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by

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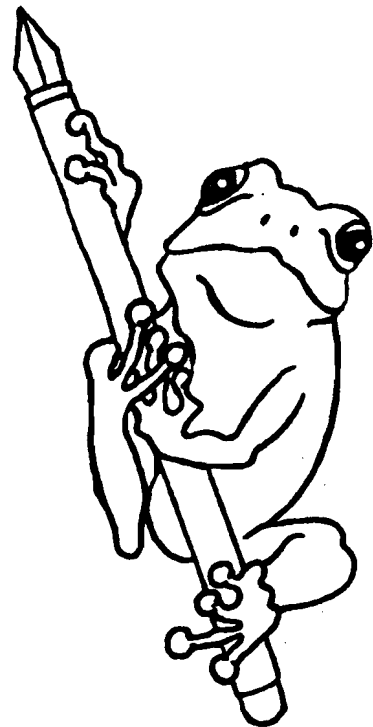
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Abstract

A three-dimensional analytic solution is derived for the fluid flow in a deformable porous medium. It is assumed that the deformation of the medium is described by Hooke's law and the flow of the fluid by Darcy's law, i.e. the theory of poroelasticity applies. The governing equations are completed by suitable boundary conditions such that a compressed saturated cubic sponge is modelled. The solution is a three-dimensional generalisation of the one-dimensional solution of Terzaghi. A two-dimensional plain strain solution is derived also. Both solutions give excellent possibilities to test numerical codes.

Keywords: poroelasticity, porous medium, Darcy, Hooke, Terzaghi, analytic solution.

1 Introduction

The theory of poroelasticity describes fluid flow in a deformable porous medium. The flow of the fluid and the deformation of the solid matrix are coupled. Poroelasticity has applications in soil science, filtration and biology (see references in [1, 2, 3, 4]). Numerical codes have been developed to solve the poroelastic equations for various geometries. Analytic solutions can be used to test the accuracy of these codes. However, the known analytic solutions are one-dimensional or two-dimensional on semi-infinite regions as half-spaces and infinite layers (see references in [1, 2, 3, 4]). In order to test numerical codes two- and three-dimensional analytic solutions on finite domains are needed.

Recently, Barry and Mercer [1, 2, 3, 4] have derived some analytic solutions on finite domains. Unfortunately, these solutions lack immediate applicability to physical poroelastic flows. In this paper a three-dimensional and a corresponding two-dimensional solution are derived that have a closer connection with physical reality. It is assumed that the porous medium is saturated with a Newtonian fluid. The deformation of the medium can be described by the theory of linear elasticity, i.e. Hooke's law applies, and the flow of the fluid by Darcy's law. The set of governing equations is completed with suitably chosen boundary conditions.

In fact, a saturated cubic rubber sponge is modelled. Water is being squeezed out of its pores due to a load applied to it. The boundary conditions are chosen such that the sponge remains cubic during compression. The compression is assumed to be equal to the expelled fluid flux.

The obtained solution is a three-dimensional generalisation of the famous one-dimensional solution of Terzaghi [7, 6, 8]. A two-dimensional plain strain solution is derived also. Both solutions give excellent possibilities to test numerical codes.

2 Governing equations

Mixture theory assumes that the different phases are mixed and exist simultaneously at each point in space. Here only two phases are considered, a single fluid and a solid, respectively denoted by the indices f and s.

The true density of the β-phase, where $\beta = f, s$, is denoted by $\rho_T^\beta = dm^\beta/dV^\beta$, where dm^β is the mass of the β-phase in a small elementary volume dV^β containing only this phase. The porous medium is assumed to be initially homogeneous. Both the fluid as the solid phase are considered intrinsically incompressible. The assumptions imply that ρ_T^β is constant and uniform for both phases. Compression of the medium arises only because of a redistribution of the fluid and solid components.

The apparent density of the β-phase in the mixture is defined as $\rho^\beta = dm^\beta/dV$, where dV is a representative elementary volume [5] of the medium. Let the volume fraction of the β-phase be defined as $\phi^\beta = dV^\beta/dV$, then $\rho^\beta = \phi^\beta \rho_T^\beta$. For a binary porous medium $\phi = \phi^f$ is the porosity and thus $\phi^s = 1 - \phi$. Since the medium is initially homogeneous ϕ is initially uniform.

Conservation of mass for each phase implies

$$\frac{\partial \phi^\beta}{\partial t} + \nabla \cdot \left(\phi^\beta \mathbf{v}^\beta \right) = 0, \ \beta = f, s,$$

where it is assumed that no sources or sinks exist. Here $\mathbf{v}^\beta$ is the average velocity [5] of the β-phase. Addition of these two equations gives

$$\nabla \cdot (\mathbf{q} + \mathbf{v}^s) = 0, \tag{1}$$

where the specific discharge relative to the solid matrix is defined as

$$\mathbf{q} = \phi \left(\mathbf{v}^f - \mathbf{v}^s \right). \tag{2}$$

If no body forces such as gravity are assumed and inertial terms are neglected, the momentum balance for each phase is

$$\nabla \cdot \mathbf{T}^\beta + \boldsymbol{\pi}^\beta = \mathbf{0}, \ \beta = f, s, \tag{3}$$

where $\mathbf{T}^\beta$ is the stress tensor and $\boldsymbol{\pi}^\beta$ is the momentum supply for each phase. It is assumed that the solid matrix is entirely elastic and initially isotropic. The deformations are considered to be small enough such that the infinitesimal elastic assumption is valid. These assumptions lead to the linear stress-strain relation

$$\mathbf{T}^s = -\phi^s p \mathbf{I} + 2\mu \boldsymbol{\epsilon} + \lambda \operatorname{trace}(\boldsymbol{\epsilon}) \mathbf{I}, \tag{4}$$

where p is the fluid pressure, μ and λ are the Lamé stress constants and $\mathbf{I}$ is the identity tensor. The strain tensor is given by

$$\boldsymbol{\epsilon} = \frac{1}{2} \left(\nabla \mathbf{u} + (\nabla \mathbf{u})^T \right), \tag{5}$$

where $\mathbf{u}$ is the displacement vector of the solid matrix. Note that $\operatorname{trace}(\boldsymbol{\epsilon}) = \nabla \cdot \mathbf{u}$. From the infinitesimal elastic assumption it follows that the boundary conditions are applied at the original position of the material.
The fluid phase is assumed to be a Newtonian viscous fluid. This assumption leads to

$$\mathbf{T}^f = -\phi^f p \mathbf{I} + 2\mu^f \left(\mathbf{d} - \frac{1}{3} \operatorname{trace}(\mathbf{d}) \mathbf{I} \right), \tag{6}$$

where μ^f is the viscous stress constant. The rate of strain tensor is given by

$$\mathbf{d} = \frac{1}{2} \left(\nabla \mathbf{v}^f + \left(\nabla \mathbf{v}^f \right)^T \right).$$

It is now assumed that the internal fluid viscosity is negligible compared with the momentum transfer, hence $\mu^f = 0$ is set in (6). With this assumption (3), (4) and (6) lead to

$$\nabla \cdot \boldsymbol{\sigma} = \nabla p, \tag{7}$$

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where the effective stress tensor is defined by

$$\boldsymbol{\sigma} = 2\mu\boldsymbol{\epsilon} + \lambda \operatorname{trace}(\boldsymbol{\epsilon})\, \mathbf{I}. \tag{8}$$

Note that $\operatorname{trace}(\boldsymbol{\sigma}) = (2\mu + 3\lambda)\operatorname{trace}(\boldsymbol{\epsilon})$.

Equation (7) is equivalent to the equilibrium equation of linear elasticity, i.e. Hooke's law applies, with a forcing term given by the gradient of the fluid pressure.

The momentum transfer in (3) is modelled by the linear relation

$$\boldsymbol{\pi}^s = -\boldsymbol{\pi}^f = k\left(\mathbf{v}^f - \mathbf{v}^s\right),$$

where k is assumed to be a constant and uniform scalar. Again, with the assumption that the internal viscosity is negligible compared with the momentum transfer, (2), (3) for $\beta = f$ and (6) lead to

$$\mathbf{q} = -K\nabla p, \tag{9}$$

where the permeability is defined by $K = \left(\phi^f\right)^2 / k$. This equation can be interpreted as Darcy's law relative to the motion of the solid matrix.

The velocity and displacement of the solid phase are related by

$$\mathbf{v}^s = \frac{\partial \mathbf{u}}{\partial t}. \tag{10}$$

While there are many ways of formulating the poroelastic equations, here (1), (5) and (7–10) are used as the governing set. This set consists of 22 coupled equations with the unknowns p, $\mathbf{q}$, $\mathbf{u}$, $\mathbf{v}^s$, $\boldsymbol{\epsilon}$ and $\boldsymbol{\sigma}$.

3 A cubic sponge

Let a rubber sponge being saturated with water. A load applied to it will produce a gradual compression, depending on the rate at which water is being squeezed out of the pores. It is assumed that all assumptions of the previous section are fulfilled.

Here a cubic sponge is considered with edges of length $2L$. For elegance of calculus the origin is placed in the centre of the sponge, and the co-ordinate axes parallel to its edges. Perfectly porous plates push on each side with equal forces. Thus the sponge remains cubic during compression.

In order to model this situation the governing set of equations (1), (5) and (7–10) is supplemented with suitable boundary conditions. For each face $\mathcal{F}$ of the cube, $\boldsymbol{\nu}$ is defined to be the outward unit normal vector and $\boldsymbol{\tau}$ symbolizes the tangential unit vectors. Consider the following boundary conditions:

- $\mathbf{u} \cdot \boldsymbol{\nu}$ is an unknown function of time only,

- $\int_{\mathcal{F}} (\boldsymbol{\nu} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{\nu} - p)\, ds = -F$, where F is a given function of time such that F goes to a constant limit value F_∞ for infinite time,

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- $\boldsymbol{\tau} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{\nu} = 0$, i.e. perfect slip applies,

- $(\mathbf{q} + \mathbf{v}^s) \cdot \boldsymbol{\nu} = 0$, i.e. the compression is equal to the expelled fluid flux.

For infinite time the set of equations reduces to

$$
\begin{aligned}
&\nabla \cdot \mathbf{q} = 0, \\
&\boldsymbol{\epsilon} = \frac{1}{2}\left(\nabla \mathbf{u} + (\nabla \mathbf{u})^T\right), \\
&\nabla \cdot \boldsymbol{\sigma} = \nabla p, \\
&\boldsymbol{\sigma} = 2\mu\boldsymbol{\epsilon} + \lambda \, \mathrm{trace}\,(\boldsymbol{\epsilon}), \\
&\mathbf{q} = -K\nabla p, \\
&\mathbf{v}^s = \mathbf{0}.
\end{aligned}
$$

From the boundary conditions it follows that the solution of the governing set of equations is $p = 0$ (in fact, p is equal to some arbitrary constant value), $\mathbf{q} = \mathbf{0}$ and

$$
\begin{aligned}
\boldsymbol{\sigma} &= -\frac{F_\infty}{4L^2}\mathbf{I}, \\
\boldsymbol{\epsilon} &= \frac{1}{2\mu + 3\lambda}\boldsymbol{\sigma} = -\frac{F_\infty}{(2\mu + 3\lambda)\,4L^2}\mathbf{I}, \\
\mathbf{u} &= -\frac{F_\infty}{(2\mu + 3\lambda)\,4L^2}\mathbf{x}.
\end{aligned}
\tag{11}
$$

Fore finite time the derivation of the solution is more elaborated. Only a suitable choice for the applied force F leads to a derivable solution. Two 'Ansätze' will be made that lead to an analytic solution. It will be checked that this solution fulfils all necessary properties. As the initial condition it is assumed that all unknowns are equal to zero.
The first 'Ansatz' is that the displacement fulfils

$$
\mathbf{u}(\mathbf{x}, t) = \alpha(t)\left(\beta'(x_1), \beta'(x_2), \beta'(x_3)\right)^T,
\tag{12}
$$

where α and β are smooth scalar functions such that $\beta'(-x) = \beta'(x)$ and $\beta'(L) \neq 0$, which are reasonable consequences of the symmetry and boundary conditions. From (12) it follows that $\mathbf{u} \cdot \boldsymbol{\nu}$ is an unknown function of time only at each boundary face. It follows directly from (10) that

$$
\mathbf{v}^s(\mathbf{x}, t) = \alpha'(t)\left(\beta'(x_1), \beta'(x_2), \beta'(x_3)\right)^T.
$$

Now the specific discharge needs to be chosen such that (1) holds. The second 'Ansatz' is

$$
\mathbf{q}(\mathbf{x}, t) = -\alpha'(t)\left(\beta'(x_1), \beta'(x_2), \beta'(x_3)\right)^T.
$$

From (9) it follows that

$$
p(\mathbf{x}, t) = \frac{1}{K}\alpha'(t)\left(\beta(x_1) + \beta(x_2) + \beta(x_3)\right).
$$

The strain tensor is given by (5), i.e.

$$\boldsymbol{\epsilon}(\mathbf{x}, t) = \alpha(t) \operatorname{diag} \left(\beta''(x_1), \beta''(x_2), \beta''(x_3) \right),$$

and the effective stress tensor by (8), i.e.

$$\boldsymbol{\sigma}(\mathbf{x}, t) = 2\mu\alpha(t) \operatorname{diag} \left(\beta''(x_1), \beta''(x_2), \beta''(x_3) \right) + \lambda\alpha(t) \left(\beta''(x_1) + \beta''(x_2) + \beta''(x_3) \right) \mathbf{I}.$$

The equilibrium equation (7) leads to

$$(2\mu + \lambda) \, \alpha(t)\beta'''(x) - \frac{1}{K}\alpha'(t)\beta'(x) = 0.$$

If α' and β''' are not equal to zero, this equation results into

$$(2\mu + \lambda) \, K \frac{\alpha(t)}{\alpha'(t)} = \frac{\beta'(x)}{\beta'''(x)}. \tag{13}$$

Define the constant C by $(2\mu + \lambda) \, K = L^2/C$, then the dimension of C is time. The following collection of solutions is obtained from (13) and the conditions for β':

$$\left.\begin{aligned} \alpha_k(t) &= \exp\left(-\frac{(2k-1)^2 \pi^2 t}{4C} \right) \\ \beta_k'(x) &= L \sin\left(\frac{(2k-1)\pi x}{2L} \right) \end{aligned}\right\} \, k = 1, 2, \ldots$$

However, any summation $\sum_{k=1}^{\infty} \gamma_k \alpha_k(t)\beta_k'(x)$ goes to zero for infinite time which conflicts with (11). Therefore a particular solution need to be added, i.e.

$$\begin{aligned} \alpha_0(t) &= -\frac{F_\infty}{(2\mu + 3\lambda) \, 4L^2}, \\ \beta_0'(x) &= x. \end{aligned}$$

3.1   Displacement

It is stated that instead of 'Ansatz' (12) the displacement is given by

$$u_i(\mathbf{x}, t) = -\frac{F_\infty}{(2\mu + 3\lambda) \, 4L^2}x_i + L \sum_{k=1}^{\infty} \gamma_k \sin\left(\frac{(2k-1)\pi x_i}{2L} \right) \exp\left(-\frac{(2k-1)^2 \pi^2 t}{4C} \right).$$

The initial condition leads to

$$0 = u_i(\mathbf{x}, 0) = -\frac{F_\infty}{(2\mu + 3\lambda) \, 4L^2}x_i + L \sum_{k=1}^{\infty} \gamma_k \sin\left(\frac{(2k-1)\pi x_i}{2L} \right). \tag{14}$$

6

Integration from 0 to L gives

$$-\frac{F_\infty}{(2\mu + 3\lambda)\, 4L^2}\int_0^L x\sin\left(\frac{(2k-1)\pi x}{2L}\right)\,dx + \frac{L^2}{2}\gamma_k = 0.$$

Partial integration results into

$$\gamma_k = -\frac{F_\infty}{(2\mu + 3\lambda)\, 4L^2}\frac{8}{(2k-1)^2\pi^2}\,(-1)^k\,.$$

It follows that

$$\mathbf{u}(\mathbf{x},t) \to -\frac{F_\infty}{(2\mu + 3\lambda)\, 4L^2}\mathbf{x}\ \text{for } t\to\infty.$$

If $x_i = L$, then

$$u_i = -\frac{F_\infty}{(2\mu + 3\lambda)\, 4L^2}\left(1 - \sum_{k=1}^{\infty}\frac{8}{(2k-1)^2\pi^2}\exp\left(-\frac{(2k-1)^2\pi^2 t}{4C}\right)\right).$$

Indeed, if also $t = 0$, then

$$u_i = -\frac{F_\infty}{(2\mu + 3\lambda)\, 4L^2}\left(1 - \sum_{k=1}^{\infty}\frac{8}{(2k-1)^2\pi^2}\right) = 0.$$

3.2 Velocity of the solid phase

It follows directly from (10) that

$$v_i^s(\mathbf{x},t) = -L\sum_{k=1}^{\infty}\gamma_k\frac{(2k-1)^2\pi^2}{4C}\sin\left(\frac{(2k-1)\pi x_i}{2L}\right)\exp\left(-\frac{(2k-1)^2\pi^2 t}{4C}\right).$$

Double differentiation of (14) leads to the conclusion that

$$v_i^s(\mathbf{x},0) = -L\sum_{k=1}^{\infty}\gamma_k\frac{(2k-1)^2\pi^2}{4C}\sin\left(\frac{(2k-1)\pi x_i}{2L}\right) = 0\ \text{if}\ -L < x_i < L.$$

It follows that $\mathbf{v}^s(\mathbf{x},t) \to \mathbf{0}$ for $t\to\infty$. If $x_i = L$, then

$$v_i^s = -\frac{F_\infty}{(2\mu + 3\lambda)\, 2LC}\sum_{k=1}^{\infty}\exp\left(-\frac{(2k-1)^2\pi^2 t}{4C}\right).$$

If also $t = 0$, then $v_i^s = -\infty$, i.e. the initial solid phase speed at the boundary is infinitely large.

7

3.3 Specific discharge

It is assumed that $\mathbf{q}(\mathbf{x}, t) = -\mathbf{v}^s(\mathbf{x}, t)$. The total outward flux is equal to

$$Q(t) = \frac{12 F_\infty L}{(2\mu + 3\lambda) C} \sum_{k=1}^{\infty} \exp\left(-\frac{(2k-1)^2 \pi^2 t}{4C}\right),$$

and therefore the expelled fluid volume is given by

$$\int_0^{\infty} Q(t) \, dt = \frac{6 F_\infty L}{2\mu + 3\lambda} \sum_{k=1}^{\infty} \frac{8}{(2k-1)^2 \pi^2} = \frac{6 F_\infty L}{2\mu + 3\lambda}.$$

This result agrees with the final displacement in (11), because the expelled fluid volume has to be equal to the volume reduction of the sponge.

3.4 Fluid pressure

From (9) it follows that

$$p(\mathbf{x}, t) = (2\mu + \lambda) \sum_{k=1}^{\infty} \gamma_k \frac{(2k-1)\pi}{2} \sum_{i=1}^{3} \cos\left(\frac{(2k-1)\pi x_i}{2L}\right) \exp\left(-\frac{(2k-1)^2 \pi^2 t}{4C}\right).$$

Differentiation of (14) leads to the conclusion that

$$p(\mathbf{x}, 0) = (2\mu + \lambda) \sum_{k=1}^{\infty} \gamma_k \frac{(2k-1)\pi}{2} \sum_{i=1}^{3} \cos\left(\frac{(2k-1)\pi x_i}{2L}\right) = \frac{2\mu + \lambda}{2\mu + 3\lambda} \frac{3 F_\infty}{4L^2}.$$

It follows that $p(\mathbf{x}, t) \to 0$ for $t \to \infty$. Furthermore,

$$\begin{aligned}
p(0, t) &= 3 (2\mu + \lambda) \sum_{k=1}^{\infty} \gamma_k \frac{(2k-1)\pi}{2} \exp\left(-\frac{(2k-1)^2 \pi^2 t}{4C}\right) \\
&= -\frac{2\mu + \lambda}{2\mu + 3\lambda} \frac{3 F_\infty}{4L^2} \sum_{k=1}^{\infty} \frac{4}{(2k-1)\pi} (-1)^k \exp\left(-\frac{(2k-1)^2 \pi^2 t}{4C}\right).
\end{aligned}$$

In particular,

$$p(0, 0) = -\frac{2\mu + \lambda}{2\mu + 3\lambda} \frac{3F}{4L^2} \sum_{k=1}^{\infty} \frac{4}{(2k-1)\pi} (-1)^k = \frac{2\mu + \lambda}{2\mu + 3\lambda} \frac{3 F_\infty}{4L^2}.$$

3.5 Strain tensor

The strain tensor is given by (5), i.e.

$$
\begin{aligned}
\epsilon(\mathbf{x},t) \;=\; & -\frac{F_\infty}{(2\mu+3\lambda)\,4L^2}\mathbf{I} + \sum_{k=1}^{\infty} \gamma_k \frac{(2k-1)\,\pi}{2}\exp\left(-\frac{(2k-1)^2\,\pi^2 t}{4C}\right) \\
& \times \;\operatorname{diag}\left(\cos\left(\frac{(2k-1)\,\pi x_1}{2L}\right),\cos\left(\frac{(2k-1)\,\pi x_2}{2L}\right),\cos\left(\frac{(2k-1)\,\pi x_3}{2L}\right)\right).
\end{aligned}
$$

Differentiation of (14) leads to the conclusion that $\epsilon(\mathbf{x},0)=\mathbf{0}$. It follows that

$$
\epsilon(\mathbf{x},t) \rightarrow -\frac{F_\infty}{(2\mu+3\lambda)\,4L^2}\mathbf{I} \ \text{for}\ t\rightarrow\infty.
$$

Furthermore,

$$
\epsilon(\mathbf{0},t) = -\frac{F_\infty}{(2\mu+3\lambda)\,4L^2}\left(1+\sum_{k=1}^{\infty}\frac{4}{(2k-1)\,\pi}(-1)^k\exp\left(-\frac{(2k-1)^2\,\pi^2 t}{4C}\right)\right)\mathbf{I}.
$$

In particular,

$$
\epsilon(\mathbf{0},0) = -\frac{F_\infty}{(2\mu+3\lambda)\,4L^2}\left(1+\sum_{k=1}^{\infty}\frac{4}{(2k-1)\,\pi}(-1)^k\right)\mathbf{I} = \mathbf{0}.
$$

If $x_i = L$, then

$$
\epsilon_{ii} = \frac{F_\infty}{(2\mu+3\lambda)\,4L^2}.
$$

3.6 Effective stress tensor

The effective stress tensor is given by (8), i.e.

$$
\begin{aligned}
\sigma(\mathbf{x},t) \;=\; & -\frac{F_\infty}{4L^2}\mathbf{I} + 2\mu\sum_{k=1}^{\infty}\gamma_k\frac{(2k-1)\,\pi}{2}\exp\left(-\frac{(2k-1)^2\,\pi^2 t}{4C}\right) \\
& \times \;\operatorname{diag}\left(\cos\left(\frac{(2k-1)\,\pi x_1}{2L}\right),\cos\left(\frac{(2k-1)\,\pi x_2}{2L}\right),\cos\left(\frac{(2k-1)\,\pi x_3}{2L}\right)\right) \\
& + \;\lambda\sum_{k=1}^{\infty}\gamma_k\frac{(2k-1)\,\pi}{2}\sum_{i=1}^{3}\cos\left(\frac{(2k-1)\,\pi x_i}{2L}\right)\exp\left(-\frac{(2k-1)^2\,\pi^2 t}{4C}\right)\mathbf{I}.
\end{aligned}
$$

Differentiation of (14) leads to the conclusion that $\sigma(\mathbf{x},0)=\mathbf{0}$. It follows that

$$
\sigma(\mathbf{x},t) \rightarrow -\frac{F_\infty}{4L^2}\mathbf{I} \ \text{for}\ t\rightarrow\infty.
$$

3.7 Force

The force on each face of the cubic sponge is given by

$$
\begin{aligned}
F(t) &= F_\infty + 8\mu L \sum_{k=1}^{\infty} \gamma_k \left(2k-1\right) \pi \exp\left(-\frac{\left(2k-1\right)^2 \pi^2 t}{4C}\right) \int_0^L \cos\left(\frac{\left(2k-1\right)\pi x}{2L}\right) dx \\
&= F_\infty \left(1 + \frac{4\mu}{2\mu + 3\lambda} \sum_{k=1}^{\infty} \frac{8}{\left(2k-1\right)^2 \pi^2} \exp\left(-\frac{\left(2k-1\right)^2 \pi^2 t}{4C}\right)\right).
\end{aligned}
$$

It follows that

$$
F(0) = \frac{2\mu + \lambda}{2\mu + 3\lambda} 3 F_\infty,
$$

and $F(t) \to F_\infty$ for $t \to \infty$.

4 Plain strain

The three-dimensional analytic solution reduces to an essentially two-dimensional solution if $u_3 = 0$ and $q_3 = 0$. This implies $\epsilon_{3i} = \epsilon_{i3} = 0$, i.e. plain strain is dealt with. The essentially two-dimensional solution is obtained if the boundary conditions of the previous section hold only at the vertical faces of the cubic sponge, i.e. if $x_1 = \pm L$ or $x_2 = \pm L$. At the horizontal faces of the sponge, i.e. if $x_3 = \pm L$, homogeneous boundary conditions are supposed, i.e. $u_3 = 0$ and $q_3 = 0$. This means that the sponge is clamped by impervious horizontal plates.

For infinite time the solution of the governing set of equations (1), (5) and (7–10) is $p = 0$, $\mathbf{q} = \mathbf{0}$ and

$$
\begin{aligned}
\boldsymbol{\sigma} &= -\frac{F_\infty}{4L^2} \operatorname{diag}\left(1, 1, \frac{\lambda}{\mu + \lambda}\right), \\
\boldsymbol{\epsilon} &= -\frac{F_\infty}{\left(\mu + \lambda\right) 8L^2} \operatorname{diag}\left(1, 1, 0\right), \\
\mathbf{u} &= -\frac{F_\infty}{\left(\mu + \lambda\right) 8L^2} \left(x_1, x_2, 0\right)^T.
\end{aligned}
$$

For finite time the displacement is given by

$$
u_i(\mathbf{x}, t) = -\frac{F_\infty}{\left(\mu + \lambda\right) 8L^2} x_i + L \sum_{k=1}^{\infty} \gamma_k \sin\left(\frac{\left(2k-1\right)\pi x_i}{2L}\right) \exp\left(-\frac{\left(2k-1\right)^2 \pi^2 t}{4C}\right), \quad i = 1, 2,
$$

where

$$
\gamma_k = -\frac{F_\infty}{\left(\mu + \lambda\right) 8L^2} \frac{8}{\left(2k-1\right)^2 \pi^2} \left(-1\right)^k.
$$

It follows directly from (10) that

$$v_i^s(\mathbf{x}, t) = -L \sum_{k=1}^{\infty} \gamma_k \frac{(2k-1)^2 \pi^2}{4C} \sin\left(\frac{(2k-1)\pi x_i}{2L}\right) \exp\left(-\frac{(2k-1)^2 \pi^2 t}{4C}\right), \quad i = 1, 2.$$

It is assumed that $\mathbf{q}(\mathbf{x}, t) = -\mathbf{v}^s(\mathbf{x}, t)$. The total outward flux is equal to

$$Q(t) = \frac{4F_\infty L}{(\mu + \lambda) C} \sum_{k=1}^{\infty} \exp\left(-\frac{(2k-1)^2 \pi^2 t}{4C}\right),$$

and therefore the expelled fluid volume is given by

$$\int_0^\infty Q(t)\, dt = \frac{2F_\infty L}{\mu + \lambda}.$$

From (9) it follows that

$$p(\mathbf{x}, t) = (2\mu + \lambda) \sum_{k=1}^{\infty} \gamma_k \frac{(2k-1)\pi}{2} \sum_{i=1}^{2} \cos\left(\frac{(2k-1)\pi x_i}{2L}\right) \exp\left(-\frac{(2k-1)^2 \pi^2 t}{4C}\right),$$

and therefore

$$p(\mathbf{x}, 0) = \frac{2\mu + \lambda}{\mu + \lambda} \frac{F_\infty}{4L^2}.$$

The strain tensor is given by (5), i.e.

$$
\begin{aligned}
\boldsymbol{\epsilon}(\mathbf{x}, t) &= -\frac{F_\infty}{(\mu + \lambda) 8L^2} \operatorname{diag}(1, 1, 0) + \sum_{k=1}^{\infty} \gamma_k \frac{(2k-1)\pi}{2} \exp\left(-\frac{(2k-1)^2 \pi^2 t}{4C}\right) \\
&\quad \times \operatorname{diag}\left(\cos\left(\frac{(2k-1)\pi x_1}{2L}\right), \cos\left(\frac{(2k-1)\pi x_2}{2L}\right), 0\right).
\end{aligned}
$$

The effective stress tensor is given by (8), i.e.

$$
\begin{aligned}
\boldsymbol{\sigma}(\mathbf{x}, t) &= -\frac{F_\infty}{4L^2} \operatorname{diag}\left(1, 1, \frac{\lambda}{\mu + \lambda}\right) + 2\mu \sum_{k=1}^{\infty} \gamma_k \frac{(2k-1)\pi}{2} \exp\left(-\frac{(2k-1)^2 \pi^2 t}{4C}\right) \\
&\quad \times \operatorname{diag}\left(\cos\left(\frac{(2k-1)\pi x_1}{2L}\right), \cos\left(\frac{(2k-1)\pi x_2}{2L}\right), 0\right) \\
&\quad + \lambda \sum_{k=1}^{\infty} \gamma_k \frac{(2k-1)\pi}{2} \sum_{i=1}^{2} \cos\left(\frac{(2k-1)\pi x_i}{2L}\right) \exp\left(-\frac{(2k-1)^2 \pi^2 t}{4C}\right) \mathbf{I}.
\end{aligned}
$$

The force on each vertical face of the cubic sponge is given by

$$F_v(t) = F_\infty \left(1 + \frac{\mu}{\mu + \lambda} \sum_{k=1}^{\infty} \frac{8}{(2k-1)^2 \pi^2} \exp\left(-\frac{(2k-1)^2 \pi^2 t}{4C}\right)\right).$$

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It follows that

$$F_v(0) = \frac{2\mu + \lambda}{\mu + \lambda} F_\infty.$$

The force on each horizontal face is given by

$$F_h(t) = \frac{\lambda}{\mu + \lambda} F_\infty \left(1 + \frac{2\mu}{\lambda} \sum_{k=1}^\infty \frac{8}{(2k-1)^2 \pi^2} \exp\left(-\frac{(2k-1)^2 \pi^2 t}{4C} \right) \right).$$

It follows that $F_h(0) = F_v(0)$.
Note that the plain strain problem is commonly described in terms of the set of equations (1), (5) and (7–10) in the two-dimensional space.

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