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Renewal theory and level passage by subordinators

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Abstract

Limit theorems in renewal processes generated by infinitely divisible life times easily yield formulas for the limit distributions of the 'undershoot' and 'overshoot' at the passage of a level by subordinators.

1. Introduction and summary

There is a natural relationship between renewal theory and Lévy processes. In fact, Lévy processes were introduced as continuous-time analogues of partial-sum processes, which is what renewal theory is about. Compound Poisson processes can be seen as partial-sum processes 'sampled' at Poisson-process points. Partial- sum processes with infinitely divisible (inf div) summands can be regarded as Lévy processes sampled at equidistant points. In this way a Lévy process can be viewed as the limiting case of such a partial-sum process, where the distance between sample moments has gone to zero.

In this note we use this relationship to 'guess' formulas for the limit distributions of the jump, the 'undershoot' and the 'overshoot' at the passage of a level x by a subordinator as $x \to \infty$.

In Section 2 we give the necessary formulas for subordinators. Section 3 contains some facts from renewal theory, especially for the case where the underlying ('life time') distribution is inf div. This leads to a conjecture for the limiting distributions connected to level passage. In section 4 this conjecture is formulated as a theorem and proved by means of results for Lévy processes. The analogy between these limiting results and those for renewal theory is emphasized.

We shall use the following notation. The distribution function of a random variable X is denoted by F_X. The Laplace-Stieltjes transform of F is denoted by $\hat{F}$, so

$$\hat{F}_X(s) = \int_0^\infty e^{-sx} F(dx).$$

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2. Subordinators

Subordinators are nondecreasing Lévy processes. We assume that the reader is familiar with such processes, and we only give the information that we will need later. For details we refer to Bertoin (1996). A subordinator $X(\cdot)$ can be regarded as a set of inf div random variables:$X(\cdot) = \{X(t) : t \geq 0\}$, with Laplace-Stieltjes transform (canonical representation) of the form

$$\hat{F}_{X(t)}(s) \; = \; \exp\left(-t \int_0^\infty (1 - e^{-sx}) x^{-1} K(dx)\right), \tag{1}$$

where K is a finite measure satisfying

$$\int_1^\infty x^{-1} K(dx) < \infty.$$

We shall sometimes write

$$K(x) \; = \; \mathrm{d} + \int_0^x y \Pi(dy),$$

where $\mathrm{d} = K(0)$ is the drift coefficient. The process $X(\cdot)$ is compound Poisson if and only if $\mathrm{d} = 0$ and Π is bounded, $\Pi([0, \infty)) = \lambda$, say. In that case we can write

$$\hat{F}_{X(t)}(s) = \exp[-\lambda t(1 - \hat{G}(s))],$$

where $G := \Pi/\lambda$ is a distribution function on $(0, \infty)$. We say that $X(\cdot)$ is arithmetic if it is compound Poisson and G is arithmetic; otherwise we say that $X(\cdot)$ is non-arithmetic. If $EX(1) = \mu < \infty$, then

$$\mu \; = \; K(\infty) \; = \; \mathrm{d} + \int_0^\infty y \Pi(dy).$$

From here on we assume that $0 < \mu < \infty$.

3. Renewal theory with inf div life times

Consider the renewal proces generated by X, i.e., let $(X_n)_1^\infty$ be a sequence of nonnegative, independent random variables distributed as X with $F_X = F$ and $0 < \mu := EX < \infty$. Further let $S_0 = 0$ and $S_n = \sum_{k=1}^n X_k$,

$$\begin{aligned}
N_x &= \inf\{n \in I\!\!N : S_n > x\} \\
V_x &= x - S_{N_x - 1}, \\
W_x &= S_{N_x} - x, \\
Z_x &= X_{Nx} = V_x + W_x.
\end{aligned}$$

The following result is well known (see Feller (1971) and Winter (1989) for the concluding statement).

Lemma 1 *Let F be non-arithmetic. Then $Z_x \to^d Z$ and $(V_x, W_x) \to^d (V, W)$ as $x \to \infty$, with*

$$
\begin{aligned}
F_Z(z) &= \mu^{-1} \int_0^z (1 - F(y))dy \\
P(V > v, W > w) &= \mu^{-1} \int_{v+w}^\infty yF(dy),
\end{aligned}
$$

or, equivalently,

$$
(V, W) =^d (UZ, (1 - U)Z),
$$

where U and Z are independent and U is uniformly distributed on $(0, 1)$.

The following property is proved in van Harn and Steutel (1995).

Proposition 1 *Let $X(\cdot)$ be a non-arithmetic subordinator, and consider the renewal process generated by $X(a)$ for some $a > 0$. Denote the analogues of V, W and Z by $V(a), W(a)$, and $Z(a)$. Then (see Section 2),*

$$
Z(a) =^d X(a) + Y,
$$

with $X(a)$ and Y independent and $F_Y = K/\mu$. Further

$$
(V(a), W(a)) =^d (UZ(a), (1 - U)Z(a)),
$$

with U as in Lemma 1.

Corollary 1 *The weak limits $Z(0)$ and $(V(0), W(0))$ of $Z(a)$ and $(V(a), W(a))$ exists as $a \to 0$ and*
$$
Z(0) =^d Y \quad , \quad (V(0), W(0)) =^d (UY, (1 - U)Y)
$$
with U and Y as before. Equivalently,

$$
\begin{aligned}
F_{Z(0)}(z) &= K(z)/\mu = \mu^{-1} \int_0^z y\Pi(dy), \\
P(V(0) > v, W(0) > w) &= \mu^{-1} \int_{v+w}^\infty \Pi((y, \infty))dy.
\end{aligned}
$$

Proof. The first statement is immediate from Proposition 1; the final statement is an easy translation of the second.

4. Limit distributions connected to level passage

Here we show that the limiting random variables $V(0), W(0)$ and $Z(0)$ have the same distribution as the limiting random variables connected with the passage by a subordinator of a level x for $x \to \infty$. This is not trivial, since a direct proof would involve the interchanging of two limit operations: $x \to \infty$ and $a \to 0$ as in Section 3. We choose our notation such

that the analogy between renewal theory and subordinators stands out very clearly. So, let $X(\cdot)$ be a non-arithmetic subordinator as presented in Section 2. Define

$$
\begin{aligned}
T_x &= \inf\{t > 0 : X(t) > x\} \\
V_x &= x - X(T_x-) \\
W_x &= X(T_x) - x \\
Z_x &= V_x + W_x = X(T_x) - X(T_x-).
\end{aligned}
$$

We shall need a result from Bertoin (1996). To this end we need some properties of a 'renewal function' which is slightly different from the function that is used in classical renewal theory:

$$
U(x) = \int_0^\infty F_{X(t)}(x)dt.
$$

We formulate these properties in a lemma.

Lemma 2 *The function U is nondecreasing and has the following properties.*

$$
U(x) \sim x/\mu \quad (x \to \infty); \tag{2}
$$

$$
U(t + h) - U(t) \to h/\mu \quad (t \to \infty) \tag{3}
$$

for all h;

$$
U(t + h) - U(t) \le U(h) \quad (all\ t, h > 0). \tag{4}
$$

Proposition 2 *For $x \ge 0$ and $0 \le y \le x < u$, we have*

$$
P(X(T_x-) \in dy, X(T_x) \in du) = U(dy)\Pi(du - y)
$$

with Π as in Section 2 and U as above.

We are now ready to prove the main result of this note.

Theorem 1 *Let $X(\cdot)$ be a non-arithmetic subordinator and with $0 < \mu := EX(1) < \infty$. Then the weak limits (V_∞, W_∞) and Z_∞ of (V_x, W_x) and Z_x exist as $x \to \infty$, and (cf. Corollary 1)*

$$
Z_\infty =^d Z(0), \ (V_\infty, W_\infty) =^d (V(0), W(0)) =^d (UZ_\infty, (1 - U)Z_\infty)
$$

i.e.,

$$
P(Z_\infty > z) = \mu^{-1} \int_{(z,\infty)} t\Pi(dt), \tag{5}
$$

$$
P(V_\infty \ge v, W_\infty > w) = \mu^{-1} \int_{v+w}^\infty \Pi((t,\infty))dt. \tag{6}
$$

4

Proof. By Proposition 2 we have

$$
\begin{aligned}
& P(V_x \geq v, W_x > w) \\
= {} & \int \int_{0 \leq y \leq x < u, \ y \leq x-v, \ u > x+w} U(dy)\Pi(du - y) \\
= {} & \int \int_{x+w-t < y \leq x-v} U(dy)\Pi(dt) \\
= {} & \int_{v+w}^{x+w} \Pi(dt)\{U(x-v) - U(x+w-t)\} + U(x-v)\Pi((x+w, \inf ty)) \\
= {} & \int_{v+w}^{x+w} \Pi(dt)\{U(x-v) - U(x+w-t)\} + o(1) \quad (x \to \infty),
\end{aligned}
$$

since $U(t) \sim t/\mu$ by (2), and $\int_x^\infty t\Pi(dt) \to 0$ as $x \to \infty$, if $EX < \infty$. By (3) and (4) the integral tends to

$$
\mu^{-1} \int_{v+w}^{\infty} (t - v - w)\Pi(dt) = \mu^{-1} \int_{v+w}^{\in fty} \Pi((t, \infty)),
$$

where the final equality follows on integration by parts. Equation (5) can be proved similarly, or follows from (6).

REMARK 1: For compound Poisson processes these results are trivial; they follow directly from Lemma 1.

REMARK 2: From (6) it follows that V_∞ and W_∞ have absolutely continuous distributions if d = 0, i.e., if $\int t\Pi(dt) = \mu$. Otherwise, $\int t\Pi(dt) = \mu - $ d and, putting $v = 0$ in (6), we obtain $P(V_\infty = 0) = P(W_\infty = 0) = P(Z_\infty = 0) = $ d$/\mu$, as can also be proved directly from from properties of the Lévy process. A similar remark can be made in connection with Corollary 1.

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