Edgeworth expansions with exact cumulants for two-sample linear rank statistics

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Abstract

We show how to compute exact cumulants of two-sample linear rank statistics. In order to approximate tail-probabilities of these statistics, we consider Edgeworth expansions using these exact cumulants instead of asymptotic cumulants. Finally, we use exact tail-probabilities to show that in several cases Edgeworth expansions with exact cumulants provides a significantly better approximation than other existing methods.

Keywords Two-sample linear rank tests, Edgeworth expansions, generating functions, exact cumulants, computer algebra.

1 Introduction

Several authors have considered Edgeworth expansions of two-sample linear rank statistics in order to obtain an approximation formula for tail-probabilities of these statistics under the null-hypothesis. We refer to [1] and [4] in which uniformly valid expansions and conditions for two-sample linear rank statistics were obtained. In [1] certain alternative hypotheses are also discussed. In [2] the authors obtain explicit asymptotic expansions for the Klotz, Van der Waerden and Wilcoxon type statistics, after verifying that the assumptions in [4] are fulfilled. They compare tail-probabilities resulting from the Edgeworth expansion and from the standard normal approximation with exact tail-probabilities. They conclude that the Edgeworth approximations are clearly omly an improvement of the normal approximation for the Klotz test at the 5% level of significance. These authors use approximation formulas for the cumulants. Therefore, we hope to improve the approximations by using the exact cumulants of the test statistics. For one special case, the Wilcoxon statistic, this was done before in [6].
Closed formulas for higher moments and cumulants were derived for this statistic (see [3]) as well as for the Freund-Ansari-Bradley statistic (see [13]). These statistics have one-dimensional probability generating functions and these generating functions are used to find closed formulas. No one-dimensional probability generating function is known for arbitrary two-sample linear rank statistics. However, there exists a two-dimensional generating function for arbitrary two-sample linear rank statistics. We use this to for computing exact cumulants. Because of the required symbolic computations we implemented the method in the computer algebra system Mathematica. We refer to [8] for a survey of computer algebra in statistics.

We compare various methods for computing tail-probabilities (exact, Edgeworth approximation with exact cumulants, Edgeworth approximation with approximated cumulants and standard normal approximation) on the base of accuracy and computing time. This results in recommendations concerning which approximation is suitable for which cases of sample sizes and test statistics.

2 Generating function

Suppose two independent random samples $X_1, \ldots, X_m$ and $Y_1, \ldots, Y_n$ with distribution functions F and G, respectively, are given. We define a general two-sample linear rank statistic:

$$T_{m,n} = \sum_{i=1}^{N} a(i)\, Z_i, \tag{1}$$

where $a : \{1, \ldots, N\} \to \mathbf{R}$ is said to be a rank score function, $N = m + n$, $Z_i = 1$ if the ith score is assigned to an X-observation and $Z_i = 0$ otherwise. Note that we do not exclude the possibility of the presence of ties. Let $H_{T_{m,n}}(x, y)$ be the two-dimensional probability generating function of $T_{m,n}$ with respect to the sample size m and the value of the test statistic t, so

$$H_{T_{m,n}}(x, y) = \sum_{m=1}^{N} \sum_{t} \Pr(T_{m,n} = t)\, x^t y^m.$$

It is known that under H_0

$$H_{T_{m,n}}(x, y) = 1/\binom{N}{m} \prod_{i=1}^{N} (1 + x^{a(i)}\, y). \tag{2}$$

This formula was already known to Euler (see [5]) in the context of generating the frequencies of all outcomes of the sum of m integers that form a subset of N integers. It is rediscovered in [11] where it is derived under the unnecessary assumption that the scores are nonnegative integers. In fact, (2) holds for any set of real scores.

3 Exact cumulants

In this section we show how to compute exact cumulants of $T_{m,n}$. We recall that cumulants are the coefficients in the cumulant generating function which is the logarithm of the moment generating function. We use these cumulants for the Edgeworth expansions in the next section. The method is very suitable for implementation in a computer algebra package like Mathematica, because it involves symbolic computation.

The first step is to compute the factorial moments of $T_{m,n}$. We extend the method used in [3] for computing (factorial) moments with the one-dimensional probability generating function of the Wilcoxon statistic to our two-dimensional case. The coefficient of y^m in $H_{T_{m,n}}(x, y)$, denoted by $H_{T_{m,n}}(x, y)[y^m]$, is obviously the one-dimensional probability generating function of $T_{m,n}$. Elementary properties of ordinary probability generating functions yield

$$E(\prod_{i=0}^{k} T_{m,n} - i) = \Big(\frac{d^k}{dx^k} H_{T_{m,n}}(x, y)[y^m]\Big)\Big|_{x=1} = \Big(\frac{d^k}{dx^k} H_{T_{m,n}}(x, y)\Big|_{x=1}\Big)[y^m]. \tag{3}$$

Since $H_{T_{m,n}}(x, y)$ has a product form it is convenient to use its logarithm for computing derivatives in an efficient way. Therefore, we define

$$L_{T_{m,n}}(x, y) = \log\Big(H_{T_{m,n}}(x, y)\Big). \tag{4}$$

Then,

$$\frac{d}{dx} H_{T_{m,n}}(x, y) = H_{T_{m,n}}(x, y) \frac{d}{dx} L_{T_{m,n}}(x, y). \tag{5}$$

With (5) we express any derivative of $H_{T_{m,n}}(x,y)$ in terms of $H_{T_{m,n}}(x,y)$ and derivatives of $L_{T_{m,n}}(x,y)$ by repeatedly using the product rule for differentation and eliminating lower derivatives of $H_{T_{m,n}}(x,y)$. It is now fairly simple to compute the kth factorial moment: find the first k derivatives of $L_{T_{m,n}}(x,y)$, substitute $x=1$ into these derivatives and into $H_{T_{m,n}}(x,y)$ and substitute the results into the expression for the kth derivative of $H_{T_{m,n}}(x,y)$. Finally, expand the resulting polynomial in y and take the coefficient of y^m which is the desired result.

From the factorial moments we compute the ordinary moments of $T_{m,n}$. Let $S_2(k,\ell), k \geq 0, \ell = 1,\ldots,k$ denote the Stirling numbers of the second kind, then (see [10, p. 44])

$$E(T_{m,n}^k) = \sum_{\ell=1}^{k} S_2(k,\ell) E(\prod_{i=0}^{\ell} T_{m,n} - i). \tag{6}$$

The last step is to get from the moments to the cumulants. Using the fact that the cumulant generating function is the logarithm of the moment generating function we are able to express the kth cumulant as a linear combination of the first k moments as in [7, p. 72].

4 Edgeworth expansions

We use the fourth and sixth order Edgeworth approximation based on the formula in [7, p. 171]. The first one is the most common one. We denote the ith *exact* cumulant by κ_i. Its value depends on the set of rank scores and the sample sizes m and n. The Edgeworth expansion formulas are

$$\begin{aligned} G_{m,n}(x) &= \Phi(y) - \phi(y)\left(\frac{\kappa_3}{6}H_2(y) + \frac{\kappa_4}{24}H_3(y) + \frac{\kappa_3^2}{72}H_5(y)\right) \\ \hat{G}_{m,n}(x) &= \Phi(y) - \phi(y)\left(\frac{\kappa_3}{6}H_2(y) + \frac{\kappa_4}{24}H_3(y) + \frac{\kappa_5}{120}H_4(y) + \frac{10\kappa_3^2 + \kappa_6}{720}H_5(y)\right), \end{aligned} \tag{7}$$

where y is the standardized value of x, Φ is the standard normal c.d.f., ϕ is the standard normal p.d.f. and $H_i(y)$ are the Hermite polynomials which are orthogonal to ϕ, so

$$H_2(y) = y^2 - 1, \quad H_3(y) = y^3 - 3y, \quad H_4(y) = y^4 - 6y^2 + 3, \quad H_5(y) = y^5 - 10y^3 + 15y.$$

5 Criteria for utility

Each method for computing tail-probabilities under the null-hypothesis should be judged on the following two criteria

1. the accuracy of that method
2. the computing time that is needed using that method

The first criterion is of course best satisfied by exact methods (see [9] or [12]), however they are often very time consuming for larger sample sizes. How much computing time is needed depends on the rank scores. If one uses scores that are in general non-rational (e.g. normal scores), computing times increase quickly when the total sample size N grows. For rational scores (e.g. Wilcoxon and Mood scores) the effect is less dramatic. We roughly found that, using a Pentium 200 Mhz PC, the exact methods are practically too time consuming for $N > 30$ in the case of rational scores and for $N > 20$ in the case of non-rational scores.

The computing time criterion is best satisfied by the Edgeworth expansion with approximated cumulants, because this method provides explicit formulas for the cumulants and tail-probabilities. From the previous sections we conclude that the computing time needed by the Edgeworth expansion with exact cumulants is basically determined by the computing time needed to compute

	6	5	4	3	2	1
A	0	0	0	1	1	25
B	0	0	1	2	3	14
C	1	1	1	1	2	8

Table 1: The tie structures used for the Wilcoxon statistic

the factorial moments, because the rest of the computations is handled with explicit formulas. Since Mathematica deals very well with the differentiation of $L_{T_{m,n}}(x, y)$ and the expansion of the expression in y, we found that the Edgeworth expansion with exact cumulants uses less than one minute computing time on a Pentium 200Mhz PC for $N \leq 100$. Therefore, it is useful to compare the two Edgeworth expansion methods on the first criterion.

6 Finite sample achievements

To find out whether the formulas in (7) are an improvement with respect to criterion 1. upon existing approximations we compare the various approximations with the exact probabilities for values of the sample sizes for which exact computations can be executed within reasonable time. We perform the comparisons for four commonly used test statistics: the Van der Waerden statistic, the Mood statistic, the Klotz statistic and the Wilcoxon statistic with tied scores. We refer to [6] for the results on the Wilcoxon statistic with untied scores. We computed exact critical values and tail-probabilities of the test statistics (see [9] or [12]) for significance levels $\alpha = 1\%$ and $\alpha = 5\%$ and for various sample sizes. For these critical values we compute approximate tail-probabilities with the approximation formulas. The absolute value of the difference between the approximated and the exact tail-probability is a measure of the accuracy of the approximation method for the concerning significance level, sample sizes and test statistic. Therefore, we consider the average deviations over the various samples as a measure of correctness of the approximation method for that significance level and test statistic.

For the Van der Waerden, Mood and Klotz test statistics we compared our approximation with the standard normal approximation and with the approximation based on the Edgeworth expansions as in [4] and [2]. For the Wilcoxon statistic with ties we compared our method with the standard normal approximation. The results are in the tables except for the Van der Waerden statistic. For this statistic we found that all approximations based on Edgeworth expansion are very good and they are only slightly better than the standard normal approximation.

One should read the tables as follows. For each pair m and $N = m + n$ the exact critical values c_α are given for the left significance levels $\alpha = 1\%$ and $\alpha = 5\%$ and, below these, for the right significance levels $\alpha = 5\%$ and $\alpha = 1\%$. The critical values are followed by the exact left and right tail-probabilities, the approximations based on the Edgeworth expansion with approximate cumulants ($F_{m,n}$), the approximations based on the fourth order Edgeworth expansion with exact cumulants ($G_{m,n}$), the approximations based on the sixth order Edgeworth expansion with exact cumulants ($\hat{G}_{m,n}$) and the standard normal approximations (Φ). For practical reasons all these tail-probabilities are given in percentages. For the Wilcoxon statistic with ties we dealt with three cases A, B and C as given in Table 1. The integers in the top row are tie sizes, the other integers are the number of ties for the relevant tie structure of the size given in the relevant column.

7 Conclusion

For the Klotz statistic we conclude from Tables 2 and 3 that the approximations based on Edgeworth expansions with exact cumulants are a definite improvement on the two other approxima-

tions for the $\alpha = 1\%$ levels. This true for all considered pairs of sample sizes. The approximation $\hat{G}_{m,n}$ is slightly better than $G_{m,n}$ in these cases. For $\alpha = 5\%$, left, the approximations based on Edgeworth expansions perform approximately equally well and perform better than the standard normal approximation. For $\alpha = 5\%$, right, $G_{m,n}$ is the best on the average and for almost all pairs of sample sizes.
For the Mood statistic we reach the same conclusion for the $\alpha = 1\%$ levels, although the improvement with regard to the existing approximations is less dramatic for the right critical value. For $\alpha = 5\%$ the approximations based on Edgeworth expansions perform approximately equally well and better than the standard normal approximation. The improvement with regard to the standard normal approximation for the Wilcoxon test with ties is small.

In general we observe a significant improvement for the $\alpha = 1\%$ levels when we use $\hat{G}_{m,n}$ or $G_{m,n}$ instead of $F_{m,n}$ or Φ. For the $\alpha = 5\%$ level we found one case in which $\hat{G}_{m,n}$ performs clearly better than the other approximations.

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Tables

Explanation: Section 6

m	N	c_α	exact	$F_{m,n}$	$G_{m,n}$	$\hat{G}_{m,n}$	Φ
6	16	1.11	1.00	0.25	1.55	1.34	2.18
		1.78	4.96	4.95	5.22	4.96	5.56
		7.03	4.85	5.13	4.70	4.69	4.20
		7.99	0.95	0.53	1.04	0.95	0.98
7	16	1.58	0.94	0.20	1.26	1.14	1.70
		2.35	4.88	4.82	4.85	4.68	5.00
		7.76	4.95	5.15	4.78	4.74	4.50
		8.70	0.94	0.36	1.05	0.98	1.14
8	16	2.17	0.98	0.35	1.21	1.12	1.46
		3.02	4.96	5.15	4.91	4.82	4.84
		8.46	4.96	5.00	4.79	4.70	4.73
		9.31	0.98	0.30	1.17	1.08	1.42
6	17	1.11	0.98	0.33	1.67	1.43	2.37
		1.75	4.92	4.69	5.14	4.84	5.56
		7.18	4.94	5.38	4.9	4.91	4.30
		8.27	0.97	0.57	0.98	0.89	0.86
7	17	1.57	0.99	0.28	1.36	1.21	1.86
		2.33	4.93	4.73	4.88	4.68	5.12
		7.98	4.87	5.10	4.72	4.70	4.36
		9.02	0.95	0.39	0.97	0.90	1.00
8	17	2.14	0.99	0.39	1.17	1.18	1.59
		2.97	4.87	4.88	4.79	4.67	4.83
		8.69	4.97	5.16	4.85	4.79	4.69
		9.66	0.97	0.34	1.08	1.00	1.25
6	18	1.07	0.94	0.25	1.64	1.38	2.40
		1.75	4.96	4.73	4.95	5.16	5.76
		7.35	4.99	5.35	4.88	4.91	4.22
		8.51	0.89	0.65	0.96	0.87	0.80
7	18	1.45	0.95	0.01	1.15	1.01	1.72
		2.33	4.91	4.80	5.03	4.81	5.34
		8.16	4.98	5.22	4.80	4.45	4.36
		9.23	0.99	0.58	1.06	0.85	1.02
8	18	2.08	0.98	0.33	1.24	1.14	1.64
		2.96	4.95	4.91	4.90	4.70	5.02
		8.92	4.92	5.11	4.79	4.79	4.54
		9.95	0.97	0.45	1.07	1.03	1.17
9	18	2.67	0.96	0.38	1.14	1.06	1.38
		3.66	4.98	5.23	4.99	4.91	4.92

m	N	c_α	exact	$F_{m,n}$	$G_{m,n}$	$\hat{G}_{m,n}$	Φ
9	18	9.62	4.98	5.13	4.91	4.83	4.85
		10.60	0.96	0.38	1.13	1.06	1.37
6	19	1.07	0.99	0.30	1.73	1.45	2.56
		1.72	4.92	4.47	5.16	4.81	5.73
		7.59	4.94	4.78	4.92	4.80	4.18
		8.73	0.99	0.73	0.95	0.98	0.75
7	19	1.54	0.96	0.34	1.48	1.30	2.11
		2.31	4.94	4.68	5.03	4.78	5.40
		8.34	4.99	5.21	4.80	4.81	4.28
		9.49	0.97	0.64	1.02	0.94	0.93
8	19	2.04	0.96	0.31	1.25	1.14	1.71
		2.93	4.94	4.76	4.87	4.70	5.07
		9.12	4.96	5.16	4.8	4.79	4.48
		10.20	0.98	0.58	1.02	1.03	1.05
9	19	2.63	0.97	0.39	1.17	1.08	1.47
		3.61	4.99	4.98	4.87	4.76	4.90
		9.85	4.97	5.17	4.90	4.84	4.75
		10.90	0.99	0.48	1.09	1.06	1.25
6	20	1.05	0.97	0.27	1.75	1.45	2.63
		1.72	4.87	4.50	5.26	4.90	5.88
		7.64	4.97	5.38	4.94	4.99	4.14
		8.91	0.99	0.83	0.98	0.88	0.73
7	20	1.49	0.93	0.25	1.43	1.24	2.12
		2.30	4.95	4.64	5.08	4.81	5.52
		8.49	4.98	5.32	4.88	4.91	4.29
		9.71	1.00	0.72	1.03	0.95	0.89
8	20	2.00	0.94	0.28	1.25	1.12	1.77
		2.94	5.00	4.91	5.06	4.87	5.32
		9.31	4.97	5.16	4.79	4.79	4.40
		10.50	0.98	0.58	1.02	0.95	1.00
9	20	2.59	1.00	0.38	1.19	1.10	1.55
		3.59	4.98	4.94	4.91	4.78	5.02
		10.10	4.98	4.94	4.90	4.64	4.67
		11.20	0.98	0.53	1.05	1.02	1.15
10	20	3.22	0.99	0.45	1.13	1.06	1.36
		4.30	4.98	5.13	4.93	4.86	4.87
		10.80	4.98	5.07	4.89	4.81	4.83
		11.90	0.99	0.40	1.10	1.03	1.33

Table 2: Klotz, exact and approximated tail-probabilities (in %)

α	$F_{m,n}$	$G_{m,n}$	$\hat{G}_{m,n}$	Φ
L,1.00 %	0.67	0.39	0.24	0.90
L,5.00 %	0.17	0.13	0.14	0.35
R,5.00 %	0.22	0.12	0.28	0.49
R,1.00 %	0.46	0.08	0.05	0.18

Table 3: Klotz, absolute average deviations from exact probabilities

Tables

Explanation: Section 6

m	N	c_α	exact	$F_{m,n}$	$G_{m,n}$	$\hat{G}_{m,n}$	Φ
12	25	354.00	1.00	1.97	0.99	0.98	1.12
		429.00	4.91	4.99	4.97	4.95	4.96
		820.00	4.99	4.96	4.94	4.93	4.88
		896.00	0.99	0.91	0.99	0.98	1.07
10	25	260.50	1.00	0	1.04	1.02	1.26
		329.50	4.92	4.94	4.91	4.87	0.50
		713.50	4.95	4.97	4.95	4.95	4.76
		789.50	0.99	0.90	1.00	0.99	1.01
9	25	215.75	0.98	0	1.04	1.01	1.32
		282.75	4.94	0.50	4.97	4.92	0.52
		658.75	4.97	4.92	4.92	4.92	4.66
		734.75	0.99	0.86	0.98	0.96	0.95
8	25	172.00	0.97	0	1.01	0.98	1.36
		236.00	4.97	4.95	4.91	4.83	0.52
		602.00	4.97	4.93	4.93	4.94	4.61
		677.00	0.98	0.85	0.98	0.96	0.91

m	N	c_α	exact	$F_{m,n}$	$G_{m,n}$	$\hat{G}_{m,n}$	Φ
15	30	695.75	1.00	0.93	0.99	0.98	1.08
		813.75	4.89	4.85	4.83	4.82	4.81
		1433.75	4.89	4.85	4.83	4.82	4.81
		1551.75	1.00	0.93	0.99	0.98	1.08
13	30	554.25	0.98	0	0.99	0.98	1.15
		670.25	4.93	4.97	4.94	4.92	5.00
		1281.25	4.98	4.94	4.92	4.92	4.80
		1403.25	0.99	0.90	0.98	0.97	1.00
11	30	420.75	0.98	0	1.00	0.98	1.24
		528.75	4.89	4.87	4.84	4.80	0.5
		1124.75	4.99	4.93	4.92	4.92	4.70
		1246.75	0.97	0.87	0.96	0.95	0.93
10	30	356.50	0.97	0	0.99	0.96	1.27
		462.50	4.97	4.96	4.92	4.87	0.51
		1044.50	4.97	4.91	4.91	4.91	4.64
		1164.50	0.98	0.86	0.97	0.95	0.90

Table 4: Mood, exact and approximated tail-probabilities (in %)

ties	m	N	c_α	exact	$G_{m,n}$	$\hat{G}_{m,n}$	Φ
A	15	30	176.5	0.95	0.93	0.93	1.01
			192.5	4.98	4.87	4.86	4.85
			272.5	4.98	4.87	4.86	4.85
			288.5	0.95	0.93	0.93	1.01
B	15	30	172	0.97	1.07	1.06	1.15
			186.5	4.88	4.77	4.76	4.75
			266.5	4.88	4.77	4.76	4.75
			282	0.97	0.95	0.94	1.00
C	15	30	166	0.96	0.94	0.94	1.01
			181	4.81	4.69	4.68	4.68
			259	4.81	4.69	4.68	4.68
			274	0.96	0.94	0.94	1.01

ties	m	N	c_α	exact	$G_{m,n}$	$\hat{G}_{m,n}$	Φ
A	10	30	102.5	0.97	0.95	0.95	1.04
			117	4.86	4.74	4.73	4.72
			193	4.87	4.74	4.73	4.72
			207.5	0.97	0.96	0.95	1.04
B	10	30	99	0.97	0.95	0.94	1.07
			113.5	4.94	4.82	4.81	4.85
			189	4.80	4.69	4.68	4.63
			203.5	0.97	0.95	0.94	1.01
C	10	30	96	0.97	0.95	0.94	1.04
			110	4.81	4.72	4.71	4.72
			183.5	4.79	4.68	4.67	4.65
			197.5	0.96	0.95	0.94	1.02

Table 5: Wilcoxon with ties, exact and approximated tail-probabilities (in %)

α	$F_{m,n}$	$G_{m,n}$	$\hat{G}_{m,n}$	Φ
L,1.00 %	0.87	0.03	0.01	0.24
L,5.00 %	0.04	0.04	0.06	0.13
R,5.00 %	0.04	0.05	0.05	0.23
R,1.00 %	0.10	0.01	0.02	0.05

Table 6: Mood, absolute average deviations from exact probabilities

α	$G_{m,n}$	$\hat{G}_{m,n}$	Φ
L,1.00 %	0.03	0.04	0.09
L,5.00 %	0.11	0.12	0.12
R,5.00 %	0.12	0.13	0.14
R,1.00 %	0.02	0.03	0.06

Table 7: Wilcoxon with ties, absolute average deviations from exact probabilities