Note on the approximation of distribution on Z+ by mixtures of negative binomial distributions

Memorandum COSOR 96-13, 1996, Eindhoven University of Technology

Note on the approximation of distributions on $\mathbb{Z}_+$ by mixtures of negative binomial distributions

F.W. Steutel and M.J.A. van Eenige

Abstract

It is shown that the distributions on $\mathbb{Z}_+$ that can be approximated by mixtures of negative binomial distributions, are precisely the so-called Poisson mixtures, i.e., mixtures of Poisson distributions.

1 Introduction

It is well known that every distribution on $\mathbb{R}_+$ is the weak limit of a sequence of distributions with rational Laplace-Stieltjes transforms (LSt's), in fact, of mixtures (with positive weights) of Gamma distributions. More precisely, if $\hat{F}$ is the LSt of a distribution function (df) on $\mathbb{R}_+$, then there are $p_{k,n} > 0$, $\lambda_{k,n} \in (0, \infty)$ and $r_{k,n} \in \mathbb{N}$ such that

$$\hat{F}(s) = \lim_{n\to\infty} \sum_{k=1}^{n} p_{k,n} \left(\frac{\lambda_{k,n}}{\lambda_{k,n} + s} \right)^{r_{k,n}} \tag{1}$$

(see e.g. p. 32 in Schassberger (1973) and pp. 78-79 in Neuts (1981)). An easy proof of (1) consists of the following two facts.

(i) Every df on $\mathbb{R}_+$ can be approximated by step functions, i.e., by mixtures of degenerate df's.

(ii) A degenerate df concentrated at c can be approximated by a Gamma distribution: the law of large numbers implies that

$$\frac{c}{n}(X_1 + \cdots + X_n) \stackrel{w}{\longrightarrow} c,$$

where $X_1, \cdots, X_n$ are i.i.d. and exponentially distributed with mean 1.

A natural question to ask is, 'What distributions on $\mathbb{Z}_+$ can be approximated by mixtures of negative binomial distributions, the analogues of Gamma distributions?'. This question will be answered in the next section.

2 Distributions on $\mathbb{Z}_+$

The analogue of (1) for the generating function P of a distribution on $\mathbb{Z}_+$ would be

$$P(z) = \lim_{n\to\infty} \sum_{k=1}^{n} p_{k,n} \Big(\frac{1-q_{k,n}}{1-q_{k,n}z} \Big)^{r_{k,n}}, \tag{2}$$

with $p_{k,n} > 0$, $q_{k,n} \in (0,1)$ and $r_{k,n} \in \mathbb{N}$. The question is whether (2) holds for all probability generating functions (pgf's).

Looking for analogues of (i) and (ii) in Section 1, we see that (i) is automatically taken care of, but that (ii) does not apply: where for a distribution concentrated at $c \in \mathbb{R}_+$ we have

$$e^{-cs} = \lim_{n\to\infty} \Big(\frac{1}{1+\frac{c}{n}s} \Big)^{n}, \tag{3}$$

the pgf z^k of a distribution concentrated at $k \in \mathbb{N}_+$ cannot be approximated by the right-hand side of (2) for the following reason. Since the function

$$\Big(\frac{1-q}{1-qz} \Big)^{k} = \Big(\frac{1}{1-\lambda(1-z)} \Big)^{k}, \tag{4}$$

with $\lambda = q/(1-q)$, is completely monotone (has alternating derivatives) in $w := 1-z$ for $w \in \mathbb{R}_+$, and complete monotonicity is preserved under the taking of pointwise limits, the right-hand side of (2) is completely monotone. This means that complete monotonicity in $1-z$ is necessary for P to satisfy (2). Since, clearly, the function $z^k = (1-w)^k$ is not completely monotone in w on $\mathbb{R}_+$, it cannot be obtained as (2).

We now turn to the sufficiency of the complete monotonicity in $1-z$. By Bernstein's theorem (Feller (1971)) a function h is completely monotone in s if and only if it can be represented as

$$h(s) = \int_{[0,\infty)} e^{-sx}\, dH(x),$$

where H is nondecreasing. It follows that a pgf P satisfying (2) must be of the form

$$P(z) = \int_{[0,\infty)} e^{-x(1-z)}\, dF(x), \tag{5}$$

where F is a df, i.e., P must be of the form $P(z) = \hat{F}(1-z)$. But then (2) follows from (1), and we find that complete monotonicity of P as a function of $1-z$ is not only necessary to have (2), but also sufficient. Summarizing we have the following result.

Theorem 1 *A distribution* $(p_k)_0^\infty$ *on* $\mathbb{Z}_+$ *can be approximated by mixtures of negative binomial distributions if and only if* (p_k) *is a Poisson mixture, i.e., if and only if the pgf* P *of* (p_k) *satisfies (5).*

3 Remarks and an example

A distribution on $\mathbb{Z}_+$ satisfying equation (5) is called a Poisson mixture; the corresponding random variable, N say, is of the form

$$N \stackrel{d}{=} N(X),$$

where $N(\cdot)$ is a unit Poisson process and X is an $\mathbb{R}_+$-valued random variable independent of $N(\cdot)$. A detailed discussion of Poisson mixtures can be found in Puri and Goldie (1979). Between the moments of N and X we have the following relations.

$$EN = EX, \qquad EN(N-1) = EX^2,$$

and hence

$$\text{var}(N) = EN + \text{var}(X).$$

This means that N is not a Poisson mixture, and hence cannot be approximated by mixtures of negative binomial distributions if $EN > \text{var}(N)$. The example in Section 2 demonstrates this; when $P(N = k) = 1$, then $EN = k > 0 = \text{var}(N)$.

That mixtures of degenerate distributions on $\mathbb{R}_+$ have mixtures of Poisson distributions as their analogues on $\mathbb{Z}_+$ results from the fact that, in many respects, the Poisson distributions themselves are the analogues on $\mathbb{Z}_+$ of the degenerate distributions on $\mathbb{R}_+$. The analogue of (3) is obtained by putting $k = n$, $\lambda = \frac{c}{n}$ in (4) and letting n tend to ∞; this yields the Poisson pgf $exp(c(z-1))$ as a limit. Similarly, the Gamma mixtures are taken into negative binomial mixtures by the following simple transformation.

$$\Big(\frac{1}{1+c(1-z)}\Big)^k = \frac{1}{k!}\int_0^\infty e^{-c(1-z)x}x^k e^{-x}dx.$$

For another analogy between Poisson distributions and degenerate distributions we refer to Steutel and van Harn (1979).

References

FELLER, W. (1971), *An Introduction to Probability Theory and Its Applications*, Volume II, (second ed.), John Wiley & Sons, New York.

NEUTS, M.F. (1981), *Matrix-Geometric Solutions in Stochastic Models: An Algorithmic Approach*, The Johns Hopkins University Press, Baltimore.

PURI, P.S. AND C.M. GOLDIE (1979), Poisson mixtures and quasi-infinite divisibility of distributions, *Journal of Applied Probability* **16**, 138-153.

SCHASSBERGER, R. (1973), *Warteschlangen*, Springer-Verlag, Vienna.

STEUTEL, F.W. AND K. VAN HARN (1979), Discrete analogues of self-decomposability and stability, *Annals of Probability* **7**, 893-899.