Cost-effective control policies for multi-echelon distribution systems

tu̸

Eindhoven University
of Technology

Department of Mathematics and Computing Science

Memorandum COSOR 96-35

Cost-effective control policies
for multi-echelon distribution systems

E.B. Diks
A.G. de Kok

Cost-effective control policies for multi-echelon distribution systems

E.B. Diks
Department of Mathematics and Computing Science
Eindhoven University of Technology
P.O. Box 513, 5600 MB Eindhoven, The Netherlands
e-mail: diks@win.tue.nl

A.G. de Kok
Department of Technology Management
P.O. Box 513, 5600 MB Eindhoven, The Netherlands
e-mail: a.g.d.kok@tm.tue.nl

Abstract

In this paper we consider a divergent multi-echelon inventory system, e.g., a distribution system or a production system. At every stockpoint orders are placed periodically. The order arrives after a fixed lead time. At the end of each period linear costs may be incurred at each stockpoint for holding inventory. Also, linear penalty costs are incurred at the most downstream facilities for backorders. The objective is to minimise the expected holding and penalty costs per period. Diks & De Kok [1996a] developed a decomposition algorithm in order to determine the control parameters of a near cost-optimal replenishment policy. Since this algorithm cannot be applied on divergent multi-echelon systems in which at some stockpoints no value is added to the product (e.g. major parts of distribution systems), we developed an extension of the algorithm in order to deal with these systems as well. A simulation study of a divergent 3-echelon system reveals that this extended algorithm performs well.

Keywords: Inventory, Allocation, Logistics

1 Introduction

The research of multi-echelon models has gained importance over the last decade because integrated control of supply chains, consisting of a number of processing and distribution stages, has become feasible through modern information technology. Multi-echelon inventory systems provide a means of modelling such supply chains, thereby enabling quantitative analysis and characterisation of optimal control policies (cf. Clark & Scarf [1960], Federgruen & Zipkin [1984], Rosling [1989] and Langenhoff & Zijm [1990]).

The start of research on multi-echelon inventory models is mostly allotted to Clark & Scarf [1960], who study an N-echelon serial system without lot sizing. They introduced the concept of echelon stock for a given stockpoint to prove that the optimal control policies for the N-echelon serial system with discounted penalty and holding costs, are characterised by N so-called echelon order-up-to-levels. The echelon stock of a stockpoint equals all stock at this stockpoint plus in transit to or on hand at any of its downstream stockpoints minus the backorders at its downstream stockpoints. Like Van Houtum & Zijm [1991a] and Zijm & Van Houtum [1994] we define the echelon inventory position of a stockpoint as its echelon stock plus all material in transfer to that stockpoint.

1

In this paper we analyse a divergent N-echelon inventory system in which every stockpoint is allowed to hold stock. Every stockpoint places replenishment orders periodically. The order arrives after a fixed lead time. Then it is decided how much stock to retain at this stockpoint, and in what way the remaining stock is allocated among its successors. Only the unfilled demands at the end-stockpoints are backordered. Penalty costs proportional to the amount short at every end-stockpoint are incurred at the end of each period. Also holding costs proportional to the inventory on hand are incurred at the end of each period. The objective is to minimise the average costs per period on the long run.

The analysis presented here can be regarded as an extension of Langenhoff & Zijm [1990] and Van Houtum & Zijm [1991b]. Langenhoff & Zijm [1990] prove exact decomposition results for a two-echelon assembly system, a two-echelon serial system and a divergent two-echelon system (which is more thoroughly analysed in Van Houtum & Zijm [1991b]). Diks & De Kok [1996b] prove exact decomposition results for the divergent N-echelon system given the *balance assumption*. Under this assumption the rationing rule always allocates non-negative stock quantities. In Eppen & Schrage [1981], Langenhoff & Zijm [1990] and De Kok, Lagodimos & Seidel [1994] similar assumptions are made. This balance assumption is *not* required if immediately after taking a rationing decision there is a sufficiently large 'demandless' period (e.g. week-end) in which products are transshipped from the stockpoints with negative allocation quantities to those with positive allocation quantities. Diks & De Kok [1996a] developed a decomposition algorithm to determine a near cost-optimal policy within a class of practically useful policies. Unfortunately, the algorithm can only be applied if every stockpoint adds positive value to the product. In this paper we extend the algorithm in order to deal with systems in which no value is added in some stockpoints. Typically, such situations occur when components or subassemblies are shipped from the out-bound stockpoint of a supplier to the in-bound stockpoint of a customer (see Figure 1). Note that such a situation as in Figure 1 occurs in many real-world supply chains.
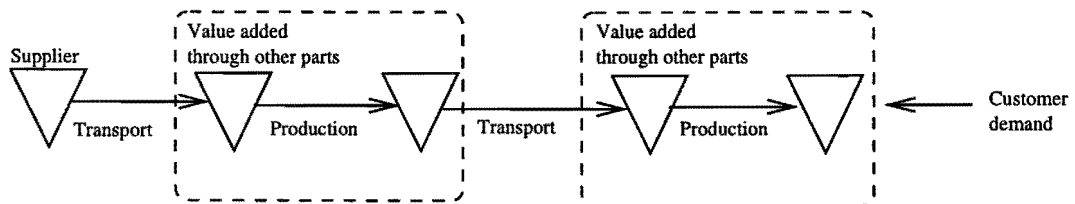
Figure 1: Supply chain with alternating value adding.

Verrijdt & De Kok [1995] study a similar divergent N-echelon system, where intermediate stock is not allowed. The control parameters of the replenishment policy are determined so as to meet the predetermined target service levels (fill rates) at the end-stockpoints (also see De Kok [1990] and Lagodimos [1992]).

The paper is organised as follows. In Section 2 we describe the model under consideration. In Section 3 we illustrate why the decomposition algorithm of Diks & De Kok [1996a] cannot be applied straightforwardly on some divergent systems. By considering an example, the extension of the algorithm is explained. In Section 4 an adaptation of the Balanced Stock (BS) rationing policy of Van der Heijden [1996] is developed. This policy is used to derive an extension of the algorithm of Diks & De Kok [1996a] to deal with stockpoints where no value is added. This new algorithm is presented in Section 5. In Section 6 we present some numerical results obtained by applying the algorithm to a 3-echelon system. These results are validated by a simulation study. Finally, in Section 7 we give a few concluding remarks.

2

2 Model description

Consider a discrete-time multi-echelon inventory system where every stockpoint is allowed to hold stock. The system has an *arborescent* structure, i.e., each location has a unique supplier. We refer to these kind of systems as divergent multi-echelon systems. Notice that a divergent multi-echelon system can be described by a directed graph (see for example Figure 2).

Figure 2: Schematic representation of a divergent 3-echelon inventory system.

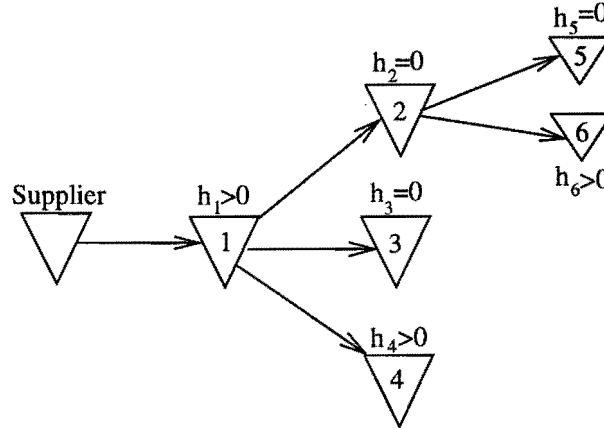For our convenience we introduce a low level code for every stockpoint. By definition the low level code of an end-stockpoint equals one. For an intermediate stockpoint it equals one plus the maximum low level code of its successors. Furthermore, we introduce the following notation:

$ech(i)$:= Set of stockpoints that constitute the echelon of stockpoint i (e.g. $ech(2) = \{2, 5, 6\}$),

U_i := Set of stockpoints on path from supplier to stockpoint i (e.g. $U_1 = \varnothing$ and $U_5 = \{1, 2\}$),

V_i := All stockpoints which are supplied by i (e.g. $V_1 = \{2, 3, 4\}$),

W_i := All stockpoints with low level code i (e.g. $W_1 = \{3, 4, 5, 6\}$),

N := Number of stages in inventory system (e.g. $N = 3$).

The examples between the brackets refer to the situation of Figure 2.

The most upstream stockpoint can place orders at an external supplier which has an infinite capacity, i.e., this supplier can always meet demand. The inventory in this multi-echelon system is controlled by periodic review policies. Every R periods the most upstream stockpoint, i say, issues a replenishment order. The replenishment order arrives after L_i periods, where L_i is a fixed, non-negative integer. Then the physical stock at stockpoint i (or part of it) is allocated immediately to its successors. There are two possibilities:

(*i*). The physical stock is sufficient to raise the echelon inventory position of each successor to its order-up-to-level. Then the required amounts are sent to the successors and excess stock is kept at stockpoint i to be allocated in the next occasion.

(*ii*). The physical stock is *not* sufficient to reach the successors' order-up-to-levels. Then material rationing is required to allocate the available physical stock over its successors appropriately. For this purpose we introduce rationing functions.

A similar allocation procedure is applied at the other intermediate stockpoints when a replenishment order arrives.

Without loss of generality we assume that only the end-stockpoints face external customer demand. In case an intermediate stockpoint i faces external demand, we redirect this demand to a new successor j with lead time $L_j := 0$. By definition this successor j is an end-stockpoint. During one period the

3

demand between end-stockpoints may be correlated. However, the demands at each end-stockpoint in subsequent periods are i.i.d.. With respect to the customer demand process, we assume that all demand which cannot be satisfied immediately is backordered.

At the end of each review period of a stockpoint costs are incurred. For each backlogged product at end-stockpoint i a penalty costs p_i is incurred. For a product at stockpoint i or in transfer to one of its successors the holding costs equals $h_i + \sum_{k \in U_i} h_k$. Notice that h_i can be regarded as an additional holding cost due to value added in stockpoint i. No fixed ordering costs are assumed. Note that because all excess customer demand is backordered, linear variable ordering costs do not influence any control policy and can therefore be omitted. The objective of the analysis is to determine a long-run average cost-optimal replenishment policy.

3 Extension of the decomposition algorithm

As indicated in the introduction in many situations (intermediate) products are shipped from one stockpoint to its successor(s) without adding any value (i.e., adding other components). For these situations the decomposition algorithm of Diks & De Kok [1996a] cannot be applied, since they assume that in every stockpoint value is added. By considering an example, we illustrate how the algorithm needs to be adapted in order to deal with stockpoints with no added value.

Before discussing the example we briefly explain the decomposition algorithm of Diks & De Kok [1996a]. The algorithm decomposes the problem into several more simple problems. The control parameters of the stockpoints with low level code $1, 2, \ldots, N$ are determined, successively. The control parameters of a stockpoint i are

    (i). the order-up-to-level y_i,

    (ii). and, if i represents an intermediate stockpoint, the allocation-fraction q_j to every successor j of the linear rationing function $z_j[x]$,

$$z_j[x] = y_j - q_j \left(\sum_{n \in V_i} y_n - x \right).$$ (1)

These control parameters are determined such that

    (i). the attained non-stock out probability equals $\hat{\alpha}_k^i$, where

$$\hat{\alpha}_k^i := \frac{\sum_{j \in U_i} h_j + p_k}{h_k + \sum_{j \in U_k} h_j + p_k} \quad \text{for } every \text{ end-stockpoint } k \in ech(i).$$ (2)

    (ii). and, if i represents an intermediate stockpoint, it is required that $\sum_{j \in V_i} q_j = 1$ with $q_j > 0$.

For more details of the algorithm we refer to Diks & De Kok [1996a].

Let us apply this algorithm to the distribution system depicted in Figure 2. Then it becomes clear where an extension of the algorithm is required. First, the algorithm defines $\hat{\alpha}_3^3$, $\hat{\alpha}_4^4$, $\hat{\alpha}_5^5$ and $\hat{\alpha}_6^6$ from (2). For every end-stockpoint the order-up-to-level is determined so as to meet this target non-stock out probability. Since $h_3 = h_5 = 0$ from (2) it follows that $\hat{\alpha}_3^3 = \hat{\alpha}_5^5 = 1$. Hence, the order-up-to-levels of end-stockpoints 3 and 5 are infinity, i.e., $y_3 = y_5 = \infty$. On the other hand h_4 and h_6 are positive, therefore $\hat{\alpha}_4^4$ and $\hat{\alpha}_6^6$ are less than 1. Hence, the order-up-to-levels of stockpoint 4 and 6 equals some finite value y_4 and y_6, respectively.

Second, the algorithm considers stockpoint 2 (with low level code 2). From (2) we determine $\hat{\alpha}_5^2$ and $\hat{\alpha}_6^2$. Since stockpoint 2 does not add any value to the product we have that $\hat{\alpha}_5^2 = \hat{\alpha}_5^5$ and $\hat{\alpha}_6^2 = \hat{\alpha}_6^6$.

Hence, $y_2 = \infty$. Since the algorithm uses the linear rationing functions of (1), it is not clear how to determine appropriate allocation-fractions q_5 and q_6.

Finally, we consider the most upstream stockpoint 1 (with low level code 3). Again, the target non-stock out probabilities $\hat{\alpha}_k^1$ for $k \in \{3, \ldots, 6\}$ are defined by (2). Since stockpoint 1 has at least one successor with an infinite order-up-to-level, it follows that upon arrival of an order at stockpoint 1, these products are immediately ordered by stockpoint 2 and 3. This means that stockpoint 1 never holds any stock, so $y_1 := y_2 + y_3 + y_4$. Using similar arguments we have $y_2 = y_5 + y_6$. So $\Delta_1 = \Delta_2 = 0$, where $\Delta_i := y_i - \sum_{j \in V_i} y_j$. Using sample path arguments the remaining control parameters need to be determined such that

$$
\hat{\alpha}_k^1 = \begin{cases} Pr(y_k - q_k D_{L_1}^1 - D_{L_k+R}^k \geq 0) & k \in \{3, 4\} \\[2mm] Pr(y_k - q_k(D_{L_2}^2 + q_2 D_{L_1}^1) - D_{L_k+R}^k \geq 0) & k \in \{5, 6\}, \end{cases} \tag{3}
$$

where random variable D_L^i denotes the demand of all end-stockpoints in $ech(i)$ during L periods. From equation (3) with $k = 4$ the allocation-fraction q_4 can be determined. The cost-optimal conditions does not impose any constraints on q_2 and q_3. We have one degree of freedom in choosing these allocation-fractions q_2 and q_3, since the allocation-fractions of stockpoint 1 have to sum up to 1. We suggest to use this degree of freedom to choose these fractions so as to minimise the expected imbalance at stockpoint 1 as much as possible, since the decomposition approach requires that there is no imbalance at every stockpoint in the system. The so-called balanced stock rationing policy introduced by Van der Heijden [1996] tries to establish this. In Section 4 we distinguish between two variants of this balanced stock rationing policy, referred to as BS1 and BS2. By applying one of these two variants appropriate q_2 and q_3 are determined. Substitution of q_2 and y_6 in (3) with $k = 6$ yields q_6. Next, $q_5 = 1 - q_6$. Similarly, substitution of q_2 and q_5 in (3) with $k = 5$ yields y_5. Finally, substituting q_3 in (3) with $k = 3$ yields y_3.

4 Balanced Stock rationing

In this section we address two rationing policies, indicated by BS1 and BS2. They are similar to the ones developed in Van der Heijden [1996], i.e., the allocation-fractions are determined so as to minimise the imbalance as much as possible.

In BS1 the allocation-fractions are determined such that a surrogate measure for the expected amount of imbalance is minimised as much as possible. By using a normal approximation, Van der Heijden [1996] showed that the expected amount of imbalance caused by a successor of stockpoint i, say j, equals

$$
E[\Omega_j] \approx \sigma_{\Omega_j} \phi\left(\frac{\mu_{\Omega_j}}{\sigma_{\Omega_j}}\right) + \mu_{\Omega_j} \Phi\left(\frac{\mu_{\Omega_j}}{\sigma_{\Omega_j}}\right), \tag{4}
$$

where

$$
\mu_{\Omega_j} = -R\mu_j, \qquad \sigma_{\Omega_j}^2 = 2q_j^2 T_i \sum_{k \in V_i} \sigma_k^2 + (R - 2q_j T_i)\sigma_j^2 \quad \text{and} \quad T_i := \min\{R, L_i\}.
$$

Van der Heijden [1996] determines *all* the allocation-fractions of stockpoint i so as to minimise the mean imbalance at stockpoint i, i.e., $E[\sum_{j \in V_i} \Omega_j]$. However, in our situation some of the allocation-fractions may already be chosen in order to minimise the expected total costs. For this purpose we introduce the set A_i denoting those successors of stockpoint i for which the allocation-fractions result from the cost minimisation, while the other successors are denoted by B_i. So, for every successor $j \in A_i$

5

the allocation-fraction q_j is already known. Let $C_i := \sum_{j \in A_i} q_j$. The allocation-fractions $\{q_j\}_{j \in B_i}$ are determined such that

$$\frac{dE(\Omega_j)}{dq_j} = \frac{\phi\left(\frac{\mu_{\Omega_j}}{\sigma_{\Omega_j}}\right)}{\sigma_{\Omega_j}} \, T_i \, (2q_j \sum_{k \in V_i} \sigma_k^2 - \sigma_j^2) = \lambda_i. \tag{5}$$

We can use bisection to find λ_i such that the allocation-fractions $\{q_j\}_{j \in B_i}$ sum up to $1 - C_i$. In each step of the bisection, the corresponding values for $\{q_j\}$ are found by another bisection, where

$$q_j \in \begin{cases} \left[0, \sigma_j^2 / (2 \sum_{k \in V_i} \sigma_k^2)\right) & C_i > 0.5 \\[2em] \left[\sigma_j^2 / (2 \sum_{k \in V_i} \sigma_k^2), 1\right] & C_i \leq 0.5. \end{cases}$$

Another variant of the balance stock rationing policy is referred to as BS2. Instead of minimising the mean imbalance as much as possible, we could also choose to minimise $\sum_{j \in B_i} \sigma_{\Omega_j}^2$ subject to $\sum_{j \in B_i} q_j = 1 - C_i$. The Lagrange multiplier technique yields

$$q_j := \frac{\sigma_j^2}{2 \sum_{k \in V_i} \sigma_k^2} + \frac{1}{|B_i|} \left(1 - C_i - \frac{\sum_{k \in B_i} \sigma_k^2}{2 \sum_{k \in V_i} \sigma_k^2} \right). \tag{6}$$

For the special case $A_i = \emptyset$ Van Donselaar [1996] was the first to define the allocation-fractions as in (6). Both the BS1 and BS2 heuristics are tested in Section 6.

5 Algorithm

In Section 3 the extension of the decomposition algorithm is explained by applying it to the system depicted in Figure 2. In this section we formalise the extension by presenting the algorithm (including its extension) in pseudo-code. Figure 3 depicts the main procedure, determining the order in which the control parameters are computed.

Figure 4 depicts procedure COMPUTELOCALPARAMETERS, which determines the control parameters of an intermediate stockpoint i, if $h_i > 0$ and $h_j > 0$ for $j \in V_i$. Figure 5 depicts procedure COMPUTEPARAMETERS, which determines the control parameters of a stockpoint i and the unknown parameters of its downstream stockpoints. The computational effort of this procedure is dominated by the effort to solve several one-dimensional service equations. These equations can be solved by a bisection scheme on the unknown parameter, e.g., in equation (3) we solve $\alpha_4^1(q_4) = \hat{\alpha}_4^1$ by bisection on $q_4 \in [0, 1]$. Note that COMPUTEPARAMETERS is a recursive procedure, since it needs to determine the control parameters of downstream stockpoints with no added value.

The parameters determined by procedure MAIN are not exact, since both COMPUTELOCALPARAMETERS and COMPUTEPARAMETERS determine the allocation-fraction q_j simply by averaging $q_{j(k)}$. As already mentioned in Diks & De Kok [1996a] this is only justifiable when the differences between the values of $q_{j(k)}$ for the different end-stockpoints k are small. Because otherwise averaging these allocation-fractions $q_{j(k)}$ implies that for some end-stockpoints the defined value q_j is too large and consequently the resulting service performance is too low, or the defined q_j is too small and consequently the resulting service performance is too large.

6

procedure MAIN
begin
  $n := 1$;
  while $n < N$ **do**
  begin
    for $i \in W_n$ **do**
    begin
      if $h_i > 0$ **then**
      begin
        for $k \in ech(i) \cap W_1$ **do** $\hat{\alpha}_k^i := (\sum_{j \in U_i} h_j + p_k)/(h_k + \sum_{j \in U_k} h_j + p_k)$;
        COMPUTEPARAMETERS(i)
      end
    end
    n:=n+1;
  end
end

Figure 3: Decomposition algorithm.

procedure COMPUTELOCALPARAMETERS($y_i, \{q_j\}_{j \in V_i}$)
begin
  initialise y_i, ε, *incr*, and *decr*;
  repeat
    $\Delta_i := y_i - \sum_{j \in V_i} y_j$;
    for $j \in V_i$ **do**
    begin
      for $k \in ech(j) \cap W_1$ **do** determine $q_{j(k)}$ by service-equation $\alpha_k^i(\Delta_i, q_{j(k)}) = \hat{\alpha}_k^i$;
      $q_j := \sum_{k \in ech(j) \cap W_1} q_{j(k)}/|ech(j) \cap W_1|$
    end
    if $\sum_{j \in V_i} q_j \leq 1 - \varepsilon$ **then** $y_i := y_i + incr$;
    if $\sum_{j \in V_i} q_j \geq 1 - \varepsilon$ **then** $y_i := y_i - decr$;
  until $|\sum_{j \in V_i} q_j - 1| < \varepsilon$
end

Figure 4: Procedure to determine all the control parameters of an intermediate stockpoint i, when this stockpoint and all its successors have positive added values.

procedure COMPUTEPARAMETERS(i)
begin
  if $i \in W_1$ **then** determine y_i by service-equation $\alpha_i^i(y_i) = \hat{\alpha}_i^i$ **else**
  begin
    if $B_i = \emptyset$ **then** COMPUTELOCALPARAMETERS($y_i, \{q_j\}_{j \in V_i}$) **else**
    begin
      $\Delta_i := 0$;
      for $j \in A_i$ **do**
      begin
        for $k \in ech(j) \cap W_1$ **do** determine $q_{j(k)}$ by service-equation $\alpha_k^i(\Delta_i, q_{j(k)}) = \hat{\alpha}_k^i$;
        $q_j := \sum_{k \in ech(j) \cap W_1} q_{j(k)} / |ech(j) \cap W_1|$
      end
      $C_i := \sum_{j \in A_i} q_j$;
      for $j \in B_i$ **do** determine q_j by Balanced Stock rationing such that $\sum_{j \in B_i} q_j = 1 - C_i$;
      for $j \in B_i$ **do** COMPUTEPARAMETERS(j);
      $y_i := \sum_{j \in V_i} y_j$
    end
  end
end

Figure 5: Procedure to determine all the control parameters of an intermediate stockpoint i and all the unknown parameters of downstream stockpoints

6 Numerical Results

In this section the performance of the extension of the decomposition algorithm of Diks & De Kok [1996a] is tested by considering 500 instances of the 3-echelon distribution system as depicted in Figure 6. The lead time of each intermediate stockpoint is drawn from a uniform distribution on $\{1, \ldots, 8\}$,
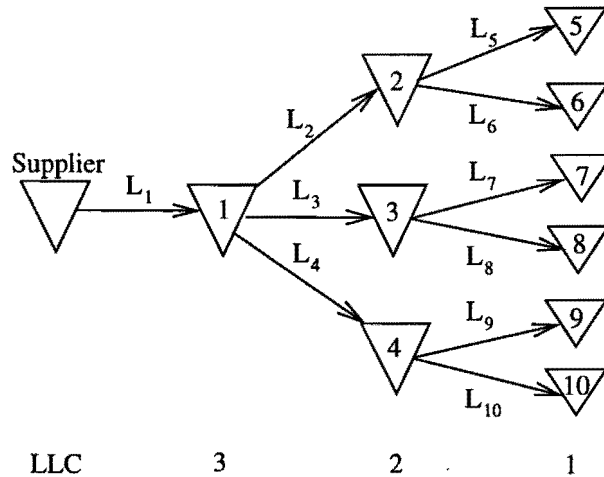

Figure 6: Divergent 3-echelon inventory system.

and the lead time of an end-stockpoint is drawn from a uniform distribution on $\{1, \ldots, 5\}$. The mean demand and squared coefficient of variation per review period at an end-stockpoint is drawn from a uniform distribution on [10,25] and [0.5, 1.5], respectively. Diks & De Kok [1996b] proved that if

8

every stockpoint uses the optimal order-up-to-level and the optimal rationing policy, the attained non-stock out probability at end-stockpoint k equals

$$\alpha_k = \frac{p_k}{h_k + \sum_{j \in U_k} h_j + p_k}. \tag{7}$$

Rewriting (7) yields

$$p_k = \frac{\alpha_k}{1 - \alpha_k} H_k, \tag{8}$$

where H_k equals the holding costs of one end-product k. In this numerical study we assume that all end-stockpoints (except for the most upstream stockpoint) have no added value. Hence, without loss of generality $H_k := 1$. In practice the required service level at an end-stockpoint usually is large, therefore we draw α_k from a uniform distribution on [0.85, 0.99]. So by setting $p_k := \alpha_k/(1 - \alpha_k)$ we know that if every stockpoint is controlled cost-optimally the attained service level at end-stockpoint k equals α_k.

The algorithm of Section 5 approximates the cost-optimal policy, since (1) the optimal rationing functions are approximated by the linear rationing functions (1), and (2) in procedure COMPUTELO-CALPARAMETERS we simply determine q_j by averaging $q_{j(k)}$, and (3) the behaviour of the inventory positions are determined by using the same approximation scheme of Van Houtum & Zijm [1991a]. Therefore the non-stock out probability in stockpoint k resulting from the algorithm, denoted by α_k^A, generally differs from α_k. Next, every instance is simulated with the control parameters obtained by the algorithm. In case the imbalance would not affect the attained non-stock out probability at stockpoint k, it would be equal to α_k^A. However, usually the phenomenon of imbalance does have some impact on the stock out probability. Hence, stockpoint k attains a non-stock out probability of α_k^S. Since we do not know the cost-optimal policy of the 3-echelon system of Figure 6, we validate the algorithm by comparing α_k with α_k^A. If for every k holds $\alpha_k = \alpha_k^A$ then the algorithm yields the cost-optimal policy. By comparing α_k^A with α_k^S we get some insight into the impact of the imbalance assumption.

Figure 7a and b depict the absolute differences $\alpha_k^A - \alpha_k$, $\alpha_k^S - \alpha_k^A$ and $\alpha_k^S - \alpha_k$ for an end-stockpoint k with $\alpha_k \leq 0.95$ and $\alpha_k > 0.95$, respectively. Notice that $\alpha_k^A - \alpha_k$ represents the 'algorithmic error' (Alg. error) due to the three aforementioned approximate steps, and $\alpha_k^S - \alpha_k^A$ represents the 'Imbalance error' (Imb. error) due to the violation of the balance assumption. We distinguish between these two cases $\alpha_k \leq 0.95$ and $\alpha_k > 0.95$, since an absolute error of 0.01 is acceptable in case α_k is not to large (e.g. 0.85), although, when α_k is large (e.g. 0.99) such an error is intolerable.

For both cases the 'algorithmic error' of BS1 and BS2 are almost identical and very small. This would advocate to use BS2 since it is much easier than BS1, which requires a nested bisection scheme. However, it turns out that the variability of the 'imbalance error' is smaller when applying BS1 instead of BS2. Comparing Figure 7a and b suggests that the effect becomes stronger for α_k not to large. Furthermore, this figure shows that the 'algorithmic error' is considerably smaller than the 'imbalance error'. Hence in order to improve the performance of the algorithm it is probably more efficient to focus our attention to the latter error. A way to reduce this error is by keeping some stock in every intermediate stockpoint i for which the algorithm suggest not to hold any stock, i.e., $\Delta_i = 0$. This means that we increase Δ_i in order to keep some products in stock. In Figure 8 this adaptation of the algorithm is denoted by $\Delta > 0$, while the original algorithm is denoted by $\Delta = 0$. Figure 8 depicts the mean of the expected costs per period for all 500 instances, for both BS1 and BS2. It turns out that differences in costs between BS1 and BS2 are negligible. Furthermore, the difference between the expected costs per period computed by the analysis and resulting from the simulation is much smaller in the case $\Delta > 0$ than $\Delta = 0$. This is due to the reduction of the imbalance in the case $\Delta > 0$. Finally, Figure 8 shows that the expected costs per period by adapting the algorithm increases.

9
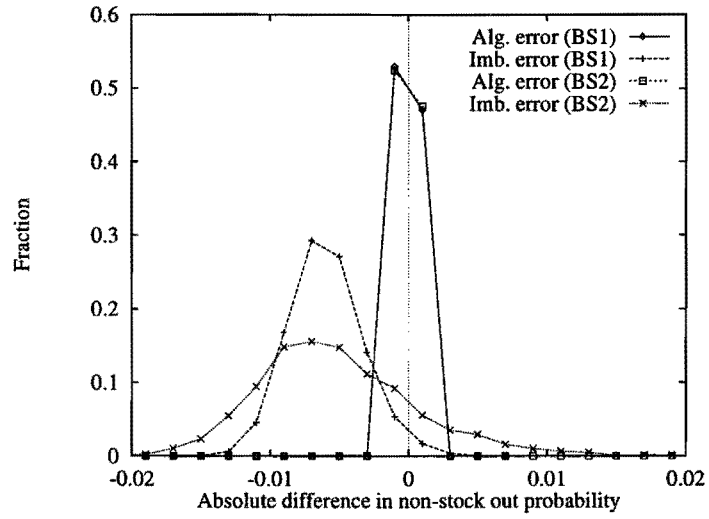
(a) $\alpha_k \leq 0.95$.
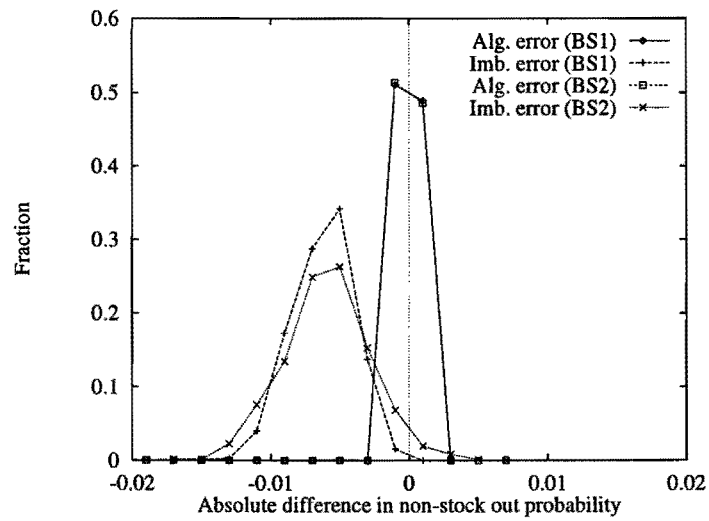
(b) $\alpha_k > 0.95$
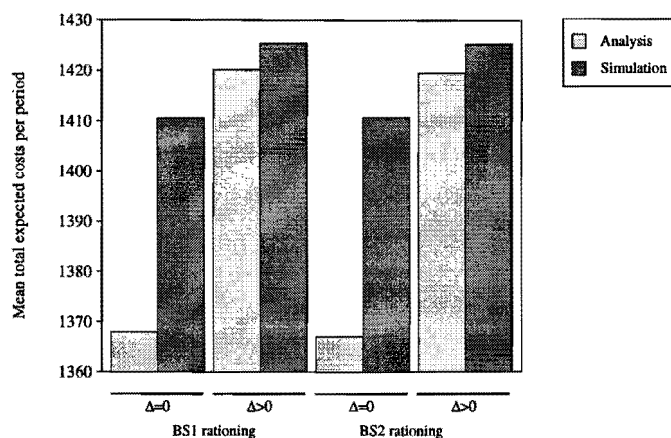
Figure 7: Performance of algorithm.

Figure 8: Mean of the expected total costs per period of the 500 instances for BS1 and BS2 rationing.

7 Conclusions

Diks & De Kok [1996a] developed a decomposition algorithm to compute the control parameters of a specific class of divergent multi-echelon inventory systems. Specifically, systems in which every stockpoint adds positive value to the product. This may be a reasonable assumption in production systems. However, for distribution systems generally this does not hold. In this paper we developed an extension of their algorithm in order to deal with those systems for which in some stockpoints no value is added. An adapted version of the Balanced Stock rationing policy of Van der Heijden [1996] (developed in Section 4) plays an important role in this extension. We considered two variants of this policy, denoted by BS1 and BS2. For both policies the algorithm performs very well, i.e, the control parameters yield a near cost-optimal policy. The BS2 policy results in a bit more imbalance in the system than the BS1 policy, although, it turns out that the total expected costs per period are (almost) identical. From a practical point of view we recommend to use BS2 rationing, since it is very simple to implement (it can even be used in spreadsheet applications). Another advantage is that the running time of an instance is very low. The 500 instances of Section 6 only takes a few seconds on a SPARC station 5.

References

DIKS, E.B., AND A.G. DE KOK [1996a], *Computational results for the control of a divergent N-echelon inventory system*, Memorandum COSOR 96-28, Department of Mathematics and Computing Science, Eindhoven University of Technology, The Netherlands, submitted for publication.

CLARK, A.J., AND H. SCARF [1960], Optimal policies for a multi-echelon inventory problem, *Management Science* 6, 475–490.

DIKS, E.B., AND A.G. DE KOK [1996b], *Optimal control of a divergent N-echelon inventory system*, Memorandum COSOR 96-09, Department of Mathematics and Computing Science, Eindhoven University of Technology, The Netherlands, submitted for publication.

DONSELAAR, K. VAN [1996], Personal communication.

EPPEN, G., AND L. SCHRAGE [1981], Centralized ordering policies in a multi-warehouse system with lead times and random demand, *Management Science* 16, 51–67.

FEDERGRUEN, A., AND P. ZIPKIN [1984], Approximations of dynamic, multilocation production and inventory problems, *Management Science* 30, 69–84.

HEIJDEN, M.C. VAN DER [1996], Supply rationing in multi-echelon divergent systems, *European Journal of Operational Research*, To appear.

HOUTUM, G.J. VAN, AND W.H.M. ZIJM [1991a], Computational procedures for stochastic multi-echelon production systems, *International Journal of Production Economics* **23**, 223–237.

HOUTUM, G.J. VAN, AND W.H.M. ZIJM [1991b], Theoretische en numerieke analyse van multi-echelon voorraadsystemen met distributiestruktuur, (In Dutch).

KOK, A.G. DE [1990], Hierarchical production planning for consumer goods, *European Journal of Operational Research* **45**, 55–69.

KOK, A.G. DE, A.G. LAGODIMOS, AND H.P. SEIDEL [1994], *Stock allocation in a 2-echelon distribution network under service-constraints*, TUE/BDK/LBS 94-03, Eindhoven University of Technology, The Netherlands.

LAGODIMOS, A.G. [1992], Multi-echelon service models for inventory systems under different rationing policies, *International Journal of Production Research* **30**, 939–958.

LANGENHOFF, L.J.G., AND W.H.M. ZIJM [1990], An analytical theory of multi-echelon production/ distribution systems, *Statistica Neerlandica* **44**, 149–174.

ROSLING, K. [1989], Optimal inventory policies for assembly systems under random demands, *Operations Research* **37**, 565–579.

VERRIJDT, J.H.C.M., AND A.G. DE KOK [1995], Distribution planning for a divergent N-echelon network without intermediate stocks under service restrictions, *International Journal of Production Economics* **38**, 225–243.

ZIJM, W.H.M., AND G.J. VAN HOUTUM [1994], On multi-stage production/inventory systems under stochastic demand, *International Journal of Production Economics* **35**, 391–400.