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FOR MARKOV PROCESSES

PROEFSCHRIFT

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CONTENTS

PREFACE

This thesis is divided into three chapters. The first chapter contains the basic material which is needed in the next two chapters. The sections 1.1 - 1.4 are well known, and presented here for later reference. In section 1.5 a systematical outline is given of the Markov processes induced by a measurable transformation. In particular, a representation theorem for the backward processes is obtained (cf. theorem 1.5.1).

In the second chapter some properties of Markov measures on product spaces and of Markov shifts are studied. The first section collects some well known facts on conservativity of Markov processes, and treats conservativity for backward processes induced by ergodic measure preserving transformations in a probability space (cf. theorem 2.1.1).

Section 2.2 mainly deals with Markov measures on two-sided product spaces. Using a method due to Kakutani [14], a criterion is derived for the singularity of two Markov probabilities for the same process on the two-sided product space (cf. theorem 2.2.1 and theorem 2.2.2).

In section 2.3 the connection between the conservative part of a Markov process and the conservative part of the corresponding Markov shift is studied. The results, given in theorems 2.3.1 and 2.3.2, extend a result of Harris and Robbins [7] in the measure preserving case.

In the last section of chapter II the results of the previous sections are applied to shift spaces for the Markov processes induced by a transformation. In particular, an example is obtained of an invertible transformation in a probability space for which there exists an algebra of recurrence sets, which generates the σ-algebra, while nevertheless the transformation is dissipative.

The last chapter is largely independent of chapter II. It deals with a general definition of periodicity and aperiodicity for Markov processes. Since these concepts depend on the class of invariant sets and the so called deterministic σ-algebra, the first two sections are devoted to a study of these subjects. In particular, in section 3.1 a characterization of the essential part by means of invariant sets is given.

Section 3.3 collects some facts on Markov chains. In section 3.4 the various existing definitions of periodicity are compared. It turns out that Moy's definition [17] of periodicity for irreducible Markov processes does not always agree with the definition of periodicity for ergodic transformations, and moreover is not always applicable. Therefore another definition of periodicity is given (cf. definition 3.4.3) which can be applied to all Markov processes and reduces to the existing definitions for Markov chains and transformations. Under this definition in general we have to distinguish between period 1 and aperiodicity. Some properties of periodic Markov processes are derived. Finally, in section 3.5 the limit behaviour of periodic Markov processes for which a subinvariant equivalent measure exists is studied. The limit theorem 3.5.1 is obtained by a method given by Foguel [4] and reduces to the well known limit theorem for Markov chains.

CHAPTER I

BASIC CONCEPTS

1.1. PRELIMINARIES

Let $(X,\mathcal{R})$ be a measurable space, i.e. $\mathcal{R}$ is a σ-algebra of subsets of a non empty set X. As far as measure theoretic concepts are concerned, we shall adhere to the terminology used in Halmos [5] and Neveu [18]. In addition, let us agree on the following conventions: If not stated otherwise, a measure will mean a non negative extended real valued σ-additive function on $\mathcal{R}$. Statements about subsets of a measure space $(X,\mathcal{R},\mu)$ will have to be interpreted modulo μ-null sets in $\mathcal{R}$, and statements on functions on $(X,\mathcal{R},\mu)$ will hold μ-almost everywhere on X. $M^+(X,\mathcal{R},\mu)$ will stand for the space of (equivalence classes of) non negative extended real valued $\mathcal{R}$-measurable functions on X.

For a proof of the following proposition the reader is referred to [11], 19.27 and 19.44.

PROPOSITION 1.1.1 (Radon-Nikodym). Let $(X,\mathcal{R},\mu)$ be a σ-finite measure space. The relation

$$\nu(A) = \int_A f d\mu \qquad \text{for all } A \in \mathcal{R}$$

establishes a one-to-one correspondence between the class of measures on $(X,\mathcal{R})$ which are absolutely continuous with respect to μ and the space $M^+(X,\mathcal{R},\mu)$. If we denote the function $f \in M^+(X,\mathcal{R},\mu)$ corresponding to the measure $\nu << \mu$ by $\frac{d\nu}{d\mu}$, then the following statements hold:

i) ν is finite if and only if $\frac{d\nu}{d\mu} \in \mathcal{L}_1^+(X,\mathcal{R},\mu)$.

ii) ν is σ-finite if and only if $\frac{d\nu}{d\mu} < \infty$.

iii) $\nu \approx \mu$ if and only if $\frac{d\nu}{d\mu} > 0$.

iv) If $\nu_0 << \nu_1$, $\nu_1 << \mu$ and ν_1 is σ-finite, then

$$\frac{d\nu_0}{d\nu_1}\frac{d\nu_1}{d\mu} = \frac{d\nu_0}{d\mu} .$$

If P is a linear operator in $\mathcal{L}_1(X,\mathcal{R},\mu)$ then the image of a function $f \in \mathcal{L}_1(X,\mathcal{R},\mu)$ under P will be denoted by fP. Similarly, if Q is a linear operator in $\mathcal{L}_\infty(X,\mathcal{R},\mu)$, then the image of a function $g \in \mathcal{L}_\infty(X,\mathcal{R},\mu)$ will be denoted by Qg.

Let P be a bounded linear operator in $\mathcal{L}_1(X,\mathcal{R},\mu)$. For every $g \in \mathcal{L}_\infty(X,\mathcal{R},\mu)$ the functional φ_g defined by

$$\varphi_g(f) = \int_X (fP)g\,d\mu \qquad \text{for all } f \in \mathcal{L}_1(X,\mathcal{R},\mu)$$

is a bounded linear functional on $\mathcal{L}_1(X,\mathcal{R},\mu)$. It follows from [2], IV.8.5 that there exists a unique function $Pg \in \mathcal{L}_\infty(X,\mathcal{R},\mu)$ such that

$$(1) \qquad \int (fP)g\,d\mu = \int f(Pg)\,d\mu \qquad \text{for all } f \in \mathcal{L}_1(X,\mathcal{R},\mu) \text{ and for all } g \in \mathcal{L}_\infty(X,\mathcal{R},\mu) .$$

The mapping $g \to Pg$ for all $g \in \mathcal{L}_\infty(X,\mathcal{R},\mu)$ is said to be the adjoint operator of the operator P in $\mathcal{L}_1(X,\mathcal{R},\mu)$, and will again be denoted by P, but now written to the left of the functions. The adjoint operator in $\mathcal{L}_\infty(X,\mathcal{R},\mu)$ is bounded and linear, but in general not every bounded linear operator in $\mathcal{L}_\infty(X,\mathcal{R},\mu)$ is the adjoint of a bounded linear operator in $\mathcal{L}_1(X,\mathcal{R},\mu)$.

PROPOSITION 1.1.2. A bounded positive linear operator P in $\mathcal{L}_\infty(X,\mathcal{R},\mu)$ is the adjoint of a bounded positive linear operator in $\mathcal{L}_1(X,\mathcal{R},\mu)$ if and only if for every sequence $(g_n)_{n=1}^\infty$ in $\mathcal{L}_\infty(X,\mathcal{R},\mu)$ such that $g_n \downarrow 0$ if $n \to \infty$, we have $\lim_{n\to\infty} Pg_n = 0$.

Here and henceforth, by $\lim_{n\to\infty} f_n$ we mean the pointwise limit of the sequence of functions $(f_n)_{n=1}^\infty$.

We only sketch the proof of this proposition. The necessity of the condition follows from relation (1) and the dominated convergence theorem. Conversely, if a bounded linear operator P in $\mathcal{L}_\infty(X,\mathcal{R},\mu)$ satisfies the condition of the proposition, then for every $f \in \mathcal{L}_1(X,\mathcal{R},\mu)$ the set function ν defined on $\mathcal{R}$ by

$$\nu(A) = \int f(P1_A)d\mu \qquad \text{for all } A \in \mathcal{R}$$

turns out to be a finite signed measure (cf. [5], § 28) such that $\nu \ll \mu$. It follows from the Radon-Nikodym theorem that there exists a unique function $fP = \frac{d\nu}{d\mu} \in \mathcal{L}_1(X,\mathcal{R},\mu)$ such that

$$\int_X (fP)1_A \, d\mu = \int_X f(P1_A)d\mu \qquad \text{for all } f \in \mathcal{L}_1(X,\mathcal{R},\mu) \text{ and all } A \in \mathcal{R},$$

from which we easily deduce relation (1). From this relation it follows that the mapping $f \to fP$ for all $f \in \mathcal{L}_1(X,\mathcal{R},\mu)$ is a bounded linear operator in $\mathcal{L}_1(X,\mathcal{R},\mu)$.

Let $B(X,\mathcal{R})$ be the Banach space of the bounded $\mathcal{R}$-measurable functions on X with the supremum norm $\|f\| = \sup_{x \in X} |f(x)|$ (cf. [2], IV.2.12).

For every $A \in \mathcal{R}$ define the operator I_A in $B(X,\mathcal{R})$ by

$$I_A f = 1_A f \qquad \text{for all } f \in B(X,\mathcal{R}) .$$

Obviously, I_A is a positive linear operator in $B(X,\mathcal{R})$ satisfying $\|I_A f\| \le \|f\|$ for all $f \in B(X,\mathcal{R})$. For every measure μ on $(X,\mathcal{R})$ the operator I_A induces an operator in $\mathcal{L}_\infty(X,\mathcal{R},\mu)$, which we again denote by I_A. This operator I_A is the adjoint of an operator in $\mathcal{L}_1(X,\mathcal{R},\mu)$ given again by

$$fI_A = 1_A f \qquad \text{for all } f \in \mathcal{L}_1(X,\mathcal{R},\mu) .$$

To conclude this section, we recall the concept of the conditional expectation operator. Let $(X,\mathcal{R},\mu)$ be a measure space and let $\mathcal{R}_0$ be a sub σ-algebra of $\mathcal{R}$ such that the measure space $(X,\mathcal{R}_0,\mu)$ is σ-finite. Choose $f \in M^+(X,\mathcal{R},\mu)$ and define the measure ν on $\mathcal{R}_0$ by

$$\nu(A) = \int_A f d\mu \qquad \text{for all } A \in \mathcal{R}_0 .$$

Then $\nu \ll \mu$ and by proposition 1.1.1 applied to the σ-finite measure space $(X,\mathcal{R}_0,\mu)$ there exists a unique function $E^\mu_{\mathcal{R}_0} f \in M^+(X,\mathcal{R}_0,\mu)$ such that for all $A \in \mathcal{R}_0$ we have

$$\int_A (E^\mu_{\mathcal{R}_0} f) d\mu = \int_A f d\mu .$$

This equality can be easily extended to

$$\int_A g(E^{\mu}_{\mathcal{R}_0} f)d\mu = \int_A gfd\mu \quad \text{for every } g \in M^+(X,\mathcal{R}_0,\mu) \text{ and every } A \in \mathcal{R}_0 ,$$

from which we derive

$$E^{\mu}_{\mathcal{R}_0}(gf) = g(E^{\mu}_{\mathcal{R}_0} f) \quad \text{for all } f \in M^+(X,\mathcal{R},\mu) \text{ and all } g \in M^+(X,\mathcal{R}_0,\mu) .$$

For every $f \in \mathcal{L}_1(X,\mathcal{R},\mu)$ we define

$$fE^{\mu}_{\mathcal{R}_0} = E^{\mu}_{\mathcal{R}_0} f^+ - E^{\mu}_{\mathcal{R}_0} f^-$$

where $f^+ = \max(f,0)$ and $f^- = \max(-f,0)$.

For every $g \in \mathcal{L}_\infty(X,\mathcal{R},\mu)$ we define

$$E^{\mu}_{\mathcal{R}_0} g = E^{\mu}_{\mathcal{R}_0} g^+ - E^{\mu}_{\mathcal{R}_0} g^- .$$

Now it is easily verified that the operators $E^{\mu}_{\mathcal{R}_0}$ defined in $\mathcal{L}_1(X,\mathcal{R},\mu)$ and $\mathcal{L}_\infty(X,\mathcal{R},\mu)$ are linear, bounded and positive, and satisfy

$$\int_X (fE^{\mu}_{\mathcal{R}_0})gd\mu = \int_X f(E^{\mu}_{\mathcal{R}_0} g)d\mu \quad \text{for all } f \in \mathcal{L}_1(X,\mathcal{R},\mu) \text{ and all } g \in \mathcal{L}_\infty(X,\mathcal{R},\mu) .$$

1.2. TRANSITION PROBABILITIES

In this section we collect some well known facts on transition probabilities. Most of this material can be found in Neveu [18], III.2 and V.

DEFINITION 1.2.1. Let $(X,\mathcal{R})$ and $(X',\mathcal{R}')$ be measurable spaces. A transition probability P from $(X,\mathcal{R})$ to $(X',\mathcal{R}')$ is a function P on $X \times \mathcal{R}'$ such that

i) for all $A \in \mathcal{R}'$ $P(\cdot,A)$ is an $\mathcal{R}$-measurable function on X;
ii) for all $x \in X$ $P(x,\cdot)$ is a probability on $\mathcal{R}'$.

PROPOSITION 1.2.1. For every measure μ on $(X,\mathcal{R})$ there exists a measure M on $(Y,\mathcal{T}) = (X,\mathcal{R}) \times (X',\mathcal{R}')$ such that

$$M(A) = \int_X \mu(dx) \int_{X'} 1_A(x,x')P(x,dx')$$

for every $A \in \mathcal{T}$.

PROOF. This proposition is an obvious extension of proposition III.2.1 in Neveu [18].

By a transition probability on $(X,\mathcal{R})$ we shall mean a transition probability from $(X,\mathcal{R})$ to $(X,\mathcal{R})$. A transition probability on $(X,\mathcal{R})$ gives rise to two operators, one acting on the class of measures on $(X,\mathcal{R})$, the other one acting in $B(X,\mathcal{R})$.

DEFINITION 1.2.2. Let P be a transition probability on $(X,\mathcal{R})$. For every measure μ on $(X,\mathcal{R})$ the measure μP is defined for every $A \in \mathcal{R}$ by

$$(\mu P)(A) = \int P(x,A)\mu(dx) \ .$$

For every $f \in B(X,\mathcal{R})$ the function Pf is defined by

$$(Pf)(x) = \int f(y)P(x,dy) \quad \text{for every } x \in X \ .$$

If we interprete $P(x,A)$ as the probability that a process will move in one transition from state x into event A, then the measure μP will be the measure on $(X,\mathcal{R})$ at time 1, if the measure on $(X,\mathcal{R})$ at time 0 was given by μ.

PROPOSITION 1.2.2. P is a positive linear operator in $B(X,\mathcal{R})$ satisfying $P1 = 1$. Moreover, for every sequence $(f_n)_{n=1}^{\infty}$ in $B(X,\mathcal{R})$ for which $f_n \downarrow 0$ if $n \to \infty$, we have $Pf_n \downarrow 0$ if $n \to \infty$.

For every measure μ on $(X,\mathcal{R})$ and for every non negative function $f \in B(X,\mathcal{R})$ we have $\mu P(f) = \mu(Pf)$, where $\mu(f) = \int f d\mu$.

PROOF. The first statement follows from the definition of P and the dominated convergence theorem. The second statement is by definition true for characteristic functions. The general validity then follows by monotone approximation.

DEFINITION 1.2.3. For every integer t let $(X_t,\mathcal{R}_t)$ be a copy of $(X,\mathcal{R})$, and define

$$(\Omega',\mathcal{A}') = \prod_{t=0}^{\infty} (X_t,\mathcal{R}_t) .$$

For every t let π'_t be the projection of Ω' on X_t, and define $\mathcal{A}'_t = \pi'^{-1}_t \mathcal{R}_t$. For $0 \le n < m \le \infty$ let $\mathcal{A}'_{nm}$ be the σ-algebra generated by the σ-algebras $\mathcal{A}_t$ for $n \le t \le m$.

The shift S' is the mapping $S' : \Omega' \to \Omega'$ defined by $\pi'_t S'\omega' = \pi'_{t+1}\omega'$ for all t and all $\omega' \in \Omega'$.

The notation $(\Omega,\mathcal{A})$ without primes will be used to denote the two-sided product space, which we shall meet in the sequel more frequently than the one-sided product space.

PROPOSITION 1.2.3 (Ionescu Tulcea). For every $x \in X$ let the set function P_x for every rectangle $\prod_{t=0}^{\infty} A_t$ (i.e. $A_t \ne X_t$ for only finitely many t) be defined by

$$P_x\left(\prod_{t=0}^{\infty} A_t\right) = (I_{A_0} P I_{A_1} P \ldots P I_{A_{T-1}} P I_{A_T})(x)$$

where T is chosen so large that $A_t = X_t$ for $t > T$. Then P_x can be extended to a probability on $\mathcal{O}\!\mathcal{L}'$. For every $A \in \mathcal{O}\!\mathcal{L}'$ the function $P_x(A)$ is an $\mathcal{R}$-measurable function of x.

PROOF. See Neveu [18], proposition V.1.1.

A point ω' of Ω' can be considered as a realization of a random process. Then $P_x(A)$ is the probability that a realization of the process of which the transition probabilities are time independent and given by P, will be an element of $A \in \mathcal{O}\!\mathcal{L}'$, if the process at time 0 is in the state x.

DEFINITION 1.2.4. For every measure μ_0 on $(X,\mathcal{R})$ the Markov measure M_0' on $(\Omega',\mathcal{O}\!\mathcal{L}')$ is defined by

$$M_0'(A) = \int P_x(A)\mu_0(dx) \qquad \text{for every } A \in \mathcal{O}\!\mathcal{L}' \ .$$

The system $(\Omega',\mathcal{O}\!\mathcal{L}',M_0',S')$ is said to be the one-sided shift space for P with initial measure μ_0.

PROPOSITION 1.2.4. Let $(\Omega',\mathcal{O}\!\mathcal{L}',M_0',S')$ be the one-sided shift space for P with initial measure μ_0. Let for every n the marginal measure μ_n be defined by

$$\mu_n(A) = M_0'(\pi_n'^{-1} A) \qquad \text{for all } A \in \mathcal{R} \ .$$

Then the following statements hold:

i) $\mu_{n+1} = \mu_n P$ for every $n \geq 0$.

ii) $M_0'(\Omega') = \mu_0(X)$;
M_0' is σ-finite if and only if μ_0 is σ-finite.

iii) $P(P_\cdot(A))(x) = P_x(S'^{-1}A)$ for all $A \in \mathcal{O}\!\mathcal{L}'$.

PROOF.

i) For all $A \in \mathcal{R}$ we have by definition

$$\mu_{n+1}(A) = M_0'(\pi_{n+1}'^{-1} A) = \int P^{n+1} 1_A d\mu_0 = (\mu_0 P^n)(P 1_A)$$

by proposition 1.2.2. The statement now easily follows by induction on n.

ii)
$$M_0'(\Omega') = \int P_x(\Omega')\mu_0(dx) = \int 1 d\mu_0 = \mu_0(X) \ .$$

Assume μ_0 is σ-finite. Let $(A_1, A_2, \ldots)$ be an $\mathcal{R}$-measurable partition of X such that $\mu_0(A_i) < \infty$ for every i. Define $A_i' = \{\omega' \mid \pi_0'(\omega') \in A_i\}$ for every i, then $(A_1', A_2', \ldots)$ forms a partition of Ω' such that

$$M_0'(A_i') = \int P_x(A_i') d\mu_0 = \int 1_{A_i} d\mu_0 = \mu_0(A_i) < \infty$$

for every i. Hence M_0' is σ-finite.
If μ_0 is not σ-finite, then there exists a set $A \in \mathcal{R}$, $\mu_0(A) > 0$ such that for all $B \subset A$, $B \in \mathcal{R}$ we have $\mu_0(B) = 0$ or $\mu_0(B) = \infty$. Put $A' = \{\omega' \mid \pi_0'(\omega') \in A\}$, then $A' \in \mathcal{O}\!\mathcal{L}'$, $P_x(A') = 1_A(x)$. It follows that for every $B' \subset A'$, $B' \in \mathcal{O}\!\mathcal{L}'$ we have $P_x(B') = 0$ on $X \setminus A$, and therefore $M_0'(B') = 0$ or $M_0'(B') = \infty$. Since $M_0'(A') = \mu_0(A) = \infty$. the measure M_0' is not σ-finite.

iii) Let $\mathcal{S}'$ be the class of all sets in $\mathcal{O}\!\mathcal{L}'$ for which statement iii) holds. Let $(A_n)_{n=1}^{\infty}$ be an increasing or de-

creasing sequence of sets in $\mathcal{B}'$ converging to $A \in \mathcal{O}\!\mathcal{C}'$. Then, since for every x P_x is a probability, the sequences $(P_x(A_n))_{n=1}^{\infty}$ and $(P_x(S'^{-1}A_n))_{n=1}^{\infty}$ converge to $P_x(A)$ and $P_x(S'^{-1}A)$ respectively.
For every n define $f_n(x) = P_x(A_n)$, then $(f_n)_{n=1}^{\infty}$ is a monotone sequence of functions in $B(X,\mathcal{R})$, satisfying $0 \le f_n \le 1$ for all n and converging to $P_x(A)$. It easily follows from proposition 1.2.2 that $(Pf_n)(x)$ converges to $P(P_{\bullet}(A))(x)$ for every x. Since $(Pf_n)(x) = P_x(S'^{-1}A_n)$ by hypothesis, we obtain $P(P_{\bullet}(A))(x) = P_x(S'^{-1}A)$, and therefore $A \in \mathcal{B}'$.
Moreover, it is an immediate consequence of the definitions that every rectangle belongs to $\mathcal{B}'$, and therefore also every finite union of pairwise disjoint rectangles belongs to $\mathcal{B}'$. Therefore $\mathcal{B}'$ is a monotone class containing an algebra which generates $\mathcal{O}\!\mathcal{C}'$. It follows by the monotone class theorem ([5], § 6 theorem B) that $\mathcal{B}' = \mathcal{O}\!\mathcal{C}'$.

1.3. MARKOV PROCESSES

Throughout this section, $(X,\mathcal{R},\mu)$ will be a σ-finite measure space.

DEFINITION 1.3.1. A Markov operator P in $\mathcal{L}_{\infty}(X,\mathcal{R},\mu)$ is a positive linear operator such that

i) $P1 \le 1$;

ii) for every sequence $(f_n)_{n=1}^{\infty}$ in $\mathcal{L}_{\infty}(X,\mathcal{R},\mu)$ with $f_n \downarrow 0$ if $n \to \infty$, we have $Pf_n \downarrow 0$ if $n \to \infty$.

The operator P in $\mathcal{L}_\infty(X,\mathcal{A},\mu)$ strongly resembles the operator P in $B(X,\mathcal{A})$ determined by a transition probability as in definition 1.2.2. Indeed, if P is a Markov operator in $\mathcal{L}_\infty(X,\mathcal{A},\mu)$, then e.g. for every sequence $(A_n)_{n=1}^\infty$ in $\mathcal{A}$ of pairwise disjoint sets with union $A \in \mathcal{A}$, we have

$$\sum_{n=1}^{\infty} P1_{A_n}(x) = P1_A(x) \quad \text{for } \mu\text{-almost all } x \in X .$$

However, the null set for which this relation does not hold, in general will depend on the choice of the sequence $(A_n)_{n=1}^\infty$. It will therefore in general not be possible to assert the existence of a null set N such that for all x outside N $P1_A(x)$ is a finite measure on $\mathcal{A}$.

On the other hand, if P is the operator in $B(X,\mathcal{A})$ determined by a transition probability as in definition 1.2.2, then P does not automatically induce a Markov operator in $\mathcal{L}_\infty(X,\mathcal{A},\mu)$. For later reference we state the following trivial technical result here.

LEMMA 1.3.1. Let P be the operator in $B(X,\mathcal{A})$ determined by a transition probability on $(X,\mathcal{A})$. Then P induces a Markov operator in $\mathcal{L}_\infty(X,\mathcal{A},\mu)$ if and only if for every set $A \in \mathcal{A}$ with $\mu(A) = 0$ we have $\mu(\{x \mid P(x,A) > 0\}) = 0$.

Definition 1.3.1 agrees with the definition in Foguel [4]. However, for instance Neveu [18], V.4 calls the operator P in definition 1.3.1 a sub-Markov operator in $\mathcal{L}_\infty(X,\mathcal{A},\mu)$ and he uses the term Markov operator in $\mathcal{L}_\infty(X,\mathcal{A},\mu)$ if the sub-Markov operator satisfies $P1 = 1$.

We can extend the domain of definition of the Markov operator P in $\mathcal{L}_\infty(X,\mathcal{A},\mu)$ to the space $M^+(X,\mathcal{A},\mu)$ in the following way.

PROPOSITION 1.3.1. Let P be a Markov operator in $\mathcal{L}_\infty(X,\mathcal{R},\mu)$. For every $f \in M^+(X,\mathcal{R},\mu)$ let $(f_n)_{n=1}^\infty$ be a sequence in $\mathcal{L}_\infty(X,\mathcal{R},\mu)$ such that $f_n \uparrow f$ if $n \to \infty$. If we define $Qf = \lim_{n\to\infty} Pf_n$, then the following statements hold.

Q is a well defined mapping of $M^+(X,\mathcal{R},\mu)$ into itself such that

1) $Q(\alpha f + \beta g) = \alpha Qf + \beta Qg$ for all $\alpha,\beta \geq 0$ and all $f,g \in M^+(X,\mathcal{R},\mu)$.

2) $Q\left(\sum_{n=1}^\infty f_n\right) = \sum_{n=1}^\infty Qf_n$ for every sequence $(f_n)_{n=1}^\infty$ in $M^+(X,\mathcal{R},\mu)$.

3) $Q1 \leq 1$.

Moreover, the restriction of Q to the space $\mathcal{L}_\infty^+(X,\mathcal{R},\mu)$ coincides with the restriction of P to the space $\mathcal{L}_\infty^+(X,\mathcal{R},\mu)$.

Conversely, if Q is a mapping of $M^+(X,\mathcal{R},\mu)$ into itself such that conditions 1), 2) and 3) hold, then there exists a unique Markov operator P in $\mathcal{L}_\infty(X,\mathcal{R},\mu)$ such that the restriction of P to $\mathcal{L}_\infty^+(X,\mathcal{R},\mu)$ coincides with the restriction of Q to $\mathcal{L}_\infty^+(X,\mathcal{R},\mu)$.

PROOF. If $(f_n)_{n=1}^\infty$ is an increasing sequence in $\mathcal{L}_\infty(X,\mathcal{R},\mu)$, then by the positivity of P also $(Pf_n)_{n=1}^\infty$ is an increasing sequence in $\mathcal{L}_\infty(X,\mathcal{R},\mu)$, hence $\lim_{n\to\infty} Pf_n$ exists.

We first show that the definition of Qf is independent of the choice of the sequence in $\mathcal{L}_\infty(X,\mathcal{R},\mu)$, increasing towards f. Indeed, if $(f'_n)_{n=1}^\infty$ is another sequence in $\mathcal{L}_\infty(X,\mathcal{R},\mu)$ such that $f'_n \uparrow f$ if $n \to \infty$, then for all $A \in \mathcal{R}$ with $\mu(A) < \infty$ we have

$$\int_A (\lim_{n\to\infty} Pf_n)d\mu = \lim_{n\to\infty} \int_A Pf_n\, d\mu = \lim_{n\to\infty} \int (1_A P)f_n\, d\mu = \int (1_A P)fd\mu \ ;$$

$$\int_A (\lim_{n\to\infty} Pf'_n)d\mu = \lim_{n\to\infty} \int_A Pf'_n\, d\mu = \lim_{n\to\infty} \int (1_A P)f'_n\, d\mu = \int (1_A P)fd\mu \ ,$$

hence $\lim_{n\to\infty} Pf_n = \lim_{n\to\infty} Pf'_n$. Now the properties 1) and 3) are trivial consequences of the definition of Q, as well as the fact that Pf = Qf for all $f \in \mathcal{L}^+_\infty(X,\mathcal{R},\mu)$. In order to prove 2), it suffices to show that for every increasing sequence $(f_n)_{n=1}^\infty$ in $M^+(X,\mathcal{R},\mu)$ converging to f we have $Qf_n \uparrow Qf$ if $n \to \infty$. We easily construct a sequence $(g_n)_{n=1}^\infty$ in $\mathcal{L}_\infty(X,\mathcal{R},\mu)$ such that $g_n \uparrow f$ if $n \to \infty$, and for all n we have $g_n \le f_n \le f$. Hence $Qg_n \le Qf_n \le Qf$ for all n, and since $Qg_n \uparrow Qf$ if $n \to \infty$, we obtain $Qf_n \uparrow Qf$ if $n \to \infty$.

Conversely, let the operator Q in $M^+(X,\mathcal{R},\mu)$ be given such that the conditions 1), 2) and 3) are satisfied. Define for every $f \in \mathcal{L}_\infty(X,\mathcal{R},\mu)$ $Pf = Pf^+ - Pf^-$. If also $f = f_1 - f_2$, where $f_i \in \mathcal{L}^+_\infty(X,\mathcal{R},\mu)$ for i = 1,2, then $f^+ + f_2 = f^- + f_1$, hence $Pf^+ + Pf_2 = Pf^- + Pf_1$ and $Pf = Pf_1 - Pf_2$. Now we easily verify that P is a positive linear operator in $\mathcal{L}_\infty(X,\mathcal{R},\mu)$ satisfying $P1 \le 1$. Let $(f_n)_{n=1}^\infty$ be a sequence in $\mathcal{L}_\infty(X,\mathcal{R},\mu)$ such that $f_n \downarrow 0$ if $n \to \infty$. For every n we have

$$Pf_1 = P(f_1 - f_n) + Pf_n \ .$$

Since $f_1 - f_n \in M^+(X,\mathcal{R},\mu)$, $f_1 - f_n \uparrow f_1$ if $n \to \infty$, we have $P(f_1 - f_n) \uparrow Pf_1$, and because of $Pf_1 < \infty$, we obtain $Pf_n \downarrow 0$ if $n \to \infty$. Hence P is a Markov operator in $\mathcal{L}_\infty(X,\mathcal{R},\mu)$.

The operator Q in $M^+(X,\mathcal{R},\mu)$ is said to be the extension to $M^+(X,\mathcal{R},\mu)$ of the Markov operator P in $\mathcal{L}_\infty(X,\mathcal{R},\mu)$. In the sequel, we shall denote this extension also by P. Since it originates from an operator in $\mathcal{L}_\infty(X,\mathcal{R},\mu)$ we shall always write P to the left of the functions on which it operates. Whether we have to consider P as a Markov operator in $\mathcal{L}_\infty(X,\mathcal{R},\mu)$ or as the extension to $M^+(X,\mathcal{R},\mu)$ will be clear from the given domain of definition.

DEFINITION 1.3.2. A Markov operator in $\mathcal{L}_1(X,\mathcal{R},\mu)$ is a positive linear contraction in $\mathcal{L}_1(X,\mathcal{R},\mu)$.

Because of proposition 1.1.2 it follows that every Markov operator in $\mathcal{L}_\infty(X,\mathcal{R},\mu)$ is the adjoint of a linear operator in $\mathcal{L}_1(X,\mathcal{R},\mu)$. In accordance with the notation introduced in section 1.1 we shall denote this operator also by P, but we shall write P to the right of the functions on which it acts.

PROPOSITION 1.3.2. The relation

(1) $$\int (fP)g\,d\mu = \int f(Pg)\,d\mu \qquad \text{for all } f \in \mathcal{L}_1(X,\mathcal{R},\mu) \text{ and for all } g \in \mathcal{L}_\infty(X,\mathcal{R},\mu)$$

establishes a one-to-one correspondence between the Markov operators in $\mathcal{L}_1(X,\mathcal{R},\mu)$ and the Markov operators in $\mathcal{L}_\infty(X,\mathcal{R},\mu)$.

PROOF. If P is a Markov operator in $\mathcal{L}_\infty(X,\mathcal{R},\mu)$, then P is the adjoint of a linear operator P in $\mathcal{L}_1(X,\mathcal{R},\mu)$ such that the relation (1) holds. From this relation we easily deduce that the linear mapping $f \to fP$ for all $f \in \mathcal{L}_1(X,\mathcal{R},\mu)$ must be positive, and for all $f \in \mathcal{L}_1^+(X,\mathcal{R},\mu)$ we have

$$\int fP\,d\mu = \int f(P1)\,d\mu \le \int f\,d\mu \;.$$

Hence the Markov operator P in $\mathcal{L}_\infty(X,\mathcal{R},\mu)$ is the adjoint of a Markov operator in $\mathcal{L}_1(X,\mathcal{R},\mu)$.

Conversely, let P be a Markov operator in $\mathcal{L}_1(X,\mathcal{R},\mu)$. Then there exists an adjoint linear operator P in $\mathcal{L}_\infty(X,\mathcal{R},\mu)$ such that the relation (1) holds. Now it easily follows that this adjoint operator actually is a Markov operator in $\mathcal{L}_\infty(X,\mathcal{R},\mu)$.

As for Markov operators in $\mathcal{L}_\infty(X,\mathcal{R},\mu)$, the domain of definition of a Markov operator in $\mathcal{L}_1(X,\mathcal{R},\mu)$ can be extended to $M^+(X,\mathcal{R},\mu)$. The proof of the next proposition is similar to the proof of proposition 1.3.1 and is therefore omitted.

PROPOSITION 1.3.3. Let P be a Markov operator in $\mathcal{L}_1(X,\mathcal{R},\mu)$. For every $f \in M^+(X,\mathcal{R},\mu)$ let $(f_n)_{n=1}^\infty$ be a sequence in $\mathcal{L}_1(X,\mathcal{R},\mu)$ such that $f_n \uparrow f$ if $n \to \infty$. If we define $fQ = \lim_{n\to\infty} f_n P$, then the following statements hold.

Q is a well defined mapping of $M^+(X,\mathcal{R},\mu)$ into itself such that

1) $(\alpha f + \beta g)Q = \alpha fQ + \beta gQ$ for all $\alpha,\beta \geq 0$ and all $f,g \in M^+(X,\mathcal{R},\mu)$.

2) $\left(\sum_{n=1}^\infty f_n\right)Q = \sum_{n=1}^\infty f_n Q$ for every sequence $(f_n)_{n=1}^\infty$ in $M^+(X,\mathcal{R},\mu)$.

3) $\int fQ\, d\mu = \int f d\mu$ for all $f \in M^+(X,\mathcal{R},\mu)$.

Moreover, the restriction of Q to the space $\mathcal{L}_1^+(X,\mathcal{R},\mu)$ coincides with the restriction of P to the space $\mathcal{L}_1^+(X,\mathcal{R},\mu)$.

Conversely, if Q is a mapping of $M^+(X,\mathcal{R},\mu)$ into itself such that conditions 1), 2) and 3) hold, then there exists a unique Markov operator P in $\mathcal{L}_1(X,\mathcal{R},\mu)$ such that the restriction of P to $\mathcal{L}_1^+(X,\mathcal{R},\mu)$ coincides with the restriction of Q to $\mathcal{L}_1^+(X,\mathcal{R},\mu)$.

The operator Q in $M^+(X,\mathcal{R},\mu)$ is said to be the extension to $M^+(X,\mathcal{R},\mu)$ of the Markov operator P in $\mathcal{L}_1(X,\mathcal{R},\mu)$. In the sequel, we shall denote this extension also by P. Since it originates from an operator in $\mathcal{L}_1(X,\mathcal{R},\mu)$, we shall always write P to the right of the functions on which it acts. Again we distinguish between a Markov operator in $\mathcal{L}_1(X,\mathcal{R},\mu)$ and the extension to $M^+(X,\mathcal{R},\mu)$ by indicating the domain of definition.

Because of the one-to-one correspondence between the Markov operators in $\mathcal{L}_\infty(X,\mathcal{R},\mu)$, the Markov operators in $\mathcal{L}_1(X,\mathcal{R},\mu)$ and their extensions to $M^+(X,\mathcal{R},\mu)$, we shall usually speak of a Markov process P on $(X,\mathcal{R},\mu)$. It will be clear from the notation and the given domain of definition how we have to interprete the operator P. It easily follows from the previous propositions that for a given Markov process P on $(X,\mathcal{R},\mu)$ for all $f \in M^+(X,\mathcal{R},\mu)$ and for all $g \in M^+(X,\mathcal{R},\mu)$ the relation $\int (fP)g\,d\mu = \int f(Pg)\,d\mu$ holds.

A Markov process P on a σ-finite measure space $(X,\mathcal{R},\mu)$ only depends on the class of μ-null sets of $\mathcal{R}$. In fact, consider P as a Markov operator in $\mathcal{L}_\infty(X,\mathcal{R},\mu)$. If we replace the measure μ by an equivalent σ-finite measure ν on $(X,\mathcal{R})$, then neither the $\mathcal{L}_\infty$-space, nor the conditions in definition 1.3.1 are influenced, and P may also be considered as a Markov operator in $\mathcal{L}_\infty(X,\mathcal{R},\nu)$. In fact, written to the left P acts on a function in the same way whether it is considered as an element of $\mathcal{L}_\infty(X,\mathcal{R},\mu)$ $(M^+(X,\mathcal{R},\mu))$ or of $\mathcal{L}_\infty(X,\mathcal{R},\nu)$ $(M^+(X,\mathcal{R},\nu))$.

The situation is essentially different if we consider the corresponding Markov operators in $\mathcal{L}_1(X,\mathcal{R},\mu)$ and $\mathcal{L}_1(X,\mathcal{R},\nu)$. Actually, for every σ-finite measure ν equivalent to μ there exists a unique Markov operator P_ν in $\mathcal{L}_1(X,\mathcal{R},\nu)$ such that the adjoint operator in $\mathcal{L}_\infty(X,\mathcal{R},\nu) = \mathcal{L}_\infty(X,\mathcal{R},\mu)$ coincides with P. Each of these Markov operators P_ν in $\mathcal{L}_1(X,\mathcal{R},\nu)$ has an extension P_ν to $M^+(X,\mathcal{R},\nu) = M^+(X,\mathcal{R},\mu)$. We shall agree to the convention that if P is a Markov process on $(X,\mathcal{R},\mu)$, the operator P written to the left of the function will be the Markov operator in $\mathcal{L}_\infty(X,\mathcal{R},\mu)$ or its extension to $M^+(X,\mathcal{R},\mu)$ and the operator P without subscript written to the right of the function will be the Markov operator in $\mathcal{L}_1(X,\mathcal{R},\mu)$ or its extension to $M^+(X,\mathcal{R},\mu)$.

The next proposition gives the relationship between the various operators P_ν.

PROPOSITION 1.3.4. Let P be a Markov process on a σ-finite measure space $(X,\mathcal{R},\mu)$ and let ν be a σ-finite measure on $(X,\mathcal{R})$ equivalent to μ. Let P_ν be the Markov operator in $\mathcal{L}_1(X,\mathcal{R},\nu)$ corresponding to P. Then the mapping $f \to f\,\frac{d\mu}{d\nu}$ defines an isometry of $\mathcal{L}_1(X,\mathcal{R},\mu)$ onto $\mathcal{L}_1(X,\mathcal{R},\nu)$ such that

$$fP_\nu = ((f\,\frac{d\nu}{d\mu})P)\,\frac{d\mu}{d\nu} \qquad \text{for all } f \in \mathcal{L}_1(X,\mathcal{R},\nu)\ .$$

For the extension of P_ν to $M^+(X,\mathcal{R},\mu)$ we have

$$fP_\nu = ((f\,\frac{d\nu}{d\mu})P)\,\frac{d\mu}{d\nu} \qquad \text{for all } f \in M^+(X,\mathcal{R},\mu)\ .$$

PROOF. Because of proposition 1.1.1 we have $0 < \frac{d\nu}{d\mu} < \infty$, and $\frac{d\mu}{d\nu}\,\frac{d\nu}{d\mu} = \frac{d\mu}{d\mu} = 1$. The first statement is now obvious.

Choose $f \in \mathcal{L}_1(X,\mathcal{R},\nu)$. Then for every $A \in \mathcal{R}$ we have

$$\int_A (fP_\nu)d\nu = \int f(P1_A)d\nu = \int (f\,\frac{d\nu}{d\mu})(P1_A)d\mu =$$

$$= \int ((f\,\frac{d\nu}{d\mu})P)1_A\, d\mu = \int_A ((f\,\frac{d\nu}{d\mu})P)\,\frac{d\mu}{d\nu}\,d\nu$$

from which the second statement follows.

Finally, choose $f \in M^+(X,\mathcal{R},\mu)$ and let $(f_n)_{n=1}^\infty$ be a sequence in $\mathcal{L}_1^+(X,\mathcal{R},\nu)$ such that $f_n \uparrow f$ if $n \to \infty$. Then by proposition 1.3.3 we obtain

$$fP_\nu = \lim_{n\to\infty} f_nP_\nu = \lim_{n\to\infty} ((f_n\,\frac{d\nu}{d\mu})P)\,\frac{d\mu}{d\nu} = (f\,\frac{d\nu}{d\mu})P\,\frac{d\mu}{d\nu}\ .$$

For any function f let supp f denote the set $\{x \mid f(x) \neq 0\}$.

PROPOSITION 1.3.5. For any $f \in M^+(X,\mathcal{R},\mu)$ put $A = \text{supp } f$. Then supp Pf = supp $P1_A$ and supp fP = supp 1_AP.

PROOF. Suppose $B = \text{supp } Pf \setminus \text{supp } P1_A$ has positive measure. Then

$$0 < \int 1_B(Pf)d\mu = \int (1_BP)fd\mu\ .$$

Since supp $f = A$, we then have

$$0 < \int (1_BP)1_A\, d\mu = \int 1_B(P1_A)d\mu = 0\ .$$

Contradiction, hence $\mu(B) = 0$. In the same way we show $\mu(\text{supp } P1_A \setminus \text{supp } Pf) = 0$, and therefore supp $P1_A$ = supp Pf. The proof of supp 1_AP = supp fP is analogous.

DEFINITION 1.3.3. For every $A \in \mathcal{R}$ we define

$$P^{-1}A = \text{supp } P1_A \ , \quad PA = \text{supp } 1_A P \ .$$

While $P1_A(x)$ is in general not a transition probability, it strongly resembles a transition probability in several respects. With this restriction in mind, the set $P^{-1}A$ may be interpreted as the class of states which have positive probability to enter the set A in one transition.

Similarly, the set PA can be considered as the class of states which we can reach from A in one transition. More precisely: $P1_{X \setminus PA} = 0$ on A and for every subset B of PA with $\mu(B) > 0$ there exists a subset A_0 of A, $\mu(A_0) > 0$ such that $P1_B > 0$ on A_0. In fact

$$\int_A P1_{X\setminus PA} \, d\mu = \int_{X\setminus PA} 1_A P \, d\mu = 0 \ ,$$

hence $P1_{X\setminus PA} = 0$ on A and $X \setminus PA$ cannot be reached in one transition from A. Let B be a subset of PA of positive measure, then

$$0 < \int_B 1_A P \, d\mu = \int_A P1_B \, d\mu \ ,$$

hence there exists a subset A_0 of A of positive measure such that $P1_B > 0$ on A_0.

We now show that the sets $P^{-1}A$ and PA are not influenced when we replace the measure μ by an equivalent σ-finite measure ν. For the set $P^{-1}A$ this is an immediate consequence of the fact that a Markov process depends on the class of null sets $\mathcal{R}$ rather than on the measure μ. For the set PA this follows from the previous propositions in the following way.

By proposition 1.3.4 we have

$$1_A P_\nu = ((1_A \frac{d\nu}{d\mu})P) \frac{d\mu}{d\nu} .$$

Now $\frac{d\nu}{d\mu} > 0$, $\frac{d\mu}{d\nu} > 0$, hence by proposition 1.3.5

$$\text{supp } 1_A P_\nu = \text{supp}(1_A \frac{d\nu}{d\mu})P = \text{supp } 1_A P_\mu .$$

The next proposition is an easy consequence of definition 1.3.3 and proposition 1.3.5.

PROPOSITION 1.3.6. For every $A \in \mathcal{R}$ and every $n \geq 0$ we have

$$P^{n+1}A = P(P^n A) , \quad P^{-(n+1)}A = P^{-1}(P^{-n}A) .$$

For every sequence $(A_n)_{n=1}^{\infty}$ in $\mathcal{R}$ we have

$$P\left(\bigcup_{n=1}^{\infty} A_n\right) = \bigcup_{n=1}^{\infty} PA_n , \quad P^{-1}\left(\bigcup_{n=1}^{\infty} A_n\right) = \bigcup_{n=1}^{\infty} P^{-1}A_n$$

$$P\left(\bigcap_{n=1}^{\infty} A_n\right) \subset \bigcap_{n=1}^{\infty} PA_n , \quad P^{-1}\left(\bigcap_{n=1}^{\infty} A_n\right) \subset \bigcap_{n=1}^{\infty} P^{-1}A_n .$$

PROPOSITION 1.3.7. The condition $1P > 0$ is equivalent to the condition $\forall_{A\in\mathcal{R}} (\mu(A) > 0 \Rightarrow \mu(P^{-1}A) > 0)$.

The condition $P1 > 0$ is equivalent to the condition $\forall_{A\in\mathcal{R}} (\mu(A) > 0 \Rightarrow \mu(PA) > 0)$.

PROOF. We only prove the first statement, the second statement being proved similarly.

Suppose $1P > 0$ and $\mu(A) > 0$. Then

$$0 < \int_A 1P \, d\mu = \int P1_A \, d\mu ,$$

hence $\mu(P^{-1}A) > 0$. Conversely, suppose for all $A \in \mathcal{R}$ we have $\mu(A) > 0 \Rightarrow \mu(P^{-1}A) > 0$. Put $A = \{x \mid 1P(x) = 0\}$, then

$$0 = \int_A 1P \, d\mu = \int P 1_A \, d\mu ,$$

hence $\mu(P^{-1}A) = 0$. It follows $\mu(A) = 0$, and therefore $1P > 0$.

1.4. BACKWARD AND ADJOINT PROCESSES

Without further mentioning P will be a Markov process on a σ-finite measure space $(X,\mathcal{R},\mu)$.

DEFINITION 1.4.1. Let μ_0 be a measure on $(X,\mathcal{R})$ such that $\mu_0 << \mu$. For every rectangle $A \times B$ in $(X,\mathcal{R}) \times (X,\mathcal{R})$ we define

$$M_{01}(A \times B) = \int 1_A (P1_B) d\mu_0 .$$

For every $B \in \mathcal{R}$ the measure μ_1 on $(X,\mathcal{R})$ is defined by

$$\mu_1(B) = M_{01}(X \times B) .$$

Since for every $B \in \mathcal{R}$ we have $\mu_1(B) = \int P1_B \, d\mu_0$ it is easily seen that μ_1 is indeed a measure on $(X,\mathcal{R})$. If P is given by a transition probability on $(X,\mathcal{R})$, then the measure μ_1 is the same as the measure $\mu_0 P$ in definition 1.2.2.

PROPOSITION 1.4.1.

i) $\mu_1 << \mu$ and $\dfrac{d\mu_0}{d\mu} P = \dfrac{d\mu_1}{d\mu}$.

ii) $\mu_1(X) \leq \mu_0(X)$. If $P1 > 0$ and μ_1 is σ-finite, then μ_0 is σ-finite.

iii) If $1P > 0$ and $\mu_0 \approx \mu$, then $\mu_1 \approx \mu$.

PROOF.

i) If $\mu(A) = 0$, then $P1_A = 0$, and $\mu_1(A) = 0$. For all $A \in \mathcal{R}$ we have

$$\mu_1(A) = \int P1_A \, d\mu_0 = \int \frac{d\mu_0}{d\mu} (P1_A) d\mu = \int_A (\frac{d\mu_0}{d\mu} P) d\mu$$

from which the second assertion in i) follows.

ii) $$\mu_1(X) = \int P1 \, d\mu_0 \leq \int d\mu_0 = \mu_0(X) .$$

Let $A \in \mathcal{R}$ be a set such that for all $B \subset A$, $B \in \mathcal{R}$ we have $\mu_0(B) = 0$ or $\mu_0(B) = \infty$. Let $(A_1, A_2, \ldots)$ be an $\mathcal{R}$-measurable partition of X such that $\mu_1(A_i) < \infty$ for all i. Since $\mu_1(A_i) = \int P1_{A_i} d\mu_0$, it follows that $P1_{A_i} = 0$ and therefore

$$\sum_{i=1}^{\infty} P1_{A_i} = P1 = 0 \quad \mu_0\text{-almost everywhere on } A.$$

Since $P1 > 0$ we find $\mu_0(A) = 0$, hence μ_0 is a σ-finite measure on $(X, \mathcal{R})$.

iii) In i) we have shown $\mu_1 << \mu$. Now suppose $\mu_1(A) = 0$. Then $P1_A = 0$ μ_0-almost everywhere, and since $\mu_0 \approx \mu$, $P1_A = 0$ μ-almost everywhere. It follows that

$$0 = \int P1_A \, d\mu = \int_A 1P \, d\mu ,$$

hence $\mu(A) = 0$.

The measure μ_1 can be seen as the measure on $(X,\mathcal{R})$ at time 1 if we had the measure μ_0 on $(X,\mathcal{R})$ at time 0. This motivates the following definition.

DEFINITION 1.4.2. A measure $\mu_0 \ll \mu$ is said to be invariant under P if $\frac{d\mu_0}{d\mu} P = \frac{d\mu_0}{d\mu}$ and subinvariant if $\frac{d\mu_0}{d\mu} P \le \frac{d\mu_0}{d\mu}$.

It is easily seen that the set function M_{01} defined on the rectangles of $(Y,\mathcal{T}) = (X,\mathcal{R}) \times (X,\mathcal{R})$ as in definition 1.4.1 can be extended to a finitely additive set function on the algebra of finite unions of rectangles in $\mathcal{T}$. However, M_{01} need not to be extendable to a measure on $(Y,\mathcal{T})$.

PROPOSITION 1.4.2. Assume for every $A \in \mathcal{R}$ we can find a representative $P(\cdot,A)$ for the equivalence class $P1_A \in \mathcal{L}_\infty(X,\mathcal{R},\mu)$ such that for μ_0-almost all $x \in X$ $P(x,\cdot)$ is a measure on $\mathcal{R}$. Then the set function M_{01} can be extended to a measure on $(X,\mathcal{R}) \times (X,\mathcal{R})$.

PROOF. Let N be the μ_0-null-set of points x for which $P(x,\cdot)$ is not a finite measure. Let μ' be a probability equivalent to μ on $\mathcal{R}$. Then define for all $A \in \mathcal{R}$

$$P'(x,A) = \begin{cases} P(x,A) & \text{if } x \in X \setminus N \\ \mu'(A) & \text{if } x \in N . \end{cases}$$

Put $(Y,\mathcal{T}) = (X,\mathcal{R}) \times (X,\mathcal{R})$. If we define for all $A' \in \mathcal{T}$

$$M(A') = \int_X \mu_0(dx_0) \int_X 1_{A'}(x_0,x_1)P'(x_0,dx_1) ,$$

then M is a measure on $(Y,\mathcal{T})$. The proof is almost identical to the proof of proposition 1.2.1, and therefore omitted. Now for every rectangle we obtain

$$M(A \times B) = \int_A \mu_0(dx_0) \int_B P'(x_0,dx_1) = \int_A P1_B \, d\mu_0 = M_{01}(A \times B)$$

from which the statement follows.

A particular case of this situation is given by the Markov processes which are determined by a transition probability in the following way.

PROPOSITION 1.4.3. Let P be the operator on $B(X,\mathcal{R})$ associated with a transition probability $(X,\mathcal{R})$. For every σ-finite measure μ_0 there exists a probability μ on $(X,\mathcal{R})$ such that $\mu_0 P^n \ll \mu$ for every $n \geq 0$, and the operator P on $B(X,\mathcal{R})$ induces a Markov process on $(X,\mathcal{R},\mu)$.

PROOF. Since the measure μ_0 is σ-finite, we can if necessary replace it by an equivalent probability μ_0'. Then for every $n \geq 0$ we have $\mu_0 P^n \approx \mu_0' P^n$. Define

$$\mu = \sum_{n=1}^{\infty} \frac{1}{2^n} \mu_0' \, P^{n-1} ,$$

then μ is a probability and $\mu_0 P^n \ll \mu$ for every $n \geq 0$. Let $A \in \mathcal{R}$ be a set such that $\mu(A) = 0$, and suppose $\mu(\{x \mid P(x,A) > 0\}) > 0$. Then there exists an integer n such that $\mu_0' \, P^{n-1}(\{x \mid P(x,A) > 0\}) > 0$, hence $(\mu_0' \, P^{n-1})(P1_A) =$ $= \mu_0' \, P^n(A) > 0$ by proposition 1.2.2. It follows that $\mu(A) > 0$. Contradiction, and therefore $\mu(\{x \mid P(x,A) > 0\}) = 0$. By lemma 1.3.1 the operator P on $B(X,\mathcal{R})$ induces a Markov process on $(X,\mathcal{R},\mu)$.

Consider the one-sided shift space $(\Omega',\mathcal{O}\!\mathcal{C}',M_0',S')$ for a transition probability P on $(X,\mathcal{R})$ with initial σ-finite measure μ_0, as in definition 1.2.4. Let μ be a probability corresponding to μ_0 as in proposition 1.4.3, then P induces a Markov process on $(X,\mathcal{R},\mu)$ which we again shall denote by P.

Let $(\mu_n)_{n=0}^{\infty}$ be the sequence of marginal measures, as in proposition 1.2.4. Then it follows from the previous propositions that for every $n \geq 0$ we have $\mu_n \ll \mu$ and $\frac{d\mu_n}{d\mu} = \frac{d\mu_0}{d\mu} P^n$. From proposition 1.2.4 we easily deduce that for every rectangle $\prod_{t=0}^{\infty} A_t$, where $A_t = X_t$ for all $t < n$ and for all $t > m$, $n < m$, we have

$$M_0'\left(\prod_{t=0}^{\infty} A_t\right) = \int I_{A_n} P \ldots P 1_{A_m} d\mu_n .$$

Note that in this formula the operators I_A and P are operators in $B(X,\mathcal{R})$. If we would choose now to read I_A and P as Markov operators in $\mathcal{L}_\infty(X,\mathcal{R},\mu)$ (which is legal because of the choice of μ), then the function $I_{A_n} P \ldots P 1_{A_m}$ is only defined modulo μ, but since $\mu_n \ll \mu$ we arrive at the same value of the integral as in the original interpretation. It follows that we may consider the operators I_A and P in this formula both as operators in $B(X,\mathcal{R})$ and as Markov operators in $\mathcal{L}_\infty(X,\mathcal{R},\mu)$.

PROPOSITION 1.4.4. Let P be a Markov process on a σ-finite measure space $(X,\mathcal{R},\mu)$ satisfying $1P > 0$. Let μ_0 be a σ-finite measure on $(X,\mathcal{R})$ with $\mu_0 \approx \mu$. There exists a Markov process $\overleftarrow{P}_{\mu_0}$ on $(X,\mathcal{R},\mu)$ such that for all $A \in \mathcal{R}$ and for all $B \in \mathcal{R}$

$$M_{01}(A \times B) = \int 1_B(\overleftarrow{P}_{\mu_0} 1_A) d\mu_1$$

if and only if μ_1 is σ-finite. The process is then uniquely defined by the measure μ_0 and satisfies

$$\overleftarrow{P}_{\mu_0} f = \frac{(f \frac{d\mu_0}{d\mu})P}{\frac{d\mu_0}{d\mu} P} \qquad \text{for all } f \in M^+(X,\mathcal{R},\mu)$$

$$f\overleftarrow{P}_{\mu_0} = \frac{d\mu_0}{d\mu} \left[P\left(\frac{f}{\frac{d\mu_0}{d\mu} P}\right)\right] \qquad \text{for all } f \in M^+(X,\mathcal{R},\mu) \ .$$

PROOF. Suppose there exists a process $\overleftarrow{P}_{\mu_0}$. Let $B \in \mathcal{R}$ be a set such that all $\mathcal{R}$-measurable subsets have μ_1-measure 0 or μ_1-measure ∞. Let $(A_1, A_2, \ldots)$ be an $\mathcal{R}$-measurable partition of X such that $\mu_0(A_n) < \infty$ for every n. Then

$$\int_B \overleftarrow{P}_{\mu_0} 1_{A_n} d\mu_1 = M_{01}(A_n \times B) < \infty$$

and therefore $\overleftarrow{P}_{\mu_0} 1_{A_n} = 0$ μ_1-almost everywhere on B. Since $\overleftarrow{P}_{\mu_0}$ is a Markov process, we have $\overleftarrow{P}_{\mu_0} 1 = \sum_{n=1}^{\infty} \overleftarrow{P}_{\mu_0} 1_{A_n}$ μ-almost everywhere on X, and therefore μ_1-almost everywhere on X. It follows that $\overleftarrow{P}_{\mu_0} 1 = 0$ μ_1-almost everywhere on B, hence

$$\mu_1(B) = M_{01}(X \times B) = \int_B \overleftarrow{P}_{\mu_0} 1 \, d\mu_1 = 0 \ .$$

This proves the σ-finiteness of μ_1.

Now we show that if μ_1 is σ-finite and $1P > 0$, then the process $\overleftarrow{P}_{\mu_0}$ exists. Because of proposition 1.4.1 we have $\mu_1 \approx \mu$, and since μ_1 is σ-finite $0 < \frac{d\mu_1}{d\mu} = \frac{d\mu_0}{d\mu} P < \infty$. The process $\overleftarrow{P}_{\mu_0}$ must satisfy for all $A \in \mathcal{R}$, $B \in \mathcal{R}$

$$\int 1_A(P1_B)d\mu_0 = \int 1_B(\overleftarrow{P}_{\mu_0}1_A)d\mu_1$$

$$\int_B (1_A \frac{d\mu_0}{d\mu})Pd\mu = \int_B \frac{d\mu_1}{d\mu} (\overleftarrow{P}_{\mu_0}1_A)d\mu$$

$$\overleftarrow{P}_{\mu_0}1_A = \frac{(1_A \frac{d\mu_0}{d\mu})P}{\frac{d\mu_0}{d\mu} P} \qquad \text{for all } A \in \mathcal{R} .$$

Using proposition 1.3.3 we see that the operator $\overleftarrow{P}_{\mu_0}$ in this way defined for every characteristic function, can be extended to $M^+(X,\mathcal{R},\mu)$ such that

$$\overleftarrow{P}_{\mu_0}f = \frac{(f \frac{d\mu_0}{d\mu})P}{\frac{d\mu_0}{d\mu} P} \qquad \text{for all } f \in M^+(X,\mathcal{R},\mu) .$$

This operator $\overleftarrow{P}_{\mu_0}$ satisfies the conditions of proposition 1.3.1 and therefore determines a Markov process on $(X,\mathcal{R},\mu)$.

The uniqueness of $\overleftarrow{P}_{\mu_0}$ is trivial. Moreover, the process $\overleftarrow{P}_{\mu_0}$ is independent of the measure μ. In fact, let ν be a σ-finite measure on $(X,\mathcal{R})$ equivalent to μ and let P_ν be the extension to $M^+(X,\mathcal{R},\mu)$ of the corresponding Markov operator in $\mathcal{L}_1(X,\mathcal{R},\nu)$. Then we have

$$\frac{(1_A \frac{d\mu_0}{d\nu})P_\nu}{\frac{d\mu_0}{d\nu} P_\nu} = \frac{((1_A \frac{d\mu_0}{d\nu} \frac{d\nu}{d\mu})P) \frac{d\mu}{d\nu}}{((\frac{d\mu_0}{d\nu} \frac{d\nu}{d\mu})P)\frac{d\mu}{d\nu}} = \frac{(1_A \frac{d\mu_0}{d\mu})P}{\frac{d\mu_0}{d\mu} P} \qquad \text{for all } A \in \mathcal{R} .$$

Finally, choose $f \in M^+(X,\mathcal{R},\mu)$. Then for all $A \in \mathcal{R}$ we obtain

$$\int_A f\overleftarrow{P}_{\mu_0}\,d\mu = \int f(\overleftarrow{P}_{\mu_0} 1_A)\,d\mu = \int [(1_A \frac{d\mu_0}{d\mu})P] \frac{f}{\frac{d\mu_0}{d\mu}P}\,d\mu =$$

$$= \int_A \frac{d\mu_0}{d\mu} P\left(\frac{f}{\frac{d\mu_0}{d\mu}P}\right) d\mu$$

from which the last statement follows.

DEFINITION 1.4.3. The Markov process $\overleftarrow{P}_{\mu_0}$ on $(X,\mathcal{R},\mu)$ introduced in proposition 1.4.4 is said to be the backward process for P corresponding to the measure μ_0.

Note that the backward process $\overleftarrow{P}_{\mu_0}$ exists for all finite measures $\mu_0 \approx \mu$, and certain σ-finite measures equivalent to μ. In the next section we shall give an example of a Markov process P on $(X,\mathcal{R},\mu)$ where μ is σ-finite and μP is not, and therefore for this process the backward process $\overleftarrow{P}_{\mu}$ does not exist.

A backward process really works in the opposite direction as P does; it follows from proposition 1.4.4 that for all $A \in \mathcal{R}$ we have

$$PA = \overleftarrow{P}_{\mu_0}^{-1}A\ , \qquad P^{-1}A = \overleftarrow{P}_{\mu_0}A\ .$$

PROPOSITION 1.4.5. Let P be a Markov process on a σ-finite measure space $(X,\mathcal{R},\mu)$. Let μ_0 be a σ-finite subinvariant measure equivalent to μ. Then the formula

$$P^*_{\mu_0} f = ((f \frac{d\mu_0}{d\mu})P) \frac{d\mu}{d\mu_0} \qquad \text{for all } f \in M^+(X,\mathcal{R},\mu)$$

defines a Markov process on $(X,\mathcal{R},\mu)$. This process $P^*_{\mu_0}$ is determined uniquely by the measure μ_0, and satisfies

$$fP^*_{\mu_0} = \frac{d\mu_0}{d\mu} (P(f \frac{d\mu}{d\mu_0})) \qquad \text{for all } f \in M^+(X,\mathcal{R},\mu) .$$

PROOF. It follows from proposition 1.3.3 that the formula

$$P^*_{\mu_0} f = ((f \frac{d\mu_0}{d\mu})P) \frac{d\mu}{d\mu_0} \qquad \text{for all } f \in M^+(X,\mathcal{R},\mu)$$

defines an operator $P^*_{\mu_0}$ on $M^+(X,\mathcal{R},\mu)$ satisfying the conditions of proposition 1.3.1, and therefore determines a Markov process $P^*_{\mu_0}$ on $(X,\mathcal{R},\mu)$.

Again, the definition of $P^*_{\mu_0}$ is independent of the choice of the measure ν. Indeed, if ν is a σ-finite measure on $(X,\mathcal{R})$ equivalent to μ, and if P_ν denotes the extension to $M^+(X,\mathcal{R},\mu)$ of the corresponding Markov operator in $\mathcal{L}_1(X,\mathcal{R},\nu)$, then we have for all $f \in M^+(X,\mathcal{R},\mu)$

$$((f \frac{d\mu_0}{d\nu})P_\nu) \frac{d\nu}{d\mu_0} = ((f \frac{d\mu_0}{d\nu} \frac{d\nu}{d\mu})P) \frac{d\mu}{d\nu} \frac{d\nu}{d\mu_0} = P^*_{\mu_0} f .$$

Finally, choose $f \in M^+(X,\mathcal{R},\mu)$. Then for all $A \in \mathcal{R}$ we have

$$\int_A fP^*_{\mu_0} d\mu = \int f(P^*_{\mu_0} 1_A) d\mu = \int ((1_A \frac{d\mu_0}{d\mu})P)(\frac{d\mu}{d\mu_0} f) d\mu =$$

$$= \int_A \frac{d\mu_0}{d\mu} (P(\frac{d\mu}{d\mu_0} f))d\mu ,$$

from which the last statement follows.

DEFINITION 1.4.4. The process $P^*_{\mu_0}$ on $(X,\mathcal{R},\mu)$ introduced in proposition 1.4.5 is said to be the adjoint process of P with respect to the subinvariant measure μ_0.

Note that also the adjoint process works in the backward direction; for all $A \in \mathcal{R}$ we have

$$PA = P^{*-1}_{\mu_0}A \quad \text{and} \quad P^{-1}A = P^*_{\mu_0} A .$$

Further, it follows from proposition 1.4.5 that $P^*_{\mu_0} 1 = 1$ if and only if μ_0 is invariant under P. In this case we have $1P > 0$, and $\mu_1 = \mu_0$, hence the backward process $\overleftarrow{P}_{\mu_0}$ exists, and for all $f \in M^+(X,\mathcal{R},\mu)$

$$\overleftarrow{P}_{\mu_0} f = \frac{(f \frac{d\mu_0}{d\mu})P}{\frac{d\mu_0}{d\mu} P} = ((f \frac{d\mu_0}{d\mu})P) \frac{d\mu}{d\mu_0} = P^*_{\mu_0} f .$$

Since for every backward process we have $\overleftarrow{P}_{\mu_0} 1 = 1$, we see that an adjoint process $P^*_{\mu_0}$ is a backward process if and only if μ_0 is invariant; in this case we have $P^*_{\mu_0} = \overleftarrow{P}_{\mu_0}$.

We conclude this section with the following well known property of an adjoint process, see e.g. Foguel [4], chapter VII. The proof is a straightforward verification, using proposition 1.4.5.

PROPOSITION 1.4.6. Let $P^*_{\mu_0}$ be the adjoint process of P with respect to the subinvariant σ-finite measure μ_0 equivalent to μ. Then μ_0 is subinvariant under $P^*_{\mu_0}$, and $P^{**}_{\mu_0\mu_0} = P$.

1.5. MARKOV PROCESSES INDUCED BY A MEASURABLE TRANSFORMATION

In this section we shall give a systematical outline of the Markov process associated with a measurable transformation T in a σ-finite measure space $(X,\mathcal{R},\mu)$.

The natural way to define an operator on $M^+(X,\mathcal{R},\mu)$ associated with T is to put for every $f \in M^+(X,\mathcal{R},\mu)$

$$Pf = f \circ T .$$

Under a non singularity condition for T the operator P turns out to be a Markov process on $(X,\mathcal{R},\mu)$. This process is said to be the forward process associated with T, since it corresponds to the transition $x \to Tx$. In fact we have $P1_A = 1_{T^{-1}A}$, which means that the probability of entering the set A under P is 1 or 0 whether or not the state is in $T^{-1}A$.

We shall identify the backward processes for P as defined in the previous section. The result will be that every backward process is a Markov process associated with T corresponding to the transition $x \to T^{-1}\{x\}$ as introduced by Hopf [13], see also [9], on a suitable σ-finite measure space $(X,\mathcal{R},\mu)$.

We start with some definitions.

DEFINITION 1.5.1. A measurable transformation in a measurable space $(X,\mathcal{R})$ is a mapping $T : X \to X$ such that $T^{-1}A \in \mathcal{R}$ for all $A \in \mathcal{R}$. A measurable transformation T on $(X,\mathcal{R})$ is said to be invertible if T is one-to-one, $TA \in \mathcal{R}$ for all $A \in \mathcal{R}$ and $TX = X$.

A measure μ on $(X,\mathcal{R})$ is said to be invariant under T if $\mu(T^{-1}A) = \mu(A)$ for all $A \in \mathcal{R}$. The transformation T is said to be negatively non singular with respect to a measure μ on $(X,\mathcal{R})$ if $\mu(T^{-1}A) = 0$ for every $A \in \mathcal{R}$ with $\mu(A) = 0$; positively non singular with respect to μ if $\mu(A) = 0$ for every set $A \in \mathcal{R}$ for which $\mu(T^{-1}A) = 0$, and non singular with respect to μ if $\mu(A) = 0$ if and only if $\mu(T^{-1}A) = 0$ for all $A \in \mathcal{R}$.

LEMMA 1.5.1. Let T be a negatively non singular measurable transformation on a σ-finite measure space $(X,\mathcal{R},\mu)$. Then the set function μT^{-1} on $\mathcal{R}$, defined for all $A \in \mathcal{R}$ by

$$(\mu T^{-1})(A) = \mu(T^{-1}A)$$

is a measure on $(X,\mathcal{R})$ such that $\mu T^{-1} \ll \mu$. Let $\frac{d\mu T^{-1}}{d\mu}$ be the Radon-Nikodym derivative in $M^+(X,\mathcal{R},\mu)$. Then for all $f \in M^+(X,\mathcal{R},\mu)$ we have

$$\int \frac{d\mu T^{-1}}{d\mu} f d\mu = \int f \circ T \, d\mu .$$

Let T be a positively non singular measurable transformation in a σ-finite measure space $(X,\mathcal{R},\mu)$. Let the set function μT on $T^{-1}\mathcal{R}$ be defined by

$$(\mu T)(T^{-1}A) = \mu(A) \qquad \text{for all } T^{-1}A \in T^{-1}\mathcal{R} .$$

Then μT is a measure on $(X,T^{-1}\mathcal{R})$ such that $\mu T \ll \mu$.

If $(X,T^{-1}\mathcal{R},\mu)$ is a σ-finite measure space, then the

Radon-Nikodym derivative $\frac{d\mu T}{d\mu} \in M^+(X,T^{-1}\mathcal{R},\mu)$ exists, and for all $f \in M^+(X,\mathcal{R},\mu)$ the following relation holds:

$$\int \frac{d\mu T}{d\mu}\,(f\circ T)d\mu = \int f d\mu \ .$$

PROOF. We only prove the statements concerning μT; the proof for μT^{-1} is similar.

The proof that μT is a measure on $(X,\mathcal{R})$ such that $\mu T << \mu$ amounts to a simple verification. Then, if $(X,T^{-1}\mathcal{R},\mu)$ is σ-finite, we have for every $A \in \mathcal{R}$ by definition

$$\int \frac{d\mu T}{d\mu}\,(1_A\circ T)d\mu = \int 1_A\, d\mu$$

from which the last statement easily follows.

DEFINITION 1.5.2. Let T be a measurable transformation on $(X,\mathcal{R})$. The formula

$$P(x,A) = 1_{T^{-1}A}(x) \qquad \text{for all } A \in \mathcal{R} \text{ and all } x \in X$$

defines a transition probability on $(X,\mathcal{R})$. If μ is a σ-finite measure on $(X,\mathcal{R})$, then the corresponding operator P on $B(X,\mathcal{R})$ induces a Markov process on $(X,\mathcal{R},\mu)$ if and only if T is negatively non singular with respect to μ. This process is said to be the forward process associated with T on $(X,\mathcal{R},\mu)$ and satisfies $Pf = f\circ T$ for all $f \in M^+(X,\mathcal{R},\mu)$.

PROPOSITION 1.5.1. Let P be the forward process associated with a negatively non singular measurable transformation on a σ-finite measure space $(X,\mathcal{R},\mu)$. Then for all $A \in \mathcal{R}$ we have $P^{-1}A = T^{-1}A$, and PA is the modulo μ smallest set $B \in \mathcal{R}$ such that $\mu(A \cap T^{-1}B) = \mu(A)$.

PROOF. Without loss of generality we may assume $\mu(X) = 1$. By definition we have $P^{-1}A = \text{supp } P1_A = T^{-1}A$. Put $B = PA$, then

$$\int 1_A(P1_{X\setminus B})d\mu = 0 , \qquad \mu(A \cap T^{-1}(X \setminus B)) = 0 ,$$

hence

$$\mu(A \cap T^{-1}B) = \mu(A) .$$

Assume $B_1 \in \mathcal{R}$ satisfies $\mu(A \cap T^{-1}B_1) = \mu(A)$. Then define $B_2 = B \setminus B_1$. We obtain

$$\mu(A \cap T^{-1}B_2) = \mu(A \cap T^{-1}B) - \mu(A \cap T^{-1}B \cap T^{-1}B_1) =$$

$$= \mu(A) - \mu(A) = 0 ,$$

$$\int 1_A(P1_{B_2})d\mu = \int_{B_2} 1_A P \, d\mu = 0 .$$

Since $B_2 \subset B$ it follows that $\mu(B_2) = 0$, and therefore $B \subset B_1$.

From this proposition we easily see that if T is non singular and if $TA \in \mathcal{R}$, then $PA = TA$.

Let μ_0 be any measure on $(X,\mathcal{R})$ such that $\mu_0 \ll \mu$. Define the measure μ_1 by $\frac{d\mu_0}{d\mu} P = \frac{d\mu_1}{d\mu}$. Then for every $A \in \mathcal{R}$ we have

$$\mu_1(A) = \int_A (\frac{d\mu_0}{d\mu} P)d\mu = \int \frac{d\mu_0}{d\mu} (P1_A)d\mu = \mu_0(T^{-1}A) ,$$

from which we easily derive that the measure μ_0 is invariant under T if and only if it is invariant under the forward process associated with T.

We shall now give an example of a measurable transformation T in a σ-finite measure space $(X,\mathcal{R},\mu)$ such that the measure μT^{-1} is not σ-finite. If P is the forward process associated with this transformation, then it follows that the measure μP is not σ-finite, and we have the example which already was announced in the previous section.

EXAMPLE. Let X be the set of natural numbers, $\mathcal{R}$ be the σ-algebra of all subsets of X and μ the counting measure on $\mathcal{R}$, i.e. $\mu(A)$ is the number of elements of the set A. Obviously, $(X,\mathcal{R},\mu)$ is a σ-finite measure space. Let $(N_1,N_2,\ldots)$ be a countable partition of X such that $\mu(N_i) = \infty$ for every i. The transformation T in $(X,\mathcal{R},\mu)$ is defined by $Tn = i$ if $n \in N_i$. It is easily seen that T is measurable and non singular. Let P be the forward process associated with T. For every non empty set A we have $\mu P(A) = \mu(T^{-1}A) = \infty$, since $T^{-1}A$ contains at least one partition element N_i. It follows $1P = \infty$ on X.

Let P be the forward process associated with a negatively non singular measurable transformation T on a σ-finite measure space $(X,\mathcal{R},\mu)$. We shall consider now the Markov operators P_{μ_0} on $\mathcal{L}_1(X,\mathcal{R},\mu_0)$ corresponding to P, where μ_0 is a σ-finite measure equivalent to μ. First we remark that if a function f is $T^{-1}\mathcal{R}$-measurable, it must be constant on every set $T^{-1}\{x\}$. Therefore we can define for every $T^{-1}\mathcal{R}$-measurable function f the function $f\circ T^{-1}$ on TX by $(f\circ T^{-1})(x) = f(y)$ where y is chosen such that $Ty = x$. If moreover $TX \in \mathcal{R}$ then the function $f\circ T^{-1}$ is $\mathcal{R}$-measurable.

Since $T^{-1}(X \setminus TX) = \emptyset$, the function $f\circ T^{-1}$ is defined μ-almost everywhere if T is positively non singular. If T is negatively non singular, it might happen that $\mu(X \setminus TX) > 0$.

In accordance to the fact that $\frac{d\mu T^{-1}}{d\mu} = 0$ on $X \setminus TX$, we shall define $\frac{d\mu T^{-1}}{d\mu}(f \circ T^{-1}) = 0$ on $X \setminus TX$ for every $T^{-1}\mathcal{R}$-measurable function f.

The next proposition slightly extends one of the results in theorem 1 of [9].

PROPOSITION 1.5.2. Let P be the forward process associated with a negatively non singular measurable transformation T on a σ-finite measure space $(X,\mathcal{R},\mu)$ and assume $TX \in \mathcal{R}$. Let ν be a measure equivalent to μ such that $(X,T^{-1}\mathcal{R},\nu)$ is a σ-finite measure space. Then the corresponding Markov operator P_ν on $\mathcal{L}_1(X,\mathcal{R},\nu)$ is given by

$$fP_\nu = \frac{d\nu T^{-1}}{d\nu}[(E^\nu_{T^{-1}\mathcal{R}} f)\circ T^{-1}] \quad \text{for all } f \in \mathcal{L}_1(X,\mathcal{R},\nu) \ .$$

PROOF. Using lemma 1.5.1 we find for every $A \in \mathcal{R}$

$$\int_A \frac{d\nu T^{-1}}{d\nu}[(E^\nu_{T^{-1}\mathcal{R}} f)\circ T^{-1}]d\nu = \int (1_A \circ T)(E^\nu_{T^{-1}\mathcal{R}} f\circ T^{-1}\circ T)d\nu =$$

$$= \int (1_A \circ T)E^\nu_{T^{-1}\mathcal{R}} f \, d\nu = \int (1_A \circ T) f d\nu = \int f(P1_A)d\nu = \int_A (fP_\nu)d\nu$$

from which the assertion follows.

We now turn to the backward processes associated with P.

PROPOSITION 1.5.3. Let T be a non singular measurable transformation on a σ-finite measure space $(X,\mathcal{R},\mu)$ and let P be the forward process associated with T. Let μ_0 be a measure equivalent to μ. There exists a backward process $\overleftarrow{P}_{\mu_0}$ on

$(X,\mathcal{R},\mu)$ corresponding to the measure μ_0 if and only if $(X,T^{-1}\mathcal{R},\mu_0)$ is a σ-finite measure space.

The process $\overleftarrow{P}_{\mu_0}$ then satisfies

$$f\overleftarrow{P}_{\mu_0} = \frac{d\mu_0}{d\mu}\left(\frac{d\mu}{d\mu_0 T^{-1}}\circ T\right)(f\circ T) \quad \text{for all } f \in M^+(X,\mathcal{R},\mu) \ .$$

If the transformation T also satisfies $TX \in \mathcal{R}$, we have

$$\overleftarrow{P}_{\mu_0} f = (E^{\mu_0}_{T^{-1}\mathcal{R}} f)\circ T^{-1} \quad \text{for all } f \in M^+(X,\mathcal{R},\mu) \ .$$

PROOF. We shall apply proposition 1.4.4. Since T is non singular and $P^{-1}A = T^{-1}A$, by proposition 1.3.7 the condition $1P > 0$ is satisfied. It follows that the backward process $\overleftarrow{P}_{\mu_0}$ exists if and only if the measure μ_1 determined by $\frac{d\mu_1}{d\mu} = \frac{d\mu_0}{d\mu}P$ is σ-finite. The measure μ_1 is the same as the measure $\mu_0 T^{-1}$, hence the backward process $\overleftarrow{P}_{\mu_0}$ exists if and only if $(X,\mathcal{R},\mu_0 T^{-1})$ is a σ-finite measure space, which in turn is the case if and only if the measure space $(X,T^{-1}\mathcal{R},\mu_0)$ is σ-finite.

From the non singularity of T we conclude that the measure $\mu_0 T^{-1}$ is equivalent to μ. Hence, if the backward process $\overleftarrow{P}_{\mu_0}$ exists, we have

$$\frac{d\mu_0 T^{-1}}{d\mu}\,\frac{d\mu}{d\mu_0 T^{-1}} = 1$$

and therefore

$$\left(\frac{d\mu_0 T^{-1}}{d\mu}\circ T\right)\left(\frac{d\mu}{d\mu_0 T^{-1}}\circ T\right) = 1$$

and we conclude from proposition 1.4.4 for every $f \in M^+(X,\mathcal{R},\mu)$

$$fP^{\leftarrow}_{\mu_0} = \frac{d\mu_0}{d\mu}\,\frac{f\circ T}{\frac{d\mu_0 T^{-1}}{d\mu}\circ T} = \frac{d\mu_0}{d\mu}\left(\frac{d\mu}{d\mu_0 T^{-1}}\circ T\right)(f\circ T) .$$

Now assume $TX \in \mathcal{R}$ and $P^{\leftarrow}_{\mu_0}$ exists. It follows from proposition 1.4.4 and from proposition 1.5.2 that for all $f \in M^+(X,\mathcal{R},\mu)$ we have

$$P^{\leftarrow}_{\mu_0} f = \frac{fP_{\mu_0}}{1P_{\mu_0}} = \frac{\frac{d\mu_0 T^{-1}}{d\mu_0}\left(E^{\mu_0}_{T^{-1}\mathcal{R}}\, f\circ T^{-1}\right)}{\frac{d\mu_0 T^{-1}}{d\mu_0}} = E^{\mu_0}_{T^{-1}\mathcal{R}}\, f\circ T^{-1} .$$

In the sequel, when we are speaking of a backward process $P^{\leftarrow}_{\mu_0}$ for a Markov process P on $(X,\mathcal{R},\mu)$, satisfying $1P > 0$, we shall always assume that $\mu_0 \approx \mu$ and that the backward process for this measure μ_0 really exist.

PROPOSITION 1.5.4. Let $P^{\leftarrow}_{\mu_0}$ be a backward process associated with a non singular measurable transformation T on a σ-finite measure space $(X,\mathcal{R},\mu)$. Let ν_0 be a σ-finite measure on $(X,\mathcal{R})$ equivalent to μ. Then the backward process $(P^{\leftarrow}_{\mu_0})^{\leftarrow}_{\nu_0}$ for $P^{\leftarrow}_{\mu_0}$ corresponding to the measure ν_0 exists, and $(P^{\leftarrow}_{\mu_0})^{\leftarrow}_{\nu_0} = P$, where P is the forward process associated with the transformation T.

PROOF. The measures μ_0, ν_0 and $\mu_0 T^{-1}$ are σ-finite measures on $(X,\mathcal{R})$ and equivalent to μ. Therefore the Radon-Nikodym derivatives $\frac{d\mu_0}{d\mu}$, $\frac{d\nu_0}{d\mu}$ and $\frac{d\mu}{d\mu_0 T^{-1}}$ are positive and finite. By proposition 1.4.4 the backward process $(\overleftarrow{P}_{\mu_0})^{\leftarrow}_{\nu_0}$ for $\overleftarrow{P}_{\mu_0}$ corresponding to the measure ν_0 exists if $1\overleftarrow{P}_{\mu_0} > 0$ and $\frac{d\nu_0}{d\mu}\overleftarrow{P}_{\mu_0} < \infty$, and it easily follows from proposition 1.5.3 that these conditions are satisfied. Then for all $f \in M^+(X,\mathcal{R},\mu)$ we have

$$(\overleftarrow{P}_{\mu_0})^{\leftarrow}_{\nu_0} f = \frac{(f\,\frac{d\nu_0}{d\mu})\overleftarrow{P}_{\mu_0}}{\frac{d\nu_0}{d\mu}\overleftarrow{P}_{\mu_0}} = \frac{\frac{d\mu_0}{d\mu}\,(\frac{d\mu}{d\mu_0 T^{-1}}\circ T)(f\circ T)(\frac{d\nu_0}{d\mu}\circ T)}{\frac{d\mu_0}{d\mu}\,(\frac{d\mu}{d\mu_0 T^{-1}}\circ T)(\frac{d\nu_0}{d\mu}\circ T)} = Pf\ ,$$

hence $(\overleftarrow{P}_{\mu_0})^{\leftarrow}_{\nu_0} = P$.

The next proposition gives a condition under which two backward processes $\overleftarrow{P}_{\mu_0}$ and $\overleftarrow{P}_{\nu_0}$ associated with T are equal.

PROPOSITION 1.5.5. Let T be a non singular measurable transformation in a σ-finite measure space $(X,\mathcal{R},\mu)$. Let μ_0 and ν_0 be measures on $(X,\mathcal{R})$ equivalent to μ such that the backward processes $\overleftarrow{P}_{\mu_0}$ and $\overleftarrow{P}_{\nu_0}$ exist. Then $\overleftarrow{P}_{\mu_0} = \overleftarrow{P}_{\nu_0}$ if and only if $\frac{d\nu_0}{d\mu_0}$ is $T^{-1}\mathcal{R}$-measurable.

PROOF. It follows from proposition 1.5.3 that the Markov operator in $\mathcal{L}_1(X,\mathcal{R},\mu_0)$ corresponding to the process $\overleftarrow{P}_{\mu_0}$ is given by

$$fP^{\leftarrow}_{\mu_0} = (\frac{d\mu_0}{d\mu_0 T^{-1}} \circ T)(f \circ T) \qquad \text{for all } f \in \mathcal{L}_1(X,\mathcal{R},\mu_0) \ .$$

Similarly, the Markov operator in $\mathcal{L}_1(X,\mathcal{R},\nu_0)$ corresponding to the process $P^{\leftarrow}_{\nu_0}$ is given by

$$fP^{\leftarrow}_{\nu_0} = (\frac{d\nu_0}{d\nu_0 T^{-1}} \circ T)(f \circ T) \qquad \text{for all } f \in \mathcal{L}_1(X,\mathcal{R},\nu_0) \ .$$

Using proposition 1.3.4 we see that the processes P_{μ_0} and $P^{\leftarrow}_{\nu_0}$ are equal if and only if for all $f \in \mathcal{L}_1(X,\mathcal{R},\mu_0)$ we have

$$(\frac{d\mu_0}{d\mu_0 T^{-1}} \circ T)(f \circ T) = (f \circ T)(\frac{d\mu_0}{d\nu_0} \circ T)(\frac{d\nu_0}{d\nu_0 T^{-1}} \circ T)\ \frac{d\nu_0}{d\mu_0} \ ,$$

hence if and only if

$$(1) \qquad \frac{d\mu_0}{d\mu_0 T^{-1}} \circ T = (\frac{d\mu_0}{d\nu_0} \circ T)(\frac{d\nu_0}{d\nu_0 T^{-1}} \circ T)\ \frac{d\nu_0}{d\mu_0} \ .$$

If the processes $P^{\leftarrow}_{\mu_0}$ and $P^{\leftarrow}_{\nu_0}$ are equal, then, since each of the Radon-Nikodym derivatives is positive, it follows from (1) that $\frac{d\nu_0}{d\mu_0}$ is $T^{-1}\mathcal{R}$-measurable.

Conversely, for every set $T^{-1}A \in T^{-1}\mathcal{R}$ we have, using lemma 1.5.1

$$\int_{T^{-1}A} \frac{d\mu_0}{d\mu_0 T^{-1}} \circ T \, d\mu_0 = \int_A \frac{d\mu_0}{d\mu_0 T^{-1}} \frac{d\mu_0 T^{-1}}{d\mu_0} \, d\mu_0 = \mu_0(A) \ ;$$

$$\int_{T^{-1}A} \frac{d\mu_0}{d\nu_0} \circ T \; \frac{d\nu_0}{d\nu_0 T^{-1}} \circ T \; \frac{d\nu_0}{d\mu_0} \, d\mu_0 =$$

$$= \int_A \frac{d\mu_0}{d\nu_0} \frac{d\nu_0}{d\nu_0 T^{-1}} \frac{d\nu_0 T^{-1}}{d\nu_0} \, d\nu_0 = \mu_0(A)$$

hence, if $\frac{d\nu_0}{d\mu_0}$ is $T^{-1}\mathcal{R}$-measurable, then relation (1) holds and $\overleftarrow{P}_{\mu_0} = \overleftarrow{P}_{\nu_0}$.

We conclude this section with a representation theorem for the backward processes associated with a transformation.

<u>THEOREM 1.5.1</u>. Let T be a non singular measurable transformation in a measure space $(X,\mathcal{R},\mu)$, and assume that $(X,T^{-1}\mathcal{R},\mu)$ is a σ-finite measure space. Let $\overleftarrow{P}_{\mu_0}$ be a backward process associated with T corresponding to the measure $\mu_0 \approx \mu$. Then the mapping

$$\overleftarrow{P}_{\mu_0} \to h = \frac{\frac{d\mu_0}{d\mu}}{E^{\mu}_{T^{-1}\mathcal{R}} \frac{d\mu_0}{d\mu}}$$

establishes a one-to-one correspondence between the class of backward processes associated with T and the class of functions $h \in M^+(X,\mathcal{R},\mu)$ such that $E^{\mu}_{T^{-1}\mathcal{R}} h = 1$ and $h > 0$.

If the function h corresponds to the process $\overleftarrow{P}_{\mu_0}$, then

$$f\overleftarrow{P}_{\mu_0} = \frac{d\mu T}{d\mu} h(f\circ T) \qquad \text{for all } f \in M^+(X,\mathcal{R},\mu) \; .$$

PROOF. Assume $\overleftarrow{P}_{\mu_0}$ exists. Since μ_0 and μ are σ-finite equivalent measures on $(X, T^{-1}\mathcal{R})$, we have $0 < E^{\mu}_{T^{-1}\mathcal{R}} \frac{d\mu_0}{d\mu} < \infty$. Define

$$h = \frac{\frac{d\mu_0}{d\mu}}{E^{\mu}_{T^{-1}\mathcal{R}} \frac{d\mu_0}{d\mu}},$$

then $h > 0$ since also $\frac{d\mu_0}{d\mu} > 0$, and

$$E^{\mu}_{T^{-1}\mathcal{R}} h = \frac{E^{\mu}_{T^{-1}\mathcal{R}} \frac{d\mu_0}{d\mu}}{E^{\mu}_{T^{-1}\mathcal{R}} \frac{d\mu_0}{d\mu}} = 1 .$$

Let P_{ν_0} be another backward process which is mapped onto the same function h. Then we have

$$\frac{\frac{d\mu_0}{d\mu}}{E^{\mu}_{T^{-1}\mathcal{R}} \frac{d\mu_0}{d\mu}} = \frac{\frac{d\nu_0}{d\mu}}{E^{\mu}_{T^{-1}\mathcal{R}} \frac{d\nu_0}{d\mu}}$$

$$\frac{d\nu_0}{d\mu_0} = \frac{d\nu_0}{d\mu} \frac{d\mu}{d\mu_0} = \frac{E^{\mu}_{T^{-1}\mathcal{R}} \frac{d\nu_0}{d\mu}}{E^{\mu}_{T^{-1}\mathcal{R}} \frac{d\mu_0}{d\mu}}$$

and it follows by proposition 1.5.5 that $\overleftarrow{P}_{\mu_0} = \overleftarrow{P}_{\nu_0}$.

Now let $h \in M^{+}(X, \mathcal{R}, \mu)$ be a function such that $h > 0$ and $E^{\mu}_{T^{-1}\mathcal{R}} h = 1$. Define the measure μ_0 on $(X, \mathcal{R})$ by

$$\mu_0(A) = \int_A h d\mu \qquad \text{for all } A \in \mathcal{R} .$$

Since the measure space $(X, T^{-1}\mathcal{R}, \mu)$ is σ-finite, there exists an $\mathcal{R}$-measurable partition $(X_1, X_2, \ldots)$ of X such that $\mu(T^{-1}X_i) < \infty$ for every i. Then for every i

$$\mu_0(T^{-1}X_i) = \int_{T^{-1}X_i} h d\mu = \int_{T^{-1}X_i} E^{\mu}_{T^{-1}\mathcal{R}} h \, d\mu = \mu(T^{-1}X_i) < \infty .$$

It follows that also the measure space $(X, T^{-1}\mathcal{R}, \mu_0)$ is σ-finite. From $h > 0$ we deduce $\mu_0 \approx \mu$ and by proposition 1.5.3 the backward process $\overleftarrow{P}_{\mu_0}$ exists. It is clear from the construction of μ_0 that the backward process $\overleftarrow{P}_{\mu_0}$ is mapped onto the function h.

Finally, let $\overleftarrow{P}_{\mu_0}$ be any backward process associated with T. Then the relationship between h and μ_0 is given by

$$\frac{d\mu_0}{d\mu} = h \, E^{\mu}_{T^{-1}\mathcal{R}} \frac{d\mu_0}{d\mu} .$$

For all $f \in M^+(X, \mathcal{R}, \mu)$ we have

$$f\overleftarrow{P}_{\mu_0} = \frac{d\mu_0}{d\mu} \left(\frac{d\mu}{d\mu_0 T^{-1}} \circ T\right)(f \circ T) .$$

Define

$$r = \frac{d\mu_0}{d\mu} \left(\frac{d\mu}{d\mu_0 T^{-1}} \circ T\right) .$$

For all $A \in \mathcal{R}$ we obtain by lemma 1.5.1

$$\int_{T^{-1}A} r\,d\mu = \int (1_A \circ T)\left(\frac{d\mu}{d\mu_0 T^{-1}} \circ T\right) d\mu_0 = \int_A \frac{d\mu_0 T^{-1}}{d\mu_0} \frac{d\mu}{d\mu_0 T^{-1}}\, d\mu_0 =$$

$$= \mu(A) = \int_{T^{-1}A} \frac{d\mu T}{d\mu}\, d\mu$$

hence $E^{\mu}_{T^{-1}\mathcal{R}}\, r = \frac{d\mu T}{d\mu}$.

On the other hand we have

$$E^{\mu}_{T^{-1}\mathcal{R}}\, r = \left(\frac{d\mu}{d\mu_0 T^{-1}} \circ T\right)\left(E^{\mu}_{T^{-1}\mathcal{R}} \frac{d\mu_0}{d\mu}\right) = \left(\frac{d\mu}{d\mu_0 T^{-1}} \circ T\right) \frac{d\mu_0}{d\mu} \frac{1}{h} ,$$

hence

$$\frac{d\mu}{d\mu_0 T^{-1}} \circ T = \frac{d\mu T}{d\mu}\, h\, \frac{d\mu}{d\mu_0}$$

and therefore $r = \frac{d\mu T}{d\mu} h$.

It follows that the backward process $\overleftarrow{P}_{\mu_0}$ is given by

$$f\overleftarrow{P}_{\mu_0} = \frac{d\mu T}{d\mu}\, h(f \circ T) \qquad \text{for all } f \in M^{+}(X,\mathcal{R},\mu) .$$

COROLLARY. If T is a non singular measurable transformation in a probability space $(X,\mathcal{R},\mu)$, then the process $\overleftarrow{P}$ corresponding to the transition $x \to T^{-1}\{x\}$ as introduced by Hopf [13], § 6, see also [9], § 4, is the backward process $\overleftarrow{P}_{\mu}$. This is easily seen by taking $h = 1$.

CHAPTER II

RECURRENCE PROPERTIES OF THE MARKOV SHIFT

2.1. CONSERVATIVITY FOR BACKWARD PROCESSES

For an invertible measurable transformation T in a σ-finite measure space $(X,\mathcal{R},\mu)$ E. Hopf [12] in 1937 has shown that X can be split up in a modulo μ unique way into two parts C and D, named the conservative and the dissipative part respectively, such that for all subsets $B \in \mathcal{R}$ of C μ-almost all points of B return to B under the action of T, and D is a countable union of wandering sets (a set $W \in \mathcal{R}$ is said to be wandering if $W \cap T^{-k}W = \emptyset$ for all $k \geq 1$).

Helmberg [8], cf. also [22], has shown that the condition of invertibility of T can be dropped. The conservative part is then characterized by the recurrence property for every subset and the dissipative part is again a countable union of wandering sets.

In 1954, E. Hopf [13] has shown how to decompose the space X with respect to a Markov process P on $(X,\mathcal{R},\mu)$. In [9], theorem 4, it is shown that the conservative part of X with respect to a transformation coincides with the conservative part of X with respect to the corresponding forward process, provided that the transformation T is negatively non singular.

In this section we shall study the relationship between the conservative part of X with respect to a Markov process P on $(X,\mathcal{R},\mu)$ and the conservative parts of X for the corresponding backward processes. In particular we shall consider the case when P is the forward process associated with a non singular measurable transformation on $(X,\mathcal{R},\mu)$.

DEFINITION 2.1.1. Let P be a Markov process on a σ-finite measure space $(X,\mathcal{R},\mu)$. The conservative part of X with respect to P is the modulo μ unique set C such that for all $f \in \mathcal{L}_1^+(X,\mathcal{R},\mu)$ we have

$$\sum_{k=0}^{\infty} fP^k = \begin{cases} 0 \text{ or } \infty \text{ on } C \\ < \infty \quad \text{ on } X \setminus C . \end{cases}$$

The dissipative part of X with respect to P is the set $D = X \setminus C$.

The existence of the partition as described in definition 2.1.1 has been shown by Hopf [13]. For a very elegant treatment of this subject the reader is referred to Foguel [4], chapter II.

A Markov process P on $(X,\mathcal{R},\mu)$ is said to be conservative on a set $A \in \mathcal{R}$ if $A \subset C$. The following proposition shows that the conservative part of X can also be characterized in terms of the Markov operator on $\mathcal{L}_\infty(X,\mathcal{R},\mu)$, and therefore that the conservative part of X with respect to P is not changed if we replace the measure μ by an equivalent σ-finite measure ν on $(X,\mathcal{R})$.

PROPOSITION 2.1.1. The following statements are equivalent for every set $A \in \mathcal{R}$.

i) P is conservative on A.

ii) For all $B \subset A$, $B \in \mathcal{R}$ we have $\sum_{k=0}^{\infty} I_B(PI_{X\setminus B})^k P1_B = 1_B$.

iii) For all $B \subset A$, $B \in \mathcal{R}$ the sum $\sum_{k=0}^{\infty} P^k 1_B$ is μ-essentially unbounded if $\mu(B) > 0$.

PROOF. See Feldman [3], theorem 2.1 and corollary 1b.

PROPOSITION 2.1.2. Let P be a Markov process on a σ-finite measure space $(X,\mathcal{R},\mu)$. Then for all $A \subset C$ we have $\sum_{k=0}^{\infty} P^k 1_A = \infty$ on A, and there exists a partition $(D_1,D_2,\dots)$ of D such that $\sum_{k=0}^{\infty} P^k 1_{D_i} \in \mathcal{L}_\infty(X,\mathcal{R},\mu)$ for all i.

PROOF. Let $A \in \mathcal{R}$ be a subset of C, and suppose $\sum_{k=0}^{\infty} P^k 1_A \le K < \infty$ on $B \subset A$, $\mu(B) < \infty$. Then

$$\infty > \int_B (\sum_{k=0}^{\infty} P^k 1_A) d\mu = \sum_{k=0}^{\infty} \int_A (1_B P^k) d\mu = \int_A (\sum_{k=0}^{\infty} 1_B P^k) d\mu ,$$

and since $\sum_{k=0}^{\infty} 1_B P^k = \infty$ on B, we conclude $\mu(B) = 0$. It follows that $\sum_{k=0}^{\infty} P^k 1_A = \infty$ on A.

The second statement follows from proposition 2.1.1 by an exhaustion procedure.

The following property characterizes the conservative part of X by the non existence of on C subinvariant functions.

PROPOSITION 2.1.3. For every $f \in M^+(X,\mathcal{R},\mu)$ with $f < \infty$ we have $Pf \le f$ on $C \Rightarrow Pf = f$ on C, $fP \le f$ on $C \Rightarrow fP = f$ on C.

PROOF. See Foguel [4], chapter II theorem B and (2.10).

PROPOSITION 2.1.4. Let C be the conservative part of a Markov process P on $(X,\mathcal{R},\mu)$. Then we have $P1_C \geq 1_C$, $1_C P > 0$ on C and $1_C P = 0$ on D.

PROOF. Since $P1_C \leq 1_C$ on C, we must have $P1_C = 1$ on C, and therefore $P1_C \geq 1_C$. It follows that $P1_D = 0$ on C, $\int (1_C P) 1_D \, d\mu = 0$, hence $1_C P = 0$ on D.

Finally, suppose $1_C P = 0$ on $A \subset C$. Then by proposition 1.3.6 we have $1_C P^k = 0$ on A for every k, hence

$$0 = \int \Big(\sum_{k=0}^{\infty} 1_C P^k \Big) 1_A \, d\mu = \int 1_C \Big(\sum_{k=0}^{\infty} P^k 1_A \Big) d\mu \; .$$

Since $\sum_{k=0}^{\infty} P^k 1_A = \infty$ on A by proposition 2.1.2, it follows that $\mu(A) = 0$.

As a consequence we have $PC = C$ and $P^{-1}D \subset D$.

The implications in proposition 2.1.3 do in general not hold for on C superinvariant functions. The following counter-example has already been studied by Post [21] and is reproduced here in a slightly different way, fitting into the present context.

EXAMPLE. Let $(X,\mathcal{R},\mu)$ be the unit interval with the Borel sets and the Lebesgue measure, and consider the transformation $Tx = 2x \pmod 1$ on $(X,\mathcal{R},\mu)$. Let $\vec{P}$ and $\overleftarrow{P}_\mu$ be the forward and backward process associated with T on $(X,\mathcal{R},\mu)$. Since μ is invariant, T and therefore P is conservative, and because of [9], corollary 5.1 or theorem 2.1.1 in this section, also $\overleftarrow{P}_\mu$

is conservative. From the propositions 1.5.2 and 1.5.3 we easily deduce that for all $f \in M^+(X,\mathcal{R},\mu)$ we have

$$fP = \overset{\leftarrow}{P}_\mu f = E^\mu_{T^{-1}\mathcal{R}} f \circ T^{-1} = \tfrac{1}{2}(f(\tfrac{1}{2}x) + f(\tfrac{1}{2}x + \tfrac{1}{2})) \ ;$$

in particular we have

$$\frac{1}{x} P = \overset{\leftarrow}{P} \frac{1}{x} = \frac{1}{x} + \frac{1}{x+1} > \frac{1}{x} \quad \text{on } X \ .$$

Let us denote by $C(P)$ the conservative part of X with respect to a Markov process P on $(X,\mathcal{R},\mu)$.

PROPOSITION 2.1.5. For every n we have $C(P) = C(P^n)$.

PROOF. Choose $f \in \mathcal{L}_1^+(X,\mathcal{R},\mu)$ such that $f > 0$. Then

$$\sum_{k=0}^{\infty} fP^k = \sum_{k=0}^{\infty} \Big(\sum_{i=0}^{n-1} fP^i\Big)P^{kn} = \begin{cases} \infty \text{ on } C(P) \\ < \infty \text{ on } X \setminus C(P) \ . \end{cases}$$

Since $\sum_{i=0}^{n-1} fP^i \in \mathcal{L}_1^+(X,\mathcal{R},\mu)$ and $\sum_{i=0}^{n-1} fP^i > 0$, we obtain $C(P^n) = C(P)$.

PROPOSITION 2.1.6. Let P be a Markov process on a σ-finite measure space $(X,\mathcal{R},\mu)$ and let μ_0 be a subinvariant measure for P equivalent to μ. Let $P^*_{\mu_0}$ be the adjoint process of P with respect to μ_0. Then $C(P^*_{\mu_0}) = C(P)$.

PROOF. See Foguel [4], (7.2).

PROPOSITION 2.1.7. Let P be the forward process associated with a non singular measurable transformation on a σ-finite measure space $(X,\mathcal{R},\mu)$. Let μ_0 be a measure equivalent to μ

such that the backward process $\overleftarrow{P}_{\mu_0}$ exists. Then $C(\overleftarrow{P}_{\mu_0}) \subset C(P)$.

PROOF. The set $X \setminus C(P)$ is a countable union of wandering sets. Let W be a wandering set such that $\mu(W) < \infty$, then $1_W \in \mathcal{L}_1(X,\mathcal{R},\mu)$. Proposition 1.5.3 shows that $1_W\overleftarrow{P}^n_{\mu_0}$ can be written as a finite product of functions, one of which is $1_W \circ T^n$. Since W is wandering, it follows that $1_W\overleftarrow{P}^n_{\mu_0} = 0$ on W, hence $\sum_{n=1}^{\infty} 1_W\overleftarrow{P}^n_{\mu_0} < \infty$ on W, $W \subset X \setminus C(\overleftarrow{P}_{\mu_0})$.

Tsurumi [23] has shown that this inclusion may be strict by producing an example of an ergodic measure preserving transformation on a probability space with a dissipative backward process. Let Q be this backward process, and $\overleftarrow{Q}_{\mu_0}$ any backward process of Q. Then by proposition 1.5.4 we have $\overleftarrow{Q}_{\mu_0} = P$, and $C(Q) = \emptyset$, $C(\overleftarrow{Q}_{\mu_0}) = X$. It follows that proposition 2.1.7 does in general not hold for all Markov processes on $(X,\mathcal{R},\mu)$.

We shall show now that Tsurumi's example is a special case of a general situation. In fact, if T is a measurable ergodic measure preserving transformation in a probability space $(X,\mathcal{R},\mu)$, then all the backward processes, except the backward process corresponding to the invariant probability μ, are dissipative.

We start with a lemma which will be needed in the proof.

LEMMA 2.1.1. Let $(a_n)_{n=1}^{\infty}$ be a sequence of real numbers, with $a_n > 0$ for all n. If $\lim_{n\to\infty} \frac{\log a_n}{n} < 0$, then $\sum_{n=1}^{\infty} a_n < \infty$.

PROOF. Put $b_n = \sqrt[n]{a_n}$, then $\lim_{n\to\infty} \log b_n = \lim_{n\to\infty} \frac{\log a_n}{n} < 0$, hence $\lim_{n\to\infty} b_n < 1$ and $\sum_{n=1}^{\infty} a_n < \infty$.

THEOREM 2.1.1. Let T be an ergodic measure preserving transformation in a probability space $(X,\mathcal{R},\mu)$. Then all backward processes associated with T are dissipative except the backward process associated with the invariant probability μ, which is conservative.

PROOF. We use theorem 1.5.1. Since μ is invariant we have $\frac{d\mu T}{d\mu} = 1$, and every backward process is given by

$$f\overleftarrow{P}_h = h(f\circ T) \qquad \text{for all } f \in M^+(X,\mathcal{R},\mu) \ ,$$

where $h > 0$, $h \in M^+(X,\mathcal{R},\mu)$ and $E^{\mu}_{T^{-1}\mathcal{R}} h = 1$.

Since $1 \in \mathcal{L}_1(X,\mathcal{R},\mu)$, it suffices to show $\sum_{n=1}^{\infty} 1\overleftarrow{P}_h^{n} < \infty$, unless $h = 1$. For every n we have

$$1P_h^n = h(h\circ T)\ \dots\ (h\circ T^{n-1})$$

$$\frac{\log 1P_h^n}{n} = \frac{1}{n}\sum_{i=0}^{n-1} (\log h)\circ T^i \ .$$

We now have to consider two cases.

i) $\log h \in \mathcal{L}_1(X,\mathcal{R},\mu)$. Then because of the individual ergodic theorem and the fact that the only invariant functions for an ergodic transformation are constants (cf. [6], p.18 and p.25), we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{\infty} \log h\circ T^i = \int \log h \, d\mu \; .$$

Since $\log h \le h-1$, we have

$$\int \log h \, d\mu \le \int h d\mu - 1 = 0$$

and the equality sign holds if and only if $h = 1$.
Hence, if h is not identically 1, then because of lemma 2.1.1 we have $\sum_{n=0}^{\infty} 1P_h^{\leftarrow n} < \infty$ and P_h is dissipative.
If $h = 1$, then $1P_1^{\leftarrow} = 1$, $\sum_{n=0}^{\infty} 1P_1^{\leftarrow n} = \infty$, and $P_1^{\leftarrow}$ is conservative. Obviously, the case $h = 1$ corresponds to the backward process $P_\mu^{\leftarrow}$.

ii) $\log h \notin \mathcal{L}_1(X,\mathcal{R},\mu)$. Since $0 \le (\log h)^+ < h$, we have $(\log h)^+ \in \mathcal{L}_1(X,\mathcal{R},\mu)$. Define for every k
$f_k = \max(\log h,-k)$, then $f_k \in \mathcal{L}_1(X,\mathcal{R},\mu)$ and $f_k \downarrow \log h$.
For every k we obtain

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log h\circ T^i \le \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f_k\circ T^i = \int f_k \, d\mu$$

and therefore, if $k \to \infty$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log h\circ T^i = -\infty \; .$$

Again by lemma 2.1.1 it follows that $\sum_{n=0}^{\infty} 1P_h^{\leftarrow n} < \infty$, and $P_h^{\leftarrow}$ is dissipative.

2.2. SINGULARITY OF MARKOV MEASURES

Throughout this section we shall assume that P is the operator in $B(X,\mathcal{R})$ given by a transition probability as in definition 1.2.2, and $(\Omega',\mathfrak{A}',M_0',S')$ is the corresponding one-sided shift space with initial measure μ_0.

DEFINITION 2.2.1. Let T and T' be measurable transformations on the σ-finite measure spaces $(X,\mathcal{R},\mu)$ and $(X',\mathcal{R}',\mu')$, respectively. An isomorphism φ from $\mathcal{R}$ onto $\mathcal{R}'$ is said to be an isomorphism from $(X,\mathcal{R},\mu,T)$ to $(X',\mathcal{R}',\mu',T')$ if φ satisfies $\mu'(\varphi A) = \mu(A)$ and $\varphi(T^{-1}A) = T'^{-1}(\varphi A)$ for all $A \in \mathcal{R}$. In this case the systems $(X,\mathcal{R},\mu,T)$ and $(X',\mathcal{R}',\mu',T')$ are said to be isomorphic.

PROPOSITION 2.2.1. Let $(\Omega',\mathfrak{A}',M_0',S')$ be the one-sided shift space for P with initial measure μ_0. Let $(\mu_n)_{n=0}^{\infty}$ be the sequence of marginal measures on $(X,\mathcal{R})$ and let for every n M_n' be the Markov measure on $(\Omega',\mathfrak{A}')$ with initial measure μ_n. Then S'^{-n} is an isomorphism from $(\Omega',\mathfrak{A}',M_n',S')$ to $(\Omega',\mathfrak{A}_{n,\infty}',M_0',S')$.

PROOF. Obviously, S'^{-n} is an isomorphism of $\mathfrak{A}'$ onto $\mathfrak{A}_{n,\infty}'$ such that for all $A \in \mathfrak{A}'$ we have $S'^{-n}(S'^{-1}A) = S'^{-1}(S'^{-n}A)$, and by proposition 1.2.2 and proposition 1.2.4

$$M_n'(A) = \int P_x(A)\mu_n(dx) = \int P^n(P_\cdot(A))d\mu_0 =$$

$$= \int P_x(S'^{-n}A)\mu_0\, dx = M_0'(S'^{-n}A) \ .$$

PROPOSITION 2.2.2. Let N_0' and M_0' be the Markov measures on

$(\Omega', \mathcal{O}\mathcal{L}')$ corresponding to the initial measures ν_0 and μ_0 on $(X, \mathcal{R})$ respectively. Then $N_0' \ll M_0'$ if and only if $\nu_0 \ll \mu_0$. If μ_0 is σ-finite and $\nu_0 \ll \mu_0$, then we have

$$\frac{dN_0'}{dM_0'}(\omega') = \frac{d\nu_0}{d\mu_0}(\pi_0' \, \omega') \quad \text{for } M_0'\text{-almost all } \omega' \in \Omega' .$$

PROOF. Assume $\nu_0 \ll \mu_0$. If for some $A \in \mathcal{O}\mathcal{L}'$ we have $M_0'(A) = 0$, then $P_x(A) = 0$ for μ_0-almost all $x \in X$, and therefore for ν_0-almost all $x \in X$. It follows that also $N_0'(A) = 0$, hence $N_0' \ll M_0'$.

If μ_0 is σ-finite, then by proposition 1.2.4 also M_0' is σ-finite. It easily follows from the definition of M_0' that for every rectangle $A = \prod_{i=0}^{\infty} A_i$, where $A_i = X_i$ if $i > n$ we have

$$\int_A \frac{d\nu_0}{d\mu_0}(\pi_0' \, \omega') M_0'(d\omega') =$$

$$= \int \frac{d\nu_0}{d\mu_0}(x_0)(I_{A_0} P \ldots P 1_{A_n})(x_0)\mu_0(dx_0) = N_0'(A) .$$

From this relation we conclude $\frac{dN_0'}{dM_0'}(\omega') = \frac{d\nu_0}{d\mu_0}(\pi_0' \, \omega')$ for M_0'-almost all $\omega' \in \Omega'$.

Now suppose $N_0' \ll M_0'$. Define for every $A \in \mathcal{R}$ the set $A' \in \mathcal{O}\mathcal{L}'$ by $A' = \{\omega' \mid \pi_0' \, \omega' \in A\}$, then we have $\nu_0(A) = N_0'(A')$, $\mu_0(A) = M_0'(A')$. It follows that $\nu_0 \ll \mu_0$.

Now we define the two-sided shift space for a transition probability P on $(X, \mathcal{R})$.

DEFINITION 2.2.2. For every integer t let $(X_t, \mathcal{R}_t)$ be a copy of $(X, \mathcal{R})$, and put $(\Omega, \mathfrak{X}) = \prod_{t=-\infty}^{+\infty} (X_t, \mathcal{R}_t)$. Let π_t be the projection of Ω onto X_t and $\mathfrak{X}_t$ the σ-algebra $\pi_t^{-1} \mathcal{R}_t$. For $-\infty \leq n < m \leq \infty$ let $\mathfrak{X}_{nm}$ be the σ-algebra generated by the σ-algebras $\mathfrak{X}_t$ for $n \leq t \leq m$.

For every measure M on $(\Omega, \mathfrak{X})$ and every integer n the marginal measure μ_n on $(X, \mathcal{R})$ is defined to be the measure $M\pi_n^{-1}$.

A measure M on $(\Omega, \mathfrak{X})$ is said to be a Markov measure for P if for every rectangle $\prod_{i=-\infty}^{+\infty} A_i$, where $A_i = X_i$ if $i < n$ or $i > m$ we have

$$M\left(\prod_{i=-\infty}^{+\infty} A_i\right) = \int I_{A_n} P \ldots P 1_{A_m} d\mu_n .$$

The shift S on the space $(\Omega, \mathfrak{X})$ is the mapping $S : \Omega \to \Omega$ defined by $\pi_t S\omega = \pi_{t+1}\omega$ for every t and all $\omega \in \Omega$.

The system $(\Omega, \mathfrak{X}, M, S)$ is said to be a two-sided shift space for P if M is a Markov measure for P.

Note that if $(\Omega, \mathfrak{X}, M, S)$ is a two-sided shift space for P with marginal measures $(\mu_n)_{n=-\infty}^{+\infty}$, we have $\mu_n P = \mu_{n+1}$ for all n. If the measure μ_0 is given, this relation uniquely defines the measures μ_n with $n > 0$. However, the Markov measure M is in general, in contrast with the one-sided shift space, not uniquely defined by μ_0. In section 2.4 we shall consider some examples. It also may happen that no two-sided shift space for a transition probability P on $(X, \mathcal{R})$ exists.

EXAMPLE. Let $(X, \mathcal{R})$ be the unit interval with the Borel sets, and define the transition probability P by

$$P(x,A) = 1_A(\tfrac{1}{2}x) \qquad \text{for all } x \in X \text{ and all } A \in \mathcal{R} .$$

Suppose there exists a two-sided shift space for P with marginal measures $(\mu_n)_{n=-\infty}^{+\infty}$. Then for every n there exists a measure μ_{-n} on $(X,\mathcal{R})$ such that $\mu_{-n}P^n = \mu_0$. It follows from the definition of P that therefore we must have $\mu_0([\frac{1}{2^n}, 1)) = 0$ for every n, hence $\mu_0(X) = M(\Omega) = 0$. Contradiction.

PROPOSITION 2.2.3. Let $(\Omega, \mathfrak{A}, M, S)$ be a two-sided shift space for P with marginal measures $(\mu_n)_{n=-\infty}^{+\infty}$. Let M_n' be the Markov measure on $(\Omega', \mathfrak{A}')$ corresponding to the initial measure μ_n. Then for every integer n the mapping $\varphi_n : \mathfrak{A}' \to \mathfrak{A}_{n,\infty}$ defined by $\varphi_n(A') = \{\omega \mid (\pi_n\omega, \pi_{n+1}\omega, \ldots) \in A'\}$ for all $A' \in \mathfrak{A}'$ is an isomorphism from $(\Omega', \mathfrak{A}', M_n', S')$ to $(\Omega, \mathfrak{A}_{n,\infty}, M, S)$.

PROOF. Obviously, φ_n is an isomorphism of $\mathfrak{A}'$ onto $\mathfrak{A}_{n,\infty}$ such that for all $A' \in \mathfrak{A}'$ we have $\varphi_n(S'^{-1}A') = S^{-1}(\varphi_n A')$. Moreover, for every rectangle $\prod_{i=0}^{\infty} A_i \in \mathfrak{A}'$, where $A_i = X_i$ if $i > m$, we have

$$M_n'\left(\prod_{i=0}^{\infty} A_i\right) = \int I_{A_0} P \ldots P 1_{A_m} d\mu_n = M\left(\varphi_n\left(\prod_{i=0}^{\infty} A_i\right)\right) ,$$

from which we deduce $M_n'(A') = M(\varphi_n A')$ for all $A' \in \mathfrak{A}'$.

Now we shall study some properties of Markov probabilities for P on the two-sided product space $(\Omega, \mathfrak{A})$. In particular, we shall derive a necessary and sufficient condition on the marginal measures under which the Markov probabilities are singular.

DEFINITION 2.2.3. Let $(\Omega, \mathcal{O}\!\mathcal{L})$ be a measurable space, and M and N measures on $(\Omega, \mathcal{O}\!\mathcal{L})$. The measures M and N are said to be singular (with respect to each other), in notation $M \perp N$, if there exists a set $A \in \mathcal{O}\!\mathcal{L}$ such that $N(A) = 0$ and $M(\Omega \setminus A) = 0$.

PROPOSITION 2.2.4. Let M and N be Markov probabilities for P on $(\Omega, \mathcal{O}\!\mathcal{L})$ with marginal probabilities $(\mu_n)_{n=-\infty}^{+\infty}$, $(\nu_n)_{n=-\infty}^{+\infty}$ respectively. Assume $0 < \delta \leq 1$. If for every $\varepsilon > 0$ there exists an integer n and a set $A_n \in \mathcal{R}_n$ such that $\mu_n(A_n) > \delta - \varepsilon$ and $\nu_n(A_n) < \varepsilon$, then there exists a set $A \in \mathcal{O}\!\mathcal{L}$ such that $M(A) \geq \delta$ and $N(A) = 0$. Conversely, if there exists a set $A \in \mathcal{O}\!\mathcal{L}$ such that $M(A) \geq \delta$ and $N(A) = 0$, then for every integer k and every $\varepsilon > 0$ there exists an integer n with $n < k$ and a set $A_n \in \mathcal{R}_n$ such that $\mu_n(A_n) > \dfrac{\frac{1}{2}(\delta - \varepsilon)}{1 - \frac{1}{2}(\delta - \varepsilon)}$ and $\nu_n(A_n) < \dfrac{2\varepsilon}{\delta - \varepsilon}$.

PROOF. For every positive integer i there exists an integer n_i and a set $A_{n_i} \in \mathcal{R}_{n_i}$ such that $\mu_{n_i}(A_{n_i}) > \delta - \frac{1}{2^i}$ and $\nu_{n_i}(A_{n_i}) < \frac{1}{2^i}$. Define $\bar{A}_i = \{\omega \mid \pi_{n_i}(\omega) \in A_{n_i}\}$, then $M(\bar{A}_i) > \delta - \frac{1}{2^i}$ and $N(A_i) < \frac{1}{2^i}$. Put $A = \bigcap_{k=1}^{\infty} \bigcup_{i=k}^{\infty} \bar{A}_i$, then it follows that $M(A) \geq \delta$ and $N(A) = 0$.

Conversely, assume there exists a set $A \in \mathcal{O}\!\mathcal{L}$ such that $M(A) \geq \delta$ and $N(A) = 0$. For every integer k define $\mathcal{B}_k = \bigcup_{n=-\infty}^{k} \mathcal{O}\!\mathcal{L}_{n,\infty}$, then $\mathcal{B}_k$ is an algebra generating $\mathcal{O}\!\mathcal{L}$. Since $M + N$ is a finite measure on $(\Omega, \mathcal{O}\!\mathcal{L})$ for every $\varepsilon > 0$ there exists a set $A' \in \mathcal{B}_k$ such that $(M + N)(A \,\Delta\, A') < \varepsilon$ (cf. [5] § 13 theorem D), hence $M(A') > \delta - \varepsilon$ and $N(A') < \varepsilon$. There exists an integer $n < k$ such that $A' \in \mathcal{O}\!\mathcal{L}_{n,\infty}$.

Let the mapping $\varphi : \Omega \to \Omega'$ be defined by

$$\varphi(\omega) = (\pi_n(\omega), \pi_{n+1}(\omega), \ldots) \quad \text{for all } \omega \in \Omega \ ,$$

then, by proposition 2.2.3 with $\varphi_n = \varphi^{-1}$, we obtain for all $A \in \mathcal{O}\!\mathcal{l}_{n,\infty}$ $M(A) = M'_n(\varphi A)$ and $N(A) = N'_n(\varphi A)$, where M'_n and N'_n are the Markov measures for P on $(\Omega', \mathcal{X}')$ with initial measure μ_n and ν_n respectively. Put $A_n = \{x \mid P_x(\varphi A') > \frac{1}{2}(\delta - \varepsilon)\}$, then we obtain

$$\delta - \varepsilon < M(A') = \int_{A_n} P_x(\varphi A')\mu_n(dx) + \int_{X \setminus A_n} P_x(\varphi A')\mu_n(dx) <$$

$$< \mu_n(A_n) + \tfrac{1}{2}(\delta - \varepsilon)(1 - \mu_n(A_n)) \ ,$$

hence

$$\mu_n(A_n) > \frac{\frac{1}{2}(\delta - \varepsilon)}{1 - \frac{1}{2}(\delta - \varepsilon)} \ ;$$

$$\varepsilon > N(A') \geq \int_{A_n} P_x(\varphi A')\nu_n(dx) \geq \tfrac{1}{2}(\delta - \varepsilon)\nu_n(A_n) \ ,$$

hence

$$\nu_n(A_n) < \frac{2\varepsilon}{\delta - \varepsilon} \ .$$

This proposition gives (for $\delta = 1$) the following criterion for the singularity of M and N: $M \perp N$ if and only if there exists a decreasing sequence $(n_i)_{i=1}^{\infty}$ of integers and for every i a set $A_{n_i} \in \mathcal{R}_{n_i}$ such that $\lim_{i\to\infty} \mu_{n_i}(A_{n_i}) = 1$ and $\lim_{i\to\infty} \nu_{n_i}(A_{n_i}) = 0$.

However, we can improve this result by using a martingale theorem. To this end, we shall use the ρ-function, which has been introduced by Kakutani [14] in order to show that two product probabilities in a product space, of which the marginal probabilities are pairwise equivalent, are either equivalent or singular.

DEFINITION 2.2.3. Let $\mathcal{P}$ be the class of probabilities on $(X,\mathcal{R})$. For every $\mu_0 \in \mathcal{P}$ and every $\nu_0 \in \mathcal{P}$ we define

$$\rho(\mu_0,\nu_0) = \int \sqrt{\frac{d\mu_0}{d\mu}} \sqrt{\frac{d\nu_0}{d\mu}}\, d\mu$$

where $\mu \in \mathcal{P}$ is chosen such that $\mu_0 << \mu$ and $\nu_0 << \mu$.

Such a probability μ certainly exists, one can take for instance $\frac{1}{2}(\mu_0 + \nu_0)$. It is easily verified (cf. [14]) that the value of $\rho(\mu_0,\nu_0)$ does not depend on the choice of μ. For convenience of the reader we also give a proof of the following proposition due to Kakutani.

PROPOSITION 2.2.5. $\mu_0 \perp \nu_0$ if and only if $\rho(\mu_0,\nu_0) = 0$.

PROOF. Choose $\mu \in \mathcal{P}$ such that $\mu_0 << \mu$ and $\nu_0 << \mu$, and define $A = \operatorname{supp} \frac{d\mu_0}{d\mu}$, $B = \operatorname{supp} \frac{d\nu_0}{d\mu}$.

If $\mu_0 \perp \nu_0$ then there exists a set $A_0 \in \mathcal{R}$ such that $\mu_0(X \setminus A_0) = 0$ and $\nu_0(A_0) = 0$. This implies

$$\int_{X\setminus A_0} \frac{d\mu_0}{d\mu}\, d\mu = 0 \,, \quad \text{hence} \quad A \subset A_0 \ [\mu]$$

$$\int_{A_0} \frac{d\nu_0}{d\mu}\, d\mu = 0 \quad , \quad \text{hence} \quad B \subset X \setminus A_0 \ [\mu] \,.$$

It follows that $\mu(A \cap B) = 0$, and therefore $\rho(\mu_0,\nu_0) = 0$.

Conversely, if $\rho(\mu_0,\nu_0) = 0$, then we necessarily have $\mu(A \cap B) = 0$, hence $\mu_0(A) = 1$, $\nu_0(A) = 0$ and $\mu_0 \perp \nu_0$.

In the next theorem ρ_X will mean the ρ-function as in definition 2.2.3 on the class of probabilities on $(X,\mathcal{R})$, and ρ_Ω means the ρ-function on the class of probabilities on $(\Omega,\mathcal{O})$.

THEOREM 2.2.1. Let M and N be Markov probabilities for P on $(\Omega,\mathcal{O})$ with marginal probabilities $(\mu_n)_{n=-\infty}^{+\infty}$, $(\nu_n)_{n=-\infty}^{+\infty}$, respectively. Then

$$\rho_\Omega(M,N) = \lim_{n\to-\infty} \rho_X(\mu_n,\nu_n) .$$

In the proof of this theorem we need the following result.

LEMMA 2.2.1. Let $(X,\mathcal{R},\mu)$ be a σ-finite measure space, and $(f_n)_{n=1}^{\infty}$ and $(g_n)_{n=1}^{\infty}$ be sequences in $\mathcal{L}_1^+(X,\mathcal{R},\mu)$ converging in $\mathcal{L}_1(X,\mathcal{R},\mu)$ to the functions f and g respectively. Then

$$\lim_{n\to\infty} \int \sqrt{f_n}\,\sqrt{g_n}\,d\mu = \int \sqrt{f}\,\sqrt{g}\,d\mu .$$

PROOF. Since

$$|\sqrt{f} - \sqrt{f_n}|^2 = |f - f_n| \frac{|f - f_n|}{(\sqrt{f} + \sqrt{f_n})^2} \le |f - f_n| ,$$

we see that the sequence $(\sqrt{f_n})_{n=1}^{\infty}$ converges in $\mathcal{L}_2(X,\mathcal{R},\mu)$ to $\sqrt{f}$, and similarly, the sequence $(\sqrt{g_n})_{n=1}^{\infty}$ converges in $\mathcal{L}_2(X,\mathcal{R},\mu)$ to $\sqrt{g}$. The statement now easily follows from the continuity of the inner product in $\mathcal{L}_2(X,\mathcal{R},\mu)$.

PROOF of theorem 2.2.1. Put $L = \frac{1}{2}(M + N)$, then also L is a Markov probability for P on $(\Omega,\mathcal{O})$, with marginal probabilities $\lambda_n = \frac{1}{2}(\mu_n + \nu_n)$, $-\infty < n < \infty$. Define on $(\Omega,\mathcal{O},L)$ the functions

$$f = \frac{dM}{dL}, \qquad f_n = \frac{d\mu_{-n}}{d\lambda_{-n}} \circ \pi_{-n}$$

$$g = \frac{dN}{dL}, \qquad g_n = \frac{d\nu_{-n}}{d\lambda_{-n}} \circ \pi_{-n} .$$

For every n and for every $A \in \mathcal{O}\!\mathcal{L}_{-n,\infty}$ we have

$$\int_A E^L_{\mathcal{O}\!\mathcal{L}_{-n,\infty}} f\, dL = \int_A \frac{dM}{dL}\, dL = M(A) ;$$

on the other hand, we also have in view of the propositions 2.2.2 and 2.2.3 for all $A \in \mathcal{O}\!\mathcal{L}_{-n,\infty}$

$$\int_A f_n\, dL = \int_A \frac{d\mu_{-n}}{d\lambda_{-n}} \circ \pi_{-n}(\omega) L(d\omega) = \int_A M(d\omega) = M(A) .$$

It follows that $E^L_{\mathcal{O}\!\mathcal{L}_{-n,\infty}} f = f_n$ for every n; the sequence $(f_n)_{n=1}^{\infty}$ is a martingale on $(\Omega, \mathcal{O}\!\mathcal{L}, L)$ with respect to the sequence of σ-algebras $(\mathcal{O}\!\mathcal{L}_{-n,\infty})_{n=1}^{\infty}$. Applying proposition IV.5.6 in [18], we see that the sequence $(f_n)_{n=1}^{\infty}$ converges in $\mathcal{L}_1(\Omega, \mathcal{O}\!\mathcal{L}, S)$ to some function h. Applying the proposition again, we obtain for every n and for every $A \in \mathcal{O}\!\mathcal{L}_{-n,\infty}$ $\int_A f dL = \int_A h dL$, and therefore also for every $A \in \mathcal{O}\!\mathcal{L}$ this relation holds. Hence f = h. Similarly, we show that the sequence $(g_n)_{n=1}^{\infty}$ converges in $\mathcal{L}_1(\Omega, \mathcal{O}\!\mathcal{L}, L)$ to g. Hence by lemma 2.2.1

$$\lim_{n\to\infty} \int \sqrt{f_n}\, \sqrt{g_n}\, dL = \int \sqrt{f}\, \sqrt{g}\, dL .$$

Since

$$\int_{\Omega} \sqrt{\frac{d\mu_{-n}}{d\lambda_{-n}} \circ \pi_{-n}} \sqrt{\frac{d\nu_{-n}}{d\lambda_{-n}} \circ \pi_{-n}}\, dL =$$

$$= \int_{X} \sqrt{\frac{d\mu_{-n}}{d\lambda_{-n}}} \sqrt{\frac{d\nu_{-n}}{d\lambda_{-n}}}\, d\lambda_{-n} = \rho_X(\mu_{-n},\nu_{-n})$$

we obtain the statement of the theorem.

It follows from this theorem that two Markov probabilities M and N for P are singular if and only if for the corresponding marginal probabilities $(\mu_n)_{n=-\infty}^{+\infty}$ and $(\nu_n)_{n=-\infty}^{+\infty}$ respectively we have $\lim_{n\to-\infty} \rho(\mu_n,\nu_n) = 0$.

<u>THEOREM 2.2.2</u>. Let M and N be Markov probabilities for P on $(\Omega,\mathcal{O})$ with marginal probabilities $(\mu_n)_{n=-\infty}^{+\infty}$ and $(\nu_n)_{n=-\infty}^{+\infty}$ respectively. Then $M \perp N$ if and only if for every $\varepsilon > 0$ there exists an integer N such that for all $n < N$ we can find a set $A_n \in \mathcal{R}_n$ such that $\mu_n(A_n) > 1-\varepsilon$ and $\nu_n(A_n) < \varepsilon$.

PROOF. The sufficiency of the condition follows (with $\delta = 1$) from proposition 2.2.4. The necessity is a consequence of theorem 2.2.1 and the next lemma.

LEMMA 2.2.2. Let $\mathcal{P}$ be the class of probabilities on $(X,\mathcal{R})$. For every $\mu_0 \in \mathcal{P}$ and for every $\nu_0 \in \mathcal{P}$ there exists a set $A \in \mathcal{R}$ such that $\mu_0(A) \geq 1 - \rho(\mu_0,\nu_0)$ and $\nu_0(A) \leq \rho(\mu_0,\nu_0)$.

PROOF. Put $A = \{x \mid \frac{d\mu_0}{d\mu} \geq \frac{d\nu_0}{d\mu}\}$, where $\mu \in \mathcal{P}$ is chosen such that $\mu_0 \ll \mu$, $\nu_0 \ll \mu$. Then it follows that $\rho(\mu_0,\nu_0) \geq \nu_0(A)$, $\rho(\mu_0,\nu_0) \geq \mu_0(X \setminus A)$, and therefore $\mu_0(A) \geq 1 - \rho(\mu_0,\nu_0)$.

2.3. CONSERVATIVITY FOR MARKOV SHIFTS

Throughout this section P will be the operator on $B(X,\mathcal{R})$ determined by a transition probability. $(\Omega',\mathcal{O}\!\mathcal{C}')$ will be the one-sided product space of $(X,\mathcal{R})$.

Let $(\Omega,\mathcal{O}\!\mathcal{C},M,S)$ be a two-sided shift space for P. Harris and Robbins [7] have shown that if the marginal measures $(\mu_n)_{n=-\infty}^{+\infty}$ are all equal to a σ-finite measure μ_0 on $(X,\mathcal{R})$ and if statement ii) of the next proposition holds with $A = X$, then the shift S is conservative on $(\Omega,\mathcal{O}\!\mathcal{C},M)$. In this section we shall generalize this result (cf. theorem 2.3.1 and theorem 2.3.2).

PROPOSITION 2.3.1. For every $B \in \mathcal{R}$ define

$$B_1 = \{\omega' \in \Omega' \mid \pi_n' \omega' \in B \text{ for at least one } n \geq 1\}$$

$$B_\infty = \{\omega' \in \Omega' \mid \pi_n' \omega' \in B \text{ for infinitely many } n\} .$$

Let μ_0 be a σ-finite measure on $(X,\mathcal{R})$. Then the following statements are equivalent for every set $A \in \mathcal{R}$.

i) $\forall_{B\in\mathcal{R},B\subset A}\ P_x(B_1) = 1$ for μ_0-almost all $x \in B$;

ii) $\forall_{B\in\mathcal{R},B\subset A}\ P_x(B_\infty) = 1$ for μ_0-almost all $x \in B$;

iii) for every σ-finite measure μ on $(X,\mathcal{R})$ such that $\mu_0 \ll \mu$ and such that P induces a Markov process on $(X,\mathcal{R},\mu)$ we have $\mu_0(A \setminus C) = 0$, where C is the conservative part of the Markov process on $(X,\mathcal{R},\mu)$.

In the proof of this proposition we need the following lemma.

LEMMA 2.3.1. Let $(X,\mathcal{R})$ be a measurable space, and let μ_0 and

μ be measures on $(X,\mathcal{R})$ such that $\mu_0 << \mu$. Assume μ is σ-finite. Then there exists a set $N \in \mathcal{R}$ such that $\mu_0(N) = 0$, and the restrictions of the measures μ and μ_0 to $X \setminus N$ are equivalent.

PROOF. The class $\mathcal{B} = \{B \in \mathcal{R} \mid \mu_0(B) = 0, \mu(B) > 0\}$ is closed under the operation of taking countable unions. Let N be the modulo μ largest set of $\mathcal{B}$, then $\mu_0(N) = 0$, and on $X \setminus N$ we have $\mu_0 \approx \mu$.

PROOF of proposition 2.3.1.

i) $\Rightarrow$ iii). Let N be the μ_0-null set as in lemma 2.3.1. If $\mu_0(A \setminus C) > 0$, then also $\mu_0(A \setminus (C \cup N)) > 0$, and therefore $\mu(A \setminus (C \cup N)) > 0$. By proposition 2.1.1 there exists a subset B of $A \setminus (C \cup N)$ such that

$$\mu\{x \in B \mid (\sum_{k=0}^{\infty} I_B(PI_{X\setminus B})^k P1_B)(x) \neq 1\} > 0 .$$

Since for all $x \in B$ we have

$$P_x(B_1) = (\sum_{k=0}^{\infty} I_B(PI_{X\setminus B})^k P1_B)(x)$$

and $\mu_0 \approx \mu$ on B, we obtain $\mu_0\{x \in B \mid P_x(B_1) \neq 1\} > 0$, contradiction.

iii) $\Rightarrow$ ii). For every set $B \in \mathcal{R}$ define the set B^n by

$$B^n = \{\omega' \in \Omega' \mid \pi_i' \omega' \notin B \text{ for all } i \geq n, \pi_{n-1}' \omega' \in B\} ,$$

then for all $x \in B$ we have

$$P_x(B_\infty) = 1 - \sum_{n=1}^{\infty} P_x(B^n) .$$

For every n define $f_n = (I_{X\setminus B}P)^n 1_{X\setminus B}$, then, since $I_{X\setminus B}P$ is a positive linear operator on $B(X,\mathcal{R})$ and $f_1 \le f_0$, $(f_n)_{n=0}^{\infty}$ is a decreasing sequence of non negative functions in $B(X,\mathcal{R})$ and $\lim_{n\to\infty} f_n = f$ exists. Since $f_{n+1} \le Pf_n \le 1$ for all n, the function f satisfies $f \le Pf \le 1$.

For every n and for all $x \in X$ we have

$$P_x(B^n) = \lim_{k\to\infty} P^{n-1}I_B P(I_{X\setminus B}P)^k 1_{X\setminus B})(x) = (P^{n-1}I_B Pf)(x) .$$

Let μ be a σ-finite measure on $(X,\mathcal{R})$ such that $\mu_0 \ll \mu$ and such that the operator P may be considered as a Markov operator in $\mathcal{L}_\infty(X,\mathcal{R},\mu)$. Such a measure exists by proposition 1.4.3. Let C be the conservative part of X with respect to P.

By applying proposition 2.1.3 on the function $1 - f$, we see that $Pf = f = I_{X\setminus B}f$ μ-almost everywhere on C. Hence, if B is a subset of C, we obtain $P_x(B^n) = 0$ for μ-almost all $x \in X$ for every n, and therefore $P_x(B_\infty) = 1$ for μ-almost all $x \in B$.

Let N be the μ_0-null set as in lemma 2.3.1. If $\mu_0(A \setminus C) = 0$ then for every subset B of A we have $\mu_0(B \setminus (C \setminus N)) = 0$. Since obviously $(B \cap (C \setminus N))_\infty \subset B_\infty$, for all $x \in X$ we have

$$P_x(B \cap (C \setminus N))_\infty \le P_x(B_\infty) \le 1 .$$

We have just shown that for μ-almost all $x \in B \cap (C \setminus N)$ we have $P_x(B \cap (C \setminus N))_\infty = 1$. Since on $B \cap (C \setminus N)$ the measures μ and μ_0 are equivalent, we obtain that for μ_0-almost all $x \in B \cap (C \setminus N)$, and therefore for μ_0-almost all $x \in B$, we have $P_x(B_\infty) = 1$.

ii) $\Rightarrow$ i). Since for every $B \in \mathcal{R}$ we have $B_\infty \subset B_1$, we have for every $x \in X$ $P_x(B_\infty) \le P_x(B_1) \le 1$, from which the statement follows.

DEFINITION 2.3.1. For every $n \ge 1$ define

$$(X^n,\mathcal{R}^n) = \prod_{t=0}^{n-1} (X_t,\mathcal{R}_t) , \quad (\Omega'',\mathfrak{A}'') = \prod_{t=0}^{\infty} (X_t^n,\mathcal{R}_t^n) ,$$

where $(X_t^n,\mathcal{R}_t^n) = (X^n,\mathcal{R}^n)$ for all t. Let the mapping $\varphi : \Omega' \to X^n$ be defined by

$$\varphi(\omega') = (\pi_0'(\omega'),\dots,\pi_{n-1}'(\omega')) \quad \text{for all } \omega' \in \Omega' .$$

For every measure μ_0 on $(X,\mathcal{R})$ let M_0' be the corresponding Markov measure on $(\Omega',\mathfrak{A}')$. Then the measure μ_0^n on $(X^n,\mathcal{R}^n)$ is defined by

$$\mu_0^n(A) = M_0'(\varphi^{-1}A) \quad \text{for all } A \in \mathcal{R}^n .$$

The function P' on $(X^n \times \mathcal{R}^n)$ is defined by

$$P'((x_0,\dots,x_{n-1}),A) = P_{x_{n-1}}(S'^{-1}\varphi^{-1}A)$$

for all $(x_0,\dots,x_{n-1}) \in X^n$ and all $A \in \mathcal{R}^n$, where S' is the shift in $(\Omega',\mathfrak{A}')$.

PROPOSITION 2.3.2. The function P' on $X^n \times \mathcal{R}^n$ is a transition probability on $(X^n,\mathcal{R}^n)$. If M_0'' is the Markov measure on $(\Omega'',\mathfrak{A}'')$ for P' with initial measure μ_0^n, and S'' is the shift in $(\Omega'',\mathfrak{A}'')$, then the mapping $\tau : \mathfrak{A}'' \to \mathfrak{A}'$ defined by

$$\tau(A'') = \{\omega' \mid ((\pi_0'\omega',\dots,\pi_{n-1}'\omega'),(\pi_n'\omega',\dots,\pi_{2n-1}'\omega'),\dots) \in A''\}$$

for all $A'' \in \mathfrak{A}''$, is an isomorphism from $(\Omega'',\mathfrak{A}'',M_0'',S'')$ to

$(\Omega', \mathcal{O}\!\mathcal{C}', M_0', S'^n)$.

PROOF. The fact that P' is a transition probability on $(X^n, \mathcal{B}^n)$ is a consequence of proposition 1.2.3. Obviously τ is an isomorphism of $\mathcal{O}\!\mathcal{C}''$ onto $\mathcal{O}\!\mathcal{C}'$ such that $\tau S''^{-1}A = S'^{-n}\tau A$ for all $A \in \mathcal{O}\!\mathcal{C}''$.

It follows from the definition of P' that for every n-tuple $(f_0,\ldots,f_{n-1})$ of non negative functions in $B(X,\mathcal{B})$ we have

$$P'(f_0(y_0)f_1(y_1) \ldots f_{n-1}(y_{n-1}))(x_0,\ldots,x_{n-1}) =$$

$$= (P(f_0 \ldots P(f_{n-2}(Pf_{n-1})) \ldots))(x_{n-1}) .$$

Consider the rectangle $A = \prod_{t=0}^{\infty} A_t$, where $A_t = X_t$ if $t > rn-1$.

$$M_0''(\tau^{-1}A) = \int I_{A_0\times\ldots\times A_{n-1}} P' \ldots P'1_{A_{(r-1)n}\times\ldots\times A_{rn-1}} d\mu_0^n =$$

$$= \int_{A_0\times\ldots\times A_{n-1}} (PI_{A_n} P \ldots P1_{A_{rn-1}})(x_{n-1})\mu_0^n(dx_0,\ldots,dx_{n-1}) =$$

$$= \int I_{A_0} P \ldots P1_{A_{rn-1}} d\mu_0 = M_0'(A) .$$

We conclude that the mapping $\tau : \mathcal{O}\!\mathcal{C}'' \to \mathcal{O}\!\mathcal{C}'$ is also measure preserving.

PROPOSITION 2.3.3. Let μ_0 be a σ-finite measure on $(X,\mathcal{B})$ such that P can be considered to be a Markov process on $(X,\mathcal{B},\mu_0)$. Let C be the conservative part of this Markov process. Let M_0' be the Markov measure on $(\Omega', \mathcal{O}\!\mathcal{C}')$ for P with initial measure μ_0, and let $(\mu_n)_{n=0}^{\infty}$ be the sequence of marginal measures.

Then the following statements hold:

i) For all n we have $\mu_n >> \mu_{n+1}$. On C we have $\mu_0 \approx \mu_n$ for every n.

ii) The shift S' is negatively non singular on $(\Omega', \mathcal{O}\mathcal{L}', M_0')$. The restriction of S' to $C_\infty' = \{\omega' \mid \pi_n' \omega' \in C \text{ for all } n\}$ is non singular.

PROOF.

i) Suppose $\mu_n(A) = 0$, then $P^n 1_A = 0$ μ_0-almost everywhere. Since P can be considered as a Markov operator in $\mathcal{L}_\infty(X, \mathcal{R}, \mu_0)$, it follows that also $P^{n+1} 1_A = 0$ μ_0-almost everywhere, hence $\mu_{n+1}(A) = 0$ and $\mu_n >> \mu_{n+1}$. If $A \subset C$ and $\mu_0(A) > 0$, then by proposition 2.1.2 $\sum_{k=0}^{\infty} P^k 1_A = \infty$ μ_0-almost everywhere on A, hence for every n $P^n 1_A > 0$ on a set of positive μ_0-measure and therefore $\mu_n(A) > 0$ for every n.

ii) Suppose for some $A \in \mathcal{O}\mathcal{L}'$ we have $M_0'(A) = 0$. Then $P_x(A) = 0$ for μ_0-almost all $x \in X$. Since P can be considered as a Markov operator in $\mathcal{L}_\infty(X, \mathcal{R}, \mu_0)$, we obtain by proposition 1.2.4

$$M_0'(S'^{-1}A) = \int P_x(S'^{-1}A)\mu_0(dx) = \int P(P_\cdot(A))d\mu_0 = 0 .$$

Now suppose $A \subset C_\infty'$ and $M_0'(A) > 0$. Then $P_x(A) > 0$ on a subset of C of positive μ_0-measure. Then

$$M_0'(S'^{-1}A) = \int P_x(A)\mu_1(dx) > 0$$

since on C μ_0 and μ_1 are equivalent. This shows that the restriction of S' to C'_∞ is non singular.

PROPOSITION 2.3.4. Let μ_0 be a σ-finite measure on $(X,\mathcal{R})$ such that the operator P on $B(X,\mathcal{R})$ induces a Markov process on $(X,\mathcal{R},\mu_0)$. Let C be the conservative part of this process. Let P', $(X^n,\mathcal{R}^n)$, μ_0^n be as in definition 2.3.1. Then the operator P' on $B(X^n,\mathcal{R}^n)$ induces a Markov process on $(X^n,\mathcal{R}^n,\mu_0^n)$ of which the conservative part is the set

$$C^n = \{(x_0,\dots,x_{n-1}) \mid x_i \in C \text{ for } 0 \le i < n\} .$$

PROOF. Let $\varphi : \Omega' \to X^n$ be the mapping as in definition 2.3.1. Assume for some $A \in \mathcal{R}^n$ we have $\mu_0^n(A) = 0$. Then

$$\int P_x(\varphi^{-1}A)\mu_0(dx) = 0 ,$$

hence $P_x(\varphi^{-1}A) = 0$ for μ_0-almost all $x \in X$.

$$\int_{X^n} (P'1_A)(x_{n-1})\mu_0^n(dx_0,\dots,dx_{n-1}) = \int_X P_{x_{n-1}}(S'^{-1}\varphi^{-1}A)\mu_{n-1}(dx_{n-1})$$

$$= \int P(P_.(\varphi^{-1}A))d\mu_{n-1} = \int P_x(\varphi^{-1}A)d\mu_n = 0 ,$$

since by the previous proposition we have $\mu_n \ll \mu_0$. It follows by lemma 1.3.1 that the operator P' induces a Markov process on $(X^n,\mathcal{R}^n,\mu_0^n)$.

Suppose P' is not conservative on C^n. Then by proposition 2.1.1 there exists a set $A \subset C^n$ with $\mu_0^n(A) > 0$ such that

(1) $$\sum_{k=0}^{\infty} P'^k 1_A \text{ is } \mu_0^n\text{-essentially bounded on } (X^n,\mathcal{R}^n).$$

Let the function $g \in B(X,\mathcal{R})$ be defined by

$$g(x) = P_x(S'^{-1}\varphi^{-1}A) \qquad \text{for all } x \in X ,$$

then

$$g(x_{n-1}) = (P'1_A)(x_0,\dots,x_{n-1}) \qquad \text{for all } (x_0,\dots,x_{n-1}) \in X^n .$$

Because of $\mu_0^n(A) > 0$ and $A \subset C^n$, we have $P_\cdot(\varphi^{-1}A) > 0$ on a subset of C of positive μ_0-measure. It follows by proposition 2.1.4 that

$$(2) \qquad \int_C g\,d\mu_0 = \int_C 1_C P(P_\cdot(\varphi^{-1}A))\,d\mu_0 = \int_X (1_C P)(P_\cdot(\varphi^{-1}A))\,d\mu_0 > 0 .$$

Define for every set $B \in \mathcal{R}$ $B' = X_0 \times \dots \times X_{n-2} \times B$. Then by definition 2.3.1

$$(P'1_{B'})(x_0,\dots,x_{n-1}) = P_{x_{n-1}}(\{\omega' \mid \pi_n'\omega' \in B\}) = (P^n 1_B)(x_{n-1}) .$$

Define for every non negative $f \in B(X,\mathcal{R})$ $f'(x_0,\dots,x_{n-1}) =$ $= f(x_{n-1})$, then we deduce

$$(3) \qquad (P'f')(x_0,\dots,x_{n-1}) = (P^n f)(x_{n-1}) .$$

Applying this property to the function g, we obtain

$$\sum_{k=1}^{\infty} P'^k 1_A(x_0,\dots,x_{n-1}) = \sum_{k=0}^{\infty} P^{kn} g(x_{n-1}) .$$

Denote the second member of this equality by $s(x_{n-1})$. Since by (1) the first member is μ_0^n-essentially bounded, s is a μ_{n-1}-essentially bounded function of x_{n-1}.

Put $s_C = I_C s$, and $s_D = I_D s$, then by proposition 2.3.3 $s_C \in \mathcal{L}_\infty(X,\mathcal{R},\mu_0)$. Moreover, by proposition 2.1.4, $PI_D s = 0$

μ_0-almost everywhere on C. It follows that μ_0-almost everywhere on C

$$\sum_{k=0}^{\infty} P^k g = s + \dots + P^{n-1}s = s_C + Ps_C + \dots + P^{n-1}s_C < \infty ,$$

hence, as a consequence of proposition 2.1.2, we obtain $\mu_0(C \cap \operatorname{supp} g) = 0$, which contradicts (2).

The set C^n therefore must be a subset of the conservative part of X^n with respect to P'.

Now define for $0 \le i \le n-1$ the sets $B_i \in \mathcal{R}^n$ by

$$B_i = \{(x_0,\dots,x_{n-1}) \mid x_j \in C \text{ if } j < i,\ x_i \in D\} .$$

Since $P1_D = 0$ μ_0-almost everywhere on C, we obtain $\mu_0^n(B_i) = 0$ for all $i > 0$.

Consider the set B_0. Let $(D_i)_{i=1}^{\infty}$ be a partition of D such that for all i $\sum_{k=0}^{\infty} P^k 1_{D_i} \in \mathcal{L}_\infty(X,\mathcal{R},\mu_0)$, and define $D_i' = \{(x_0,\dots,x_{n-1}) \mid x_0 \in D_i\}$, then $(D_i')_{i=1}^{\infty}$ is a partition of B_0. For every i we have

$$(P'1_{D_i'})(x_0,\dots,x_{n-1}) = P_{x_{n-1}}(\{\omega' \mid \pi_1'(\omega') \in D_i\}) = (P1_{D_i})(x_{n-1}) ,$$

hence

$$(\sum_{k=1}^{\infty} P'^k 1_{D_i'})(x_0,\dots,x_{n-1}) = (\sum_{k=0}^{\infty} P^{kn+1} 1_{D_i})(x_{n-1}) \quad \text{by (3)}.$$

Since the right-hand side is μ_0-essentially bounded, and therefore μ_{n-1}-essentially bounded, we obtain

$$\sum_{k=1}^{\infty} P'^k 1_{D_i'} \text{ is } \mu_0^n\text{-essentially bounded on } (X^n,\mathcal{R}^n) ,$$

hence, for every i, D_i', and therefore B_0, belongs to the dissipative part of X^n with respect to P'.

Since $(C^n,B_0,\ldots,B_{n-1})$ is a partition of X^n, the proposition is proved.

We now shall discuss some recurrence properties for the shift in the one-sided and the two-sided shift space. We first give the following definition.

DEFINITION 2.3.2. Let T be a measurable transformation in a measure space $(X,\mathcal{R},\mu)$. A set $A \in \mathcal{R}$ is said to be a recurrence set if for μ-almost $x \in A$ we have $T^n x \in A$ for infinitely many n.

The next proposition is essentially part of the proof of theorem 1 in the paper of Harris and Robbins [7].

PROPOSITION 2.3.5. Let T be a measure preserving transformation in a σ-finite measure space $(X,\mathcal{R},\mu)$. Let $\mathcal{O}\!\mathcal{L}$ be an algebra generating $\mathcal{R}$ such that every $A \in \mathcal{O}\!\mathcal{L}$ is a recurrence set. Then T is conservative on X.

PROOF. Let W be a wandering set of finite measure. Choose $\varepsilon > 0$ and $A \in \mathcal{O}\!\mathcal{L}$ such that $\mu(A \,\Delta\, W) < \varepsilon$. Since $A \subset \bigcup_{n=1}^{\infty} T^{-1}A$, there exists an integer N such that $\mu(A \setminus \bigcup_{n=1}^{N} T^{-n}A) < \varepsilon$.

Then

$$0 = \mu(T^{-N}W \cap \bigcup_{i=0}^{N-1} T^{-i}W) > \mu(T^{-N}A \cap \bigcup_{i=0}^{N-1} T^{-1}W) - \varepsilon =$$

$$= \sum_{i=0}^{N-1} \mu(T^{-N}A \cap T^{-i}W) - \varepsilon = \sum_{i=0}^{N-1} \mu(T^{i-N}A \cap W) - \varepsilon \geq$$

$$\geq \mu(\bigcup_{i=1}^{N} T^{-i}A \cap W) - \varepsilon > \mu(A \cap W) - 2\varepsilon ,$$

hence $\mu(W) \leq \mu(A \cap W) + \mu(A \Delta W) < 3\varepsilon$. It follows that $\mu(W) = 0$, and T is conservative on X.

For a proof of the following proposition the reader is referred to Wright [24].

PROPOSITION 2.3.6. Let T be a measurable transformation in a measure space $(X,\mathcal{R},\mu)$. Let $A \in \mathcal{R}$ be such that μ-almost all $x \in A$ return to A under the action of T only a finite number of times. Then A is (modulo μ) a countable union of wandering sets.

THEOREM 2.3.1. Let $(\Omega',\mathcal{X}',M_0',S')$ be the one-sided shift space for P with initial measure μ_0. Let μ be a σ-finite measure on $(X,\mathcal{R})$ such that P induces a Markov process on $(X,\mathcal{R},\mu)$ and $\mu_0 << \mu$. Let C be the conservative part of this process, and define $C' = \{\omega' \mid \pi_n'(\omega') \in C \text{ for all } n\}$. Then the following statements hold.

i) For every n ($< \infty$) and every $A \in \mathcal{X}_{0n}'$, $A \cap C_\infty'$ is a recurrence set for S'.

ii) The set $\Omega' \setminus C_\infty'$ belongs to the dissipative part of Ω' with respect to S'.

iii) If there exists a σ-finite measure ν such that $\nu << \mu$, $\operatorname{supp} \frac{d\nu}{d\mu} = C$ and $\frac{d\nu}{d\mu} P = \frac{d\nu}{d\mu}$, then C_∞' is the conservative part of Ω' with respect to S'.

PROOF. Let M' be the Markov measure corresponding to the initial measure μ, then by proposition 2.2.2 we have $M_0' \ll M'$. Hence, it suffices to prove the theorem for the one-sided shift space $(\Omega', \mathcal{O}\!\mathcal{C}', M', S')$.

Define for every $n \geq 0$ the set X_n by

$$X_n = \{\omega' \mid \pi_i'(\omega') \in C \text{ if } i < n,\ \pi_n'(\omega') \in D\} ,$$

then $(X_n)_{n=0}^{\infty}$ is a partition of $\Omega \setminus C_\infty'$. Since $I_C P 1_D = 0$ μ-almost everywhere, $M'(X_n) = 0$ for every $n \geq 1$, and in order to show ii), we only have to show that the set X_0 belongs to the dissipative part of Ω'.

Let $(D_i)_{i=1}^{\infty}$ be a partition of D such that for every i

$$\sum_{k=1}^{\infty} P^k 1_{D_i} \in \mathcal{L}_\infty(X, \mathcal{A}, \mu) .$$

We may assume $\mu(D_i) < \infty$ for every i.

Define $D_i' = \{\omega' \mid \pi_0'(\omega') \in D_i\}$, then $(D_i')_{i=1}^{\infty}$ is a partition of X_0. Let $D_{i,n}'$ be the set of points ω' of D_i' for which there exists an integer $k \geq n$ such that $S'^k\, \omega' \in D_i'$, then

$$D_{i,n}' = \bigcup_{k=n}^{\infty} \{\omega' \mid \pi_0'(\omega') \in D_i,\ \pi_k'(\omega) \in D_i\} ,$$

$$M'(D_{i,n}') \leq \sum_{k=n}^{\infty} \int_{D_i} P^k 1_{D_i} d\mu = \int_{D_i} (\sum_{k=n}^{\infty} P^k 1_{D_i}) d\mu$$

which, by the dominated convergence theorem, tends to 0 if $n \to \infty$. It follows that M'-almost all points of D_i' return to D_i' under S' at most a finite number of times, hence by proposition 2.3.6 every set D_i', and therefore X_0, must be a subset of the dissipative part of Ω' with respect to S'.

Suppose $B' = \{\omega' \mid \pi_0'(\omega') \in B\} \in \mathcal{O}\!\mathcal{C}_0'$, and $B \subset C$. Then by proposition 2.3.1 M'-almost all points of B' return to B' under S' infinitely often. Since $M'(B' \cap X_0) = 0$, also M-almost all points of $B' \cap C_\infty'$ return to B' under S' infinitely often. Since, if these points return to B', they obviously return to $B' \cap C_\infty'$, it follows that $B' \cap C_\infty'$ is a recurrence set for S'.

Suppose $A \in \mathcal{O}\!\mathcal{C}_{0,n-1}' \cap C_\infty'$. Consider the isomorphism between $(\Omega'', \mathcal{O}\!\mathcal{C}'', M'', S'')$ and $(\Omega', \mathcal{O}\!\mathcal{C}', M', S'^n)$ as in proposition 2.3.2. Under this isomorphism, the σ-algebra $\mathcal{O}\!\mathcal{C}_{0,n-1}' \subset \mathcal{O}\!\mathcal{C}'$ corresponds to the σ-algebra $\mathcal{O}\!\mathcal{C}_0''$. Let the set C_∞' correspond to the set C_∞'', then by proposition 2.3.4 the set C_∞'' indeed is the product of the conservative part of X^n with respect to P' and the measure μ^n. Now it follows that M'-almost all points of A return to A infinitely often under S'^n, hence A is a recurrence set.

Finally, let the measure ν on $(X, \mathcal{R})$ satisfy the conditions in iii), and let N' be the Markov measure on $(\Omega', \mathcal{O}\!\mathcal{C}')$ for P with initial measure ν, then $N' << M'$. If $A \in \mathcal{O}\!\mathcal{C}'$, $A \subset C_\infty'$, has positive M'-measure, then $P_x(A) > 0$ on a subset of C of positive μ-measure. Since $\mu \approx \nu$ on C we also have $P_x(A) > 0$ on a subset of C of positive ν-measure, and therefore $N'(A) > 0$. It follows that the measures M' and N' are equivalent on $(C_\infty', \mathcal{O}\!\mathcal{C}' \cap C_\infty')$. Moreover, for every $A \in \mathcal{O}\!\mathcal{C}' \cap C_\infty'$ we have

$$N'(S'^{-1}A) = \int P_x(S'^{-1}A)d\nu = \int \frac{d\nu}{d\mu} P(P_\cdot(A))d\mu =$$

$$= \int (P_\cdot(A))d\nu = N'(A) \quad ,$$

hence S' is a measure preserving transformation in

$(C'_\infty, \mathfrak{A}' \cap C'_\infty, N')$.

Since by the equivalence of N' and M' every set in $\mathfrak{A}'_{0,n} \cap C'_\infty$ is a recurrence set, and $\bigcup_{n=0}^{\infty} \mathfrak{A}'_{0,n} \cap C'_\infty$ is an algebra generating $\mathfrak{A}$, it follows by proposition 2.3.5 that S' is conservative on $(C'_\infty, \mathfrak{A}' \cap C'_\infty, N')$, and therefore on $(C'_\infty, \mathfrak{A}' \cap C'_\infty, M')$.

The next theorem is an immediate consequence of theorem 2.3.1 and proposition 2.2.3.

<u>THEOREM 2.3.2</u>. Let $(\Omega, \mathfrak{A}, M, S)$ be a two-sided shift space for P with marginal measures $(\mu_n)_{n=-\infty}^{+\infty}$. Let μ be a σ-finite measure such that $\mu_n \ll \mu$ for all n and P induces a Markov process on $(X, \mathcal{B}, \mu)$, of which the conservative part is C. Define $C_\infty = \{\omega \mid \pi_n(\omega) \in C \text{ for all } n\}$, then

i) for every $A \in \mathfrak{A}_{n,m}$, $-\infty < n \le m < \infty$, the set $A \cap C_\infty$ is a recurrence set for S.

ii) $\Omega \setminus C_\infty$ is a subset of the dissipative part of Ω with respect to S.

The analogon of property iii) in theorem 2.3.1 would be the statement: if there exists a σ-finite measure $\nu \ll \mu$ such that for all n $\mu_n \ll \nu$, $\frac{d\nu}{d\mu} P = \frac{d\nu}{d\mu}$, and supp $\frac{d\nu}{d\mu} = C$, then C_∞ is the conservative part of Ω with respect to S.

However, in the next section we shall give an example that this statement is false.

2.4. SHIFT SPACES FOR THE MARKOV PROCESSES INDUCED BY A MEASURABLE TRANSFORMATION

Throughout this section, T will be a non singular measurable transformation on a σ-finite measure space $(X,\mathcal{R},\mu)$; P will be the forward process on $(X,\mathcal{R},\mu)$ associated with T and $\overleftarrow{P}_\nu$ will be the backward process associated with T for a measure $\nu \approx \mu$ such that $(X,T^{-1}\mathcal{R},\nu)$ is σ-finite.

We shall make the assumption that, just as the process P, the process $\overleftarrow{P}_\nu$ is given by a transition probability, and therefore that for every initial measure there exists a one-sided shift space for $\overleftarrow{P}_\nu$. This assumption is not a very strong one: e.g. if T satisfies $TX \in \mathcal{R}$, where X is a complete separable metric space and $\mathcal{R}$ the σ-algebra of Borel sets of X, then it follows from proposition 1.5.3 and theorem 7.1 in the book of Parthasarathy [20] that the assumption is fulfilled.

If P and Q are Markov processes on $(X,\mathcal{R},\mu)$, then PQ will denote the Markov process which is determined by

$$(PQ)(f) = P(Qf) \qquad \text{for all } f \in \mathcal{L}_\infty(X,\mathcal{R},\mu) .$$

It is easily seen that the same relation holds for all $f \in M^+(X,\mathcal{R},\mu)$, and that for all $f \in \mathcal{L}_1(X,\mathcal{R},\mu)$ and for all $f \in M^+(X,\mathcal{R},\mu)$ we have $f(PQ) = (fP)Q$.

PROPOSITION 2.4.1. Suppose $n > 0$ and $(X,T^{-n}\mathcal{R},\mu)$ is a σ-finite measure space. Then we have

$$\overleftarrow{P}^n_\nu P^n = I \ ; \qquad P^n \overleftarrow{P}^n_\nu E^\mu_{T^{-n}\mathcal{R}} = E^\mu_{T^{-n}\mathcal{R}} .$$

PROOF. Let h be the function corresponding to the backward process $\overleftarrow{P}_\nu$ as in theorem 1.5.1. Then for all $f \in M^+(X,\mathcal{R},\mu)$ and for all $A \in \mathcal{R}$ we have

$$\int_A \overleftarrow{P}_\nu Pf\, d\mu = \int (1_A \overleftarrow{P}_\nu)(f\circ T)d\mu = \int \frac{d\mu T}{d\mu} h(1_A\circ T)(f\circ T)d\mu =$$

$$= \int (E^\mu_{T^{-1}\mathcal{R}} h)\, \frac{d\mu T}{d\mu}\, (1_A\circ T)(f\circ T)d\mu = \int_A f d\mu \ ,$$

hence $\overleftarrow{P}_\nu P = I$ and therefore $\overleftarrow{P}^n_\nu P^n = I$.

The second statement actually says that every function $f \in M^+(X,T^{-n}\mathcal{R},\mu)$ is invariant under $P^n\, \overleftarrow{P}^n$, written to the left of f. For every $f \in M^+(X,T^{-n}\mathcal{R},\mu)$ there exists a unique function $g \in M^+(X,\mathcal{R},\mu)$ such that $f = g\circ T^n$. Then we have

$$P^n\, \overleftarrow{P}^n_\nu f = P^n\, \overleftarrow{P}^n_\nu\, P^n g = P^n g = f \ .$$

LEMMA 2.4.1. Let $(\mu_i)^n_{i=0}$ be a sequence of measures on $(X,\mathcal{R})$ such that for every i $\mu_i << \mu$ and

$$\frac{d\mu_0}{d\mu}\, \overleftarrow{P}^i_\nu = \frac{d\mu_i}{d\mu}\ , \qquad \frac{d\mu_n}{d\mu}\, P^i = \frac{d\mu_{n-i}}{d\mu}\ .$$

Then for every sequence $(A_i)^n_{i=0}$ in $\mathcal{R}$ we have

$$\int I_{A_0} \overleftarrow{P}_\nu\, I_{A_1} \overleftarrow{P}_\nu \ldots \overleftarrow{P}_\nu 1_{A_n} d\mu_0 = \int I_{A_n} P \ldots P I_{A_1} P 1_{A_0} d\mu_n \ .$$

PROOF. By proposition 1.3.4 we may assume that μ is a σ-finite measure on $(X,T^{-1}\mathcal{R})$. Let h be the function corresponding to $\overleftarrow{P}_\nu$ as in theorem 1.5.1. For all $f \in M^+(X,\mathcal{R},\mu)$ and for all $g \in M^+(X,\mathcal{R},\mu)$ we have for every measure $\mu_0 << \mu$

$$\int f(\overleftarrow{P}_\nu\, g)\,d\mu_0 = \int ((f\,\frac{d\mu_0}{d\mu})\overleftarrow{P}_\nu)g\,d\mu =$$

$$= \int \frac{d\mu T}{d\mu}\, h(f\circ T)(\frac{d\mu_0}{d\mu}\circ T)g\,d\mu = \int g(Pf)(\frac{d\mu_0}{d\mu}\,\overleftarrow{P}_\nu)d\mu \ .$$

Using this formula, we obtain

$$\int I_{A_0}\overleftarrow{P}_\nu\, I_{A_1}\overleftarrow{P}_\nu \ \dots\ \overleftarrow{P}_\nu 1_{A_n} d\mu_0 =$$

$$= \int (I_{A_1}\overleftarrow{P}_\nu \ \dots\ \overleftarrow{P}_\nu 1_{A_n})(P1_{A_0})(\frac{d\mu_0}{d\mu}\,\overleftarrow{P}_\nu)d\mu =$$

$$= \int (\overleftarrow{P}_\nu(I_{A_2}\overleftarrow{P}_\nu \ \dots\ \overleftarrow{P}_\nu 1_{A_n}))(I_{A_1}P1_{A_0})d\mu_1 =$$

$$= \dots =$$

$$= \int I_{A_n}P \ \dots\ PI_{A_1}P1_{A_0}d\mu_n \ .$$

THEOREM 2.4.1. Let $(\Omega'',\mathfrak{A}'',M_0'',S'')$ be the one-sided shift space for the process $\overleftarrow{P}_\nu$ with initial σ-finite measure $\mu_0 \ll \mu$. Then there exists a unique measure M on $(\Omega,\mathfrak{A})$ such that $(\Omega,\mathfrak{A},M,S)$ is a two-sided shift space for P and the mapping $\varphi : \mathfrak{A}'' \to \mathfrak{A}$ defined by

$$\varphi(A'') = \{\omega \mid (\pi_0\omega,\pi_{-1}\omega,\dots) \in A''\}$$

for all $A'' \in \mathfrak{A}''$ is an isomorphism from $(\Omega'',\mathfrak{A}'',M_0'',S'')$ to $(\Omega,\mathfrak{A}_{-\infty,0},M,S^{-1})$.

Let N be another Markov measure for P on $(\Omega,\mathfrak{A})$ with marginal measures $(\nu_n)_{n=-\infty}^{+\infty}$, such that φ is an isomorphism from $(\Omega'',\mathfrak{A}'',N_0'',S'')$ to $(\Omega,\mathfrak{A}_{-\infty,0},N,S^{-1})$, where $(\Omega'',\mathfrak{A}'',N_0'',S'')$

is the one-sided shift space for $\overleftarrow{P}_{\nu}$ with initial measure ν_0. Then $N << M$ if and only if $\nu_0 << \mu_0$. In this case we have

$$\frac{dN}{dM}(\omega) = \frac{d\nu_0}{d\mu_0}(\pi_0\omega) \qquad \text{for } M\text{-almost all } \omega \in \Omega \ .$$

PROOF. Let $(\Omega',\mathcal{O}\!\mathcal{C}',M_0',S')$ be the one-sided shift space for P with initial measure μ_0. Let for every $x \in X$ P_x be the probability on $\mathcal{O}\!\mathcal{C}'$ as in proposition 1.2.3. If we define for every $\omega'' \in \Omega''$ and every $A' \in \mathcal{O}\!\mathcal{C}'$

$$Q(\omega'',A') = P_{\pi_0''\omega''}(S'^{-1}A') \ ,$$

then Q is a transition probability from $(\Omega'',\mathcal{O}\!\mathcal{C}'')$ to $(\Omega',\mathcal{O}\!\mathcal{C}')$.

Let M^* be the measure on $(\Omega'',\mathcal{O}\!\mathcal{C}'') \times (\Omega',\mathcal{O}\!\mathcal{C}')$, defined as in proposition 1.2.1 with the measure M_0'' on $(\Omega'',\mathcal{O}\!\mathcal{C}'')$ and the transition probability Q.

Define the mapping ψ of $(\Omega'',\mathcal{O}\!\mathcal{C}'') \times (\Omega',\mathcal{O}\!\mathcal{C}')$ onto $(\Omega,\mathcal{O}\!\mathcal{C})$ by

$$\pi_n\psi(\omega'',\omega') = \begin{cases} \pi''_{-n}(\omega'') & \text{if } n \le 0 \ , \\ \pi'_{n-1}(\omega') & \text{if } n > 0 \ , \end{cases}$$

then ψ is a one-to-one and bimeasurable mapping. Hence, the set function M on $(\Omega,\mathcal{O}\!\mathcal{C})$ defined by $M(A) = M^*(\psi^{-1}A)$ for all $A \in \mathcal{O}\!\mathcal{C}$ indeed is a measure on $(\Omega,\mathcal{O}\!\mathcal{C})$.

Let $(\mu''_n)_{n=0}^{\infty}$ be the sequence of the marginal measures of M_0''. If f is a non negative function in $B(X,\mathcal{R})$ and if A'' is the rectangle $\prod_{t=0}^{\infty} A_t$, where $A_t = X_t$ if $t > n$, then it follows from lemma 2.4.1 that

$$\int 1_{A''}(\omega'')f(\pi_0''\,\omega'')M_0''(d\omega'') = \int I_{A_n} PI_{A_{n-1}} \ldots PI_{A_0} f\, d\mu_n''$$

since for every $i \geq 0$ we have $\frac{d\mu_i''}{d\mu} P_\nu^{\leftarrow} = \frac{d\mu_{i+1}''}{d\mu}$, and $\frac{d\mu_{i+1}''}{d\mu} P = \frac{d\mu_i''}{d\mu} P_\nu^{\leftarrow} P = \frac{d\mu_i''}{d\mu}$ by proposition 2.4.1.

Now consider the rectangle $A = \prod_{t=-\infty}^{+\infty} A_t$, where $A_t = X_t$ if $t < n$ or $t > m$. We may assume $n < 0$ and $m > 0$. Put $A'' = \prod_{t=0}^{\infty} A_t''$, where $A_t'' = A_{-t}$ for all $t \geq 0$, and $A' = \prod_{t=0}^{\infty} A_t'$, where $A_t' = A_{t+1}$ for all $t \geq 0$. Then

$$M(A) = M^*(A'' \times A') = \int 1_{A''}(\omega'')Q_{\pi_0''\omega''}(A')M_0''(d\omega'') =$$

$$= \int I_{A_n} P \ldots PI_{A_m} d\mu_{-n}'' \quad \text{(cf. proposition 1.2.3).}$$

It follows that the measure M on $(\Omega,\mathfrak{A})$ indeed is a Markov measure for P on $(\Omega,\mathfrak{A})$ with marginal measures $\mu_n = \mu_{-n}''$ if $n \leq 0$, and $\frac{d\mu_n}{d\mu} = \frac{d\mu_0}{d\mu} P^n$ if $n > 0$.

It is clear from the construction of M that the mapping $\varphi : \mathfrak{A}'' \to \mathfrak{A}_{-\infty,0}$ as defined in the theorem, is an isomorphism from $(\Omega'',\mathfrak{A}'',M_0'',S'')$ to $(\Omega,\mathfrak{A}_{-\infty,0},M,S^{-1})$.

The uniqueness of M follows from the fact that a Markov measure for P on $(\Omega,\mathfrak{A})$ is uniquely defined by the sequence $(\mu_n)_{n=-\infty}^{+\infty}$ of the marginal measures. In this case, these marginal measures must satisfy

$$\frac{d\mu_n}{d\mu} = \frac{d\mu_0}{d\mu} P^n \quad \text{if } n > 0 \;; \qquad \frac{d\mu_n}{d\mu} = \frac{d\mu_0}{d\mu} P_\nu^{\leftarrow -n} \quad \text{if } n \leq 0 \;,$$

and therefore indeed are unique.

Finally, assume $\nu_0 << \mu_0$, and let N be the measure on $(\Omega,\mathcal{O}\mathcal{C})$ constructed as above with initial measure ν_0 instead of μ_0. Let A be the rectangle $\prod_{t=-\infty}^{+\infty} A_t$, where $A_t = X_t$ if $t < n < 0$ or $t > m > 0$. Again, put $A'' = \prod_{t=0}^{\infty} A''_t$ with $A''_t = A_{-t}$ for all $t \geq 0$, and $A' = \prod_{t=0}^{\infty} A'_t$ with $A'_t = A_{t+1}$ for all $t \geq 0$. Then it follows from the construction of M and N and prcposition 2.2.2 that

$$\int 1_A(\omega) \frac{d\nu_0}{d\mu_0} (\pi_0\omega)M(d\omega) =$$

$$= \int 1_{A''}(\omega'') \frac{d\nu_0}{d\mu_0} (\pi''_0 \omega'')Q_{\pi''_0\omega''}(A')M''_0(d\omega'') =$$

$$= \int 1_{A''}(\omega'')Q_{\pi''_0\omega''}(A')N''_0(d\omega'') = N(A) ,$$

hence $N << M$.

Conversely, if $N << M$, then in particular we must have $N''_0 << M''_0$ and therefore by proposition 2.2.2 $\nu_0 << \mu_0$.

PROPOSITION 2.4.2. Let T be a measurable measure preserving transformation in a σ-finite measure space $(X,\mathcal{R},\mu)$. Let P be the forward process and let $\overleftarrow{P}_\mu$ be the backward process for μ corresponding to T. Then there exists a Markov measure M for P on $(\Omega,\mathcal{O}\mathcal{C})$ with marginal measure $\mu_n = \mu$ for all n such that $(\Omega,\mathcal{O}\mathcal{C},M,S)$ is a two-sided shift space for P and $(\Omega,\mathcal{O}\mathcal{C},M,S^{-1})$ is a two-sided shift space for $\overleftarrow{P}_\mu$.

PROOF. Let $(\Omega,\mathcal{O}\mathcal{C},M,S)$ be the two-sided shift space for P as constructed in theorem 2.4.1 with initial measure μ and such that $(\Omega, \mathcal{O}\!\mathcal{L}_{-\infty,0}, M, S^{-1})$ is isomorphic to a one-sided shift space for $\overleftarrow{P}_\mu$. Because of the invariance of μ we have $1P = 1$, and since also $\frac{d\mu T}{d\mu} = 1$, by theorem 1.5.1 we obtain $1\overleftarrow{P}_\mu = 1$. It follows that every marginal measure μ_n of M equals μ. Then by lemma 2.4.1 we have for every rectangle $A = \prod_{t=-\infty}^{+\infty} A_t$ where $A_t = X_t$ if $t < n$ or $t > m$

$$M(A) = \int I_{A_n} P I_{A_{n+1}} P \ldots P 1_{A_m} d\mu_n = \int I_{A_m} \overleftarrow{P}_\mu \ldots \overleftarrow{P}_\mu 1_{A_n} d\mu_m ,$$

from which the last statement follows.

To conclude this section we shall give a more detailed description of the Markov shifts associated with the transformation $Tx = 2x \pmod 1$ on the unit interval.

EXAMPLE. Let $(X, \mathcal{B}, \mu)$ be the unit interval with the Borel sets and the Lebesgue measure. Let T be the transformation $Tx = 2x \pmod 1$. It is well-known (see e.g. [6], p. 29) that T is measure preserving and ergodic.

Let H be the set of (equivalence classes of modulo μ equal) positive $\mathcal{B}$-measurable functions h such that $E^{\mu}_{T^{-1}\mathcal{B}} h = 1$. Then for every $h \in H$ we may assume that h is everywhere defined and satisfies $h(\frac{1}{2}x) + h(\frac{1}{2}x + \frac{1}{2}) = 2$ for all $x \in [0,1]$.

According to theorem 1.5.1 there exists a one-to-one correspondence between the class of backward processes associated with T and H. Let for every $h \in H$ $\overleftarrow{P}_h$ be the backward process on $(X, \mathcal{B}, \mu)$ corresponding to h. Then for all $f \in M^+(X, \mathcal{B}, \mu)$ we have $f\overleftarrow{P}_h = h(f \circ T)$.

If for every set $A \in \mathcal{R}$ we denote by A^- the set $A \cap [0,\frac{1}{2})$ and by A^+ the set $A \cap [\frac{1}{2},1]$, then

$$(\overleftarrow{P}_h 1_A)(x) = \tfrac{1}{2}h(\tfrac{1}{2}x)1_{TA^-}(x) + \tfrac{1}{2}h(\tfrac{1}{2}x + \tfrac{1}{2})1_{TA^+}(x) .$$

Indeed, for every $f \in M^+(X,\mathcal{R},\mu)$ we have

$$\int_0^1 f(\overleftarrow{P}_h 1_A)d\mu = \tfrac{1}{2}\int_0^1 f(x)h(\tfrac{1}{2}x)1_{TA^-}(x)dx + \tfrac{1}{2}\int_0^1 f(x)h(\tfrac{1}{2}x+\tfrac{1}{2})1_{TA^+}(x)dx =$$

$$= \int_0^{\frac{1}{2}} f(2t)h(t)1_{A^-}(t)dt + \int_{\frac{1}{2}}^1 f(2t-1)h(t)1_{A^+}(t)dt = \int_A h(f\circ T)d\mu .$$

Note that the process $\overleftarrow{P}_h$ indeed is given by a transition probability. The interpretation of $\overleftarrow{P}_h$ is clear: if under the action of the transformation T we have arrived in the state x, this state can have been reached only from the states $\frac{1}{2}x$ and $\frac{1}{2}x + \frac{1}{2}$. The formula for $\overleftarrow{P}_h$ now says that the probability that we came from $\frac{1}{2}x$ is $\frac{1}{2}h(\frac{1}{2}x)$, and the probability that we came from $\frac{1}{2}x + \frac{1}{2}$ is $\frac{1}{2}h(\frac{1}{2}x + \frac{1}{2})$.

Let for every $h \in H$ $(\Omega,\mathfrak{A},M_h,S)$ be the two-sided shift space for P such that $(\Omega,\mathfrak{A}_{-\infty,0},M_h,S^{-1})$ is isomorphic to the one-sided shift space for $\overleftarrow{P}_h$ with initial measure μ. Every measure M_h therefore is a probability on $(\Omega,\mathfrak{A})$ and the marginal measure μ_n^h equals μ if $n \geq 0$, and satisfies

$$\frac{d\mu_{-n}^h}{d\mu} = h(h\circ T)\ \ldots\ (h\circ T^{n-1}) \qquad \text{for all } n > 0 .$$

It follows that all marginal measures are equivalent to μ.

PROPERTY 1. For every integer n and every $h \in H$ the shift S is conservative on $(\Omega, \mathfrak{A}_{n,\infty}, M_h)$.

PROOF. The system $(\Omega, \mathfrak{A}_{n,\infty}, M_h, S)$ is isomorphic with the one-sided shift space for P with initial measure μ_n^h by proposition 2.2.3. Since P is conservative on X, $\mu_n^h \ll \mu$ and $1P = 1$ it follows by theorem 2.3.1 that the shift on this one-sided shift space is conservative and therefore that S is conservative on $(\Omega, \mathfrak{A}_{n,\infty}, M_h)$.

PROPERTY 2. For every $h \in H$ such that $h \not\equiv 1$, the shift S is dissipative on $(\Omega, \mathfrak{A}, M_h)$. If $h \equiv 1$, then S is conservative on $(\Omega, \mathfrak{A}, M_1)$.

PROOF. If $h = 1$ on X, then M_1 is an invariant probability under S, and therefore S is conservative on $(\Omega, \mathfrak{A}, M_1)$.

Now assume $h \not\equiv 1$. Then $(\Omega, \mathfrak{A}_{-\infty,0}, M_h, S^{-1})$ is isomorphic with the one-sided shift space for $\overleftarrow{P_h}$ with initial measure μ. By theorem 2.1.1, $\overleftarrow{P_h}$ is dissipative, and therefore by theorem 2.3.1 the shift in this one-sided shift space is dissipative. Hence there exists a partition $(W_i)_{i=1}^{\infty}$ of Ω such that for every i $W_i \in \mathfrak{A}_{-\infty,0}$ and $W_i \cap S^k W_i = \emptyset$ for every $k > 0$. It follows that W_i is a wandering set under S, and therefore that S is dissipative on $(\Omega, \mathfrak{A}, M_h)$.

Note that although there exists an algebra of recurrence sets generating $\mathfrak{A}$, and despite of the fact that all marginal measures of M_h are equivalent with an invariant measure for P, the shift S is dissipative on $(\Omega, \mathfrak{A}, M_h)$. In particular, it follows that if $h \equiv 1$ we have $M_h \perp M_1$. The next property might one lead to conjecture that any two measures M_{h_1} and M_{h_2} are mutually singular.

PROPERTY 3. If $f \in H$, $g \in H$ satisfy the conditions:

i) there exists a constant $\delta_1 > 0$ such that $f > \delta_1$ and $g > \delta_1$,

ii) there exists a constant $\delta_2 > 0$ and a dyadic interval $(\frac{p}{2^n}, \frac{p+1}{2^n}) = I$ such that $|f - g| > \delta_2$ on I,

then $M_f \perp M_g$.

PROOF. First we note that if a and b are real numbers such that $0 < a \le 2$, $0 < b \le 2$ and $|a - b| \ge \delta_2$, then

$$(a + b)^2 - (2\sqrt{ab})^2 \ge \delta_2^2$$

$$(a + b) - 2\sqrt{ab} \ge \frac{1}{8}\,\delta_2^2$$

$$\sqrt{ab} \le \frac{a + b}{2} - \frac{1}{16}\,\delta_2^2 .$$

Since $f \in H$ and $g \in H$ there exists an integer j with $0 < j \le 2^{n-1}$ such that $|f(x) - g(x)| > \delta_2$ on $(\frac{j-1}{2^n}, \frac{j}{2^n}) \cup (\frac{j-1}{2^n} + \frac{1}{2}, \frac{j}{2^n} + \frac{1}{2})$.

For every real x define the function k(x) by

$$k(x) = \sqrt{f(x(\mathrm{mod}\ 1))g(x(\mathrm{mod}\ 1))} .$$

For reasons that will soon become clear, we first estimate the sum

$$A = \sum_{i=1}^{2^n} \frac{1}{2^n}\, k(\frac{x}{2^n} + \frac{i-1}{2^n})k(\frac{x}{2^{n-1}} + \frac{i-1}{2^{n-1}}) \ldots k(\frac{x}{2} + \frac{i-1}{2}) .$$

Since the function k has period 1, we obtain

$$A = \sum_{i=1}^{2^{n-1}} \frac{1}{2^{n-1}} k(\frac{x}{2^{n-1}} + \frac{i-1}{2^{n-1}}) \ldots k(\frac{x}{2} + \frac{i-1}{2}) \cdot$$

$$\cdot \tfrac{1}{2}[k(\frac{x}{2^n} + \frac{i-1}{2^n}) + k(\frac{x}{2^n} + \frac{i-1}{2^n} + \tfrac{1}{2})] .$$

For every i and every $x \in [0,1]$ we have

$$\tfrac{1}{2}(k(\frac{x}{2^n} + \frac{i-1}{2^n}) + k(\frac{x}{2^n} + \frac{i-1}{2^n} + \tfrac{1}{2})) \leq$$

$$\leq \tfrac{1}{4}(f(\frac{x}{2^n} + \frac{i-1}{2^n}) + f(\frac{x}{2^n} + \frac{i-1}{2^n} + \tfrac{1}{2}) +$$

$$+ g(\frac{x}{2^n} + \frac{i-1}{2^n}) + g(\frac{x}{2^n} + \frac{i-1}{2^n} + \tfrac{1}{2})) \leq 1 .$$

For $i = j$ we obtain for μ-almost all $x \in [0,1]$

$$\tfrac{1}{2}(k(\frac{x}{2^n} + \frac{j-1}{2^n}) + k(\frac{x}{2^n} + \frac{j-1}{2^n} + \tfrac{1}{2})) \leq 1 - \frac{1}{16} \delta_2^2 .$$

Hence

$$A \leq \sum_{i=1}^{2^{n-1}} \frac{1}{2^{n-1}} k(\frac{x}{2^{n-1}} + \frac{i-1}{2^{n-1}}) \ldots k(\frac{x}{2} + \frac{i-1}{2}) -$$

$$- \frac{1}{16} \delta_2^2 \frac{1}{2^{n-1}} k(\frac{x}{2^{n-1}} + \frac{j-1}{2^{n-1}}) \ldots k(\frac{x}{2} + \frac{j-1}{2}) ,$$

$$A \leq \sum_{i=1}^{2^{n-1}} \frac{1}{2^{n-1}} k(\frac{x}{2^{n-1}} + \frac{i-1}{2^{n-1}}) \ldots k(\frac{x}{2} + \frac{i-1}{2}) - \frac{1}{16} \delta_2^2 \left(\frac{\delta_1}{2}\right)^{n-1} .$$

In the same way we show

$$\sum_{i=1}^{2^{n-1}} \frac{1}{2^{n-1}} k(\frac{x}{2^{n-1}} + \frac{i-1}{2^{n-1}}) \ldots k(\frac{x}{2} + \frac{i-1}{2}) \leq$$

$$\leq \sum_{i=1}^{2^{n-2}} \frac{1}{2^{n-2}} k\left(\frac{x}{2^{n-2}} + \frac{i-1}{2^{n-2}}\right) \ldots k\left(\frac{x}{2} + \frac{i-1}{2}\right) \leq$$

$$\leq \ldots \leq \tfrac{1}{2}\left(k\left(\tfrac{x}{2}\right) + k\left(\tfrac{x}{2} + \tfrac{1}{2}\right)\right) \leq 1 \qquad \text{for all } x \text{ ,}$$

hence

$$0 < A \leq 1 - \frac{1}{16}\, \delta_2^2 \left(\frac{\delta_1}{2}\right)^{n-1} .$$

Because of theorem 2.2.1 we have $M_f \perp M_g$ if

$$\lim_{p\to\infty} \rho(\mu^f_{-p}, \mu^g_{-p}) = 0 \text{ ,}$$

hence if

$$\lim_{p\to\infty} \int_0^1 k(x) \ldots k(T^{p-1}x)dx = 0 .$$

If $I_p = \int_0^1 k(x) \ldots k(T^p x)dx$, then for all $p > n$ we have

$$I_p = \sum_{i=1}^{2^n} \int_{\frac{i-1}{2^n}}^{\frac{i}{2^n}} k(x) \ldots k(T^p x)dx =$$

$$= \sum_{i=1}^{2^n} \frac{1}{2^n} \int_0^1 k\left(\frac{x}{2^n} + \frac{i-1}{2^n}\right) k\left(\frac{x}{2^{n-1}} + \frac{i-1}{2^{n-1}}\right) \ldots k(T^{p-n}x)dx =$$

$$= \int_0^1 k(x) \ \ldots \ k(T^{p-n}x)\left[\frac{1}{2^n}\sum_{i=1}^{2^n} k\left(\frac{x}{2^n} + \frac{i-1}{2^n}\right) \ \ldots \ k\left(\frac{x}{2} + \frac{i-1}{2}\right)\right]dx \le$$

$$\le \left(1 - \frac{1}{16}\,\delta_2^2\left(\frac{\delta_1}{2}\right)^{n-1}\right) I_{p-n} \ .$$

It follows that $\lim_{p\to\infty} I_p = 0$, and therefore $M_f \perp M_g$.

CHAPTER III

PERIODICITY FOR MARKOV PROCESSES

3.1. THE ESSENTIAL PART OF A MARKOV PROCESS

Throughout this section P will be a Markov process on a σ-finite measure space $(X,\mathcal{R},\mu)$.

DEFINITION 3.1.1. A set $A \in \mathcal{R}$ is said to be invariant under P if $PA \subset A$. The class of invariant sets will be denoted by $\mathcal{R}_i$.

Sometimes an invariant set is said to be closed. Since PA is the set of states which can be reached from A, invariance of the set A means that if the process has entered the set A, it cannot get out of A anymore. Obviously, $\emptyset$ and X are invariant sets. By proposition 2.1.4 also the conservative part C of X with respect to C is invariant. However, in general the dissipative part is not invariant.

PROPOSITION 3.1.1. A set $A \in \mathcal{R}$ is invariant if and only if $P1_A = P1$ on A.

PROOF. Let A be invariant. For all $B \subset A$ we have $PB \subset PA \subset A$, hence

$$\int_B P1_A \, d\mu = \int (1_B P) 1_A \, d\mu = \int 1_B P \, d\mu = \int_B P1 \, d\mu \ ,$$

hence $P1_A = P1$ on A. Conversely, suppose for some set $A \in \mathcal{R}$ we have $P1_A = P1$ on A. Then $P1_{X\setminus A} = 0$ on A,

$$0 = \int_A P1_{X\setminus A}\, d\mu = \int_{X\setminus A} 1_A P\, d\mu \;,$$

hence supp $1_A P \subset A$.

DEFINITION 3.1.2. The restriction P_A of P to an invariant set A is for all $f \in M^+(A,\mathcal{R} \cap A,\mu)$ defined by

$$(P_A f)(x) = (Pf_A)(x) \qquad \text{for all } x \in A \;,$$

where $f_A(x) = f(x)$ if $x \in A$ and $f_A(x) = 0$ if $x \in X \setminus A$.

PROPOSITION 3.1.2. The operator P_A as defined in definition 3.1.2 is the extension to $M^+(A,\mathcal{R} \cap A,\mu)$ of a Markov operator in $\mathcal{L}_\infty(A,\mathcal{R} \cap A,\mu)$ and therefore determines a Markov process P_A on $(A,\mathcal{R} \cap A,\mu)$. For every integer n and every $f \in M^+(A,\mathcal{R} \cap A,\mu)$ we have

$$P_A^n\, f = P^n f_A \qquad \text{on } A$$

$$fP_A^n = f_A P^n \qquad \text{on } A\;.$$

PROOF. The proof of the fact that the operator P_A determines a Markov process on $(A,\mathcal{R} \cap A,\mu)$ amounts to a simple verification. For every $f \in M^+(X,\mathcal{R},\mu)$ we have $PI_A f = Pf$ on A. In fact, since for every $B \subset A$ we have supp $1_B P \subset A$, we have

$$\int_B PI_A f\, d\mu = \int (1_B P) I_A f\, d\mu = \int (1_B P) f d\mu = \int_B Pf\, d\mu \;.$$

The second statement now follows from $P_A^n\, f = (PI_A)^{n-1}\, Pf_A$ on

A for all $f \in M^+(A, \mathcal{R} \cap A, \mu)$.

Finally, for every $B \subset A$ and every $f \in M^+(A, \mathcal{R} \cap A, \mu)$ we have

$$\int_B f P_A^n \, d\mu = \int_A f(P_A^n 1_B) d\mu = \int_X f_A (P^n 1_B) d\mu = \int_B f_A P^n \, d\mu$$

from which the last relation follows.

PROPOSITION 3.1.3. If $(A_n)_{n=1}^{\infty}$ is a sequence in $\mathcal{R}_i$, then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{R}_i$ and $\bigcap_{n=1}^{\infty} A_n \in \mathcal{R}_i$.

PROOF. Using proposition 1.3.6 we obtain

$$P\left(\bigcup_{n=1}^{\infty} A_n\right) = \bigcup_{n=1}^{\infty} PA_n \subset \bigcup_{n=1}^{\infty} A_n$$

$$P\left(\bigcap_{n=1}^{\infty} A_n\right) \subset \bigcap_{n=1}^{\infty} PA_n \subset \bigcap_{n=1}^{\infty} A_n .$$

We have already remarked that if A and B are invariant sets, the set $A \setminus B$ is not necessarily invariant.

DEFINITION 3.1.3. An invariant set $A \in \mathcal{R}$ is said to be properly invariant if for all invariant sets $B \in \mathcal{R}$ with $B \subset A$ the set $A \setminus B$ is invariant.

PROPOSITION 3.1.4. An invariant set $A \in \mathcal{R}$ is properly invariant if and only if for all invariant sets $B \subset A$ we have $I_A P 1_B = I_B P 1$.

PROOF. Suppose A is properly invariant, and B is an invariant subset of A. Then since $A \setminus B$ is invariant, we have $P 1_{A \setminus B} = P 1$

on $A \setminus B$, and therefore $P1_B = 0$ on $A \setminus B$, hence $I_A P1_B = I_B P1$.

Conversely, if A is invariant and for all invariant $B \subset A$ we have $I_A P1_B = I_B P1$, then it follows from $P1_A = P1_B + P1_{A \setminus B}$ that for every invariant subset B of A we have $I_{A \setminus B} P1 = I_{A \setminus B} P1_A = I_{A \setminus B} P1_{A \setminus B}$, hence $A \setminus B$ is invariant.

The conservative part C is an example of a properly invariant set. Indeed, if B is an invariant subset of C, then we have $P1_B = P1 = 1$ on B, hence $P1_B \geq 1_B$ on C. Proposition 2.1.3, applied to the function $1 - 1_B$ then implies $P1_B = 1_B$ on C.

PROPOSITION 3.1.5. The properly invariant sets form a σ-ring.

PROOF. As a consequence of the definition we have that every invariant subset of a properly invariant set is properly invariant. Therefore, if A is properly invariant and if B is invariant, then $A \setminus B$ is also properly invariant.

Let $(A_n)_{n=1}^{\infty}$ be a sequence of properly invariant sets, and put $A = \bigcup_{n=1}^{\infty} A_n$. If $A'_n = A_n \setminus \bigcup_{i=1}^{n-1} A_i$, then $(A'_n)_{n=1}^{\infty}$ is a sequence of pairwise disjoint properly invariant sets with union A; hence we may assume that the sets $(A_n)_{n=1}^{\infty}$ are pairwise disjoint. By proposition 3.1.3 A is invariant. Let B be an invariant subset of A, then for every n $B_n = A_n \cap B$ is invariant, and therefore $A_n \setminus B_n$ is invariant. Since $A \setminus B = \bigcup_{n=1}^{\infty} (A_n \setminus B_n)$, $A \setminus B$ is invariant and A is properly invariant.

DEFINITION 3.1.4. The essential part E of X with respect to P is the modulo μ largest properly invariant set.

Note that if we consider P on the measure space $(X,\mathcal{R},\nu)$ instead of on $(X,\mathcal{R},\mu)$, where ν is a σ-finite measure equivalent to μ, the concepts of invariant set and properly invariant set are not influenced. Hence also the essential part of X with respect to P is independent of the particular choice of μ.

PROPOSITION 3.1.6. Let E be the essential part of X for a Markov process P on $(X,\mathcal{R},\mu)$. Let A be a subset of E. If $\bar{A}$ denotes the smallest invariant set containing A, then, if $P1 > 0$ on $\bar{A}$, we have for every $n \geq 0$

$$\bar{A} = \bigcup_{k=n}^{\infty} P^k A = E \cap \left(\bigcup_{k=n}^{\infty} P^{-k} A \right) .$$

PROOF. Obviously we have $\bar{A} = \bigcup_{k=0}^{\infty} P^k A \subset E$. Put $B = \bigcup_{k=n}^{\infty} P^k A$, then $B \subset \bar{A}$, B is invariant, and therefore also $\bar{A} \setminus B$ in invariant. It follows that $P^n(\bar{A} \setminus B) \subset \bar{A} \setminus B$. On the other hand, $P^n(\bar{A} \setminus B) \subset \bigcup_{k=n}^{\infty} P^k A = B$, hence $\mu P^n(\bar{A} \setminus B) = 0$. From $P1 > 0$ on $\bar{A}$ and the invariance of $\bar{A}$ we conclude $P1_{\bar{A}} > 0$ on $\bar{A}$, and therefore by proposition 1.3.5 $P^n 1_{\bar{A}} > 0$ on $\bar{A}$. Since $\mu P^n(\bar{A} \setminus B) = 0$, we have $\int 1_{\bar{A}\setminus B}(P^n 1_{\bar{A}}) d\mu = 0$, and therefore $\mu(\bar{A} \setminus B) = 0$. It follows that $B = \bar{A}$.

From $P1_{\bar{A}} > 0$ on $\bar{A}$, $P1_{\bar{A}} = 0$ on $E \setminus \bar{A}$ and $P1_{X\setminus E} = 0$ on E we conclude $P^k 1_{\bar{A}} = 0$ on $E \setminus \bar{A}$ for every k, and therefore $E \cap \left(\bigcup_{k=n}^{\infty} P^{-k} A \right) \subset \bar{A}$. Put $B = \bar{A} \setminus \left(\bigcup_{k=n}^{\infty} P^{-k} A \right)$, and let $\bar{B}$ be the smallest invariant set containing B. Then $\bar{B} \subset \bar{A}$,

$$0 = \int 1_B(\sum_{k=n}^{\infty} P^k 1_A)d\mu = \int (\sum_{k=n}^{\infty} 1_B P^k) 1_A \, d\mu .$$

It follows that $\mu(A \cap \bar{B}) = 0$, and since we may assume $\mu(A) > 0$, $\mu(\bar{B}) = 0$, and therefore $\mu(B) = 0$.

THEOREM 3.1.1. Assume $P1 > 0$ on an invariant set F. The set F is properly invariant if and only if for every two sets $A \subset F$ and $B \subset F$ for which there exists an integer $n > 0$ with $\mu(A \cap P^{-n}B) > 0$, there also exists an integer $m > 0$ such that $\mu(B \cap P^{-m}A) > 0$.

Roughly speaking: the property "if there is a positive probability that we can ever reach the set B from the set A, then there is a positive probability that we can ever reach the set A from the set B" characterizes the essential part of the process P.

PROOF. Let F be properly invariant, and suppose $\mu(A \cap P^{-n}B) > 0$. Then by proposition 3.1.6 we have $\mu(A \cap \bar{B}) > 0$, and therefore there exists an integer $m > 0$ such that $\mu(A \cap P^{m}B) > 0$. Then

$$\int (1_B P^m) 1_A \, d\mu = \int 1_B (P^m 1_A) d\mu > 0 ,$$

hence $\mu(B \cap P^{-m}A) > 0$.

Conversely, if F is not properly invariant, then there exists an invariant set $B \subset F$ such that $P1_B > 0$ on a subset A of $F \setminus B$, $\mu(A) > 0$. Then $\mu(A \cap P^{-1}B) > 0$. However, for every $n > 0$ we have $\mu(B \cap P^{-n}A) = 0$, since $\int (1_B P^n) 1_A \, d\mu = 0$ for every n because of the invariance of B.

Let T be a measurable negatively non singular transformation in a σ-finite measure space $(X,\mathcal{R},\mu)$. A set $A \in \mathcal{R}$ is said to be invariant under T if $T^{-1}A \supset A$. If P is the forward process for T, then $P1_A = 1$ on A, hence A is invariant under P. Conversely, if A is invariant under P, then we have $P1_A = P1 = 1$ on A, hence $T^{-1}A \supset A$, and A is invariant under T. Hence, just as for conservativity (cf. section 2.1), the concept of an invariant set is the same for a transformation and for the forward process associated with the transformation.

PROPOSITION 3.1.7. Let T be a negatively non singular measurable transformation on a σ-finite measure space $(X,\mathcal{R},\mu)$. Let P be the forward process associated with T. Then the essential part of X with respect to P is the conservative part of X with respect to P.

PROOF. Since C is properly invariant, we have $C \subset E$. Let A be an invariant subset of D with $\mu(A) > 0$. Since D is a countable union of wandering sets, there exists a wandering set $W \subset A$ with $\mu(W) > 0$. If we define $\hat{W} = \bigcup_{k=0}^{\infty} T^{-k}W$, then

$$W \subset T^{-1}(A \setminus \hat{W}) = T^{-1}A \setminus \bigcup_{k=1}^{\infty} T^{-k}W ,$$

hence $\mu(W \cap P^{-1}(A \setminus \hat{W})) > 0$. On the other hand we have $\mu((A \setminus \hat{W}) \cap P^{-n}W) = 0$ for every $n > 0$; hence the set A cannot belong to the essential part of X. It follows that C = E.

DEFINITION 3.1.5. A Markov process P on a σ-finite measure space $(X,\mathcal{R},\mu)$ is said to be irreducible if $\mathcal{R}_i = \{\emptyset,X\}$.

Since in this case the invariant sets form a σ-algebra, for every irreducible Markov process we have $E = X$.

For later reference we also note the following property.

LEMMA 3.1.1. If P is an irreducible Markov process on $(X,\mathcal{R},\mu)$ and if $\mathcal{R}$ is not trivial, then $P1 > 0$ and $1P > 0$.

PROOF. If $P1 = 0$ everywhere, then for every set $A \in \mathcal{R}$ we have $PA = \emptyset$. Therefore every $A \in \mathcal{R}$ is invariant which is impossible because of the irreducibility of P. Hence $P1 > 0$ on a subset of X of positive measure.

Suppose $P1 = 0$ on A. Then $PA = \emptyset$, A is invariant and therefore $A = \emptyset$ or $A = X$. Since $A = X$ is impossible, it follows that $A = \emptyset$, and therefore $P1 > 0$.

Finally, by proposition 1.3.5 we have $P^2X \subset PX$, hence PX is invariant. Because of $P1 > 0$ we have $\mu(PX) > 0$, hence $PX = X$ and $1P > 0$.

PROPOSITION 3.1.8. Let P be the forward process associated with a negatively non singular measurable transformation T on $(X,\mathcal{R},\mu)$. Then P is irreducible if and only if T is conservative and ergodic.

PROOF. If P is irreducible, then by proposition 3.1.7 we must have $C = E = X$. Moreover, for every invariant set A we have $\mu(A) = 0$ or $\mu(X \setminus A) = 0$, hence T is ergodic.

Conversely, if T is conservative, then $E = C = X$. If T moreover is ergodic, for every invariant set we have $\mu(A) = 0$ or $\mu(X \setminus A) = 0$, hence $\mathcal{R}_i = \{\emptyset, X\}$.

<u>THEOREM 3.1.2</u>. Suppose $P1 > 0$. For any positive integer n let E^n be the essential part of X with respect to P^n. Then $E^n = E$.

PROOF. First let A be a P^n-properly invariant set. Then $P^n(PA) = P(P^nA) \subset PA$, hence PA is P^n-invariant. Let B_0 be a P^n-invariant subset of PA, and put $B_1 = PA \setminus B_0$. The set $P^{n-1}B_0$ is a P^n-invariant subset of A. Put $B_2 = A \setminus P^{n-1}B_0$, then since A is P^n-properly invariant, also the set B_2 is P^n-invariant. Suppose $\mu(PB_2 \cap B_0) > 0$. Then it follows from the propositions 1.3.6 and 1.3.7 that

$$0 < \mu(P^{n-1}(PB_2 \cap B_0)) \leq \mu(B_2 \cap P^{n-1}B_0) = 0 .$$

Contradiction, hence $\mu(PB_2 \cap B_0) = 0$. Since

$$PA = P(P^{n-1}B_0) \cup PB_2 ,$$

we obtain $PB_2 = B_1$, and therefore $P^{n-1}PB_2 = P^{n-1}B_1 \subset B_2$, $P^nB_1 \subset PB_2 = B_1$. Consequently, the set B_1 is P^n-invariant. It follows that if A is P^n-properly invariant, also the set PA is P^n-properly invariant. In particular, PE^n is P^n-properly invariant, and therefore we must have $PE^n \subset E^n$, for otherwise $E^n \cup PE^n$ would be a P^n-properly invariant set larger than E^n. Hence E^n is P-invariant.

Now let A be a P-invariant subset of E^n, and put $B = E^n \setminus A$. Since A obviously is P^n-invariant, also B is P^n-invariant. If $\mu(A \cap PB) > 0$, then $0 < \mu(P^{n-1}A \cap P^nB) \leq$ $\leq \mu(A \cap B) = 0$, hence $\mu(A \cap PB) = 0$ and $PB \subset B$. It follows that E^n is P-properly invariant, hence $E^n \subset E$.

We show now that also $E \subset E^n$. Obviously, E is P^n-invariant. Let A be a P^n-invariant subset of E. For every j, let $(P^jA)^1$ denote the set P^jA, and $(P^jA)^0$ denote the set $E \setminus P^jA$. By $[i_0,\ldots,i_{n-1}]$ we shall denote the set

$$A^{i_0} \cap (PA)^{i_1} \cap \ldots \cap (P^{n-1}A)^{i_{n-1}} \cap E ,$$

where for every j $i_j = 0$ or $i_j = 1$. The sets $([i_0,\ldots,i_{n-1}])$ form a partition of E.

For $k = 0,\ldots,n$ let B_k be the union of those sets $[i_0,\ldots,i_{n-1}]$ of which exactly k of the numbers $(i_0,\ldots,i_{n-1})$ are 0. Then $(B_0,\ldots,B_n)$ is a partition of E such that $PB_k \subset \bigcup_{i=0}^{k} B_i$ for every k by proposition 1.3.6. Since $PB_0 \subset B_0$, the set B_0, and therefore the set $E \setminus B_0$, is invariant under P. Therefore we must have $PB_1 \cap B_0 = \emptyset$, hence $PB_1 \subset B_1$. Inductively, it follows that $PB_k \subset B_k$ for every k. Combining this property with proposition 1.3.6, we obtain $P[i_0,\ldots,i_{n-1}] \subset [i_{n-1},i_0,\ldots,i_{n-2}]$, and therefore $P^n[i_0,\ldots,i_{n-1}] \subset [i_0,\ldots,i_{n-1}]$.

Since $E \setminus A$ is the union of the sets $[i_0,i_1,\ldots,i_{n-1}]$ for which $i_0 = 0$, it follows $P^n(E \setminus A) \subset E \setminus A$. Hence E is P^n-properly invariant, and $E \subset E^n$.

<u>THEOREM 3.1.3</u>. Let P be a Markov process on $(X,\mathcal{R},\mu)$ such that X is properly invariant. Let Q be a backward process or an adjoint process for P. Then $\mathcal{R}_i(P) = \mathcal{R}_i(Q)$, and consequently X is also properly invariant with respect to Q.

PROOF. If $A \in \mathcal{R}_i(P)$, then by proposition 3.1.4 we have $P^{-1}A \subset A$, and since $P^{-1}A = QA$ (section 1.4), $A \in \mathcal{R}_i(Q)$. Conversely, if $A \in \mathcal{R}_i(Q)$, then $P^{-1}A \subset A$, hence $P1_{X\setminus A} = P1$ on $X \setminus A$, and $X \setminus A \in \mathcal{R}_i(P)$. Since X is properly invariant, we also have $A \in \mathcal{R}_i(P)$.

Without the assumption that X is properly invariant under P, $\mathcal{R}_i(P)$ and $\mathcal{R}_i(Q)$ need not coincide. In fact, the essential part of X with respect to P need not even be invariant with respect to Q.

3.2. THE DETERMINISTIC σ-ALGEBRA

Let P be a Markov process on a σ-finite measure space $(X,\mathcal{R},\mu)$. If P is the forward process associated with a transformation T, we have for every sequence $(A_n)_{n=1}^{\infty}$ in $\mathcal{R}$ $P^{-1}(\bigcap_{n=1}^{\infty} A_n) = \bigcap_{n=1}^{\infty} P^{-1}A_n$. This property does in general not hold for every Markov process P. However, there exists a sub-σ-algebra $\mathcal{R}_0$ of $\mathcal{R}$ such that this property holds for every sequence of sets in $\mathcal{R}_0$.

LEMMA 3.2.1. The class

$$\mathcal{R}_0 = \{A \in \mathcal{R} \mid P^n 1_A = I_{P^{-n}A} P^n 1 \text{ for all } n \geq 0\}$$

is a σ-algebra.

PROOF. Put $\mathcal{R}_0^n = \{A \in \mathcal{R} \mid P^n 1_A = I_{P^{-n}A} P^n 1\}$, then $\mathcal{R}_0 = \bigcap_{n=0}^{\infty} \mathcal{R}_0^n$, and it suffices to show that $\mathcal{R}_0^n$ is a σ-algebra. Obviously $X \in \mathcal{R}_0^n$. If $A \in \mathcal{R}_0^n$ and $B \in \mathcal{R}_0^n$, then we have

$$I_{P^{-n}(A\cup B)} P^n 1 \geq P^n 1_{A\cup B} \geq$$

$$\geq \max(I_{P^{-n}A} P^n 1, I_{P^{-n}B} P^n 1) = I_{P^{-n}A\cup P^{-n}B} P^n 1 .$$

It follows that $P^n 1_{A\cup B} = I_{P^{-n}A\cup P^{-n}B} P^n 1 = I_{P^{-n}(A\cup B)} P^n 1$, hence $A \cup B \in \mathcal{R}_0^n$.

In order to show that $A \setminus B \in \mathcal{R}_0^n$, we may assume $B \subset A$, since $A \setminus B = (A \cup B) \setminus B$. Then we have

$$P^n 1_{A\setminus B} = P^n 1_A - P^n 1_B = I_{P^{-n}A} P^n 1 - I_{P^{-n}B} P^n 1 = I_{P^{-n}A \setminus P^{-n}B} P^n 1 .$$

It follows that $A \setminus B \in \mathcal{R}_0^n$ and $P^{-n}(A \setminus B) = (P^{-n}A \setminus P^{-n}B) \cap P^{-n}X$. Since $P^{-n}A \subset P^{-n}X$, we obtain $P^{-n}(A \setminus B) = P^{-n}A \setminus P^{-n}B$.

Finally, let $(A_k)_{k=1}^{\infty}$ be a sequence in $\mathcal{R}_0$, and define $A = \bigcup_{k=1}^{\infty} A_k$. We may assume $A_k \uparrow A$ if $k \to \infty$. Then we have

$$P^n 1_A = \lim_{k\to\infty} P^n 1_{A_k} = \lim_{k\to\infty} I_{P^{-n}A_k} P^n 1 = I_{\bigcup_{k=1}^{\infty} P^{-n}A_k} P^n 1 .$$

It follows that $A \in \mathcal{R}_0^n$.

DEFINITION 3.2.1. The σ-algebra $\mathcal{R}_0$ introduced in lemma 3.2.1 is said to be the deterministic σ-algebra for P.

As a consequence of the proof of lemma 3.2.1 we note that for every n the mapping P^{-n} is an isomorphism from the σ-algebra $\mathcal{R}_0$ onto the σ-ring $P^{-n}\mathcal{R}_0$. It easily follows from the definition that if P is the forward process associated with a transformation T, then $\mathcal{R}_0 = \mathcal{R}$.

In order to enforce the class $\mathcal{R}_0$ to be a σ-algebra, the definition given here slightly differs from the definition given in Foguel [4], p. 7. The definitions coincide if the process satisfies $P1 = 1$.

PROPOSITION 3.2.1. $\mathcal{R}_0 = \{A \in \mathcal{R} \mid P^n P^{-n} A \subset A \text{ for all } n \geq 0\}$.

PROOF. If $A \in \mathcal{R}_0$, then $P^n 1_{X \setminus A} = 0$ on $P^{-n}A$, hence

$$\int (1_{P^{-n}A} P^n) 1_{X\setminus A} \, d\mu = 0 \ , \qquad P^n P^{-n} A \subset A \ .$$

Conversely, if A satisfies $P^n P^{-n} A \subset A$ for all $n \geq 0$, then

$$\int (1_{P^{-n}A} P^n) 1_{X\setminus A} \, d\mu = 0 \ ,$$

hence $P^n 1_{X\setminus A} = 0$ on $P^{-n}A$, and therefore $P^n 1_A = P^n 1$ on $P^{-n}A$, hence $P^n 1_A = I_{P^{-n}A} P^n 1$ for all $n \geq 0$ and $A \in \mathcal{R}_0$.

PROPOSITION 3.2.2. If $P1 > 0$, then $P^{-1}\mathcal{R}_0 \subset \mathcal{R}_0$.

PROOF. Suppose $A \in \mathcal{R}_0$, and let $B \in \mathcal{R}$ satisfy $PB \subset A$. Then

$$0 = \int 1_B (P 1_{X\setminus A}) d\mu = \int 1_B 1_{P^{-1}X \setminus P^{-1}A} (P1) d\mu \ .$$

Since $P1 > 0$, we conclude $P^{-1}X = X$ and $\mu(B \cap (X \setminus P^{-1}A)) = 0$, $B \subset P^{-1}A$. Now for every n we have by proposition 3.2.1 $PP^n P^{-n} P^{-1} A \subset A$, hence $P^n P^{-n}(P^{-1}A) \subset P^{-1}A$, $P^{-1}A \in \mathcal{R}_0$.

<u>THEOREM 3.2.1</u>. Suppose $1P > 0$. Let $\mathcal{R}_0(P^n)$ be the deterministic σ-algebra of the Markov process P^n on $(X,\mathcal{R},\mu)$. Then $\mathcal{R}_0(P^n) = \mathcal{R}_0$.

PROOF. It follows by applying proposition 3.2.1 that $\mathcal{R}_0 \subset \mathcal{R}_0(P^n)$.

From the condition $1P > 0$ we conclude $P^k P^{-k} A \supset A$ for all $k \geq 0$ and all $A \in \mathcal{R}$. Indeed, by proposition 1.3.5 we have for every $k \geq 0$ $1P^k > 0$. Then for all $B \subset A$ with $\mu(B) > 0$ we have $P^{-k}B \subset P^{-k}A$, and by proposition 1.3.7 $\mu(P^{-k}B) > 0$. Hence

$$\int (1_{P^{-k}A} P^k) 1_B \, d\mu > 0, \quad \mu(B \cap P^k P^{-k} A) > 0 \quad \text{and} \quad P^k P^{-k} A \supset A.$$

Now assume $A \in \mathcal{R}_0(P^n)$. For every integer $k > 0$ choose α such that $\alpha n > k$. Then we have

$$P^{\alpha n-k} P^{k-\alpha n} P^{-k} A \supset P^{-k} A$$

$$P^k P^{\alpha n-k} P^{k-\alpha n} P^{-k} A \supset P^k P^{-k} A \supset A .$$

Since $A \in \mathcal{R}_0(P^n)$, the left hand side is contained in A, and therefore we have $P^k P^{-k} A = A$ for all k, and consequently $A \in \mathcal{R}_0(P)$.

In this proof we have shown that if the condition $1P > 0$ is satisfied, we have for all $A \in \mathcal{R}$ and all $n \geq 0$ $P^n P^{-n} A \supset A$. Then in particular it follows from proposition 3.2.1 that if $1P > 0$, we also have $\mathcal{R}_0 = \{A \in \mathcal{R} \mid P^n P^{-n} A = A \text{ for all } n \geq 0\}$.

A theorem similar to theorem 3.2.1 does in general not hold for backward or adjoint processes instead of P^n.

EXAMPLE 1. Let $(X,\mathcal{R},\mu)$ be the unit interval with the Borel sets and the Lebesgue measure. Let P be the forward process associated with the transformation $Tx = 2x \pmod 1$. Consider the backward process $\overleftarrow{P}_\mu$. Since μ is invariant, the process $\overleftarrow{P}_\mu$ is at the same time the adjoint process P^*_μ (see p. 30). Since $\overleftarrow{P}_\mu$ acts in opposite direction as P, and since by proposition 1.5.1 we have $PA = TA$, it follows by proposition 3.2.1 that

$$\mathcal{R}_0(\overleftarrow{P}_\mu) = \{A \mid T^{-n} T^n A \subset A \text{ for all } n > 0\} .$$

Suppose $A \in \mathcal{R}_0(\overset{\leftarrow}{P}_\mu)$, $\mu(A) > 0$. Then for every $\alpha < 1$ there exists an open interval $I = (\frac{p}{2^n}, \frac{p+1}{2^n})$ such that $\mu(A \cap I) > \frac{\alpha}{2^n}$ (cf. [5], § 16 theorem A). It follows that $\mu(T^n A) \geq$ $\geq \mu(T^n(A \cap I)) > \alpha$, hence $\mu(A) \geq \mu(T^{-n}T^n A) > \alpha$. Since α is arbitrary, we obtain $\mu(A) = 1$. Hence $\mathcal{R}_0(\overset{\leftarrow}{P}_\mu) = \{\emptyset, X\}$, while $\mathcal{R}_0 = \mathcal{R}$.

PROPOSITION 3.2.3. Let P be a Markov process on $(X, \mathcal{R}, \mu)$, and let X be properly invariant. Then $\mathcal{R}_i \subset \mathcal{R}_0$.

Suppose $A \in \mathcal{R}_i$. Then for all $B \subset A$ we have

$$PB = P_A B \qquad ; \qquad P^{-1}B = P_A^{-1}B$$

$$\mathcal{R}_i(P_A) = \mathcal{R}_i \cap A \; ; \qquad \mathcal{R}_0(P_A) = \mathcal{R}_0 \cap A \; .$$

PROOF. If $A \in \mathcal{R}_i$, then by proposition 3.1.4 we have $P1_A = I_A P1$, hence $P^{-1}A \subset A$, and therefore $P^{-n}A \subset A$ for every n. Similarly we show $P^{-n}(X \setminus A) \subset X \setminus A$. Then from $P^n 1 = P^n 1_A + P^n 1_{X\setminus A}$ we conclude $P^n 1_A = I_A P^n 1$ for every n, hence $A \in \mathcal{R}_0$, $\mathcal{R}_i \subset \mathcal{R}_0$.

Let A be invariant, and suppose $B \subset A$. Since $P^{-1}A \subset A$, $PA \subset A$, we obtain $P_A^{-1}B = P^{-1}B$ and $P_A B = PB$ by proposition 3.1.2. In particular this implies $\mathcal{R}_i(P_A) = \mathcal{R}_i \cap A$. Finally, for a subset B of A we have $P^n P^{-n} B \subset B$ if and only if $P_A^n P_A^{-n} B \subset B$, that is by proposition 3.2.1 $B \in \mathcal{R}_0(P_A)$ if and only if $B \in \mathcal{R}_0 \cap A$.

DEFINITION 3.2.2. A cycle of length n is a sequence of pairwise disjoint non empty sets $(A_1, \ldots, A_n)$ in $\mathcal{R}$ such that $PA_i \subset A_{i+1}$ for $1 \leq i \leq n$, where $A_{n+1} = A_1$.

PROPOSITION 3.2.4. Let $\alpha = (A_1,\ldots,A_n)$ be a partition of X. Then the following statements are equivalent:

i) α is a cycle.

ii) $P^{-1}A_i \subset A_{i-1 \pmod n}$ for $1 \le i \le n$.

iii) $P^k 1_{A_i} = I_{A_{i-k \pmod n}} P^k 1$ for every $k \ge 0$ and for every i, $1 \le i \le n$.

iv) $1_{A_i} P^k = 1P^k I_{A_{i+k \pmod n}}$ for every $k \ge 0$ and for every i, $1 \le i \le n$.

If $1P > 0$, then each of the statements i),...,iv) is equivalent to

v) $PA_i = A_{i+1 \pmod n}$ for $1 \le i \le n$.

If $P1 > 0$, then each of the statements i),...,iv) is equivalent to

vi) $P^{-1}A_i = A_{i-1 \pmod n}$ for $1 \le i \le n$.

PROOF.

i) $\Rightarrow$ ii). For every i we have $\int 1_{A_j}(P1_{A_i})d\mu = 0$ if $j \ne i-1 \pmod n$, hence $P^{-1}A_i \subset A_{i-1 \pmod n}$.

ii) $\Rightarrow$ iii). For every $k \ge 0$ we have for every i, $1 \le i \le n$, $P^{-k}A_i \subset A_{i-k \pmod n}$, hence $P^k 1_{A_i} \le I_{A_{i-k \pmod n}} P^k 1$. Then

$$P^k 1 = \sum_{i=1}^{n} P^k 1_{A_i} \le \sum_{i=1}^{n} I_{A_{i-k \pmod n}} P^k 1 = P^k 1 ,$$

hence for every i $P^k 1_{A_i} = I_{A_{i-k \pmod n}} P^k 1$.

iii) $\Rightarrow$ iv). From iii) we conclude that $\int (1_{A_i} P^k) 1_{A_j} d\mu = 0$ if $j \neq i+k \pmod n$, hence $P^k A_i \subset A_{i+k \pmod n}$ and $1_{A_i} P^k \leq$ $\leq 1 P^k I_{A_{i+k \pmod n}}$. Then

$$1P^k = \sum_{i=1}^{n} 1_{A_i} P^k \leq \sum_{i=1}^{n} 1P^k I_{A_{i+k \pmod n}} = 1P^k$$

and $1_{A_i} P^k = 1P^k I_{A_{i+k \pmod n}}$ for $1 \leq i \leq n$.

iv) $\Rightarrow$ i). For $k = 1$ we have $1_{A_i} P = 1PI_{A_{i+1 \pmod n}}$, hence $PA_i \subset A_{i+1 \pmod n}$ and α is a cycle.

Now suppose $1P > 0$, then $\sum_{i=1}^{n} 1_{A_i} P > 0$ on X. If α is a cycle, we have supp $1_{A_i} P \subset A_{i+1}$ for every i, and we conclude supp $1_{A_i} P = A_{i+1}$ for every i, and therefore v). The implication v) $\Rightarrow$ i) is obvious.

A similar reasoning shows the equivalence of ii) and vi) if the condition $P1 > 0$ holds.

PROPOSITION 3.2.5. If $(A_1, \ldots, A_n)$ is a cycle, then $\bigcup_{i=1}^{n} A_i$ is invariant. If moreover X is properly invariant, then $A_i \in \mathcal{R}_0$ for $1 \leq i \leq n$.

PROOF. Define $A = \bigcup_{i=1}^{n} A_i$, then $PA = \bigcup_{i=1}^{n} PA_i \subset A$, A is invariant. Then by propositions 3.2.3 and 3.2.4 we have for every i, $1 \leq i \leq n$, and all $k \geq 0$

$$P^{-k}A_i = P_A^{-k}A_i \subset A_{i-k(\mathrm{mod}\ n)},$$

$$P^kP^{-k}A_i \subset P^kA_{i-k(\mathrm{mod}\ n)} \subset A_i .$$

By proposition 3.2.1 it follows that $A_i \in \mathcal{R}_0$ for $1 \le i \le n$.

For later reference we now give an example of a transformation T in a probability space $(X,\mathcal{R},\mu)$ such that for every n there exists a partition $(A_1,\ldots,A_{2^n})$ of X such that $T^{-1}A_i = A_{i-1}$ for $1 \le i \le 2^n$, $A_0 = A_{2^n}$. This transformation turns out to be conservative and ergodic as well. Hence the forward process P associated with T is irreducible, while for every n there exists a partition of X forming a cycle of length greater than n.

EXAMPLE 2. Let $(X,\mathcal{R},\mu)$ be the unit interval with the Borel sets and the Lebesgue measure. Apart from the null set of the dyadic points, which we shall ignore in the sequel, every point $x \in X$ has a unique dyadic expansion $x = \sum_{i=1}^{\infty} a_i 2^{-i}$, where $a_i = 0$ or $a_i = 1$ for every i. Let N be the least integer such that $a_N = 1$. Then $Tx = \sum_{i=1}^{\infty} b_i 2^{-i}$, where $b_i = 1$ if $i < N$, $b_N = 0$, and $b_i = a_i$ if $i > N$.

We shall denote the interval of points of which the dyadic expansion starts with $a_1,\ldots,a_n$ by $[a_1 \ldots a_n]$.

PROPERTY 1. For $1 \le p \le n$ put $A = [0 \ldots 0\ 1\ a_{p+1} \ldots a_n]$. Then for $1 \le i \le 2^{p-1}$ the sets T^iA are pairwise disjoint. Moreover we have

$$T^{i}A \quad = [b_1 \ \dots \ b_{p-1} \ 0 \ a_{p+1} \ \dots \ a_n] \ , \ 1 \le i < 2^{p-1}$$

$$T^{2^{p-1}}A = [0 \ \dots \ 0 \quad 0 \ a_{p+1} \ \dots \ a_n] \ .$$

PROOF. If $p = 1$, then the assertion is trivial. Suppose the assertion has been proved for $p = q-1$ and let now $p = q$. Applying the assertion consecutively to TA, T^2A, T^4A, ..., $T^{2^{q-2}}$, we get

$$\begin{aligned}
TA &= [1\ 1\ 1\ 1\ \dots\ 1\ 1\ 0\ a_{q+1}\ \dots\ a_n] \\
T^2A &= [0\ 1\ 1\ 1\ \dots\ 1\ 1\ 0\ a_{q+1}\ \dots\ a_n] \\
T^3A &= [1\ 0\ 1\ 1\ \dots\ 1\ 1\ 0\ a_{q+1}\ \dots\ a_n] \\
T^4A &= [0\ 0\ 1\ 1\ \dots\ 1\ 1\ 0\ a_{q+1}\ \dots\ a_n] \\
&\vdots \\
T^8A &= [0\ 0\ 0\ 1\ \dots\ 1\ 1\ 0\ a_{q+1}\ \dots\ a_n] \\
&\vdots \\
T^{2^{q-2}}A &= [0\ 0\ 0\ 0\ \dots\ 0\ 1\ 0\ a_{q+1}\ \dots\ a_n] \\
&\vdots \\
T^{2^{q-1}}A &= [0\ 0\ 0\ 0\ \dots\ 0\ 0\ 0\ a_{q+1}\ \dots\ a_n] \ .
\end{aligned}$$

Note that for TA, T^2A, ..., $T^{2^{q-2}}$ the place number of the first digit 1 assumes consecutively the values 1,2,3,..., q-1 such that the inductive hypothesis may be applied. At the same time we see that for $1 \le i \le 2^k$ $(k \le q-2)$ the (k+1)st digit in T^iA is 1, while for $2^k < i \le 2^{k+1}$ the (k+1)st digit in T^iA is 0. These last sets, however, are pairwise disjoint by the inductive hypothesis applied to $T^{2^k}A$.

PROPERTY 2. The transformation T in $(X,\mathcal{R},\mu)$ is measurable, invertible and measure preserving. For every dyadic interval of A of length 2^{-n}, the sets T^iA, $0 \le i < 2^n$ for a partition of X which is a cycle.

PROOF. Let A be the interval $[a_1 \ldots a_n]$ where $a_i = 1$ for $1 \le i \le n$. By property 1, we obtain the following sequence of pairwise disjoint images of A:

$$[0\ 1\ 1\ \ldots\ 1] = TA$$

$$\ldots \quad . \quad .$$

$$[0\ 0\ 1\ \ldots\ 1] = T^{1+2}A$$

$$\ldots \quad . \quad .$$

$$[0\ 0\ 0\ \ldots\ 0] = T^{1+2+\ldots+2^{n-1}}A$$

$$[1\ 1\ 1\ \ldots\ 1] = T^{2^n}A = A\ .$$

It follows that the sequence $(T^iA)_{i=0}^{2^n-1}$ is a cycle of length 2^n. Since each of the elements of the cycle is a dyadic interval of length 2^{-n}, the sets $(T^iA)_{i=0}^{2^n-1}$ form a partition of X.

Obviously, T is one-to-one. Since both the image and the pre-image of a dyadic interval is a dyadic interval of the same length, we conclude that T is measurable, invertible and measure preserving.

PROPERTY 3. T is conservative and ergodic.

PROOF. The conservativity of T follows from the fact that T is a measure preserving transformation in a finite measure space.

Let B be a set such that $TB = B$, and $\mu(B) > 0$. For every

$\alpha < 1$ there exists a dyadic interval A such that $\mu(B \cap A) > \alpha\mu(A)$.

Let 2^{-n} be the length of A, then the sets T^iA, $0 \le i < 2^n$ are pairwise disjoint with union X. Since B is invariant and T is measure preserving, we obtain for $0 \le i < 2^n$ $\mu(B \cap T^iA) > \alpha\mu(T^iA)$ and after summation over i $\mu(B) > \alpha$. Hence $\mu(B) = 1$ and T is ergodic.

3.3. MARKOV CHAINS

In this section we shall recall some well known facts from the theory of Markov chains and show how they fit into the theory of Markov processes as presented in the previous sections. We mainly shall follow the book of Chung [1]. However, we adapt the notation to that used in this thesis.

DEFINITION 3.3.1. A Markov chain is a Markov process P on a measure space $(X,\mathcal{R},\mu)$ satisfying $P1 = 1$, where

X is a finite or countably infinite set,

$\mathcal{R}$ is the σ-algebra of all subsets of X,

μ is the counting measure, i.e. the measure that assigns to every set its number of elements.

We shall assume that $X = \{1,\dots,n\}$ if X is finite, and that X is the set of the natural numbers if X is countably infinite.

Since the only null set is $\emptyset$, we have $B(X,\mathcal{R}) = \mathcal{L}_\infty(X,\mathcal{R},\mu)$, and every element of $\mathcal{L}_\infty(X,\mathcal{R},\mu)$ is a bounded sequence of real numbers. Define $p_{ij} = P1_{\{j\}}(i)$, then for every $\underline{a} = (a_1,a_2,\dots) \in \mathcal{L}_\infty(X,\mathcal{R},\mu)$ we have

$$(1) \qquad P\underline{a} = \begin{pmatrix} p_{11} & p_{12} & p_{13} & \cdots \\ p_{21} & p_{22} & p_{23} & \cdots \\ p_{31} & p_{32} & p_{33} & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \end{pmatrix}$$

where the matrix satisfies $0 \leq p_{ij} \leq 1$ and $\sum_{j \in X} p_{ij} = 1$ for all $i \in X$, $j \in X$. Conversely, for every matrix of transition probabilities the relation (1) determines a Markov operator P on $\mathcal{L}_\infty(X,\mathcal{R},\mu)$ satisfying $P1 = 1$.

We shall denote the matrix (p_{ij}) corresponding to P by relation (1) also by P.

Let $p_{ij}^{(n)}$ be the element of the matrix P^n in the i-th row and the j-th column, then $P_{ij}^{(n)} = P^n 1_{\{j\}}(i)$ can be interpreted as the probability that in n transitions the process will move from state i to state j.

Following Chung [1], I.3, we say that a state i leads to a state j, in notation $i \to j$, if there exists an integer n such that $p_{ij}^{(n)} > 0$, which means $j \in P^n\{i\}$ or $i \in P^{-n}\{j\}$. The state i is said to communicate with the state j if $i \to j$ and $j \to i$; a state i is said to be essential if it communicates with every state it leads to.

Let E be the essential part of X with respect to P as defined in definition 3.1.4 and assume $i \in E$. Let A be the least invariant set containing i, then by proposition 3.1.6 A consists of all states j for which $i \to j$. Then by theorem 3.1.1 we have $j \to i$ under P.

Hence, every state i of E is essential in the sense of Chung, and the class to which i belongs (the set of states communicating with i) is the least invariant set containing i.

Now assume $i \notin E$. Again $A = \{j \mid i \to j\}$ is the least invariant set containing i. Since A is not a subset of E, there must be an invariant subset B of A, $\mu(B) > 0$ such that $A \setminus B$ is not invariant. Because of the minimality of A we must have $i \in A \setminus B$. Then for all $j \in B$ we have $i \to j$, but not $j \to i$. Hence i is inessential in the terminology of Chung, and the set of essential states coincides with the essential part of X with respect to P.

Since $C \subset E$ (cf. p. 93), $X \setminus E$ is a subset of the dissipative part of X with respect to P. It follows from proposition 2.1.2 that for every state $j \in X \setminus E$ we have $\sum_{k=0}^{\infty} P^k 1_{\{j\}} \in \mathcal{L}_\infty(X,\mathcal{R},\mu)$, hence $\lim_{n\to\infty} P^n 1_{\{j\}}(i) = 0$ for every $j \in X \setminus E$ and every $i \in X$.

We shall discuss now the behaviour of P on E. Since E is invariant, we can restrict the process to E. Therefore, we shall assume $X = E$. Then the class $\mathcal{R}_i$ of invariant sets is a σ-algebra which is atomic, and every atom is an essential class in the terminology of Chung. In order to describe the deterministic σ-algebra for P, because of proposition 3.2.3 it suffices to describe this σ-algebra on each of the atoms of $\mathcal{R}_i$ for the restriction of P to this atom. Obviously this restriction is an irreducible Markov chain.

PROPOSITION 3.3.1. Let P be an irreducible Markov chain and let $\mathcal{R}_0$ be the deterministic σ-algebra. There exists a partition $(A_1,\ldots,A_d)$ of X forming a cycle such that $\mathcal{R}_0$ is generated by the sets $A_1,\ldots,A_d$.

PROOF. For the proof of proposition 3.3.2 it is important to note that in this proof we only use $P1 > 0$, but not $P1 = 1$.

The σ-algebra $\mathcal{R}_0$ is atomic. Let A be an atom of $\mathcal{R}_0$, then by proposition 3.2.2 $P^{-n}A \in \mathcal{R}_0$ for every $n \geq 0$. By proposition 3.1.6 the set $\bigcup_{n=1}^{\infty} P^{-n}A$ is invariant. By lemma 3.1.1 we have $1P > 0$, and therefore by proposition 1.3.7 $\mu(P^{-1}A) > 0$. Hence $\bigcup_{n=1}^{\infty} P^{-n}A = X$ and there exists a least integer d such that $\mu(A \cap P^{-d}A) > 0$. Since A is an atom of $\mathcal{R}_0$ we conclude $A \subset P^{-d}A$, hence $P^d 1_A = P^d 1$ on A and A is invariant under P^d. By theorem 3.1.2 we have that X is P^d-properly invariant, hence $P^d 1_A = I_A P^d 1$, and $P^{-d}A = A$. Hence the sets $P^{-d}A,\ldots,P^{-1}A$ form a cycle of length d, and since P is irreducible, the union of the sets $P^{-d}A,\ldots,P^{-1}A$ is X.

Finally, suppose that for some i the set $P^{-i}A$ is not an atom of $\mathcal{R}_0$. Then there exist two disjoint subsets $B_1 \in \mathcal{R}_0$ and $B_2 \in \mathcal{R}_0$ of $P^{-i}A$ such that $\mu(B_1) > 0$ and $\mu(B_2) > 0$. Then $P^{-(d-i)}B_1$ and $P^{-(d-i)}B_2$ are disjoint $\mathcal{R}_0$-measurable subsets of $P^{-d}A = A$ of positive measure (see p.101), which contradicts the fact that A is an atom of $\mathcal{R}_0$.

PROPOSITION 3.3.2. Let P be an irreducible Markov chain and let Q be any backward or adjoint process for P. Let $\mathcal{R}_0(P)$ and $\mathcal{R}_0(Q)$ be the deterministic σ-algebras of P and Q respectively. Then $\mathcal{R}_0(P) = \mathcal{R}_0(Q)$.

PROOF. By theorem 3.1.3 also Q is an irreducible Markov process on $(X,\mathcal{R},\mu)$ which satisfies $Q1 > 0$ by lemma 3.1.1.

Let $A_1,\ldots,A_d$ be the atoms of $\mathcal{R}_0(P)$ as in proposition 3.3.1. Then for every $n \geq 0$ and every A_i we have $P^nA_i \subset A_{i+n(\text{mod } d)}$, $P^{-n}P^nA_i \subset A_i$, and therefore $Q^nQ^{-n}A_i \subset A_i$. It follows that $A_i \in \mathcal{R}_0(Q)$, and therefore $\mathcal{R}_0(P) \subset \mathcal{R}_0(Q)$.

Conversely, since in the proof of proposition 3.3.1 we only used the fact that $P1 > 0$, it follows that there exists a partition $(A_1', \ldots, A_{d'}')$ of X forming a cycle for Q such that each of the sets A_i' is an atom of $\mathcal{R}_0(Q)$. Then a similar reasoning shows that $\mathcal{R}_0(Q) \subset \mathcal{R}_0(P)$.

If P is an irreducible Markov chain and $A_1, \ldots, A_d$ are the atoms of $\mathcal{R}_0$, then d is said to be the period of P. It follows from proposition 3.2.5 that d is the maximal length of a cycle.

This definition agrees with the usual definition of a period for an irreducible Markov chain. To show this, let d be the period of P as defined in I.3 of [1], and consider theorem I.3.4 in [1]. Because of relation (1) in that theorem the elements of the partition $(C_1, \ldots, C_d)$ of X satisfy $P1_{C_{r+1}} = 1$ on C_r for $1 \le r \le d$, $C_{d+1} = C_1$, hence $P1_{C_{r+1}} \ge 1_{C_r}$. Then we have

$$1 = P1 = \sum_{r=1}^{d} P1_{C_{r+1}} \ge \sum_{r=1}^{d} 1_{C_r} = 1 ,$$

hence for every r $P1_{C_{r+1}} = 1_{C_r}$, and by proposition 3.2.4 the sets $C_1, \ldots, C_d$ form a cycle of length d. Let $(D_1, \ldots, D_e)$ be another cycle in X. Then by proposition 3.2.4 we have for every $n \ge 0$ and every r, $1 \le r \le e$, $P^n 1_{D_{r+n \pmod e}} = 1_{D_r}$, hence

$$\sum_{j \in D_{r+n \pmod e}} p_{ij}^{(n)} = 1 \quad \text{if } i \in D_r .$$

Again by theorem I.3.4 we have $e \le d$, and therefore d is the maximal length of a cycle.

In the literature we sometimes meet the expressions non cyclic or aperiodic for irreducible Markov chains of period 1. However, since in the sequel we want to distinguish between period 1 and aperiodicity, we shall not use this terminology.

Again following Chung [1], I.4, a state i is said to be recurrent if

$$(\sum_{k=0}^{\infty} (PI_{X\setminus\{i\}})^k \, P1_{\{i\}})(i) = 1 \; .$$

If the state i is not recurrent, then it is said to be transient. It follows from proposition 2.1.1 that the class of recurrent states coincides with the conservative part of X with respect to P.

For every recurrent state i the mean recurrence time m_i is defined by

$$m_i = (\sum_{n=1}^{\infty} n(PI_{X\setminus\{i\}})^{n-1} \, P1_{\{i\}})(i) \; .$$

The recurrent state i is said to be a positive state if $m_i < \infty$, and a null state if $m_i = \infty$. If P is an irreducible conservative Markov chain, then by [1], theorem I.6.2, either all states are positive or all of them are null.

We now turn to the limit behaviour of irreducible Markov chains. If the chain is dissipative, then we can show as we did for inessential states, that $\lim_{n\to\infty} P^n 1_{\{j\}}(i) = 0$ for every state $i \in X$ and $j \in X$. Now assume the chain to be recurrent. Fix $j \in X$. Let d be the period of P and $(A_1,\ldots,A_d)$ the corresponding cycle of atoms of $\mathcal{R}_0$. We may assume $j \in A_d$. Then we have by [1], theorem I.6.4b

$$\lim_{n\to\infty} P^{nd+r} 1_{\{j\}}(i) = \begin{cases} \frac{d}{m_j} & \text{if } i \in A_{d-r} \\ 0 & \text{if } i \notin A_{d-r} \end{cases}$$

where $0 \le r < d$. In particular, if the chain is a null chain, we have $\lim_{n\to\infty} P^n 1_{\{j\}}(i) = 0$ for all $i \in X$, $j \in X$.

Let P be an irreducible positive chain with period d. Let $(A_1,\ldots,A_d)$ be the cycle of atoms of $\mathcal{R}_0$. Define the measure ν on $(X,\mathcal{R})$ by $\nu\{i\} = m_i^{-1}$ for every $i \in X$, then by [1], theorem I.7.1, we have $\nu(A_k) = \frac{1}{d}$ for $1 \le k \le d$, and ν is the unique invariant probability for P.

Note that the limit theorem in this case may be written as

$$\lim_{n\to\infty} P^{nd+r} 1_{\{j\}} = E^{\nu}_{\mathcal{R}_0} P^r 1_{\{j\}} \quad \text{for every } j \in X .$$

Indeed, $E^{\nu}_{\mathcal{R}_0} P^r 1_{\{j\}}$ must be constant on each of the sets $A_1,\ldots,A_d$. By proposition 3.2.4 we have $P^r 1_{\{j\}} = 0$ on A_k for all $k \ne d-r$, hence

$$E^{\nu}_{\mathcal{R}_0} P^r 1_{\{j\}} = 0 \quad \text{on } A_k \quad \text{for } k \ne d-r .$$

Since ν is invariant, we have $1P_\nu = 1$, and therefore by proposition 3.2.4

$$\int_{A_{d-r}} P^r 1_{\{j\}}\, d\nu = \int (1_{A_{d-r}} P^r_\nu) 1_{\{j\}}\, d\nu = \int 1_{A_d} 1_{\{j\}}\, d\nu = \frac{1}{m_j} ,$$

hence

$$E^{\nu}_{\mathcal{R}_0} P^r 1_{\{j\}} = \frac{1}{m_j} \frac{1}{\nu(A_{d-r})} = \frac{d}{m_j} \quad \text{on } A_{d-r} .$$

To conclude this section we remark that if P is a irreducible null chain, then by [15], theorem 6.9, there also exists a unique invariant equivalent measure ν. In this case, the measure ν is σ-finite and satisfies $\nu(X) = \infty$.

3.4. SOME CONCEPTS OF PERIODICITY

In this section we shall give a definition of periodicity for a Markov process P on a σ-finite measure space $(X,\mathcal{R},\mu)$, which reduces to the concept of periodicity for a Markov chain if P is a Markov chain. Moreover, if T is a measurable transformation on $(X,\mathcal{R},\mu)$, then the forward process associated with T will be periodic on an invariant set A if and only if T is periodic with the same period on A.

DEFINITION 3.4.1. Let T be a non singular measurable transformation in a σ-finite measure space $(X,\mathcal{R},\mu)$. The transformation T is said to be periodic with period d on a set A if there exists a partition $(A_1,\ldots,A_d)$ of A such that for $1 \le i \le d$ we have $T^{-1}A_i \supset A_{i-1}$ $(A_0 = A_d)$ and for all $B \subset A$ we have $T^{-d}B \supset B$.

The transformation is said to be aperiodic if there are no invariant sets of positive measure on which T is periodic.

This definition conforms with the concepts of strict periodicity and aperiodicity as in definition 1.1 of [10], even if it formally differs from it. For reasons of uniform terminology we have dropped the adjective strict.

Since $T^{-1}A = \bigcup_{i=1}^{d} T^{-i}A_i \supset A$, A is invariant under T, and moreover, since for all $B \subset A$ we have $T^{-d}B \supset B$, for every wandering set $W \subset A$ we must have $\mu(W) = 0$, hence T is conservative on A.

If T_A is the restriction of T to A, and if P_A denotes the restriction of the forward process P associated with T, then obviously P_A is the forward process associated with T_A on $(A, \mathcal{R} \cap A, \mu)$.

If the forward process associated with T is an irreducible Markov chain with period d, then, since $\mathcal{R}_0(P) = \mathcal{R}$, the space X must consist of d points $1,\dots,d$, and the transformation T must be defined by $Tx = x+1 \pmod d$. It follows that then also T is periodic with period d.

PROPOSITION 3.4.1. Let T be a non singular measurable transformation in a σ-finite measure space $(X,\mathcal{R},\mu)$, and let P be the corresponding forward process. The transformation T is periodic with period d on X if and only if the process P satisfies the following conditions:

i) $P^{-d}A = A$ for all $A \in \mathcal{R}$;

ii) there exists a partition $(A_1,\dots,A_d)$ of X forming a cycle;

iii) $P^{-n}P^nA = A$ for all $A \in \mathcal{R}$ and all $n \geq 0$.

PROOF. Actually, already conditions i) and ii) are equivalent with the conditions in definition 3.4.1. Hence we only have to show iii) for a measurable non singular transformation with period d on X. It follows from proposition 1.5.1 that we have $P^nA = P^{-(d-n)(\text{mod } d)}A$, hence $P^{-n}P^nA = A$ for all $n \geq 0$.

Note that because of proposition 3.2.1 condition iii) is equivalent to $\mathcal{R}_0(Q) = \mathcal{R}$ for every backward or adjoint process for P.

PROPOSITION 3.4.2. Let T be a non singular measurable transformation in a σ-finite measure space $(X,\mathcal{R},\mu)$ with period d. Then there exists an invariant probability ν equivalent to μ.

PROOF. Let μ' be a probability on $(X,\mathcal{R})$ equivalent to μ, and define for every $A \in \mathcal{R}$ $\nu(A) = \frac{1}{d} \sum_{i=1}^{d} \mu'(T^{-i}A)$, then ν is an equivalent invariant probability.

We now turn to the concept of periodicity for irreducible Markov processes as introduced by Moy [17]. Since Moy uses the term aperiodic for what we prefer to call periodic with period 1 (cf. p. 115), we shall adapt her terminology.

DEFINITION 3.4.2 (Moy [17]). Let P be an irreducible Markov process on a σ-finite measure space $(X,\mathcal{R},\mu)$. If there exists a maximal number d such that we can find a partition $(A_1,\ldots,A_d)$ of X forming a cycle, then P is said to be periodic with period d.

It follows from proposition 3.2.5 that if P is a Markov chain, definition 3.4.2 reduces to the definition following proposition 3.3.2.

However, Moy's definition does not apply for all irreducible Markov processes. Consider for instance the forward process P associated with the transformation T considered in example 2 of section 3.2. Since T is ergodic and conservative, by proposition 3.1.8 P is irreducible. For every n there exists a partition of X forming a cycle of length 2^n; hence definition 3.4.2 cannot be applied.

Moreover, if definition 3.4.2 can be applied, it is for a forward process induced by a measurable transformation not always in accordance with the concept of periodicity for transformations. For instance, let for $i = -1$, $i = 1$, I_i be the interval $\{(x,i) \mid 0 \le x \le 1\}$, $\mathcal{R}_i$ the Borel sets in I_i and λ_i the Lebesgue measure on $(I_i,\mathcal{R}_i)$. Define $X = I_{-1} \cup I_1$, $\mathcal{R} = \{A \mid A \cap I_i \in \mathcal{R}_i \text{ for } i = -1,\ i = 1\}$, and $\mu(A) = \lambda_{-1}(A \cap I_{-1}) + \lambda_1(A \cap I_1)$ for all $A \in \mathcal{R}$, then $(X,\mathcal{R},\mu)$ is a finite measure space.

The transformation T on $(X,\mathcal{R},\mu)$ defined by

$$T(x,y) = (2x (\mathrm{mod}\ 1), -y) \qquad \text{for all } (x,y) \in X$$

is ergodic and conservative, hence the forward process P is irreducible. Since (I_{-1},I_1) is a cycle of length 2 and no cycles of length > 2 exist, P is periodic with period 2 according to definition 3.4.2, while T is aperiodic.

Now we want to give a definition of periodicity and aperiodicity which can be applied to all Markov processes, corresponds with the concept of periodicity for Markov chains and is such that a transformation T and the corresponding forward process P are periodic with the same period on the same sets. Because of the last example above, such a definition will not agree in all cases with definition 3.4.2.

There are, however, two more aspects which bear upon the choice of the definition of periodicity for a Markov process.

First we note that if P is the forward process associated with a measurable non singular transformation of period d or an irreducible Markov chain with period d, the mapping $A \to PA$ for all $A \in \mathcal{R}_0$ and the mapping $A \to P^{-1}A$ for all $A \in \mathcal{R}_0$ are isomorphisms from $\mathcal{R}_0$ onto itself such that P^d and P^{-d} are the identity map.

Since for any backward process Q of P we have $PA = Q^{-1}A$

and $P^{-1}A = QA$ for all $A \in \mathcal{R}$, it seems reasonable that a Markov process P and any corresponding backward process have the same period.

Moreover, it seems desirable to obtain a limit theorem for periodic Markov processes which for Markov chains reduces to the limit theorem mentioned in section 3.3.

We shall show that the next definition meets these requirements.

DEFINITION 3.4.3. A Markov process P on a σ-finite measure space $(X,\mathcal{R},\mu)$ is said to be periodic with period d on X if

i) For every $A \in \mathcal{R}_i$ with $\mu(A) > 0$, d is the least integer ≥ 1 such that $\mathcal{R}_0(P_A^d) = \mathcal{R}_i(P_A^d)$.

ii) $\{A \in \mathcal{R} \mid P^{-n}P^nA \subset A \text{ for all } n \geq 0\} = \mathcal{R}_0(P)$.

A Markov process P on a σ-finite measure space $(X,\mathcal{R},\mu)$ is said to be periodic with period d on a set $A \in \mathcal{R}_i$ if the restriction P_A of P to $(A,\mathcal{R} \cap A,)$ is periodic with period d on A.

We postpone the proof that this definition agrees with the definition of periodicity for irreducible Markov chains and for transformations, and first examine some consequences of this definition.

PROPOSITION 3.4.3. Let P be a Markov process on $(X,\mathcal{R},\mu)$ which is periodic with period d on X. Then X is properly invariant, $P^{-1}X = PX$, and $P^{-1}X = PX = X$ if $d \geq 2$.

PROOF. Since $\mathcal{R}_0(P^d) = \mathcal{R}_i(P^d)$ the invariant sets under P^d form a σ-algebra, hence X is P^d-properly invariant.

Define $A_0 = X \setminus P^{-1}X$, then for all $B \subset A_0$ we have $\int 1_B(P1)d\mu = 0$, hence $PB = \emptyset$. It follows that $B \in \mathcal{R}_i$ and $\mathcal{R}_i \cap A_0 = \mathcal{R} \cap A_0$. Since $P1 = 0$ on A_0, we have $P^n 1 = 0$ on A_0 for every n, and therefore for every $B \subset A_0$ by proposition 3.1.2 $P^n_{A_0} 1_B = P^n 1_B = 0$ on A_0, $P^n_{A_0} 1_B = I_B P^n_{A_0} 1 = 0$, hence $B \in \mathcal{R}_0(P_{A_0})$. We obtain $\mathcal{R}_0(P_{A_0}) = \mathcal{R} \cap A_0 = \mathcal{R}_i(P_{A_0})$.

If $d \geq 2$, by definition 3.4.3 we therefore must have $\mu(A_0) = 0$, and therefore $P^{-1}X = X$, $P1 > 0$. By theorem 3.1.2 X is properly invariant.

Since A_0 is invariant, so is $A_1 = P^{-1}X$. From $1P = 1_{A_0}P + 1_{A_1}P$ we conclude, since $PA_0 = \emptyset$, that $1P = 1_{A_1}P$, and therefore $1P = 0$ on A_0. Since PX is invariant, so is $A_1 \setminus PX$, hence $P(A_1 \setminus PX) \subset (A_1 \setminus PX)$. This implies $\mu(P(A_1 \setminus PX)) = 0$, and therefore $\int 1_{A_1 \setminus PX}(P1)d\mu = 0$.

Since $P1 > 0$ on A_1 we conclude $\mu(A_1 \setminus PX) = 0$, hence $1P > 0$ on A_1, $A_1 = PX = P^{-1}X$.

COROLLARY. If P is periodic with period d on X, then P is periodic with period d on every set $A \in \mathcal{R}_i$ for which $\mu(A) > 0$.

PROOF. For $d = 1$, the statement follows from proposition 3.2.3. For $d > 1$, the statement follows from proposition 3.2.3 and theorem 3.2.1.

PROPOSITION 3.4.4. Let P be a Markov process on $(X,\mathcal{R},\mu)$ which is periodic with period d on X and which satisfies $P1 > 0$ (this condition is automatically satisfied if $d \geq 2$ by proposition 3.4.3). Then the mappings $B \to PB$ and $B \to P^{-1}B$ for all $B \in \mathcal{R}_0$, which in the sequel shall be denoted by P and P^{-1}, are isomorphisms of $\mathcal{R}_0$ onto itself such that P^d and P^{-d} are the identity maps on $\mathcal{R}_0$, and P and P^{-1} are each others inverse on $\mathcal{R}_0$.

PROOF. By definition, for all $B \in \mathcal{R}_0$ we have $B \in \mathcal{R}_i(P^d)$. Since X is P^d-properly invariant we conclude by proposition 3.1.4 $1_B P^d 1 = P^d 1_B$, hence $P^{-d}B = B$ for all $B \in \mathcal{R}_0$. Then by proposition 3.2.2 we have $\mathcal{R}_0 \supset P^{-1}\mathcal{R}_0 \supset \ldots \supset P^{-d}\mathcal{R}_0 = \mathcal{R}_0$, hence $P^{-1}\mathcal{R}_0 = \mathcal{R}_0$. On page 101 we already noted that P^{-1} is an isomorphism of $\mathcal{R}_0$ onto $P^{-1}\mathcal{R}_0$, hence in this case P^{-1} is an isomorphism of $\mathcal{R}_0$ onto itself, such that P^{-d} is the identity map on $\mathcal{R}_0$.

Since by proposition 3.4.3 we have $1P > 0$, it follows that $\mathcal{R}_0 = \{A \mid P^n P^{-n} A = A \text{ for all } n \geq 0\}$ (cf. p. 102). In particular, we have $PP^{-1}B = B$ for all $B \in \mathcal{R}_0$, and therefore PP^{-1} is the identity map on $\mathcal{R}_0$ and P is an isomorphism of $\mathcal{R}_0$ onto itself. Then $P^{-1}PP^{-1}B = P^{-1}B$ for all $B \in \mathcal{R}_0$, and also $P^{-1}P$ is the identity map of $\mathcal{R}_0$ onto itself.

PROPOSITION 3.4.5. Let P be a Markov process on $(X,\mathcal{R},\mu)$ which is periodic with period d on X. Then there exists a partition $(A_1,\ldots,A_d)$ of X forming a cycle.

PROOF. For d = 1 the statement is obvious. Hence we may assume $d \geq 2$. Since X is properly invariant, the elements of a cycle must belong to $\mathcal{R}_0$ by proposition 3.2.5.

Let m be the maximal length of a cycle, then by proposition 3.4.4 we must have $m \leq d$. Suppose $(A'_1,\ldots,A'_m)$ is a cycle of length m, and suppose $m < d$. Put $A = \bigcup_{i=1}^{m} A'_i$, then $A \in \mathcal{R}_i$.

From $d \geq 2$ we conclude $1P > 0$, and therefore by proposition 3.2.3 and 3.2.4 $PA_i' = A'_{i+1 (\text{mod } n)}$. Then by proposition 3.4.4 we have for every i $P(\mathcal{R}_0 \cap A_i') = \mathcal{R}_0 \cap A'_{i+1(\text{mod } n)}$.

If for all $B \in \mathcal{R}_0 \cap A_1'$ we had $P^m B = B$, then it would follow that even for all $B \in \mathcal{R}_0 \cap A$ we had $P^m B = B$. Then by proposition 3.2.3 and definition 3.4.3 we would have $\mu(A) = 0$, contradiction. Hence there exists a set $B \in \mathcal{R}_0 \cap A_1'$ such that $P^m B \neq B$.

Now note that for $B' \in \mathcal{R}_0$ and $n \geq 0$ $P^n B' \supset B'$ implies $P^n B' = B'$. In fact, we have $B' \subset P^n B' \subset P^{2n} B' \subset \ldots \subset P^{dn} B' = B'$.

It follows that if we put $B_0 = B \setminus P^m B$, we have $\mu(B_0) > 0$, and the sets $P^i B_0$, $0 \leq i < 2m$, are pairwise disjoint. If $P^{2m} B_0 = B_0$, then we have a cycle of length 2m; if $P^{2m} B_0 \neq B_0$, then put $B_1 = B_0 \setminus P^{2m} B_0$, then $\mu(B_1) > 0$ and the sets $P^i B_1$, $0 \leq i < 3m$, are pairwise disjoint, etc.

Since for every $B \in \mathcal{R}_0$ we have $P^d B = B$, by this construction we obtain a cycle of length at least 2m and at most d, which contradicts the fact that the maximal cycle length was m. Hence, if P is periodic with period d, there exists a cycle of length d. Using the corollary of proposition 3.4.3 it follows that every invariant set of positive measure contains a cycle of length d. The partition $(A_1, \ldots, A_d)$ of X forming a cycle is now obtained by an exhaustion procedure.

PROPOSITION 3.4.6. Let P be a Markov process on $(X, \mathcal{R}, \mu)$ which is periodic with period d on X, and satisfies $1P > 0$. If Q is any backward or adjoint process for P, then also Q is periodic on X with period d.

PROOF. By theorem 3.1.3 we have $\mathcal{R}_i(P) = \mathcal{R}_i(Q)$. Since for all

$A \in \mathcal{R}$ we have $P^{-1}A = QA$ and $Q^{-1}A = PA$, condition ii) in definition 3.4.3 yields by proposition 3.2.1 $\mathcal{R}_0(P) = \mathcal{R}_0(Q)$. First we show that for every $A \in \mathcal{R}_i(Q)$ with $\mu(A) > 0$ we have $\mathcal{R}_0(Q_A^d) = \mathcal{R}_i(Q_A^d)$.

From $1P > 0$ we conclude by proposition 3.4.3 that also $P1 > 0$, therefore $Q1 > 0$, $1Q > 0$ and $Q_A 1 > 0$, $1Q_A > 0$ on A. By proposition 3.2.3 we have $\mathcal{R}_i(Q_A) = \mathcal{R}_i(Q) \cap A$, $\mathcal{R}_0(Q_A) = \mathcal{R}_0(Q) \cap A$ and $Q_A = Q = P^{-1}$ on $\mathcal{R}_0(Q) \cap A$.

Since $\mathcal{R}_i(Q_A)$ is a σ-algebra, A is properly invariant under Q_A, and therefore by theorem 3.1.2 properly invariant under Q_A^d. By proposition 3.2.3 we obtain $\mathcal{R}_i(Q_A^d) \subset \mathcal{R}_0(Q_A^d)$.

Conversely, by theorem 3.2.1 the latter σ-algebra equals $\mathcal{R}_0(Q_A)$. For all $B \in \mathcal{R}_0(Q_A)$ we have $Q_A^d B = P^{-d}B = B$ by proposition 3.4.4, hence $B \in \mathcal{R}_i(Q_A^d)$. This shows $\mathcal{R}_0(Q_A^d) = \mathcal{R}_i(Q_A^d)$.

Let $(A_1,\dots,A_d)$ be the partition of X as in proposition 3.4.5. If $A \in \mathcal{R}_i(Q)$, $\mu(A) > 0$, then $(A_1 \cap A,\dots,A_d \cap A)$ is a cycle for Q_A of length d, and therefore d is the least number such that $\mathcal{R}_0(Q_A^d) = \mathcal{R}_i(Q_A^d)$.

Finally,

$$\{A \in \mathcal{R} \mid Q^{-n}Q^{n}A \subset A \text{ for all } n \geq 0\} =$$

$$= \{A \in \mathcal{R} \mid P^{n}P^{-n}A \subset A \text{ for all } n \geq 0\} =$$

$$= \mathcal{R}_0(P) = \mathcal{R}_0(Q) \ .$$

Hence Q is periodic on X with period d.

Now we verify that definition 3.4.3 is in accordance with the definitions of periodicity for Markov chains and for transformations.

First, let P be an irreducible Markov chain with period d. Then $\mathcal{R}_i = \{\emptyset, X\}$, and the only invariant set of positive measure is X. By proposition 3.3.1 d is the least integer such that $\mathcal{R}_0(P^d) = \mathcal{R}_i(P^d)$, hence condition i) of definition 3.4.3 is fulfilled.

By lemma 3.1.1 we have $1P > 0$, and therefore there exists a backward process Q. Then

$$\{A \in \mathcal{R} \mid P^{-n}P^n A \subset A \text{ for all } n \geq 0\} = \mathcal{R}_0(Q) = \mathcal{R}_0(P)$$

by proposition 3.3.2, and also condition ii) of definition 3.4.3 is satisfied.

Conversely, let P be an irreducible Markov chain which is periodic with period d according to definition 3.4.3. Then by proposition 3.4.4 and 3.4.5 d is the maximal length of a cycle.

Now let T be a non singular measurable transformation on $(X, \mathcal{R}, \mu)$ which is periodic with period d on X, and let P be the forward process associated with T. Then it follows from proposition 3.4.1 that for every invariant set A with $\mu(A) > 0$ d is the least number such that $\mathcal{R}_0(P_A^d) = \mathcal{R}_i(P_A^d) = \mathcal{R} \cap A$. Moreover, by proposition 3.4.1 we have

$$\{A \mid P^{-n}P^n A \subset A \text{ for all } n \geq 0\} = \mathcal{R} = \mathcal{R}_0(P) \ ,$$

hence P is periodic with period d.

Conversely, if the forward process P is periodic with period d, then conditions i), ii) and iii) of proposition 3.4.1 are satisfied because of proposition 3.4.5, 3.4.4 and definition 3.4.3, respectively, and T is periodic on X with period d.

DEFINITION 3.4.4. A Markov process P on a σ-finite measure space $(X,\mathcal{R},\mu)$ is said to be aperiodic on an invariant set A if there are no invariant subsets of positive measure of A on which P is periodic.

<u>THEOREM 3.4.1</u>. Let P be a Markov process on a σ-finite measure space $(X,\mathcal{R},\mu)$, and suppose X is properly invariant and $P1 > 0$. There exists a finite or countably infinite partition $(X_0,X_1,\ldots)$ of X into invariant sets and an increasing sequence (n_i) of positive numbers such that P is aperiodic on X_0 and periodic with period n_i on X_i for $i \geq 1$.

PROOF. Let $(A_n)_{n=1}^{\infty}$ be a sequence of invariant sets of positive measure such that for every n P is periodic on A_n with period d. Put $A = \bigcup_{n=1}^{\infty} A_n$, then A is invariant, $\mu(A) > 0$. Let B be an invariant subset of A of positive measure. First we show that $\mathcal{R}_0(P_B^d) = \mathcal{R}_i(P_B^d)$.

Since $P1 > 0$ we also have $P_B 1 > 0$. For every n $B \cap A_n$ is invariant under P and therefore under P_{A_n}. Since by proposition 3.4.3 $1P_{A_n} > 0$, we obtain $1_{B\cap A_n} P_{A_n} > 0$ on $B \cap A_n$. Then by proposition 3.2.3 $1_{B\cap A_n} P_B = 1_{B\cap A_n} P_{A_n} > 0$ on $B \cap A_n$, hence $1P_B > 0$.

Again by proposition 3.2.3 we have $\mathcal{R}_i(P_B) = \mathcal{R}_i \cap B$, $\mathcal{R}_0(P_B) = \mathcal{R}_0 \cap B$. Since $\mathcal{R}_i(P_B)$ is a σ-algebra, B is P_B-properly invariant and therefore by theorem 3.1.2 P_B^d-properly invariant. Then by proposition 3.2.3 we have $\mathcal{R}_i(P_B^d) \subset \mathcal{R}_0(P_B^d)$.

Conversely, by theorem 3.2.1 we have $\mathcal{R}_0(P_B^d) = \mathcal{R}_0(P_B)$. For all $B_0 \in \mathcal{R}_0(P_B)$ and all n we have $B_0 \cap A_n \in \mathcal{R}_0 \cap B \cap A_n \subset$ $\subset \mathcal{R}_0 \cap A_n$, $B_0 \cap A_n \in \mathcal{R}_0(P_{A_n})$. It follows that

$$P_B^d(B_0) = \bigcup_{n=1}^{\infty} P_B^d(B_0 \cap A_n) = \bigcup_{n=1}^{\infty} P_{A_n}^d(B_0 \cap A_n) = \bigcup_{n=1}^{\infty} B_0 \cap A_n = B_0$$

hence $B_0 \in \mathcal{R}_i(P_B^d)$. This shows $\mathcal{R}_0(P_B^d) = \mathcal{R}_i(P_B^d)$.

If $\mathcal{R}_0(P_B^k) = \mathcal{R}_i(P_B^k)$, then, if n is chosen such that $\mu(B \cap A_n) > 0$, we have $\mathcal{R}_0(P_{B\cap A_n}^k) = \mathcal{R}_i(P_{B\cap A_n}^k)$, and since P is periodic with period d on A_n, we have $k \geq d$. Hence d is the least positive number such that $\mathcal{R}_0(P_B^d) = \mathcal{R}_i(P_B^d)$ for all invariant sets $B \subset A$ with $\mu(B) > 0$, and condition i) of definition 3.4.3 is satisfied.

From proposition 3.1.2 and the fact that X is properly invariant we deduce that for every $A \in \mathcal{R}_i$ and every $B \in \mathcal{R}$ we have $P_A^n(A \cap B) = (P^n B) \cap A$ and $P_A^{-n}(A \cap B) = (P^{-n}B) \cap A$.

Now suppose for some set $B \subset A$ we have $P_A^{-k}P_A^k B \subset B$ for all $k \geq 0$. Then for every $n \geq 0$

$$P_{A_n}^{-k}P_{A_n}^k(B \cap A_n) = P_{A_n}^{-k}((P_A^k B) \cap A_n) = P_A^{-k}P_A^k B \cap A_n \subset B \cap A_n .$$

Since P_{A_n} is periodic, we obtain $B \cap A_n \in \mathcal{R}_0 \cap A_n$ for every $n \geq 0$, and therefore $B \in \mathcal{R}_0 \cap A$.

Conversely, let $B \in \mathcal{R}_0 \cap A$ be given. Then for every $k \geq 0$ we have

$$P_A^{-k}P_A^k B = P_A^{-k}P_A^k\Big(\bigcup_{n=1}^{\infty} (B \cap A_n)\Big) =$$

$$= \bigcup_{n=1}^{\infty} P_A^{-k}P_A^k(B \cap A_n) =$$

$$= \bigcup_{n=1}^{\infty} P_{A_n}^{-k}P_{A_n}^k(B \cap A_n) \subset \bigcup_{n=1}^{\infty} (B \cap A_n) = B .$$

Therefore we have $\{B \subset A \mid P_A^{-k} P_A^k B \subset B \text{ for all } k \geq 0\} =$
$= \mathcal{R}_0 \cap A = \mathcal{R}_0(P_A)$, and P is periodic with period d on A.

Let (n_i) be the sequence of positive numbers such that for every i there exists an invariant set A of positive measure on which P is periodic with period n_i, and define X_i to be the mod μ largest invariant set on which P is periodic with period n_i. Then $X_0 = X \setminus \bigcup_i X_i$ is invariant, and P is aperiodic on X_0.

REMARK. Suppose that the Markov process P is periodic with period d on $A \in \mathcal{R}_i$. Let B be an invariant subset of A, then $A \setminus B$ is invariant under P_A, that is, $1_{A\setminus B} P_A = 0$ on B. Then it follows from proposition 3.1.2 and the invariance of A that $1_{A\setminus B} P = 0$ on $B \cup (X \setminus A)$, hence $A \setminus B$ is invariant under P, and A is properly invariant. It follows that every invariant subset on which P is periodic must be contained in the essential part. Therefore, if we agree to call P aperiodic on the (not necessarily invariant) set $E \setminus X$, theorem 3.4.1 also holds without the assumption that X is properly invariant, but then of course, the set X_0 on which P is aperiodic need not be invariant.

3.5. CONVERGENCE OF PERIODIC MARKOV PROCESSES ADMITTING A SUBINVARIANT EQUIVALENT σ-FINITE MEASURE

In this section we shall derive a limit theorem for the behaviour of P^n if $n \to \infty$, where P is a periodic Markov process on a σ-finite measure space $(X,\mathcal{R},\mu)$ such that $1P \leq 1$.

To this end, we start with considering the so-called positive part of X with respect to a given Markov process P on $(X,\mathcal{R},\mu)$. Various characterizations have been given, among others by Neveu [19], Krengel [16]. For our purposes the following definition suffices.

DEFINITION 3.5.1. Let P be a Markov process on a σ-finite measure space $(X,\mathcal{R},\mu)$. The positive part F of X with respect to P is the maximal support of a finite invariant measure $\nu << \mu$.

It is easy to see that such a maximal support exists. Indeed, let $(A_i)_{i=1}^{\infty}$ be a sequence of sets such that for every i there exists an invariant probability $\nu_i << \mu$ with support A_i. If $f_i = \frac{d\nu_i}{d\mu}$, then supp $f_i = A_i$ and $f_iP = f_i$. Put $f = \sum_{i=1}^{\infty} 2^{-i} f_i \in \mathcal{L}_1^+(X,\mathcal{R},\mu)$, then $fP = f$, hence there exists an invariant probability absolutely continuous with respect to μ on $\bigcup_{i=1}^{\infty} A_i$. The positive part F is then the modulo μ largest set in the class of supports of finite invariant measures $\nu << \mu$.

PROPOSITION 3.5.1. For all $n \geq 0$ we have $F(P) = F(P^n)$.

PROOF. If $\nu << \mu$ is a finite invariant measure with support $F(P)$, then ν is also P^n-invariant, hence $F(P) \subset F(P^n)$.

Conversely, let $\lambda << \mu$ be a finite P^n-invariant measure with support $F(P^n)$. If $g = \frac{d\lambda}{d\mu}$, then $gP^n = g$ and $g \in \mathcal{L}_1^+(X,\mathcal{R},\mu)$. Define $f = g + \ldots + P^{n-1}g$, then $f \in \mathcal{L}_1^+(X,\mathcal{R},\mu)$, $fP = P$ and supp $f \supset F(P^n)$. It follows that $F(P^n) \subset F(P)$, and therefore $F(P) = F(P^n)$.

Obviously, we have $F \subset C$ and $PF = F$. Let $f \in M^+(X,\mathcal{R},\mu)$ satisfy $fP \leq f < \infty$. Then $fP = fI_F P + fI_{X\setminus F}P$, and because of the invariance of F $fI_F P \leq fI_F$. By proposition 2.1.3 we conclude $fI_F P = fI_F$, and therefore $fI_{X\setminus F}P = 0$ on F.

Let ν be a subinvariant σ-finite measure equivalent to μ. Then there also exists a σ-finite subinvariant measure $\nu' \approx \mu$ such that $\nu'(F) < \infty$. Indeed, put $f = \frac{d\nu}{d\mu}$, then $fP \leq f < \infty$. Let $g \in \mathcal{L}_1^+(X,\mathcal{R},\mu)$ satisfy supp $g = F$ and $gP = g$. Define $h = g + fI_{X\setminus F}$, then $hP = gP + fI_{X\setminus F}P \leq g + fI_{X\setminus F} = h$. It follows that the measure ν' defined by $\frac{d\nu'}{d\mu} = h$ is σ-finite and subinvariant, and $\nu'(F') < \infty$.

Let P be a Markov process on $(X,\mathcal{R},\mu)$ admitting an equivalent σ-finite subinvariant measure. Without loss of generality we then may assume that μ is that subinvariant measure, hence $1P \leq 1$, and moreover we may assume $\mu(F) < \infty$.

Let P^* be the adjoint process of P with respect to the subinvariant measure μ. Then by proposition 1.4.5 we have $P^*f = fP$, $fP^* = Pf$ for all $f \in M^+(X,\mathcal{R},\mu)$, and therefore for all $f \in M^+(X,\mathcal{R},\mu)$ and all $g \in M^+(X,\mathcal{R},\mu)$

$$\int f(Pg)d\mu = \int g(P^*f)d\mu \ .$$

Consider the restriction of the Markov operator P in $\mathcal{L}_\infty(X,\mathcal{R},\mu)$ to the set of all integrable step functions $h = \sum_{i=1}^{n} \alpha_i 1_{A_i}$. Then for every step function h we have

$$\|Ph\|_1 = \sum_{i=1}^{n} |\alpha_i| \int (1P)1_{A_i} d\mu \leq \sum_{i=1}^{n} |\alpha_i| \mu(A_i) = \|h\|_1 \ .$$

Since also $\|Ph\|_\infty \leq \|h\|_\infty$, it follows by the Riesz convexity theorem ([2], theorem VI.10.11), that the restriction of the Markov operator P in $\mathcal{L}_\infty(X,\mathcal{R},\mu)$ to the set of all integrable step functions h satisfies $\|Ph\|_2 \leq \|h\|_2$ and therefore can be extended to a bounded linear operator P' in $\mathcal{L}_2(X,\mathcal{R},\mu)$.

Since for every $f \in \mathcal{L}_2^+(X,\mathcal{R},\mu)$ there exists a sequence of non negative integrable step functions $(h_n)_{n=1}^\infty$ such that $h_n \uparrow h$ both pointwise and in $\mathcal{L}_2$-norm, we obtain $P'f = Pf$ for all $f \in \mathcal{L}_2^+(X,\mathcal{R},\mu)$, and therefore $P'f = Pf^+ - Pf^-$ for all $f \in \mathcal{L}_2(X,\mathcal{R},\mu)$. In the sequel we shall drop the prime and speak of a Markov operator P in $\mathcal{L}_2(X,\mathcal{R},\mu)$.

By proposition 1.4.6 the measure μ is also subinvariant under the adjoint process P^*. Hence the formula

$$P^*f = P^*f^+ - P^*f^- \qquad \text{for all } f \in \mathcal{L}_2(X,\mathcal{R},\mu)$$

defines a Markov operator in $\mathcal{L}_2(X,\mathcal{R},\mu)$. Then for all $f \in \mathcal{L}_2(X,\mathcal{R},\mu)$ and for all $g \in \mathcal{L}_2(X,\mathcal{R},\mu)$ we have

$$\int f(Pg)d\mu = \int f^+(Pg^+)d\mu - \int f^-(Pg^+)d\mu - \int f^+(Pg^-)d\mu + \int f^-(Pg^-)d\mu =$$

$$= \int g^+(P^*f^+)d\mu - \int g^+(P^*f^-)d\mu - \int g^-(P^*f^+)d\mu + \int g^-(P^*f^-)d\mu =$$

$$= \int g(P^*f)d\mu \ .$$

Therefore, the adjoint operator of a Markov operator P in $\mathcal{L}_2(X,\mathcal{R},\mu)$ is indeed the Markov operator P^* in $\mathcal{L}_2(X,\mathcal{R},\mu)$ for the adjoint process.

The limit theorem for periodic Markov processes, which we are going to derive now, follows from the theory of positive linear contractions in $\mathcal{L}_2$-spaces. The results we need are due to Foguel [4]. We shall give a brief outline here;

for details and proofs the reader is referred to Foguel's book [4], chapter VIII.

Unless explicitly stated otherwise, in the sequel $\|f\|$ will mean the $\mathcal{L}_2$-norm of f, and (f,g) will be the inner product of the functions $f \in \mathcal{L}_2(X,\mathcal{R},\mu)$ and $g \in \mathcal{L}_2(X,\mathcal{R},\mu)$.

Consider the set $H \subset \mathcal{L}_2(X,\mathcal{R},\mu)$ of functions f for which for every n $\|P^n f\| = \|P^{*n} f\| = \|f\|$. The set H is then a closed linear subspace of $\mathcal{L}_2(X,\mathcal{R},\mu)$, and could have been defined also as the set of functions $f \in \mathcal{L}_2(X,\mathcal{R},\mu)$ for which for every $n \geq 0$ $P^{*n}P^n f = P^n P^{*n} f = f$.

The limit theorem is based on the following property:

(α) If $g \perp H$, then for every $f \in \mathcal{L}_2(X,\mathcal{R},\mu)$ we have

$$\lim_{n\to\infty} (f,P^n g) = \lim_{n\to\infty} (f,P^{*n} g) = 0 .$$

Let $\mathcal{R}'$ be the class of sets $A \in \mathcal{R}$ for which $1_A \in H$. Then both $P1_A$ and $P^* 1_A$ are characteristic functions of sets in $\mathcal{R}'$, and the set of functions $\{1_A \mid A \in \mathcal{R}'\}$ spans H.

We now identify the space H and the class $\mathcal{R}'$.

PROPOSITION 3.5.2. Let P be a periodic Markov process on $(X,\mathcal{R},\mu)$ with period d such that $0 < 1P \leq 1$ and $\mu(F) < \infty$. Then

$$\mathcal{R}' = \mathcal{R}_0 \cap F ,$$

$$H = \{f \in \mathcal{L}_2(X,\mathcal{R},\mu) \mid P^d f = P^{*d} f = f\} .$$

PROOF. If $A \in \mathcal{R}'$, then for all $n \geq 0$ we have $P^{*n}P^n 1_A = 1_A$, hence $P^n P^{-n} A = A$ and $A \in \mathcal{R}_0$. Therefore by proposition 3.4.4 we have $1_A P^d = 1_A$, and the restriction of μ to A is a finite measure and invariant for P^d. It follows $A \subset F(P^d)$, and by proposition 3.5.1 $A \subset F$. Hence $\mathcal{R}' \subset \mathcal{R}_0 \cap F$.

It follows that for every $A \in \mathcal{R}'$ we have $P^d 1_A = 1_A$ and $P^{*d} 1_A = 1_A$. Since $\{1_A \mid A \in \mathcal{R}'\}$ spans H, we have for every $f \in H$ $P^d f = P^{*d} f = f$. Conversely, if $f \in \mathcal{L}_2(X,\mathcal{R},\mu)$ satisfies $P^d f = P^{*d} f = f$, then for every n we have $\|f\| \geq \|P^n f\| \geq \geq \|P^{nd} f\| = \|f\|$, $\|f\| \geq \|P^{*n} f\| \geq \|P^{*nd} f\| = \|f\|$, hence $f \in H$. This shows

$$H = \{f \in \mathcal{L}_2(X,\mathcal{R},\mu) \mid P^d f = P^{*d} f = f\} .$$

For every $A \in \mathcal{R}_0 \cap F$ we have $\mu(A) < \infty$, $P^{-d} A = A$ and $P^d A = A$, hence $P^d 1_A \leq 1_A$ and $1_A P^d \leq 1_A$ since $1P \leq 1$. Then by propositions 2.1.5 and 2.1.3 it follows that $P^d 1_A = 1_A$, $1_A P^d = = P^{*d} 1_A = 1_A$, and therefore $1_A \in H$, $A \in \mathcal{R}'$. This completes the proof.

In the last part of the previous proposition we actually have shown that for every $A \in \mathcal{R}_0 \cap C$ with $\mu(A) < \infty$, we have $A \subset F$. Hence for all $A \in \mathcal{R}_0 \cap (C \setminus F)$ we have $\mu(A) = 0$ or $\mu(A) = \infty$, and in general the measure space $(X,\mathcal{R}_0,\mu)$ will not be σ-finite. Therefore the conditional expectation operator $E^{\mu}_{\mathcal{R}_0}$ not necessarily exists.

For every $g \in \mathcal{L}_2(X,\mathcal{R},\mu)$ the function $E^F_{\mathcal{R}_0} g$ is defined uniquely by the following conditions:

i) $E^F_{\mathcal{R}_0} g$ is $\mathcal{R}_0$-measurable.

ii) $E^F_{\mathcal{R}_0} g = 0$ on $X \setminus F$.

iii) $\int_A E^F_{\mathcal{R}_0} g \, d\mu = \int_A g d\mu$ for all $A \in \mathcal{R}_0 \cap F$.

It is easily verified that $E^F_{\mathcal{R}_0}$ is a positive linear contraction in $\mathcal{L}_2(X,\mathcal{R},\mu)$.

THEOREM 3.5.1. Let P be a periodic Markov process on a σ-finite measure space $(X,\mathcal{R},\mu)$ with period d, and suppose $0 < 1P \le 1$ and $\mu(F) < \infty$. Then for every $g \in \mathcal{L}_2(X,\mathcal{R},\mu)$ and every $r \ge 0$ the sequences $(P^{nd+r}g)_{n=1}^{\infty}$ and $(P^{*nd+r}g)_{n=1}^{\infty}$ converge in $\mathcal{L}_2(X,\mathcal{R},\mu)$ weakly to $E^F_{\mathcal{R}_0}P^r g = P^r E^F_{\mathcal{R}_0} g$, respectively $E^F_{\mathcal{R}_0}P^{*r}g = P^{*r}E^F_{\mathcal{R}_0}g$.

PROOF. Define $h = P^t g - E^F_{\mathcal{R}_0}P^t g$, then $h \in \mathcal{L}_2(X,\mathcal{R},\mu)$. For all $A \in \mathcal{R}_0 \cap F$ we have

$$(1_A,h) = \int 1_A(P^t g)d\mu - \int 1_A(E^F_{\mathcal{R}_0}P^t g)d\mu = \int 1_A(P^t g)d\mu - \int 1_A(P^t g)d\mu = 0 ,$$

hence by proposition 3.5.2 $h \perp H$.

Moreover, since $E^F_{\mathcal{R}_0}P^t g$ is $\mathcal{R}_0$-measurable and has support in F, the function $E^F_{\mathcal{R}_0}P^t g$ is invariant under P^d. Hence we have for every $n \ge 0$

$$P^{nd}h = P^{nd+t}g - E^F_{\mathcal{R}_0}P^t g .$$

For every $f \in \mathcal{L}_2(X,\mathcal{R},\mu)$ we have $fP^s = P^{*s}f \in \mathcal{L}_2(X,\mathcal{R},\mu)$. Therefore by property (α) for all $f \in \mathcal{L}_2(X,\mathcal{R},\mu)$

$$\lim_{n\to\infty} \int (fP^s)(P^{nd}h)d\mu =$$

$$= \lim_{n\to\infty} \int (fP^s)(P^{nd+t}g)d\mu - \int (fP^s)(E^F_{\mathcal{R}_0}P^t g)d\mu = 0 .$$

If we take $s = 0$, $t = r$, then we obtain that the sequence $(P^{nd+r}g)_{n=1}^{\infty}$ converges in $\mathcal{L}_2(X,\mathcal{R},\mu)$ weakly to $E_{\mathcal{R}_0}^F P^r g$, and if we take $s = r$, $t = 0$, then the same sequence converges weakly to $P^r E_{\mathcal{R}_0}^F g$. It follows that $P^r E_{\mathcal{R}_0}^F g = E_{\mathcal{R}_0}^F P^r g$.

The second statement is proved in the same way by replacing P by P^*.

Theorem 3.5.1 indeed implies the convergence theorem for Markov chains.

Let P be an irreducible Markov chain with period d, and let ν be the unique invariant measure for P.

If all states are null states, then $\nu(X) = \infty$, $\nu(F) = 0$, and for all states $i \in X$, $j \in X$ we have

$$\lim_{n\to\infty} \int 1_{\{i\}} P^{nd+r} 1_{\{j\}} \, d\nu = 0 \quad \text{i.e.} \quad \lim_{n\to\infty} p_{ij}^{(nd+r)} = 0 .$$

If all states are positive, then $\nu(X) < \infty$, $F = X$, and for all states $i \in X$, $j \in X$ we have

$$\lim_{n\to\infty} \int 1_{\{i\}} P^{nd+r} 1_{\{j\}} \, d\nu = \int 1_{\{i\}} E_{\mathcal{R}_0}^{\nu} P^r 1_{\{j\}} \, d\nu ,$$

$$\lim_{n\to\infty} p_{ij}^{(nd+r)} \nu\{i\} = (E_{\mathcal{R}_0}^{\nu} P^r 1_{\{j\}})(i)\nu\{i\} ,$$

$$\lim_{n\to\infty} p_{ij}^{(nd+r)} = E_{\mathcal{R}_0}^{\nu} 1_{\{j\}}(i) .$$

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SAMENVATTING

In dit proefschrift komen twee onderwerpen uit de theorie van de Markov processen aan de orde.

Het eerste onderwerp heeft betrekking op terugkeereigenschappen van de Markov shift. In 1953 hebben Harris en Robbins [7] aangetoond dat voor een conservatief Markov proces de shift in de tweezijdige produktruimte conservatief is als de Markov maat op de produktruimte invariant is onder de shift. In dit proefschrift wordt het verband onderzocht tussen het conservatieve gedeelte van het Markov proces en het conservatieve gedeelte van de Markov shift, zowel in de eenzijdige als in de tweezijdige produktruimte. De resultaten zijn geformuleerd in de stellingen 2.3.1 en 2.3.2.

Tevens wordt het verband onderzocht tussen de Markov maten voor eenzelfde proces op de tweezijdige produktruimte; in het bijzonder wordt een noodzakelijke en voldoende voorwaarde gegeven voor het singulier zijn van twee Markov waarschijnlijkheden (stelling 2.2.2). De hierbij gevolgde methode is afkomstig van Kakutani [14].

Met iedere niet singuliere meetbare transformatie correspondeert een "forward Markov process", en in het algemeen

een klasse van "backward Markov processes". Voor deze backward processes wordt in paragraaf 1.5 een representatiestelling afgeleid. Tevens wordt in paragraaf 2.1 aangetoond dat als de transformatie ergodisch en maatbewarend is op een waarschijnlijkheidsruimte, alle backward processes dissipatief zijn, behalve het backward process dat correspondeert met de invariante waarschijnlijkheid.

Met behulp van deze backward processes kunnen we Markov maten op de tweezijdige produktruimte construeren voor het forward process corresponderend met de transformatie. Deze constructie, beschreven in paragraaf 2.4, levert een voorbeeld van een dissipatieve inverteerbare transformatie in een waarschijnlijkheidsruimte waarvoor er een dichtliggende algebra van terugkeerverzamelingen bestaat.

Het tweede onderwerp in dit proefschrift betreft de definitie van periodiciteit voor Markov processen. Het concept periodiciteit is algemeen bekend voor Markov ketens. Bovendien bestaat er een definitie gegeven door Moy [17] voor irreducibele Markov processen, die samenvalt met de definitie voor irreducibele Markov ketens, en een periodiciteitsbegrip voor transformaties. Het blijkt dat Moy's definitie niet in overeenstemming is met het periodiciteitsbegrip voor transformaties. In paragraaf 3.4 wordt een definitie van periodiciteit gegeven die toepasbaar is op alle Markov processen, voor Markov ketens samenvalt met de daar gebruikelijke definitie en voor de Markov processen corresponderend met een meetbare transformatie periodiciteit oplevert dan en slechts dan als de transformatie periodiek is met dezelfde periode.

De diverse definities van periodiciteit blijken alle af te hangen van de klasse der invariante verzamelingen en de deterministische σ-algebra. Daarom worden in de eerste twee paragrafen van hoofdstuk III enige eigenschappen van deze verzamelingen besproken. In het bijzonder wordt in paragraaf 3.1 een karakterisering gegeven van het essentiële gedeelte met behulp van invariante verzamelingen. Tenslotte wordt in de laatste paragraaf een limietstelling voor periodieke Markov processen afgeleid. Deze afleiding berust op een methode, aangegeven door Foguel [4]. Toegepast op Markov ketens levert deze stelling de bekende limietstelling op.

CURRICULUM VITAE

De schrijver van dit proefschrift werd in 1939 in Amsterdam geboren. In 1957 behaalde hij het einddiploma H.B.S.-b. Daarna studeerde hij wis- en natuurkunde aan de Universiteit van Amsterdam, waar hij in 1965 cum laude het doctoraal examen wiskunde behaalde met specialisatie lineaire analyse. Van 1961 tot 1964 was hij werkzaam als assistent van Prof.dr. A. Heyting en van 1964-1965 als assistent van Prof.dr. C.G. Lekkerkerker. Sinds 1965 is hij als wetenschappelijk medewerker verbonden aan de Technische Hogeschool te Eindhoven.

STELLINGEN

I

Zij P een overgangswaarschijnlijkheid op een meetbare ruimte $(X,\mathcal{B})$ en $(\Omega,\mathfrak{A},M,S)$ de eenzijdige shift ruimte voor P met betrekking tot een σ-eindige beginverdeling μ_0.

De σ-algebra der asymptotische gebeurtenissen wordt gedefinieerd door

$$\mathfrak{A}_\infty = \bigcap_{n=0}^{\infty} S^{-n}\mathfrak{A}.$$

Voor iedere $A \in \mathfrak{A}_\infty$ bestaat er een rij $(A_n)_{n=0}^{\infty}$ in $\mathcal{B}$ zo dat

$$\lim_{n\to\infty} M(A \,\Delta\, \Pi_n^{-1} A_n) = 0$$

waarin Π_n de projectie in Ω op de n-de coördinaat voorstelt.

II

Stel T is een meetbare transformatie op een meetbare ruimte $(X,\mathcal{B})$. Laat de overgangswaarschijnlijkheid P op $(X,\mathcal{B})$ gedefinieerd zijn door

$$P(x,A) = 1_{T^{-1}A}(x) \quad \text{voor alle } x \in X \text{ en voor alle } A \in \mathcal{B}.$$

Zij $(\Omega,\mathfrak{A},M,S)$ de eenzijdige shift ruimte voor P met betrekking tot een σ-eindige beginmaat μ_0, en $\mathfrak{A}_\infty$ de σ-algebra der asymptotische gebeurtenissen. Dan geldt

$$\mathfrak{A}_\infty = \prod_{n=0}^{\infty} \mathcal{R}_\infty [M] ,$$

waarin

$$\mathcal{R}_\infty = \bigcap_{n=0}^{\infty} T^{-n} \mathcal{R} .$$

III

Stel T is een meetbare niet singuliere transformatie in een σ-eindige maatruimte $(X,\mathcal{R},\mu)$. De definitie "T heet essentieel inverteerbaar als $T^{-1}\mathcal{R} = \mathcal{R}[\mu]$" lijkt natuurlijker dan de definitie van het concept van essentiële inverteerbaarheid zoals gegeven door Helmberg en Simons.

Literatuur: Helmberg, G. en F.H. Simons: A dualization of Kac's recurrence theorem.
Proc. K.N.A.W. Series A, 69, 608-615 (1966).

IV

Stel C is het conservatieve gedeelte van X met betrekking tot een niet singuliere meetbare transformatie T in een σ-eindige maatruimte $(X,\mathcal{R},\mu)$. Als $T^{-1}\mathcal{R} = \mathcal{R}[\mu]$, dan is C het conservatieve gedeelte van X met betrekking tot het terugproces behorend bij T.

Literatuur: Helmberg, G. en F.H. Simons: On the conservative parts of the Markov processes induced by a measurable transformation, corollary 5.1.
Z. Wahrscheinlichkeitstheorie verw. Geb. 11, 165-180 (1969).

V

Zij P een conservatief Markov proces op een σ-eindige maatruimte $(X,\mathcal{R},\mu)$. Voor iedere $A \in \mathcal{R}$ wordt de terugkeertijd gedefinieerd door

$$r_A = \sum_{n=1}^{\infty} n(PI_{X\setminus A})^{n-1} PI_A .$$

Als μ invariant is onder P, dan geldt

$$\int_A r_A \, d\mu = \mu(A^*)$$

waarin A^* de kleinste invariante verzameling is die A omvat.

VI

Gegeven zijn n laden met in iedere la n ballen van dezelfde kleur. Twee ballen uit verschillende laden hebben verschillende kleuren. Een verwisseling is het ruilen van een zeker aantal ballen uit een la met hetzelfde aantal ballen uit een andere la. Dan is het kleinste aantal verwisselingen om in iedere la van iedere kleur een bal te krijgen groter dan of gelijk aan $\frac{n}{2} \cdot {}^2\log n$. Dit minimum wordt aangenomen als n een macht van 2 is.

VII

Er bestaat voor $n \geq 2$ geen perfecte 3-error-correcting Lee code voor woordlengte n over een alfabet van 5 letters.

Literatuur: Golomb, S.W. en L.R. Welch: Algebraic coding and the Lee metric, in: Error correcting codes. Proceedings of a Symposium at the University of Wisconsin, Madison.
John Wiley & Sons, Inc. - New York.

VIII

Het besluit om het afvalwater uit de veenkoloniën ongezuiverd in de Eemsmonding te lozen, waarbij systematisch de waarschuwingen van de zijde der biologen werden genegeerd, toont aan hoe weinig serieus de overheid het probleem van de watervervuiling opvat.

IX

Gevreesd moet worden dat de opstellers van de Wet verontreiniging oppervlaktewateren het zelfreinigend vermogen van zeewater sterk overschatten.

X

Het in Nederland gevolgde systeem van kinderbijslag wekt ten onrechte de indruk dat Nederland een onderbevolkt gebied is.

Eindhoven, 25 juni 1971 F.H. Simons