Stabilization of irrational transfer functions with internal loop

Stabilization of irrational transfer functions by controllers with internal loop

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Abstract: Transfer functions are called well-posed if they are bounded and analytic on some right half-plane. Two concepts of L^2-stabilization are analyzed for well-posed transfer functions: the usual one, as in Figure 3, and a more general one called stabilization with internal loop, see Figure 4. In both cases we obtain a complete parametrization of all stabilizing controllers in terms of a doubly coprime factorization of the original transfer function. Moreover, the connection between the two stabilization concepts is clarified. We analyze two special subclasses of stabilizing controllers with internal loop, called canonical and dual canonical controllers. We show that, in a certain sense, every stabilizing controller with internal loop is equivalent to a canonical controller, and also to a dual canonical controller.

1. An intriguing control problem

We consider the following problem: For an unstable plant with transfer function

$$\mathbf{P}(s) = 3\frac{s+1}{s-1},$$

design a controller that performs the following two tasks:
(1) it stabilizes $\mathbf{P}$,

(2) even if an external (unpredictable) disturbance d is added to the input u of the plant, the controller *predicts exactly* the output y of the plant one second ahead. In other words, at some auxiliary output of our controller we can at any moment of time t read off the value $y(t+1)$.

This might seem to be impossible at first sight since, as we said, d is not predictable. The problem is illustrated in Figure 1.

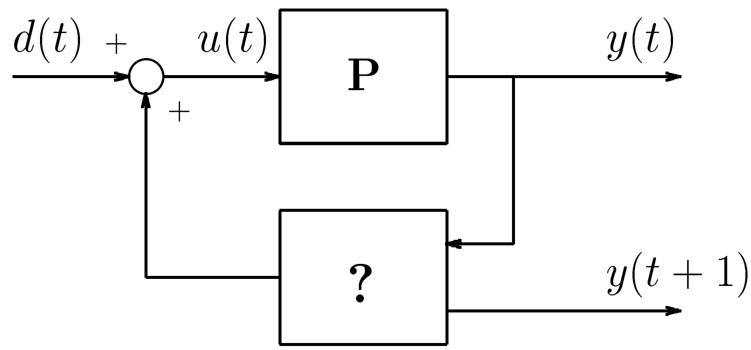

Figure 1: A seemingly impossible problem

Let us connect our plant to a system consisting of a delay line and a summation point, as shown in Figure 2. This interconnection has two inputs and two outputs: one input is the disturbance d and one output is the output y of the plant. The other input is denoted by ζ_i and the other output is denoted by ζ_o. Realizing this interconnection changes hardly anything: the plant still works in open loop, with the input $u = d + \zeta_i$. The transfer function from ζ_i to ζ_o is $\mathbf{H}(s) = 1 - e^{-s} + \mathbf{P}(s)$.

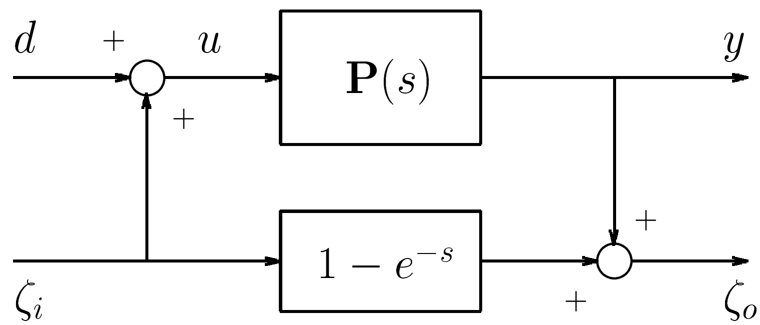

Figure 2: The plant and the controller, before closing the feedback loop

Now let us see what happens if we close a feedback loop from ζ_o to ζ_i, i.e., we set $\zeta_i = \zeta_o$. Closing this loop is possible, because $1 - \mathbf{H}$ is invertible and moreover, its inverse is in H^∞. We invite the reader to verify that now $\zeta_o(t) = y(t+1)$ and the overall system is well-posed and stable. By this we mean that we may inject additional input signals at any point, and the transfer function from any such input signal to any other signal will be in H^∞. Thus, we have solved our original problem.

We emphasize the importance of proceeding in stages: first connecting the device to the plant, and only afterwards closing the feedback loop. Proceeding in the wrong order, i.e., trying to connect ζ_i to ζ_o first, will not work: it is an ill-posed feedback. In closed loop, the transfer function from y to ζ_o is e^s, but realizing this transfer function in open loop (i.e., without being connected to the plant) is clearly impossible. This is an example of a so-called *controller with internal loop*, introduced in Weiss and Curtain [17]. We study the structure of such controllers in the next sections, and we return to this example at the end of Section 4.

2. Stabilization with internal loop

In this paper, we investigate the concept of a stabilizing controller, as well as the problem of parametrizing all stabilizing controllers for a well-posed (possibly irrational) transfer function. Our analysis is entirely in the frequency domain.

Throughout this paper, U and Y are Hilbert spaces, called the *input space* and the *output space*, respectively. An $\mathcal{L}(U,Y)$-valued transfer function is called *well-posed* if it is bounded and analytic on some right half-plane (where $\mathrm{Re}\ s > \alpha$). This class of functions is a natural generalization of the proper rational transfer functions and moreover, it coincides with the transfer functions of well-posed linear systems, a class for which there exists a well developed state space theory (see Salamon [7, 8], Staffans [10, 11, 12], Weiss [15, 16] and Weiss and Rebarber [18]). We do not distinguish between two transfer functions defined on two different right half-planes if one function is a restriction of the other (thus, by a transfer function we mean, in fact, an equivalence class of analytic functions). We mention that matrix-valued transfer functions with entries in the factor space of H^∞, used for example in Georgiou and Smith [6] and in Foias, Özbay and Tannenbaum [4], are neither contained in, nor do they contain the matrix-valued well-posed transfer functions.

For any Banach space Z, we denote by $H^\infty(Z)$ the Banach space of Z-valued bounded analytic functions on the usual right half-plane (where $\mathrm{Re}\, s > 0$), with the sup norm. We write H^∞ instead of $H^\infty(Z)$, if the space Z is clear from the context. An $\mathcal{L}(U,Y)$-valued transfer function $\mathbf{P}$ is called *stable* if $\mathbf{P} \in H^\infty(\mathcal{L}(U,Y))$. This concept of stability is also called *L^2-stability* (or *input-output stability*), because it is equivalent to the property that input functions $u \in L^2([0,\infty),U)$ are mapped into output functions $y \in L^2([0,\infty),Y)$, via the formula $\hat{y} = \mathbf{P}\hat{u}$, where a hat is used to indicate the Laplace transformation.

Definition 2.1. Assume that $\mathbf{P}$ and $\mathbf{C}$ are well-posed transfer functions with values in $\mathcal{L}(U,Y)$ and $\mathcal{L}(Y,U)$, respectively. We say that $\mathbf{C}$ is an *admissible feedback transfer function* for $\mathbf{P}$ if $I - \mathbf{CP}$ (or equivalently $I - \mathbf{PC}$) has a well-posed inverse. In particular, we say that $\mathbf{C}$ *stabilizes* $\mathbf{P}$ if

$$\begin{bmatrix} I & -\mathbf{C} \\ -\mathbf{P} & I \end{bmatrix}^{-1} \in H^\infty(\mathcal{L}(U \times Y)).$$

The intuitive interpretation of the last condition can be given by the block diagram in Figure 3, where the transfer function from $\begin{bmatrix} v_p \\ v_c \end{bmatrix}$ to $\begin{bmatrix} u_p \\ u_c \end{bmatrix}$ is

$$\mathbf{T_{P,C}} = \begin{bmatrix} I & -\mathbf{C} \\ -\mathbf{P} & I \end{bmatrix}^{-1} = \begin{bmatrix} (I - \mathbf{CP})^{-1} & \mathbf{C}(I - \mathbf{PC})^{-1} \\ \mathbf{P}(I - \mathbf{CP})^{-1} & (I - \mathbf{PC})^{-1} \end{bmatrix}. \tag{2.1}$$

A few words about the connection with state space theory: If **P** and **C** are the transfer functions of two well-posed linear systems and their feedback connection is exponentially stable, then **C** stabilizes **P**, as is easy to see. The more surprizing fact is that the converse is true as well, provided that the two systems satisfy certain natural assumptions, such as regularity, stabilizability and detectability, see Section 4 of Weiss and Curtain [17]. Recently, these assumptions have been replaced by the weaker conditions of optimizability and detectability (without assuming regularity), see Section 6 of Weiss and Rebarber [18]. For the state space theory of coprime factorizations we refer to Staffans [10] and to our paper [3].

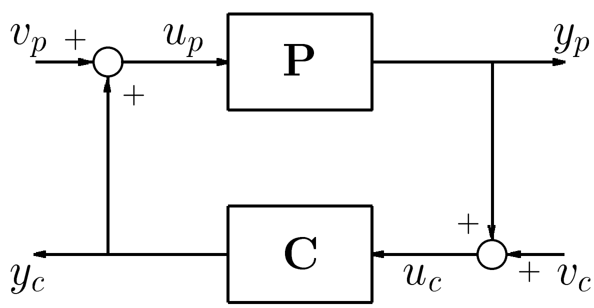

Figure 3: The feedback connection of **P** and **C**

While extending the theory of dynamic stabilization to regular linear systems (a subclass of the well-posed linear systems) in [17], it became apparent that a more general concept of stabilizing controller is needed. Indeed, it was shown in Example 6.5 of [17] that even the standard observer-based controller is not, in general, a stabilizing controller in the usual sense; that is, it is not a well-posed linear system and, correspondingly, its transfer function is not well-posed. However, the observer-based controller may be regarded in a natural way as a stable well-posed linear system with two inputs and two outputs. First we connect one input and one output to the plant, obtaining a new well-posed linear system. Next we connect the remaining input of the controller to its remaining output, so obtaining a stable closed-loop system. For the details of this procedure see [17].

Motivated by this fact, a new type of controller was introduced, the so-called *stabilizing controller with internal loop*, see Definition 4.10 in [17]. This was later used in [3], Townley *et al* [13] and [18]. We now formulate the frequency domain counterpart of this concept. In fact, we introduce two new concepts, which generalize the two concepts from Definition 2.1.

Definition 2.2. Let U, Y and R be Hilbert spaces. Let $\mathbf{P}$ and $\mathbf{K} = \begin{bmatrix} \mathbf{K}_{11} & \mathbf{K}_{12} \\ \mathbf{K}_{21} & \mathbf{K}_{22} \end{bmatrix}$ be well-posed transfer functions with values in $\mathcal{L}(U,Y)$ and in $\mathcal{L}(Y\times R, U\times R)$, respectively, and denote $\mathbf{G} = \begin{bmatrix} \mathbf{P} & 0 \\ 0 & \mathbf{K} \end{bmatrix}$. We say that $\mathbf{K}$ is an *admissible feedback transfer function with internal loop for* $\mathbf{P}$ if $I - F\mathbf{G}$ has a well-posed inverse, where

$$F = \begin{bmatrix} 0 & I & 0 \\ I & 0 & 0 \\ 0 & 0 & I \end{bmatrix}, \qquad F \in \mathcal{L}(Y\times U\times R, U\times Y\times R). \tag{2.2}$$

In particular, we say that $\mathbf{K}$ *stabilizes* $\mathbf{P}$ *with internal loop* if

$$(I - F\mathbf{G})^{-1} \in H^\infty(\mathcal{L}(U\times Y\times R)).$$

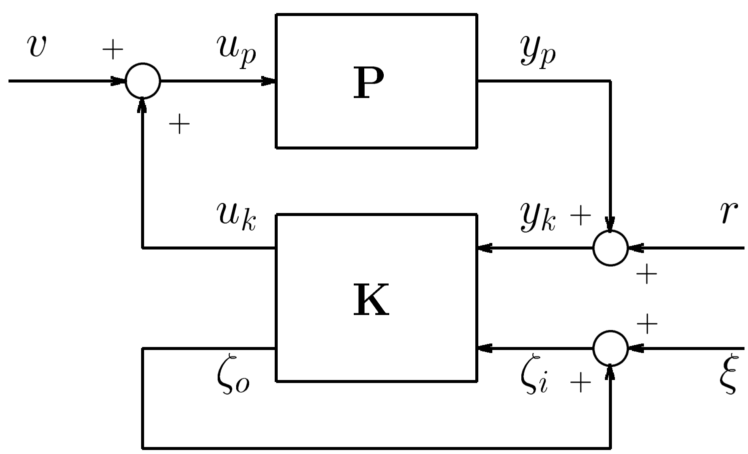

Figure 4: The plant $\mathbf{P}$ connected to a controller $\mathbf{K}$ with internal loop

The intuitive interpretation of Definition 2.2 is the following: $\mathbf{P}$ represents the plant and $\mathbf{K}$ is the transfer function of the controller in Figure 4 from $\begin{bmatrix} y_k \\ \zeta_i \end{bmatrix}$ to $\begin{bmatrix} u_k \\ \zeta_o \end{bmatrix}$, when all the connections are open. The connection from ζ_o to ζ_i is the so-called *internal loop*. The admissibility part of the definition means that it is possible to close all the connections in Figure 4, and obtain well-posed transfer functions from the three external inputs (v, r and ξ) to all the other signals. The stabilization part of the definition means that in fact all these closed-loop transfer functions are stable. To understand that this is indeed what the definition says, it is helpful to redraw Figure 4 by clearly indicating the role of F as a feedback operator for $\mathbf{G}$, the parallel connection of $\mathbf{P}$ and $\mathbf{C}$, as is done in Figure 5.

If $I - \mathbf{K}_{22}$ has a well-posed inverse, then the two concepts introduced in Definition 2.2 reduce to those introduced in Definition 2.1. Indeed, in this case the internal

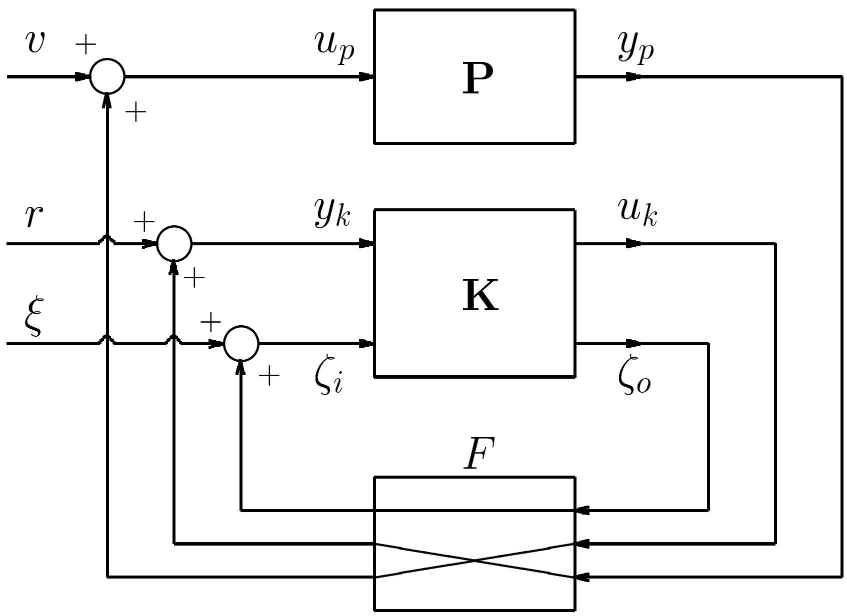

Figure 5: The parallel connection of $\mathbf{P}$ and $\mathbf{K}$ with the feedback operator F. This diagram is equivalent to the one in Figure 4.

loop can be closed first (i.e., before connecting the controller to the plant), to obtain a conventional controller with the following transfer function from y_k to u_k:

$$\mathbf{C} = \mathbf{K}_{11} + \mathbf{K}_{12}(I - \mathbf{K}_{22})^{-1}\mathbf{K}_{21}\,. \tag{2.3}$$

In many cases, the controller $\mathbf{C}$ defined by (2.3) will not be well-posed in the sense of our definition: for example, it could be improper or even anticausal. However, the internal loop construction is not merely an artifice for allowing improper or anticausal controllers. It includes controllers for which the expresssion (2.3) *is not defined at all* (this can happen if $I - \mathbf{K}_{22}$ is nowhere invertible). A finite-dimensional single input-single output (SISO) and physically meaningful example of a stabilizing controller with internal loop for which $\mathbf{C}$ does not exist was given in Section 1 of [17]. Another, infinite-dimensional SISO example follows in this section.

The connection of the stabilization concept in Definition 2.2 to the state space theory of stabilizing controllers with internal loop (see Definition 4.10 in [17]) is similar to the one we explained for the stabilization concept in Definition 2.1: If $\mathbf{P}$ is the transfer function of a well-posed plant and $\mathbf{K}$ is the transfer function of a stabilizing controller with internal loop for the same plant, then $\mathbf{K}$ stabilizes $\mathbf{P}$ with internal loop. The converse holds whenever the two systems satisfy certain natural assumptions, see Proposition 4.11 in [17] (where the product $F\mathbf{G}$ is denoted by $\mathbf{L}$). A stronger version of this result from [17] is Theorem 6.4 in [18], where the assumptions are optimizability and estimatability (but not regularity).

For the remainder of this paper, if $\mathbf{P}$ and $\mathbf{K}$ are as in Definition 2.2, we shall call $\mathbf{K}$ a *stabilizing controller with internal loop for* $\mathbf{P}$. This clearly contradicts the terminology of [17], since $\mathbf{K}$ is only the transfer function of a controller, and not the controller itself. However, since our discussion here is only in terms of transfer functions, this will (hopefully) not lead to confusions.

Example 2.3. We take $U = Y = R = \mathbb{C}$ and

$$\mathbf{P}(s) = \frac{2}{1+e^{-2s}}\,, \qquad \mathbf{K} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}.$$

It is easy to see that the transfer function (2.3) of the controller is undefined since $I - \mathbf{K}_{22} = 0$. However, it is not difficult to check that $\mathbf{K}$ stabilizes $\mathbf{P}$ with internal loop (this verification can be simplified considerably by using Theorem 4.2). The plant is unstable: $\mathbf{P}$ has infinitely many poles on the imaginary axis. All these facts can be understood intuitively from the following physical interpretation.

Consider a circuit composed of a unit resistor connected in series with a lossless transmission line of unit length, with unit distributed capacitance and unit distributed inductance. One end of the transmission line is earthed, as shown in Figure 6. The input of this system is the current through the resistor denoted by $u(t)$, whereas the observation is chosen to be the input voltage of the circuit $y(t)$.

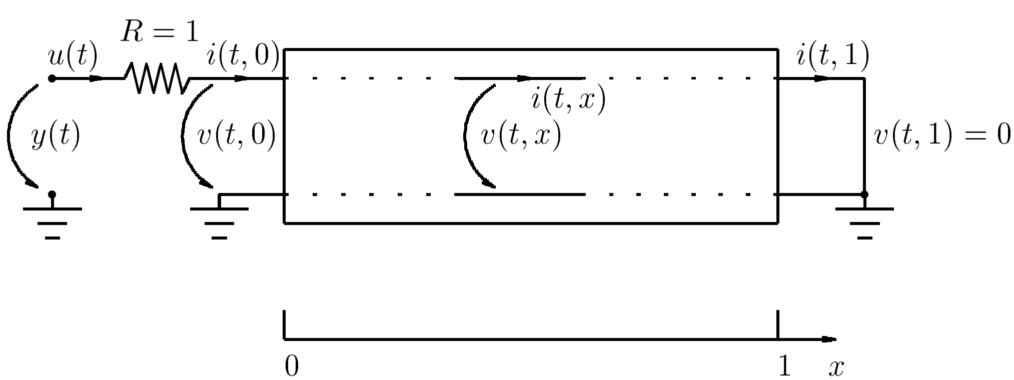

Figure 6: Transmission line with resistor

The local current $i(t,x)$ through and the local voltage $v(t,x)$ across the transmission line are related by the equations

$$\frac{\partial i}{\partial t} = -\frac{\partial v}{\partial x}\,, \quad \frac{\partial v}{\partial t} = -\frac{\partial i}{\partial x}\,, \tag{2.4}$$

with the boundary conditions

$$\begin{aligned} i(t,0) &= u(t), \\ v(t,1) &= 0 \quad \text{(because of the earthing)}, \\ y(t) &= u(t) + v(t,0) \quad (u(t) \text{ is added because of the resistor}). \end{aligned}$$

It is readily verified that the transfer function from u to y, the impedance of the circuit, is $\mathbf{P}(s) = \frac{2}{1+e^{-2s}}$. The infinite number of poles on the imaginary axis can be explained by the fact that disconnecting the input of the circuit, i.e., $u(t) = 0$, creates infinitely many undamped oscillatory modes on the line, due to the lossless nature of the line. The only dissipative element, the resistor, cannot dissipate the energy of these oscillations if no current traverses it. A simple interpretation of the proposed controller is the earthing of the input of the circuit. This causes $y(t) = 0$, something that is clearly not achievable by a conventional controller. It is clear that the system does become stable under the new boundary condition, since by earthing the input, the resistor will now be able to dissipate the energy from the circuit. All these intuitive conclusions will be justified rigorously in Section 2.

Another motivation for introducing controllers with internal loop is to obtain a clean and elegant Youla parametrization. In the Youla parametrization of conventional controllers, if the plant is not strictly proper, it is difficult to see a priori how to choose the parameter in such a way that the resulting controller will be well-posed. Even if we choose to ignore well-posedness, as some researchers do, we still have to ensure that the denominator in the Youla parametrization is invertible. Thus, the parametrization is not clean; there is always an extra condition on the parameter. This in turn makes it awkward to use this parametrization to solve, for example, the H^∞-control problem for well-posed linear systems. By contrast, we do obtain a clean parametrization for all stabilizing canonical or dual canonical controllers. A stabilizing controller with internal loop $\mathbf{K}$ is called *canonical* if $\mathbf{K}_{11} = 0$, $\mathbf{K}_{12} = I$ and $\mathbf{K}_{21}, \mathbf{K}_{22} \in H^\infty$. It is called *dual canonical* if $\mathbf{K}_{11} = 0$, $\mathbf{K}_{21} = I$ and $\mathbf{K}_{12}, \mathbf{K}_{22} \in H^\infty$. We shall prove that, in a certain sense, it is sufficient to consider canonical or dual canonical controllers (Theorem 5.2).

The organization of the paper is as follows. In Section 3, we recall the usual Youla parameterization of all stabilizing controllers in terms of a doubly coprime factorization (cf. Francis [5], Vidyasagar [14] and Baras [1]). However, since we consider the input and output spaces to be Hilbert spaces, certain technical difficulties arise and we actually need to redo the proofs for this situation. We also give two examples to illustrate the invertibility problems with this parametrization.

In Section 4 we introduce canonical and dual canonical controllers. We show that a plant $\mathbf{P}$ is stabilizable with internal loop by a canonical (dual canonical) controller if and only if $\mathbf{P}$ has a right-coprime (left-coprime) factorization. We give a complete parameterization of all (dual) canonical stabilizing controllers with internal loop. Moreover, the relationship between (conventional) stabilization and stabilization with internal loop by a (dual) canonical controller is clarified.

In Section 5 we introduce the concept of equivalence for stabilizing controllers with internal loop. We show that if the plant $\mathbf{P}$ has a doubly coprime factorization, then any stabilizing controller with internal loop for $\mathbf{P}$ is equivalent to a canonical one as well as to a dual canonical one.

3. Coprime factorization

This section contains more or less well-known material on doubly coprime factorizations, only the statements are slightly different, and the proofs are a little more involved, because we consider operator-valued functions. Thus, we must be careful to distinguish between right and left invertibility, to assume the existence of coprime factorizations, etc. The reader who is not interested in such technicalities may skip to the examples at the end of the section.

Definition 3.1. Let $\mathbf{P}$ be a well-posed transfer function. A *right-coprime factorization of* $\mathbf{P}$ *over* H^∞ is an ordered set of four operator-valued functions M, N, $\tilde{R}$, $\tilde{S}$ in H^∞ such that M has a well-posed inverse, $\mathbf{P} = NM^{-1}$, and $\tilde{S}M - \tilde{R}N = I$.

A *left-coprime factorization of* $\mathbf{P}$ *over* H^∞ is an ordered set of four operator-valued functions $\tilde{M}$, $\tilde{N}$, R, S in H^∞ such that $\tilde{M}$ has a well-posed inverse, $\mathbf{P} = \tilde{M}^{-1}\tilde{N}$, and $\tilde{M}S - \tilde{N}R = I$.

A *doubly coprime factorization of* $\mathbf{P}$ *over* H^∞ is an ordered set of eight operator-valued functions M, N, R, S, $\tilde{M}$, $\tilde{N}$, $\tilde{R}$, $\tilde{S}$ in H^∞ such that, on some right half-plane, M and $\tilde{M}$ have well-posed inverses, $\mathbf{P}$ has the factorizations

$$\mathbf{P} \;=\; NM^{-1} \;=\; \tilde{M}^{-1}\tilde{N}, \tag{3.1}$$

and

$$\begin{bmatrix} M & R \\ N & S \end{bmatrix}^{-1} = \begin{bmatrix} \tilde{S} & -\tilde{R} \\ -\tilde{N} & \tilde{M} \end{bmatrix}. \tag{3.2}$$

If a left- and a right-coprime factorization of $\mathbf{P}$ are given, as in (3.1), then it is easy to construct $R, S, \tilde{R}, \tilde{S} \in H^\infty$ such that we have a doubly coprime factorization, as in (3.2), see Theorem 60 in Vidyasagar [14] or Lemma 4.3 in Staffans [10]. For a matrix-valued function, (3.2) is often written in the form

$$\begin{bmatrix} \tilde{S} & -\tilde{R} \\ -\tilde{N} & \tilde{M} \end{bmatrix}\begin{bmatrix} M & R \\ N & S \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}. \tag{3.3}$$

Indeed, for matrix-valued functions, (3.3) is equivalent to (3.2), but in general it is weaker than (3.2). Most arguments about doubly coprime factorizations rely on (3.3), but occasionally we need the factors from (3.3) in reversed order, see for example the proof of Theorem 3.4.

Notice that Definition 3.1 extends easily to transfer functions that are not necessarily well-posed, but are matrix-valued with components in the quotient field of H^∞, denoted by H^∞/H^∞. Smith [9] showed that a transfer function in this class is stabilizable if and only if it has left- and right-coprime factorizations. For operator-valued transfer functions, no similar result seems to be known. If a matrix-valued well-posed transfer function has a left- or a right-coprime factorization, or if it is stabilizable, then obviously its entries are in H^∞/H^∞.

Explicit formulas for coprime factorizations in terms of state space realizations are known for a wide class of infinite-dimensional systems called regular linear systems, see our paper [3]. More general but (unavoidably) somewhat less explicit formulas for well-posed linear systems were given by Staffans [10]. A modest, but useful formula in terms of transfer functions is the following.

Proposition 3.2. *Let* $\mathbf{P}$ *and* $\mathbf{C}$ *be well-posed transfer functions with values in* $\mathcal{L}(U,Y)$ *and* $\mathcal{L}(Y,U)$ *respectively. If*

$$\mathbf{C} \in H^\infty \qquad \text{and} \qquad \mathbf{P}(I-\mathbf{CP})^{-1} \in H^\infty\,, \tag{3.4}$$

then denoting

$$\mathbf{L} = \begin{bmatrix} \mathbf{C} \\ I \end{bmatrix} \mathbf{P}(I-\mathbf{CP})^{-1}[I \;\; -\mathbf{C}],$$

the following is a doubly coprime factorization of $\mathbf{P}$*:*

$$\begin{bmatrix} M & R \\ N & S \end{bmatrix} = I + \mathbf{L}\,, \qquad \begin{bmatrix} \tilde{S} & -\tilde{R} \\ -\tilde{N} & \tilde{M} \end{bmatrix} = I - \mathbf{L}\,. \tag{3.5}$$

Proof. It is easy to see that $\mathbf{L}^2 = 0$, so that $(I+\mathbf{L})(I-\mathbf{L}) = (I-\mathbf{L})(I+\mathbf{L}) = I$, which is (3.2). To check (3.1), we note that $M = (I-\mathbf{CP})^{-1}$ and $N = \mathbf{P}(I-\mathbf{CP})^{-1}$, whence $\mathbf{P} = NM^{-1}$. The identity $\mathbf{P} = \tilde{M}^{-1}\tilde{N}$ is verified similarly. ■

Lemma 3.3. *Let* $\mathbf{P}$ *and* $\mathbf{C}$ *be well-posed transfer functions with values in* $\mathcal{L}(U,Y)$ *and* $\mathcal{L}(Y,U)$ *respectively and suppose that* $\mathbf{P}$ *has a left coprime factorization* $\mathbf{P} = \tilde{M}^{-1}\tilde{N}$ *and* $\mathbf{C}$ *has a right-coprime factorization* $\mathbf{C} = WV^{-1}$*. Then* $\mathbf{C}$ *stabilizes* $\mathbf{P}$ *if and only if* $\Delta = \tilde{M}V - \tilde{N}W$ *is invertible over* H^∞*.*

There is an obvious dual statement in which right is replaced by left.

Proof. Let R, S, $\tilde{R}_c$, $\tilde{S}_c$ be such that

$$\tilde{M}S - \tilde{N}R = I \qquad \text{and} \qquad \tilde{S}_c W - \tilde{R}_c V = I\,. \tag{3.6}$$

Notice that $\Delta = \tilde{M}V - \tilde{N}W = \tilde{M}(I-\mathbf{PC})V$.

Suppose first that $\mathbf{C}$ stabilizes $\mathbf{P}$. Then clearly Δ^{-1} exists and

$$\Delta^{-1} = V^{-1}(I-\mathbf{PC})^{-1}\tilde{M}^{-1}.$$

It is readily verified that, with the notation from (2.1),

$$\begin{aligned} \mathbf{T}_{\mathbf{P},\mathbf{C}} &= \begin{bmatrix} I+\mathbf{C}(I-\mathbf{PC})^{-1}\mathbf{P} & \mathbf{C}(I-\mathbf{PC})^{-1} \\ (I-\mathbf{PC})^{-1}\mathbf{P} & (I-\mathbf{PC})^{-1} \end{bmatrix} \\ &= \begin{bmatrix} I+W\Delta^{-1}\tilde{N} & W\Delta^{-1}\tilde{M} \\ V\Delta^{-1}\tilde{N} & V\Delta^{-1}\tilde{M} \end{bmatrix} \end{aligned} \tag{3.7}$$

and since $\mathbf{T}_{\mathbf{P},\mathbf{C}} \in H^\infty$, all components are in H^∞.

Now post-multiply the second identity in (3.6) by $\Delta^{-1}\tilde{M}$ to obtain

$$\tilde{S}_c W \Delta^{-1}\tilde{M} - \tilde{R}_c V \Delta^{-1}\tilde{M} \;=\; \Delta^{-1}\tilde{M},$$

which shows that $\Delta^{-1}\tilde{M} \in H^\infty$. Similarly, post-multiplying the second identity in (3.6) by $\Delta^{-1}\tilde{N}$ shows that $\Delta^{-1}\tilde{N} \in H^\infty$. Next, pre-multiplying the first identity in (3.6) by Δ^{-1} gives

$$\Delta^{-1}\tilde{M}S - \Delta^{-1}\tilde{N}R \;=\; \Delta^{-1}$$

and hence $\Delta^{-1} \in H^\infty$.

Conversely, if $\Delta^{-1} \in H^\infty$, then we see from $\Delta = \tilde{M}(I - \mathbf{PC})V$ that $(I - \mathbf{PC})^{-1}$ exists, and (3.7) shows that $\mathbf{C}$ stabilizes $\mathbf{P}$. ■

The following theorem is the well known Youla-Bongiorno parameterization.

Theorem 3.4. *Suppose that the well-posed transfer function* $\mathbf{P}$ *has a doubly coprime factorization over* H^∞*, as in* (3.1) and (3.2)*.*

(1) *Assume that* $\mathbf{C}$ *is a well-posed transfer function that stabilizes* $\mathbf{P}$*. Then* $\mathbf{C}$ *has a right-coprime factorization over* H^∞ *if and only if* $\mathbf{C}$ *has a left-coprime factorization over* H^∞*. In this case, there exists a unique* $Q \in H^\infty$ *such that* $S+NQ$ *and* $\tilde{S} + Q\tilde{N}$ *have well-posed inverses and*

$$\mathbf{C} = (R + MQ)(S + NQ)^{-1} = (\tilde{S} + Q\tilde{N})^{-1}(\tilde{R} + Q\tilde{M}) \tag{3.8}$$

are right- and left-coprime factorizations of $\mathbf{C}$*.*

(2) *Conversely, if* $Q \in H^\infty$ *is such that one of* $S + NQ$ *or* $\tilde{S} + Q\tilde{N}$ *has a well-posed inverse, then the other also has a well-posed inverse and the equality in* (3.8) *holds. In this case, the transfer function* $\mathbf{C}$ *defined in* (3.8) *stabilizes* $\mathbf{P}$*.*

Proof. We prove part (1). Suppose that $\mathbf{C}$ stabilizes $\mathbf{P}$ and $\mathbf{C} = WV^{-1}$ is a right-coprime factorization. We show that $\mathbf{C}$ can be represented as in the first factorization from (3.8). By Lemma 3.3, $\Delta = \tilde{M}V - \tilde{N}W$ is invertible over H^∞, so

$$Q \;=\; M^{-1}(W\Delta^{-1} - R) \tag{3.9}$$

is well defined and, moreover,

$$R + MQ \;=\; W\Delta^{-1}. \tag{3.10}$$

We calculate

$$\begin{aligned}
S + NQ &= S + NM^{-1}(W\Delta^{-1} - R) && \text{(from (3.9))}\\
&= S + \tilde{M}^{-1}\tilde{N}(W\Delta^{-1} - R) && \text{(since } \tilde{M}^{-1}\tilde{N} = NM^{-1}\text{)}\\
&= \tilde{M}^{-1}(\tilde{M}S + \tilde{N}W\Delta^{-1} - \tilde{N}R) &&\\
&= \tilde{M}^{-1}(I + \tilde{N}W\Delta^{-1}) && \text{(from (3.2))}\\
&= \tilde{M}^{-1}(\tilde{M}V - \tilde{N}W + \tilde{N}W)\Delta^{-1} && \text{(by the definition of } \Delta\text{).}
\end{aligned}$$

Thus

$$S + NQ \;=\; V\Delta^{-1} \tag{3.11}$$

and $S + NQ$ has a well-posed inverse, since V has one and $\Delta \in H^\infty$.

Furthermore, from (3.10) and (3.11) we obtain

$$\begin{aligned}\mathbf{C} &= WV^{-1} = W\Delta^{-1}\Delta V^{-1} \\ &= (R + MQ)(S + NQ)^{-1} .\end{aligned}$$

We have to prove that $Q \in H^\infty$. This follows from

$$\begin{aligned}Q &= (\tilde{S}M - \tilde{R}N)Q && \text{(from (3.2))} \\ &= \tilde{S}(W\Delta^{-1} - R) - \tilde{R}(V\Delta^{-1} - S) && \text{(from (3.10) and (3.11))} \\ &= (\tilde{S}W - \tilde{R}V)\Delta^{-1} && \text{(from (3.2))}.\end{aligned}$$

Next, we show that $\tilde{S} + Q\tilde{N}$ has a well-posed inverse and the two factorizations in (3.8) are equal. From (3.3), pre-multiplying by $\begin{bmatrix} I & -Q \\ 0 & I \end{bmatrix}$ and post-multiplying by $\begin{bmatrix} I & Q \\ 0 & I \end{bmatrix}$, we obtain

$$\begin{bmatrix} \tilde{S} + Q\tilde{N} & -\tilde{R} - Q\tilde{M} \\ -\tilde{N} & \tilde{M} \end{bmatrix} \begin{bmatrix} M & R + MQ \\ N & S + NQ \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}. \tag{3.12}$$

By reversing the order of the factors in (3.3) and using a similar device we can obtain a version of (3.12) with the factors in the reverse order. This proves that $X = \begin{bmatrix} \tilde{S} + Q\tilde{N} & -\tilde{R} - Q\tilde{M} \\ -\tilde{N} & \tilde{M} \end{bmatrix}$ has a well-posed inverse. (Note that this does not follow from (3.12) alone, since we work on infinite-dimensional spaces.)

Using (3.3), (3.7), the definition of Δ and the expression $Q = (\tilde{S}W - \tilde{R}V)\Delta^{-1}$, it is readily verified that

$$X\mathbf{T}_{\mathbf{P},\mathbf{C}} = \begin{bmatrix} \tilde{S} + Q\tilde{N} & 0 \\ 0 & \tilde{M} \end{bmatrix}. \tag{3.13}$$

Since X and $\mathbf{T}_{\mathbf{P},\mathbf{C}}$ have well-posed inverses, $\tilde{S} + Q\tilde{N}$ must have one. The (1,2) block of (3.12) shows that the two factorizations in (3.8) are equal. Now we have to prove the coprimeness of the factorizations in (3.8). The following identity which follows from (3.2) shows that $R + MQ$ and $S + NQ$ are right-coprime:

$$\tilde{M}(S + NQ) - \tilde{N}(R + MQ) = I. \tag{3.14}$$

The fact that the other factorization is left-coprime is proved similarly.

Finally, to prove the uniqueness of Q, first notice that Q from (3.9) is independent of the choice of the factors in the right-coprime factorization $\mathbf{C} = WV^{-1}$. Assume that there is a $Q_1 \in H^\infty$ such that $\mathbf{C} = (R + MQ_1)(S + NQ_1)^{-1}$. Denote $W_1 = R + MQ_1$ and $V_1 = S + NQ_1$ to obtain $\Delta_1 = \tilde{M}V_1 - \tilde{N}W_1 = I$. Substituting this into (3.9) shows that $Q = Q_1$. Thus we have proved part (1), assuming the existence of a right-coprime factorization for $\mathbf{C}$. A similar argument proves (1) if we start from the assumption that $\mathbf{C}$ has a left-coprime factorization.

To prove part (2), we assume that $Q \in H^\infty$ is such that $S+NQ$ has a well-posed inverse, and we define $\mathbf{C} = (R+MQ)(S+NQ)^{-1}$. The identity (3.14) obtained from (3.2) shows that this factorization of $\mathbf{C}$ is right-coprime. We apply Lemma 3.3 with $V = S+NQ$, $W = R+MQ$ and $\Delta = I$ to conclude that $\mathbf{C}$ stabilizes $\mathbf{P}$. The remaining statements in part (2) follow from part (1). If we start from a $Q \in H^\infty$ such that $\tilde{S} + Q\tilde{N}$ has a well-posed inverse, the proof of part (2) is similar. ■

Let us apply this theorem to the transfer function $\mathbf{P}$ from Example 2.3.

Example 3.5. A doubly coprime factorization of $\mathbf{P}(s) = \frac{2}{1+e^{-2s}}$ is given by $N(s) = \tilde{N}(s) = 2$, $M(s) = \tilde{M}(s) = 1+e^{-2s}$, $S(s) = \tilde{S}(s) = 1$, $R(s) = \tilde{R}(s) = \frac{1}{2}e^{-2s}$. The Youla parametrization (3.8) yields the controller

$$\mathbf{C}(s) = \left(\frac{1}{2}e^{-2s} + (1+e^{-2s})Q(s)\right)(1+2Q(s))^{-1}$$

for any $Q \in H^\infty$, and introducing $\tilde{Q} = 1+2Q$ we obtain the more transparent form

$$\mathbf{C}(s) = \frac{1}{2}(1+e^{-2s}) - \frac{1}{2}\tilde{Q}^{-1}. \tag{3.15}$$

The above formula yields well-posed controllers provided that $\tilde{Q}$ has a well-posed inverse. On the other hand, the controller suggested in Example 2.3 from physical considerations is not accounted for by (3.15) (see also Example 4.7).

In the above example, it was easy to see which parameters $\tilde{Q}$ should be chosen to obtain a well-posed controller. However, in general it is not a priori clear which $Q \in H^\infty$ will guarantee that $S+NQ$ has a well-posed inverse. We illustrate this point by another example.

Example 3.6. This is the frequency domain version of Example 6.5 in [17]. We take $U = Y = \ell^2$, the space of square summable sequences $x = (x_1, x_2, \ldots)$. The transfer function of the plant is $\mathbf{P}(s) = \operatorname{diag}(P_n(s))$, $n = 1,2,3,\ldots$, where

$$P_n(s) = \frac{(0.64n-1)s - n}{s(s+n)}.$$

This has poles of multiplicity one at $s = -n$, $n = 1,2,3,\ldots$, a pole of infinite multiplicity at $s = 0$ and $\mathbf{P}$ is well-posed. An observer-based controller for this plant has the transfer function $\mathbf{C}(s) = \operatorname{diag}(C_n(s))$, where

$$C_n(s) = \frac{(1-0.64n)s+n}{s^2 + s(2-0.28n)+2n},$$

as computed in [17]. At first glance this seems fine; $C_n(s)$ is a strictly proper function for every n. However, the real parts of the poles of $C_n(s)$ converge to $+\infty$, as is easy to see. Consequently, $\mathbf{C}(s)$ cannot be uniformly bounded in norm on any right half-plane and it is therefore not well-posed.

One can use this observer-based controller to obtain a doubly coprime factorization $M = \tilde{M} = \text{diag}\,(M_n)$, $N = \tilde{N} = \text{diag}\,(N_n)$, $R = \tilde{R} = \text{diag}\,(R_n)$, $S = \tilde{S} = \text{diag}\,(S_n)$, with $N_n(s) = \tilde{N}_n(s) = \frac{(0.64n-1)s-n}{s^2+s(1+0.36n)+2n}$, $M_n = \tilde{M}_n = 1 + N_n$, $S_n = \tilde{S}_n = 1 - N_n$, $R_n = \tilde{R}_n = -N_n$. According to the Youla parametrization (3.8) a set of stabilizing controllers for $\mathbf{P}$ is given by $\mathbf{C}(s) = \text{diag}\,(C_n(s))$, where

$$C_n = (R_n + M_n Q)(S_n + N_n Q)^{-1}, \tag{3.16}$$

and $Q \in H^\infty$ is such that $(S + NQ)^{-1}$ is well-posed. As we have seen in the case of the observer-based controller ($Q = 0$), this is not always the case. With some care, we can choose the parameter $Q \in H^\infty$ such that $\mathbf{C}$ is well-posed. For example, $Q(s) = 1$ yields $\mathbf{C}(s) = I_{\ell^2}$. However, it is not a priori clear which $Q \in H^\infty$ will lead to a well-posed $\mathbf{C}$.

4. Canonical and dual canonical controllers

In [17], a procedure was developed to design stabilizing controllers with internal loop (in the state-space framework, assuming regularity, stabilizability and detectability). The transfer functions of the controllers obtained there were of the form

$$\mathbf{K} = \begin{bmatrix} 0 & I \\ \mathbf{K}_{21} & \mathbf{K}_{22} \end{bmatrix}, \quad \text{with } \mathbf{K}_{21}, \mathbf{K}_{22} \in H^\infty. \tag{4.1}$$

Since this class plays a special role, we call the controllers of the form (4.1) *canonical controllers*. Analogously, controllers of the form

$$\mathbf{K} = \begin{bmatrix} 0 & \mathbf{K}_{12} \\ I & \mathbf{K}_{22} \end{bmatrix}, \quad \text{with } \mathbf{K}_{12}, \mathbf{K}_{22} \in H^\infty, \tag{4.2}$$

will be called *dual canonical controllers* (they can be obtained by a dual design procedure, as explained in Section 5 of [17]).

We analyze the properties of (dual) canonical controllers in some detail. First we recall Proposition 4.8 from [17].

Proposition 4.1. *Using the notation of Definition* 2.2*, suppose that* $\mathbf{K}_{11}$ *is an admissible feedback transfer function for* $\mathbf{P}$*. Then the following two conditions are equivalent:*

(1) $I - \mathcal{F}(\mathbf{K}, \mathbf{P})$ *has a well-posed inverse, where*

$$\mathcal{F}(\mathbf{K}, \mathbf{P}) = \mathbf{K}_{22} + \mathbf{K}_{21}\mathbf{P}(I - \mathbf{K}_{11}\mathbf{P})^{-1}\mathbf{K}_{12}.$$

(2) $\mathbf{K}$ *is an admissible feedback transfer function with internal loop for* $\mathbf{P}$*.*

In fact, Proposition 4.8 from [17] only states that (1) implies (2), but the converse is also true, with practically the same proof.

Theorem 4.2. *The canonical controller* $\mathbf{K}$ *stabilizes* $\mathbf{P}$ *with internal loop iff*

$$\Delta = I - \mathbf{K}_{22} - \mathbf{K}_{21}\mathbf{P}$$

is invertible on some right half-plane and Δ^{-1}, $\mathbf{P}\Delta^{-1} \in H^\infty$.

If $\mathbf{P}$ *has a right-coprime factorization* $\mathbf{P} = NM^{-1}$, *then* $\mathbf{K}$ *stabilizes* $\mathbf{P}$ *with internal loop iff* $D = M - \mathbf{K}_{22}M - \mathbf{K}_{21}N$ *is invertible over* H^∞.

Note that Δ and Δ^{-1} might be defined only on some right half-plane of the form $\mathrm{Re}\, s > \alpha$ with $\alpha > 0$, but this is not a problem if Δ^{-1} has an analytic extension to a function in H^∞ (see our convention at the beginning of Section 2).

Proof. First we prove the first statement. According to Proposition 4.1, $I - F\mathbf{G}$ has a well-posed inverse iff $I - \mathcal{F}(\mathbf{K}, \mathbf{P}) = \Delta$ has a well-posed inverse. Moreover,

$$(I - F\mathbf{G})^{-1} = \begin{bmatrix} I + \Delta^{-1}\mathbf{K}_{21}\mathbf{P} & \Delta^{-1}\mathbf{K}_{21} & \Delta^{-1} \\ \mathbf{P}(I + \Delta^{-1}\mathbf{K}_{21}\mathbf{P}) & I + \mathbf{P}\Delta^{-1}\mathbf{K}_{21} & \mathbf{P}\Delta^{-1} \\ \Delta^{-1}\mathbf{K}_{21}\mathbf{P} & \Delta^{-1}\mathbf{K}_{21} & \Delta^{-1} \end{bmatrix}$$

and so $(I - F\mathbf{G})^{-1} \in H^\infty$ iff the expressions

$$\Delta^{-1}, \ \mathbf{P}\Delta^{-1}, \ \Delta^{-1}\mathbf{K}_{21}\mathbf{P}, \ \mathbf{P}(I + \Delta^{-1}\mathbf{K}_{21}\mathbf{P})$$

are all in H^∞. We show that if the first two of these expressions are in H^∞, then so are the other two. We have

$$\begin{aligned} I + \Delta^{-1}\mathbf{K}_{21}\mathbf{P} &= \Delta^{-1}(\Delta + \mathbf{K}_{21}\mathbf{P}) \\ &= \Delta^{-1}(I - \mathbf{K}_{22}) \in H^\infty \end{aligned}$$

and

$$\mathbf{P}(I + \Delta^{-1}\mathbf{K}_{21}\mathbf{P}) = \mathbf{P}\Delta^{-1}(I - \mathbf{K}_{22}) \in H^\infty.$$

Thus, $\mathbf{K}$ stabilizes $\mathbf{P}$ iff $\Delta^{-1} \in H^\infty$ and $\mathbf{P}\Delta^{-1} \in H^\infty$.

Let us prove the second assertion in the theorem. If $\mathbf{P} = NM^{-1}$ and $D^{-1} \in H^\infty$, then from the formulae

$$\Delta^{-1} = MD^{-1}\ , \quad \mathbf{P}\Delta^{-1} = ND^{-1},$$

we see that $\Delta^{-1}, \mathbf{P}\Delta^{-1} \in H^\infty$. Conversely, assuming that M, N, $\tilde{R}$ and $\tilde{S}$ are as in Definition 3.1 and $\Delta^{-1}, \mathbf{P}\Delta^{-1} \in H^\infty$, then

$$\tilde{S}\Delta^{-1} - \tilde{R}\mathbf{P}\Delta^{-1} = (\tilde{S}M - \tilde{R}N)(M - \mathbf{K}_{22}M - \mathbf{K}_{21}N)^{-1} = D^{-1}.$$

Since $\tilde{S}$ and $\tilde{R}$ are in H^∞, we see that $D^{-1} \in H^\infty$. ■

We have the following interesting connection with right-coprime factorizations.

Corollary 4.3. $\mathbf{P}$ *has a right-coprime factorization if and only if* $\mathbf{P}$ *is stabilizable with internal loop by a canonical controller.*

Proof. If $\mathbf{K}$ stabilizes $\mathbf{P}$, then by Theorem 4.2, Δ^{-1} and $\mathbf{P}\Delta^{-1}$ are in H^∞. So $M = \Delta^{-1}$ and $N = \mathbf{P}\Delta^{-1}$ is a right-coprime factorization, since

$$(I - \mathbf{K}_{22})M - \mathbf{K}_{21}N = I.$$

Conversely, if $\mathbf{P} = NM^{-1}$ is a right-coprime factorization such that $\tilde{S}M - \tilde{R}N = I$, then $\mathbf{K} = \begin{bmatrix} 0 & I \\ \tilde{R} & I - \tilde{S} \end{bmatrix}$ stabilizes $\mathbf{P}$ with internal loop. Indeed, from $\Delta = I - (I - \tilde{S}) - \tilde{R}NM^{-1} = (\tilde{S}M - \tilde{R}N)M^{-1} = M^{-1}$, we see that $\Delta^{-1} = M$ and $\mathbf{P}\Delta^{-1} = N$, and we can apply Theorem 4.2. ■

Proposition 4.4. *Suppose that* $\mathbf{P}$ *has a doubly coprime factorization as in* (3.1) *and* (3.2). *Then all canonical controllers that stabilize* $\mathbf{P}$ *with internal loop are parameterized by*

$$\mathbf{K} = \begin{bmatrix} 0 & I \\ \tilde{E}(\tilde{R} + Q\tilde{M}) & I - \tilde{E}(\tilde{S} + Q\tilde{N}) \end{bmatrix},$$

where $Q, \tilde{E} \in H^\infty$ *and* $\tilde{E}$ *is invertible over* H^∞.

Proof. For $\mathbf{K}$ as above we show that it stabilizes $\mathbf{P}$ with internal loop. We compute, using (3.2),

$$\begin{aligned} D &= M - \mathbf{K}_{22}M - \mathbf{K}_{21}N \\ &= \tilde{E}(\tilde{S} + Q\tilde{N})M - \tilde{E}(\tilde{R} + Q\tilde{M})N \\ &= \tilde{E}(\tilde{S}M - \tilde{R}N + Q(\tilde{N}M - \tilde{M}N)) = \tilde{E}. \end{aligned}$$

Since $\tilde{E}$ is invertible over H^∞, Theorem 4.2 shows that $\mathbf{K}$ stabilizes $\mathbf{P}$.

Conversely, suppose that $\mathbf{K} = \begin{bmatrix} 0 & I \\ \mathbf{K}_{21} & \mathbf{K}_{22} \end{bmatrix}$ stabilizes $\mathbf{P}$ with internal loop. Theorem 4.2 shows that $D = M - \mathbf{K}_{22}M - \mathbf{K}_{21}N$ is invertible over H^∞. Defining

$$Q = (D^{-1}\mathbf{K}_{21} - \tilde{R})\tilde{M}^{-1}$$

we evaluate

$$\begin{aligned} \tilde{S} + Q\tilde{N} &= \tilde{S} + D^{-1}\mathbf{K}_{21}\tilde{M}^{-1}\tilde{N} - \tilde{R}\tilde{M}^{-1}\tilde{N} \\ &= (\tilde{S}M - \tilde{R}N)M^{-1} + D^{-1}\mathbf{K}_{21}NM^{-1} \\ &= D^{-1}(\mathbf{K}_{21}N + D)M^{-1} \\ &= D^{-1}(M - \mathbf{K}_{22}M)M^{-1} \\ &= D^{-1}(I - \mathbf{K}_{22}) \end{aligned}$$

and

$$\tilde{R} + Q\tilde{M} = \tilde{R} + (D^{-1}\mathbf{K}_{21} - \tilde{R}) = D^{-1}\mathbf{K}_{21}.$$

Thus, denoting $\tilde{E} = D$, we have that $\mathbf{K}_{21} = \tilde{E}(\tilde{R}+Q\tilde{M})$ and $\mathbf{K}_{22} = I - \tilde{E}(\tilde{S}+Q\tilde{N})$ as claimed. It remains to prove that $Q \in H^\infty$. From (3.2) and our formulae for $\tilde{R} + Q\tilde{M}$ and $\tilde{S} + Q\tilde{N}$, we have

$$\begin{aligned} Q &= Q(\tilde{M}S - \tilde{N}R) \\ &= (D^{-1}\mathbf{K}_{21} - \tilde{R})S - [D^{-1}(I - \mathbf{K}_{22}) - \tilde{S}]R \\ &= D^{-1}\mathbf{K}_{21}S - D^{-1}(I - \mathbf{K}_{22})R, \end{aligned}$$

and it is clear that both terms are in H^∞. ■

We call the function Q appearing in Proposition 4.4, the *Youla parameter* of $\mathbf{K}$. We mention that by using (one half of) Proposition 4.4, it is possible to give an alternative, shorter proof of Theorem 5.4 (the main result) in [17].

The following corollary contains the dual statements of Theorem 4.2, Corollary 4.3 and Proposition 4.4.

Corollary 4.5. *Let $\mathbf{P}$ be a well-posed transfer function.*

a. The dual canonical controller $\mathbf{K}$ stabilizes $\mathbf{P}$ with internal loop iff $\Delta = I - \mathbf{K}_{22} - \mathbf{P}\mathbf{K}_{12}$ is invertible on some right half-plane and Δ^{-1}, $\Delta^{-1}\mathbf{P} \in H^\infty$.

b. If $\mathbf{P}$ has a left-coprime factorization $\mathbf{P} = \tilde{M}^{-1}\tilde{N}$, then $\mathbf{K}$ stabilizes $\mathbf{P}$ with internal loop iff $D = \tilde{M} - \tilde{M}\mathbf{K}_{22} - \tilde{N}\mathbf{K}_{12}$ is invertible over H^∞.

c. $\mathbf{P}$ has a left-coprime factorization if and only if $\mathbf{P}$ is stabilizable with internal loop by a dual canonical controller.

d. If $\mathbf{P}$ has a doubly coprime factorization as in (3.1) *and* (3.2)*, then all dual canonical controllers which stabilize $\mathbf{P}$ with internal loop are given by*

$$\mathbf{K} = \begin{bmatrix} 0 & (R + MQ)E \\ I & I - (S + NQ)E \end{bmatrix},$$

where $Q, E \in H^\infty$ and E is invertible over H^∞.

Again, we call Q appearing in Corollary 4.5 the *Youla parameter* of $\mathbf{K}$. The formulae for the parameterization of all stabilizing (dual) canonical controllers are reminiscent of the Youla parameterization (3.8). Consequently, we expect a strong relationship between stabilization with internal loop and the usual concept of stabilization. This is clarified in the following proposition.

Proposition 4.6. *Let $\mathbf{P}$ and $\mathbf{C}$ be well-posed transfer functions with values in $\mathcal{L}(U, Y)$ and $\mathcal{L}(Y, U)$, respectively. Assume that $\mathbf{C}$ has a left-coprime factorization $\mathbf{C} = (I - \mathbf{K}_{22})^{-1}\mathbf{K}_{21}$. Then $\mathbf{P}$ is stabilized by $\mathbf{C}$ iff $\mathbf{P}$ is stabilized with internal loop by the canonical controller* $\mathbf{K} = \begin{bmatrix} 0 & I \\ \mathbf{K}_{21} & \mathbf{K}_{22} \end{bmatrix}$.

Proof. Suppose that $\mathbf{C} = (I - \mathbf{K}_{22})^{-1}\mathbf{K}_{21}$ stabilizes $\mathbf{P}$. Define

$$\begin{aligned} \Delta &= I - \mathbf{K}_{22} - \mathbf{K}_{21}\mathbf{P} \\ &= (I - \mathbf{K}_{22})(I - \mathbf{CP}). \end{aligned}$$

Then, according to Definition 2.1 and formula (2.1),

$$\Delta^{-1}\mathbf{K}_{21} = (I - \mathbf{CP})^{-1}\mathbf{C} \in H^\infty,$$

$$\Delta^{-1}(I - \mathbf{K}_{22}) = (I - \mathbf{CP})^{-1} \in H^\infty.$$

Since $\mathbf{K}_{21}$ and $I - \mathbf{K}_{22}$ are left-coprime, there exist R and S in H^∞ such that $(I - \mathbf{K}_{22})S - \mathbf{K}_{21}R = I$, whence

$$\Delta^{-1}(I - \mathbf{K}_{22})S - \Delta^{-1}\mathbf{K}_{21}R = \Delta^{-1}.$$

Since both terms on the left-hand side are in H^∞, we see that $\Delta^{-1} \in H^\infty$.

Similarly, $\mathbf{P}\Delta^{-1} \in H^\infty$ follows from coprimeness and the facts that

$$\mathbf{P}\Delta^{-1}(I - \mathbf{K}_{22}) = \mathbf{P}(I - \mathbf{CP})^{-1} \in H^\infty,$$

$$\mathbf{P}\Delta^{-1}\mathbf{K}_{21} = \mathbf{P}(I - \mathbf{CP})^{-1}\mathbf{C} \in H^\infty.$$

According to Theorem 4.2, $\mathbf{K}$ stabilizes $\mathbf{P}$ with internal loop.

Conversely, suppose that $\mathbf{K}$ stabilizes $\mathbf{P}$ with internal loop. Then, from

$$\begin{aligned} (I - \mathbf{CP})^{-1} &= \Delta^{-1}(I - \mathbf{K}_{22}) \\ \mathbf{P}(I - \mathbf{CP})^{-1} &= \mathbf{P}\Delta^{-1}(I - \mathbf{K}_{22}) \\ (I - \mathbf{CP})^{-1}\mathbf{C} &= \Delta^{-1}\mathbf{K}_{21} \\ (I - \mathbf{PC})^{-1} &= I + \mathbf{P}(I - \mathbf{CP})^{-1}\mathbf{C} = I + \mathbf{P}\Delta^{-1}\mathbf{K}_{21} \end{aligned}$$

and from Theorem 4.2 we see that all entries of $\mathbf{T}_{\mathbf{P},\mathbf{C}}$ from (2.1) are in H^∞. ■

Notice that if a canonical $\mathbf{K}$ stabilizes $\mathbf{P}$ with internal loop, then $\mathbf{K}_{21}$ and $I - \mathbf{K}_{22}$ are left-coprime, since $(I - \mathbf{K}_{22})\Delta^{-1} - \mathbf{K}_{21}\mathbf{P}\Delta^{-1} = I$. Naturally, Proposition 4.6 has a dual statement for right-coprime factorizations of $\mathbf{C}$.

An interesting feature of the last result is that it remains true without the requirement that $\mathbf{C}$ be well-posed, if we modify Definitions 2.1 and 3.1 accordingly, replacing everywhere well-posedness by analyticity.

We complete this section by returning to two earlier examples.

Example 4.7. Using the results in Example 3.5 and Proposition 4.4, we obtain that all canonical controllers that stabilize $\mathbf{P}(s) = \frac{2}{1+e^{-2s}}$ are

$$\mathbf{K}(s) = \begin{bmatrix} 0 & 1 \\ \frac{1}{2}\tilde{E}(s)\left[-1 + (1 + e^{-2s})\tilde{Q}(s)\right] & 1 - \tilde{E}(s)\tilde{Q}(s) \end{bmatrix},$$

where $\tilde{E}, \tilde{E}^{-1}, \tilde{Q} \in H^\infty$. (We have used, as in Example 3.5, $\tilde{Q} = 1 + 2Q$.) If $\tilde{Q}^{-1}$ is well-posed, then we obtain the controllers (3.15) from Example 3.5, and if $\tilde{E}(s) = 2$ and $\tilde{Q} = 0$, then we obtain the controller suggested in Example 2.3.

Example 4.8. The plant in the "intriguing control problem" of Section 1 and its controller are determined by the following transfer functions:

$$\mathbf{P}(s) = 3\frac{s+1}{s-1}, \qquad \mathbf{K}(s) = \begin{bmatrix} 0 & 1 \\ 1 & 1-e^{-s} \end{bmatrix}.$$

Note that $\mathbf{K}$ is canonical (and also dual canonical). To check that $\mathbf{K}$ stabilizes $\mathbf{P}$ with internal loop, we use Theorem 4.2. We compute $\Delta = e^{-s} - 3\frac{s+1}{s-1}$, so that

$$\Delta^{-1} = -\frac{1}{3} \cdot \frac{s-1}{(1-\frac{e^{-s}}{3})s + (1+\frac{e^{-s}}{3})}.$$

First we show that Δ^{-1} has no poles in the closed right half-plane. Indeed, if such a pole p were to exist, it would have to satisfy

$$p = -\frac{1+\frac{e^{-p}}{3}}{1-\frac{e^{-p}}{3}}.$$

But for Re $p \geq 0$, the right-hand side above is in the open left half-plane, which is a contradiction. Thus, Δ^{-1} is analytic on an open set containing the closed right half-plane. To prove that $\Delta^{-1} \in H^\infty$, it only remains to show that it does not behave badly at infinity. It is easy to see that $|\Delta^{-1}(s)| < 1$ for $|s|$ sufficiently large (we consider only s with Re $s > 0$). The fact that $\mathbf{P}\Delta^{-1} \in H^\infty$ follows similarly. Computing the transfer function from d to y and from d to ζ_o, we see that the first is equal to the second delayed by 1 (i.e., multiplied by e^{-s}).

5. Reduction to a (dual) canonical controller

In this section, we show that the canonical controllers are not as special as they seem; any stabilizing controller with internal loop is, in a certain sense, equivalent to a canonical controller. To do this, let us analyze the connection in Figure 4 in more detail. We use the auxiliary inputs v and r from Figure 4, so that $u_p = u_k + v$ and $y_k = y_p + r$, while assuming that $\xi = 0$. If $\mathbf{K}$ stabilizes $\mathbf{P}$ with internal loop, then we can define the transfer function $\mathbf{T}_{\mathbf{P},\mathbf{K}} \in H^\infty$ by

$$\begin{bmatrix} \hat{u}_p \\ \hat{y}_k \end{bmatrix} = \mathbf{T}_{\mathbf{P},\mathbf{K}} \begin{bmatrix} \hat{v} \\ \hat{r} \end{bmatrix}.$$

We call $\mathbf{T}_{\mathbf{P},\mathbf{K}}$ the *compensation operator* of the pair $(\mathbf{P}, \mathbf{K})$. This is analogous to the transfer function $\mathbf{T}_{\mathbf{P},\mathbf{C}}$ introduced in (2.1). In fact, $\mathbf{T}_{\mathbf{P},\mathbf{K}}$ is the left upper 2×2 corner of the matrix $(I - F\mathbf{G})^{-1}$ from Definition 2.2.

Definition 5.1. Let $\mathbf{P}$ be well-posed and suppose that $\mathbf{K}_1$ and $\mathbf{K}_2$ are stabilizing controllers with internal loop for $\mathbf{P}$. We call $\mathbf{K}_1$ and $\mathbf{K}_2$ *equivalent* if the compensation operators of $(\mathbf{P}, \mathbf{K}_1)$ and $(\mathbf{P}, \mathbf{K}_2)$ are equal.

The main result of this section is the following.

Theorem 5.2. *Suppose that* $\mathbf{P}$ *has a doubly coprime factorization. If* $\mathbf{K}$ *is a stabilizing controller with internal loop for* $\mathbf{P}$*, then there exists a canonical controller* $\tilde{\mathbf{K}}$ *such that* $\tilde{\mathbf{K}}$ *stabilizes* $\mathbf{P}$ *with internal loop and* $\tilde{\mathbf{K}}$ *is equivalent to* $\mathbf{K}$*.*

Proof. We use the notation from (3.1) and (3.2). By Proposition 4.4, for every $Q \in H^\infty$, the following $\tilde{\mathbf{K}}$ stabilizes $\mathbf{P}$ with internal loop:

$$\tilde{\mathbf{K}} = \begin{bmatrix} 0 & I \\ \tilde{R} + Q\tilde{M} & I - (\tilde{S} + Q\tilde{N}) \end{bmatrix}. \tag{5.1}$$

We shall find a Youla parameter $Q \in H^\infty$ such that the compensation operator of $(\mathbf{P}, \tilde{\mathbf{K}})$ coincides with that of $(\mathbf{P}, \mathbf{K})$. First, we explain how this Q will be constructed. We consider the extended transfer function

$$\tilde{\mathbf{P}} = \begin{bmatrix} -M^{-1}R & M^{-1} \\ \tilde{M}^{-1} & \mathbf{P} \end{bmatrix}.$$

We denote the inputs of $\tilde{\mathbf{P}}$ by $\begin{bmatrix} \tilde{w}_p \\ \tilde{u}_p \end{bmatrix}$ and its outputs by $\begin{bmatrix} \tilde{v}_p \\ \tilde{y}_p \end{bmatrix}$, as in Figure 7.

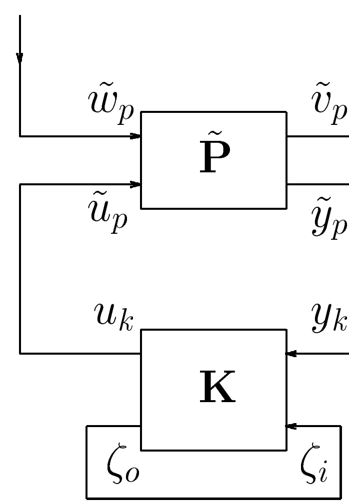

Figure 7: The extended system $\tilde{\mathbf{P}}$ connected to the controller $\mathbf{K}$ with internal loop. The transfer function from $\tilde{w}_p$ to $\tilde{v}_p$ is the desired Youla parameter.

We shall prove that if $\mathbf{K}$ stabilizes $\mathbf{P}$, then the interconnection of $\tilde{\mathbf{P}}$ and $\mathbf{K}$ shown in Figure 7 is stable. Then we take Q to be the H^∞ transfer function from $\tilde{w}_p$ to $\tilde{v}_p$ in the interconnection in Figure 7. There are two assertions that have to be proven in order to complete the proof of the theorem:

Assertion 1: If $\mathbf{K}$ stabilizes $\mathbf{P}$, then the system in Figure 7 is stable.

Assertion 2: If Q is chosen to be the transfer function from $\tilde{w}_p$ to $\tilde{v}_p$ in the interconnection in Figure 7, and $\tilde{\mathbf{K}}$ is given by (5.1), then $\mathbf{T}_{\mathbf{P},\mathbf{K}} = \mathbf{T}_{\tilde{\mathbf{P}},\tilde{\mathbf{K}}}$.

Proof of Assertion 1: Suppose that $\mathbf{K}$ stabilizes $\mathbf{P}$ with internal loop. According to Definition 2.2, this means that the parallel connection of $\mathbf{P}$ and $\mathbf{K}$ as independent channels, denoted by $\mathbf{G}$ in the definition, is such that $(I - F\mathbf{G})^{-1} \in H^\infty$. Since F is invertible, this is equivalent to the fact that $\mathbf{G}$ is stabilized by the static output feedback

$$\begin{bmatrix} u_p \\ y_k \\ \zeta_i \end{bmatrix} = \begin{bmatrix} 0 & I & 0 \\ I & 0 & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} y_p \\ u_k \\ \zeta_o \end{bmatrix} = F \begin{bmatrix} y_p \\ u_k \\ \zeta_o \end{bmatrix}.$$

We have to prove that the parallel connection of $\tilde{\mathbf{P}}$ and $\mathbf{K}$,

$$\tilde{\mathbf{G}} = \begin{bmatrix} \tilde{\mathbf{P}} & 0 \\ 0 & \mathbf{K} \end{bmatrix} = \begin{bmatrix} -M^{-1}R & M^{-1} & 0 & 0 \\ \tilde{M}^{-1} & \mathbf{P} & 0 & 0 \\ 0 & 0 & \mathbf{K}_{11} & \mathbf{K}_{12} \\ 0 & 0 & \mathbf{K}_{21} & \mathbf{K}_{22} \end{bmatrix},$$

is stabilized by the static output feedback

$$\begin{bmatrix} \tilde{w}_p \\ \tilde{u}_p \\ y_k \\ \zeta_i \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \begin{bmatrix} \tilde{v}_p \\ \tilde{y}_p \\ u_k \\ \zeta_o \end{bmatrix} = \tilde{F} \begin{bmatrix} \tilde{v}_p \\ \tilde{y}_p \\ u_k \\ \zeta_o \end{bmatrix}.$$

Equivalently, we have to prove that the four transfer functions from (2.1), $(I - \tilde{\mathbf{G}}\tilde{F})^{-1}$, $(I - \tilde{\mathbf{G}}\tilde{F})^{-1}\tilde{\mathbf{G}}$, $\tilde{F}(I - \tilde{\mathbf{G}}\tilde{F})^{-1}$ and $(I - \tilde{F}\tilde{\mathbf{G}})^{-1}$, are in H^∞. Because $\tilde{F}$ is a constant transfer function, it is enough to prove that the first two are in H^∞, and the other two will follow immediately (for the last one, use the identity $(I - \tilde{F}\tilde{\mathbf{G}})^{-1} = I + \tilde{F}(I - \tilde{\mathbf{G}}\tilde{F})^{-1}\tilde{\mathbf{G}}$). Notice that

$$\tilde{\mathbf{G}} = \left[\begin{array}{c|c} -M^{-1}R & \begin{bmatrix} M^{-1} & 0 & 0 \end{bmatrix} \\ \hline \begin{bmatrix} \tilde{M}^{-1} \\ 0 \\ 0 \end{bmatrix} & \mathbf{G} \end{array}\right]$$

and

$$\tilde{F} = \left[\begin{array}{c|c} 0 & \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \\ \hline \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} & F \end{array}\right].$$

It is easy to see that

$$(I - \tilde{\mathbf{G}}\tilde{F})^{-1} = \left[\begin{array}{c|c} I & \begin{bmatrix} 0 & M^{-1} & 0 \end{bmatrix}(I - \mathbf{G}F)^{-1} \\ \hline \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} & (I - \mathbf{G}F)^{-1} \end{array}\right].$$

Since F stabilizes $\mathbf{G}$, $(I-\mathbf{G}F)^{-1}$ is in H^∞. Therefore, $(I-\tilde{\mathbf{G}}\tilde{F})^{-1}$ is in H^∞, if we show that $\begin{bmatrix} 0 & M^{-1} & 0 \end{bmatrix}(I-\mathbf{G}F)^{-1}$ is in H^∞. Let us denote by T_{jk} the constituent blocks of $(I-\mathbf{G}F)^{-1}$, i.e.,

$$(I-\mathbf{G}F)\begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{13} & T_{23} & T_{33} \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}.$$

¿From the upper row of this equality, we deduce that

$$T_{11} - I = \mathbf{P}T_{21}\ , \quad T_{12} = \mathbf{P}T_{22}\ , \quad T_{13} = \mathbf{P}T_{23}\ .$$

Recall from (3.2) that $\tilde{S}M - \tilde{R}N = I$ and compute

$$\begin{aligned} \tilde{S}T_{21} - \tilde{R}(T_{11}-I) &= \tilde{S}T_{21} - \tilde{R}NM^{-1}T_{21} \\ &= (\tilde{S}M - \tilde{R}N)M^{-1}T_{21} \\ &= M^{-1}T_{21}. \end{aligned} \tag{5.2}$$

Similarly,

$$\tilde{S}T_{22} - \tilde{R}T_{12} = \tilde{S}T_{22} - \tilde{R}NM^{-1}T_{22} = M^{-1}T_{22} \tag{5.3}$$

and

$$\tilde{S}T_{23} - \tilde{R}T_{13} = \tilde{S}T_{23} - \tilde{R}NM^{-1}T_{23} = M^{-1}T_{23}. \tag{5.4}$$

This shows that

$$\begin{bmatrix} 0 & M^{-1} & 0 \end{bmatrix}(I-\mathbf{G}F)^{-1} = M^{-1}\begin{bmatrix} T_{21} & T_{22} & T_{23} \end{bmatrix} \in H^\infty.$$

So we have shown that $(I-\tilde{\mathbf{G}}\tilde{F})^{-1}$ is in H^∞.

Let us now show that $(I-\tilde{\mathbf{G}}\tilde{F})^{-1}\tilde{\mathbf{G}}$ is in H^∞. We can compute, using the notation introduced before, that

$$(I-\tilde{\mathbf{G}}\tilde{F})^{-1}\tilde{\mathbf{G}} =$$

$$= \left[\begin{array}{c|c} -M^{-1}R + M^{-1}T_{21}\tilde{M}^{-1} & \begin{bmatrix} M^{-1} & 0 & 0 \end{bmatrix} + M^{-1}\begin{bmatrix} T_{21} & T_{22} & T_{23} \end{bmatrix}\mathbf{G} \\ \hline (I-\mathbf{G}F)^{-1}\begin{bmatrix} \tilde{M}^{-1} \\ 0 \\ 0 \end{bmatrix} & (I-\mathbf{G}F)^{-1}\mathbf{G} \end{array}\right]. \tag{5.5}$$

Since F stabilizes $\mathbf{G}$, we have that $(I-\mathbf{G}F)^{-1}\mathbf{G}$ is in H^∞. We have to check that the other three blocks in (5.5) are in H^∞. For this, we consider the identity

$$\begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{13} & T_{23} & T_{33} \end{bmatrix}(I-\mathbf{G}F) = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}.$$

¿From the second row and the second column of the previous identity, we deduce

$$T_{12} = T_{11}\mathbf{P} \ , \quad T_{22} - I = T_{21}\mathbf{P} \ , \quad T_{32} = T_{31}\mathbf{P} \ , \tag{5.6}$$

$$T_{22}\mathbf{K}_{11} + T_{23}\mathbf{K}_{21} = T_{21} \ , \quad T_{22}\mathbf{K}_{12} + T_{23}(\mathbf{K}_{22} - I) = 0 \ . \tag{5.7}$$

We can now proceed to show that the remaining three blocks in (5.5) are in H^∞.

Block (2,1): We have to prove that

$$(I - \mathbf{G}F)^{-1} \begin{bmatrix} \tilde{M}^{-1} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} T_{11}\tilde{M}^{-1} \\ T_{21}\tilde{M}^{-1} \\ T_{31}\tilde{M}^{-1} \end{bmatrix} \in H^\infty.$$

Recall from (3.2) that $\tilde{M}S - \tilde{N}R = I$ and use the identities (5.6) to compute

$$\begin{aligned} T_{11}S - T_{12}R &= T_{11}S - T_{11}\mathbf{P}R \\ &= T_{11}\tilde{M}^{-1}(\tilde{M}S - \tilde{N}R) = T_{11}\tilde{M}^{-1}, \end{aligned}$$

and similarly

$$\begin{aligned} T_{21}S + (I - T_{22})R &= T_{21}S - T_{21}\mathbf{P}R \\ &= T_{21}\tilde{M}^{-1}(\tilde{M}S - \tilde{N}R) = T_{21}\tilde{M}^{-1}, \end{aligned}$$

$$T_{31}S - T_{32}R = T_{31}S - T_{31}\mathbf{P}R = T_{31}\tilde{M}^{-1}.$$

These identities show that block (2,1) in (5.5) is in H^∞.

Block (1,1): We have, using identity (5.2) and (3.2), that

$$\begin{aligned} -M^{-1}R + M^{-1}T_{21}\tilde{M}^{-1} &= -M^{-1}R + (\tilde{S}T_{21} - \tilde{R}T_{11} + \tilde{R})\tilde{M}^{-1} \\ &= \tilde{S}T_{21}\tilde{M}^{-1} - \tilde{R}T_{11}\tilde{M}^{-1}. \end{aligned}$$

As we previously proved, $T_{21}\tilde{M}^{-1}$ and $T_{11}\tilde{M}^{-1}$ are in H^∞, and so block (1,1) in (5.5) is in H^∞.

Block (1,2): A little computation shows that block (2,1) is equal to

$$\begin{bmatrix} M^{-1} + M^{-1}T_{21}\mathbf{P} & M^{-1}(T_{22}\mathbf{K}_{11} + T_{23}\mathbf{K}_{21}) & M^{-1}(T_{22}\mathbf{K}_{12} + T_{23}\mathbf{K}_{22}) \end{bmatrix}.$$

Using (5.6), we have

$$M^{-1} + M^{-1}T_{21}\mathbf{P} = M^{-1} + M^{-1}(T_{22} - I) = M^{-1}T_{22},$$

which is in H^∞ by (5.3). Using (5.7), we obtain

$$M^{-1}(T_{22}\mathbf{K}_{11} + T_{23}\mathbf{K}_{21}) = M^{-1}T_{21},$$

which is in H^∞ by (5.2), and

$$M^{-1}(T_{22}\mathbf{K}_{12} + T_{23}\mathbf{K}_{22}) = M^{-1}T_{23},$$

which is in H^∞ by (5.4). This concludes the proof of Assertion 1.

Proof of Assertion 2: We have chosen Q to be the transfer function from $\tilde{w}_p$ to $\tilde{v}_p$ in Figure 7. By Assertion 1, this is in H^∞. We must show that, if we choose $\tilde{\mathbf{K}}$ from (5.1) with this Q, then $\mathbf{T}_{\mathbf{P},\mathbf{K}} = \mathbf{T}_{\mathbf{P},\tilde{\mathbf{K}}}$.

$\tilde{\mathbf{K}}$ from (5.1) can be realized via the system $\mathbf{K}_G$ with outputs $\tilde{u}_k$, $\tilde{w}_k$, $\tilde{\zeta}_o$, inputs $\tilde{y}_k$, $\tilde{v}_k$, $\tilde{\zeta}_i$ and connection $\tilde{v}_k = Q\tilde{w}_k$, as in Figure 8, with

$$\mathbf{K}_G = \begin{bmatrix} 0 & 0 & I \\ \tilde{M} & 0 & -\tilde{N} \\ \tilde{R} & I & I-\tilde{S} \end{bmatrix}.$$

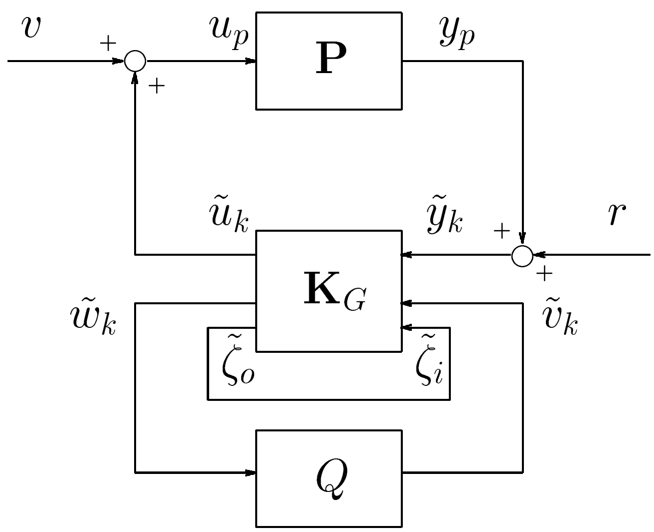

Figure 8: The system in Figure 4, with $\tilde{\mathbf{K}}$ in place of $\mathbf{K}$, after the realization of $\tilde{\mathbf{K}}$ as the feedback interconnection of $\mathbf{K}_G$ with Q (and with $\xi = 0$)

To see this, consider

$$\begin{bmatrix} \tilde{u}_k \\ \tilde{w}_k \\ \tilde{\zeta}_o \end{bmatrix} = \begin{bmatrix} 0 & 0 & I \\ \tilde{M} & 0 & -\tilde{N} \\ \tilde{R} & I & I-\tilde{S} \end{bmatrix} \begin{bmatrix} \tilde{y}_k \\ \tilde{v}_k \\ \tilde{\zeta}_i \end{bmatrix}, \qquad \tilde{v}_k = Q\tilde{w}_k.$$

These yield $\tilde{u}_k = \tilde{\zeta}_i$ and $\tilde{\zeta}_o = (\tilde{R} + Q\tilde{M})\tilde{y}_k + (I - \tilde{S} - Q\tilde{N})\tilde{\zeta}_i$, i.e.,

$$\begin{bmatrix} \tilde{u}_k \\ \tilde{\zeta}_o \end{bmatrix} = \tilde{\mathbf{K}} \begin{bmatrix} \tilde{y}_k \\ \tilde{\zeta}_i \end{bmatrix}.$$

The realization of $\tilde{\mathbf{K}}$ as in Figure 8 is more convenient to show that $(\mathbf{P}, \tilde{\mathbf{K}})$ has the same compensation operator as $(\mathbf{P}, \mathbf{K})$, i.e.,

$$\begin{bmatrix} u_p \\ \tilde{y}_k \end{bmatrix} = \mathbf{T}_{\mathbf{P},\mathbf{K}} \begin{bmatrix} v \\ r \end{bmatrix}.$$

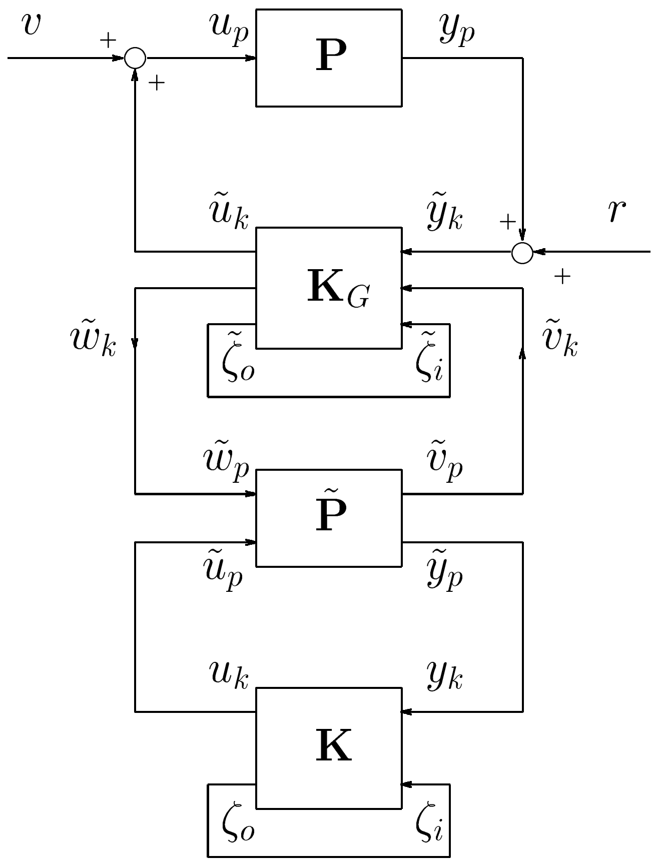

Figure 9: Expanding Q in Figure 8, using that Q is obtained in Figure 7 as the transfer function from $\tilde{w}_p$ to $\tilde{v}_p$

Let us expand Q in Figure 8 into its components to obtain Figure 9.

The idea is now to prove that $y_k = \tilde{y}_k$ and $u_k = \tilde{u}_k$. If this is so, then $y_k = y_p + r$ and $u_p = u_k + v$, and this implies that the transfer function from $\begin{bmatrix} v \\ r \end{bmatrix}$ to $\begin{bmatrix} u_p \\ y_k \end{bmatrix}$ is $\mathbf{T}_{\mathbf{P},\mathbf{K}}$. Since $y_k = \tilde{y}_k$, this is enough to prove our claim. So it remains only to prove that $y_k = \tilde{y}_k$ and $u_k = \tilde{u}_k$ in Figure 9.

First we show that the connection between $\mathbf{K}_G$ and $\tilde{\mathbf{P}}$ in Figure 9 (represented separately in Figure 10) is well-posed, i.e., the output feedback $\bar{F}$ given by

$$\begin{bmatrix} \tilde{y}_k \\ \tilde{v}_k \\ \tilde{\zeta}_i \\ \tilde{w}_p \\ \tilde{u}_p \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{u}_k \\ \tilde{w}_k \\ \tilde{\zeta}_o \\ \tilde{v}_p \\ \tilde{y}_p \end{bmatrix} = \bar{F} \begin{bmatrix} \tilde{u}_k \\ \tilde{w}_k \\ \tilde{\zeta}_o \\ \tilde{v}_p \\ \tilde{y}_p \end{bmatrix}$$

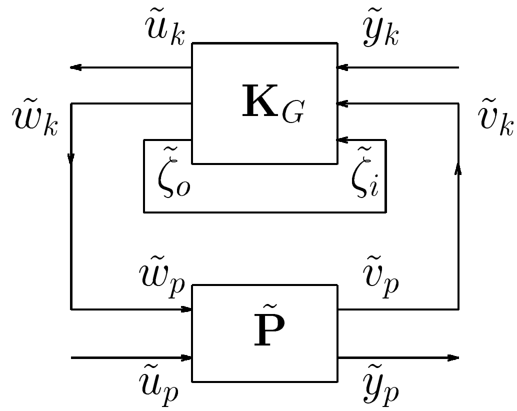

Figure 10: Connection between $\mathbf{K}_G$ and $\tilde{\mathbf{P}}$ in Figure 9

is admissible for the parallel connection of $\mathbf{K}_G$ and $\tilde{\mathbf{P}}$,

$$\bar{\mathbf{G}} = \begin{bmatrix} \mathbf{K}_G & 0 \\ 0 & \tilde{\mathbf{P}} \end{bmatrix}.$$

Straightforward computations using the identity (3.2) yield

$$(I - \bar{\mathbf{G}}\bar{F})^{-1} = \begin{bmatrix} I & -R & M & M & 0 \\ 0 & I + \tilde{N}R & -\tilde{N}M & -\tilde{N}M & 0 \\ 0 & -R & M & M & 0 \\ 0 & -\tilde{S}R & -I + \tilde{S}M & \tilde{S}M & 0 \\ 0 & S & -N & -N & I \end{bmatrix}.$$

So we have well-posedness and, in addition, $(I - \bar{\mathbf{G}}\bar{F})^{-1} \in H^\infty$. Moreover,

$$(I - \bar{\mathbf{G}}\bar{F})^{-1}\bar{\mathbf{G}} = \begin{bmatrix} 0 & * & * & * & I \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ I & * & * & * & 0 \end{bmatrix},$$

in which a star denotes an irrelevant entry. This shows that

$$\begin{bmatrix} \tilde{y}_k \\ \tilde{u}_p \end{bmatrix} = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \begin{bmatrix} \tilde{u}_k \\ \tilde{y}_p \end{bmatrix}$$

and hence $y_k = \tilde{y}_k$ and $u_k = \tilde{u}_k$, as claimed. ■

Example 5.3. It is interesting to calculate the canonical equivalent to a dual canonical controller. Suppose that $\mathbf{K} = \begin{bmatrix} 0 & \mathbf{K}_{12} \\ I & \mathbf{K}_{22} \end{bmatrix}$ stabilizes the plant $\mathbf{P}$ with internal loop. If $\mathbf{P}$ admits a doubly coprime factorization, then the procedure in the proof of Theorem 5.2 produces the canonical controller

$$\tilde{\mathbf{K}} = \begin{bmatrix} 0 & I \\ M^{-1}\mathbf{K}_{12}\tilde{\Delta}^{-1} & I - M^{-1}(I + \mathbf{K}_{12}\tilde{\Delta}^{-1}\mathbf{P}) \end{bmatrix},$$

where $\tilde{\Delta} = I - \mathbf{K}_{22} - \mathbf{P}\mathbf{K}_{12}$, and M comes from the coprime factorization (3.1).

It is clear that the following dual to Theorem 5.2 holds.

Corollary 5.4. *Suppose that* $\mathbf{P}$ *has a doubly coprime factorization. If* $\mathbf{K}$ *is a stabilizing controller with internal loop for* $\mathbf{P}$*, then there exists a dual canonical controller* $\tilde{\mathbf{K}}$ *such that* $\tilde{\mathbf{K}}$ *stabilizes* $\mathbf{P}$ *with internal loop and* $\tilde{\mathbf{K}}$ *is equivalent to* $\mathbf{K}$.

Theorem 5.5. *Suppose that* $\mathbf{P}$ *has a doubly coprime factorization as in* (3.1), (3.2)*, and* $\mathbf{K}$ *is a stabilizing controller with internal loop for* $\mathbf{P}$*. Then there exists a unique* $Q \in H^\infty$ *such that* $\mathbf{K}$ *is equivalent to the canonical controller*

$$\mathbf{K}_l = \begin{bmatrix} 0 & I \\ \tilde{R} + Q\tilde{M} & I - (\tilde{S} + Q\tilde{N}) \end{bmatrix},$$

and is also equivalent to the dual canonical controller

$$\mathbf{K}_r = \begin{bmatrix} 0 & R + MQ \\ I & I - (S + NQ) \end{bmatrix}.$$

The proof is the same as for Theorem 5.2.

6. Conclusions

In this paper, we have examined two concepts of stabilization for the class of well-posed transfer functions. The motivation for choosing this class was that its members have realizations as well-posed linear systems (see Salamon [8], Staffans [12]). We have proposed a new and more general type of controller, controllers with an internal loop. The need for this extension is demonstrated by several examples of such controllers, which cannot be reduced to conventional controllers. We have introduced canonical and dual canonical controllers, which are controllers with internal loop of a special (simple) structure. We have found that these are closely related to (doubly) coprime factorizations of the plant transfer function. For the case that $\mathbf{P}$ has a doubly coprime factorization, we have given a complete parameterization of all stabilizing controllers with internal loop which are (dual) canonical. We have proven that any stabilizing controller with internal loop is equivalent to a canonical one and also to a dual canonical one. Finally, we remark that, although our motivation was to develop a new theory for the well-posed class of irrational transfer functions, the concept of stabilization with internal loop is new even for rational transfer functions.

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