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Abstract

This paper gives a characterization of the minimal order of a dynamic state feedback that solves the model matching problem for a given nonlinear SISO-system and a given linear SISO-model.

1 Introduction and problem statement

Input-output linearization methods are among the most commonly used methods in practical nonlinear control systems design. Among the input-output linearizaton methods, the method of linear model matching plays an important role. The linear model matching problem for SISO-systems is defined as follows ([5]). Consider an analytic SISO-system Σ of the form

$$\Sigma \left\{ \begin{array}{rcll} \dot{x} & = & f(x)+g(x)u & ,x \in I\!R^n,\ u \in I\!R \\ y & = & h(x) & ,y \in I\!R \end{array} \right. \tag{1}$$

around a point $x_0 \in I\!R^n$, together with a strictly proper transfer function $g(s) = \frac{p(s)}{q(s)}$, where $p, q \in I\!R[s]$ are monic and coprime. Then the *linear model matching problem* (LMMP) is said to be solvable for Σ and g around x_0 if for Σ around x_0 there exists a dynamic state feedback Q of the form

$$Q \left\{ \begin{array}{rcl} \dot{\xi} & = & \alpha(x,\xi)+\beta(x,\xi)v, \quad \xi \in I\!R^{\nu},\ v \in I\!R \\ u & = & \gamma(x,\xi)+\delta(x,\xi)v \end{array} \right. \tag{2}$$

such that the (linear) input-output behavior of $\Sigma \circ Q$ around x_0 is described by g, i.e., given v, the output y of $\Sigma \circ Q$ satisfies the linear differential equation

$$q(\frac{d}{dt})y = p(\frac{d}{dt})v \tag{3}$$

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Define

$$d := \deg(p) \tag{4}$$

$$\tilde{r} := \deg(q) - d \tag{5}$$

and let r denote the relative degree ([6]) of h for Σ. We will assume throughout that r is well-defined around x_0. For SISO-systems, the solvability conditions for the LMMP take a particularly simple form: the LMMP is solvable for Σ and g around x_0 if and only if (see e.g. [5],[7])

$$\tilde{r} \geq r \tag{6}$$

A drawback of the dynamic state feedbacks proposed in e.g. [5] to solve the LMMP is that typically their order equals $\tilde{r}+d$, which may be unnecessarily large. Indeed, a simple argument already shows that if (6) holds, there exists a dynamic state feedback of order $\tilde{r} - r + d$ that solves the LMMP for Σ and g. (We will not give this argument here; the stated result is an immediate consequence of the results developed in this paper.) It goes without saying that in practical nonlinear control systems design it is of importance to know what is the minimal order of a dynamic state feedback solving the LMMP for Σ and g. It is the purpose of the present paper to characterize this minimal order. The paper employs the same sort of methods as the paper [3], where, amongst others, necessary and sufficient conditions for the existence of a *static* state feedback solving the LMMP for Σ and g were given (for an alternative approach to the LMMP via static state feedback, see e.g. [8]).

The paper is organized as follows. In the next section, we introduce the notion of relative degree of a one-form. Further, we introduce a system associated with Σ and g, and derive some properties of this associated system. Using these properties, we characterize the minimal order of a dynamic state feedback solving the LMMP for Σ and g in Section 3. Moreover, in Section 3 we illustrate the developed results by means of an example. Finally, in Section 4 some conclusions are drawn.

2 Preliminaries

2.1 Relative degree of one-forms

In this subsection we give a differential geometric treatment of the notion of relative degree of a one-form. This notion was introduced in [1] in an algebraic framework, and put into a differential geometric framework in [3]. The material presented in this subsection is taken almost *verbatim* from [3].

Consider the system Σ. Define the manifold $M_0 := \mathbb{R}^n$ with local coordinates x, and the manifolds $M_k := M_{k-1} \times \mathbb{R}$ with local coordinates $(x, u, \cdots, u^{(k-1)})$ $(k = 1, \cdots, 2n+1)$. Then M_k is an embedded submanifold of M_ℓ $(k = 0, \cdots, 2n;\ \ell = k+1, \cdots, 2n+1)$, with the natural embedding $i_{k\ell} : M_k \to M_\ell$ defined by

$$i_{k\ell}(x, u, \cdots, u^{(k-1)}) = (x, u, \cdots, u^{(k-1)}, 0, \cdots, 0)$$

Let Ξ_k denote the codistribution span$\{dx\}$ on M_k $(k = 0, \cdots, 2n+1)$. On M_{2n+1}, we define the vector field

$$f^e := (f + gu)\frac{\partial}{\partial x} + \sum_{i=0}^{2n} u^{(i+1)} \frac{\partial}{\partial u^{(i)}} \tag{7}$$

For a one-form ω on M_k, we define $\omega^{(\ell)}$ on M_{2n+1} by

$$\omega^{(\ell)} := \mathcal{L}^{\ell}_{f^e}((i_{k2n+1})_* \omega) \qquad (\omega \in M_k;\ k = 0, \cdots, n+1;\ \ell = 0, \cdots, 2n+1-k) \tag{8}$$

Then $\omega^{(\ell)}$ may be interpreted as a one-form on $M_{k+\ell}$, in the sense that

$$(i_{k+\ell 2n+1})_*(i_{k+\ell 2n+1})^* \omega^{(\ell)} = \omega^{(\ell)} \qquad (\omega \in M_k;\ k = 0, \cdots, n+1;\ \ell = 0, \cdots, 2n+1-k) \tag{9}$$

Let $\omega \in M_k$ $(k = 0, \cdots, n)$ and assume that there exists an $\ell \in \{1, \cdots, n\}$ such that $\omega^{(\ell)} \notin \Xi_{2n+1}$. Then the smallest such ℓ is called the *relative degree* of ω, to be denoted by r_ω. If for all $\ell \in \{1, \cdots, n\}$ we have $\omega^{(\ell)} \in \Xi_{2n+1}$, we define $r_\omega := +\infty$. For a function ϕ satisfying $d\phi \in \Xi_k$, we define its relative degree by $r_\phi := r_{d\phi}$. Define codistributions $\mathcal{H}^{\ell}_k$ by

$$\mathcal{H}^{\ell}_k := \{\omega \in \Xi_\ell \mid r_\omega \geq k\} \quad (k = 1, \cdots, n; \ell = k-1, \cdots, 2n+1-k) \tag{10}$$

It may then be shown that $\mathcal{H}^{\ell}_k$ may be identified with $\mathcal{H}^{k-1}_k$, in the sense that

$$(i_{k-1\ell})_*(i_{k-1\ell})^* \mathcal{H}^{\ell}_k = (i_{k-1\ell})_* \mathcal{H}^{k-1}_k \quad (k = 1, \cdots, n;\ \ell = k-1, \cdots, 2n+1-k) \tag{11}$$

We further define the codistribution $\mathcal{H}^n_\infty$ by

$$\mathcal{H}^n_\infty := \{\omega \in \Xi_n \mid r_\omega = +\infty\} \tag{12}$$

Now define

$$\mathcal{H}_k := (i_{k-12n+1})_* \mathcal{H}^{k-1}_k \quad (k = 1, \cdots, n) \tag{13}$$

$$\mathcal{H}_\infty := (i_{n2n+1})_* \mathcal{H}^n_\infty \tag{14}$$

The codistributions defined in (13), (14) have the following properties (for a proof, see (*mutatis mutandis*) [1]).

Lemma 2.1 *Let $x_0 \in \mathbb{R}^n$ be given, and assume that the codistributions $\mathcal{H}_k$ $(k \in \{1, \cdots, n, \infty\})$ have constant dimension around x_0. Then around x_0 these codistributions have the following properties:*

(i) $\mathcal{H}_1 \supset \mathcal{H}_2 \supset \cdots \supset \mathcal{H}_n \supset \mathcal{H}_\infty$

(ii) $\mathcal{H}_\infty$ *is integrable.*

(iii) Σ *is strongly accessible if and only if* $\mathcal{H}_\infty = \{0\}$.

(iv) $\mathcal{H}_k = \{\omega \in \mathcal{H}_{k-1} \mid ((i_{k-22n+1})^*\omega)^{(1)} \in \mathcal{H}_k\}$ $(k = 1, \cdots, n)$

(v) $\mathcal{H}_\infty = \{\omega \in \mathcal{H}_n \mid ((i_{n-12n+1})^*\omega)^{(1)} \in \mathcal{H}_n\}$

(vi) Define

$$\sigma := n + 1 - \dim(\mathcal{H}_\infty) \tag{15}$$

Then

$$\dim(\mathcal{H}_k) = n + 1 - k \quad (k = 1, \cdots, \sigma) \tag{16}$$

and

$$\mathcal{H}_k = \mathcal{H}_\infty \quad (k = \sigma, \cdots, n) \tag{17}$$

(vii) Let $\lambda \in \mathcal{H}_{\sigma-1} \backslash \mathcal{H}_\infty$. *Then*

$$\mathcal{H}_k = \mathcal{H}_\infty \oplus \text{span}\{((i_{n-22n+1})^*\lambda)^{(\ell)} \mid \ell = 0, \cdots, \sigma - 1 - k\} \quad (k = 1, \cdots, \sigma - 1) \tag{18}$$

(viii) $\mathcal{H}_k$ $(k \in \{1, \cdots, n, \infty\})$ *is invariant under regular static state feedback.* ■

Remark 2.2 Consider a nonlinear control system Σ of the form (1). In the sequel, we will encounter extensions of Σ of the following form:

$$\left\{ \begin{array}{rcl} \dot{x} & = & f(x) + g(x)u \\ \dot{\eta} & = & v(x, \eta) + w(x, \eta)u \quad , \eta \in \mathbb{R}^\ell \end{array} \right. \tag{19}$$

Similarly to what has been done above for Σ, one may define for (19) codistributions $\mathcal{H}_k^e$ consisting of one-forms in span$\{dx, dz\}$ having relative degree $\geq k$ $(k \in \{1, \cdots, n + \nu, \infty\})$. These codistributions will then be codistributions on the manifold $M^e := \mathbb{R}^{n+\ell} \times \mathbb{R}^{2(n+\ell)+1}$, with local coordinates $(x, \eta, u, \cdots, u^{(2(n+\ell))})$. The manifold M that was defined above, is an embedded submanifold of M^e, with the natural embedding $i : M \to M^e$ defined by

$$i(x, u, \cdots, u^{(2n)}) := (x, 0, u, \cdots, u^{(2n)}, 0, \cdots, 0)$$

Let Ξ^e denote the codistribution span$\{dx\}$ on M^e. Consider the codistributions $\mathcal{H}_k$ $(k \in \{1, \cdots, n, \infty\})$ that were defined above for Σ. It then follows from the form of (19) that

$$i_*\mathcal{H}_k \subset \mathcal{H}_k^e \quad (k \in \{1, \cdots, n\}) \tag{20}$$

$$i_*\mathcal{H}_\infty \subset \mathcal{H}_k^e \quad (k \in \{n + 1, \cdots, n + \ell, \infty\}) \tag{21}$$

$$\mathcal{H}_k^e \cap \Xi^e = i_*\mathcal{H}_k \quad (k \in \{1, \cdots, n\}) \tag{22}$$

$$\mathcal{H}_k^e \cap \Xi^e = i_*\mathcal{H}_\infty \quad (k \in \{n + 1, \cdots, n + \ell, \infty\}) \tag{23}$$

In the sequel, we will frequently apply some abuse of notation, in that we will write $\mathcal{H}_k$ instead of $i_*\mathcal{H}_k$. Further, we will make no explicit distinction between the codistribution span$\{dx\}$ on M and M^e.

2.2 Associated system

Let Σ as in (1) and a strictly proper transfer function $g(s) = \frac{p(s)}{q(s)}$ as in Section 1 be given. In the solution of the minimal order LMMP that will be presented in Section 3, we make use of a system Σ^p that is associated with Σ and g in the following way. Write

$$p(s) = s^d + \sum_{k=0}^{d-1} p_k s^k \tag{24}$$

and define Σ^p by

$$\Sigma^p \left\{ \begin{array}{rcl} \dot{x} & = & f(x) + g(x)u \\ \dot{z}_1 & = & z_2 \\ & \vdots & \\ \dot{z}_{d-1} & = & z_d \\ \dot{z}_d & = & h(x) - \sum_{k=1}^{d} p_{k-1} z_k \end{array} \right. \tag{25}$$

Similarly to what has been done in Section 2.1, we define a sequence of codistributions $\mathcal{H}_k^p$ $(k \in \{1, \cdots, n+d, \infty\})$ for Σ^p. Note that Σ^p is of the form (19).

In what follows, the following result on the structure of $\mathcal{H}_\infty^p$ is of importance.

Proposition 2.3 *Let $x_0 \in I\!R^n$ be given, and assume that Σ is strongly accessible around x_0. Further, assume that the codistributions $\mathcal{H}_k$ $(k \in \{1, \cdots, n, \infty\})$ have constant dimension around x_0. Define*

$$\epsilon := \dim(\mathcal{H}_\infty^p) \tag{26}$$

Then there exist functions $\phi_1, \cdots, \phi_\epsilon$, $\alpha_{ik} \in I\!R$ $(i = 1, \cdots, \epsilon;\ k = 1, \cdots, d)$ and $a_k \in I\!R$ $(k = 0, \cdots, \epsilon - 1)$ such that

$$\mathcal{H}_\infty^p = \text{span}\{d\phi_i - \sum_{k=1}^{d-\epsilon} \alpha_{ik} dz_k - dz_{d-\epsilon+i} \mid i = 1, \cdots, \epsilon\} \tag{27}$$

$$d\phi_i \in \text{span}\{dx\} \quad (i = 1, \cdots, \epsilon) \tag{28}$$

$$r_{\phi_i} = r + \epsilon - i + 1 \quad (i = 1, \cdots, \epsilon) \tag{29}$$

and

$$dh = d\phi_1^{(\epsilon)} + \sum_{k=0}^{\epsilon-1} a_k d\phi_1^{(k)} \tag{30}$$

Proof See Appendix. ■

3 Minimal order linear model matching

Let Σ and g be given as in Section 1. In this section, we derive the minimal order of a dynamic state feedback that solves the LMMP for Σ and g.

We first consider the case that

$$\tilde{r} > r \tag{31}$$

Let Q be a dynamic state feedback of the form (2) that solves the LMMP for Σ and g. Since the relative degree of h for $\Sigma \circ Q$ equals $\tilde{r}$, we have for $\Sigma \circ Q$ that the differentials $dy, \cdots, dy^{(\tilde{r}-1)}$ are independent, while also (see e.g. [4])

$$\dim(\mathrm{span}\{dx, dy^{(r)}, \cdots, dy^{(\tilde{r}-1)}\}) = n + \tilde{r} - r \tag{32}$$

This implies that for $\Sigma \circ Q$ there exist new coordinates $(x, \bar{\xi}(x,\xi), \tilde{\xi}(x,\xi))$ with

$$\bar{\xi}_i = y^{(r+i-1)} \quad (i = 1, \cdots, \tilde{r} - r) \tag{33}$$

so that in these new coordinates $\Sigma \circ Q$ takes the form

$$\left\{\begin{array}{rcl} \dot{x} & = & f(x) + g(x)\bar{\gamma}(x,\bar{\xi},\tilde{\xi}) \\ \dot{\bar{\xi}}_1 & = & \bar{\xi}_2 \\ & \vdots & \\ \dot{\bar{\xi}}_{\tilde{r}-r-1} & = & \bar{\xi}_{\tilde{r}-r} \\ \dot{\bar{\xi}}_{\tilde{r}-r} & = & \bar{\alpha}(x,\bar{\xi},\tilde{\xi}) + \bar{\beta}(x,\bar{\xi},\tilde{\xi})v \\ \dot{\tilde{\xi}} & = & \tilde{\alpha}(x,\bar{\xi},\tilde{\xi}) + \tilde{\beta}(x,\bar{\xi},\tilde{\xi})v \\ y & = & h(x) \end{array}\right. \tag{34}$$

where

$$\bar{\gamma}(x,\bar{\xi},\tilde{\xi}) = (\mathcal{L}_g\mathcal{L}_f^{r-1}h(x))^{-1}(\bar{\xi}_1 - \mathcal{L}_f^r h(x)) \tag{35}$$

Define the system $\bar{\Sigma}$ by

$$\bar{\Sigma} \left\{\begin{array}{rcl} \dot{x} & = & f(x) + g(x)\bar{\gamma}(x,\bar{\xi}) \\ \dot{\bar{\xi}}_1 & = & \bar{\xi}_2 \\ & \vdots & \\ \dot{\bar{\xi}}_{\tilde{r}-r-1} & = & \bar{\xi}_{\tilde{r}-r} \\ \dot{\bar{\xi}}_{\tilde{r}-r} & = & \bar{u} \\ y & = & h(x) \end{array}\right. \tag{36}$$

If $\tilde{r} = r$, we define $\bar{\Sigma}$ to be equal to Σ. From the discussion above, we then obtain the following result.

Proposition 3.1 *There exists a dynamic state feedback of order ν that solves the LMMP for Σ and g if and only if there exists a dynamic state feedback of order $\nu - \tilde{r} + r$ that solves the LMMP for $\bar{\Sigma}$ and g.* ∎

Note that the relative degree of h for $\bar{\Sigma}$ equals $\tilde{r}$. It thus follows that, in order to characterize the minimal order of a dynamic state feedback that solves the LMMP for Σ and g, it suffices to consider the case

$$\tilde{r} = r \tag{37}$$

We first derive a lower bound for the order of a dynamic state feedback that solves the LMMP for Σ and g, using the following result from [3].

Theorem 3.2 *Let $x_0 \in \mathbb{R}^n$ be given. Assume that (37) holds, and that Σ is strongly accessible around x_0. Further, assume that the codistributions $\mathcal{H}_k$ ($k \in \{1, \cdots, n, \infty\}$) have constant dimension around x_0. Then there exists a* static *state feedback that solves the LMMP for Σ and g if and only if*

$$\mathcal{H}^p_\infty = \mathcal{H}^p_{n+1} \tag{38}$$

■

Remark 3.3 If Σ is not strongly accessible, (38) still is a necessary condition for the existence of a static state feedback that solves the LMMP for Σ and g. However, it is not a sufficient condition any more (cf. [3]). Further, note that it follows from Lemma 2.1 that (38) is equivalent to

$$\dim(\mathcal{H}^p_\infty) = d + \dim(\mathcal{H}_\infty) \tag{39}$$

Theorem 3.4 *Let $x_0 \in \mathbb{R}^n$ be given. Assume that (37) holds, and that Σ is strongly accessible around x_0. Further, assume that the codistributions $\mathcal{H}_k$ ($k = 1, \cdots, n, \infty\}$) have constant dimension around x_0. Let Q be a dynamic state feedback of the form (2) that solves the LMMP for Σ around x_0. Then*

$$\nu \geq d - \dim(\mathcal{H}^p_\infty) \tag{40}$$

Proof Since (37) holds, and Q solves the LMMP for Σ and g, the relative degree of h for $\Sigma \circ Q$ equals r. This implies that $\delta(x, \xi) \neq 0$, and thus

$$v = \delta(x, \xi)^{-1}(u - \gamma(x, \xi)) \tag{41}$$

Consider the system $\tilde{\Sigma}$ given by

$$\tilde{\Sigma} \left\{ \begin{array}{rcl} \dot{x} & = & f(x) + g(x)u \\ \dot{\xi} & = & (\alpha(x, \xi) - \delta(x, \xi)^{-1}\gamma(x, \xi)) + \beta(x, \xi)\delta(x, \xi)^{-1}u \\ y & = & h(x) \end{array} \right. \tag{42}$$

It then follows from (41) and the fact that Q solves the LMMP for Σ and g, that there exists a static state feedback that solves the LMMP for $\tilde{\Sigma}$ and g. Define the system $\tilde{\Sigma}^p$ associated with $\tilde{\Sigma}$ and g as in (25), and define sequences of codistributions $\tilde{\mathcal{H}}_k$ ($k \in \{1, \cdots, n+\nu, \infty$), $\tilde{\mathcal{H}}^p_k$

($k \in \{1, \cdots, n+\nu+d, \infty\}$), associated with $\tilde{\Sigma}$, $\tilde{\Sigma}^p$ respectively, as in Section 2.1. It follows from the form of $\tilde{\Sigma}$ and the fact that Σ is strongly accessible, that

$$\tilde{\mathcal{H}}_\infty \cap \operatorname{span}\{dx\} = \{0\} \tag{43}$$

Further, by the form of $\tilde{\Sigma}^p$ we have that

$$\mathcal{H}^p_\infty = \tilde{\mathcal{H}}^p_\infty \cap \operatorname{span}\{dx, dz\} \tag{44}$$

which gives that there exists a codistribution $\tilde{\mathcal{H}}^c_\infty$ satisfying

$$\tilde{\mathcal{H}}^p_\infty = \tilde{\mathcal{H}}_\infty \oplus \mathcal{H}^p_\infty \oplus \tilde{\mathcal{H}}^c_\infty \tag{45}$$

and

$$\tilde{\mathcal{H}}^c_\infty \cap \operatorname{span}\{dx, dz\} = \{0\} \tag{46}$$

From (46), it follows in particular that

$$\dim(\tilde{\mathcal{H}}^c_\infty) \leq \nu \tag{47}$$

By Theorem 3.2, Remark 3.3 and the fact that there exists a static state feedback that solves the LMMP for $\tilde{\Sigma}$, we have

$$\dim(\tilde{\mathcal{H}}^p_\infty) = d \tag{48}$$

We now obtain

$$\nu \overset{(47)}{\geq} \dim(\tilde{\mathcal{H}}^c_\infty) \overset{(45)}{=} \dim(\tilde{\mathcal{H}}^p_\infty) - \dim(\tilde{\mathcal{H}}_\infty) - \dim(\mathcal{H}^p_\infty) \overset{(39)}{=} d - \dim(\mathcal{H}^p_\infty) \tag{49}$$

which establishes our claim. ∎

We next show that the lower bound given in (40) is sharp, i.e., we are going to show that there exists a dynamic state feedback of order $d - \dim(\mathcal{H}^p_\infty)$ that solves the LMMP for Σ and g.

Theorem 3.5 *Let $x_0 \in I\!R^n$ be given. Assume that (37) holds and that Σ is strongly accessible around x_0. Further, assume that the codistributions $\mathcal{H}_k$ ($k \in \{1, \cdots, n, \infty\}$) have constant dimension around x_0. Let ϵ be defined by (26). Then there exists a dynamic state feedback of dimension $d - \epsilon$ that solves the LMMP for Σ and g.*

Proof Consider the function ϕ_1 and constants $\alpha_{11}, \cdots, \alpha_{1d-\epsilon}$ from Proposition 2.3, and define the following extended system $\tilde{\Sigma}$:

$$\tilde{\Sigma} \left\{ \begin{array}{rcl} \dot{x} & = & f(x) + g(x)u \\ \dot{\xi}_1 & = & \xi_2 \\ & \vdots & \\ \dot{\xi}_{d-\epsilon-1} & = & \xi_{d-\epsilon} \\ \dot{\xi}_{d-\epsilon} & = & \phi_1(x) - \sum_{k=1}^{d-\epsilon} \alpha_{1k}\xi_k \end{array} \right. \tag{50}$$

Further define an associated system $\tilde{\Sigma}^p$ analogously to (25). Let for $\tilde{\Sigma}^p$ the codistribution consisting of one-forms having infinite relative degree be denoted by $\tilde{\mathcal{H}}^p_\infty$. Comparing the forms of Σ^p and $\tilde{\Sigma}^p$ we obtain

$$\mathcal{H}^p_\infty \subset \tilde{\mathcal{H}}^p_\infty \tag{51}$$

Define the functions ψ_i by

$$\psi_i := \xi_i - z_i \quad (i = 1, \cdots, d - \epsilon) \tag{52}$$

Note that

$$\text{span}\{d\psi_1, \cdots, d\psi_{d-\epsilon}\} \cap \mathcal{H}^p_\infty = \{0\} \tag{53}$$

We have

$$\dot{\psi}_i = \psi_{i+1} \quad (i = 1, \cdots, d - \epsilon - 1) \tag{54}$$

and

$$\dot{\psi}_{d-\epsilon} = \phi_1 - \sum_{k=1}^{d-\epsilon} \alpha_{1k} z_k - z_{d-\epsilon+1} - \sum_{k=1}^{d-\epsilon} \alpha_{1k} \psi_k \tag{55}$$

From Proposition 2.3 it follows that

$$d\phi_1 - \sum_{k=1}^{d-\epsilon} \alpha_{1k} dz_k - dz_{d-\epsilon+1} \in \mathcal{H}^p_\infty \subset \tilde{\mathcal{H}}^p_\infty \tag{56}$$

It then follows from (55),(56) that

$$d\psi_i \in \tilde{\mathcal{H}}^p_\infty \quad (i = 1, \cdots, d - \epsilon) \tag{57}$$

which, together with (51) and (53), gives that

$$\dim(\tilde{\mathcal{H}}^p_\infty) = d \tag{58}$$

Note that it follows from the form of $\tilde{\Sigma}^p$ and (29) that $r_{\xi_i} = r + d - i + 1$ $(i = 1, \cdots, d - \epsilon)$, which gives that $\tilde{\Sigma}^p$ is strongly accessible. Then (39),(58) give that there exists a static state feedback that solves the LMMP for $\tilde{\Sigma}$ and g. This establishes our claim. ■

Remark 3.6 Since $r_{\phi_1} = r + \epsilon$, we have for $\tilde{\Sigma}$ that $r_{\xi_1} = r + d$. From [3], it then follows that a static state feedback solving the LMMP for $\tilde{\Sigma}$ and g is given by

$$u = \tilde{b}(x, \xi)^{-1}(v - \tilde{a}(x, \xi) - \sum_{k=1}^{r+d} q_{k-1} \xi_1^{(k-1)}) \tag{59}$$

where $\tilde{a}$,$\tilde{b}$ satisfy

$$\xi_1^{(r+d)} = \tilde{a}(x, \xi) + \tilde{b}(x, \xi) u \tag{60}$$

and $q_1, \cdots, q_{r+d-1}$ are such that

$$q(s) = s^{r+d} + \sum_{k=0}^{r+d} q_k s^k \tag{61}$$

Since the feedback (59) solves the LMMP for $\tilde{\Sigma}$ and g, we have, after (59) has been applied,

$$q(\frac{d}{dt})y = p(\frac{d}{dt})v \tag{62}$$

and, by (59),(60),

$$q(\frac{d}{dt})\xi_1 = v \tag{63}$$

which gives

$$y = p(\frac{d}{dt})\xi_1 \tag{64}$$

Define polynomials $a(s)$, $b(s)$ by

$$a(s) := \sum_{k=0}^{\epsilon-1} a_k s^k + s^\epsilon \tag{65}$$

$$b(s) := \sum_{k=0}^{d-\epsilon-1} \alpha_{1k+1} s^k + s^{d-\epsilon} \tag{66}$$

with $a_0, \cdots, a_{\epsilon-1}$, $\alpha_{11}, \cdots, \alpha_{1d-\epsilon}$ as in Proposition 2.3. Then it follows from (30),(50) that

$$y = a(\frac{d}{dt})\phi_1 \tag{67}$$

$$\phi_1 = b(\frac{d}{dt})\xi_1 \tag{68}$$

which gives

$$y = a(\frac{d}{dt})b(\frac{d}{dt})\xi_1 \tag{69}$$

From (64),(69) it then follows that

$$p(s) = a(s)b(s) \tag{70}$$

Let $w \in I\!R[s]$ be such that $\deg(w) = r + \epsilon$ and w and a are coprime. It then follows from (67), the fact that $r_{\phi_1} = r + \epsilon$ and [3] that for Σ there exists a static state feedback $Q_s: \ u = \alpha(x) + \beta(x)v$ such that the input-output behavior of $\Sigma \circ Q_s$ is described by $\frac{a(s)}{w(s)}$. Given this observation and (70), the result of Theorem 3.5 may be interpreted as follows:

(i) There always exists a dynamic state feedback of order d that solves the LMMP for Σ and g, and

(ii) there exists a dynamic state feedback of order less than d that solves the LMMP for Σ and g *only if* Σ itself is able to reproduce some of the zeros of $g(s)$.

Analogously to (25), let $\bar{\Sigma}^p$ be the system associated with $\bar{\Sigma}$ and g. Let $\bar{\mathcal{H}}^p_\infty$ denote the codistribution consisting of one-forms having infinite relative degree for $\bar{\Sigma}^p$. Combining Proposition 3.1 and Theorems 3.4 and 3.5, we then arrive at the following result.

Theorem 3.7 *Let $x_0 \in \mathbb{R}^n$ be given. Assume that Σ is strongly accessible around x_0, and that the codistributions $\mathcal{H}_k$ $(k \in \{1, \cdots, n, \infty\})$ have constant dimension around x_0. Further, assume that (6) holds. Then the minimal order of a dynamic state feedback solving the LMMP for Σ and g is given by*

$$\tilde{r} - r + d - \dim(\bar{\mathcal{H}}^p_\infty) \tag{71}$$

■

We illustrate the theory developed with an example.

Example 3.8 Consider on $\{x \in \mathbb{R}^4 \mid x_1 > 0\}$ the SISO-system Σ given by

$$\Sigma \left\{ \begin{array}{rcl} \dot{x}_1 &=& x_1x_2 - x_1 \\ \dot{x}_2 &=& 2x_2 - x_2^2 - 1 + \frac{1}{x_1}u \\ \dot{x}_3 &=& 3x_1 + x_3 - 3x_1^2 - 2x_1x_2 + 2x_1^2x_2 \\ \dot{x}_4 &=& -x_1^4 + x_1^3 + 2x_1^2 - x_3^2 - x_1x_3 + 2x_1^2x_3 \\ y &=& x_1x_2 \end{array} \right.$$

Further, consider

$$g(s) = \frac{s^3 + 4s^2 + 5s + 2}{(s+3)^4}$$

Note that we have $\tilde{r} = r = 1$. We find

$$\mathcal{H}^p_\infty = \text{span}\{dx_1 - 2dz_1 - 3dz_2 - dz_3\}$$

It then follows from Theorem 3.7 that the minimal order of a dynamic state feedback solving the LMMP for Σ and g equals 2. It may be checked that the following dynamic state feedback indeed solves the LMMP for Σ and g:

$$\left\{ \begin{array}{rcl} \dot{\xi}_1 &=& \xi_2 \\ \dot{\xi}_2 &=& x_1 - 2\xi_1 - 3\xi_2 \\ u &=& -9x_1x_2 - 16x_1 - 31\xi_1 - 15\xi_2 \end{array} \right.$$

Next, consider

$$g(s) = \frac{s^3 + s^2 - s - 1}{(s+3)^4}$$

We now find

$$\mathcal{H}^p_\infty = \text{span}\{d(x_1^2 - 2x_1 - x_3) - dz_1 - dz_2, dx_1 + dz_1 - dz_3\}$$

which gives by Theorem 3.7 that the minimal order of a dynamic state feedback that solves the LMMP for Σ and g equals 1. In this case, it may be checked that the following dynamic state feedback solves the LMMP for Σ and g:

$$\left\{ \begin{array}{rcl} \dot{\xi} &=& x_1^2 - 2x_1 - x_3 - \xi \\ u &=& -12x_1x_2 + 197x_1 - 120x_1^2 + 120x_3 - 16\xi \end{array} \right.$$

4 Conclusions

In this paper we have characterized the minimal order of a dynamic state feedback that solves the model matching problem for a given nonlinear SISO-system and a given linear SISO-model. The design of a minimal order dynamic state feedback that solves the LMMP in the vein of the proof of Theorem 3.5 and Remark 3.6 is completely constructive up to finding a function ϕ_1 satisfying $\tilde{\omega}_1 = d\phi_1$. However, since this only involves integration, this will be not too big a problem in the practical implementation of a minimal order controller.

In this paper, we have restricted to SISO-systems. We expect that an extension of the results in the paper to MISO-systems is possible. Also an extension to MIMO-systems (at least for square systems having an invertible decoupling matrix) seems possible. These remain topics for future research.

References

[1] E. Aranda-Bricaire, E., C.H. Moog and J.B. Pomet, *A linear algebraic framework for dynamic feedback linearization*, IEEE Trans. Automat. Control **40** (1995) 127-132.

[2] R.L. Bryant, S.S. Chern, R.B. Gardner, H.L. Goldschmidt and P.A. Griffiths, *Exterior Differential Systems* (Springer-Verlag, New York, 1991).

[3] H.J.C. Huijberts, *Characterization of static feedback realizable transfer functions for nonlinear control systems*, RANA-report 97-05, Department of Mathematics and Computing Science, Eindhoven University of Technology. To appear in Intern. J. Nonl. Robust Control.

[4] H.J.C. Huijberts, H. Nijmeijer and L.L.M. van der Wegen, *Minimality of dynamic input-output decoupling for nonlinear control systems*, Syst. Control Lett. **18** (1992) 435-443.

[5] A. Isidori, *Nonlinear control systems: an introduction* LNCIS 72 (Springer-Verlag, Berlin, 1989).

[6] A. Isidori, *Nonlinear control systems* (second edition) (Springer-Verlag, Berlin, 1989).

[7] H. Nijmeijer and A.J. van der Schaft, *Nonlinear dynamical control systems* (Springer-Verlag, New York, 1990).

[8] P.N. Paraskevopoulos, A.S. Tsirikos and E.A. Karagianni, *Robust tracking of an inverted pendulum via a new linear exact model matching technique* (Proceedings CDC 1995, New Orleans, USA) 1676-1683.

Appendix: Proof of Proposition 2.3

In this Appendix we give a proof of Proposition 2.3. We first state and prove some lemmas.

Lemma 4.1 *Let $x_0 \in I\!R^n$ be given,and assume that Σ is strongly accessible around x_0. Further, assume that the codistributions $\mathcal{H}_k$ $(k \in \{1, \cdots, n, \infty\})$ have constant dimension around*

x_0. *Let ϵ be defined by (26). Then there exist one-forms $\tilde{\omega}_1, \cdots, \tilde{\omega}_\epsilon \in \mathrm{span}\{dx\}$ and functions α_{ik} $(i = 1, \cdots, \epsilon;\ k = 1, \cdots, d)$ such that*

$$\mathcal{H}^p_\infty = \mathrm{span}\{\tilde{\omega}_i - \sum_{k=1}^{d} \alpha_{ik} dz_k \mid i = 1, \cdots, \epsilon\} \tag{72}$$

$$d\tilde{\omega}_i \in \mathrm{span}\{\pi \wedge \rho \mid \pi, \rho \in \mathrm{span}\{dx, du, \cdots, du^{(2n)}\}\} \quad (i = 1, \cdots, \epsilon) \tag{73}$$

$$d\alpha_{ik} \in \mathrm{span}\{dx, du, \cdots, du^{(2n)}\} \quad (i = 1, \cdots, \epsilon;\ k = 1, \cdots, d) \tag{74}$$

and

$$(\sum_{k=1}^{d} \alpha_{1k} dz_k) \wedge \cdots \wedge (\sum_{k=1}^{d} \alpha_{\epsilon k} dz_k) \neq 0 \tag{75}$$

■

Proof It follows from Lemma 2.1, (23), and the fact that Σ is strongly accessible, that

$$\dim(\mathcal{H}^p_{n+k}) = d - k + 1 \quad (k = 1, \cdots, d - \epsilon + 1) \tag{76}$$

$$\mathcal{H}_{n+d-\epsilon+1} = \mathcal{H}^p_\infty \tag{77}$$

and

$$\mathcal{H}^p_{n+k} \cap \mathrm{span}\{dx\} = \{0\} \quad (k = 1, \cdots, d - \epsilon + 1) \tag{78}$$

From (76),(78) it follows in particular that there exist one-forms $\omega_1, \cdots, \omega_d \in \mathrm{span}\{dx\}$ such that

$$r_{\omega_i} = r + d - i + 1 \quad (i = 1, \cdots, d) \tag{79}$$

and

$$\mathcal{H}^p_{n+1} = \mathrm{span}\{\omega_1 - dz_1, \cdots, \omega_d - dz_d\} \tag{80}$$

Further, it follows from Lemma 2.2 in [3] that

$$d\omega_i \in \mathrm{span}\{\pi \wedge \rho \mid \pi, \rho \in \mathrm{span}\{dx, du, \cdots, du^{(2n)}\}\} \quad (i = 1, \cdots, d) \tag{81}$$

Combining items *(i)*,*(vi)*,*(vii)* in Lemma 2.1, we also have that there exists a $\lambda \in \mathcal{H}_n - \{0\}$ such that

$$\mathcal{H}^p_n = \mathrm{span}\{\lambda\} \oplus \mathrm{span}\{\omega_1 - dz_1, \cdots, \omega_d - dz_d\} \tag{82}$$

We have

$$(\omega_i - dz_i)^{(1)} = \dot{\omega}_i - dz_{i+1} = \dot{\omega}_i - \omega_{i+1} + (\omega_{i+1} - dz_{i+1}) \quad (i = 1, \cdots, d-1) \tag{83}$$

and

$$\begin{aligned}(\omega_d - dz_d)^{(1)} = \ & \dot{\omega}_d - dh + \sum_{k=1}^{d} p_{k-1} dz_k = \\ & \dot{\omega}_d - dh + \sum_{k=1}^{d} p_{k-1}\omega_k - \sum_{k=1}^{d} p_{k-1}(\omega_k - dz_k)\end{aligned} \tag{84}$$

It then follows from Lemma 2.1.*(iv)* and (82),(83),(84) that there exist functions $\beta_1, \cdots, \beta_d$ satisfying

$$d\beta_i \in \text{span}\{dx, du, \cdots, du^{(2n)}\} \ \ (i = 1, \cdots, d) \tag{85}$$

and

$$\dot{\omega}_i = \omega_{i+1} + \beta_i \lambda \ \ (i = 1, \cdots, d-1) \tag{86}$$

$$\dot{\omega}_d = dh - \sum_{k=1}^{d} p_{k-1}\omega_k + \beta_d \lambda \tag{87}$$

Next, consider $\omega \in \mathcal{H}_{n+2}$. Since $\mathcal{H}_{n+2} \subset \mathcal{H}_{n+1}$, there exist functions $\alpha_1, \cdots, \alpha_d$ such that

$$\omega = \sum_{k=1}^{d} \alpha_k(\omega_k - dz_k) \tag{88}$$

By Lemma 2.1.*(iv)*, we have

$$\mathcal{H}^p_{n+1} \ni \dot{\omega} = \sum_{k=1}^{d} (\dot{\alpha}_k(\omega_k - dz_k) + \alpha_k(\dot{\omega}_k - d\dot{z}_k)) \tag{89}$$

and hence

$$\begin{aligned} \mathcal{H}^p_{n+1} \ni \ \ & \sum_{k=1}^{d} \alpha_k(\dot{\omega}_k - d\dot{z}_k) = \sum_{k=1}^{d-1} \alpha_k(\omega_{k+1} + \beta_k\lambda - dz_{k+1}) + \\ & \alpha_d(dh - \sum_{k=1}^{d} p_{k-1}\omega_k + \beta_d\lambda - dh + \sum_{k=1}^{d} p_{k-1}dz_k) = \\ & -\alpha_d p_0(\omega_1 - dz_1) + \sum_{k=2}^{d} (\alpha_{k-1} - p_{k-1}\alpha_d)(\omega_k - dz_k) + \sum_{k=1}^{d} \alpha_k\beta_k\lambda \end{aligned} \tag{90}$$

which gives that $\alpha_1, \cdots, \alpha_d$ have to satisfy

$$\sum_{k=1}^{d} \alpha_k \beta_k = 0 \tag{91}$$

From (85),(91) it then follows that $\mathcal{H}^p_{n+2}$ has the following form:

$$\mathcal{H}^p_{n+2} = \text{span}\{\pi_1^2, \cdots, \pi_{d-1}^2\} \tag{92}$$

where

$$\pi_i^2 = \sum_{k=1}^{d} \gamma_{ik}^2(\omega_k - dz_k) \ \ (i = 1, \cdots, d-1) \tag{93}$$

$$\pi_1^2 \wedge \cdots \wedge \pi_{d-1}^2 \neq 0 \tag{94}$$

and

$$d\gamma_{ik}^2 \in \text{span}\{dx, du, \cdots, du^{(2n)}\} \ \ (i = 1, \cdots, d-1; \ k = 1, \cdots, d) \tag{95}$$

Next, let $\ell \in \{2, \cdots, d - \epsilon\}$, and assume that

$$\mathcal{H}^p_{n+\ell} = \text{span}\{\pi^\ell_1, \cdots, \pi^\ell_{d-\ell+1}\} \tag{96}$$

where

$$\pi^\ell_i = \sum_{k=1}^{d} \gamma^\ell_{ik}(\omega_k - dz_k) \quad (i = 1, \cdots, d - \ell + 1) \tag{97}$$

$$\pi^\ell_1 \wedge \cdots \wedge \pi^\ell_{d-\ell+1} \neq 0 \tag{98}$$

and

$$d\gamma^\ell_{ik} \in \text{span}\{dx, du, \cdots, du^{(2n)}\} \quad (i = 1, \cdots, d - \ell + 1; \; k = 1, \cdots, d) \tag{99}$$

Let $\omega \in \mathcal{H}^p_{n+\ell+1}$. Since $\mathcal{H}^p_{n+\ell+1} \subset \mathcal{H}^p_{n+\ell}$, there exist functions $\alpha_1, \cdots, \alpha_{d-\ell+1}$ such that

$$\omega = \sum_{k=1}^{d-\ell+1} \alpha_k \pi^\ell_k = \sum_{k=1}^{d-\ell+1} \alpha_k (\sum_{i=1}^{d} \gamma^\ell_{ki}(\omega_i - dz_i)) \tag{100}$$

Analogously to (90), we must now have that

$$\begin{aligned} \mathcal{H}^p_{n+\ell} \ni \quad & \sum_{k=1}^{d-\ell+1} \alpha_k (\sum_{i=1}^{d} (\dot{\gamma}^\ell_{ki}(\omega_i - dz_i) + \gamma^\ell_{ki}(\dot{\omega}_i - d\dot{z}_i))) = \\ & \sum_{k=1}^{d-\ell+1} \alpha_k (\sum_{i=1}^{d} \dot{\gamma}^\ell_{ki}(\omega_i - dz_i) + \gamma^\ell_{ki}\beta_i\lambda) \end{aligned} \tag{101}$$

Note that since $\mathcal{H}^p_{n+\ell} \subset \mathcal{H}^p_{n+2}$, we have

$$\sum_{i=1}^{d} \gamma^\ell_{ki}\beta_i = 0 \tag{102}$$

It then follows from (101),(102) that there should exist functions $\delta_1, \cdots, \delta_{d-\ell+1}$ such that

$$\sum_{i=1}^{d} (\sum_{k=1}^{d-\ell+1} (\alpha_k \dot{\gamma}^\ell_{ki} - \delta_k \gamma^\ell_{ki}))(\omega_i - dz_i) = 0 \tag{103}$$

From (99), it follows that

$$d\dot{\gamma}^\ell_{ki} \in \text{span}\{dx, du, \cdots, du^{(2n)}\} \quad (k = 1, \cdots, d - \ell + 1; \; i = 1, \cdots, d) \tag{104}$$

It then follows from (103),(104) that without loss of generality we may assume that also

$$d\alpha_k \in \text{span}\{dx, du, \cdots, du^{(2n)}\} \quad (k = 1, \cdots, d - \ell + 1) \tag{105}$$

which establishes our claim. ∎

and

$$dh = d\dot{\phi}_\epsilon + (p_{d-\epsilon} - \alpha_{\epsilon d-\epsilon})d\phi_1 + \sum_{i=2}^{\epsilon} p_{d-\epsilon+i-1} d\phi_i \tag{123}$$

Equality (122) then gives

$$d\phi_i = d\phi_1^{(i-1)} - \sum_{\ell=1}^{i-1} \alpha_{i-\ell d-\epsilon} d\phi_1^{(\ell-1)} \tag{124}$$

Combining (123) and (124), we obtain

$$\begin{aligned} dh = \; & d\phi_1^{(\epsilon)} - \sum_{\ell=1}^{\epsilon-1} \alpha_{\epsilon-\ell d-\epsilon} d\phi_1^{(\ell)} + (p_{d-\epsilon} - \alpha_{\epsilon d-\epsilon})d\phi_1 + \\ & \sum_{i=2}^{\epsilon} p_{d-\epsilon+i-1}(d\phi_1^{(i-1)} - \sum_{\ell=1}^{i-1} \alpha_{i-\ell d-\epsilon} d\phi_1^{(\ell-1)}) = \cdots = \\ & \sum_{\ell=1}^{\epsilon}(p_{d-\epsilon+\ell-1} - \alpha_{\epsilon-\ell+1 d-\epsilon} - \sum_{i=\ell+1}^{\epsilon} p_{d-\epsilon+i-1}\alpha_{i-\ell d-\epsilon}) d\phi_1^{(\ell-1)} + d\phi_1^{(\epsilon)} \end{aligned} \tag{125}$$

Defining

$$a_k := p_{d-\epsilon+k} - \alpha_{\epsilon-k d-\epsilon} - \sum_{i=k+2}^{\epsilon} p_{d-\epsilon+i-1}\alpha_{i-k-1 d-\epsilon} \quad (k = 0, \cdots, \epsilon - 1) \tag{126}$$

this establishes (30). ∎

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