^* \subset \bigcup_{x \in \text{supp}(\rho)^*} Q_{x,\varepsilon} .$$

Let $k \in \mathbb{N}$. The set $\text{supp}(\rho)^* \cap [-k,k]$ is bounded in $\mathbb{R}$. With Besicovitch covering's Lemma ([WZ], p.185) it follows that there is a countable set $\{x_1,x_2,...\}$ such that

$$\text{supp}(\rho)^* \cap [-k,k] \subset \bigcup_{i=1}^{\infty} Q_{x_i,\varepsilon_i} .$$

Hence

$$\rho(\text{supp}(\rho)^* \cap [-k,k]) = 0 .$$

Since $k \in \mathbb{N}$ is arbitrary, $\text{supp}(\rho)^*$ itself is a set of measure zero. □

There is another charaterization of supp(ρ).

(2.3) Lemma

supp(ρ) is the complement of the largest open set $\mathcal{O}$ for which $\rho(\mathcal{O}) = 0$.

Proof

Let $\text{supp}_1(\rho)$ denote the complement of the largest measurable open null set, the set $\text{supp}_1(\rho)$ is well defined (see [Bou], p. 16). Suppose $x \notin \text{supp}_1(\rho)$. Then there exists $\varepsilon > 0$ such that the interval

$[x-\varepsilon, x+\varepsilon] \subset \mathrm{supp}_1(\rho)^*$. So $\rho([x-\varepsilon, x+\varepsilon]) = 0$, and $x \notin \mathrm{supp}(\rho)$. Conversely, suppose $x \notin \mathrm{supp}(\rho)$. Then there exists $\varepsilon > 0$ such that $\rho([x-\varepsilon, x+\varepsilon]) = 0$. This implies that $(x-\varepsilon, x+\varepsilon) \subset \mathrm{supp}_1(\rho)^*$. Hence $x \notin \mathrm{supp}_1(\rho)$, completing the proof. □

(2.4) Definition

The Borel measure ν is absolutely continuous with respect to the Borel measure μ, notation $\nu \ll \mu$, if for every Borel set N with $\mu(N) = 0$, also $\nu(N) = 0$.

The Borel measures ν and μ are equivalent, $\nu \sim \mu$, if $\nu \ll \mu$ and $\mu \ll \nu$.

It is clear that $\nu \sim \mu$ implies $\mathrm{supp}(\nu) = \mathrm{supp}(\mu)$. So it makes sense to write $\mathrm{supp}(\langle\nu\rangle)$ meaning the support of each ν in the equivalence class $\langle\nu\rangle$.

(2.5) Definition

Two equivalence classes $\langle\nu\rangle$ and $\langle\mu\rangle$ are called mutually disjoint if

$$\nu(\mathrm{supp}\langle\nu\rangle \cap \mathrm{supp}\langle\mu\rangle) = \mu(\mathrm{supp}\langle\nu\rangle \cap \mathrm{supp}\langle\mu\rangle) = 0 .$$

If one wants a canonical listing of the eigenvalues of a matrix it is natural to list all eigenvalues of multiplicity one, two, etc. We need a way of saying that an operator is of uniform multiplicity one, two, etc. To this end we introduce

(2.6) Definition

A self-adjoint operator T is said to be of uniform multiplicity m, $1 \le m \le \infty$, if T is unitarily equivalent to multiplication by the identity function in $L_2(\mathbb{R},\mu) \oplus \ldots \oplus L_2(\mathbb{R},\mu)$, where there are m terms in the sum and μ is a finite Borel measure.

This definition makes sense because if T is also unitarily equivalent to multiplication by the identity function on $L_2(\mathbb{R},\nu) \oplus \ldots \oplus L_2(\mathbb{R},\nu)$ (n times), then m = n and $\mu \sim \nu$ (see [Br]).

(2.7) Theorem (Commutative multiplicity theorem)

Let T be a self-adjoint operator in a Hilbert space X. Then there exists a decomposition $X = X_\infty \oplus X_1 \oplus \ldots \oplus X_m \oplus \ldots$ so that

(i) T acts invariantly in each X_m

(ii) $T \restriction X_m$ has uniform multiplicity m

(iii) The measure classes $\langle\mu_m\rangle$ associated with the spectral representation of $T \restriction X_m$ are mutually disjoint.

Further, the subspaces $X_\infty, X_1, X_2, \ldots$ (some of which may be zero) and the measure classes $\langle\mu_\infty\rangle, \langle\mu_1\rangle, \langle\mu_2\rangle, \ldots$ are uniquely determined by (i), (ii) and (iii).

Proof

For a proof see Nelson, [Ne_2] ch. VI, Brown, [Br], or [RS]. □

3. A total set of generalized eigenfunctions for the self-adjoint operator T

(3.1) Definition

A set $\Gamma \subset X$ is called cyclic with respect to T if

$$X = \bigoplus_{\gamma\in\Gamma} X_\gamma .$$

Since X is separable, Γ consists of an at most countable number of elements. If Γ can be choosen such that it consists of one element only, this element is called a cyclic vector and the operator T a cyclic operator. The cyclic set Γ is not uniquely determined. The commutative multiplicity theorem brings in some uniqueness.

(3.2) Lemma

T has uniform multiplicity one if and only if T is cyclic. (see Definition 2.6)

By Theorem (2.7) X can be splitted into a countable direct sum,

$$X = X_\infty \oplus X_1 \oplus X_2 \oplus \ldots .$$

The restricted operator $T \upharpoonright X_m$, $1 \le m \le \infty$, is unitarily equivalent to multiplication by the identity function in

$$L_2(\mathbb{R},\mu_m) \oplus \ldots \oplus L_2(\mathbb{R},\mu_m) \quad , \quad (m \text{ times}).$$

By X_{mj}, $j = 1,\ldots,m$, we denote the orthogonal subspace of X_m, which corresponds to the j-th term in the direct sum. Since $T \upharpoonright X_{mj}$ obviously has uniform multiplicity one, there exists a cyclic vector $\gamma_j^{(m)}$ for $T \upharpoonright X_{mj}$. Thus we obtain a set Γ,

$$\Gamma := \{\gamma_j^{(m)} \mid 1 \le j < m+1 \ , \ 1 \le m \le \infty\} ,$$

which is cyclic for T. Note that $1 \le m \le \infty$ means $m = \infty,1,2,\ldots$.

Let m, $1 \le m \le \infty$, be fixed so that $X_m \neq \{0\}$, and let j, $1 \le j < m+1$ be fixed. Further, let $\rho_j^{(m)}$ denote the finite Borel measure $\left(\left(H_\lambda \gamma_j^{(m)}, \gamma_j^{(m)} \right)\right)_{\lambda\in\mathbb{R}}$. The projection from X onto X_{mj} is denoted by $P_j^{(m)}$ and the unitary operator from X_{mj} onto $L_2(\mathbb{R},\rho_{\gamma_j^{(m)}})$ by $U_j^{(m)}$. Finally, put $\hat{v}_{k,j}^{(m)} = U_j^{(m)} P_j^{(m)} v_k$.

From Theorem (1.5) we obtain sets $N_j^{(m)}$ of measure zero with respect to $\rho_j^{(m)}$, $m = \infty, 1, 2, \ldots$, such that for each $\sigma \in \text{supp}(\rho_j^{(m)}) \backslash N_j^{(m)}$

$$G_{\sigma,j}^{(m)} : t \to \sum_{k=1}^{\infty} e^{-\lambda_k t} \, \overline{v_{k,j}^{(m)}(\sigma)} \, v_k$$

is in $T_{X,A}$, and

$$\mathcal{T} G_{\sigma,j}^{(m)} = \sigma \, G_{\sigma,j}^{(m)} .$$

Following Theorem (2.7) $\rho_i^{(m)} \sim \rho_j^{(m)}$ for all i, $1 \le i < m+1$, i.e. the set $N_j^{(m)}$ is a null set with respect to each $\rho_i^{(m)}$. Put $N^{(m)} = \bigcup_{j=1}^{m} N_j^{(m)}$.

(3.3) Theorem

Let m, $1 \le m \le \infty$, be taken such that $X_m \ne \{0\}$. Then there exists a null set $N^{(m)}$ with respect to $<\mu_m>$ with the property that for every $\sigma \in \text{supp}(<\mu_m>) \backslash N^{(m)}$ there are at least m independent generalized eigenfunctions with eigenvalue σ. Further, the set

$$\{G_{\sigma,j}^{(m)} \mid 1 \le j < m+1, \ 1 \le m \le \infty, \ \sigma \in \text{supp}(<\mu_m>) \backslash N^{(m)}\}$$

is total.

Proof

Suppose $\sum_{j=1}^{m} \alpha_j \, G_{\sigma,j}^{(m)} = 0$. Then for all $f \in S_{X,A}$

$$\sum_{j=1}^{m} \alpha_j \, \hat{f}_j^{(m)}(\sigma) = 0 .$$

Since $S_{X,A}$ is dense in X, this leads to a contradiction.

A set $V \subset T_{X,A}$ is said to be total if $\forall_{F \in V} <g,F> = 0 \Rightarrow g = 0$.

So suppose,

$$<g, G_{\sigma,j}^{(m)}> = 0 .$$

for $1 \leq j < m+1$, $1 \leq m \leq \infty$ and $\sigma \in \operatorname{supp}(\langle\mu_m\rangle)\backslash N^{(m)}$. Then it immediately follows that $(U_j^{(m)} P_j^{(m)} g)(\sigma) = 0$ almost everywhere with respect to $\langle\mu_m\rangle$, with $1 \leq j < m+1$ and $1 \leq m \leq \infty$. So $g = 0$.

(3.4) Lemma

Let $\sigma(T)$ be the spectrum of T. Then

$$\sigma(T) = \overline{\bigcup_{m \in \mathbb{N} \cup \{\infty\}} \operatorname{supp}(\langle\mu_m\rangle)} .$$

Proof

If $x \notin \sigma(T)$, then there exists $\varepsilon > 0$ such that

$$H([x-\varepsilon , x+\varepsilon]) = 0 .$$

So for all m, $1 \leq m \leq \infty$,

$$\mu_m([x-\varepsilon , x+\varepsilon]) = 0 .$$

This implies $(x-\varepsilon/2 , x+\varepsilon/2) \not\subset \operatorname{supp}(\mu_m)$ and hence

$$x \notin \overline{\bigcup_{1 \leq m \leq \infty} \operatorname{supp}(\langle\mu_m\rangle)} .$$

Conversely, suppose $x \notin \overline{\bigcup_{1 \leq m \leq \infty} \operatorname{supp}(\langle\mu_m\rangle)}$. Then there exists $\delta > 0$ such that $(x-\delta , x+\delta) \not\subset \operatorname{supp}(\langle\mu_m\rangle)$, $1 \leq m \leq \infty$. Hence $H([x-\delta , x+\delta])\gamma_j^{(m)} = 0$ for all $m \in \mathbb{N} \cup \{\infty\}$, $1 \leq j < m+1$. This implies $H([x-\delta , x+\delta]) = 0$. So $x \notin \sigma(T)$. □

We finish this section with two examples.

(3.5) Example

Let $\lambda_0 \in \sigma(T)$ be an eigenvalue of multiplicity m_0. Then $H(\{\lambda_0\})$ is a non-zero projection on X, and for j, $1 \le j < m_0 + 1$ fixed, we have

$$G_{\lambda_{0,j}}^{(m_0)} = \lim_{h \downarrow 0} \left\{ \frac{Q_h(\lambda_0) \int dH_\lambda \gamma_j^{(m_0)}}{Q_h(\lambda_0) \int d(H_\lambda \gamma_j^{(m_0)}, \gamma_j^{(m_0)})} \right\} = \frac{H(\{\lambda_0\})\gamma_j^{(m_0)}}{\| H\{\lambda_0\}\gamma_j^{(m_0)} \|^2}$$

Hence $G_{\lambda_{0,j}}^{(m_0)} \in X$.

(3.6) Example

Let C be a self-adjoint compact operator on X. Then the vectors

$$\gamma_j^{(m)} := \sum_{k=1}^{\infty} 2^{-k} e_{j,k}^{(m)} \quad , \quad 1 \le j \le m, \ 1 \le m < \infty ,$$

where the series may be a finite sum, establish a cyclic set for C. Here $(e_{j,k}^{(m)})$ is an orthonormal basis of eigenvectors for C; $e_{j,k}^{(m)}$ is the j-th eigenvector, $1 \le j \le m$, with eigenvalue $\mu_k^{(m)}$ of multiplicity m, $1 \le m < \infty$.

4. The case of n-commuting self-adjoint operators

In this section we shall extend the theory of the first part of this paper to the case of n commuting self-adjoint operators, where n is a natural number. We only discuss the frame work of this extension, because there really is no essential difference with the theory of one self-adjoint operator.

Let $(T_1, T_2, \ldots, T_n)$ be an n-set of commuting self-adjoint operators in X, which map $S_{X,A}$ continuously into itself. Let $(\lambda_i)_{\lambda_i \in \mathbb{R}}$, $i = 1,\ldots,n$, denote their respective spectral resolutions. For $f \in X$, the Hilbert space X_f is the closure in X of the linear span

$$\langle\{H_1(\Delta_1) \ldots H_n(\Delta_n) f \mid \Delta_i \subset \mathbb{R} \text{ a Borel set, } i = 1,\ldots,n\}\rangle .$$

The Hilbert space X_f is unitarily equivalent to $L_2(\mathbb{R}^n,\rho_f)$, where ρ_f is the well-defined finite measure

$$\rho_f(\Delta_1,\ldots,\Delta_n) = (H_1(\Delta_1)\ldots H_n(\Delta_n)f, f)$$

over the Borel subsets of $\mathbb{R}^n$. For every $g \in X_f$ there exists $\hat{g} \in L_2(\mathbb{R}^n,\rho_f)$ with the properties

$$g = \int_{\mathbb{R}^n} \hat{g}\, d\, H_1(\lambda_1)\ldots H_n(\lambda_n) f$$

$$\| g \|^2 = \int_{\mathbb{R}^n} |\hat{g}|^2\, d\,\rho_f .$$

The n-set restricted to X_f, $(T_1,\ldots,T_n) \restriction X_f$ is unitarily equivalent to the n-set $(Q_1,\ldots,Q_n)$, where Q_i denotes multiplication by λ_i in $L_2(\mathbb{R}^n,\rho_f)$.

For $x \in \mathbb{R}^n$ and $h > 0$, we define the cube $Q_h(x)$ by

$$Q_h(x) := \{\xi \in \mathbb{R}^n \mid |x_i - \xi_i| \le h,\ i = 1,\ldots,n\} .$$

Further we define the set $P \subset \mathbb{R}^n$ by

$$P := \{x \in \mathbb{R}^n \mid \forall_{h>0} : \rho_f(Q_h(x)) > 0\} .$$

Then in case of the n-set $(T_1,\ldots,T_n)$, Theorem (1.3) can be reformulated as follows

(4.1) Theorem

For $x \in P$, define

$$G_{x,h}(t) := \mathrm{emb}(\rho_\delta(Q_h(x)))^{-1}\left(\int_{Q_h(x)} d\,H_1(\lambda_1)\dots H_n(\lambda_n)\delta\right)(t)$$

There exists a null set N with respect to ρ_δ such that for all $x \in P\backslash N$

(i) $G_x(t) := \lim_{h\downarrow 0} G_{x,h}(t)$ exists in X for all $t > 0$

(ii) $G_x : t \mapsto G_x(t) \in T_{X,A}$ and $G_x \neq 0$

(iii) $T_i G_x = x_i G_x$.

Proof

cf. the proof of Theorem 1.3. □

The measure theoretical part of section 2 can be adapted in the usual way to measures in $\mathbb{R}^n$, cf. Definition (2.1), (2.4), (2.5) and (2.6) and Lemma (2.2) and (2.3).

For a better understanding of the commutative multiplicity theorem for an n-set of self-adjoint commuting operators, we introduce the notion of (generalized) eigentuple of multiplicity m, $1 \le m \le \infty$.

(4.2) Definition

An n-tuple $\lambda = (\lambda_1,\dots,\lambda_n) \in \mathbb{R}^n$ is an eigentuple of the n-set $(T_1,\dots,T_n)$ of multiplicity m if there exist m orthonormal simultaneous eigenvectors $e^{(m)}_{\lambda,j}$ such that

$$T_i\, e^{(m)}_{\lambda,j} = \lambda_i\, e^{(m)}_{\lambda,j} \quad , \quad 1 \le j < m+1,\ 1 \le i \le n .$$

Similarly, the notion generalized eigentuple can be introduced.

If one wants a canonical listing of the eigentuples of an n-set of commuting matrices it is natural to list all eigentuples of multiplicity one, two,... . We need a way of saying that an n-set of commuting self-adjoint operators is of uniform multiplicity one, two, etc.

(4.3) Definition

An n-set $(T_1,\ldots,T_n)$ of commuting self-adjoint operators is said to be of uniform multiplicity m if each T_i is unitarily equivalent to multiplication by λ_i in $L_2(\mathbb{R}^n,\mu) \oplus \ldots \oplus L_2(\mathbb{R}^n,\mu)$, where there are m terms in the sum and where μ is a finite Borel measure in $\mathbb{R}^n$.

The formulation of the commutative multiplicity theorem for an n-set of commuting self-adjoint operators is quite evident.

(4.4) Theorem

Let $(T_1,\ldots,T_n)$ be an n-set of commuting self-adjoint operators in X. Then there exists a decomposition

$$X = X_\infty \oplus X_1 \oplus X_2 \oplus \ldots$$

such that

(i) The n-set $(T_1,\ldots,T_n)$ acts invariantly in each X_m, $1 \le m \le \infty$.

(ii) The n-set $(T_1,\ldots,T_n)$ restricted to X_m has uniform multiplicity m.

(iii) The measure classes $\langle\mu_m\rangle$ associated with $(T_1,\ldots,T_n) \upharpoonright X_m$ are mutually disjoint.

Further, the subspaces $X_\infty, X_1, X_2,\ldots$ (some of which may be zero) and the classes $\langle\mu_\infty\rangle, \langle\mu_1\rangle,\ldots$ are uniquely determined by (i), (ii) and (iii).

The proof of this theorem can be derived from the proof in the one dimensional case and is essentially the same (see [Ne_2], [Br]).

(4.5) Definition

A set $\Gamma \subset X$ is called cyclic with respect to $(T_1,\ldots,T_n)$ if

$$X = \bigoplus_{\gamma\in\Gamma} X_\gamma .$$

Note that Γ is at most countable.

If Γ consists of one element, this element is called cyclic vector. Lemma 3.1 can be replaced by

(4.6) Lemma

The n-set $(T_1,\ldots,T_n)$ is of uniform multiplicity one if and only if it has a cyclic vector.

Following Theorem (4.4) X can be splitted into a direct sum $X = X_\infty \oplus X_1 \oplus X_2 \oplus \ldots$. Each of the restricted operators $T_i \restriction X_m$, $1 \le i < m+1$ is unitarily equivalent to multiplication by λ_i in

$$L_2(\mathbb{R}^n,\mu_m) \oplus \ldots \oplus L_2(\mathbb{R}^n,\mu_m) \quad , \quad \text{m-times} .$$

Let X_{mj}, $1 \le j < m+1$ be the orthogonal subspace of X_m, which corresponds to the j-th term in the sum above. Then $(T_1,\ldots,T_n) \restriction X_{mj}$ has a cyclic vector $\gamma_j^{(m)}$, say. In this way a set Γ is obtained

$$\Gamma = \{\gamma_j^{(m)} \mid 1 \le j < m+1, \ 1 \le m \le \infty\}$$

which is cyclic for $(T_1,\ldots,T_n)$.

(4.7) Theorem

Take m, $1 \leq m \leq \infty$, such that $X_m \neq \{0\}$. Then there exists a null set $N^{(m)}$ with respect to $<\mu_m>$, such that for all $\lambda \in \mathrm{supp}(<\mu_m>)\backslash N^{(m)}$, there are at least m independent simultaneous generalized eigenfunctions of $(T_1,\ldots,T_n)$ with generalized eigentuple $\lambda = (\lambda_1,\ldots,\lambda_n)$.
Further, the set of all generalized eigenfunctions is total.

(4.8) Example

Consider $S_{X,A}$ with $X = L_2(\mathbb{R})$ and $A = \frac{1}{2}\left(-\frac{d^2}{dx^2} + x^2 + 1\right)$ and the 2-set (Φ,Q^2) where Φ denotes the parity operator and Q^2 multiplication by x^2; so

$$(Q^2 f)(x) = x^2 f(x) \quad \text{and} \quad (\Phi f)(x) = f(-x) .$$

Then the 2-set (Φ,Q^2) has uniform multiplicity 1 because it has a cyclic vector; for instance take

$$\gamma : x \mapsto (1 + x)e^{-\frac{1}{2}x^2} .$$

5. A mathematical interpretation of Dirac's formalism

In the preface to his book on the foundations of quantum mechanics von Neumann says that Dirac's formalism *is scarcely to be surpassed in brevity and elegance* but that it *in no way satisfies the requirements of mathematical rigour*. The improper functions of Dirac, the δ-function and its derivatives, have stimulated the growth of a new branch of mathematics: the theory of distributions. Yet, as far as we know, no paper on Dirac's formalism mathematically foundates the bold way in which Dirac treats the continuous spectrum of a self-adjoint operator. Most papers on this subject only solve the so called generalized eigenvalue problem by means of the rigged Hilbert space theory of Gelfand and Shilov. But Dirac's formalism has more aspects.

In this section an interpretation of the formalism is studied in terms of our distribution theory. It consists of the definition of ket and bra space, of Parseval's identity, of the Fourier expansion of kets with respect to continuous bases, of the existence and orthogonality of complete sets of eigenkets, of matrices of unbounded linear mappings with respect to continuous bases, and of some matrix computation.

We shall only consider quantum systems at a given time without superselection rules. So we do not need to specify whether we are using the Heisenberg or Schrödinger pictures. A quantum system at a given time is determined by states and observables. The space of all states is mostly supposed to be in 1-1 correspondence with the set of all one dimensional subspaces of an infinite dimensional separable Hilbert space X and the set of observables in 1-1 correspondence with the set of all self-adjoint operators in X. But in general we do not need to consider all self-adjoint operators. To describe a quantum system one can make a choice out of the set of observables, e.g. 'energy', 'momentum' and 'spin', which is sufficiently large to completely determine the quantum system and in particular all relevant observables.

In his formalism Dirac treats all points in the spectrum of a self-adjoint operator similarly. So the formalism assumes for instance that the notion multiplicity of λ for every point λ in the spectrum makes sense, and further that for each λ with multiplicity m there exist precisely m independent eigenstates. Of course, Hilbert space theory can not fulfil these wishes.

Hilbert spaces are too small. Therefore, it is natural to look for spaces, which extend Hilbert space, and with structures comparable to Hilbert space structure. For instance, the trajectory spaces $T_{X,A}$ are acceptable candidates.

In Dirac's formalism the dual space of the ket space, the so called bra space, is in 1-1 correspondence with the ket space. So the latter space ought to be self-dual. To this end distribution theory can't ever be of any help. We try to circumvent this problem by a new interpretation of Dirac's bracket notion.

Let QS be a quantum mechanical system. We assume that QS is completely determined by the set of self-adjoint operators $\{P_1,\ldots,P_n\}$ in the Hilbert space X. Further, we suppose that there exists a nuclear space $S_{X,A}$ such that each P_i maps $S_{X,A}$ continuously into itself. So the P_i, $i = 1,\ldots,n$, can be extended to continuous linear mappings on $T_{X,A}$. For instance, when the set $\{P_1,\ldots,P_n\}$ is an n-set of commuting self-adjoint operators it is possible to construct such a nuclear space (cf. Ch. III, Section 9). In our interpretation the set of observables of QS corresponds uniquely to the set of self-adjoint operators which are continuous on $S_{X,A}$. We note that the choice of the space $S_{X,A}$ depends on the self-adjoint operators $P_1,\ldots,P_n$. For the set of states we take the set of one dimensional subspaces of $T_{X,A}$.

In Dirac's terminology, the trajectories of $T_{X,A}$ are called ket vectors Therefore we introduce Dirac's bracket notation and denote them by $|G\rangle$ in the sequel. The label G in the expression $|G\rangle$ is mostly chosen such that it expresses best the properties of $|G\rangle$ which are relevant in the context. To $|G\rangle$ uniquely corresponds the bra $\langle G|$ defined by

$$\langle G| : t \mapsto \sum_{k=1}^{\infty} e^{-\lambda_k t} \langle v_k, |G\rangle\rangle\, v_k$$

where (v_k) denotes the orthonormal basis of eigenvectors of A, and where the series converges in X for each $t > 0$.

The expression $\langle F \mid G\rangle$, called the bracket of $\langle F|$ and $|G\rangle$, denotes the complex valued function

$$\langle F \mid G\rangle : t \mapsto \overline{\langle |F\rangle(t), |G\rangle\rangle} \quad , \quad t > 0 .$$

The function $\langle F \mid G\rangle$ is well defined because $|F\rangle(t) \in S_{X,A}$ for every $t > 0$. It extends to an analytic function on the open right half plane.

Let $f \in S_{X,A}$. Then obviously $\langle f \mid G\rangle(-\tau)$ exists for every $|G\rangle$ and $\tau > 0$ sufficiently small and

$$\langle f \mid G\rangle(-\tau) = \overline{\langle |f\rangle(-\tau) \, , \, |G\rangle\rangle} ;$$

similarly $\langle G \mid f\rangle(-\tau)$ exists and

$$\langle G \mid f\rangle(-\tau) = \langle |f\rangle(-\tau) \, , \, |G\rangle\rangle .$$

To emphasize this nice property of the elements in $S_{X,A}$ the kets and bras corresponding to elements in $S_{X,A}$ are called test kets and test bras.

Finally, we remark that for all $t > 0$ the function $\langle F \mid G\rangle$ satisfies

$$\langle F \mid G\rangle(t) = \langle F(t) \mid G\rangle(0) = \overline{\langle G(t) \mid F\rangle(0)} = \overline{\langle G \mid F\rangle(t)}$$

and

$$\langle F \mid G\rangle(t) = \langle F(t) \mid G\rangle(0) = \langle F \mid G(t)\rangle(0) .$$

Let $P : S_{X,A} \to S_{X,A}$ be an observable of QS. For simplicity, suppose that P is a cyclic operator in X. Then all points in $\sigma(P)$, the spectrum of P, have multiplicity one. Further, there exists a cyclic vector γ in X such that P is unitarily equivalent to multiplication by λ in the Hilbert space $L_2(\mathbb{R}, d(H_\lambda\gamma,\gamma))$. Here $(H_\lambda)_{\lambda\in\mathbb{R}}$ denotes the spectral resolution of the identity with respect to P. As in section 3, the Borel measure $d(H_\lambda\gamma,\gamma)$ is denoted by $d\rho_\gamma(\lambda)$ in the sequel.

Following the preceding sections there exists a null set N with respect to ρ_γ such that for each $\lambda \in \sigma(P)\backslash N$ there is an eigenket $|\lambda\rangle$. With the notation of section 3, $|\lambda\rangle$ has the following Fourier expansion

$$|\lambda\rangle = \sum_{k=1}^{\infty} \overline{\hat{v}_k(\lambda)}\,|v_k\rangle$$

where the series converges in $T_{X,A}$.

Let $g \in S_{X,A}$. Then $g = e^{-tA} f$ for a well chosen $f \in S_{X,A}$ and $t > 0$. Consider the following formal computation

$$\begin{aligned} g &= \sum_{k=1}^{\infty} e^{-\lambda_k t}(f,v_k)\,v_k \\ &= \sum_{k=1}^{\infty} e^{-\lambda_k t}\left(\int_{\mathbb{R}} \hat{f}(\lambda)\,\overline{\hat{v}_k(\lambda)}\,d\rho_\gamma(\lambda)\right) v_k \\ &\overset{(*)}{=} \int_{\mathbb{R}} \hat{f}(\lambda)\left(\sum_{k=0}^{\infty} e^{-\lambda_k t}\,\overline{\hat{v}_k(\lambda)}\,v_k\right) d\rho_\gamma(\lambda)\,. \end{aligned}$$

Hence

$$g = \int_{\mathbb{R}} \langle\lambda\,|\,f\rangle(0)\,|\lambda\rangle(t)\,d\rho_\gamma(\lambda)\,.$$

The only problem in this computation is the equality (*). We shall therefore prove that summation and integration can be interchanged. The following inequalities hold true

$$\sum_{k=1}^{\infty} \int_{\mathbb{R}} e^{-\lambda_k t} |\hat{f}(\lambda) \overline{\hat{v}_k(\lambda)}| \, d\rho_\gamma(\lambda) \le$$

$$\le \frac{1}{2}\left(\sum_{k=1}^{\infty} e^{-\lambda_k t} \int_{\mathbb{R}} |\hat{f}(\lambda)|^2 \, d\rho_\gamma(\lambda) + \sum_{k=1}^{\infty} e^{-\lambda_k t} \int_{\mathbb{R}} |\hat{v}_k(\lambda)|^2 \, d\rho_\gamma(\lambda) \right) =$$

$$= \frac{1}{2} (\|f\|^2 + 1) \left(\sum_{k=1}^{\infty} e^{-\lambda_k t} \right).$$

By the Fubini-Tonelli theorem equality (*) is verified.

With the aid of the above derivation, g can be written as

$$g = \int_{\mathbb{R}} \langle\lambda \mid g\rangle(-t) \, |\lambda\rangle(t) \, d\rho_\gamma(\lambda)$$

where the integral converges absolutely in X, and does not depend on the choice of $t > 0$.

(5.1) Theorem

Let $|f\rangle$ be a test ket. Then

$$|f\rangle = \int_{\mathbb{R}} \langle\lambda \mid f\rangle(0) \, |\lambda\rangle d\rho_\gamma(\lambda)$$

i.e.

$$|f\rangle(t) = \int_{\mathbb{R}} \langle\lambda \mid f\rangle(0) \, |\lambda\rangle(t) \, d\rho_\gamma(\lambda) \quad , \quad t > 0 .$$

Here we use the usual notion of integral for functions from $\mathbb{R}$ into X.

Proof

Let $t > 0$. Put $g = |f\rangle(t)$. We have seen that

$$g = \int_{\mathbb{R}} \langle\lambda \mid g\rangle(-t) \, |\lambda\rangle(t) \, d\rho_\gamma(\lambda)$$

with absolute convergence in X.

Since $\langle\lambda \mid g\rangle(-t) = \langle\lambda \mid f\rangle(0)$, the assertion follows. □

Parseval's identity is an immediate consequence of section 3

(5.2) $$\|f\|^2 = \int_{\mathbb{R}} |\hat{f}(\lambda)|^2 \, d\rho_\gamma(\lambda) = \int_{\mathbb{R}} |\langle f \mid \lambda\rangle(0)|^2 \, d\rho_\gamma(\lambda) .$$

Further, from Theorem (5.1) it is clear that

(5.3) $$P|f\rangle = \int_{\mathbb{R}} \lambda \langle\lambda \mid f\rangle(0) \, |\lambda\rangle \, d\rho_\gamma(\lambda) .$$

Let $F \in T_{X,A}$. Then for every $\tau > 0$, $F(\tau) \in S_{X,A}$ and hence

$$\text{emb}(|F\rangle(\tau)) = |F(\tau)\rangle = \int_{\mathbb{R}} \langle\lambda \mid F(\tau)\rangle(0) \; |\lambda\rangle \, d\rho_\gamma(\lambda)$$

following Theorem (5.1). Further, let $t > 0$. Then for every τ, $0 < \tau < t$

(5.4) $$|F\rangle(t) = e^{-(t-\tau)A} |F\rangle(\tau) = \int_{\mathbb{R}} \langle\lambda \mid F\rangle(\tau) \; |\lambda\rangle(t-\tau) \, d\rho_\gamma(\lambda) .$$

The integral in (5.4) does not depend on the choice of τ and converges absolutely in X. The ket $|F\rangle$ can thus be represented by

$$|F\rangle : t \to \int_{\mathbb{R}} \langle\lambda \mid F\rangle(\tau) \; |\lambda\rangle(t-\tau) \, d\rho_\gamma(\lambda) .$$

By the expression

$$\int_{\mathbb{R}} \langle\lambda \mid F\rangle \; |\lambda\rangle \, d\rho_\gamma(\lambda)$$

is meant the trajectory

$$t \mapsto \int_{\mathbb{R}} \langle\lambda \mid F\rangle(\tau) \; |\lambda\rangle(t-\tau) \, d\rho_\gamma(\lambda) .$$

Each of the integrals does not depend on the choice of τ, $0 < \tau < t$, and converges absolutely in X. We can write

$$(5.5) \qquad |F\rangle = \int_{\mathbb{R}} \langle\lambda \mid F\rangle \, |\lambda\rangle \, d\rho_\gamma(\lambda)$$

where the integral has to be understood in the interpretation that we have derived above. (Cf. the appendix.)

The result of Theorem (5.1) can be sharpened. To this end, let $f \in S_{X,A}$. Then there exists $\tau > 0$ such that $e^{\tau A} f \in S_{X,A}$. We have

$$|f\rangle = \int_{\mathbb{R}} \langle\lambda \mid f\rangle \, |\lambda\rangle \, d\rho_\gamma(\lambda) = \int_{\mathbb{R}} \langle\lambda \mid f\rangle(-\tau) \, |\lambda(\tau)\rangle \, d\rho_\gamma(\lambda)$$

where the latter integral converges in X. Since $e^{\frac{\tau}{2}A}$ is a closed operator in X, and since $\int_{\mathbb{R}} \langle\lambda \mid f\rangle(-\tau) \, |\lambda\rangle(\tau/2) \, d\rho_\gamma(\lambda)$ exists as an integral of a function from $\mathbb{R}$ into X, the integral

$$\int_{\mathbb{R}} \langle\lambda \mid f\rangle(-\tau) \, |\lambda\rangle(\tau) \, d\rho_\gamma(\lambda)$$

exists in $S_{X,A}$-sense. Hence in our interpretation

$$|f\rangle = \int_{\mathbb{R}} \langle\lambda \mid f\rangle \, |\lambda\rangle \, d\rho_\gamma(\lambda)$$

where the integral exists in $S_{X,A}$-sense. (Cf. the appendix.)

Consider the following equality

$$\langle\mu \mid \lambda\rangle(t) = \sum_{k=1}^{\infty} e^{-\lambda_k t} \, \overline{\hat{v}_k(\lambda)} \, \hat{v}_k(\mu) \quad , \quad \lambda,\mu \in \sigma(P)\backslash N, \ t > 0 \, .$$

Let δ_λ denote the function

$$\delta_\lambda : (\mu,t) \mapsto \langle\mu \mid \lambda\rangle(t)$$

and let U denote the unitary operator from X onto $Y = L_2(\mathbb{R},\rho_\gamma)$. Put $B = UAU^*$. Then $\delta_\lambda \in T_{Y,B}$ and for $\hat{f} \in S_{Y,B}$

$$<\hat{f},\delta_\lambda> = \sum_{k=1}^{\infty} \left(\int_{\mathbb{R}} \hat{f}(\mu)\, \overline{\hat{v}_k(\mu)}\, d\rho_\gamma(\mu) \right) \hat{v}_k(\lambda) = \hat{f}(\lambda) .$$

So δ_λ is Dirac's delta function in $T_{Y,B}$ and consequently we write

(5.6) $$<\mu \mid \lambda> = \delta_\lambda(\mu) .$$

Relation (5.6) expresses the generalization of the orthogonality relations for the eigenvectors of P to the eigenkets of P in agreement with Dirac's notation.

For the sake of completeness we rewrite the result (5.5) for the bras and test bras

(5.7) $$<F| = \int_{\mathbb{R}} <F \mid \lambda> <\lambda| \, d\rho_\gamma(\lambda)$$

where the integral exists in $T_{X,A}$-sense. If $<F|$ is a test bra the integral exists in $S_{X,A}$-sense.

Another aspect of Dirac's formalism is the so called closure property of a complete set of eigenkets.

(5.8) <u>Theorem</u> *(closure property)*

$$P^n = \int_{\mathbb{R}} \lambda^n \,|\lambda><\lambda|\, d\rho_\gamma(\lambda) \quad , \quad n = 0,1,2,\ldots$$

where the integral converges in $T_{X\otimes X, A\boxplus A}$. Here $|\lambda><\lambda|$ denotes the tensor product $|\lambda> \otimes |\lambda>$ ($\in T_{X\otimes X,A\boxplus A}$).

Proof

Let $t > 0$. Consider the following formal derivation

$$e^{-t(A\boxplus A)}P^n = \sum_{k,\ell} e^{-t\lambda_k}e^{-t\lambda_\ell}\,\overline{\langle v_k \otimes v_\ell\,,\,P^n\rangle}\, v_k \otimes v_\ell$$

$$= \sum_{k,\ell} e^{-\lambda_k t}e^{-\lambda_\ell t}\left(\int_{\mathbb{R}} \lambda^n \langle v_k \mid \lambda\rangle(0)\,\overline{\langle v_\ell \mid \lambda\rangle(0)}\; d\rho_\gamma(\lambda)\right) v_k \otimes v_\ell$$

$$\overset{(*)}{=} \int_{\mathbb{R}} \lambda^n \left(\sum_{k,\ell} (e^{-\lambda_k t}\,\overline{\hat{v}_k(\lambda)}\, v_k) \otimes (e^{-\lambda_\ell t}\,\hat{v}_\ell(\lambda)\, v_\ell)\right) d\rho_\gamma(\lambda)$$

$$= \int_{\mathbb{R}} \lambda^n\, |\lambda\rangle(t) \otimes |\lambda\rangle(t)\; d\rho_\gamma(\lambda)\,.$$

We shall prove that summation and integration can be interchanged. The remaining part of the proof is straight forward.

$$\sum_{k,\ell}\int_{\mathbb{R}} |e^{-\lambda_k t}e^{-\lambda_\ell t}\lambda^n\,\hat{v}_k(\lambda)\,\overline{\hat{v}_\ell(\lambda)}\,|\; d\rho_\gamma(\lambda) \le$$

$$\le \sum_{k,\ell}\int_{\mathbb{R}} \frac{1}{2}\, e^{-\lambda_k t}e^{-\lambda_\ell t}\,(\lambda^{2n}|\hat{v}_k(\lambda)|^2 + |\hat{v}_\ell(\lambda)|^2)\; d\rho_\gamma(\lambda) \le$$

$$\le \frac{1}{2}\sum_k e^{-\lambda_k t}\,(\|P^n v_k\|^2 + 1)\left(\sum_\ell e^{-\lambda_\ell t}\right) \le$$

$$\le \frac{1}{2}\,(|||P^n e^{-\frac{1}{2}tA}|||^2 + 1)\;|||e^{-\frac{1}{2}tA}|||^2\,.$$ □

Next we discuss the general case that $P : S_{X,A} \to S_{X,A}$ has a countable cyclic set. There will appear no essential difference with the case of a cyclic operator P. The same notation as in section 3 will be employed. Proofs will be omitted.

So let $\{\gamma_j^{(m)} \mid m = \infty, 1, 2, \ldots \;,\; 1 \le j < m+1\}$ be the cyclic set for P. Then X can be written as

$$X = \bigoplus_{m=1}^{m=\infty} \bigoplus_{j=1}^{m} X_{\gamma_j^{(m)}}$$

where by absence of better notations $\bigoplus_{m=1}^{m=\infty} \bigoplus_{j=1}^{m} X_{\gamma_j^{(m)}}$ will denote

$$\left(\bigoplus_{m=1}^{\infty} \bigoplus_{j=1}^{m} X_{\gamma_j^{(m)}} \right) \oplus \left(\bigoplus_{j=1}^{\infty} X_{\gamma_j^{(\infty)}} \right).$$

The Hilbert space $X_{\gamma_j^{(m)}}$ is unitarily equivalent to $L_2(\mathbb{R}, \rho_{\gamma_j^{(m)}})$ and $P \restriction X_{\gamma_j^{(m)}}$ is unitarily equivalent to multiplication by λ in $L_2(\mathbb{R}, \rho_{\gamma_j^{(m)}})$. Following section 3 there exist sets $N^{(m)}$, each of which has measure zero with respect to $\langle \rho_{\gamma_j^{(m)}} \rangle$, $m = \infty, 1, 2, \ldots$ such that for all λ in $\mathrm{supp}(\langle \rho_{\gamma_j^{(m)}} \rangle) \backslash N^{(m)}$ there are m independent eigenkets $|\lambda, m, j\rangle$, $1 \le j < m+1$. The eigenkets can be written as

$$|\lambda, m, j\rangle = \sum_{k=1}^{\infty} \overline{\hat{v}_{k,j}^{(m)}(\lambda)}\, |v_k\rangle$$

where the series converges in $T_{X,A}$. Then similar to Theorem (5.1)

(5.9) <u>Theorem</u>

Let $f \in S_{X,A}$. Then

$$|f\rangle = \sum_{m=1}^{m=\infty} \sum_{j=1}^{m} \int_{\mathbb{R}} \langle \lambda, m, j \mid f\rangle(0)\, |\lambda, m, j\rangle \, d\rho_{\gamma_j^{(m)}}(\lambda)$$

with convergence in $T_{X,A}$. Further

$$\| f \|^2 = \sum_{m=1}^{m=\infty} \sum_{j=1}^{\infty} \int_{\mathbb{R}} |\langle \lambda, m, j \mid f\rangle(0)|^2 \, d\rho_{\gamma_j^{(m)}}(\lambda)$$

(Parseval's identity) and

$$P|f\rangle = \sum_{m=1}^{m=\infty} \sum_{j=1}^{\infty} \int_{\mathbb{R}} \lambda \langle\lambda,m,j \mid f\rangle(0) \, |\lambda,m,j\rangle \, d\rho_{\gamma_j^{(m)}}(\lambda) .$$

Henceforth we will call the set $\{|\lambda,m,j\rangle \mid \lambda \in \sigma(P) \; , \; 1 \le m \le \infty, \; 1 \le j < m+1\}$ a Dirac basis. (In fact we have to exclude a null set N.)

With the same interpretation as in (5.5) we have

$$(5.10) \qquad |F\rangle = \sum_{m=1}^{m=\infty} \sum_{j=1}^{m} \int_{\mathbb{R}} \langle\lambda,m,j \mid F\rangle \, |\lambda,m,j\rangle \, d\rho_{\gamma_j^{(m)}}(\lambda)$$

with convergence in $T_{X,A}$. In particular if $|F\rangle$ in (5.10) is a test ket the convergence takes place even in $S_{X,A}$-sense.

Consider the following equality

$$\langle\mu,n,i \mid \lambda,m,j\rangle(t) = \sum_{k=1}^{\infty} e^{-\lambda_k t} \, \overline{\hat{v}_{k,j}^{(m)}(\lambda)} \, \hat{v}_{k,i}^{(n)}(\mu)$$

where $\lambda \in \operatorname{supp}(\langle\rho_{\gamma_j^{(m)}}\rangle)\backslash N^{(m)}$, $\mu \in \operatorname{supp}(\langle\rho_{\gamma_i^{(n)}}\rangle)\backslash N^{(n)}$, $1 \le j < m+1$, $1 \le i < n+1$ and $m,n = \infty,1,2,\ldots$.

Let $\delta_{\lambda,j}^{(m)}$ denote the function

$$\delta_{\lambda,j}^{(m)} : (\mu,n,i,t) \to \langle\mu,n,i \mid \lambda,m,j\rangle(t)$$

and U the unitary operator from X onto $Y = \bigoplus_{m=1}^{m=\infty} \bigoplus_{j=1}^{m} L_2(\mathbb{R},\rho_{\gamma_j^{(m)}})$. Put $B = UAU^*$. Then $\delta_{\lambda,j}^{(m)} \in T_{Y,B}$, and for $\hat{f} \in S_{Y,B}$

$$\hat{f} : (\mu,n,i) \to \hat{f}_i^{(n)}(\mu) \quad , \quad \hat{f}_i^{(n)} \in L_2(\mathbb{R},\rho_{\gamma_j^{(n)}})$$

and

$$\langle\hat{f},\delta_{\lambda,j}^{(m)}\rangle = \sum_{n=1}^{n=\infty} \sum_{i=1}^{n} \sum_{k=1}^{\infty} \left(\int_{\mathbb{R}} \hat{f}_i^{(n)}(\mu) \, \overline{\hat{v}_{k,i}^{(n)}(\mu)} \, d\rho_{\gamma_i^{(n)}}(\mu) \right) v_{k,j}^{(m)}(\lambda)$$

$$= f_j^{(m)}(\lambda) .$$

Hence

$$\langle \mu,n,i \mid \lambda,m,j\rangle = \delta_\lambda(\mu)\,\delta_{ji}\,\delta_{mn}\,.$$

Finally we give the adaptation of the closure property (5.8).

(5.11) Theorem

$$P^n = \sum_{m=1}^{m=\infty} \sum_{j=1}^{m} \int_{\mathbb{R}} \lambda^n \,|\lambda,m,j\rangle\langle\lambda,m,j|\, d\rho_{\gamma_j^{(m)}}(\lambda) \quad , \quad n = 0,1,2,\dots$$

with convergence of the integral in $T_{X\otimes X, A\boxplus A}$.

Here we do not intend to discuss the interpretation of Dirac's formalism for an n-set of commuting observables. The generalization to this case is immediate and rather trivial. All results remain valid in an adapted form. We only notice the nice way in which the definition of a complete set of commuting observables in the sense of Dirac can be expressed in our terminology.

(5.12) Proposition

The n-set $(P_1,\dots,P_n)$ is a complete set of commuting observables iff it has uniform multiplicity one.

Given an orthonormal basis in X. Every bounded linear operator B in X is uniquely represented by its matrix [B] with respect to this basis. The product of two operators B_1B_2 has matrix $[B_1B_2]$ which can be derived by formal matrix multiplication, $[B_1B_2]_{k\ell} = \sum_i [B_1]_{ki}\,[B_2]_{i\ell}$. Dirac assumes that the matrix notion can also be introduced in the case of Dirac bases, and that operating with these matrices runs similarly to the discrete case. Because of this assumption one can choose a representation so that *the representatives of the more abstract quantities occurring in the problem are as simple as possible.* Examples of such representations are the so called x- and p-representations.

Here we shall give a mathematical interpretation of this hypothesis of Dirac. We shall restrict ourselves to representations of observables with repsect to a complete set of generalized eigenfunctions of a cyclic self-adjoint operator. The general case of a non-cyclic self-adjoint operator or of a commuting n-set can be dealt with similarly.

Let $P : S_{X,A} \to S_{X,A}$ be a cyclic self-adjoint operator, and let $|\lambda>$, $\lambda \in \sigma(P)$, denote the eigenkets of P in $T_{X,A}$. The operator $P \otimes P$ is self-adjoint in $X \otimes X$, and maps $S_{X\otimes X, A \boxplus A}$ continuously into itself. Eigenkets in $T_{X\otimes X, A\boxplus A}$ of $P \otimes P$ are $|\lambda> \otimes\ |\mu>$, $\lambda, \mu \in \sigma(P)$. Following Dirac we shall denote the tensor product $|\lambda> \otimes\ |\mu>$ by $|\mu><\lambda|$ in the sequel. Every continuous linear mapping from $T_{X,A}$ into $S_{X,A}$ is derived from an element of $S_{X\otimes X, A\boxplus A}$, because of the Kernel theorem. With the methods we employed in the proof of Theorem (5.1) the following result can be shown.

(5.13) <u>Theorem</u>

Let $B \in S_{X\otimes X, A\boxplus A}$. Then

$$B = \iint_{\mathbb{R}^2} <\mu \mid B \mid \lambda>(0)\ |\mu><\lambda|\ d\rho_\gamma(\lambda)\ d\rho_\gamma(\mu)$$

where the integral converges in $T_{X\otimes X, A\boxplus A}$, and where

$$<\mu \mid B \mid \lambda>(t) = <e^{-t\,A\boxplus A}B\ ,\ |\lambda> \otimes\ |\mu>> .$$

We note that

$$e^{-tA\boxplus A}B = \iint_{\mathbb{R}^2} \langle\mu \mid B \mid \lambda\rangle(0)\ (|\mu\rangle\langle\lambda|)(t)\ d\rho_\gamma(\lambda)\ d\rho_\gamma(\mu)\quad ,\quad t > 0\ ,$$

where the integral converges absolutely in $X \otimes X$.

Similar to the one variable case $T_{X,A}$ (cf. (5.5)), Theorem (5.13) can be adapted such that it is valid for elements in $T_{X\otimes X,A\boxplus A}$.

(5.14) Theorem

Let $G \in T_{X\otimes X,A\boxplus A}$. Then we have with $\langle\mu \mid G \mid \lambda\rangle : t \mapsto \langle\mu \mid G(t) \mid \lambda\rangle$,

$$G = \iint_{\mathbb{R}^2} \langle\mu \mid G \mid \lambda\rangle\ |\mu\rangle\langle\lambda|\ d\rho_\gamma(\lambda)\ d\rho_\gamma(\mu)$$

where similarly to (5.5) the integral has to be understood in the following sense.

$$G : t \mapsto \iint_{\mathbb{R}^2} \langle\lambda \mid G \mid \mu\rangle(\tau)\ (|\mu\rangle\langle\lambda|)(t-\tau)\ d\rho_\gamma(\lambda)\ d\rho_\gamma(\mu)\ .$$

Here the integrals do not depend on the choice of τ, $0 < \tau < t$, and converge in $X \otimes X$.

With respect to the Dirac basis $(|\lambda\rangle)_{\lambda\in\sigma(P)}$ an element B, $B \in S_{X\otimes X,A\boxplus A}$, can be represented by the matrix [B] given by

(5.15) $$[B]_{\mu\lambda} = \langle\mu \mid B \mid \lambda\rangle(0)\quad ,\quad \mu,\lambda \in \sigma(P)$$

and following Theorem (5.13)

$$B = \iint_{\mathbb{R}^2} [B]_{\mu\lambda}\ |\mu\rangle\langle\lambda|\ d\rho_\gamma(\lambda)\ d\rho_\gamma(\mu)\ .$$

Further for $|F\rangle \in \mathcal{T}_{X,A}$, the ket $B|F\rangle$ is a test ket and

$$(5.16) \qquad B|F\rangle = \iint_{\mathbb{R}^2} \langle\mu \mid B \mid \lambda\rangle(-\tau) \langle\lambda \mid F\rangle(\tau) \, |\mu\rangle \, d\rho_\gamma(\lambda) \, d\rho_\gamma(\mu)$$

where $\tau > 0$ has to be taken so small that $B\, e^{\tau A} \in \mathcal{S}_{X\otimes X, A\boxplus A}$, and where the integral converges in $\mathcal{T}_{X,A}$ and does not depend on the choice of $\tau > 0$. Even convergence in $\mathcal{S}_{X,A}$ can be proved. Further

$$(5.17) \qquad \langle\mu \mid B|F\rangle(0) = \int_{\mathbb{R}} \langle\mu \mid B \mid \lambda\rangle(-\tau) \langle\lambda \mid F\rangle(\tau) \, d\rho_\gamma(\lambda)$$

where the integral converges absolutely. Note that $\langle\mu \mid B \mid \lambda\rangle(-\tau)$ exists because $B|F\rangle$ is a test ket for every ket $|F\rangle$.

The matrix notion can be extended to elements of $\mathcal{T}_{X\otimes X, A\boxplus A}$. To this end, let $G \in \mathcal{T}_{X\otimes X, A\boxplus A}$. Then with the expression $[G]$ we mean the set of functions

$$(5.18) \qquad [G]_{\mu\lambda} = \langle\mu \mid G \mid \lambda\rangle .$$

We note that $G(t) \in \mathcal{S}_{X\otimes X, A\boxplus A}$. The expression $[G]$ will be called the matrix of G. By Theorem (5.14) we have

$$G = \iint_{\mathbb{R}^2} [G]_{\mu\lambda} \, |\mu\rangle\langle\lambda| \, d\rho_\gamma(\lambda) \, d\rho_\gamma(\mu) .$$

Let $|f\rangle$ be a test ket. Then $G|f\rangle$ can be represented by

$$(5.19) \qquad G|f\rangle : t \mapsto \iint_{\mathbb{R}^2} [G]_{\mu\lambda}(\tau) \langle\lambda \mid f\rangle(-\tau) \, |\mu\rangle(t-\tau) \, d\rho_\gamma(\lambda) \, d\rho_\gamma(\mu)$$

where τ, $0 < \tau < t$, has to be taken so small that $|f\rangle(-\tau) \in \mathcal{S}_{X,A}$, and where the integrals converge absolutely in X and do not depend on the choice of $\tau > 0$. Further

$$(5.20) \qquad \langle\mu \mid G \mid f\rangle : t \to \int_{\mathbb{R}} [e^{-(t-\tau)A} G(\tau)]_{\mu\lambda} \langle\lambda \mid f\rangle(-\tau)\, d\rho_\gamma(\lambda)$$

where the integrals converge absolutely and do not depend on the choice of $\tau > 0$.

Similarly a matrix notion will be introduced for continuous linear mappings from $S_{X,A}$ into itself resp. $T_{X,A}$ into itself, or equivalently because of the Kernel theorem for elements in $T(S_{X\otimes X,\, I\otimes A}, A\otimes I)$ resp. $T(S_{X\otimes X, A\otimes I}, I\otimes A)$, i.e. the spaces $\sum'_B$ and $\sum'_A$ as introduced by De Graaf in [G], Ch. IV (cf. Ch. III).

For $R \in T(S_{X\otimes X, I\otimes A}, A\otimes I)$ the matrix representation $[R]$ is defined by

$$(5.21) \qquad [R]_{\mu\lambda} : (s,t) \mapsto \langle\mu \mid R(t) \mid \lambda\rangle(s)\,.$$

Note that $R(t) \in S_{X\otimes X, A\boxplus A}$ $t > 0$, fixed. So there exists $\sigma > 0$ such that $\langle\mu \mid R(t) \mid \lambda\rangle(-\sigma)$ is well-defined because $R(t)\,|\lambda\rangle$ is a test ket. It can be shown that

$$(5.22) \qquad R : t \mapsto R(t) = \iint_{\mathbb{R}^2} [R]_{\mu\lambda}(-\sigma,\tau)\, (|\lambda\rangle(t-\tau) \otimes |\mu\rangle(\sigma))\, d\rho_\gamma(\lambda)\, d\rho_\gamma(\mu)$$

where the integrals converge in $X \otimes X$ and do not depend on the choice of τ, $0 < \tau < t$ and of $\sigma > 0$ sufficiently small. We write

$$(5.23) \qquad R = \iint_{\mathbb{R}^2} [R]_{\mu\lambda}\, |\mu\rangle\langle\lambda|\, d\rho_\gamma(\lambda)\, d\rho_\gamma(\mu)$$

where the integral has to be interpreted in the sense of (5.22) and converges in $T_{X\otimes X, A\boxplus A}$ (even in $T(S_{X\otimes X, I\otimes A}, A\otimes I)$). Let $R' \in T(S_{X\otimes X, I\otimes A}, A\otimes I)$. Then the matrix of the product $R'R$ is given by

(5.24) $$[R'R]_{\mu\lambda} : (s,t) \mapsto \int_{\mathbb{R}} [R']_{\mu\nu}(s,\sigma)\,[R]_{\nu\lambda}(-\sigma,t)\,d\rho_\gamma(\nu)$$

where the integrals converge absolutely and do not depend on the choice of σ, and where $\sigma > 0$ has to be taken such that

$$e^{\sigma A} R(t) \in S_{X\otimes X, A\boxplus A}.$$

We write

(5.25) $$[R'R]_{\mu\lambda} = \int_{\mathbb{R}} [R']_{\mu\nu}\,[R]_{\nu\lambda}\,d\rho_\gamma(\nu)$$

where the integral converges in the indicated distributional sense. Further, let $|f\rangle$ be a test ket. Then $R|f\rangle$ is a test ket, also, and

(5.26) $$R|f\rangle = \iint_{\mathbb{R}^2} \langle\mu\,|\,R(\tau)\,|\,\lambda\rangle(0)\,\langle\lambda\,|\,f\rangle(-\tau)\,|\mu\rangle\,d\rho_\gamma(\lambda)\,d\rho_\gamma(\mu)$$

$$= \iint_{\mathbb{R}^2} [R]_{\mu\lambda}(-\sigma,\tau)\,\langle\lambda\,|\,f\rangle(-\tau)\,|\mu\rangle(\sigma)\,d\rho_\gamma(\lambda)\,d\rho_\gamma(\mu)$$

where the integral converges in $T_{X,A}$ and does not depend on the choice of $\tau > 0$ and of $\sigma > 0$ chosen sufficiently small as indicated in (5.21). Finally, we have

(5.27) $$\langle\mu\,|\,R\,|\,f\rangle : s \mapsto \int_{\mathbb{R}} [R]_{\mu\lambda}(s,\tau)\,\langle\lambda\,|\,f\rangle(-\tau)\,d\rho_\gamma(\lambda).$$

For $Q \in T(S_{X\otimes X,A\otimes I}, I\otimes A)$ its matrix $[Q]$ is defined by

(5.28) $$[Q]_{\mu\lambda} : (s,t) \to \langle\mu \mid Q(s) \mid \lambda\rangle(t) .$$

Note that $Q(s) \in S_{X\otimes X,A\boxplus A}$. So there exists $\tau > 0$ such that $\langle\mu \mid Q(s) \mid \lambda\rangle(-\tau)$ is well-defined because $Q(s)\,|\lambda\rangle$ is a test ket. It can be shown that

(5.29) $$Q : s \mapsto Q(s) = \iint_{\mathbb{R}^2} [Q]_{\mu\lambda}(\sigma,-\tau)\,(|\lambda\rangle(\tau) \otimes |\mu\rangle(s-\sigma)\,d\rho_\gamma(\lambda)\,d\rho_\gamma(\mu)$$

where σ, $0 < \sigma < s$, and where the integrals converge in $X \otimes X$ and do not depend on the choice of σ, and of $\tau > 0$ sufficiently small (cf. (5.21)). We write

(5.30) $$Q = \iint_{\mathbb{R}^2} [Q]_{\mu\lambda}\,(|\mu\rangle\langle\lambda|)\,d\rho_\gamma(\lambda)\,d\rho_\gamma(\mu)$$

where the integral has to be interpreted in the sense of (5.29) and converges in $T_{X\otimes X,A\boxplus A}$. Let $Q' \in T(S_{X\otimes X,A\otimes I}, I\otimes A)$. Then the matrix of the product $Q'Q$ is given by

(5.31) $$[Q'Q]_{\mu\lambda} : (s,t) \mapsto \int_{\mathbb{R}} [Q']_{\mu\nu}(s,-\tau)\,[Q]_{\nu\lambda}(\tau,t)\,d\rho_\gamma(\nu)$$

where the integrals converge absolutely and do not depend on the choice of τ, and where $\tau > 0$ has to be taken such that

$$Q'(t)e^{\tau A} \in S_{X\otimes X,A\boxplus A} .$$

We write

$$(5.32)\qquad [Q'Q]_{\mu\lambda} = \int_{\mathbb{R}} [Q'Q]_{\mu\nu}\,[Q]_{\nu\lambda}\,d\rho_\gamma(\nu)\ .$$

Again the integral converges in the above-mentioned distributional sense.

Further, $Q|H>$ can be represented by

$$(5.33)\qquad Q|H> \,:\, s \mapsto \iint_{\mathbb{R}^2} [Q]_{\mu\lambda}(\sigma,-\tau)\,<\lambda\mid H>(\tau)\,|\mu>(s-\sigma)\,d\rho_\gamma(\lambda)\,d\rho_\gamma(\mu)$$

where the integrals converge absolutely in X for every $s > 0$ and do not depend on the choice of σ, $0 < \sigma < s$, and $\tau > 0$, and where $\tau > 0$ has to be taken such that $Q(\sigma)e^{\tau A} \in S_{X\otimes X, A\boxplus A}$.

Finally, note that

$$(5.34)\qquad <\mu\mid Q\mid H> \,:\, s \mapsto \int_{\mathbb{R}} [Q_{\mu\lambda}](s,-\tau)\,<\lambda\mid H>(\tau)\,d\rho_\gamma(\lambda)\,.$$

Remark

The proofs of most results we gave in the last part of this section become more transparant by the following relation:

Let $B \in S_{X\otimes X, A\boxplus A}$, and let $t_1 > 0$ and $t_2 > 0$. Then

$$(e^{-t_1 A}\otimes e^{-t_2 A})B = \iint <\mu\mid B\mid \lambda>(0)\,(|\lambda>(t_1)\otimes|\mu>(t_2))\,d\rho_\gamma(\lambda)\,d\rho_\gamma(\mu)\ .$$

The proof of this relation runs analogously to the proof of Theorem (5.1).

References to this section:

[An], [Bö], [Di], [Ja], [GeVi], [Mel], [Ro].

Appendix Some integration theory

In this appendix we shall introduce the notion of integral for functions from $\mathbb{R}$ into $S_{X,A}$, and, also, from $\mathbb{R}$ into $T_{X,A}$. Therefore, we introduce so-called integrable functions.

(a.1) Definition

Let f be a function from the interval [a,b] into $S_{X,A}$. Then f is called integrable if for all $\psi \in B_+(\mathbb{R})$ the function

$$s \mapsto \psi(A)\, f(s)$$

from [a,b] into X is integrable; so ${}_a\int^b \psi(A)\, f(s)\, ds \in X$.

Let $f : [a,b] \mapsto S_{X,A}$ be integrable. By taking $\psi = 1$ in the above definition it follows that f is also integrable as a function from $\mathbb{R}$ into X. Hence ${}_a\int^b f(s)\, ds$ exists and defines an element of X. Since $\psi(A)$ is a closed operator and since ${}_a\int^b \psi(A)\, f(s)\, ds \in X$ by definition (a.1) we obtain

$$\psi(A) \left(\int_a^b f(s)\, ds \right) = \int_a^b \psi(A)\, f(s)\, ds \in X .$$

Hence $({}_a\int^b f(s)\, ds) \in \bigcap_{\psi \in B_+(\mathbb{R})} D(\psi(A)) = S_{X,A}$. Cf. Ch. I, Section 1.
We shall call ${}_a\int^b f(s)\, ds$ the integral of the integrable function f over [a,b].

(a.2) Definition

Let F be a function from the interval [a,b] into $T_{X,A}$. Then F is called integrable over [a,b] if for all $t > 0$ the function

$$s \mapsto F(s;t)$$

from [a,b] into X is integrable,

i.e.

$$\int_a^b F(s;t)\,ds \in X\ .$$

Let $F : [a,b] \mapsto T_{X,A}$ be integrable over $[a,b]$. Then for all $t > 0$, the X-integral $_a\int^b F(s;t)\,ds \in X$. Let $_a\int^b F(s)\,ds$ denote the mapping

$$\left(\int_a^b F(s)\,ds\right) : t \mapsto \int_a^b F(s;t)\,ds\ .$$

Then for all $t > 0$ and $\tau > 0$

$$e^{-\tau A} \int_a^b F(s;t)\,ds \;=\; \int_a^b F(s;t+\tau)\,ds$$

due to the continuity of $e^{-\tau A}$ on X. Hence the expression $_a\int^b F(s)\,ds$ denotes an element of $T_{X,A}$. We shall call $_a\int^b F(s)\,ds$ the integral of F over $[a,b]$.

EPILOGUE

During the research we have examined several topics which are related to the theory of analyticity spaces and trajectory spaces. Not all of them are contained in the present thesis. In Chapter one we have already mentioned the papers on the characterization of the spaces $S^{k/k+1}_{1/k+1}$, $k \in \mathbb{N}$ and on the Hankel invariant distribution spaces $S_{L_2(0,\infty),A_\alpha}$, $\alpha > -1$. In short we shall sum up some other subjects of study.

For each of the four types of linear mappings which appear in our theory, a closed graph theorem is valid. A forthcoming paper will deal with this kind of theorems, and, also, with the characterization of closed subspaces. A closed subspace of the analyticity space $S_{X,A}$ is in 1 - 1 correspondence with an array of projections $(\pi_t)_{t>0}$ from X into X satisfying

$$\pi_t = e^{-\tau A}\pi_{t+\tau} \quad , \qquad t,\tau > 0 \ .$$

It will lead to the following result.

If P is a continuous injection on $S_{X,A}(T_{X,A})$ with a closed range in $S_{X,A}(T_{X,A})$, then its dual P' is a continuous surjection on $T_{X,A}(S_{X,A})$.

Another paper, that we are prepairing, deals with groups and semigroups of elements of T^A (T_A and E_A). Here, we are mainly looking for conditions on a linear operator L in X, which imply that L generates a semigroup of one of the mentioned types. We have been able to characterize the so-called holomorphic groups in the described way. However, with respect to the semigroups we stand at a starting point. A related topic is the so-called classification problem:

Given A and X, find conditions on B and Y such that

$$S_{X,A} = S_{Y,B} \ .$$

In $[E_3]$ we have considered the case in which B is obtained from A by means of a perturbation.

Finally, we shall devote some attention to a promising result of our research: another, new set up $[E_1]$ of a theory of generalized functions. This theory is a kind of reverse of De Graaf's theory. We summarize it here.

In $[E_1]$ we start with the evolution equation

(e.1) $\quad \frac{d\varphi}{dt} = A\varphi$

where A is a nonnegative, self-adjoint operator in the Hilbert space X. A solution φ of (e.1) is called an orbit if it satisfies

(e.2.i) $\quad \forall_{t\in\mathbb{C}} : \varphi(t) \in X$

(e.2.ii) $\quad \forall_{t\in\mathbb{C}}\forall_{\tau\in\mathbb{C}} : \varphi(t+\tau) = e^{\tau A}\varphi(t)$.

Each orbit φ is uniquely determined by its value $\varphi(0)$ and hence $\varphi(0) \in D((e^A)^\infty) = \bigcap_{n=1}^{\infty} D(e^{nA})$. With the seminorms

(e.3) $\quad \varphi \mapsto \|\varphi(t)\| \quad , \quad \varphi \in \tau(X,A)$

where $t \in \mathbb{C}$, the space $\tau(X,A)$ becomes a Frêchet space. The topological structure of $\tau(X,A)$ is similar to the topological structure of $T_{X,A}$.

The asperity space $\sigma(X,A)$ consists of elements F for which there exists $t > 0$ such that $e^{-tA}F \in X$. We have

(e.4) $\quad \sigma(X,A) = \bigcup_{t>0} X_t = \bigcup_{n\in\mathbb{N}} X_n$

where X_t denotes the completion of X with respect to the norm $\|\cdot\|_t$

(e.5) $\quad \|f\|_t = \|e^{-tA} f\| \quad , \quad f \in X$.

It is not hard to see that $X_t \subset X_\tau$ for $\tau > t$. On $\sigma(X,A)$ we impose an inductive limit topology. The inductive limit is not strict. Inspired by [G] we have been able to construct a set of seminorms which generate the given topology for $\sigma(X,A)$. The topological structure of the asperity space $\sigma(X,A)$ is similar to the topological structure of the analyticity space $S_{X,A}$.

The pairing between $\tau(X,A)$ and $\sigma(X,A)$ is defined by

(e.6) $\quad \langle\varphi,\Psi\rangle = (\varphi(t), e^{-tA}\Psi) \quad , \quad \varphi \in \tau(X,A)\,,\ \Psi \in \sigma(X,A)$.

Here $t > 0$ has to be taken so large that $e^{-tA}\Psi \in X$. Due to (e.2.ii) the definition does not depend on the choice of $t > 0$. The spaces $\tau(X,A)$ and $\sigma(X,A)$ are reflexive in the given topologies.

In addition, $[E_1]$ contains the characterization of the four types of continuous linear mappings

$$\tau(X,A) \to \tau(Y,B)\,,\quad \tau(X,A) \to \sigma(Y,B)\,,\quad \sigma(X,A) \to \tau(Y,B)\,,\quad \sigma(X,A) \to \sigma(Y,B)$$

and the introduction of four topological tensor products.

$$\tau(\sigma(X\otimes Y, A\otimes I), I\otimes B),\ \sigma(X\otimes Y, A\boxplus B),\ \tau(X\otimes Y, A\boxplus B) \text{ and}$$
$$\tau(\sigma(X\otimes Y, I\otimes B), A\otimes I)\ .$$

These tensor products lead to four Kernel theorems just as in [G].

Schwartz' space of tempered distributions $S'(\mathbb{R})$ can be regarded as the strong dual of the space $\mathcal{D}(H^\infty) \subset L_2(\mathbb{R})$ where

$$(e.7) \qquad H = \frac{1}{2}\left(-\frac{d^2}{dx^2} + x^2 + 1\right) .$$

The space $\mathcal{D}(H^\infty)$ becomes an orbital space in the following natural way. Let $u_0 \in \mathcal{D}(H^\infty)$. Then define the mapping $u : \mathbb{C} \to L_2(\mathbb{R})$ by

$$(e.8) \qquad u(t) := e^{t \log H} u_0 \quad , \quad t \in \mathbb{C} .$$

Then $\mathcal{D}(H^\infty)$ corresponds to the space $\tau(L_2(\mathbb{R}), \log H)$.
So the theory on orbital spaces and asperity spaces can be looked upon as a very general theory on distributions of the tempered kind.

In $[E_1]$ we also have shown that the space of Hermite pansions, introduced by Korevaar, [K], equals the asperity space

$$(e.9) \qquad \sigma(L_2(\mathbb{R}), H)$$

with H as in (e.7). Moreover, in $[E_2]$, [EG], we have discussed the Hankel invariant distribution spaces $\sigma(L_2(0,\infty), \log A_\alpha)$ and $\sigma(L_2(0,\infty), A_\alpha)$. The corresponding test function spaces $\tau(L_2(0,\infty), \log A_\alpha)$ and $\tau(L_2(0,\infty), A_\alpha)$ are described in classical analytic terms.

We conclude this epilogue with the following quintuple:

$$\tau(X,A) \subset S_{X,A} \subset X \subset T_{X,A} \subset \sigma(X,A) \quad .$$

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INDEX OF NOTATIONS

$C^{\omega}(A)$	:	the analyticity domain of A (25)
$C^{\omega}(U)$	:	the analyticity domain of a unitary Lie group representation (26)
$(G_{\lambda})_{\lambda\in\mathbb{R}}$, $(H_{\mu})_{\mu\in\mathbb{R}}$, etc.	:	spectral resolution of the identity of C, $\mathcal{D}$, etc. (30)
emb	:	embedding (11)
δ_x, $\delta_{x,j}^{(m)}$	:	delta function concentrated in x (142)
δ_{ij}	:	Kronecker delta
$Q_h(x)$	:	cube with centre x (147)
supp(μ)	:	support of the Borel measure μ (152)
$\sigma(P)$	:	spectrum of the operator P (158)
$\|F\rangle$, $\|G\rangle$,...	:	kets (166)
$\langle F\|$, $\langle G\|$,...	:	bras (167)
$\langle F \| G\rangle$	:	the bracket of $\langle F\|$ and $\|G\rangle$ (167)
$\|G\rangle\langle F\|$	:	the tensor produkt of $\|F\rangle$ and $\|G\rangle$ (172)
$(\|\lambda\rangle)$, $(\|\lambda,m,j\rangle)$	:	Dirac basis (175)
$(\langle\mu\| B \|\lambda\rangle)$	:	matrix of B with respect to the Dirac basis $(\|\lambda\rangle)$ (177)
S_{α}^{β}	:	Gelfand-Shilov space of type S (22)
$\mathbb{H}_{\alpha}$	:	Hankel transform of order α (23)
J_{α}	:	Bessel function of order α (23)
$L_n^{(\alpha)}$	:	n-th Laguerre function of order α (23)
ψ_n	:	n-th Hermite function (142)

INDEX OF TERMS

SAMENVATTING

Geïnspireerd door het artikel [B] van De Bruijn, heeft de Graaf een theorie van gegeneraliseerde functies ontwikkeld. Deze theorie [G] wordt gekenmerkt door haar vrij strakke functionaal analytische aanpak. Zij is gebaseerd op het triplet

$$S_{X,A} \subset X \subset T_{X,A}$$

waarbij X een Hilbert-ruimte is, en A een niet-negatieve zelf-geadjungeerde operator in X. Het eerste gedeelte van [G] bevat de gebruikelijke aspekten van een distributietheorie, zoals de definitie van de testfunctieruimte ($S_{X,A}$) en van de distributieruimte ($T_{X,A}$), en van hun paring. Aldus kunnen $S_{X,A}$ en $T_{X,A}$ gezien worden als elkaars duale.

Het tweede gedeelte is minder conventioneel. De theorie [G] onderscheidt zich hierin van andere distributietheorieën. In dit gedeelte staan gedetailleerde karakterisaties van vijf soorten continue lineaire afbeeldingen. Voorts worden er vier soorten topologische tensor produkten ingevoerd, die aanleiding geven tot vier Kern-stellingen. We merken op dat een Kern-stelling (Kernel theorem) voorwaarden levert waaronder alle lineaire afbeeldingen van een bepaald type voorgesteld kunnen worden door kernen uit een geschikt gekozen topologisch tensor produkt. In gangbare distributietheorieën heeft het begrip Kern-stelling een zwakkere betekenis en wordt veelal slechts gegeven voor de continue lineaire afbeeldingen van de testfunctieruimte in de distributieruimte.

In dit proefschrift wordt de theorie van De Graaf verder uitgebouwd en in verband gebracht met andere wiskundige disciplines. Dit laatste heeft ertoe geleid ruimten van het type $S_{X,A}$ analyticiteitsruimten te noemen. Het blijkt namelijk dat de verzameling $S_{X,A}$ juist het analyticiteits domein van de operator A in X is. De elementen van $T_{X,A}$ kunnen opgevat worden als trajecten in de Hilbert-ruimte X. Vandaar dat $T_{X,A}$ de trajectruimte heet.

In Hoofdstuk I wordt de theorie [G] samengevat. Daarnaast komen enige voorbeelden van analyticiteitsruimten aan de orde.

In het tweede en derde hoofdstuk wordt het tweede gedeelte van [G] verder uitgewerkt. Zoals vermeld zijn daar de topologische tensor produkten Σ'_A, Σ'_B en Σ_A, Σ_B ingevoerd maar de beschrijving in [G] van deze ruimten is niet erg doorzichtig. Omdat de ruimten Σ'_A en Σ'_B de kernen bevatten van de continue lineaire afbeeldingen van $S_{X,A}$ in $S_{Y,B}$ resp. $T_{X,A}$ in $T_{Y,B}$, is het zaak hun topologische structuur goed in de vingers te krijgen. Daarom hebben we twee typen ruimten ingevoerd, die bepaald worden door een Hilbert-ruimte Z en een paar commuterende niet-negatieve, onbegrensde, zelf-geadjungeerde operatoren in Z. De ruimten Σ'_A en Σ'_B zijn van het ene type, Σ_A en Σ_B van het andere. Op de nieuw ingevoerde ruimten worden topologieën geïntroduceerd, wordt een paring gegeven en bovendien worden hun doorsneden gekarakteriseerd.

De verkregen resultaten worden gebruikt in de beschrijving van de operator theorie voor analyticiteitsruimten en trajectruimten. Genoemde beschrijving heeft geleid tot een (vijfde) Kern-stelling voor de zgn. uitbreidbare lineaire afbeeldingen. Verder heeft deze beschrijving de bestudering verlicht van de algebra's van continue lineaire afbeeldingen van $S_{X,A}$ in zichzelf, c.q. van $T_{X,A}$ in zichzelf en van de uitbreidbare lineaire afbeeldingen. Laatstgenoemde algebra dient in dit proefschrift als een mathematisch model voor quantum statistiek. Als de ruimte $S_{X,A}$ nucleair is, bezit iedere continue lineaire afbeelding op $S_{X,A}$ een goed gedefinieerde matrix representatie. De oneindige matrices corresponderende met deze afbeeldingen zijn zeer eenvoudig te karakteriseren (in tegenstelling tot matrices van begrensde lineaire operatoren op een Hilbert-ruimte). We krijgen aldus een groot reservoir aan voorbeelden. Het vermelden waard zijn de zgn. verschuivingsoperatoren, die nader zijn onderzocht. Tenslotte heeft de matrixcalculus geleid tot de constructie van nucleaire analyticiteitsruimten waarop een eindig aantal operatoren in X continu is.

Het vierde hoofdstuk staat tamelijk los van de overige hoofdstukken. Hierin wordt een theorie van gegeneraliseerde eigenfuncties ontwikkeld die gebaseerd is op de theorie van gegeneraliseerde functies van De Graaf. We beperken ons tot nucleaire analyticiteitsruimten $S_{X,A}$ en tot zelf-geadjungeerde operatoren P in X die continu zijn op $S_{X,A}$. De commutatieve multipliciteitstheorie voor zelf-geadjungeerde operatoren speelt een belangrijke rol. Als P continu is op de nucleaire ruimte $S_{X,A}$ en als $\lambda \in \sigma(P)$ multipliciteit m_λ heeft, dan bestaan er tenminste m_λ (gegeneraliseerde) eigenvectoren in $T_{X,A}$ met (gegeneraliseerde) eigenwaarde λ. Een gelijksoortige uitspraak geldt voor een eindig aantal commuterende zelf-geadjungeerde operatoren.

Het tweede gedeelte van dit hoofdstuk is gewijd aan een wiskundige interpretatie van het Dirac formalisme. We hebben het bracket begrip van Dirac zodanig geïnterpreteerd dat het "inwendig produkt" tussen twee gegeneraliseerde functies mathematisch zinvol wordt. Een aantal aspekten van het Dirac formalisme krijgen aldus een wiskundige betekenis. We noemen hier: Fourierontwikkeling t.a.v. Dirac bases, quasi-orthogonaliteit van eigenkets en matrixcalculus m.b.t. Dirac bases.

CURRICULUM VITAE

De schrijver van dit proefschrift is op 13 november 1956 geboren te Sint-Oedenrode. In 1974 verwierf hij het einddiploma Gymnasium-β aan het Gymnasium Bernrode te Heeswijk-Dinther. Daarna studeerde hij wiskunde aan de T.H.-Eindhoven. Zijn afstudeeronderzoek werd verricht onder leiding van Prof.dr. S.T.M. Ackermans en had betrekking op asymptotische Fuglede stellingen. In december 1979 behaalde hij het diploma wiskundig ingenieur. Sindsdien is hij werkzaam als wetenschappelijk assistent bij Prof.dr.ir. J. de Graaf, eerst in dienst van de T.H.-Eindhoven, daarna in dienst van Z.W.O.

STELLINGEN

behorende bij het proefschrift

ANALYTICITY SPACES, TRAJECTORY SPACES

AND LINEAR MAPPINGS BETWEEN THEM

door

S.J.L. van Eijndhoven

- 1 -

De door McKennon ingevoerde testfunctieruimte Z is gelijk aan $\tau(L_2(\mathbb{R}),H^{\frac{1}{2}})$ waarbij $H = \frac{1}{2}\left(\frac{-d^2}{dx^2} + x^2 + 1\right)$. Zijn bewering dat deze ruimte invariant is onder de Laplace-transformatie is onjuist.

Literatuur: [E], [McK].

- 2 -

Zij N een begrensde normale operator op een Hilbert-ruimte en zij B een begrensde lineaire operator. Dan volgt uit $N(NB-BN) - (NB-BN)N = 0$ dat $NB - BN = 0$. Deze stelling heeft asymptotische uitbreidingen in termen van een vrij algemene klasse van operatortopologieën, waaronder de uniforme, de sterke en de zwakke operatortopologie.

Literatuur: [AEM].

- 3 -

Met behulp van de wiskundige interpretatie van het Dirac-formalisme uit dit proefschrift alsmede de operatortheorie voor analyticiteits- en trajectruimten is het mogelijk de (anti-) commutatierelaties CCR en CAR voor quantumveldoperatoren wiskundig te funderen.

II.

- 4 -

Laat $0 < \rho < 1$, $k \in \mathbb{N}$ en $\ell \in \mathbb{N}$. Dan is de Gelfand-Shilov ruimte

$$S^{\frac{1}{\rho}\cdot\frac{k}{k+\ell}}_{\frac{1}{\rho}\frac{\ell}{k+\ell}} \text{ bevat in de analyticiteitsruimte } S_{L_2(\mathbb{R}),\left(\left(\frac{-d^2}{dx^2}\right)^\ell + (x^2)^k\right)^{\frac{k+\ell}{k\ell}\cdot\frac{\rho}{2}}}.$$

Voor $\ell = 1$ ēn $\rho = 1$ geldt ook het omgekeerde.

Literatuur: [EGP].

- 5 -

Veronderstel dat in de locaal convexe Hausdorff topologische vectorruimte R de continuiteit van lineaire functionalen reeds door nulrijen beschreven kan worden. Dan is een lineaire deelruimte V in R gesloten dan en slechts dan als V rijgesloten is.

- 6 -

Zij P een lineaire afbeelding van $S_{X,A}$ in $S_{Y,B}$ met een gesloten grafiek in $S_{X,A} \times S_{Y,B}$. Dan is P continu.

Zij P een continue lineaire injectie (surjectie) van $S_{X,A}$ in $S_{Y,B}$ zodat $P(S_{X,A})$ gesloten is in $S_{Y,B}$. Dan is $P' : T_{Y,B} \to T_{X,A}$ een continue surjectie (injectie met $P'(T_{Y,B})$ gesloten in $T_{X,A}$).

De Weyl quantizatie ΩK van het symbool K kan geschreven worden als

$$\Omega K = \frac{2}{\sqrt{2\pi}}\, \Phi \int_{\mathbb{R}^2} K(\alpha,\beta)\, \exp(2i(-\beta\, Q + \alpha\, P))\, d\alpha\, d\beta$$

met $P = -i\,\frac{d}{dx}$, $Q = x$ en Φ de pariteitsoperator.

Schrijf $X = L_2(\mathbb{R}^+ \times S^1, r dr d\varphi)$, $H_p = -\frac{1}{8}\left(\frac{1}{r}\frac{\partial}{\partial r}\, r\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}\right) + \frac{r^2}{2} + \frac{1}{2}$

en $M_p = -\frac{1}{2}\, i\,\frac{\partial}{\partial\varphi}$. Dan bevatten de ruimten $T(S_{X,H_p+M_p}, H_p - M_p)$ en $T(S_{X,H_p-M_p}, H_p + M_p)$ juist de Weylsymbolen van de continue lineaire afbeeldingen van $S_{L_2(\mathbb{R}),H}$ in zichzelf, resp. van $T_{L_2(\mathbb{R}),H}$ in zichzelf, waarbij $H = \frac{1}{2}\left(\frac{-d^2}{dx^2} + x^2 + 1\right)$.

Voor $K \in T(S_{X,H_p+M_p}, H_p - M_p)$ worden de matrixelementen van de operator $\Omega K : S_{L_2(\mathbb{R}),H} \to S_{L_2(\mathbb{R}),H}$ t.o.v. de Hermite basis gegeven door

$$(\Omega K)_{nm} = \overline{\langle\!\langle \Lambda_n^{(n-m)}, K \rangle\!\rangle_X} \quad , \quad n,m \in \mathbb{N} \cup \{0\}\,.$$

Hierbij is

$$\Lambda_n^{(n-m)}(r,\varphi) = \sqrt{\frac{2}{\pi}}\,(-1)^n\, e^{i(n-m)\varphi} \sqrt{\frac{m!}{n!}}\,(r\sqrt{2})^{n-m}\, e^{-r^2}\, L_m^{(n-m)}(2r^2)$$

en

$$L_m^{(n-m)}(t) = \frac{e^t}{m!\, t^{n-m}} \left(\frac{d}{dt}\right)^m (t^n e^{-t}) \quad , \quad t > 0\,.$$

Literatuur: [Da], [Pe].

- 8 -

De Hankel invariante testfunctieruimte $\mathcal{H}_\mu$ van Zemanian is gelijk aan

$$\tau\left(L_2(0,\infty)\ ,\ \log\left(\frac{-d^2}{dx^2} + x^2 + \frac{\mu^2 - \frac{1}{4}}{x^2} - 2\mu\right)\right).$$

Literatuur: [EG], [Z].

- 9 -

Laat R een positieve zelf-geadjungeerde Hilbert-Schmidt operator zijn op de Hilbert-ruimte X. Laat D een (onbegrensde) lineaire operator zijn in X, die de deelruimte $R(X)$ afbeeldt in $L_2(\mathbb{R},\mu)$ waarbij μ een finiete niet-negatieve Borelmaat is. Veronderstel dat DR een goed gedefinieerde Hilbert-Schmidt operator is van X in $L_2(\mathbb{R},\mu)$. Dan bestaat er een nulverzameling N_μ zo dat voor alle $f \in R(X)$ en alle $x \in \mathbb{R}\backslash N_\mu$

$$(Df)(x) = \lim_{h\downarrow 0} \mu(Q_h(x))^{-1} \int_{Q_h(x)} (Df)\,d\mu$$

waarbij $Q_h(x) = [x-h, x+h]$.

De lineaire functionalen

$$f \mapsto (Df)(x) \quad , \quad f \in R(X)$$

zijn continu t.a.v. de norm $\|\cdot\|_R$ op $R(X)$ gedefinieerd door

$$\|f\|_R = \|R^{-1}f\|_X \ .$$

Deze stelling is een maattheoretische generalisatie van de inbeddingstheorema's van Sobolev.

De Leidsche Bul werd vroeger op de studentensoos bij grote hoeveelheden verorberd. Dit baksel verdient daarom eerder de naam 'studentengebak' dan het oudvaderlandse 'saucijzenbroodje'. Vaak wordt de naam van dit laatste baksel al te scherpzinnig verklaard als 'panis socialis in usum studiosorum'. Maar uit niets blijkt dat studenten speciaal het saucijzenbroodje de voorkeur gaven.

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